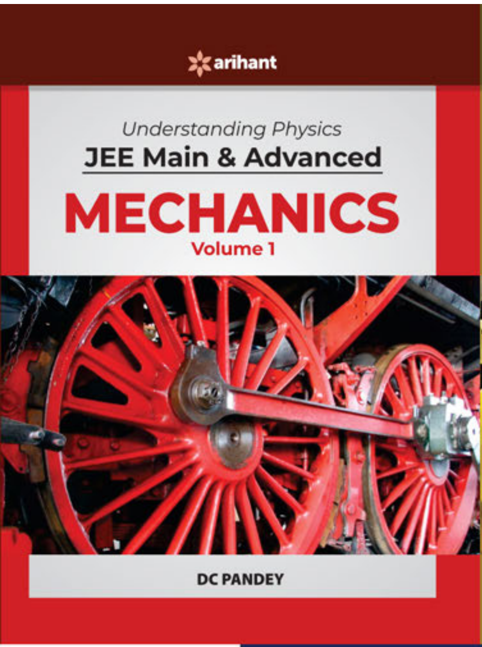

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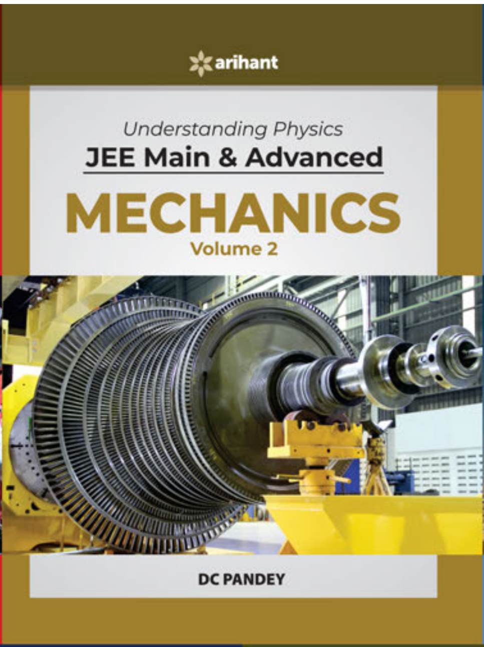

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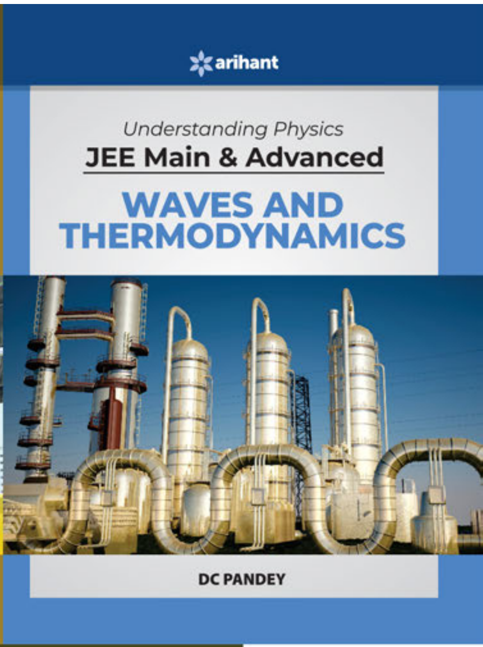

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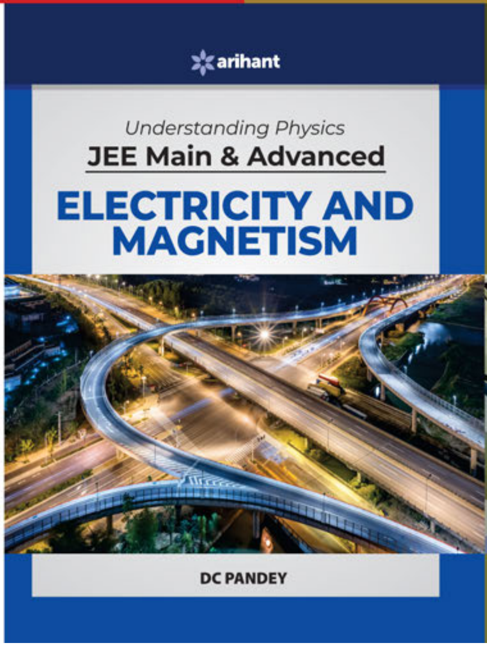

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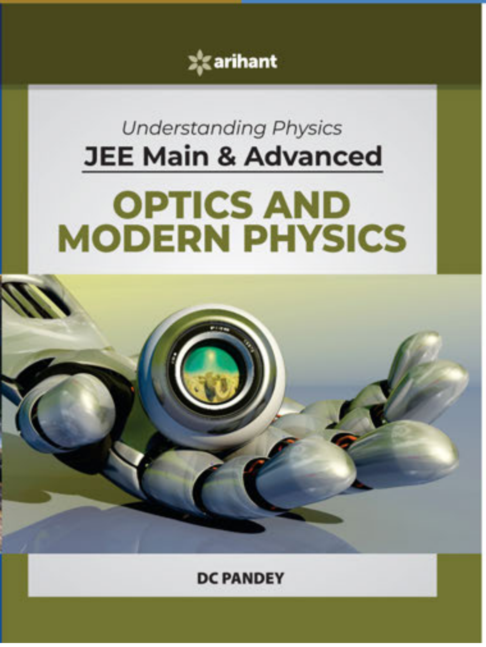

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced.
The NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions.

I am very thankful to (Dr.) Mrs. Sarita Pandey, Mr. Anoop Dhyani and Mr. Nisar Ahmad

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

Physics and Measurement

Physics, Technology and Society, SI units, Fundamental and derived units, Least count, Accuracy and Precision of measuring instruments, Errors in measurement, Dimensions of physical quantities, Dimensional analysis and its applications.

Kinematics

Frame of reference, Motion in a straight line, Position-time graph, Speed and velocity, Uniform and non-uniform motion, Average speed and instantaneous velocity, Uniformly accelerated motion, Velocity-time and position-time graphs, Relations for uniformly accelerated motion, Scalars and Vectors, vectors addition and subtraction, Zero vector, scalar and vector products, Unit vector, resolution of a vector, Relative velocity, motion in plane, Projectile motion, Uniform circular motion.

Laws of Motion

Force and inertia, Newton's first law of motion, Momentum, Newton's second law of motion, impulse, Newton's third Law of motion, law of conservation of linear momentum and its applications, Equilibrium of concurrent forces. Static and kinetic friction, Laws of friction, rolling friction. Dynamics of uniform circular motion, centripetal force and its applications.

Work, Energy and Power

Work done by a constant force and a variable force, Kinetic and potential energies, Work energy theorem, power. Potential energy of a spring, Conservation of mechanical energy, Conservative and non-conservative forces, Elastic and inelastic collisions in one and two dimensions.

Centre of Mass

Centre of mass of a two particle system, Centre of mass of a rigid body.

Experimental Skills

Vernier Callipers and its use to measure internal and external diameter and depth of a vessel. Screw gauge its use to determine thickness/diameter of thin sheet/wire.

JEE Advanced

General Physics

Units and dimensions, Dimensional analysis, Least count, Significant figures, Methods of measurement and error analysis for physical quantities pertaining to the following experiments, Experiments based on vernier callipers and screw gauge (micrometer).

Kinematics

Kinematics in one and two dimensions (Cartesian coordinates only), Projectiles, Uniform circular motion, Relative velocity.

Laws of Motion

Newton's laws of motion, Inertial and uniformly accelerated frames of reference, Static and dynamic friction.

Work, Energy and Power

Kinetic and potential energy, Work and power, Conservation of linear momentum and mechanical energy.

Centre of Mass and Collision

System of particles, Centre of mass and its motion, Impulse, Elastic and inelastic collisions.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

01

Basic Mathematics

Chapter Contents

1.1 Basic Mathematics

1.1 Basic Mathematics

The following formulae are frequently used in Physics. So, the students who have just gone in class XI are advised to remember them first.

Logarithms

(i) $e \approx 2.7183$

(ii) If $e^x = y$, then $x = \log_e y = \ln y$

(iii) If $10^x = y$, then $x = \log_{10} y$

(iv) $\log_{10} y = 0.4343 \log_e y = 0.4343 \ln y$

(v) $\log (ab) = \log (a) + \log (b)$

(vi) $\log \left(\frac{a}{b}\right) = \log (a) - \log (b)$

(vii) $\log a^n = n \log (a)$

Trigonometry

(i) $\sin^2 \theta + \cos^2 \theta = 1$

(ii) $1 + \tan^2 \theta = \sec^2 \theta$

(iii) $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

(iv) $\sin 2\theta = 2 \sin \theta \cos \theta$

(v) $\cos 2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta = \cos^2 \theta - \sin^2 \theta$

(vi) $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

(vii) $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

(viii) $\sin C + \sin D = 2 \sin \left(\frac{C+D}{2}\right) \cos \left(\frac{C-D}{2}\right)$

(ix) $\sin C - \sin D = 2 \sin \left(\frac{C-D}{2}\right) \cos \left(\frac{C+D}{2}\right)$

(x) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(xi) $\cos C - \cos D = 2 \sin \frac{D-C}{2} \sin \frac{C+D}{2}$

(xii) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

(xiii) $\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(xiv) $\sin (90° + \theta) = \cos \theta$

(xv) $\cos (90° + \theta) = -\sin \theta$

(xvi) $\tan (90° + \theta) = -\cot \theta$

(xvii) $\sin (90° - \theta) = \cos \theta$ (xviii) $\cos (90° - \theta) = \sin \theta$

(xix) $\tan (90° - \theta) = \cot \theta$

(xx) $\sin (180° - \theta) = \sin \theta$

(xxi) $\cos(180^\circ - \theta) = -\cos\theta$

(xxii) $\tan(180^\circ - \theta) = -\tan\theta$

(xxiii) $\sin(180^\circ + \theta) = -\sin\theta$

(xxiv) $\cos(180^\circ + \theta) = -\cos\theta$

(xxv) $\tan(180^\circ + \theta) = \tan\theta$

(xxvi) $\sin(-\theta) = -\sin\theta$

(xxvii) $\cos(-\theta) = \cos\theta$

(xxviii) $\tan(-\theta) = -\tan\theta$

Differentiation

(i) $\dfrac{d}{dx}(\text{constant}) = 0$

(ii) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\dfrac{d}{dx}(\log_e x)$ or $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(iv) $\dfrac{d}{dx}(\sin x) = \cos x$

(v) $\dfrac{d}{dx}(\cos x) = -\sin x$

(vi) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(vii) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(viii) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(ix) $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(x) $\dfrac{d}{dx}(e^x) = e^x$

(xi) $\dfrac{d}{dx}\{f_1(x) \,.\, f_2(x)\} = f_1(x)\dfrac{d}{dx}f_2(x) + f_2(x)\dfrac{d}{dx}f_1(x)$

(xii) $\dfrac{d}{dx}\dfrac{f_1(x)}{f_2(x)} = \dfrac{f_2(x)\dfrac{d}{dx}f_1(x) - f_1(x)\dfrac{d}{dx}f_2(x)}{\{f_2(x)\}^2}$

(xiii) $\dfrac{d}{dx}f(ax+b) = a\dfrac{d}{dx}f(X)$, where $X = ax + b$

Integration

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c\,(n \neq -1)$

(ii) $\int \dfrac{dx}{x} = \log_e x + c$ or $\ln x + c$

(iii) $\int \sin x\,dx = -\cos x + c$

(iv) $\int \cos x\,dx = \sin x + c$

(v) $\int e^x dx = e^x + c$

(vi) $\int \sec^2 x\,dx = \tan x + c$

(vii) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(viii) $\int \sec x \tan x\,dx = \sec x + c$

(ix) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(x) $\int f(ax+b)\,dx = \dfrac{1}{a}\int f(X)\,dX$, where $X = ax + b$

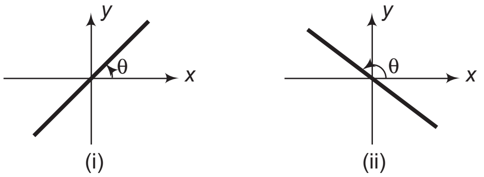
Here, c is constant of integration.

Graphs

Following graphs and their corresponding equations are frequently used in Physics.

(i) $y = mx$, represents a straight line passing through origin. Here, $m = \tan\theta$ is also called the slope of line, where θ is the angle which the line makes with positive x-axis, when drawn in anticlockwise direction from the positive x-axis towards the line.

The two possible cases are shown in Fig. 1.1. In Fig. 1.1 (i), $\theta < 90°$. Therefore, $\tan\theta$ or slope of line is positive. In Fig. 1.1 (ii), $90° < \theta < 180°$. Therefore, $\tan\theta$ or slope of line is negative.

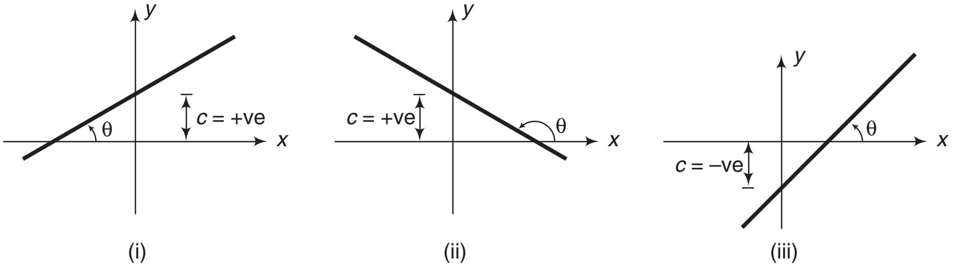


(i) (ii)

Fig. 1.1

Note *That $y = mx$ or $y \propto x$ also means that value of y becomes 2 times if x is doubled. Or it will remain $\frac{1}{4}$ th if x becomes $\frac{1}{4}$ times.*

(ii) $y = mx + c$, represents a straight line not passing through origin. Here, m is the slope of line as discussed above and c the intercept on y-axis.



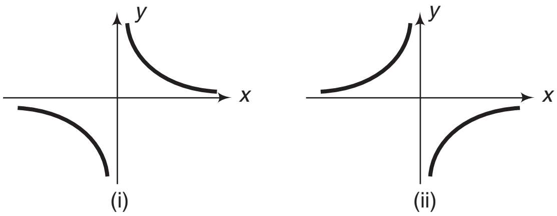
(i) (ii) (iii)

Fig. 1.2

In figure (i) : slope and intercept both are positive.
In figure (ii) : slope is negative but intercept is positive and
In figure (iii) : slope is positive but intercept is negative.

Note *That in $y = mx + c$, y does not become two times if x is doubled.*

(iii) $y \propto \frac{1}{x}$ or $y = \frac{2}{x}$ etc., represents a rectangular hyperbola in first and third quadrants. The shape of rectangular hyperbola is shown in Fig. 1.3(i).



(i) (ii)

Fig. 1.3

From the graph we can see that $y \to 0$ as $x \to \infty$ or $x \to 0$ as $y \to \infty$.

Similarly, $y = -\dfrac{4}{x}$ represents a rectangular hyperbola in second and fourth quadrants as shown in Fig. 1.3(ii).

Note *That in case of rectangular hyperbola if x is doubled y will become half.*

(iv) $y \propto x^2$ or $y = 2x^2$, etc., represents a parabola passing through origin as shown in Fig. 1.4(i).

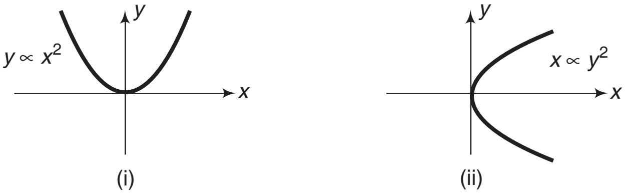


Fig. 1.4

Note *That in the parabola $y = 2x^2$ or $y \propto x^2$, if x is doubled, y will become four times.*

Graph $x \propto y^2$ or $x = 4y^2$ is again a parabola passing through origin as shown in Fig 1.4 (ii). In this case if y is doubled, x will become four times.

(v) $y = x^2 + 4$ or $x = y^2 - 6$ will represent a parabola but not passing through origin. In the first equation ($y = x^2 + 4$), if x is doubled, y will not become four times.

(vi) $y = Ae^{-Kx}$; represents exponentially decreasing graph. Value of y decreases exponentially from A to 0. The graph is shown in Fig. 1.5.

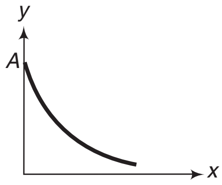


Fig. 1.5

From the graph and the equation, we can see that $y = A$ at $x = 0$ and $y \to 0$ as $x \to \infty$.

(vii) $y = A(1 - e^{-Kx})$, represents an exponentially increasing graph. Value of y increases exponentially from 0 to A. The graph is shown in Fig. 1.6.

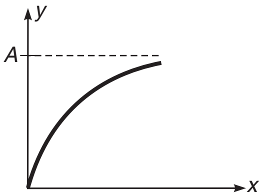


Fig. 1.6

From the graph and the equation we can see that $y = 0$ at $x = 0$ and $y \to A$ as $x \to \infty$.

Maxima and Minima

Suppose y is a function of x. Or $y = f(x)$.

Then we can draw a graph between x and y. Let the graph is as shown in Fig. 1.7.

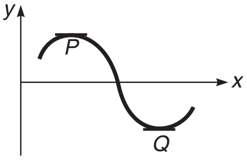


Fig. 1.7

Then from the graph we can see that at maximum or minimum value of y slope $\left(\text{or } \dfrac{dy}{dx}\right)$ to the graph is zero.

Thus, $\dfrac{dy}{dx} = 0$ at maximum or minimum value of y.

By putting $\dfrac{dy}{dx} = 0$ we will get different values of x. At these values of x, value of y is maximum if $\dfrac{d^2y}{dx^2}$ (double differentiation of y with respect to x) is negative at this value of x. Similarly y is minimum if $\dfrac{d^2y}{dx^2}$ is positive. Thus,

$$\frac{d^2y}{dx^2} = -\text{ve for maximum value of } y$$

and

$$\frac{d^2y}{dx^2} = +\text{ve for minimum value of } y$$

Note *That at constant value of y also* $\dfrac{dy}{dx} = 0$ *but in this case* $\dfrac{d^2y}{dx^2}$ *is zero.*

Example 1.1 *Differentiate the following functions with respect to x*

(a) $x^3 + 5x^2 - 2$ (b) $x \sin x$ (c) $(2x+3)^6$ (d) $\dfrac{x}{\sin x}$ (e) $e^{(5x+2)}$

Solution (a) $\dfrac{d}{dx}(x^3 + 5x^2 - 2) = \dfrac{d}{dx}(x^3) + 5\dfrac{d}{dx}(x^2) - \dfrac{d}{dx}(2)$

$$= 3x^2 + 5(2x) - 0$$

$$= 3x^2 + 10x$$

(b) $\dfrac{d}{dx}(x \sin x) = x\dfrac{d}{dx}(\sin x) + \sin x \,.\, \dfrac{d}{dx}(x)$

$$= x\cos x + \sin x\,(1)$$

$$= x\cos x + \sin x$$

(c) $\dfrac{d}{dx}(2x+3)^6 = 2\dfrac{d}{dX}(X)^6,$ where $X = 2x + 3$

$$= 2\{6X^5\} = 12X^5 = 12(2x+3)^5$$

(d) $$\frac{d}{dx}\left(\frac{x}{\sin x}\right)=\frac{\sin x\frac{d}{dx}(x)-x\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$=\frac{(\sin x)(1)-x(\cos x)}{\sin^2 x}=\frac{\sin x-x\cos x}{\sin^2 x}$$

(e) $$\frac{d}{dx}e^{(5x+2)}=5\frac{d}{dX}e^X\text{, where } X=5x+2=5e^X=5e^{5x+2}$$

Example 1.2 *Integrate the following functions with respect to x*

(a) $\int(5x^2+3x-2)\,dx$ *(b)* $\int\left(4\sin x-\frac{2}{x}\right)dx$

(c) $\int\frac{dx}{4x+5}$ *(d)* $\int(6x+2)^3\,dx$

Solution (a) $$\int(5x^2+3x-2)\,dx=5\int x^2\,dx+3\int x\,dx-2\int dx$$

$$=\frac{5x^3}{3}+\frac{3x^2}{2}-2x+c$$

(b) $$\int\left(4\sin x-\frac{2}{x}\right)dx=4\int\sin x\,dx-2\int\frac{dx}{x}$$

$$=-4\cos x-2\ln x+c$$

(c) $$\int\frac{dx}{4x+5}=\frac{1}{4}\int\frac{dX}{X},\quad\text{where}\quad X=4x+5$$

$$=\frac{1}{4}\ln X+c_1=\frac{1}{4}\ln(4x+5)+c_2$$

(d) $$\int(6x+2)^3\,dx=\frac{1}{6}\int X^3\,dX,\quad\text{where}\quad X=6x+2$$

$$=\frac{1}{6}\left(\frac{X^4}{4}\right)+c_1=\frac{(6x+2)^4}{24}+c_2$$

Example 1.3 *Draw straight lines corresponding to following equations*

(a) $y=2x$ *(b)* $y=-6x$ *(c)* $y=4x+2$ *(d)* $y=6x-4$

Solution (a) In $y=2x$, slope is 2 and intercept is zero. Hence, the graph is as shown below.

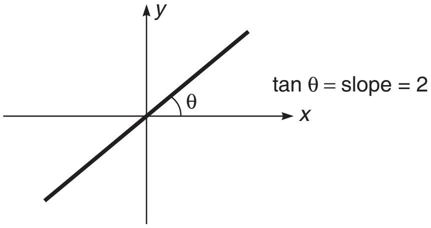


Fig. 1.8

(b) In $y = -6x$, slope is -6 and intercept is zero. Hence, the graph is as shown below.

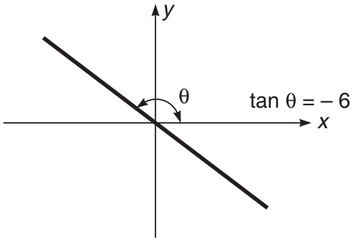


Fig. 1.9

(c) In $y = 4x + 2$, slope is $+4$ and intercept is 2. The graph is as shown below.

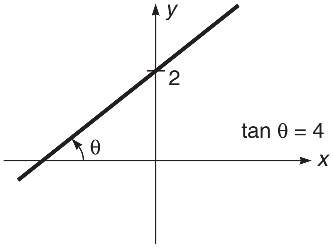


Fig. 1.10

(d) In $y = 6x - 4$, slope is $+6$ and intercept is -4. Hence, the graph is as shown below.

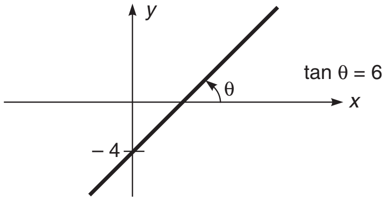


Fig. 1.11

Example 1.4 *Find maximum or minimum values of the functions*

(a) $y = 25x^2 + 5 - 10x$ *(b)* $y = 9 - (x - 3)^2$

Solution (a) For maximum and minimum value, we can put $\frac{dy}{dx} = 0$.

or $$\frac{dy}{dx} = 50x - 10 = 0$$

∴ $$x = \frac{1}{5}$$

Further, $$\frac{d^2y}{dx^2} = 50$$

or $\frac{d^2y}{dx^2}$ has positive value at $x = \frac{1}{5}$. Therefore, y has minimum value at $x = \frac{1}{5}$.

Substituting $x=\frac{1}{5}$ in given equation, we get

$$y_{\text{min}} = 25\left(\frac{1}{5}\right)^2 + 5 - 10\left(\frac{1}{5}\right) = 4$$

(b) $y = 9-(x-3)^2 = 9-x^2-9+6x$

or $$y = 6x - x^2$$

∴ $$\frac{dy}{dx} = 6-2x$$

For minimum or maximum value of y we will substitute $\frac{dy}{dx}=0$

or $$6-2x=0 \quad \text{or} \quad x=3$$

To check whether value of y is maximum or minimum at $x=3$ we will have to check whether $\frac{d^2y}{dx^2}$ is positive or negative.

$$\frac{d^2y}{dx^2} = -2$$

or $\frac{d^2y}{dx^2}$ is negative at $x=3$. Hence, value of y is maximum. This maximum value of y is,

$$y_{\text{max}} = 9-(3-3)^2 = 9$$

Exercises

Subjective Questions

Trigonometry

1. Find the value of

(a) $\cos 120°$ (b) $\sin 240°$

(c) $\tan(-60°)$ (d) $\cot 300°$

(e) $\tan 330°$ (f) $\cos(-60°)$

(g) $\sin(-150°)$ (h) $\cos(-120°)$

2. Find the value of

(a) $\sec^2\theta - \tan^2\theta$ (b) $\operatorname{cosec}^2\theta - \cot^2\theta - 1$

(c) $2\sin 45° \cos 15°$ (d) $2\sin 15° \cos 45°$

Calculus

3. Differentiate the following functions with respect to x

(a) $x^4 + 3x^2 - 2x$ (b) $x^2 \cos x$

(c) $(6x+7)^4$ (d) $e^x x^5$

(e) $\frac{(1+x)}{e^x}$

4. Integrate the following functions with respect to t

(a) $\int (3t^2 - 2t)\, dt$ (b) $\int (4\cos t + t^2)\, dt$

(c) $\int (2t-4)^{-4}\, dt$ (d) $\int \frac{dt}{(6t-1)}$

5. Integrate the following functions

(a) $\int_0^2 2t\, dt$ (b) $\int_{\pi/6}^{\pi/3} \sin x\, dx$

(c) $\int_4^{10} \frac{dx}{x}$ (d) $\int_0^{\pi} \cos x\, dx$

(e) $\int_1^2 (2t-4)\, dt$

6. Find maximum/minimum value of y in the functions given below

(a) $y = 5 - (x-1)^2$ (b) $y = 4x^2 - 4x + 7$

(c) $y = x^3 - 3x$ (d) $y = x^3 - 6x^2 + 9x + 15$

(e) $y = (\sin 2x - x)$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Graphs

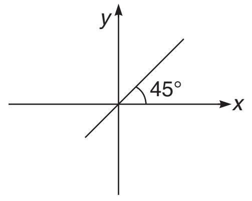
7. Draw the graphs corresponding to the equations

(a) $y = 4x$ (b) $y = -6x$

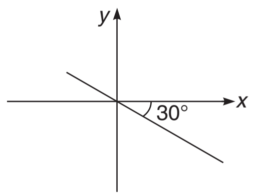
(c) $y = x + 4$ (d) $y = -2x + 4$

(e) $y = 2x - 4$ (f) $y = -4x - 6$

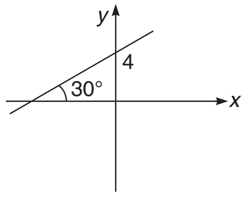
8. For the graphs given below, write down their x-y equations



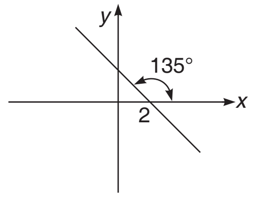
(a)



(b)



(c)



(d)

9. For the equations given below, tell the nature of graphs.

(a) $y = 2x^2$

(b) $y = -4x^2 + 6$

(c) $y = 6e^{-4x}$

(d) $y = 4(1 - e^{-2x})$

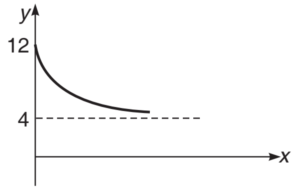
(e) $y = \frac{4}{x}$

(f) $y = -\frac{2}{x}$

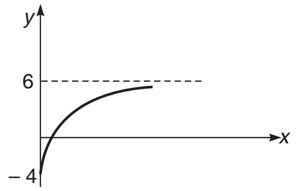
10. Value of y decreases exponentially from $y = 10$ to $y = 6$. Plot a x-y graph.

11. Value of y increases exponentially from $y = -4$ to $y = +4$. Plot a x-y graph.

12. The graph shown in figure is exponential. Write down the equation corresponding to the graph.



13. The graph shown in figure is exponential. Write down the equation corresponding to the graph.



Answers

Subjective Questions

1. (a) $-\frac{1}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$ (e) $-\frac{1}{\sqrt{3}}$ (f) $\frac{1}{2}$ (g) $-\frac{1}{2}$ (h) $-\frac{1}{2}$

2. (a) 1 (b) 0 (c) $\left(\frac{\sqrt{3}+1}{2}\right)$ (d) $\left(\frac{\sqrt{3}-1}{2}\right)$

3. (a) $4x^3 + 6x - 2$ (b) $2x\cos x - x^2 \sin x$ (c) $24(6x+7)^3$ (d) $5e^x x^4 + e^x x^5$ (e) $-xe^{-x}$

4. (a) $t^3 - t^2 + C$ (b) $4\sin t + \frac{t^3}{3} + C$ (c) $-\frac{1}{6(2t-4)^3} + C$ (d) $\frac{1}{6}\ln(6t-1) + C$

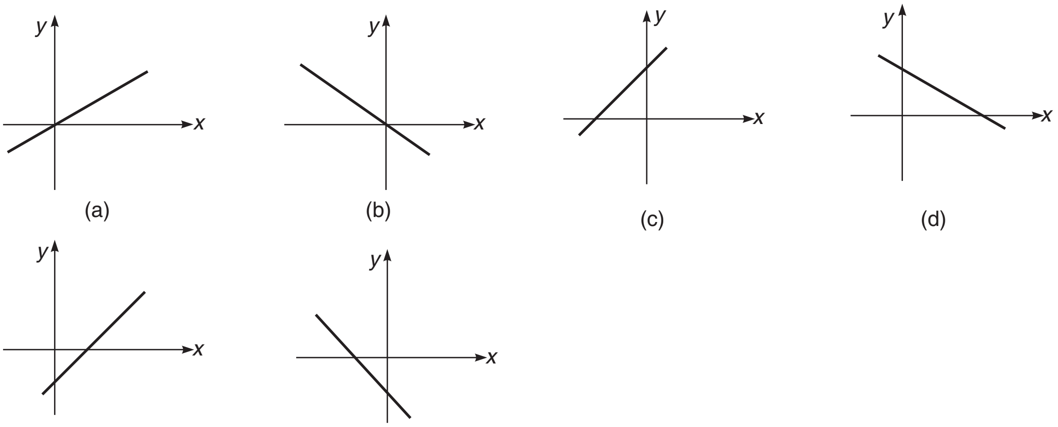
5. (a) 4 (b) $\dfrac{(\sqrt{3}-1)}{2}$ (c) ln (5/2) (d) Zero (e) -1

6. (a) $y_{\max} = 5$ at $x = 1$ (b) $y_{\min} = 6$ at $x = 1/2$ (c) $y_{\min} = -2$ at $x = 1$ and $y_{\max} = 2$ at $x = -1$

(d) $y_{\min} = 15$ at $x = 3$ and $y_{\max} = 19$ at $x = 1$ (e) $y_{\min} = -\left(\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\right)$ at $x = -\pi/6$ and

$y_{\max} = \left(\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\right)$ at $x = \pi/6$

7.



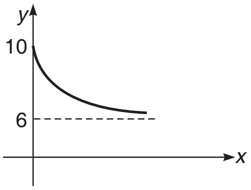
(a) (b) (c) (d)

(e) (f)

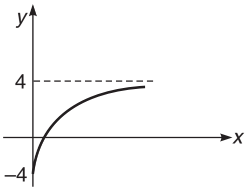
8. (a) $y = x$ (b) $y = -\dfrac{x}{\sqrt{3}}$ (c) $y = \dfrac{x}{\sqrt{3}} + 4$ (d) $y = -x + 2$

9. (a) parabola passing through origin
(b) parabola not passing through origin
(c) exponentially decreasing graph
(d) exponentially increasing graph
(e) Rectangular hyperbola in first and third quadrant
(f) Rectangular hyperbola in second and fourth quadrant

10.



11.



12. $y = 4 + 8e^{-Kx}$ Here, K is a positive constant

13. $y = -4 + 10\,(1 - e^{-Kx})$ Here, K is positive constant

02

Measurement and Errors

Chapter Contents

2.1 Errors in Measurement and Least Count
2.2 Significant Figures
2.3 Rounding off a Digit
2.4 Algebraic Operations with Significant Figures
2.5 Error Analysis

2.1 Errors in Measurement and Least Count

To get some overview of error, least count and significant figures, let us have some examples.

Example 2.1 *Let us use a centimeter scale (on which only centimeter scales are there) to measure a length AB.*

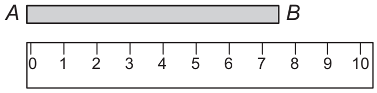


Fig. 2.1

From the figure, we can see that length AB is more than 7 cm and less than 8 cm. In this case, ***Least Count (LC)*** *of this centimeter scale is 1 cm, as it can measure accurately upto centimeters only. If we note down the length* (l) *of line AB as* $l = 7$ *cm then maximum uncertainty or maximum possible error in* l *can be 1 cm* $(= LC)$, *because this scale can measure accurately only upto 1 cm.*

Example 2.2 *Let us now use a millimeter scale (on which millimeter marks are there). This is also our normal meter scale which we use in our routine life.*

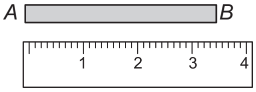


Fig. 2.2

From the figure, we can see that length AB is more than 3.3 cm and less than 3.4 cm. If we note down the length,

$$l = AB = 3.4 \text{ cm}$$

Then, this measurement has two significant figures 3 and 4 in which 3 is absolutely correct and 4 is reasonably correct (doubtful). Least count of this scale is 0.1 cm because this scale can measure accurately only upto 0.1 cm. Further, maximum uncertainty or maximum possible error in l *can also be 0.1 cm.*

INTRODUCTORY EXERCISE 2.1

1. If we measure a length $l = 6.24$ cm with the help of a vernier callipers, then
   (a) What is least count of vernier callipers ?
   (b) How many significant figures are there in the measured length ?
   (c) Which digits are absolutely correct and which is/are doubtful ?
2. If we measure a length $l = 3.267$ cm with the help of a screw gauge, then
   (a) What is maximum uncertainty or maximum possible error in l ?
   (b) How many significant figures are there in the measured length ?
   (c) Which digits are absolutely correct and which is/are doubtful ?

2.2 Significant Figures

From example 2.2, we can conclude that:

"*In a measured quantity, significant figures are the digits which are absolutely correct plus the first uncertain digit*".

Rules for Counting Significant Figures

Rule 1 All non-zero digits are significant. For example, 126.28 has five significant figures.

Rule 2 The zeros appearing between two non-zero digits are significant. For example, 6.025 has four significant figures.

Rule 3 Trailing zeros after decimal places are significant. Measurement $l = 6.400$ cm has four significant figures. Let us take an example in its support.

Table 2.1

Measurement	Accuracy	*l* lies between (in cm)	Significant figures	Remarks
l = 6.4 cm	0.1 cm	6.3 – 6.5	Two	
l = 6.40 cm	0.01 cm	6.39 – 6.41	Three	closer
l = 6.400 cm	0.001 cm	6.399 – 6.401	Four	more closer

Thus, the significant figures depend on the accuracy of measurement. More the number of significant figures, more accurate is the measurement.

Rule 4 The powers of ten are not counted as significant figures. For example, 1.4×10^{-7} has only two significant figures 1 and 4.

Rule 5 If a measurement is less than one, then all zeros occurring to the left of last non-zero digit are not significant. For example, 0.0042 has two significant figures 4 and 2.

Rule 6 Change in units of measurement of a quantity does not change the number of significant figures. Suppose a measurement was done using mm scale and we get $l = 72$ mm (two significant figures).

We can write this measurement in other units also (without changing the number of significant figures) :

7.2 cm → Two significant figures.
0.072 m → Two significant figures.
0.000072 km → Two significant figures.
7.2×10^{7} nm → Two significant figures

Rule 7 The terminal or trailing zeros in a number without a decimal point are not significant. This also sometimes arises due to change of unit.

For example, 264 m = 26400 cm = 264000 mm

All have only three significant figures 2, 6 and 4. All trailing zeros are not significant.

Zeroes at the end of a number are significant only if they are behind a decimal point as in Rule-3. Otherwise, it is impossible to tell if they are significant. For example, in the number 8200, it is not clear if the zeros are significant or not. The number of significant digits in 8200 is at least two, but could be three or four. To avoid uncertainty, use scientific notation to place significant zeros behind a decimal point

$$8.200 \times 10^{3} \text{ has four significant digits.}$$

8.20×10^3 has three significant digits.

8.2×10^3 has two significant digits.

Therefore, if it is not expressed in scientific notations, then write least number of significant digits. Hence, in the number 8200, take significant digits as two.

Rule 8 Exact measurements have infinite number of significant figures. For example,

10 bananas in a basket

46 students in a class

speed of light in vacuum $= 299{,}792{,}458$ m/s (exact)

$$\pi = \frac{22}{7} \text{ (exact)}$$

All these measurements have infinite number of significant figures.

Example 2.3

Table 2.2

Measured value	Number of significant figures	Rule number
12376 cm	5	1
6024.7 cm	5	2
0.071 cm	2	5
4100 cm	2	7
2.40 cm	3	3
1.60×10^{14} km	3	4

INTRODUCTORY EXERCISE 2.2

1. Count total number of significant figures in the following measurements:
   (a) 4.080 cm (b) 0.079 m (c) 950
   (d) 10.00 cm (e) 4.07080 (f) 7.090×10^5

2.3 Rounding Off a Digit

Following are the rules for rounding off a measurement :

Rule 1 If the number lying to the right of cut off digit is less than 5, then the cut off digit is retained as such. However, if it is more than 5, then the cut off digit is increased by 1.

For example, $x = 6.24$ is rounded off to 6.2 to two significant digits and $x = 5.328$ is rounded off to 5.33 to three significant digits.

Rule 2 If the digit to be dropped is 5 followed by digits other than zero, then the preceding digit is increased by 1.

For example, $x = 14.252$ is rounded off to $x = 14.3$ to three significant digits.

Rule 3 If the digit to be dropped is simply 5 or 5 followed by zeros, then the preceding digit it left unchanged if it is even.

For example, $x = 6.250$ or $x = 6.25$ becomes $x = 6.2$ after rounding off to two significant digits.

Rule 4 If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is raised by one if it is odd.

For example, $x = 6.350$ or $x = 6.35$ becomes $x = 6.4$ after rounding off to two significant digits.

Example 2.4

Table 2.3

Measured value	After rounding off to three significant digits	Rule
7.364	7.36	1
7.367	7.37	1
8.3251	8.33	2
9.445	9.44	3
9.4450	9.44	3
15.75	15.8	4
15.7500	15.8	4

INTRODUCTORY EXERCISE 2.3

1. Round off the following numbers to three significant figures :
   (a) 24572 (b) 24.937 (c) 36.350 (d) 42.450×10^9
2. Round 742396 to four, three and two significant digits.

2.4 Algebraic Operations with Significant Figures

The final result shall have significant figures corresponding to their number in the least accurate variable involved. To understand this, let us consider a chain of which all links are strong except the one. The chain will obviously break at the weakest link. Thus, the strength of the chain cannot be more than the strength of the weakest link in the chain.

Addition and Subtraction

Suppose, in the measured values to be added or subtracted the least number of digits after the decimal is n. Then, in the sum or difference also, the number of digits after the decimal should be n.

Example 2.5 $1.2 + 3.45 + 6.789 = 11.439 \approx 11.4$

Here, the least number of significant digits after the decimal is one. Hence, the result will be 11.4 (when rounded off to smallest number of decimal places).

Example 2.6 $12.63 - 10.2 = 2.43 \approx 2.4$

Multiplication or Division

Suppose in the measured values to be multiplied or divided the least number of significant digits be n. Then in the product or quotient, the number of significant digits should also be n.

Example 2.7 $1.2 \times 36.72 = 44.064 \approx 44$

The least number of significant digits in the measured values are two. Hence, the result when rounded off to two significant digits become 44. Therefore, the answer is 44.

Example 2.8 $\dfrac{1101\ \text{ms}^{-1}}{10.2\ \text{ms}^{-1}} = 107.94117647 \approx 108$

Example 2.9 *Find, volume of a cube of side* $a = 1.4 \times 10^{-2}$ *m.*

Solution Volume $V = a^3$

$$= (1.4 \times 10^{-2}) \times (1.4 \times 10^{-2}) \times (1.4 \times 10^{-2}) = 2.744 \times 10^{-6}\ \text{m}^3$$

Since, each value of a has two significant figures. Hence, we will round off the result to two significant figures.

$\therefore$ $$V = 2.7 \times 10^{-6}\ \text{m}^3$$ **Ans.**

Example 2.10 *Radius of a wire is* 2.50 *mm. The length of the wire is* 50.0 *cm. If mass of wire was measured as* 25 *g, then find the density of wire in correct significant figures.*
[*Given,* $\pi = 3.14$, *exact*]

Solution Given, $r = 2.50$ mm (three significant figures)

$= 0.250$ cm (three significant figures)

Note *Change in the units of measurement of a quantity does not change the number of significant figures.*

Further given that,

$l = 50.0$ cm (three significant figures)

$m = 25$ gm (two significant figures)

$\pi = 3.14$ exact (infinite significant figures)

$$\rho = \frac{m}{V} = \frac{m}{\pi r^2 l}$$

$$= \frac{25}{(3.14)(0.250)(0.250)(50.0)}$$

$$= 2.5477\ \text{g/cm}^3$$

But in the measured values, least number of significant figures are two. Hence, we will round off the result to two significant figures.

$\therefore$ $$\rho = 2.5\ \text{g/cm}^3$$ **Ans.**

INTRODUCTORY EXERCISE 2.4

1. Round to the appropriate number of significant digits
   (a) $13.214 + 234.6 + 7.0350 + 6.38$
   (b) $1247 + 134.5 + 450 + 78$
2. Simplify and round to the appropriate number of significant digits
   (a) $16.235 \times 0.217 \times 5$
   (b) 0.00435×4.6

2.5 Error Analysis

We have studied in the above articles that no measurement is perfect. Every instrument can measure upto a certain accuracy called **Least Count (LC)**.

Least Count

The smallest measurement that can be measured accurately by an instrument is called its least count.

Instrument	Its least count
mm scale	1 mm
Vernier callipers	0.1 mm
Screw gauge	0.01 mm
Stop watch	0.1 sec
Temperature thermometer	1°C

Permissible Error due to Least Count

Error in measurement due to the limitation (or least count) of the instrument is called permissible error. Least count of a millimeter scale is 1 mm. Therefore, maximum permissible error in the measurement of a length by a millimeter scale may be 1 mm.

If we measure a length $l = 26$ mm. Then, maximum value of true value may be $(26 + 1)$ mm $= 27$ mm and minimum value of true value may be $(26 - 1)$ mm $= 25$ mm.

Thus, we can write it like,

$$l = (26 \pm 1) \text{ mm}$$

If from any other instrument we measure a length $= 24.6$ mm, then the maximum permissible error (or least count) from this instrument is 0.1 mm. So, we can write the measurement like,

$$l = (24.6 \pm 0.1) \text{ mm}$$

Classification of Errors

Errors can be classified in two ways. First classification is based on the cause of error. Systematic error and random errors fall in this group. Second classification is based on the magnitude of errors. Absolute error, mean absolute error and relative (or fractional) error lie on this group. Now, let us discuss them separately.

Systematic Error

Systematic errors are the errors whose causes are known to us. Such errors can therefore be minimised. Following are few causes of these errors :

(a) Instrumental errors may be due to erroneous instruments. These errors can be reduced by using more accurate instruments and applying zero correction, when required.

(b) Sometimes errors arise on account of ignoring certain facts. For example in measuring time period of simple pendulum error may creep because no consideration is taken of air resistance. These errors can be reduced by applying proper corrections to the formula used.

(c) Change in temperature, pressure, humidity, etc., may also sometimes cause errors in the result. Relevant corrections can be made to minimise their effects.

Random Error

The causes of random errors are not known. Hence, it is not possible to remove them completely. These errors may arise due to a variety of reasons. For example the reading of a sensitive beam balance may change by the vibrations caused in the building due to persons moving in the laboratory or vehicles running nearby. The random errors can be minimized by repeating the observation a large number of times and taking the arithmetic mean of all the observations. The mean value would be very close to the most accurate reading. Thus,

$$\boxed{a_{\text{mean}} = \frac{a_1 + a_2 + \ldots + a_n}{n}}$$

Absolute Error

The difference between the true value and the measured value of a quantity is called an absolute error. Usually the mean value a_m is taken as the true value. So, if

$$a_m = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

Then by definition, absolute errors in the measured values of the quantity are,

$$\begin{aligned} \Delta a_1 &= a_m - a_1 \\ \Delta a_2 &= a_m - a_2 \\ \ldots \quad &\ldots \quad \ldots \\ \Delta a_n &= a_m - a_n \end{aligned}$$

Absolute error may be positive or negative.

Mean Absolute Error

Arithmetic mean of the magnitudes of absolute errors in all the measurements is called the mean absolute error. Thus,

$$\boxed{\Delta a_{\text{mean}} = \frac{|\Delta a_1| + |\Delta a_2| + \ldots + |\Delta a_n|}{n}}$$

The final result of measurement can be written as, $a = a_m \pm \Delta a_{mean}$

This implies that value of a is likely to lie between $a_m + \Delta a_{mean}$ and $a_m - \Delta a_{mean}$.

Relative or Fractional Error

The ratio of mean absolute error to the mean value of the quantity measured is called relative or fractional error. Thus,

$$\text{Relative error} = \frac{\Delta a_{\text{mean}}}{a_m}$$

Relative error expressed in percentage is called as the percentage error, i.e.

$$\text{Percentage error} = \frac{\Delta a_{\text{mean}}}{a_m} \times 100$$

Example 2.11 *The diameter of a wire as measured by screw gauge was found to be 2.620, 2.625, 2.630, 2.628 and 2.626 cm. Calculate*

(a) mean value of diameter
(b) absolute error in each measurement
(c) mean absolute error
(d) fractional error
(e) percentage error
(f) Express the result in terms of percentage error

Solution (a) Mean value of diameter

$$a_m = \frac{2.620 + 2.625 + 2.630 + 2.628 + 2.626}{5}$$

$$= 2.6258\,\text{cm}$$

$$= 2.626\,\text{cm} \qquad \text{(rounding off to three decimal places)}$$

(b) Taking a_m as the true value, the absolute errors in different observations are,

$$\Delta a_1 = 2.626 - 2.620 = +\,0.006\,\text{cm}$$
$$\Delta a_2 = 2.626 - 2.625 = +\,0.001\,\text{cm}$$
$$\Delta a_3 = 2.626 - 2.630 = -\,0.004\,\text{cm}$$
$$\Delta a_4 = 2.626 - 2.628 = -\,0.002\,\text{cm}$$
$$\Delta a_5 = 2.626 - 2.626 = 0.000\,\text{cm}$$

(c) Mean absolute error,

$$\Delta a_{\text{mean}} = \frac{|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + |\Delta a_4| + |\Delta a_5|}{5}$$

$$= \frac{0.006 + 0.001 + 0.004 + 0.002 + 0.000}{5}$$

$$= 0.0026 = 0.003 \qquad \text{(rounding off to three decimal places)}$$

(d) Fractional error $= \pm \dfrac{\Delta a_{\text{mean}}}{a_m} = \dfrac{\pm\, 0.003}{2.626} = \pm\, 0.001$

(e) Percentage error $= \pm\, 0.001 \times 100 = \pm\, 0.1\%$

(f) Diameter of wire can be written as,

$$d = 2.626 \pm 0.1\%$$

Combination of Errors

Errors in Sum or Difference

Let $x = a \pm b$

Further, let Δa is the absolute error in the measurement of a, Δb the absolute error in the measurement of b and Δx is the absolute error in the measurement of x.

Then,
$$x + \Delta x = (a \pm \Delta a) \pm (b \pm \Delta b)$$
$$= (a \pm b) \pm (\pm \Delta a \pm \Delta b)$$
$$= x \pm (\pm \Delta a \pm \Delta b)$$

or
$$\Delta x = \pm \Delta a \pm \Delta b$$

The four possible values of Δx are $(\Delta a - \Delta b)$, $(\Delta a + \Delta b)$, $(-\Delta a - \Delta b)$ and $(-\Delta a + \Delta b)$.

Therefore, the maximum absolute error in x is,

$$\Delta x = \pm (\Delta a + \Delta b)$$

i.e. the maximum absolute error in sum and difference of two quantities is equal to sum of the absolute errors in the individual quantities.

Example 2.12 *The volumes of two bodies are measured to be* $V_1 = (10.2 \pm 0.02)\, cm^3$ *and* $V_2 = (6.4 \pm 0.01)\, cm^3$. *Calculate sum and difference in volumes with error limits.*

Solution $V_1 = (10.2 \pm 0.02)\,\text{cm}^3$

and $$V_2 = (6.4 \pm 0.01)\,\text{cm}^3$$

$$\Delta V = \pm (\Delta V_1 + \Delta V_2)$$
$$= \pm (0.02 + 0.01)\,\text{cm}^3 = \pm 0.03\,\text{cm}^3$$

$$V_1 + V_2 = (10.2 + 6.4)\,\text{cm}^3 = 16.6\,\text{cm}^3$$

and $$V_1 - V_2 = (10.2 - 6.4)\,\text{cm}^3 = 3.8\,\text{cm}^3$$

Hence, sum of volumes = $(16.6 \pm 0.03)\,\text{cm}^3$

and difference of volumes = $(3.8 \pm 0.03)\,\text{cm}^3$

Errors in a Product

Let $x = ab$

Then, $$(x \pm \Delta x) = (a \pm \Delta a)(b \pm \Delta b)$$

or $$x\left(1 \pm \frac{\Delta x}{x}\right) = ab\left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)$$

or $$1 \pm \frac{\Delta x}{x} = 1 \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b} \qquad (\text{as } x = ab)$$

or $$\pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$$

Here, $\frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$ is a small quantity, so can be neglected.

Hence, $$\pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b}$$

Possible values of $\frac{\Delta x}{x}$ are $\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right), \left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right), \left(-\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$ and $\left(-\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$.

Hence, maximum possible value of

$$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

Therefore, maximum fractional error in product of two (or more) quantities is equal to sum of fractional errors in the individual quantities.

Errors in Division

Let $$x = \frac{a}{b}$$

Then, $$x \pm \Delta x = \frac{a \pm \Delta a}{b \pm \Delta b}$$

or $$x\left(1 \pm \frac{\Delta x}{x}\right) = \frac{a\left(1 \pm \frac{\Delta a}{a}\right)}{b\left(1 \pm \frac{\Delta b}{b}\right)}$$

or $$\left(1 \pm \frac{\Delta x}{x}\right) = \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)^{-1} \qquad \left(\text{as } x = \frac{a}{b}\right)$$

As $\frac{\Delta b}{b} << 1$, so expanding binomially, we get

$$\left(1 \pm \frac{\Delta x}{x}\right) = \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \mp \frac{\Delta b}{b}\right)$$

or $$1 \pm \frac{\Delta x}{x} = 1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$$

Here, $\frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$ is small quantity, so can be neglected. Therefore,

$$\pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

Possible values of $\frac{\Delta x}{x}$ are $\left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$, $\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$, $\left(-\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$ and $\left(-\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$. Therefore, the maximum value of

$$\boxed{\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)}$$

or the maximum value of fractional error in division of two quantities is equal to the sum of fractional errors in the individual quantities.

Error in Quantity Raised to Some Power

Let, $x = \frac{a^n}{b^m}$. Then, $\ln(x) = n \ln(a) - m \ln(b)$

Differentiating both sides, we get

$$\frac{dx}{x} = n \cdot \frac{da}{a} - m\frac{db}{b}$$

In terms of fractional error we may write,

$$\pm \frac{\Delta x}{x} = \pm n\frac{\Delta a}{a} \mp m\frac{\Delta b}{b}$$

Therefore, maximum value of

$$\boxed{\frac{\Delta x}{x} = \pm\left(n\frac{\Delta a}{a} + m\frac{\Delta b}{b}\right)}$$

Note *Errors in product and division can also be obtained by taking logarithm on both sides* $\left(\text{in } x = ab \text{ or } x = \frac{a}{b}\right)$ *and then differentiating.*

Example 2.13 *The mass and density of a solid sphere are measured to be* $(12.4 \pm 0.1)\,kg$ *and* $(4.6 \pm 0.2)\,kg/m^3$. *Calculate the volume of the sphere with error limits.*

Solution Here, $m \pm \Delta m = (12.4 \pm 0.1)$ kg

and $$\rho \pm \Delta\rho = (4.6 \pm 0.2)\ \text{kg/m}^3$$

Volume $$V = \frac{m}{\rho} = \frac{12.4}{4.6}$$

$$= 2.69\ \text{m}^3 = 2.7\ \text{m}^3 \qquad \text{(rounding off to one decimal place)}$$

Now, $$\frac{\Delta V}{V} = \pm\left(\frac{\Delta m}{m} + \frac{\Delta\rho}{\rho}\right)$$

or $$\Delta V = \pm\left(\frac{\Delta m}{m} + \frac{\Delta\rho}{\rho}\right)\times V$$

$$= \pm\left(\frac{0.1}{12.4} + \frac{0.2}{4.6}\right)\times 2.7 = \pm\, 0.14$$

∴ $$V \pm \Delta V = (2.7 \pm 0.14)\ \text{m}^3$$

Example 2.14 *Calculate percentage error in determination of time period of a pendulum*

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where, l and g are measured with $\pm\,1\%$ *and* $\pm\,2\%$.

Solution $T = 2\pi\sqrt{\frac{l}{g}}$

or $$T = (2\pi)\,(l)^{+1/2}\,(g)^{-1/2}$$

Taking logarithm of both sides, we have

$$\ln\,(T) = \ln\,(2\pi) + \frac{1}{2}(\ln\,l) - \left(\frac{1}{2}\right)\ln\,(g) \qquad \text{...(i)}$$

Here, 2π is a constant, therefore $\ln\,(2\pi)$ is also a constant.

Differentiating Eq. (i), we have

$$\frac{1}{T}\,dT = 0 + \frac{1}{2}\left(\frac{1}{l}\right)(dl) - \frac{1}{2}\left(\frac{1}{g}\right)(dg)$$

or $$\left(\frac{dT}{T}\right)_{\max} = \text{maximum value of}\left(\pm\frac{1}{2}\frac{dl}{l}\mp\frac{1}{2}\frac{dg}{g}\right)$$

$$=\frac{1}{2}\left(\frac{dl}{l}\right)+\frac{1}{2}\left(\frac{dg}{g}\right)$$

This can also be written as

$$\left(\frac{\Delta T}{T}\times 100\right)_{\max} = \frac{1}{2}\left[\frac{\Delta l}{l}\times 100\right]+\frac{1}{2}\left[\frac{\Delta g}{g}\times 100\right]$$

or percentage error in time period

$$=\pm\left[\frac{1}{2}\,(\text{percentage error in } l\,)+\frac{1}{2}\,(\text{percentage error in } g\,)\right]$$

$$=\pm\left[\frac{1}{2}\times 1+\frac{1}{2}\times 2\right]=\pm\,1.5\%$$ **Ans.**

Final Touch Points

Order of Magnitude In physics, a number of times we come across quantities which vary over a wide range. For example, size of universe, mass of sun, radius of a nucleus etc. In this case, we use the powers of ten method. In this method, each number is expressed as $n\times 10^m$, where $1\le n\le 10$ and m is a positive or negative integer.If n is less than or equal to 5, then order of number is 10^m and if n is greater than 5 then order of number is 10^{m+1}.

For example, diameter of the sun is 1.39×10^9 m. Therefore, the diameter of the sun is of the order of 10^9 m as n or $1.39\le 5$.

Solved Examples

Example 1 *Round off* 0.07284 *to four, three and two significant digits.*

Solution

0.07284	(four significant digits)
0.0728	(three significant digits)
0.073	(two significant digits)

Example 2 *Round off* 231.45 *to four, three and two significant digits.*

Solution

231.5	(four significant digits)
231	(three significant digits)
230	(two significant digits)

Example 3 *Three measurements are* $a = 483$, $b = 73.67$ *and* $c = 15.67$. *Find the value* $\frac{ab}{c}$ *to correct significant figures.*

Solution

$$\frac{ab}{c} = \frac{483 \times 73.67}{15.67}$$

$$= 2270.7472$$

$$= 2.27 \times 10^3$$ **Ans.**

Note *The result is rounded off to least number of significant figures in the given measurement i.e.* 3 (*in* 483).

Example 4 *Three measurements are,* $a = 25.6$, $b = 21.1$ *and* $c = 2.43$. *Find the value* $a - b - c$ *to correct significant figures.*

Solution

$$a - b - c = 25.6 - 21.1 - 2.43$$

$$= 2.07 = 2.1$$ **Ans.**

Note *In the measurements, least number of significant digits after the decimal is one (in 25.6 and 21.1). Hence, the result will also be rounded off to one decimal place.*

Example 5 *A thin wire has a length of* 21.7 *cm and radius* 0.46 *mm. Calculate the volume of the wire to correct significant figures.*

Solution Given, $l = 21.7$ cm, $r = 0.46$ mm $= 0.046$ cm

Volume of wire $V = \pi r^2 l$

$$= \frac{22}{7}(0.046)^2(21.7)$$

$$= 0.1443\ \text{cm}^3 = 0.14\ \text{cm}^3$$

Note *The result is rounded off to least number of significant figures in the given measurements i.e. 2 (in 0.46 mm).*

Example 6 *The radius of a sphere is measured to be* (1.2 ± 0.2) *cm. Calculate its volume with error limits.*

Solution Volume, $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\left(\frac{22}{7}\right)(1.2)^3$

$$= 7.24\ \text{cm}^3 = 7.2\ \text{cm}^3$$

Further, $\frac{\Delta V}{V} = 3\left(\frac{\Delta r}{r}\right)$

$$\therefore \qquad \Delta V = 3\left(\frac{\Delta r}{r}\right)V = \frac{3 \times 0.2 \times 7.2}{1.2}$$

$$= 3.6\ \text{cm}^3$$

$$\therefore \qquad V = (7.2 \pm 3.6)\ \text{cm}^3$$

Example 7 *Calculate equivalent resistance of two resistors* R_1 *and* R_2 *in parallel where,* $R_1 = (6 \pm 0.2)$ *ohm and* $R_2 = (3 \pm 0.1)$ *ohm*

Solution In parallel,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{...(i)}$$

or,

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{(6)(3)}{6+3} = 2\ \text{ohm}$$

Differentiating Eq. (i), we have

$$-\frac{dR}{R^2} = -\frac{dR_1}{R_1^2} - \frac{dR_2}{R_2^2}$$

Therefore, maximum permissible error in equivalent resistance may be

$$\Delta R = \left(\frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}\right)(R^2)$$

Substituting the values we get,

$$\Delta R = \left[\frac{0.2}{(6)^2} + \frac{0.1}{(3)^2}\right](2)^2$$

$$= 0.07\ \text{ohm}$$

$$\therefore \qquad R = (2 \pm 0.07)\ \text{ohm}$$

Ans.

Exercises

Objective Questions

Single Correct Option

1. The number of significant figures in 3400 is

(a) 3 (b) 1
(c) 4 (d) 2

2. The significant figures in the number 6.0023 are

(a) 2 (b) 5
(c) 4 (d) 3

3. The length and breadth of a metal sheet are 3.124 m and 3.002 m respectively. The area of this sheet upto correct significant figure is

(a) 9.378 m^2 (b) 9.37 m^2
(c) 9.4 m^2 (d) None of these

4. The length, breadth and thickness of a block are given by $l = 12$ cm, $b = 6$ cm and $t = 2.45$ cm. The volume of the block according to the idea of significant figures should be

(a) 1×10^2 cm^3 (b) 2×10^2 cm^3
(c) 1.763×10^2 cm^3 (d) None of these

5. If error in measurement of radius of a sphere is 1%, what will be the error in measurement of volume?

(a) 1% (b) $\frac{1}{3}$%
(c) 3% (d) None of these

6. The density of a cube is measured by measuring its mass and length of its sides. If the maximum error in the measurement of mass and length are 4% and 3% respectively, the maximum error in the measurement of density will be

(a) 7% (b) 9%
(c) 12% (d) 13%

7. Percentage error in the measurement of mass and speed are 2% and 3% respectively. The error in the measurement of kinetic energy obtained by measuring mass and speed will be

(a) 12% (b) 10%
(c) 8% (d) 5%

8. A force F is applied on a square plate of side L. If the percentage error in the determination of L is 2% and that in F is 4%. What is the permissible error in pressure?

(a) 8% (b) 6% (c) 4% (d) 2%

9. The heat generated in a circuit is dependent upon the resistance, current and time for which the current is flown. If the error in measuring the above are 1%, 2% and 1% respectively, then maximum error in measuring the heat is

(a) 8% (b) 6% (c) 18% (d) 12%

10. Let g be the acceleration due to gravity at the earth's surface and K the rotational kinetic energy of the earth. Suppose the earth's radius decreases by 2%. Keeping all other quantities constant, then

(a) g increases by 2% and K increases by 2%
(b) g increases by 4% and K increases by 4%
(c) g decreases by 4% and K decreases by 2%
(d) g decreases by 2% and K decreases by 4%

11. A physical quantity A is dependent on other four physical quantities p, q, r and s as given by $A = \frac{\sqrt{pq}}{r^2 s^3}$. The percentage error of measurement in p, q, r and s are 1%, 3%, 0.5% and 0.33% respectively, then the maximum percentage error in A is

(a) 2% (b) 0%
(c) 4% (d) 3%

12. The length of a simple pendulum is about 100 cm known to have an accuracy of 1 mm. Its period of oscillation is 2 s determined by measuring the time for 100 oscillations using a clock of 0.1 s resolution. What is the accuracy in the determined value of g?

(a) 0.2% (b) 0.5%
(c) 0.1% (d) 2%

13. The mass of a ball is 1.76 kg. The mass of 25 such balls is

(a) 0.44×10^3 kg (b) 44.0 kg
(c) 44 kg (d) 44.00 kg

14. The least count of a stop watch is 0.2 s. The time of 20 oscillations of a pendulum is measured to be 25 s. The percentage error in the time period is

(a) 1.2 % (b) 0.8 %
(c) 1.8 % (d) None of these

Subjective Questions

1. Write down the number of significant figures in the following

(a) 6428 (b) 62.00 m
(c) 0.00628 cm (d) 1200 N

2. Write the number of significant digits in the following

(a) 1001 (b) 100.1
(c) 100.10 (d) 0.001001

3. State the number of significant figures in the following

(a) 0.007 m^2 (b) 2.64×10^{24} kg (c) 0.2370 g/cm^{-3}

4. Round the following numbers to 2 significant digits

(a) 3472 (b) 84.16
(c) 2.55 (d) 28.5

5. Perform the following operations

(a) 703 + 7 + 0.66 (b) 2.21×0.3 (c) 12.4×84

6. Add 6.75×10^3 cm to 4.52×10^2 cm with regard to significant figures.

7. Evaluate $\frac{25.2 \times 1374}{33.3}$. All the digits in this expression are significant.

8. Solve with due regards to significant figures

$$(4.0 \times 10^{-4} - 2.5 \times 10^{-6})$$

9. The mass of a box measured by a grocer's balance is 2.300 kg. Two gold pieces of masses 20.15 g and 20.17 g are added to the box. What is (a) the total mass of the box, (b) the difference in the masses of the pieces to correct significant figures ?

10. A thin wire has length of 21.7 cm and radius 0.46 mm. Calculate the volume of the wire to correct significant figures.

11. A cube has a side of length 2.342 m. Find volume and surface area in correct significant figures.

12. Find density when a mass of 9.23 kg occupies a volume of 1.1 m^3. Take care of significant figures.

13. Length, breadth and thickness of a rectangular slab are 4.234 m, 1.005 m and 2.01 m respectively. Find volume of the slab to correct significant figures.

14. The radius of a sphere is measured to be (2.1 ± 0.5) cm. Calculate its surface area with error limits.

15. The temperature of two bodies measured by a thermometer are $(20 \pm 0.5)°$ C and $(50 \pm 0.5)°$ C. Calculate the temperature difference with error limits.

16. The resistance $R = \frac{V}{I}$, where $V = (100 \pm 5.0)$ V and $I = (10 \pm 0.2)$ A. Find the percentage error in R.

17. Find the percentage error in specific resistance given by $\rho = \frac{\pi r^2 R}{l}$ where r is the radius having value (0.2 ± 0.02) cm, R is the resistance of (60 ± 2) ohm and l is the length of (150 ± 0.1) cm.

18. A physical quantity ρ is related to four variables α, β, γ and η as

$$\rho = \frac{\alpha^3 \beta^2}{\eta \sqrt{\gamma}}$$

The percentage errors of measurements in α, β, γ and η are 1%, 3%, 4% and 2% respectively. Find the percentage error in ρ.

19. The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{L/g}$. Measured value of L is 20.0 cm known to 1 mm accuracy and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1s resolution. What is the accuracy in the determination of g ?

Answers

Introductory Exercise 2.1

1. (a) 0.01 cm (b) 3
(c) 6 and 2 are absolutely correct and 4 is doubtful.
2. (a) 0.001 cm (b) 4
(c) 3, 2 and 6 are absolutely correct and 7 is doubtful.

Introductory Exercise 2.2

1. (a) 4 (b) 2 (c) 2 (d) 4
(e) 6 (f) 4

Introductory Exercise 2.3

1. (a) 24600 (b) 24.9 (c) 36.4 (d) 42.4×10^9
2. 742400, 742000, 740000

Introductory Exercise 2.4

1. (a) 261.2 (b) 1910 2. (a) 20 (b) 0.020

Exercises

Objective Questions

1. (d) **2.** (b) **3.** (a) **4.** (b) **5.** (c) **6.** (d) **7.** (c) **8.** (a) **9.** (b) **10.** (b)
11. (c) **12.** (a) **13.** (b) **14.** (b)

Subjective Questions

1. (a) 4 (b) 4 (c) 3 (d) 2
2. (a) 4 (b) 4 (c) 5 (d) 4
3. (a) 1 (b) 3 (c) 4
4. (a) 3500 (b) 84 (c) 2.6 (d) 28
5. (a) 711 (b) 0.7 (c) 1.0×10^3
6. 7.20×10^3 cm
7. 1040
8. 4.0×10^{-4}
9. (a) 2.3 kg (b) 0.02 gm
10. 0.14 cm^3
11. Area = 5.485 m^2, Volume = 12.85 m^3
12. Density = 8.4 kg/m^3
13. Volume = 8.55 m^3
14. (55.4 ± 26.4) cm^2
15. $(30 \pm 1)^\circ$C
16. 7%
17. 23.4 %
18. 13%
19. 2.7 %

03

Experiments

Chapter Contents

3.1 Vernier Callipers
3.2 Screw Gauge
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3.7 Verification of Ohm's Law using Voltmeter and Ammeter
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3.10 Focal Length of a Concave Mirror using u-v method
3.11 Focal Length of a Convex Lens using u-v method

3.1 Vernier Callipers

Length is an elementary physical quantity. The device generally used in everyday life for measurement of length is a meter scale (we can also call it mm scale). It can be used for measurement of length with an accuracy of 1 mm. So, the least count of a meter scale is 1 mm. To measure length accurately upto 0.1 mm or 0.01 mm vernier callipers and screw gauge are used.

Vernier callipers has following three parts :

(i) Main scale It consists of a steel metallic strip *M*, graduated in cm and mm at one edge.

It carries two fixed jaws *A* and *C* as shown in figure.

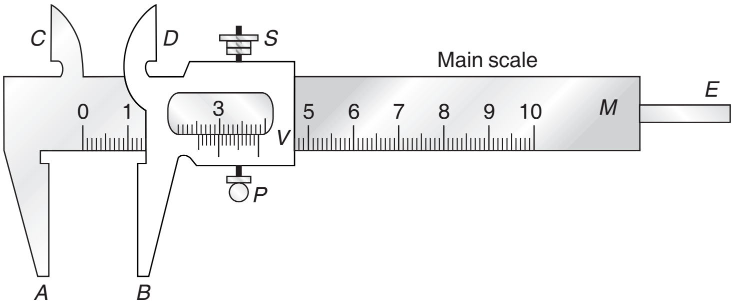


Fig. 3.1

(ii) Vernier scale Vernier scale *V* slides on metallic strip *M*. It can be fixed in any position by screw *S*. The side of the vernier scale which slide over the mm sides has ten divisions over a length of 9 mm. *B* and *D* two movable jaws are fixed with it. When vernier scale is pushed towards *A* and *C*, then *B* touches *A* and straight side of *C* will touch straight side of *D*. In this position, if the instrument is free from error, zeros of vernier scale will coincide with zeros of main scales. To measure the external diameter of an object it is held between the jaws *A* and *B*, while the straight edges of *C* and *D* are used for measuring the internal diameter of a hollow object.

(iii) Metallic strip There is a thin metallic strip *E* attached to the back side of *M* and connected with vernier scale. When jaws *A* and *B* touch each other, the edge of *E* touches the edge of *M*. When the jaws *A* and *B* are separated *E* moves outwards. This strip *E* is used for measuring the depth of a vessel.

Principle (Theory)

In the common form, the divisions on the vernier scale *V* are smaller in size than the smallest division on the main scale *M,* but in some special cases the size of the vernier division may be larger than the main scale division. Let n vernier scale divisions (VSD) coincide with $(n-1)$ main scale divisions (MSD). Then,

$$n \text{ VSD} = (n-1) \text{ MSD}$$

or

$$1 \text{ VSD} = \left(\frac{n-1}{n}\right) \text{MSD}$$

$$1 \text{ MSD} - 1 \text{ VSD} = 1 \text{ MSD} - \left(\frac{n-1}{n}\right) \text{MSD} = \frac{1}{n} \text{ MSD}$$

The difference between the values of one main scale division and one vernier scale division is known as **Vernier Constant (VC)** or the **Least Count (LC).** This is the smallest distance that can be accurately measured with the vernier scale. Thus,

$$\boxed{\text{VC} = \text{LC} = 1\,\text{MSD} - 1\,\text{VSD} = \left(\frac{1}{n}\right)\text{MSD} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}}$$

In the ordinary vernier callipers one main scale division be 1 mm and 10 vernier scale divisions coincide with 9 main scale divisions.

$$1\ \text{VSD} = \frac{9}{10}\ \text{MSD} = 0.9\ \text{mm}$$

$$\begin{aligned}\text{VC} &= 1\ \text{MSD} - 1\ \text{VSD} = 1\ \text{mm} - 0.9\ \text{mm}\\ &= 0.1\ \text{mm} = 0.01\ \text{cm}\end{aligned}$$

Reading a Vernier Callipers

If we have to measure a length AB, the end A is coincided with the zero of main scale, suppose the end B lies between 1.0 cm and 1.1 cm on the main scale. Then,

$$1.0\,\text{cm} < AB < 1.1\,\text{cm}$$

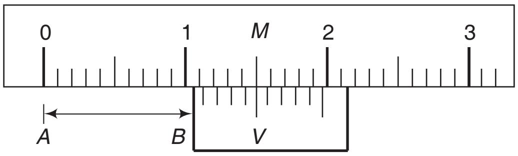


Fig. 3.2

Let 5th division of vernier scale coincides with 1.5 cm of main scale.

Then, $$AB = 1.0 + 5 \times \text{VC} = (1.0 + 5 \times 0.01)\ \text{cm} = 1.05\ \text{cm}$$

Thus, we can make the following formula,

$$\boxed{\text{Total reading} = N + n \times \text{VC}}$$

Here, N = main scale reading before on the left of the zero of the vernier scale.

n = number of vernier division which just coincides with any of the main scale division.

Note *That the main scale reading with which the vernier scale division coincides has no connection with reading.*

Zero Error and Zero Correction

If the zero of the vernier scale does not coincide with the zero of main scale when jaw B touches A and the straight edge of D touches the straight edge of C, then the instrument has an error called **zero error.** Zero error is always algebraically subtracted from measured length.

Zero correction has a magnitude equal to zero error but its sign is opposite to that of the zero error. Zero correction is always algebraically added to measured length.

$$\text{Zero error} \longrightarrow \text{algebraically subtracted}$$

$$\text{Zero correction} \longrightarrow \text{algebraically added}$$

Positive and Negative Zero Errors

If zero of vernier scale lies to the right of the main scale the zero error is positive and if it lies to the left of the main scale the zero error is negative (when jaws A and B are in contact).

$$\boxed{\text{Positive zero error} = (N + x \times \text{VC})}$$

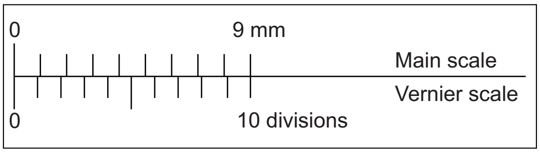
Here, N = main scale reading on the left of zero of vernier scale.

x = vernier scale division which coincides with any main scale division.

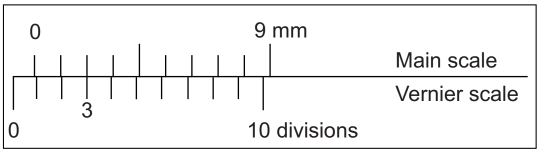
When the vernier zero lies before the main scale zero the error is said to be negative zero error. If 8th vernier scale division coincides with the main scale division, then

$$\begin{aligned}\text{Negative zero error} &= -\,[0.00\text{ cm} + 8 \times \text{VC}]\\ &= -\,[0.00\text{ cm} + 8 \times 0.01\text{ cm}]\\ &= -\,0.08\text{ cm}\end{aligned}$$

No Zero Error



Negative Error



Positive Error

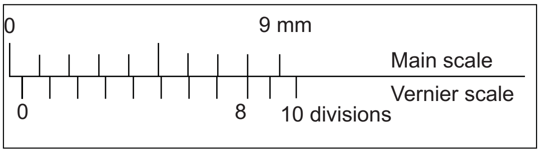


Fig. 3.3 Positive and negative zero error

Summary

1. $\text{VC} = \text{LC} = \dfrac{1\text{ MSD}}{n} = \dfrac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}} = 1\text{ MSD} - 1\text{ VSD}$
2. In ordinary vernier callipers, $1\text{ MSD} = 1$ mm and $n = 10$

   $$\therefore \qquad \text{VC} \quad \text{or} \quad \text{LC} = \frac{1}{10}\text{ mm} = 0.01\text{ cm}$$
3. Total reading $= (N + n \times \text{VC})$
4. Zero correction $= -$ zero error
5. Zero error is algebraically subtracted while the zero correction is algebraically added.

6. If zero of vernier scale lies to the right of zero of main scale the error is positive. The actual length in this case is less than observed length.
7. If zero of vernier scale lies to the left of zero of main scale the error is negative and the actual length is more than the observed length.
8. Positive zero error = $(N + x \times \text{VC})$

Example 3.1 *N divisions on the main scale of a vernier callipers coincide with $N+1$ divisions on the vernier scale. If each division on the main scale is of a units, determine the least count of the instrument.* **(JEE 2003)**

Solution $(N+1)$ divisions on the vernier scale = N divisions on main scale

$\therefore$ 1 division on vernier scale = $\dfrac{N}{N+1}$ divisions on main scale

Each division on the main scale is of a units.

$\therefore$ 1 division on vernier scale = $\left(\dfrac{N}{N+1}\right) a$ units = a' (say)

$$\text{Least count} = 1\,\text{main scale division} - 1\ \text{vernier scale division}$$
$$= a - a' = a - \left(\frac{N}{N+1}\right) a = \frac{a}{N+1}$$

Example 3.2 *In the diagram shown in figure, find the magnitude and nature of zero error.*

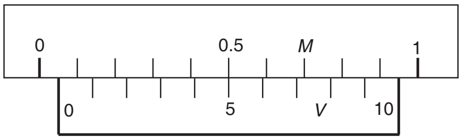


Fig. 3.4

Solution Here, zero of vernier scale lies to the right of zero of main scale, hence, it has positive zero error.

Further, $N = 0,\ x = 5,$ LC or VC $= 0.01$ cm

Hence, Zero error $= N + x \times \text{VC}$

$$= 0 + 5 \times 0.01 = 0.05\,\text{cm}$$

Zero correction = – 0.05 cm

$\therefore$ Actual length will be 0.05 cm less than the measured length.

Example 3.3 *The smallest division on main scale of a vernier callipers is 1 mm and 10 vernier divisions coincide with 9 main scale divisions. While measuring the length of a line, the zero mark of the vernier scale lies between 10.2 cm and 10.3 cm and the third division of vernier scale coincides with a main scale division.*

(a) Determine the least count of the callipers.

(b) Find the length of the line.

Solution (a) Least Count (LC) $= \dfrac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$

$$= \frac{1}{10}\,\text{mm} = 0.1\,\text{mm} = 0.01\,\text{cm}$$

(b) $L = N + n\,(\text{LC}) = (10.2 + 3 \times 0.01)\,\text{cm} = 10.23\,\text{cm}$

INTRODUCTORY EXERCISE 3.1

1. The main scale of a vernier callipers reads 10 mm in 10 divisions. Ten divisions of vernier scale coincide with nine divisions of the main scale. When the two jaws of the callipers touch each other, the fifth division of the vernier coincides with 9 main scale divisions and the zero of the vernier is to the right of zero of main scale, when a cylinder is tightly placed between the two jaws, the zero of the vernier scale lies slightly to the left of 3.2 cm and the fourth vernier division coincides with a main scale division. Find diameter of the cylinder.
2. In a vernier callipers, N divisions of the main scale coincide with $N + m$ divisions of the vernier scale. What is the value of m for which the instrument has minimum least count.

3.2 Screw Gauge

Principle of a Micrometer Screw

The least count of vernier callipers ordinarily available in the laboratory is 0.01 cm. When lengths are to be measured with greater accuracy, say upto 0.001 cm, screw gauge and spherometer are used which are based on the principle of **micrometer screw** discussed below.

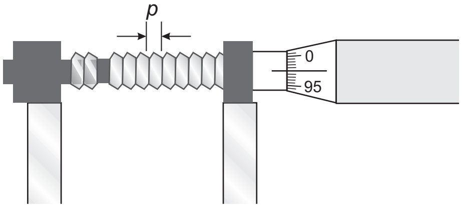


Fig. 3.5

If an accurately cut single threaded screw is rotated in a closely fitted nut, then in addition to the circular motion of the screw there is a linear motion of the screw head in the forward or backward direction, along the axis of the screw. The linear distance moved by the screw, when it is given one complete rotation is called the **pitch** (p) of the screw. This is equal to the distance between two consecutive threads as measured along the axis of the screw. In most of the cases, it is either 1 mm or 0.5 mm. A circular cap is fixed on one end of the screw and the circumference of the cap is normally divided into 100 or 50 equal parts. If it is divided into 100 equal parts, then the screw moves forward or backward by $\dfrac{1}{100}$ $\left(\text{or } \dfrac{1}{50}\right)$ of the pitch, if the circular scale (we will discuss later about circular scale) is rotated through one circular scale division. It is the minimum distance which can be accurately measured and so called the **Least Count (LC)** of the screw.

Thus,
$$\boxed{\text{Least count} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}}$$

If pitch is 1 mm and there are 100 divisions on circular scale then,

$$\text{LC} = \frac{1\,\text{mm}}{100} = 0.01\,\text{mm}$$
$$= 0.001\,\text{cm} = 10\,\mu\text{m}$$

Since, LC is of the order of 10 μm, the screw is called micrometer screw.

Screw Gauge

Screw gauge works on the principle of micrometer screw. It consists of a *U*-shaped metal frame *M*. At one end of it is fixed a small metal piece *A*. It is called stud and it has a plane face. The other end *N* of *M* carries a cylindrical hub *H*. It is graduated in millimeter and half millimeter depending upon the pitch of the screw. This scale is called **linear scale** or **pitch scale.**

A nut is threaded through the hub and the frame *N*. Through the nut moves a screw *S*. The front face *B* of the screw, facing the plane face *A* is also plane. A hollow cylindrical cap *K* is capable of rotating over the hub when screw is rotated. As the cap is rotated the screw either moves in or out. The surface *E* of the cap *K* is divided into 50 or 100 equal parts. It is called the **circular scale** or **head scale.** In an accurately adjusted instrument when the faces *A* and *B* are just touching each other. Zero of circular scale should coincide with zero of linear scale.

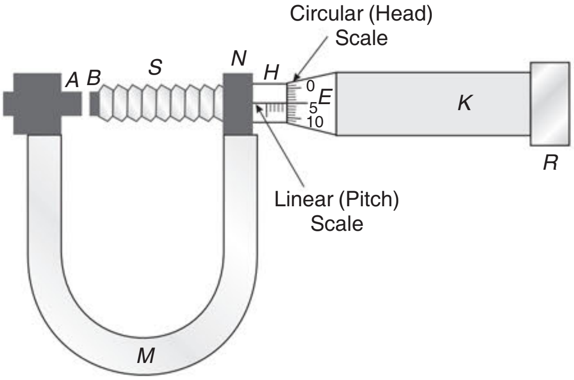


Fig. 3.6

To Measure Diameter of a Given Wire Using a Screw Gauge

If with the wire between plane faces *A* and *B*, the edge of the cap lies ahead of *N* th division of linear scale, and *n*th division of circular scale lies over reference line.

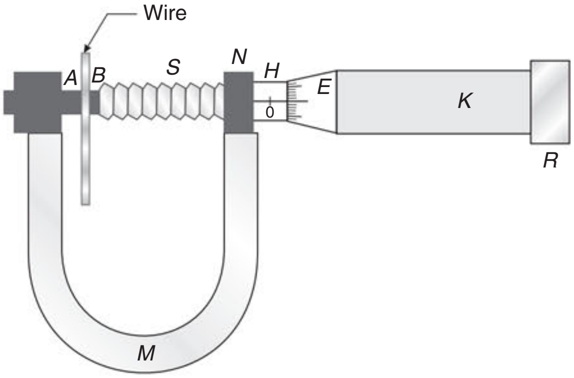


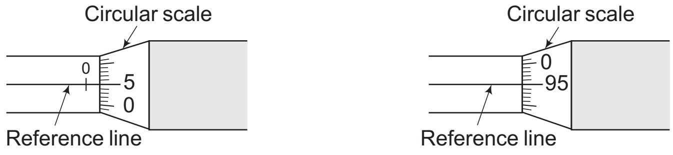
Fig. 3.7

Then,

$$\boxed{\text{Total reading} = N + n \times \text{LC}}$$

Zero Error and Zero Correction

If zero mark of circular scale does not coincide with the zero of the pitch scale when the faces A and B are just touching each other, the instrument is said to possess zero error. If the zero of the circular scale advances beyond the reference line the zero error is **negative** and zero correction is **positive.** If it is left behind the reference line the zero is **positive** and zero correction is **negative.** For example, if zero of circular scale advances beyond the reference line by 5 divisions, zero correction $= +5 \times (\text{LC})$ and if the zero of circular scale is left behind the reference line by 5 divisions, zero correction $= -5 \times (\text{LC})$.



(a) Positive zero error (b) Negative zero error

Fig. 3.8

Note *In negative zero error 95th division of the circular scale is coinciding with the reference line. Hence there are 5 divisions between zero mark on the circular scale and the reference line.*

Back Lash Error

When the sense of rotation of the screw is suddenly changed, the screw head may rotate, but the screw itself may not move forward or backwards. Thus, the scale reading may change even by the actual movement of the screw. This is known as back lash error. This error is due to loose fitting of the screw. This arises due to wear and tear of the threading due to prolonged use of the screw. To reduce this error the screw must always be rotated in the same direction for a particular set of observations.

Example 3.4 *The pitch of a screw gauge is 1 mm and there are 100 divisions on the circular scale. In measuring the diameter of a sphere there are six divisions on the linear scale and forty divisions on circular scale coincide with the reference line. Find the diameter of the sphere.*

Solution $\text{LC} = \dfrac{1}{100} = 0.01\,\text{mm}$

$$\text{Linear scale reading} = 6\,(\text{pitch}) = 6\text{ mm}$$

$$\text{Circular scale reading} = n\,(\text{LC}) = 40 \times 0.01 = 0.4\text{ mm}$$

$$\therefore \quad \text{Total reading} = (6 + 0.4) = 6.4\text{ mm}$$

Example 3.5 *The pitch of a screw gauge is 1 mm and there are 100 divisions on circular scale. When faces A and B are just touching each without putting anything between the studs 32nd division of the circular scale (below its zero) coincides with the reference line. When a glass plate is placed between the studs, the linear scale reads 4 divisions and the circular scale reads 16 divisions. Find the thickness of the glass plate. Zero of linear scale is not hidden from circular scale when A and B touches each other.*

Solution $\text{Least count (LC)} = \dfrac{\text{Pitch}}{\text{Number of divisions on circular scale}} = \dfrac{1}{100}\text{ mm}$

$$= 0.01\,\text{mm}$$

As zero is not hidden from circular scale when A and B touches each other. Hence, the screw gauge has positive error.

$$e = +n\,(\text{LC}) = 32 \times 0.01 = 0.32 \text{ mm}$$

$$\text{Linear scale reading} = 4 \times (1 \text{ mm}) = 4 \text{ mm}$$

$$\text{Circular scale reading} = 16 \times (0.01 \text{ mm}) = 0.16 \text{ mm}$$

$$\therefore \quad \text{Measured reading} = (4 + 0.16) \text{ mm} = 4.16 \text{ mm}$$

$$\therefore \quad \text{Absolute reading} = \text{Measured reading} - e$$

$$= (4.16 - 0.32) \text{ mm} = 3.84 \text{ mm}$$

Therefore, thickness of the glass plate is 3.84 mm.

INTRODUCTORY EXERCISE 3.2

1. Read the screw gauge shown below in the figure.

Given that circular scale has 100 divisions and in one complete rotation the screw advances by 1mm.

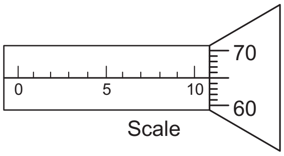


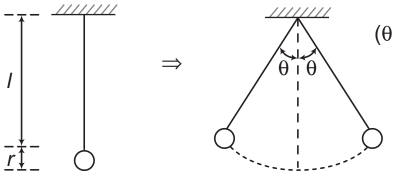
Fig. 3.9

2. The pitch of a screw gauge having 50 divisions on its circular scale is 1 mm. When the two jaws of the screw gauge are in contact with each other, the zero of the circular scale lies 6 divisions below the line of graduation. When a wire is placed between the jaws , 3 linear scale divisions are clearly visible while 31st division on the circular scale coincides with the reference line. Find diameter of the wire.

3.3 Determination of '*g*' using a Simple Pendulum

In this experiment, a small spherical bob is hanged with a cotton thread. This arrangement is called simple pendulum. The bob is displaced slightly and allowed to oscillate.

The period of small oscillations is given by



(θ should be small)

Fig. 3.10

$$T = 2\pi \sqrt{\frac{L}{g}}$$

where, $L = l + r$ (as shown in figure)

$= \text{equivalent length of pendulum}$

$$\therefore \quad \boxed{g = \frac{4\pi^2 L}{T^2}} \qquad \text{...(i)}$$

To find time period, time taken for 50 oscillations is noted using a stop watch.

$$\therefore \quad T = \frac{\text{Time taken for 50 oscillations}}{50}$$

Now, substituting the values of T and L in Eq. (i), we can easily find the value of 'g'.

Graphical Method of Finding Value of *g*

Eq. (i) can also be written as

$$T^2 = \left(\frac{4\pi^2}{g}\right) L \qquad \text{...(ii)}$$

$$\Rightarrow \qquad T^2 \propto L$$

Therefore, T^2 *versus* L graph is a straight line passing through origin with slope $= \left(\frac{4\pi^2}{g}\right)$

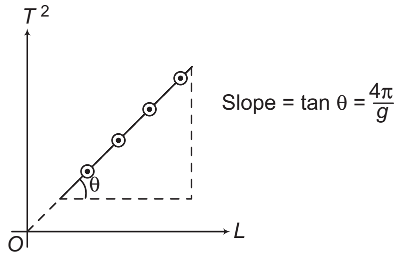


Fig. 3.11

Therefore, from the slope of this graph $(= 4\pi^2/g)$ we can determine the value of g.

Example 3.6 *In a certain observation we get* $l = 23.2$ *cm,* $r = 1.32$ *cm and time taken for* 20 *oscillations was* 20.0 *sec. Taking* $\pi^2 = 10$, *find the value of g in proper significant figures.*

Solution Equivalent length of pendulum,

$$L = 23.2 \text{ cm} + 1.32 \text{ cm} = 24.52 \text{ cm}$$

$$= 24.5 \text{ cm} \qquad \text{(according to addition rule of significant figures)}$$

Time period, $T = \frac{20.0}{20} = 1.00$ s. Time period has 3 significant figures

Now, $$g = (4\pi^2)\frac{l}{T^2} = \frac{4 \times 10 \times 24.5 \times 10^{-2}}{(1.00)^2} = 9.80 \text{ m/s}^2$$ **Ans.**

Example 3.7 *For different values of L, we get different values of* T^2. *The graph between L versus* T^2 *is as shown in figure. Find the value of 'g' from the given graph. (Take* $\pi^2 = 10$).

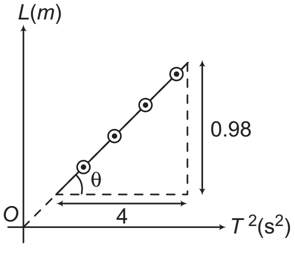


Fig. 3.12

Solution From the equation, $$T = 2\pi\sqrt{\frac{L}{g}}$$

we get, $$L = \left(\frac{g}{4\pi^2}\right) T^2 \quad \Rightarrow \quad L \propto T^2$$

i.e. L *versus* T^2 graph is a straight line passing through origin with slope $= \frac{g}{4\pi^2}$

$$\therefore \qquad \text{Slope} = \tan\theta = \frac{g}{4\pi^2} \quad \text{or} \quad g = (4\pi^2)\tan\theta$$

$$= \frac{4 \times 10 \times 0.98}{4} = 9.8 \text{ m/s}^2$$ **Ans.**

Example 3.8 *In a certain observation we got, $l = 23.2$ cm, $r = 1.32$ cm and time taken for 10 oscillations was 10.0 s. Find, maximum percentage error in determination of 'g'.*

Solution

$$l = 23.2 \text{ cm} \Rightarrow \Delta l = 0.1 \text{ cm}$$

$$r = 1.32 \text{ cm} \Rightarrow \Delta r = 0.01 \text{ cm}$$

$$t = 10.0 \text{ s} \Rightarrow \Delta t = 0.1 \text{ s}$$

Now,

$$g = 4\pi^2\left(\frac{L}{T^2}\right) = 4\pi^2\left[\frac{l+r}{(t/n)^2}\right]$$

$$g = 4\pi^2 n^2\left(\frac{l+r}{t^2}\right)$$

$\therefore$ Maximum percentage error in g will be

$$\left(\frac{\Delta g}{g}\right) \times 100 = \left[\frac{\Delta l + \Delta r}{l + r} + 2\left(\frac{\Delta t}{t}\right)\right] \times 100$$

$$= \left[\frac{0.1 + 0.01}{23.2 + 1.32} + 2 \times \frac{0.1}{10.0}\right] \times 100$$

$$= 2.4 \%$$ **Ans.**

INTRODUCTORY EXERCISE 3.3

1. What is a second's pendulum ?
2. Why should the amplitude be small for a simple pendulum experiment ?
3. Does the time period depend upon the mass, the size and the material of the bob ?
4. What type of graph do you expect between (i) L and T and (ii) L and T^2 ?
5. Why do the pendulum clocks go slow in summer and fast in winter ?
6. Why do we use Invar material for the pendulum of good clocks ?
7. A simple pendulum has a bob which is a hollow sphere full of sand and oscillates with certain period. If all that sand is drained out through a hole at its bottom, then its period
   (a) increases
   (b) decreases
   (c) remains same
   (d) is zero
8. The second's pendulum is taken from earth to moon, to keep the time period constant
   (a) the length of the second's pendulum should be decreased
   (b) the length of the second's pendulum should be increased
   (c) the amplitude should increase
   (d) the amplitude should decrease

3.4 Young's Modulus by Searle's Method

Young's modulus of a wire can be determined by an ordinary experiment as discussed below.

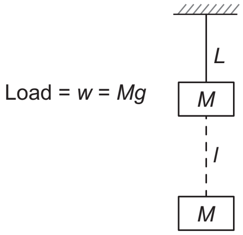


Fig. 3.13

A mass M is hanged from a wire of length L, cross sectional radius r and Young's modulus Y. Let change in length in wire is l. Then,

$$\text{Stress} = \frac{F}{A} = \frac{Mg}{\pi r^2}$$

$$\text{Strain} = \frac{l}{L}$$

and Young's modulus $Y = \frac{\text{Stress}}{\text{Strain}}$ or $Y = \frac{Mg/\pi r^2}{l/L}$

$$\Rightarrow \qquad l = \left(\frac{L}{\pi r^2 Y}\right) Mg$$

or

$$\boxed{l = \left(\frac{L}{\pi r^2 Y}\right) w}$$

$$\Rightarrow \qquad l \propto w$$

Therefore, l *versus* w graph is a straight line passing through origin with

$$\text{Slope} = \frac{L}{\pi r^2 Y} = \tan\theta$$

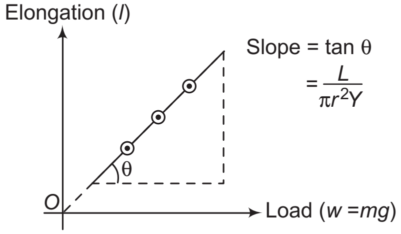


Fig. 3.14

$$\therefore \qquad Y = \frac{L}{\pi r^2 (\tan\theta)} \qquad \text{...(i)}$$

Thus, by measuring the slope (or tan θ) we can find Young's modulus Y from Eq. (i).

Note *We can also take load along y-axis and elongation along x-axis. In that case, slope* $= \frac{\pi r^2 Y}{L}$

Limitations of this Method

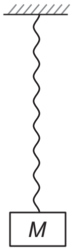


Fig. 3.15

1. For small loads, there may be some bends or kinks in the wire. So, it is better to start with some initial weight, so that wire becomes straight.
2. There is slight difference in behaviour of wire under loading and unloading load.

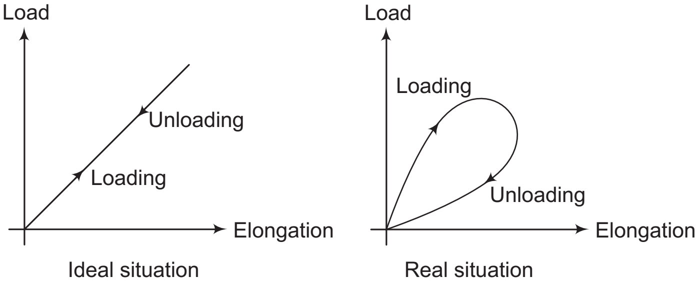


Fig. 3.16

Modification in Searle's Method

To keep the experimental wire straight and kink free we start with some dead load (say 2 kg). Now, we gradually increase the load and measure the extra elongation.

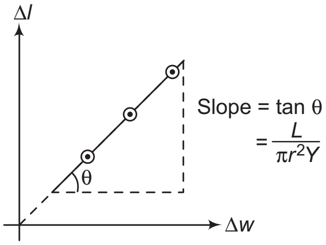


Fig. 3.17

$$l = \left(\frac{L}{\pi r^2 Y}\right) w$$

$$\Rightarrow \qquad \boxed{\Delta l = \left(\frac{L}{\pi r^2 Y}\right) \Delta w}$$

$$\Rightarrow \qquad \Delta l \propto \Delta w$$

or Δl *versus* Δw graph is again a straight line passing through origin with same slope, $\frac{L}{\pi r^2 Y}$

To measure extra elongation, compared to initial loaded position, we use a reference wire also carrying 2 kg.

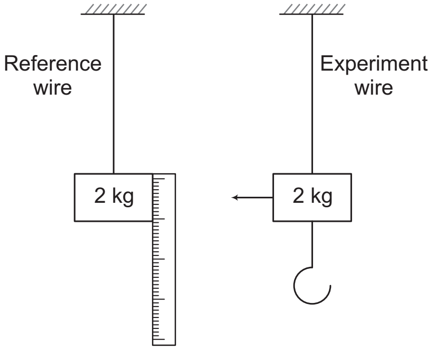


Fig. 3.18

Searle's Apparatus

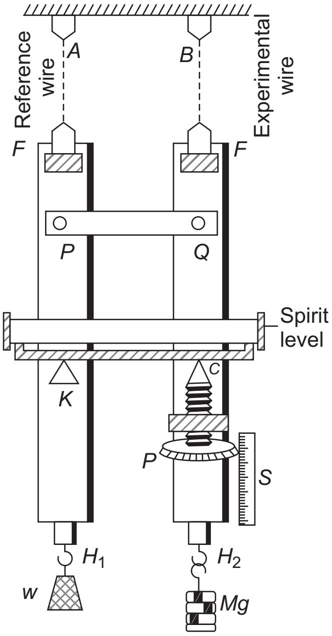


Fig. 3.19

It consists of two metal frames P and Q hinged together, such that they can have only vertical relative motion. A spirit level (S.L.) is supported at one end on a rigid cross bar frame whose other end rests on the tip of a micrometer screw C. If there is any relative motion between the two frames, the spirit level no longer remains horizontal and the bubble is displaced in the spirit level.

To bring the bubble back to its original position, the screw has to be moved up or down. The distance through which the screw has to be moved gives the relative motion between the two frames.

The frames are suspended by two identical long wires of the same material, from the same rigid horizontal support. Wire B is the experimental wire and the wire A acts simply as a reference wire. The frames are provided with hooks H_1 and H_2 at their ends from which weights are suspended. The hook H_1 attached to the frame of the reference wire carries a constant weight W to keep the wire taut. To the hook H_2 of the experimental wire (i.e. wire B), is attached a hanger over which slotted weights can be placed to apply the stretching force, Mg.

Method

Step 1 Measure the length of the experimental wire.

Step 2 Measure the diameter of the experimental wire with the help of a screw gauge at about five different places.

Step 3 Find pitch and least count of the micrometer and adjust it such that the bubble in spirit level is exactly at the centre. Also note down the initial reading of micrometer.

Step 4 Gradually increase the load on the hanger H_2 in steps of 0.5 kg. Observe the reading on the micrometer at each step after levelling the instrument with the help of spirit level. To avoid the backlash error, all the final adjustments should be made by moving the screw in the upward direction only.

Step 5 Unload the wire by removing the weights in the same order and take the reading on the micrometer screw each time. The readings during loading and unloading should agree closely.

Step 6 Plot Δl *versus* Δw graph and from its slope determine the value of Y. We have seen above that,

$$\boxed{\text{Slope} = \tan\theta = \frac{L}{\pi r^2 Y}} \quad \therefore \quad \boxed{Y = \frac{L}{(\pi r^2)\tan\theta}}$$

Observation

Initial reading $l = 0.540$ mm , Radius of the wire $= 0.200$ mm

S.No.	Extra load on hanger Δm (kg)	Extra load Δw(N)	Micrometer reading During loading (p) (mm)	Micrometer reading During unloading (q) (mm)	Mean reading $(p+q)/2$ (mm)	Extra elongation (mm)
1	0.5	5	0.555	0.561	0.558	0.018
2	1.0	10	0.565	0.571	0.568	0.028
3	1.5	15	0.576	0.580	0.578	0.038
4	2.0	20	0.587	0.593	0.590	0.050
5	2.5	25	0.597	0.603	0.600	0.060
6	3.0	30	0.608	0.612	0.610	0.070
7	3.5	35	0.620	0.622	0.621	0.081
8	4.0	40	0.630	0.632	0.631	0.091
9	4.5	45	0.641	0.643	0.642	0.102
10	5.0	50	0.652	0.652	0.652	0.112

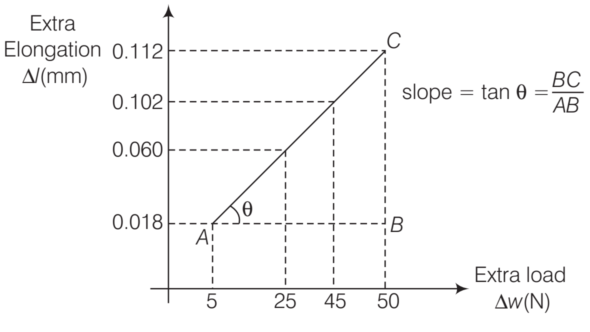


Fig. 3.20

$$\text{slope} = \frac{BC}{AB}$$

Example 3.9 *The adjacent graph shows the extension (Δl) of a wire of length 1 m suspended from the top of a roof at one end and with a load w connected to the other end. If the cross-sectional area of the wire is $10^{-6}\ m^2$, calculate from the graph the Young's modulus of the material of the wire.* **(JEE 2003)**

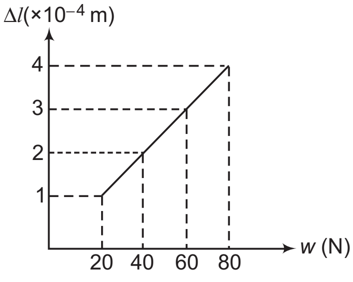


Fig 3.21

Solution $\Delta l = \left(\frac{l}{YA}\right) \cdot w \quad \Rightarrow \quad \Delta l \propto w$

i.e. Δl *versus* w graph is a straight line passing through origin (as shown in question also), the slope of which is $\frac{l}{YA}$.

$$\therefore \qquad \text{Slope} = \left(\frac{l}{YA}\right)$$

$$\therefore \qquad Y = \left(\frac{l}{A}\right)\left(\frac{1}{\text{slope}}\right)$$

$$= \left(\frac{1.0}{10^{-6}}\right)\frac{(80-20)}{(4-1)\times 10^{-4}}$$

$$= 2.0\times 10^{11}\ \text{N/m}^2$$ **Ans.**

Example 3.10 *In Searle's experiment, which is used to find Young's modulus of elasticity, the diameter of experimental wire is $D = 0.05$ cm (measured by a scale of least count 0.001 cm) and length is $L = 110$ cm (measured by a scale of least count 0.1 cm). A weight of 50 N causes an extension of $l = 0.125$ cm (measured by a micrometer of least count 0.001 cm). Find maximum possible error in the values of Young's modulus. Screw gauge and meter scale are free from error.* **(JEE 2004)**

Solution Young's modulus of elasticity is given by

$$Y = \frac{\text{stress}}{\text{strain}}$$

$$= \frac{F/A}{l/L} = \frac{FL}{lA} = \frac{FL}{l\left(\frac{\pi d^2}{4}\right)}$$

Substituting the values, we get

$$Y = \frac{50 \times 1.1 \times 4}{(1.25 \times 10^{-3}) \times \pi \times (5.0 \times 10^{-4})^2}$$

$$= 2.24 \times 10^{11} \text{ N/m}^2$$

Now,

$$\frac{\Delta Y}{Y} = \frac{\Delta L}{L} + \frac{\Delta l}{l} + 2\frac{\Delta d}{d}$$

$$= \left(\frac{0.1}{110}\right) + \left(\frac{0.001}{0.125}\right) + 2\left(\frac{0.001}{0.05}\right) = 0.0489$$

$$\Delta Y = (0.0489)Y$$

$$= (0.0489) \times (2.24 \times 10^{11}) \text{ N/m}^2$$

$$= 1.09 \times 10^{10} \text{ N/m}^2$$

Ans.

INTRODUCTORY EXERCISE 3.4

1. A student performs an experiment to determine Young's modulus of a wire, exactly 2 m long by Searle's method. In a particular reading the student measures the extension in the length of the wire to be 0.8 mm with an uncertainty of ± 0.05 mm at a load of 1.0 kg . The student also measures the diameter of the wire to be 0.4 mm with an uncertainty of ± 0.01 mm. Take $g = 9.8 \text{ m/s}^2$ (exact). Find Young's modulus of elasticity with limits of error.
2. Which of the following is wrong regarding Searle's apparatus method in finding Young's modulus of a given wire ?
   (a) Average elongation of wire will be determined with a particular load while increasing the load and decreasing the load.
   (b) Reference wire will be just taut and experimental wire will undergo for elongation.
   (c) Air bubble in the spirit level will be disturbed from the central position due to relative displacement between the wires due to elongation.
   (d) Average elongation of the wires is to be determined by increasing the load attached to both the wires.

3.5 Determination of Specific Heat

Determination of Specific Heat Capacity of a given Solid

Specific heat of a solid can be determined by the **"Method of Mixture"** using the concept of the **"Law of Heat Exchange"** i.e.

Heat lost by hot body = Heat gained by cold body

The method of mixture is based on the fact that when a hot solid body is mixed with a cold body, the hot body loses heat and the cold body absorbs heat until thermal equilibrium is attained. At equilibrium, final temperature of mixture is measured. The specific heat of the solid is calculated with the help of the law of heat exchange.

Let

Mass of solid = m_s kg
Mass of liquid = m_l kg
Mass of calorimeter = m_c kg
Initial temperature of solid = T_sK
Initial temperature of liquid = T_lK
Initial temperature of the calorimeter = T_cK
Specific heat of solid = c_s
Specific heat of liquid = c_l
Specific heat of the material of the calorimeter = c_c
Final temperature of the mixture = T K

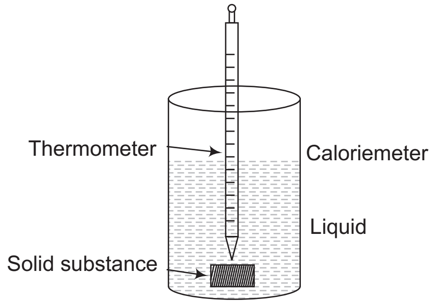


Fig. 3.22

According to the law of heat exchange

$$Q_{\text{Lost by solid}} = Q_{\text{Gained by liquid}} + Q_{\text{Gained by calorimeter}}$$

$$m_s c_s (T_s - T) = m_l c_l (T - T_l) + m_c c_c (T - T_c)$$

$$c_s = \frac{m_l c_l (T - T_1) + m_c c_c (T - T_c)}{m_s (T_s - T)}$$

Which is the required value of specific heat of solid in J/kg-K.

Determination of Specific Heat Capacity of the given Liquid by the Method of Mixtures

To determine the specific heat capacity of a liquid by the method of mixtures a solid of known specific heat capacity is taken and the given liquid is taken in the calorimeter in place of water. Suppose a solid of mass m_s and specific heat capacity c_s is heated to T_2°C and then mixed with m_1 mass of liquid of specific heat capacity c_1 at temperature T_1. The temperature of the mixture is T. Then,

Heat lost by the solid = $m_s c_s (T_2 - T)$

Heat gained by the liquid plus calorimeter = $(m_1 c_1 + m_c c_c)(T - T_1)$

By law of heat exchange,

$$\text{Heat lost} = \text{Heat gained}$$

$$\therefore \quad \boxed{m_s c_s (T_2 - T) = (m_1 c_1 + m_c c_c)(T - T_1)}$$

From this equation, we calculate the value of c_1. However, the procedure remains exactly the same as done previously.

Note *Specific heat is also called specific heat capacity and may be denoted by S, similarly temperature by* θ.

Example 3.11 *The mass, specific heat capacity and the temperature of a solid are* 1000 *g,* $\frac{1}{2}$ *cal/g-°C and* 80°*C respectively. The mass of the liquid and the calorimeter are* 900 *g and* 200 *g. Initially, both are at room temperature* 20°*C. Both calorimeter and the solid are made of same material. In the steady state, temperature of mixture is* 40°*C, then find the specific heat capacity of the unknown liquid.*

Solution m_1 = mass of solid $= 1000\,\text{g}$, S_1 = specific heat of solid $= \frac{1}{2}$ cal/g-° C

$= S_2$ or specific heat of calorimeter

m_2 = mass of calorimeter $= 200\,\text{g}$

m_3 = mass of unknown liquid $= 900\,\text{g}$

S_3 = specific heat of unknown liquid

From law of heat exchange,

Heat given by solid = Heat taken by calorimeter + Heat taken by unknown liquid

$$\therefore \qquad m_1 S_1 \, |\Delta\theta_1| = m_2 S_2 \, |\Delta\theta_2| + m_3 S_3 \, |\Delta\theta_3|$$

$$\therefore \qquad 1000 \times \frac{1}{2} \times (80 - 40) = 200 \times \frac{1}{2} (40 - 20) + 900 \times S_3 \, (40 - 20)$$

Solving this equation we get, $S_3 = 1\,\text{cal/g-° C}$ **Ans.**

Electrical Calorimeter

Figure shows an electrical calorimeter to determine specific heat capacity of an unknown liquid. We take a known quantity of liquid in an insulated calorimeter and heat it by passing a known current (i) through a heating coil immersed within the liquid. First of all, mass of empty calorimeter is measured and suppose it is m_1. Then, the unknown liquid is poured in it. Now, the combined mass (of calorimeter and liquid) is measured and let it be m_2. So, the mass of unknown liquid is $(m_2 - m_1)$. Initially, both are at room temperature (θ_0).

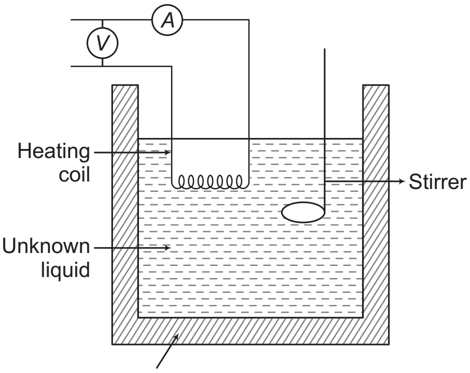


Fig. 3.23

Now, current i is passed through the heating coil at a potential difference V for time t. Due to this heat, the temperature of calorimeter and unknown liquid increase simultaneously. Suppose the final temperature is θ_f. If there is no heat loss to the surroundings, then

Heat supplied by the heating coil = heat absorbed by the liquid + heat absorbed by the calorimeter.

$$\therefore \qquad Vit = (m_2 - m_1)\, S_l \, (\theta_f - \theta_0) + m_1 S_c (\theta_f - \theta_0)$$

Here S_l = Specific heat of unknown liquid and

S_c = Specific heat of calorimeter

Solving this equation we get, $$\boxed{S_l = \left(\frac{1}{m_2 - m_1}\right)\left[\frac{Vit}{\theta_f - \theta_0} - m_1 S_c\right]}$$

Note *The sources of error in this experiment are errors due to improper connection of the heating coil and the radiation losses.*

Example 3.12 *In electrical calorimeter experiment, voltage across the heater is* 100.0 *V and current is* 10.0 *A. Heater is switched on for* $t = 700.0$ *s. Room temperature is* $\theta_0 = 10.0°C$ *and final temperature of calorimeter and unknown liquid is* $\theta_f = 73.0°C$. *Mass of empty calorimeter is* $m_1 = 1.0$ *kg and combined mass of calorimeter and unknown liquid is* $m_2 = 3.0$ *kg. Find the specific heat capacity of the unknown liquid in proper significant figures. Specific heat of calorimeter* $= 3.0 \times 10^3$ *J/kg°C*

Solution Given , $V = 100.0$ V, $i = 10.0$ A, $t = 700.0$ s, $\theta_0 = 10.0°$ C , $\theta_f = 73.0°$ C ,

$m_1 = 1.0$ kg and $m_2 = 3.0$ kg

Substituting the values in the expression,

$$S_l = \left(\frac{1}{m_2 - m_1}\right)\left[\frac{Vit}{\theta_f - \theta_0} - m_1 S_c\right]$$

we have,

$$S_l = \frac{1}{3.0 - 1.0}\left[\frac{(100.0)(10.0)(700.0)}{73.0 - 10.0} - (1.0)\,(3.0 \times 10^3\,)\right]$$

$$= 4.1 \times 10^3 \text{ J/kg° C}$$ **Ans.**

(According to the rules of significant figures)

3.6 Speed of Sound using Resonance Tube

Apparatus

Figure shows a resonance tube. It consists of a long vertical glass tube T. A metre scale S (graduated in mm) is fixed adjacent to this tube. The zero of the scale coincides with the upper end of the tube. The lower end of the tube T is connected to a reservoir R of water tube through a pipe P. The water level in the tube can be adjusted by the adjustable screws attached with the reservoir. The vertical adjustment of the tube can be made with the help of levelling screws. For fine adjustments of the water level in the tube, the pinchcock is used.

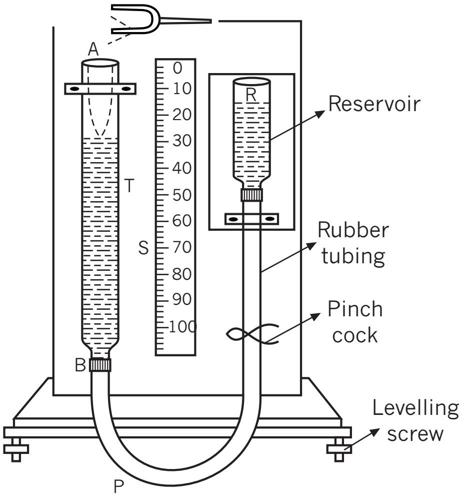


Fig. 3.24

Principle

If a vibrating tuning fork (of known frequency) is held over the open end of the resonance tube T, then resonance is obtained at some position as the level of water is lowered. If e is the end correction of the tube and l_1 is the length from the water level to the top of the tube, then

$$l_1 + e = \frac{\lambda}{4} = \frac{1}{4}\left(\frac{v}{f}\right) \qquad \text{...(i)}$$

Here, v is the speed of sound in air and f is the frequency of tuning fork (or air column). Now, the water level is further lowered until a resonance is again obtained. If l_2 is the new length of air column, Then,

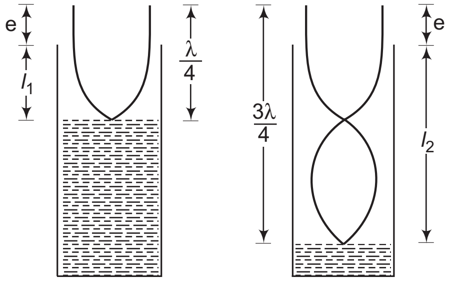


Fig. 3.25

$$l_2 + e = \frac{3\lambda}{4} = \frac{3}{4}\left(\frac{v}{f}\right) \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$l_2 - l_1 = \frac{1}{2}\left(\frac{v}{f}\right) \quad \text{or} \quad \boxed{v = 2f\,(l_2 - l_1)} \qquad \text{...(iii)}$$

So, from Eq. (iii) we can find speed of sound v.

Note *We have nothing to do with the end correction e, as far as v is concerned.*

Example 3.13 *Corresponding to given observation calculate speed of sound. Frequency of tuning fork = 340 Hz*

Resonance	Length from the water level (in cm)	
	During falling	During rising
First	23.9	24.1
Second	73.9	74.1

Solution Mean length from the water level in first resonance is

$$l_1 = \frac{23.9 + 24.1}{2}$$
$$= 24.0 \text{ cm}$$

Similarly, mean length from the water level in second resonance is

$$l_2 = \frac{73.9 + 74.1}{2}$$
$$= 74.0 \text{ cm}$$

∴ Speed of sound,

$$v = 2f\,(l_2 - l_1)$$
$$= 2 \times 340\,(0.740 - 0.240)$$
$$= 340 \text{ m/s}$$

Ans.

Example 3.14 *If a tuning fork of frequency* $(340 \pm 1\%)$ *is used in the resonance tube method and the first and second resonance lengths are* 20.0 *cm and* 74.0 *cm respectively. Find the maximum possible percentage error in speed of sound.*

Solution $l_1 = 20.0 \text{ cm}$

$$\Rightarrow \qquad \Delta l_1 = 0.1 \text{ cm}$$

$$\Rightarrow \qquad l_2 = 74.0 \text{ cm}$$

$$\Rightarrow \qquad \Delta l_2 = 0.1 \text{ cm}$$

$$v = 2f(l_2 - l_2)$$

$$\therefore \qquad \frac{\Delta v}{v} \times 100 = \frac{\Delta f}{f} \times 100 + \left(\frac{\Delta l_1 + \Delta l_2}{l_2 - l_1}\right) \times 100$$

$$= 1\% + \left(\frac{0.1 + 0.1}{74.0 - 20.0}\right) \times 100$$

$$= 1\% + 0.37\% = 1.37\,\%$$

Ans.

INTRODUCTORY EXERCISE 3.5

1. In the experiment for the determination of the speed of sound in air using the resonance column method, the length of the air column that resonates in the fundamental mode, with a tuning fork is 0.1 m. When this length is changed to 0.35 m, the same tuning fork resonates with the first overtone. Calculate the end correction. **(JEE 2003)**

(a) 0.012 m
(b) 0.025 m
(c) 0.05 m
(d) 0.024 m

2. A student is performing the experiment of resonance column. The diameter of the column tube is 4 cm. The frequency of the tuning fork is 512 Hz. The air temperature is 38° C in which the speed of sound is 336 m/s. The zero of the meter scale coincides with the top end of the resonance column tube. When the first resonance occurs, the reading of the water level in the column is **(JEE 2012)**

(a) 14.0 cm
(b) 15.2 cm
(c) 6.4 cm
(d) 17.6 cm

3.7 Verification of Ohm's Law using Voltmeter and Ammeter

Ohm's law states that the electric current I flowing through a conductor is directly proportional to the potential difference (V) across its ends provided that the physical conditions of the conductor (such as temperature, dimensions, etc.) are kept constant. Mathematically,

$$\boxed{V \propto I \quad \text{or} \quad V = IR}$$

Here, R is a constant known as resistance of the conductor and depends on the nature and dimensions of the conductor.

Circuit Diagram The circuit diagram is as shown below.

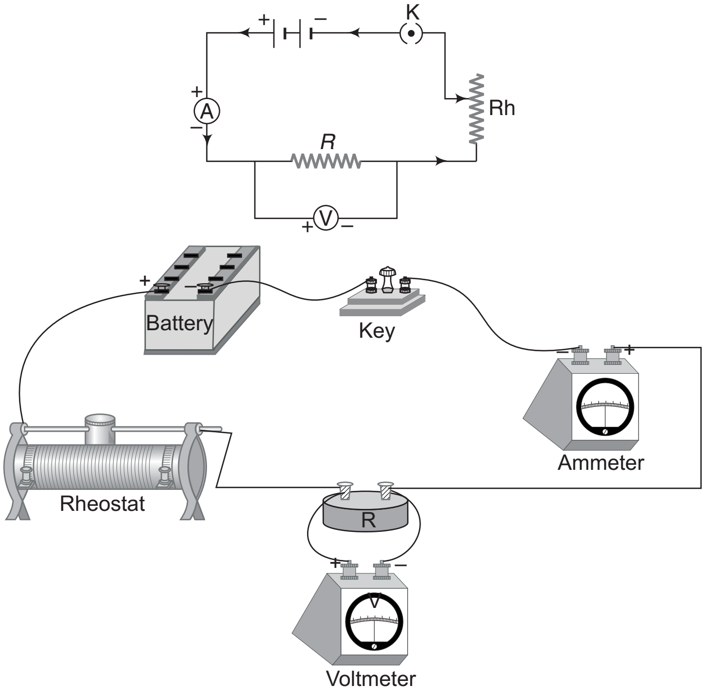


Fig. 3.26

Procedure

By shifting the rheostat contact, readings of ammeter and voltmeter are noted down. At least six set of observations are taken. Then, a graph is plotted between potential difference V and current I. The graph comes to be a straight line as shown in figure.

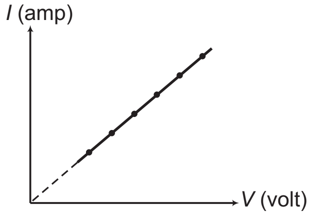


Fig. 3.27

Result

It is found from the graph that the ratio $\frac{V}{I}$ is constant. Hence, current voltage relationship is established, i.e. $V \propto I$. It means Ohm's law is established.

Precautions

1. The connections should be clean and tight.
2. Rheostat should be of low resistance.
3. Thick copper wire should be used for connections.
4. The key should be inserted only while taking observations to avoid heating of resistance.
5. The effect of finite resistance of the voltmeter can be over come by using a high resistance instrument or a potentiometer.
6. The lengths of connecting wires should be minimised as much as possible.

Error Analysis

The error in computing the ratio

$$R = \frac{V}{I} \text{ is given by}$$

$$\boxed{\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I}}$$

where, ΔV and ΔI are the order of the least counts of the instruments used.

Example 3.15 *What result do you expect in above experiment, if by mistake, voltmeter is connected in series with the resistance.*

Solution Due to high resistance of voltmeter, current (and therefore reading of ammeter) in the circuit will be very low.

Example 3.16 *What result do you expect in above experiment if by mistake, ammeter is connected in parallel with voltmeter and resistance as shown in figure?*

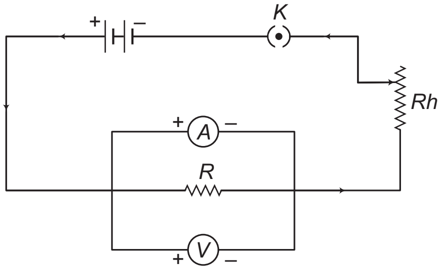


Fig. 3.28

Solution As ammeter has very low resistance, therefore most of the current will pass through the ammeter so reading of ammeter will be very large.

Example 3.17 *In the experiment of Ohm's law, when potential difference of* 10.0 *V is applied, current measured is* 1.00 *A. If length of wire is found to be* 10.0 *cm and diameter of wire* 2.50 *mm, then find maximum permissible percentage error in resistivity.*

Solution

$$R = \frac{\rho l}{A} = \frac{V}{I} \qquad \text{...(i)}$$

where, ρ = resistivity and

A = cross sectional area

Therefore, from Eq. (i)

$$\rho = \frac{AV}{lI} = \frac{\pi d^2 V}{4lI} \qquad \text{...(ii)}$$

where, $$A = \frac{\pi d^2}{4} \qquad (d = \text{diameter})$$

From Eq. (ii), we can see that maximum permissible percentage error in ρ will be

$$\frac{\Delta\rho}{\rho} \times 100 = \left[2\left(\frac{\Delta d}{d}\right) + \left(\frac{\Delta V}{V}\right) + \left(\frac{\Delta l}{l}\right) + \left(\frac{\Delta I}{I}\right)\right] \times 100$$

$$= \left[2 \times \frac{0.01}{2.50} + \frac{0.1}{10.0} + \frac{0.1}{10.0} + \frac{0.01}{1.00}\right] \times 100$$

$$= 3.8\,\%$$ **Ans.**

Example 3.18 *Draw the circuit for experimental verification of Ohm's law using a source of variable DC voltage, a main resistance of* 100 Ω, *two galvanometers and two resistances of values* 10^6 Ω *and* 10^{-3} Ω *respectively. Clearly show the positions of the voltmeter and the ammeter.* **[JEE 2004]**

Solution

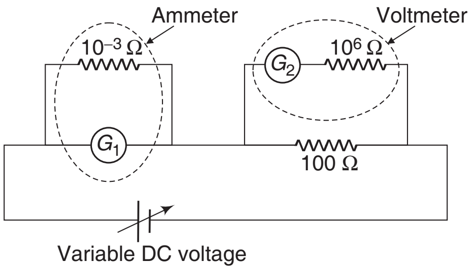
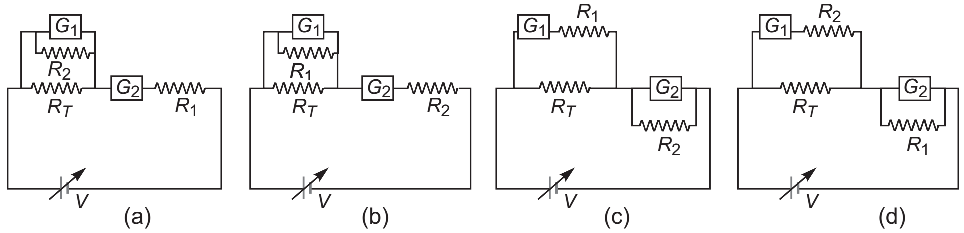


Fig. 3.29

INTRODUCTORY EXERCISE 3.6

1. In an experiment, current measured is, $I = 10.0$ A, potential difference measured is $V = 100.0$ V, length of the wire is 31.4 cm and the diameter of the wire is 2.00 mm (all in correct significant figures). Find resistivity of the wire in correct significant figures. [Take π = 3.14, exact]
2. In the previous question, find the maximum permissible percentage error in resistivity and resistance.
3. To verify Ohm's law, a student is provided with a test resistor R_T, a high resistance R_1, a small resistance R_2, two identical galvanometers G_1 and G_2, and a variable voltage source V. The correct circuit to carry out the experiment is



(a) (b) (c) (d)

Fig. 3.30

3.8 Meter Bridge Experiment

Meter bridge works on Wheat stone's bridge principle and is used to find the unknown resistance (X) and its specific resistance (or resistivity).

Theory

As the metre bridge wire AC has uniform material density and area of cross-section, its resistance is proportional to its length. Hence, AB and BC are the ratio arms and their resistances correspond to P and Q respectively.

Thus,
$$\frac{\text{Resistance of } AB}{\text{Resistance of } BC} = \frac{P}{Q} = \frac{\lambda l}{\lambda\,(100 - l)} = \frac{l}{100 - l}$$

Here, λ is the resistance per unit length of the bridge wire.

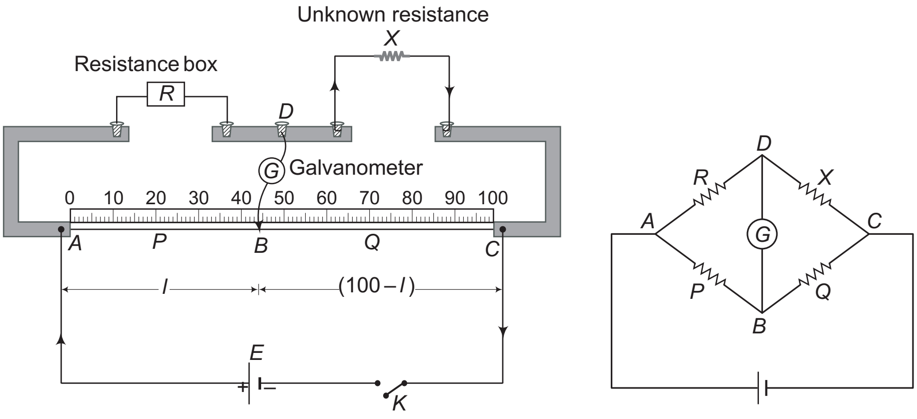


Fig. 3.31

Hence, according to Wheatstone's bridge principle,

When current through galvanometer is zero or bridge is balanced, then
$$\frac{P}{Q} = \frac{R}{X}$$

or
$$X = \frac{Q}{P} R$$

∴
$$\boxed{X = \left(\frac{100 - l}{l}\right) R} \qquad \text{...(i)}$$

So, by knowing R and l unknown resistance X can be determined.

Specific Resistance From resistance formula,
$$X = \rho \frac{L}{A}$$

or
$$\rho = \frac{XA}{L}$$

For a wire of radius r or diameter $D = 2r$,

$$A = \pi r^2 = \frac{\pi D^2}{4}$$

or $$\rho = \frac{X\pi D^2}{4L} \qquad \ldots\text{(ii)}$$

By knowing X, D and L we can find specific resistance of the given wire by Eq. (ii).

Precautions

1. The connections should be clean and tight.
2. Null point should be brought between 40 cm and 60 cm.
3. At one place, diameter of wire (D) should be measured in two mutually perpendicular directions.
4. The jockey should be moved gently over the bridge wire so that it does not rub the wire.

End Corrections

In meter bridge, some extra length (under the metallic strips) comes at points A and C. Therefore, some additional length (α and β) should be included at the ends. Here, α and β are called the end corrections. Hence in place of l we use $l + \alpha$ and in place of $100 - l$ we use $100 - l + \beta$.

To find α and β, use known resistors R_1 and R_2 in place of R and X and suppose we get null point length equal to l_1. Then,

$$\frac{R_1}{R_2} = \frac{l_1 + \alpha}{100 - l_1 + \beta} \qquad \text{...(i)}$$

Now, we interchange the positions of R_1 and R_2 and suppose the new null point length is l_2. Then,

$$\frac{R_2}{R_1} = \frac{l_2 + \alpha}{100 - l_2 + \beta} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\boxed{\begin{aligned} \alpha &= \frac{R_2 l_1 - R_1 l_2}{R_1 - R_2} \\ \text{and} \quad \beta &= \frac{R_1 l_1 - R_2 l_2}{R_1 - R_2} - 100 \end{aligned}}$$

Example 3.19 *If resistance R_1 in resistance box is 300 Ω, then the balanced length is found to be 75.0 cm from end A. The diameter of unknown wire is 1 mm and length of the unknown wire is 31.4 cm. Find the specific resistance of the unknown wire.*

Solution $$\frac{R}{X} = \frac{l}{100 - l}$$

$$\Rightarrow \qquad X = \left(\frac{100 - l}{l}\right) R = \left(\frac{100 - 75}{75}\right)(300) = 100\,\Omega$$

Now, $$X = \frac{\rho l}{A} = \frac{\rho l}{(\pi d^2/4)}$$

$$\therefore \qquad \rho = \frac{\pi d^2 X}{4l}$$

$$= \frac{(22/7)(10^{-3})^2(100)}{(4)(0.314)}$$

$$= 2.5 \times 10^{-4}\ \Omega\text{-m}$$ **Ans.**

Example 3.20 *In a meter bridge, null point is 20 cm, when the known resistance R is shunted by 10 Ω resistance, null point is found to be shifted by 10 cm. Find the unknown resistance X.*

Solution $\frac{R}{X} = \frac{l}{100 - l}$

$$\therefore \qquad X = \left(\frac{100 - l}{l}\right) R$$

$$\text{or} \qquad X = \left(\frac{100 - 20}{20}\right) R = 4R \qquad \text{...(i)}$$

When known resistance R is shunted, its net resistance will decrease. Therefore, resistance parallel to this (i.e. P) should also decrease or its new null point length should also decrease.

$$\therefore \qquad \frac{R'}{X} = \frac{l'}{100 - l'}$$

$$= \frac{20 - 10}{100 - (20 - 10)} = \frac{1}{9}$$

$$\text{or} \qquad X = 9R' \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$4R = 9R' = 9\left[\frac{10R}{10 + R}\right]$$

Solving this equation, we get

$$R = \frac{50}{4}\ \Omega$$

Now, from Eq. (i), the unknown resistance

$$X = 4R = 4\left(\frac{50}{4}\right)$$

$$\text{or} \qquad X = 50\ \Omega$$ **Ans.**

Note *R′ is resultant of R and 10 Ω in parallel.*

$$\therefore \qquad \frac{1}{R'} = \frac{1}{10} + \frac{1}{R}$$

$$\text{or} \qquad R' = \frac{10R}{10 + R}$$

Example 3.21 *If we use 100 Ω and 200 Ω in place of R and X we get null point deflection, l = 33 cm. If we interchange the resistors, the null point length is found to be 67 cm. Find end corrections α and β.*

Solution

$$\alpha = \frac{R_2 l_1 - R_1 l_2}{R_1 - R_2} = \frac{(200)(33) - (100)(67)}{100 - 200} = 1\,\text{cm}$$ **Ans.**

$$\beta = \frac{R_1 l_1 - R_2 l_2}{R_1 - R_2} - 100$$

$$= \frac{(100)(33) - (200)(67)}{100 - 200} - 100$$

$$= 1\,\text{cm}$$ **Ans.**

INTRODUCTORY EXERCISE 3.7

1. A resistance of 2 Ω is connected across one gap of a meter bridge (the length of the wire is 100 cm) and an unknown resistance, greater than 2 Ω, is connected across the other gap. When these resistance are interchanged, the balance point shifts by 20 cm. Neglecting any corrections, the unknown resistance is **(JEE 2007)**

   (a) 3 Ω (b) 4 Ω (c) 5 Ω (d) 6 Ω

2. A meter bridge is set-up as shown in figure, to determine an unknown resistance *X* using a standard 10 Ω resistor. The galvanometer shows null point when tapping-key is at 52 cm mark. The end-corrections are 1 cm and 2 cm respectively for the ends *A* and *B*. The determined value of *X* is **(JEE 2011)**

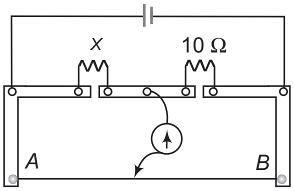


Fig. 3.32

   (a) 10.2 Ω (b) 10.6 Ω (c) 10.8 Ω (d) 11.1 Ω

3. R_1, R_2, R_3 are different values of *R*. *A*, *B* and *C* are the null points obtained corresponding to R_1, R_2 and R_3 respectively. For which resistor, the value of *X* will be the most accurate and why? **(JEE 2005)**

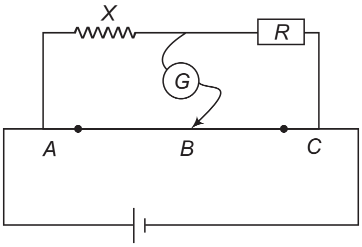


Fig. 3.33

3.9 Post Office Box

Post office box also works on the principle of Wheatstone's bridge.

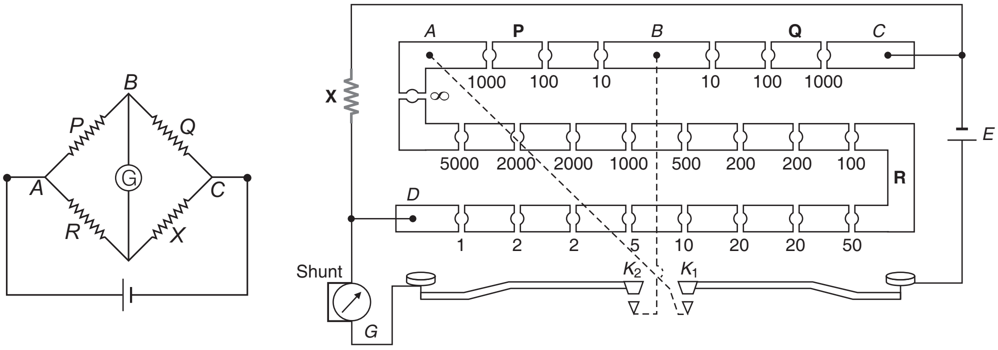


Fig. 3.34

In a Wheatstone's bridge circuit, if $\frac{P}{Q}=\frac{R}{X}$ then the bridge is balanced. So, unknown resistance $X=\frac{Q}{P}R$.

P and Q are set in arms AB and BC where we can have, $10\,\Omega, 100\,\Omega$ or $1000\,\Omega$ resistances to set any ratio $\frac{Q}{P}$.

These arms are called ratio arm, initially we take $Q=10\,\Omega$ and $P=10\,\Omega$ to set $\frac{Q}{P}=1$. The unknown resistance (X) is connected between C and D and battery is connected across A and C.

Now, put resistance in part A to D such that the bridge gets balanced. For this keep on increasing the resistance with $1\,\Omega$ interval, check the deflection in galvanometer by first pressing key K_1 then galvanometer key K_2.

Suppose at $R=4\,\Omega$, we get deflection towards left and at $R=5\,\Omega$, we get deflection towards right. Then, we can say that for balanced condition R should lie between 4 Ω to 5 Ω.

Now, $X=\frac{Q}{P}R=\frac{10}{10}R=R=4\,\Omega$ to $5\,\Omega$

Two get closer value of X, in the second observation, let us choose $\frac{Q}{P}=\frac{1}{10}$ i.e. $\left(\frac{P=100}{Q=10}\right)$

Suppose, now at $R=42$. We get deflection towards left and at $R=43$ deflection is towards right. So $R\in(42, 43)$.

Now, $X=\frac{Q}{P}R=\frac{10}{100}R=\frac{1}{10}R$, where $R\in(42, 43\,\Omega)$. Now, to get further closer value take $\frac{Q}{P}=\frac{1}{100}$ and so on.

The observation table is shown below.

S.No.	Resistance in the Ratio arm		Resistance in arm (*AD* (*R*) (ohm)	Direction of deflection	Unknown resistance $X = \frac{Q}{P} \times R$ (ohm)
	AB (*P*) (ohm)	*BC* (*Q*) (ohm)			
1	10	10	4	Left	4 to 5
			5	Right	
2	100	10	40	Left (large)	(4.2 to 4.3)
			50	Right (large)	
			42	Left	
			43	Right	
3	1000	10	420	Left	4.25
			424	Left	
			425	No deflection	
			426	Right	

So, the correct value of X is 4.25 Ω

Example 3.22 *To locate null point, deflection battery key* (K_1) *is pressed before the galvanometer key* (K_2). *Explain why?*

Solution If galvanometer key K_2 is pressed first then just after closing the battery key K_1 current suddenly increases.

So, due to self induction, a large back emf is generated in the galvanometer, which may damage the galvanometer.

Example 3.23 *What are the maximum and minimum values of unknown resistance X, which can be determined using the post office box shown in the Fig.* 3.34 *?*

Solution

$$X = \frac{QR}{P}$$

$$\therefore \quad X_{\max} = \frac{Q_{\max} R_{\max}}{P_{\min}}$$

$$= \frac{1000}{10}(11110)$$

$$= 1111 \, \text{k}\Omega \qquad \textbf{Ans.}$$

$$X_{\min} = \frac{Q_{\min} R_{\min}}{P_{\max}}$$

$$= \frac{(10)(1)}{1000}$$

$$= 0.01 \, \Omega \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 3.8

1. In post office box experiment, if $\frac{Q}{P} = \frac{1}{10}$. In R if 142 Ω is used then we get deflection towards right and if $R = 143\ \Omega$, then deflection is towards left. What is the range of unknown resistance?
2. What is the change in experiment if battery is connected between B and C and galvanometer is connected across A and C ?
3. For the post office box arrangement to determine the value of unknown resistance, the unknown resistance should be connected between **(JEE 2004)**

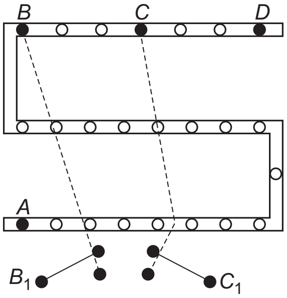


Fig. 3.35

(a) B and C (b) C and D (c) A and D (d) B_1 and C_1

3.10 Focal Length of a Concave Mirror using *u-v* Method

In this experiment, a knitting needle is used as an object O mounted in front of the concave mirror.

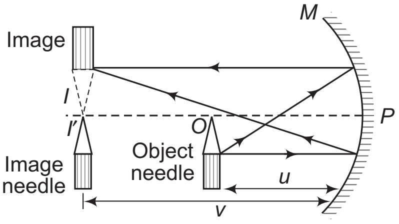


Fig. 3.36

First of all, we make a rough estimation of f. For this, make a sharp image of a far away object (like sun) on a filter paper. The image distance of the far object will be an approximate estimation of focal length f.

Now, the object needle is kept beyond F, so that its real and inverted image I can be formed. You can see this inverted image in the mirror by closing your one eye and keeping the other eye along the pole of the mirror.

To locate the position of the image use a second needle and shift this needle such that its peak coincide with the image. The second needle gives the distance of image v. This image is called image needle I. Note the object distance u and image distance v from the mm scale on optical bench.

Take some more observations in similar manner.

Determining *f* from *u-v* Observation

Method 1

Use mirror formula $\frac{1}{f}=\frac{1}{v}+\frac{1}{u}$ to find focal length from each *u-v* observation. Finally taking average of all we can find the focal length.

Method 2

The relation between object distance *u* and the image *v* from the pole of the mirror is given by

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}\cdot$$

where, f is the focal length of the mirror. The focal length of the concave mirror can be obtained from $\frac{1}{v}$ *versus* $\frac{1}{u}$ graph.

When the image is real (of course only upon then it can be obtained on screen), the object lies between focus (F) and infinity. In such a situation, *u*, *v* and f all are negative. Hence, the mirror formula,

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

becomes,

$$-\frac{1}{v}-\frac{1}{u}=-\frac{1}{f}$$

or again,

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

or

$$\frac{1}{v}=-\frac{1}{u}+\frac{1}{f}$$

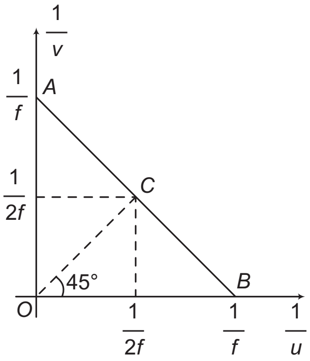


Fig. 3.37

Comparing with $y=mx+c$, the desired graph will be a straight line with slope -1 and intercept equal to $\frac{1}{f}$.

The corresponding $\frac{1}{v}$ *versus* $\frac{1}{u}$ graph is as shown in Fig. 3.37. The intercepts on the horizontal and vertical axes are equal. It is equal to $\frac{1}{f}$. A straight line *OC* at an angle 45° with the horizontal axis intersects line *AB* at *C*. The coordinates of point *C* are $\left(\frac{1}{2f},\frac{1}{2f}\right)\cdot$ The focal length of the mirror can be calculated by measuring the coordinates of either of the points *A*, *B* or *C*.

Method 3

From *u-v* curve

Relation between *u* and *v* is

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f} \quad \text{...(i)}$$

After substituting *u*, *v* and f with sign (all negative) we get the same result.

For an object kept beyond F, u-v graph is as shown in figure. If we draw a line

$$u = v \qquad \text{...(ii)}$$

then, it intersects the graph at point $P\ (2f, 2f)$.

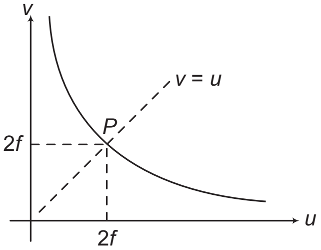


Fig. 3.38

From u-v data plot v *versus* u curve and draw a line bisecting the axis. Find the intersection point and equate them to $(2f, 2f)$

By joining u_n and v_n : Mark $u_1, u_2, u_3 \ldots\ldots u_n$ along x-axis and $v_1, v_2, v_3 \ldots\ldots v_n$ along y-axis. If we join u_1 with v_1, u_2 with v_2, u_3 with v_3 and so on then all lines intersects at a common point (f, f).

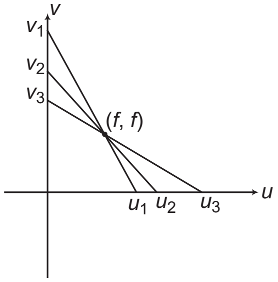


Fig. 3.39

Explanation

General equation of a line joining two points $P\ (a, 0)$ and $Q\ (0, b)$ is

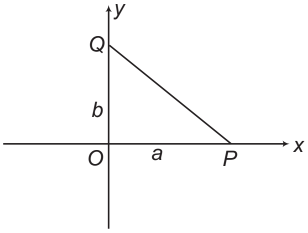


Fig. 3.40

$$y = mx + c$$

$$\Rightarrow \qquad y = \frac{-b}{a}x + b$$

$$\Rightarrow \qquad \frac{x}{a} + \frac{y}{b} = 1$$

Now, line joining u_1 and v_1 will be

$$\frac{x}{u_1} + \frac{y}{v_1} = 1 \qquad \text{...(iii)}$$

where,

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f}$$

or

$$\frac{f}{u_1} + \frac{f}{v_1} = 1 \qquad \text{...(iv)}$$

Similarly, line joining u_2 and v_2 is

$$\frac{x}{u_2}+\frac{y}{v_2}=1 \quad \text{...(v)}$$

where,

$$\frac{f}{u_2}+\frac{f}{v_2}=1 \quad \text{...(vi)}$$

and line joining u_n and v_n is

$$\frac{x}{v_n}+\frac{y}{u_n}=1 \quad \text{...(vii)}$$

where,

$$\frac{f}{u_n}+\frac{f}{v_n}=1 \quad \text{...(viii)}$$

From Eq. (iv), (vi), (viii), we can say that $x = f$ and $y = f$ will satisfy all Eq. (iii), (v), (vii). So, point (f, f) will be the common intersection point of all the lines.

From u-v data, draw $u_1, u_2 \ldots\ldots u_n$ along x-axis and $v_1, v_2, \ldots\ldots v_n$ along y-axis. Join u_1 with v_1, u_2 with v_2, …… u_n with v_n. Find common intersection point and equate it to (f, f).

Index Error

In u - v method, we require the distance between object or image from the pole P of the mirror. This is called actual distance. But practically, we measure the distance between the indices A and B. This is called the observed distance. The difference between two is called the index error (e). This is constant for every observation.

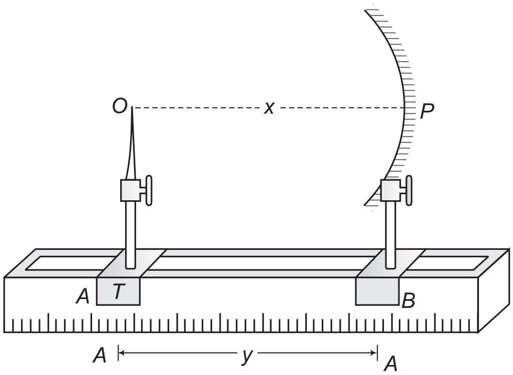


Fig. 3.41

Index error = Observed distance − Actual distance

To determine index error, mirror and object needle are placed at arbitery position. Measure the distances x and y as shown in figure.

So, index error is e = observed distance − Actual distance = $y - x$

once we get e, in every observation, we get

Actual distance = Observed distance (separation between the indices) − excess reading (e)

Example 3.24 *To find index error (e) distance between object needle and pole of the concave mirror is 20 cm. The separation between the indices of object needle and mirror was observed to be 20.2 cm. In some observation, the observed image distance is 20.2 cm and the object distance is 30.2 cm. Find*

(a) the index error e.

(b) focal length of the mirror f.

Solution (a) Index error e = observed distance – actual distance

= separation between indices – distance between object needle and pole of the mirror

$= 20.2 - 20.0 = 0.2$ cm **Ans.**

(b) $|u| = 30.2 - 0.2 = 30\,\text{cm}$

$\therefore$ $u = -30\,\text{cm}$

$|v| = 20.2 - 0.2 = 20\,\text{cm}$

$\therefore$ $v = -20\,\text{cm}$

Using the mirror formula,

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} = \frac{1}{-20} + \frac{1}{-30}$$

or $f = -12\,\text{cm}$ **Ans.**

Note *Since, it is a concave mirror, therefore focal length is negative.*

Example 3.25 *In u-v method to find focal length of a concave mirror, if object distance is found to be 10.0 cm and image distance was also found to be 10.0 cm, then find maximum permissible error in f.*

Solution Using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f} \qquad \text{...(i)}$$

we have,

$$\frac{1}{-10} + \frac{1}{-10} = \frac{1}{f}$$

$\Rightarrow$ $f = -5\,\text{cm}$ or $|f| = 5\,\text{cm}$

Now, differentiating Eq. (i).

we have,

$$\frac{-df}{f^2} = -\frac{du}{u^2} - \frac{dv}{v^2}$$

This equation can be written as

$$|\Delta f|_{\max} = \left[\frac{|\Delta u|}{u^2} + \frac{|\Delta v|}{v^2}\right](f^2)$$

Substituting the values we get,

$$|\Delta f|_{\max} = \left[\frac{0.1}{(10)^2} + \frac{0.1}{(10)^2}\right](5)^2 = 0.05\text{ cm}$$

$\therefore$ $|f| = (5 \pm 0.05)\,\text{cm}$ **Ans.**

Example 3.26 *A student performed the experiment of determination of focal length of a concave mirror by u-v method using an optical bench of length* 1.5 *m. The focal length of the mirror used is* 24 *cm. The maximum error in the location of the image can be* 0.2 *cm. The* 5 *sets of* (u, v) *values recorded by the student (in cm) are* (42, 56), (48, 48), (60, 40), (66, 33), (78, 39)*. The data set(s) that cannot come from experiment and is (are) incorrectly recorded, is (are)* **(JEE 2009)**

(a) (42, 56) *(b)* (48, 48) *(c)* (66, 33) *(d)* (78, 39)

Solution Values of options (c) and (d) do not match with the mirror formula,

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

3.11 Focal Length of a Convex Lens using *u-v* Method

In this experiment, a convex lens is fixed in position L and a needle is used as an object mounted in front of the convex lens.

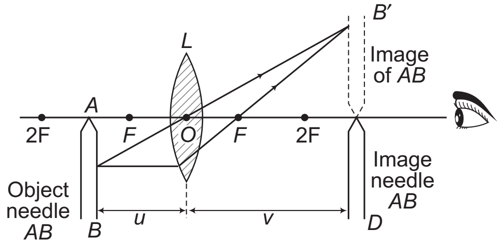


Fig. 3.42

First of all, we make a rough estimation of f. For estimating f roughly make a sharp image of a far away object (like sun) on a filter paper. The image distance of the far object will be an approximate estimation of focal length.

Now, the object needle is kept beyond F, so that its real and inverted image can be formed. To locate the position of the image, use a second needle and shift this needle such that its peak coincide with the image. The second needle gives the distance of image (v). Note the object distance u and image distance v from the mm scale on optical bench.

Take 4 to 5 more observations in similar manner.

Determining *f* from *u-v* Observations

Method 1

Use lens formula $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$ to find focal length corresponding to each u-v observation. Finally, take average of all.

Method 2

The relation between u, v and f for a convex lens is,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

Using the proper sign convention, u is negative, v and f are positive. So, we have,

$$\frac{1}{v}-\frac{1}{-u}=\frac{1}{f}$$

or

$$\frac{1}{v}=-\frac{1}{u}+\frac{1}{f}$$

Comparing with $y=mx+c$, $\frac{1}{v}$ *versus* $\frac{1}{u}$ graph is a straight line with slope -1 and intercept $\frac{1}{f}$. The corresponding graph is as shown in Fig. 3.43. Proceeding in the similar manner as discussed in case of a concave mirror the focal length of the lens can be calculated by measuring the coordinates of either of the points *A, B* and *C*.

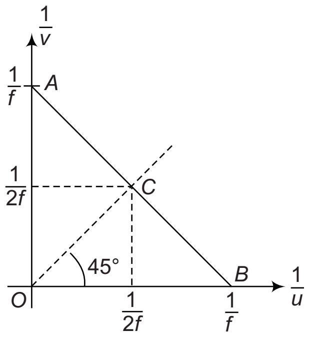


Fig. 3.43

The v *versus* u graph is as shown in the Fig. 3.44. By measuring the coordinates of point C whose coordinates are $(2f, 2f)$ we can calculate the focal length of the lens.

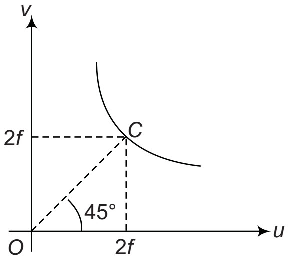


Fig. 3.44

Method 3

By joining u_n and v_n

Locate $u_1, u_2, u_3 \ldots\ldots u_n$ along x-axis and $v_1, v_2, v_3 \ldots\ldots v_n$ y-axis. If we join u_1 with v_1, u_2 with v_2, u_3 with v_3 and ………… so on. All lines intersect at a common point $(-f, f)$.

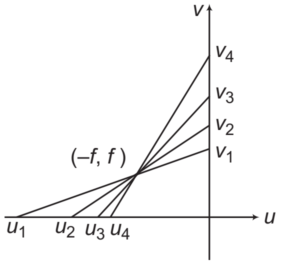


Fig. 3.45

From u-v data draw $u_1, u_2 \ldots\ldots u_n$ along x-axis and $v_1, v_2, \ldots\ldots v_n$ data on y-axis. Join u_1 and v_1, u_2 with v_2 …… u_n and v_n. Find common intersection point and equate it to $(-f, f)$.

Note *Index error is similar to the concave mirror.*

Example 3.27 *The graph between object distance u and image distance v for a lens is given below. The focal length of the lens is* **(JEE 2006)**

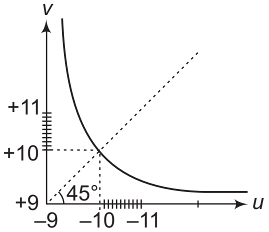


Fig. 3.46

(a) 5 ± 0.1 *(b)* 5 ± 0.05 *(c)* 0.5 ± 0.1 *(d)* 0.5 ± 0.05

Solution From the lens formula,

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u} \text{ we have,}$$

$$\frac{1}{f}=\frac{1}{10}-\frac{1}{-10} \text{ or } f=+5$$

Further, $$\Delta u = 0.1$$

and $$\Delta v = 0.1$$ (from the graph)

Now, differentiating the lens formula, we have

$$\frac{\Delta f}{f^2}=\frac{\Delta v}{v^2}+\frac{\Delta u}{u^2}$$

or $$\Delta f=\left(\frac{\Delta v}{v^2}+\frac{\Delta u}{u^2}\right)f^2$$

Substituting the values, we have

$$\Delta f=\left(\frac{0.1}{10^2}+\frac{0.1}{10^2}\right)(5)^2=0.05$$

$\therefore$ $$f \pm \Delta f = 5 \pm 0.05$$

$\therefore$ The correct option is (b).

Exercises

Objective Questions

1. For positive error, the correction is

(a) positive
(b) negative
(c) nil
(d) may be positive or negative

2. Screw gauge is said to have a negative error

(a) when circular scale zero coincides with base line of main scale
(b) when circular scale zero is above the base line of main scale
(c) when circular scale zero is below the base line of main scale
(d) None of the above

3. Vernier constant is the (One or more than one correct option may be correct) :

(a) value of one MSD divided by total number of divisions on the main scale
(b) value of one VSD divided by total number of divisions on the vernier scale
(c) total number of divisions on the main scale divided by total number of divisions on the vernier scale
(d) difference between the value of one main scale division and one vernier scale division

4. Least count of screw gauge is defined as

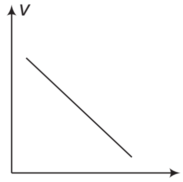
(a) $\dfrac{\text{distance moved by thimble on main scale}}{\text{number of rotation of thimble}}$

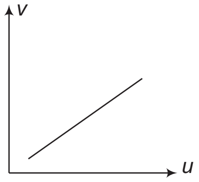
(b) $\dfrac{\text{pitch of the screw}}{\text{number of divisions on circular scale}}$

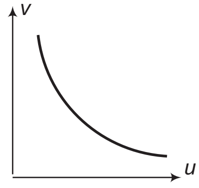
(c) $\dfrac{\text{number of rotation of thimble}}{\text{number of circular scale divisions}}$

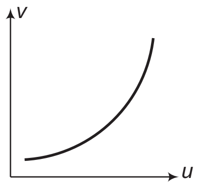
(d) None of the above

5. In an experiment to find focal length of a concave mirror, a graph is drawn between the magnitudes of u and v. The graph looks like

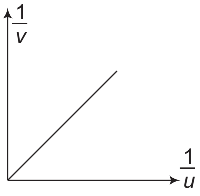
(a)


(b)


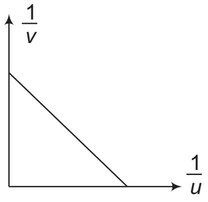
(c)


(d)


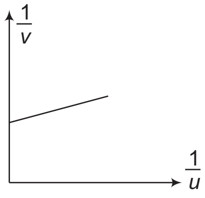
6. The graph between $\frac{1}{v}$ and $\frac{1}{u}$ for a concave mirror looks like



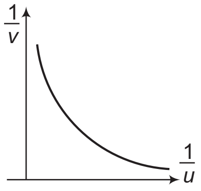
(a)



(b)

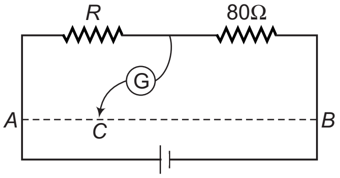


(c)



(d)

7. AB is a wire of uniform resistance. The galvanometer G shows no deflection when the length $AC = 20$ cm and $CB = 80$ cm. The resistance R is equal to



(a) 80 Ω (b) 10 Ω (c) 20 Ω (d) 40 Ω

8. Select the **incorrect** statement.

(a) If the zero of vernier scale does not coincide with the zero of the main scale, then the vernier callipers is said to be having zero error

(b) Zero correction has a magnitude equal to zero error but sign is opposite to that of zero error

(c) Zero error is positive when the zero of vernier scale lies to the left of the zero of the main scale

(d) Zero error is negative when the zero of vernier scale lies to the left of the zero of the main scale

9. In the Searle's experiment, after every step of loading, why should we wait for two minutes before taking the reading? (More than one options may be correct)

(a) So that the wire can have its desired change in length

(b) So that the wire can attain room temperature

(c) So that vertical oscillations can get subsided

(d) So that the wire has no change in its radius

10. In a meter bridge set up, which of the following should be the properties of the one meter long wire?

(a) High resistivity and low temperature coefficient

(b) Low resistivity and low temperature coefficient

(c) Low resistivity and high temperature coefficient

(d) High resistivity and high temperature coefficient

11. The mass of a copper calorimeter is 40 g and its specific heat in SI units is 4.2×10^2 J kg^{-1} °C^{-1}. The thermal capacity is

(a) 4 J °C^{-1} (b) 18.6 J

(c) 16.8 J /kg (d) 16.8 J °C^{-1}

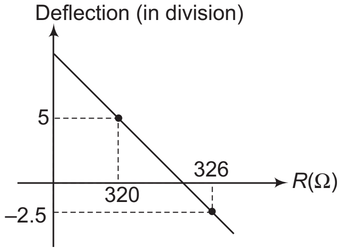
12. A graph is drawn with $\frac{1}{u}$ along x-axis and $\frac{1}{v}$ along the y-axis. If the intercept on the x-axis is $0.5\ \text{m}^{-1}$, the focal length of the lens is (in meter)

(a) 2.00 (b) 0.50

(c) 0.20 (d) 1.00

13. For a post office box, the graph of galvanometer deflection versus R (resistance pulled out of resistance box) for the ratio 100 : 1 is given as shown. Find the value of unknown resistance.

(a) 324 Ω
(b) 3.24 Ω
(c) 32.4 Ω
(d) None of the above



14. 1 cm on the main scale of a vernier callipers is divided into 10 equal parts. If 10 divisions of vernier coincide with 8 small divisions of main scale, then the least count of the calliper is

(a) 0.01 cm
(b) 0.02 cm
(c) 0.05 cm
(d) 0.005 cm

15. The vernier constant of a vernier callipers is 0.001 cm. If 49 main scale divisions coincide with 50 vernier scale divisions, then the value of 1 main scale division is

(a) 0.1 mm
(b) 0.5 mm
(c) 0.4 mm
(d) 1 mm

16. 1 cm of main scale of a vernier callipers is divided into 10 divisions. The least count of the callipers is 0.005 cm, then the vernier scale must have

(a) 10 divisions
(b) 20 divisions
(c) 25 divisions
(d) 50 divisions

17. Each division on the main scale is 1 mm. Which of the following vernier scales give vernier constant equal to 0.01 mm ?

(a) 9 mm divided into 10 divisions
(b) 90 mm divided into 100 divisions
(c) 99 mm divided into 100 divisions
(d) 9 mm divided into 100 divisions

18. A vernier callipers having 1 main scale division = 0.1 cm is designed to have a least count of 0.02 cm. If n be the number of divisions on vernier scale and m be the length of vernier scale, then

(a) $n = 10, m = 0.5$ cm
(b) $n = 9, m = 0.4$ cm
(c) $n = 10, m = 0.8$ cm
(d) $n = 10, m = 0.2$ cm

19. The length of a rectangular plate is measured by a meter scale and is found to be 10.0 cm. Its width is measured by vernier callipers as 1.00 cm. The least count of the meter scale and vernier calipers are 0.1 cm and 0.01 cm respectively. Maximum permissible error in area measurement is

(a) ± 0.2 cm^2
(b) ± 0.1 cm^2
(c) ± 0.3 cm^2
(d) zero

20. In the previous question, minimum possible error in area measurement can be

(a) ± 0.02 cm^2
(b) ± 0.01 cm^2
(c) ± 0.03 cm^2
(d) zero

21. The distance moved by the screw of a screw gauge is 2 mm in four rotations and there are 50 divisions on its cap. When nothing is put between its jaws, 20th division of circular scale coincides with reference line, and zero of linear scale is hidden from circular scale when two jaws touch each other or zero of circular scale is lying above the reference line. When plate is placed between the jaws, main scale reads 2 divisions and circular scale reads 20 divisions. Thickness of plate is

(a) 1.1 mm
(b) 1.2 mm
(c) 1.4 mm
(d) 1.5 mm

22. The end correction (e) is (l_1 = length of air column at first resonance and l_2 is length of air column at second resonance)

(a) $e = \frac{l_2 - 3l_1}{2}$ (b) $e = \frac{l_1 - 3l_2}{2}$

(c) $e = \frac{l_2 - 2l_1}{2}$ (d) $e = \frac{l_1 - 2l_2}{2}$

23. The end correction of a resonance tube is 1 cm. If shortest resonating length is 15 cm, the next resonating length will be

(a) 47 cm (b) 45 cm (c) 50 cm (d) 33 cm

24. A tuning fork of frequency 340 Hz is excited and held above a cylindrical tube of length 120 cm. It is slowly filled with water. The minimum height of water column required for resonance to be first heard (Velocity of sound = 340 ms^{-1}) is

(a) 25 cm (b) 75 cm (c) 45 cm (d) 105 cm

25. Two unknown frequency tuning forks are used in resonance column apparatus. When only first tuning fork is excited the 1st and 2nd resonating lengths noted are 10 cm and 30 cm respectively. When only second tuning fork is excited the 1st and 2nd resonating lengths noted are 30 cm and 90 cm respectively. The ratio of the frequency of the 1st to 2nd tuning fork is

(a) 1 : 3 (b) 1 : 2 (c) 3 : 1 (d) 2 : 1

26. In an experiment to determine the specific heat of aluminium, piece of aluminium weighing 500 g is heated to 100 °C. It is then quickly transferred into a copper calorimeter of mass 500 g containing 300 g of water at 30 °C. The final temperature of the mixture is found to be 46.8 °C. If specific heat of copper is 0.093 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$, then the specific heat of aluminium is

(a) 0.11 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$ (b) 0.22 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$

(c) 0.33 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$ (d) 0.44 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$

27. When 0.2 kg of brass at 100 °C is dropped into 0.5 kg of water at 20 °C, the resulting temperature is 23 °C. The specific heat of brass is

(a) $0.41 \times 10^3\ \text{Jkg}^{-1}\,{}^{\circ}\text{C}^{-1}$ (b) $0.41 \times 10^2\ \text{Jkg}^{-1}\,{}^{\circ}\text{C}^{-1}$

(c) $0.41 \times 10^4\ \text{Jkg}^{-1}\,{}^{\circ}\text{C}^{-1}$ (d) $0.41\ \text{Jkg}^{-1}\,{}^{\circ}\text{C}^{-1}$

28. In an experiment to determine the specific heat of a metal, a 0.20 kg block of the metal at 150 °C is dropped in a copper calorimeter (of water equivalent 0.025 kg) containing 150 cm^3 of water at 27 °C. The final temperature is 40 °C. The specific heat of the metal is

(a) 0.1 $\text{Jg}^{-1}\,{}^{\circ}\text{C}^{-1}$ (b) 0.2 $\text{Jg}^{-1}\,{}^{\circ}\text{C}^{-1}$

(c) 0.3 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$ (d) 0.1 cal $\text{g}^{-1}\,{}^{\circ}\text{C}^{-1}$

29. The resistance in the left and right gaps of a balanced meter bridge are R_1 and R_2. The balanced point is 50 cm. If a resistance of 24 Ω is connected in parallel to R_2, the balance point is 70 cm. The value of R_1 or R_2 is

(a) 12 Ω (b) 8 Ω (c) 16 Ω (d) 32 Ω

30. An unknown resistance R_1 is connected in series with a resistance of 10 Ω. This combination is connected to one gap of a meter bridge, while other gap is connected to another resistance R_2. The balance point is at 50 cm. Now, when the 10 Ω resistance is removed, the balance point shifts to 40 cm. Then, the value of R_1 is

(a) 60 Ω (b) 40 Ω

(c) 20 Ω (d) 10 Ω

31. Two resistances are connected in the two gaps of a meter bridge. The balance point is 20 cm from the zero end. When a resistance 15 Ω is connected in series with the smaller of two resistance, the null point shifts to 40 cm. The smaller of the two resistance has the value

(a) 8 Ω (b) 9 Ω

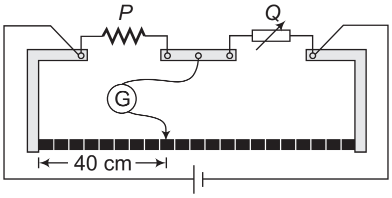
(c) 10 Ω (d) 12 Ω

32. In a meter bridge experiment, null point is obtained at 20 cm from one end of the wire when resistance X is balanced against another resistance Y. If $X < Y$, then the new position of the null point from the same end, if one decides to balance a resistance of $4X$ against Y will be at

(a) 50 cm (b) 80 cm

(c) 40 cm (d) 70 cm

33. In a metre bridge, the gaps are closed by two resistances P and Q and the balance point is obtained at 40 cm. When Q is shunted by a resistance of 10 Ω, the balance point shifts to 50 cm. The values of P and Q are



(a) $\frac{10}{3}\ \Omega, 5\ \Omega$ (b) $20\ \Omega, 30\ \Omega$

(c) $10\ \Omega, 15\ \Omega$ (d) $5\ \Omega, \frac{15}{2}\ \Omega$

Subjective Questions

1. What is the material of the wire of meter bridge ?

2. For determination of resistance of a coil, which of two methods is better Ohm's law method or meter bridge method ?

3. Which method is more accurate in the determination of f for a concave mirror.

(i) u *versus* v or (ii) $\frac{1}{u}$ *versus* $\frac{1}{v}$ graphs ?

4. Why is the second resonance found feebler than the first ?

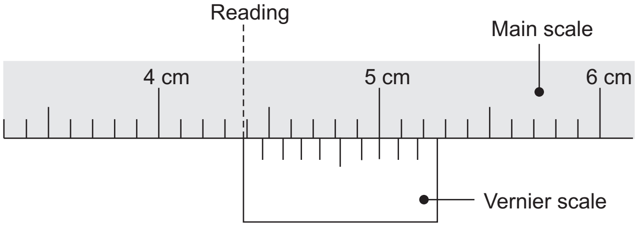
5. Why is the meter bridge suitable for resistance of moderate values only ?

6. Can we measure a resistance of the order of 0.160 Ω using a Wheatstone's bridge ? Support your answer with reasoning.

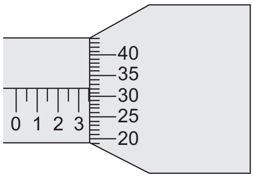
7. 19 divisions on the main scale of a vernier callipers coincide with 20 divisions on the vernier scale. If each division on the main scale is of 1 cm, determine the least count of instrument.

8. In a vernier callipers, 1 cm of the main scale is divided into 20 equal parts. 19 divisions of the main scale coincide with 20 divisions on the vernier scale. Find the least count of the instrument.

9. The diagram below shows part of the main scale and vernier scale of a vernier callipers, which is used to measure the diameter of a metal ball. Find the least count and the radius of the ball.



10. The given diagram represents a screw gauge. The circular scale is divided into 50 divisions and the linear scale is divided into millimeters. If the screw advances by 1 mm when the circular scale makes 2 complete revolutions, find the least count of the instrument and the reading of the instrument in the figure.



11. The pitch of a screw gauge is 0.5 mm and there are 50 divisions on the circular scale. In measuring the thickness of a metal plate, there are five divisions on the pitch scale (or main scale) and thirty fourth divisions coincide with the reference line. Calculate the thickness of the metal plate.

12. The pitch of a screw gauge is 1 mm and there are 50 divisions on its cap. When nothing is put in between the studs, 44th division of the circular scale coincides with the reference line and the zero of the main scale is not visible or zero of circular scale is lying above the reference line. When a glass plate is placed between the studs, the main scale reads three divisions and the circular scale reads 26 divisions. Calculate the thickness of the plate.

13. The pitch of a screw gauge is 1 mm and there are 100 divisions on its circular scale. When nothing is put in between its jaws, the zero of the circular scale lies 6 divisions below the reference line. When a wire is placed between the jaws, 2 linear scale divisions are clearly visible while 62 divisions on circular scale coincide with the reference line. Determine the diameter of the wire.

14. Least count of a vernier callipers is 0.01 cm. When the two jaws of the instrument touch each other the 5th division of the vernier scale coincide with a main scale division and the zero of the vernier scale lies to the left of the zero of the main scale. Furthermore while measuring the diameter of a sphere, the zero mark of the vernier scale lies between 2.4 cm and 2.5 cm and the 6th vernier division coincides with a main scale division. Calculate the diameter of the sphere.

15. The edge of a cube is measured using a vernier callipers. [9 divisions of the main scale is equal to 10 divisions of vernier scale and 1 main scale division is 1 mm]. The main scale division reading is 10 and 1st division of vernier scale was found to be coinciding with the main scale. The mass of the cube is 2.736 g. Calculate the density in g/cm^3 upto correct significant figures.

Answers

Introductory Exercise 3.1

1. 3.19 cm **2.** 1

Introductory Exercise 3.2

1. 10.65 mm **2.** 3.5 mm

Introductory Exercise 3.3

4. (i) Parabolic (ii) Straight line **7.** (c) **8.** (a)

Introductory Exercise 3.4

1. $(1.94 \pm 0.22) \times 10^{11}\ \mathrm{N/m^2}$ **2.** (d)

Introductory Exercise 3.5

1. (b) **2.** (b)

Introductory Exercise 3.6

1. $1.00 \times 10^{-4}\ \Omega \cdot \mathrm{m}$ **2.** 2.41 %, 1.1 % **3.** (c)

Introductory Exercise 3.7

1. (a) **2.** (b) **3.** *B* is most accurate

Introductory Exercise 3.8

1. 14.2 Ω to 14.3 Ω **3.** (c)

Exercises

Objective Questions

1.(b) **2.**(b) **3.**(d) **4.**(b) **5.**(c) **6.**(b) **7.**(c) **8.**(c) **9.**(a,b,c) **10.**(a)
11.(d) **12.**(a) **13.**(b) **14.**(b) **15.**(b) **16.**(b) **17.**(c) **18.**(c) **19.**(a) **20.**(d)
21.(d) **22.**(a) **23.**(a) **24.**(c) **25.**(c) **26.**(b) **27.**(a) **28.**(d) **29.**(d) **30.**(c)
31.(b) **32.**(a) **33.**(a)

Subjective Questions

1. Constantan **2.** Meter bridge method **3.** $\frac{1}{v}$ *versus* $\frac{1}{u}$ **4.** See the hints
5. See the hints **6.** No **7.** 0.05 cm **8.** 0.0025 cm
9. 0.01 cm, 2.18 cm **10.** 0.01 mm, 3.32 mm **11.** 2.84 mm **12.** 3.64 mm
13. 2.56 mm **14.** 2.51 cm **15.** 2.66 g/cm^3

04

Units and Dimensions

Chapter Contents

4.1 Units

4.2 Fundamental and Derived Units

4.3 Dimensions

4.4 Uses of Dimensions

4.1 Units

To measure a physical quantity we need some standard unit of that quantity. The measurement of the quantity is mentioned in two parts, the first part gives how many times of the standard unit and the second part gives the name of the unit. Thus, suppose I say that length of this wire is 5 metre. The numeric part 5 says that it is 5 times of the unit of length and the second part metre says that unit chosen here is metre.

4.2 Fundamental and Derived Units

There are a large number of physical quantities and every quantity needs a unit.
However, not all the quantities are independent. For example, if a unit of length is defined, a unit of volume is automatically obtained. Thus, we can define a set of fundamental quantities and all other quantities may be expressed in terms of the fundamental quantities. Fundamental quantities are only seven in numbers. Unit of all other quantities can be expressed in terms of the units of these seven quantities by multiplication or division.

Many different choices can be made for the fundamental quantities. For example, if we take length and time as the fundamental quantities then speed is a derived quantity and if we take speed and time as fundamental quantities then length is a derived quantity.

Several system of units are in use over the world. The units defined for the fundamental quantities are called fundamental units and those obtained for derived quantities are called the derived units.

SI Units

In 1971, General Conference on Weight and Measures held its meeting and decided a system of units which is known as the International System of Units. It is abbreviated as SI from the French name Le System International *d*' Unites. This system is widely used throughout the world. Table below gives the seven fundamental quantities and their SI units.

Table 4.1 Fundamental quantities and their SI units.

S.No.	Quantity	SI Unit	Symbol
1.	Length	metre	m
2.	Mass	kilogram	kg
3.	Time	second	s
4.	Electric current	ampere	A
5.	Thermodynamic temperature	kelvin	K
6.	Amount of substance	mole	mol
7.	Luminous intensity	candela	cd

Two **supplementary** units namely plane angle and solid angle are also defined. Their units are radian (rad) and steradian (st) respectively.

(i) **CGS System** In this system, the units of length, mass and time are centimetre (cm), gram (g) and second (s) respectively. The unit of force is dyne and that of work or energy is erg.

(ii) **FPS System** In this system, the units of length, mass and time are foot, pound and second. The unit of force in this system is poundal.

Definitions of Some Important SI Units

(i) **Metre** : $1\text{ m} = 1{,}650{,}763.73$ wavelengths in vacuum, of radiation corresponding to orange-red light of krypton-86.

(ii) **Second** : $1\text{ s} = 9{,}192{,}631{,}770$ time periods of a particular radiation from Cesium-133 atom.

(iii) **Kilogram** : $1\text{ kg} =$ mass of 1 litre volume of water at 4°C.

(iv) **Ampere** : It is the current which when flows through two infinitely long straight conductors of negligible cross-section placed at a distance of one metre in vacuum produces a force of 2×10^{-7} N/m between them.

(v) **Kelvin** : $1\text{ K} = 1/273.16$ part of the thermodynamic temperature of triple point of water.

(vi) **Mole** : It is the amount of substance of a system which contains as many elementary particles (atoms, molecules, ions etc.) as there are atoms in 12 g of carbon-12.

(vii) **Candela** : It is luminous intensity in a perpendicular direction of a surface of $\left(\frac{1}{600000}\right)\text{m}^2$ of a black body at the temperature of freezing platinum under a pressure of $1.013 \times 10^5\ \text{N/m}^2$.

(viii) **Radian** : It is the plane angle between two radii of a circle which cut-off on the circumference, an arc equal in length to the radius.

(ix) **Steradian** : The steradian is the solid angle which having its vertex at the centre of the sphere, cut-off an area of the surface of sphere equal to that of a square with sides of length equal to the radius of the sphere.

SI Prefixes

The most commonly used prefixes are given below in tabular form.

Power of 10	Prefix	Symbol
6	mega	M
3	kilo	k
– 2	centi	c
– 3	milli	m
– 6	micro	μ
– 9	nano	n

4.3 Dimensions

Dimensions of a physical quantity are the powers to which the fundamental quantities must be raised to represent the given physical quantity.

For example, density $= \dfrac{\text{mass}}{\text{volume}} = \dfrac{\text{mass}}{(\text{length})^3}$ or density = (mass) $(\text{length})^{-3}$ …(i)

Thus, the dimensions of density are 1 in mass and -3 in length. The dimensions of all other fundamental quantities are zero.For convenience, the fundamental quantities are represented by one letter symbols. Generally mass is denoted by M, length by L, time by T and electric current by A.

The thermodynamic temperature, the amount of substance and the luminous intensity are denoted by the symbols of their units K, mol and cd respectively. The physical quantity that is expressed in terms of the base quantities is enclosed in square brackets.

Thus, Eq. (i) can be written as

$$[\text{density}] = [\text{ML}^{-3}]$$

Such an expression for a physical quantity in terms of the fundamental quantities is called the dimensional formula. Here, it is worthnoting that constants such as 5, π or trigonometrical functions such as sin θ, cos θ, etc., have no units and dimensions.

$$[\sin\theta]=[\cos\theta]=[\tan\theta]=[\log x]=[e^x]=[M^0L^0T^0]$$

Table 4.2 **Dimensional formulae and SI units of some physical quantities frequently used in physics**

S.No.	Physical Quantity	SI Units	Dimensional Formula
1.	Velocity = displacement/time	m/s	$[M^0LT^{-1}]$
2.	Acceleration = velocity/time	m/s^2	$[M^0LT^{-2}]$
3.	Force = mass × acceleration	$kg\text{-}m/s^2$ = newton or N	$[MLT^{-2}]$
4.	Work = force × displacement	$kg\text{-}m^2/s^2$ = N-m = joule or J	$[ML^2T^{-2}]$
5.	Energy	J	$[ML^2T^{-2}]$
6.	Torque = force × perpendicular distance	N-m	$[ML^2T^{-2}]$
7.	Power = work/time	J/s or watt	$[ML^2T^{-3}]$
8.	Momentum = mass × velocity	kg-m/s	$[MLT^{-1}]$
9.	Impulse = force × time	N-s	$[MLT^{-1}]$
10.	Angle = arc/radius	radian or rad	$[M^0L^0T^0]$
11.	Strain = $\frac{\Delta L}{L}$ or $\frac{\Delta V}{V}$	no units	$[M^0L^0T^0]$
12.	Stress = force/area	N/m^2	$[ML^{-1}T^{-2}]$
13.	Pressure = force/area	N/m^2	$[ML^{-1}T^{-2}]$
14.	Modulus of elasticity = stress/strain	N/m^2	$[ML^{-1}T^{-2}]$
15.	Frequency = 1/time period	per sec or hertz (Hz)	$[M^0L^0T^{-1}]$
16.	Angular velocity = angle/time	rad/s	$[M^0L^0T^{-1}]$
17.	Moment of inertia = (mass) × $(\text{distance})^2$	$kg\text{-}m^2$	$[ML^2T^0]$
18.	Surface tension = force/length	N/m	$[ML^0T^{-2}]$
19.	Gravitational constant $= \frac{\text{force} \times (\text{distance})^2}{(\text{mass})^2}$	$N\text{-}m^2/kg^2$	$[M^{-1}L^3T^{-2}]$
20.	Angular momentum	$kg\text{-}m^2/s$	$[ML^2T^{-1}]$
21.	Coefficient of viscosity	$N\text{-}s/m^2$	$[ML^{-1}T^{-1}]$
22.	Planck's constant	J-s	$[ML^2T^{-1}]$
23.	Specific heat (*s*)	J/kg-K	$[L^2T^{-2}\theta^{-1}]$
24.	Coefficient of thermal conductivity (*K*)	watt/m-K	$[MLT^{-3}\theta^{-1}]$
25.	Gas constant (*R*)	J/mol-K	$[ML^2T^{-2}\theta^{-1}\,\text{mol}^{-1}]$
26.	Boltzmann constant (*k*)	J/K	$[ML^2T^{-2}\theta^{-1}]$

S.No.	Physical Quantity	SI Units	Dimensional Formula
27.	Wein's constant (*b*)	m-K	$[L\,\theta]$
28.	Stefan's constant (σ)	watt/m^2-K^4	$[MT^{-3}\theta^{-4}]$
29.	Electric charge	C	$[AT]$
30.	Electric intensity	N/C	$[MLT^{-3}A^{-1}]$
31.	Electric potential	volt	$[ML^2T^{-3}A^{-1}]$
32.	Capacitance	farad	$[M^{-1}L^{-2}T^4A^2]$
33.	Permittivity of free space	$C^2N^{-1}m^{-2}$	$[M^{-1}L^{-3}T^4A^2]$
34.	Electric dipole moment	C-m	$[LTA]$
35.	Resistance	ohm	$[ML^2T^{-3}A^{-2}]$
36.	Magnetic field	tesla (T) or weber/m^2 (Wb/m^2)	$[MT^{-2}A^{-1}]$
37.	Coefficient of self induction	henry	$[ML^2T^{-2}A^{-2}]$

Example 4.1 *Find the dimensional formula of the following quantities :*

(a) Density *(b) Velocity* *(c) Acceleration*
(d) Momentum *(e) Force* *(f) Work or energy*
(g) Power *(h) Pressure*

Solution (a) Density $= \dfrac{\text{mass}}{\text{volume}}$

$$[\text{Density}] = \frac{[\text{mass}]}{[\text{volume}]} = \frac{[M]}{[L^{-3}]} = [ML^{-3}]$$

(b) Velocity $[v] = \dfrac{\text{displacement}}{\text{time}}$

$$[v] = \frac{[\text{displacement}]}{[\text{time}]} = \frac{[L]}{[T]} = [M^0LT^{-1}]$$

(c) Acceleration $[a] = \left[\dfrac{dv}{dt}\right]$

$$[a] = \frac{dv \to \text{kind of velocity}}{dt \to \text{kind of time}} = \frac{[LT^{-1}]}{[T]} = [LT^{-2}]$$

(d) Momentum $[P] = [mv]$

$$[P] = [M][v] = [M][LT^{-1}] = [MLT^{-1}]$$

(e) Force $[F] = [ma]$

$$[F] = [m][a] = [M][LT^{-2}] = [MLT^{-2}]$$

(f) Work or Energy = force $\times$ displacement

$$[\text{Work}] = [\text{force}][\text{displacement}]$$
$$= [MLT^{-2}][L] = [ML^2T^{-2}]$$

(g) Power $= \dfrac{\text{Work}}{\text{Time}}$

$$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]} = \frac{[\text{M L}^2\text{T}^{-2}]}{[\text{T}]} = [\text{M L}^2\text{T}^{-3}]$$

(h) Pressure $= \dfrac{\text{Force}}{\text{Area}}$

$$[\text{Pressure}] = \frac{[\text{Force}]}{[\text{Area}]} = \frac{[\text{M L T}^{-2}]}{\text{L}^2}$$

$$= [\text{M L}^{-1}\text{T}^{-2}]$$

Example 4.2 *Find the dimensional formula of the following quantities :*

(a) Surface tension, T

(b) Universal constant of gravitation, G

(c) Impulse, J

(d) Torque τ

The equations involving these equations are :

$T = F/l$, $F = \dfrac{Gm_1 m_2}{r^2}$, $J = F \times t$ *and* $\tau = F \times l$

Solution (a) $T = \dfrac{F}{l}$

$$\Rightarrow \qquad [\text{T}] = \frac{[\text{F}]}{[\text{l}]} = \frac{[\text{MLT}^{-2}]}{[\text{L}]} = [\text{MT}^{-2}]$$ **Ans.**

(b) $F = \dfrac{Gm_1 m_2}{r^2} \quad \Rightarrow \quad G = \dfrac{Fr^2}{m_1 m_2}$

or $$[G] = \frac{[\text{F}][\text{r}]^2}{[\text{m}]^2} = \frac{[\text{MLT}^{-2}][\text{L}^2]}{[\text{M}^2]}$$

$$= [\text{M}^{-1}\text{L}^3\text{T}^{-2}]$$ **Ans.**

(c) $J = F \times t$

$\therefore$ $$[\text{J}] = [\text{F}][\text{t}]$$

$$= [\text{MLT}^{-2}][\text{T}]$$

$$= [\text{MLT}^{-1}]$$ **Ans.**

(d) $\tau = F \times l$

$\therefore$ $$[\tau] = [\text{F}][\text{l}]$$

$$= [\text{MLT}^{-2}][\text{L}]$$

$$= [\text{ML}^2\text{T}^{-2}]$$ **Ans.**

4.4 Uses of Dimensions

Theory of dimensions have following main uses:

1. **Conversion of units** This is based on the fact that the product of the numerical value (n) and its corresponding unit (u) is a constant, i.e.

$$n[u] = \text{constant}$$

or $$n_1 [u_1] = n_2 [u_2]$$

Suppose the dimensions of a physical quantity are a in mass, b in length and c in time. If the fundamental units in one system are M_1, L_1 and T_1 and in the other system are M_2, L_2 and T_2 respectively. Then, we can write

$$n_1 [\mathrm{M}_1^a \, \mathrm{L}_1^b \, \mathrm{T}_1^c] = n_2 [\mathrm{M}_2^a \, \mathrm{L}_2^b \, \mathrm{T}_2^c] \qquad \ldots\text{(i)}$$

Here, n_1 and n_2 are the numerical values in two systems of units respectively. Using Eq. (i), we can convert the numerical value of a physical quantity from one system of units into the other system.

Example 4.3 *The value of gravitation constant is* $G = 6.67 \times 10^{-11}$ *N-m*^{2}*/kg*^{2} *in SI units. Convert it into CGS system of units.*

Solution The dimensional formula of G is $[\mathrm{M}^{-1}\,\mathrm{L}^3\,\mathrm{T}^{-2}]$.

Using Eq. (i), i.e.

$$n_1 [\mathrm{M}_1^{-1} \mathrm{L}_1^3 \mathrm{T}_1^{-2}] = n_2 [\mathrm{M}_2^{-1} \mathrm{L}_2^3 \mathrm{T}_2^{-2}]$$

$$n_2 = n_1 \left[\frac{\mathrm{M}_1}{\mathrm{M}_2}\right]^{-1} \left[\frac{\mathrm{L}_1}{\mathrm{L}_2}\right]^{3} \left[\frac{\mathrm{T}_1}{\mathrm{T}_2}\right]^{-2}$$

Here, $$n_1 = 6.67 \times 10^{-11}$$

$$M_1 = 1\,\text{kg},\ M_2 = 1\,\text{g} = 10^{-3}\ \text{kg},\ L_1 = 1\,\text{m},\ L_2 = 1\,\text{cm} = 10^{-2}\ \text{m},\ T_1 = T_2 = 1\,\text{s}$$

Substituting in the above equation, we get

$$n_2 = 6.67 \times 10^{-11} \left[\frac{1\,\text{kg}}{10^{-3}\ \text{kg}}\right]^{-1} \left[\frac{1\,\text{m}}{10^{-2}\ \text{m}}\right]^{3} \left[\frac{1\,\text{s}}{1\,\text{s}}\right]^{-2}$$

or $$n_2 = 6.67 \times 10^{-2}$$

Thus, value of G in CGS system of units is 6.67×10^{-2} dyne cm^2/g^2.

2. **To check the dimensional correctness of a given physical equation** Every physical equation should be dimensionally balanced. This is called the 'Principle of Homogeneity'. The dimensions of each term on both sides of an equation must be the same. On this basis, we can judge whether a given equation is correct or not. But a dimensionally correct equation may or may not be physically correct.

Example 4.4 *Show that the expression of the time period* T *of a simple pendulum of length* l *given by* $T = 2\pi \sqrt{\dfrac{l}{g}}$ *is dimensionally correct.*

Solution $T = 2\pi\sqrt{\frac{l}{g}}$

Dimensionally $[\mathrm{T}] = \sqrt{\frac{[\mathrm{L}]}{[\mathrm{LT}^{-2}]}} = [\mathrm{T}]$

As in the above equation, the dimensions of both sides are same. The given formula is dimensionally correct.

Principle of Homogeneity of Dimensions

This principle states that the dimensions of all the terms in a physical expression should be same. For example, in the physical expression $s = ut + \frac{1}{2}at^2$, the dimensions of s, ut and $\frac{1}{2}at^2$ all are same.

Note *The physical quantities separated by the symbols +, −, =, >, < etc., have the same dimensions.*

Example 4.5 *The velocity v of a particle depends upon the time t according to the equation* $v = a + bt + \frac{c}{d+t}\cdot$ *Write the dimensions of a, b, c and d.*

Solution From principle of homogeneity,

$$[a] = [v] \quad \text{or} \quad [a] = [\mathrm{LT}^{-1}]$$

$$[bt] = [v] \quad \text{or} \quad [b] = \frac{[v]}{[t]} = \frac{[\mathrm{LT}^{-1}]}{[\mathrm{T}]}$$

or $$[b] = [\mathrm{LT}^{-2}]$$

Similarly, $$[d] = [t] = [\mathrm{T}]$$

Further, $$\frac{[c]}{[d+t]} = [v] \quad \text{or} \quad [c] = [v][d+t]$$

or $$[c] = [\mathrm{LT}^{-1}][\mathrm{T}] \quad \text{or} \quad [c] = [\mathrm{L}]$$

3. **To establish the relation among various physical quantities** If we know the factors on which a given physical quantity may depend, we can find a formula relating the quantity with those factors. Let us take an example.

Example 4.6 *The frequency (f) of a stretched string depends upon the tension F (dimensions of force), length l of the string and the mass per unit length μ of string. Derive the formula for frequency.*

Solution Suppose, that the frequency f depends on the tension raised to the power a, length raised to the power b and mass per unit length raised to the power c. Then,

$$f \propto [F]^a\ [l]^b\ [\mu]^c$$

or $$f = k\,[F]^a\ [l]^b\ [\mu]^c \qquad \text{...(i)}$$

Here, k is a dimensionless constant. Thus,

$$[f] = [F]^a\ [l]^b\ [\mu]^c$$

or $$[\mathrm{M}^0\mathrm{L}^0\mathrm{T}^{-1}]=[\mathrm{MLT}^{-2}]^a\ [\mathrm{L}]^b\ [\mathrm{ML}^{-1}]^c$$

or $$[\mathrm{M}^0\mathrm{L}^0\mathrm{T}^{-1}]=[\mathrm{M}^{a+c}\mathrm{L}^{a+b-c}\mathrm{T}^{-2a}]$$

For dimensional balance, the dimensions on both sides should be same.

Thus, $$a+c=0 \qquad \text{...(ii)}$$

$$a+b-c=0 \qquad \text{...(iii)}$$

and $$-2a=-1 \qquad \text{...(iv)}$$

Solving these three equations, we get

$$a=\frac{1}{2},\quad c=-\frac{1}{2}\quad \text{and}\quad b=-1$$

Substituting these values in Eq. (i), we get

$$f=k(F)^{1/2}(l)^{-1}(\mu)^{-1/2}$$

or $$f=\frac{k}{l}\sqrt{\frac{F}{\mu}}$$

Experimentally, the value of k is found to be $\frac{1}{2}$.

Hence, $$f=\frac{1}{2l}\sqrt{\frac{F}{\mu}}$$

Limitations of Dimensional Analysis

The method of dimensions has the following limitations :

(i) By this method, the value of dimensionless constant cannot be calculated.

(ii) By this method, the equation containing trigonometrical, exponential and logarithmic terms cannot be analysed.

(iii) This method is useful when a physical quantity depends on other quantities by multiplication and power relations. It cannot be used if a physical quantity depends on sum or difference of two quantities. For example we, cannot get the relation, $s=ut+\frac{1}{2}at^2$ from dimensional analysis.

Final Touch Points

1. There are some physical quantities which have the same dimensions. They are given in tabular form as below :

S.No.	Physical quantities or combination of physical quantities	Dimensions
1.	Angle, strain, sin θ, π, e^x	$[M^0 L^0 T^0]$
2.	Work, Energy, Torque, Rhc	$[ML^2 T^{-2}]$
3.	Time, $\frac{L}{R}$, CR, $\sqrt{LC}$	$[M^0 L^0 T]$
4.	Frequency, ω, $\frac{R}{L}$, $\frac{1}{CR}$, $\frac{1}{\sqrt{LC}}$, velocity gradient, Decay constant. Activity of a radioactive substance	$[M^0 L^0 T^{-1}]$
5.	Pressure, stress, modulus of elasticity, energy density (energy per unit volume), $\varepsilon_0 E^2$, $\frac{B^2}{\mu_0}$	$[ML^{-1}T^{-2}]$
6.	Angular impulse, angular momentum, Planck's constant	$[ML^2T^{-1}]$
7.	Linear momentum, linear impulse	$[MLT^{-1}]$
8.	Wavelength, radius of gyration, Light year	$[M^0 LT^0]$
9.	Velocity, $\frac{1}{\sqrt{\varepsilon_0\mu_0}}$, $\sqrt{\frac{GM}{R}}$, $\frac{E}{B}$	$[M^0 LT^{-1}]$

2. Astronomical unit 1 AU = mean distance of earth from sun $\approx 1.5 \times 10^{11}$ m

Light year 1 ly = distance travelled by light in vacuum in 1 year $= 9.46 \times 10^{15}$ m

Parsec 1 Parsec $= 3.07 \times 10^{16}$ m = 3.26 light year

X-ray unit 1 U $= 10^{-3}$ m

1 shake $= 10^{-8}$ s

1 Bar $= 10^5$ N/m^2 $= 10^5$Pa

1 torr = 1 mm of Hg = 133.3 Pa

1 barn $= 10^{-28}$ m^2

1 horse power = 746 W

1 pound = 453.6 g = 0.4536 kg

Solved Examples

Example 1 *Find the dimensional formulae of*

(a) coefficient of viscosity η *(b) charge* q

(c) potential V *(d) capacitance* C *and*

(e) resistance R

Some of the equations containing these quantities are

$$F = -\eta A\left(\frac{\Delta v}{\Delta l}\right),\ q = It,\ U = VIt,\ q = CV \text{ and } V = IR$$

where, A *denotes the area,* v *the velocity,* l *is the length,* I *the electric current,* t *the time and* U *the energy.*

Solution (a) $\eta = -\dfrac{F}{A}\dfrac{\Delta l}{\Delta v} \quad \Rightarrow \quad \therefore\ [\eta] = \dfrac{[F][l]}{[A][v]} = \dfrac{[\mathrm{MLT^{-2}}][\mathrm{L}]}{[\mathrm{L^2}][\mathrm{LT^{-1}}]} = [\mathrm{ML^{-1}T^{-1}}]$

(b) $q = It \Rightarrow \therefore\ [q] = [I]\,[t] = [\mathrm{AT}]$

(c) $U = VIt$

$$\therefore \qquad V = \frac{U}{It} \quad \text{or} \quad [V] = \frac{[U]}{[I]\,[t]} = \frac{[\mathrm{ML^2T^{-2}}]}{[\mathrm{A}][\mathrm{T}]} = [\mathrm{ML^2T^{-3}A^{-1}}]$$

(d) $q = CV$

$$\therefore \qquad C = \frac{q}{V} \quad \text{or} \quad [C] = \frac{[q]}{[V]} = \frac{[\mathrm{AT}]}{[\mathrm{ML^2T^{-3}A^{-1}}]} = [\mathrm{M^{-1}L^{-2}T^4A^2}]$$

(e) $V = IR$

$$\therefore \qquad R = \frac{V}{I} \quad \text{or} \quad [R] = \frac{[V]}{[I]} = \frac{[\mathrm{ML^2T^{-3}A^{-1}}]}{[\mathrm{A}]} = [\mathrm{ML^2T^{-3}A^{-2}}]$$

Example 2 *Write the dimensions of* a *and* b *in the relation,* $P = \dfrac{b - x^2}{at}$*, where* P *is power,* x *is distance and* t *is time.*

Solution The given equation can be written as, $Pat = b - x^2$

Now, $\qquad [Pat] = [b] = [x^2] \quad \text{or} \quad [b] = [x^2] = [\mathrm{M^0L^2T^0}]$

and $\qquad [a] = \dfrac{[x^2]}{[Pt]} = \dfrac{[\mathrm{L^2}]}{[\mathrm{ML^2T^{-3}}]\,[\mathrm{T}]} = [\mathrm{M^{-1}L^0T^2}]$

Example 3 *The centripetal force* F *acting on a particle moving uniformly in a circle may depend upon mass* (m)*, velocity* (v) *and radius* (r) *of the circle. Derive the formula for* F *using the method of dimensions.*

Solution Let $\qquad F = k\,(m)^x\,(v)^y\,(r)^z \qquad$ ...(i)

Here, k is a dimensionless constant of proportionality. Writing the dimensions of RHS and LHS in Eq. (i), we have

$$[\mathrm{MLT^{-2}}] = [\mathrm{M}]^x\,[\mathrm{LT^{-1}}]^y\,[\mathrm{L}]^z = [\mathrm{M}^x\mathrm{L}^{y+z}\mathrm{T}^{-y}]$$

Equating the powers of M, L and T of both sides, we have,

$$x = 1, \quad y = 2 \quad \text{and} \quad y + z = 1 \quad \text{or} \quad z = 1 - y = -1$$

Putting the values in Eq. (i), we get

$$F = kmv^2r^{-1} = k\,\frac{mv^2}{r}$$

or $$F = \frac{mv^2}{r} \qquad \text{(where, } k = 1\text{)}$$

Example 4 *If velocity, time and force were chosen as basic quantities, find the dimensions of mass and energy.*

Solution (i) We know that,

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$= \text{mass} \times \frac{\text{velocity}}{\text{time}}$$

$$\Rightarrow \qquad \text{mass} = \frac{\text{force} \times \text{time}}{\text{velocity}}$$

or $$[\text{mass}] = \frac{[\text{force}] \times [\text{time}]}{[\text{velocity}]}$$

$$= \frac{[\text{F}]\ [\text{T}]}{[\text{v}]}$$

$$\therefore \qquad [\text{mass}] = [\text{F T v}^{-1}]$$ **Ans.**

(ii) Dimensions of energy are same as the dimensions of kinetic energy

$$\therefore \qquad [\text{Energy}] = \left[\frac{1}{2}\text{mv}^2\right] = [\text{m}]\ [\text{v}]^2$$

$$= [\text{FTv}^{-1}]\ [\text{v}]^2$$

$$= [\text{FTv}]$$ **Ans.**

Example 5 *Force acting on a particle is 5 N. If units of length and time are doubled and unit of mass is halved then find the numerical value of force in the new system of units.*

Solution Force $= 5\ \text{N} = \dfrac{5\ \text{kg-m}}{\text{s}^2}$

If units of length and time are doubled and unit of mass is halved, then value of force in new system of units will be

$$5\left[\frac{\frac{1}{2} \times 2}{(2)^2}\right] = \frac{5}{4}$$ **Ans.**

Example 6 *Can pressure (p), density (ρ) and velocity (v) be taken as fundamental quantities?*

Solution No, they cannot be taken as fundamental quantities, as they are related to each other by the relation,

$$p = \rho v^2$$

Exercises

Assertion and Reason

Directions *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

(*e*) *If both **Assertion** and **Reason** are wrong.*

1. Assertion Velocity, volume and acceleration can be taken as fundamental quantities because

Reason All the three are independent from each other.

2. Assertion If two physical quantities have same dimensions, then they can be certainly added or subtracted because

Reason If the dimensions of both the quantities are same then both the physical quantities should be similar.

Objective Questions

Single Correct Option

1. The dimensional formula for Planck's constant and angular momentum are

(a) $[ML^2T^{-2}]$ and $[MLT^{-1}]$

(b) $[ML^2T^{-1}]$ and $[ML^2T^{-1}]$

(c) $[ML^3T^1]$ and $[ML^2T^{-2}]$

(d) $[MLT^{-1}]$ and $[MLT^{-2}]$

2. Dimension of velocity gradient is

(a) $[M^0L^0T^{-1}]$

(b) $[ML^{-1}T^{-1}]$

(c) $[M^0LT^{-1}]$

(d) $[ML^0T^{-1}]$

3. Which of the following is the dimension of the coefficient of friction?

(a) $[M^2L^2T]$

(b) $[M^0L^0T^0]$

(c) $[ML^2T^{-2}]$

(d) $[M^2L^2T^{-2}]$

4. Which of the following sets have different dimensions? **(JEE 2005)**

(a) Pressure, Young's modulus, Stress

(b) Emf, Potential difference, Electric potential

(c) Heat, Work done, Energy

(d) Dipole moment, Electric flux, Electric field

5. The viscous force F on a sphere of radius a moving in a medium with velocity v is given by $F = 6\pi\,\eta a\,v$. The dimensions of η are

(a) $[ML^{-3}]$

(b) $[MLT^{-2}]$

(c) $[MT^{-1}]$

(d) $[ML^{-1}T^{-1}]$

6. A force is given by

$$F = at + bt^2$$

where, t is the time. The dimensions of a and b are

(a) $[MLT^{-4}]$ and $[MLT]$

(b) $[MLT^{-1}]$ and $[MLT^0]$

(c) $[MLT^{-3}]$ and $[MLT^{-4}]$

(d) $[MLT^{-3}]$ and $[MLT^0]$

7. The physical quantity having the dimensions $[M^{-1}L^{-3}T^{3}A^{2}]$ is
(a) resistance (b) resistivity
(c) electrical conductivity (d) electromotive force

8. The dimensional formula for magnetic flux is
(a) $[ML^{2}T^{-2}A^{-1}]$ (b) $[ML^{3}T^{-2}A^{-2}]$ (c) $[M^{0}L^{-2}T^{-2}A^{-2}]$ (d) $[ML^{2}T^{-1}A^{2}]$

9. Choose the **wrong** statement.
(a) All quantities may be represented dimensionally in terms of the base quantities
(b) A base quantity cannot be represented dimensionally in terms of the rest of the base quantities
(c) The dimension of a base quantity in other base quantities is always zero
(d) The dimension of a derived quantity is never zero in any base quantity

10. If unit of length and time is doubled, the numerical value of g (acceleration due to gravity) will be
(a) doubled (b) halved (c) four times (d) same

11. Using mass (M), length (L), time (T) and current (A) as fundamental quantities, the dimension of permeability is
(a) $[M^{-1}LT^{-2}A]$ (b) $[ML^{-2}T^{-2}A^{-1}]$ (c) $[MLT^{-2}A^{-2}]$ (d) $[MLT^{-1}A^{-1}]$

12. The equation of a wave is given by

$$y = a \sin \omega \left(\frac{x}{v} - k \right)$$

where, ω is angular velocity and v is the linear velocity. The dimensions of k will be
(a) $[T^{2}]$ (b) $[T^{-1}]$ (c) $[T]$ (d) $[LT]$

13. If the energy (E), velocity (v) and force (F) be taken as fundamental quantities, then the dimensions of mass will be
(a) $[Fv^{-2}]$ (b) $[Fv^{-1}]$ (c) $[Ev^{-2}]$ (d) $[Ev^{2}]$

14. If force F, length L and time T are taken as fundamental units, the dimensional formula for mass will be
(a) $[FL^{-1}T^{2}]$ (b) $[FLT^{-2}]$ (c) $[FL^{-1}T^{-1}]$ (d) $[FL^{5}T^{2}]$

15. The ratio of the dimensions of Planck's constant and that of the moment of inertia is the dimension of
(a) frequency (b) velocity (c) angular momentum (d) time

16. Given that $y = A \sin \left[\left(\frac{2\pi}{\lambda} (ct - x) \right) \right]$, where y and x are measured in metres. Which of the following statements is true ?
(a) The unit of λ is same as that of x and A (b) The unit of λ is same as that of x but not of A
(c) The unit of c is same as that of $\frac{2\pi}{\lambda}$ (d) The unit of $(ct - x)$ is same as that of $\frac{2\pi}{\lambda}$

17. Which of the following sets cannot enter into the list of fundamental quantities in any system of units?
(a) length, mass and density (b) length, time and velocity
(c) mass, time and velocity (d) length, time and mass

18. In the formula $X = 3YZ^{2}$, X and Z have dimensions of capacitance and magnetic induction respectively. What are the dimensions of Y in MKSQ system? **(JEE 1995)**
(a) $[M^{-3}L^{-1}T^{3}Q^{4}]$ (b) $[M^{-3}L^{-2}T^{4}Q^{4}]$ (c) $[M^{-2}L^{-2}T^{4}Q^{4}]$ (d) $[M^{-3}L^{-2}T^{4}Q]$

19. A quantity X is given by $\varepsilon_0 L \dfrac{\Delta V}{\Delta t}$, where ε_0 is the permittivity of free space, L is a length, ΔV is a potential difference and Δt is a time interval. The dimensional formula for X is the same as that of **(JEE 2001)**

(a) resistance (b) charge
(c) voltage (d) current

20. In the relation $p = \dfrac{\alpha}{\beta} e^{-\frac{\alpha Z}{k\theta}}$, p is pressure, Z is distance, k is Boltzmann constant and θ is the temperature. The dimensional formula of β will be **(JEE 2004)**

(a) $[M^0L^2T^0]$ (b) $[ML^2T]$
(c) $[ML^0T^{-1}]$ (d) $[M^0L^2T^{-1}]$

More than One Correct Options

1. The dimensions of the quantities in one (or more) of the following pairs are the same. Identify the pair (s). **(JEE 1986)**

(a) Torque and work (b) Angular momentum and work
(c) Energy and Young's modulus (d) Light year and wavelength

2. The pairs of physical quantities that have the same dimensions is (are) **(JEE 1995)**

(a) Reynolds number and coefficient of friction
(b) Curie and frequency of a light wave
(c) Latent heat and gravitational potential
(d) Planck's constant and torque

3. The SI unit of the inductance, the henry can by written as **(JEE 1998)**

(a) weber/ampere (b) volt-second/ampere
(c) joule/(ampere)^2 (d) ohm-second

4. Let $[\varepsilon_0]$ denote the dimensional formula of the permittivity of the vacuum and $[\mu_0]$ that of the permeability of the vacuum. If M = mass, L = length, T = time and I = electric current. **(JEE 1998)**

(a) $[\varepsilon_0] = [M^{-1}L^{-3}\,T^2\,I]$ (b) $[\varepsilon_0] = [M^{-1}L^{-3}\,T^4\,I^2]$
(c) $[\mu_0] = [MLT^{-2}\,I^{-2}]$ (d) $[\mu_0] = [ML^2\,T^{-1}\,I]$

5. L, C and R represent the physical quantities inductance, capacitance and resistance respectively. The combinations which have the dimensions of frequency are **(JEE 1984)**

(a) $\dfrac{1}{RC}$ (b) $\dfrac{R}{L}$
(c) $\dfrac{1}{\sqrt{LC}}$ (d) $\dfrac{C}{L}$

Match the Columns

1. Match the two columns. **(JEE 2003)**

Column I	Column II
(a) Boltzmann constant	(p) $[ML^2\,T^{-1}]$
(b) Coefficient of viscosity	(q) $[ML^{-1}\,T^{-1}]$
(c) Planck constant	(r) $[MLT^{-3}K^{-1}]$
(d) Thermal conductivity	(s) $[ML^2\,T^{-2}K^{-1}]$

2. Match the physical quantities given in Column I with dimensions expressed in terms of mass (M), length (L), time (T), and charge (Q) given in Column II. **(JEE 1993)**

Column I	Column II
(a) Angular momentum	(p) $[ML^2T^{-2}]$
(b) Latent heat	(q) $[ML^2Q^{-2}]$
(c) Torque	(r) $[ML^2T^{-1}]$
(d) Capacitance	(s) $[ML^3T^{-1}Q^{-2}]$
(e) Inductance	(t) $[M^{-1}L^{-2}T^2Q^2]$
(f) Resistivity	(u) $[L^2T^{-2}]$

3. Column I gives three physical quantities. Select the appropriate units for the choices given in Column II. Some of the physical quantities may have more than one choice. **(JEE 1990)**

Column I	Column II
(a) Capacitance	(p) ohm-second
(b) Inductance	(q) coulomb^2-joule^{-1}
	(r) coulomb (volt)$^{-1}$,
(c) Magnetic induction	(s) newton (ampere metre)$^{-1}$,
	(t) volt-second (ampere)$^{-1}$

4. Some physical quantities are given in Column I and some possible SI units in which these quantities may be expressed are given in Column II. Match the physical quantities in Column I with the units in Column II. **(JEE 2007)**

Column I	Column II
(a) GM_eM_s G — universal gravitational constant, M_e — mass of the earth, M_s — mass of the sun.	(p) (volt) (coulomb) (metre)
(b) $\dfrac{3RT}{M}$ R — universal gas constant, T — absolute temperature, M — molar mass.	(q) (kilogram) (metre)^3 (second)$^{-2}$
(c) $\dfrac{F^2}{q^2B^2}$ F — force, q — charge, B — magnetic field.	(r) (metre)^2 (second)$^{-2}$
(d) $\dfrac{GM_e}{R_e}$ G — universal gravitational constant, M_e — mass of the earth, R_e — radius of the earth.	(s) (farad) (volt)^2 (kg)$^{-1}$

Subjective Questions

1. In the expression $y = a \sin(\omega t + \theta)$, y is the displacement and t is the time. Write the dimensions of a, ω and θ.
2. Young's modulus of steel is 2.0×10^{11} N/m^2. Express it in dyne/cm^2.
3. Surface tension of water in the CGS system is 72 dyne/cm. What is its value in SI units?
4. The relation between the energy E and the frequency v of a photon is expressed by the equation $E = hv$, where h is Planck's constant. Write down the SI units of h and its dimensions.
5. Check the correctness of the relation $s_t = u + \frac{a}{2}(2t - 1)$, where u is initial velocity, a is acceleration and s_t is the displacement of the body in t^{th} second.
6. Give the MKS units for each of the following quantities: **(JEE 1980)**
   (a) Young's modulus (b) Magnetic induction (c) Power of a lens
7. A gas bubble, from an explosion under water, oscillates with a period T proportional to $p^a d^b E^c$, where p is the static pressure, d is the density of water and E is the total energy of the explosion. Find the values of a, b and c.
8. Show dimensionally that the expression, $Y = \frac{MgL}{\pi r^2 l}$ is dimensionally correct, where Y is Young's modulus of the material of wire, L is length of wire, Mg is the weight applied on the wire and l is the increase in the length of the wire.
9. The energy E of an oscillating body in simple harmonic motion depends on its mass m, frequency n and amplitude a. Using the method of dimensional analysis find the relation between E, m, n and a. **(JEE 1981)**
10. $\frac{\alpha}{t^2} = Fv + \frac{\beta}{x^2}$. Find dimension formula for $[\alpha]$ and $[\beta]$ (here t = time, F = force, v = velocity, x = distance)
11. For n moles of gas, Van der Waals' equation is $\left(p - \frac{a}{V^2}\right)(V - b) = nRT$

    Find the dimensions of a and b, where p = pressure of gas, V = volume of gas and T = temperature of gas.
12. In the formula, $p = \frac{nRT}{V - b} e^{\frac{a}{RTV}}$, find the dimensions of a and b, where p = pressure, n = number of moles, T = temperature, V = volume and R = universal gas constant.
13. Write the dimensions of the following in terms of mass, time, length and charge **(JEE 1982)**
    (a) Magnetic flux (b) Rigidity modulus.
14. Let x and a stand for distance. Is $\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \sin^{-1} \frac{a}{x}$ dimensionally correct?
15. In the equation $\int \frac{dx}{\sqrt{2ax - x^2}} = a^n \sin^{-1}\left(\frac{x}{a} - 1\right)$. Find the value of n.
16. Taking force F, length L and time T to be the fundamental quantities, find the dimensions of
    (a) density (b) pressure (c) momentum and (d) energy.

Answers

Assertion and Reason

1. (e) **2.** (e)

Single Correct Option

1. (b) **2.** (a) **3.** (b) **4.** (d) **5.** (d) **6.** (c) **7.** (c) **8.** (a) **9.** (d) **10.** (b)
11. (c) **12.** (c) **13.** (c) **14.** (a) **15.** (a) **16.** (a) **17.** (b) **18.** (b) **19.** (d) **20.** (a)

More than One Correct Options

1. (a,d) **2.** (a,b,c) **3.** (all) **4.** (b,c) **5.** (a,b, c)

Match the Columns

1. (a) $\rightarrow$ s, (b) $\rightarrow$ q, (c) $\rightarrow$ p, (d) $\rightarrow$ r
2. (a) $\rightarrow$ r, (b) $\rightarrow$ u, (c) $\rightarrow$ p, (d) $\rightarrow$ t, (e) $\rightarrow$ q, (f) $\rightarrow$ s
3. (a) $\rightarrow$ q,r, (b) $\rightarrow$ p,t, (c) $\rightarrow$ s
4. (a) $\rightarrow$ p, q, (b) $\rightarrow$ r, s, (c) $\rightarrow$ r, s, (d) $\rightarrow$ r, s

Subjective Questions

1. $[M^0 L T^0], [M^0 L^0 T^{-1}], [M^0 L^0 T^0]$

2. 2.0×10^{12} dyne/cm^2

3. 0.072 N/m

4. J·s, $[ML^2 T^{-1}]$

5. Given equation seems to be dimensionally incorrect but it is correct

6. (a) N/m^2 (b) Tesla (c) m^{-1}

7. $a = \frac{-5}{6}, b = \frac{1}{2}, c = \frac{1}{3}$

9. $E = kmn^2a^2$ (k = a dimensionless constant)

10. $[\beta] = [ML^4 T^{-3}]$, $[\alpha] = [ML^2 T^{-1}]$

11. $[a] = [ML^5 T^{-2}]$, $[b] = [L^3]$

12. $[a] = [ML^5 T^{-2}]$, $[b] = [L]^3$

13. (a) $[ML^2 T^{-1} Q^{-1}]$ (b) $[ML^{-1} T^{-2}]$

14. No

15. zero

16. (a) $[FL^{-4} T^2]$ (b) $[FL^{-2}]$ (c) $[FT]$ (d) $[FL]$

05

Vectors

Chapter Contents

5.1 Vector and Scalar Quantities
5.2 General Points regarding Vectors
5.3 Addition and Subtraction of Two Vectors
5.4 Components of a Vector
5.5 Product of Two Vectors

5.1 Vector and Scalar Quantities

Any physical quantity is either a scalar or a vector. A scalar quantity can be described completely by its magnitude only. Addition, subtraction, division or multiplication of scalar quantities can be done according to the ordinary rules of algebra. Mass, volume, density, etc., are few examples of scalar quantities. If a physical quantity in addition to magnitude has a specified direction as well as obeys the law of parallelogram of addition, then and then only it is said to be a vector quantity. Displacement, velocity, acceleration, etc., are few examples of vectors.

Any vector quantity should have a specified direction but it is not a sufficient condition for a quantity to be a vector. For example, current flowing in a wire is shown by a direction but it is not a vector because it does not obey the law of parallelogram of vector addition. For example, in the figure shown here.

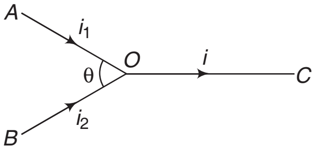


Fig. 5.1

Current flowing in wire OC = current in wire AO + current in wire BO or $i = i_1 + i_2$ was the current a vector quantity, $i \neq i_1 + i_2$

It also depends on angle θ, the angle between i_1 and i_2.

1. **Scalar quantities** Mass, volume, distance, speed, density, work, power, energy, length, gravitation constant (G), specific heat, specific gravity, charge, current, potential, time, electric or magnetic flux, pressure, surface tension, temperature.
2. **Vector quantities** Displacement, velocity, acceleration, force, weight, acceleration due to gravity (g), gravitational field strength, electric field, magnetic field, dipole moment, torque, linear momentum, angular momentum.

5.2 General Points Regarding Vectors

Vector Notation

Usually a vector is represented by a bold capital letter with an arrow over it, as **A, B**, **C**, etc.

The magnitude of a vector **A** is represented by A or $|\mathbf{A}|$ and is always positive.

Graphical Representation of a Vector

Graphically a vector is represented by an arrow drawn to a chosen scale, parallel to the direction of the vector. The length and the direction of the arrow thus represent the magnitude and the direction of the vector respectively.

Thus, the arrow in Fig. 5.2 represents a vector **A** in xy-plane making an angle θ with x-axis.

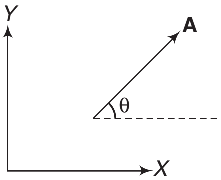


Fig. 5.2

Steps Involved Representing a Vector

(i) By choosing a proper scale, draw a line whose length is proportional to the magnitude of the vector.

(ii) By following the standard convention to show direction, indicate the direction of the vector by marking an arrow head at one end of the line.

Example To represent the displacement of a body along x-axis.

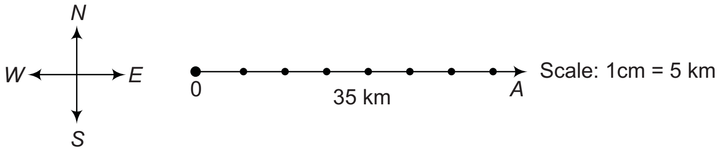


Fig. 5.3 Graphical representation of a vector

The vector represented by the directed line segment OA in Fig. 5.3 is denoted by **OA** (to be read as vector **OA**) or a simple notation as **A** (to be read as vector **A**). For vector **OA**, O is the initial point and A is the terminal point. In the figure shown, **OA** or **A** is a displacement vector of magnitude 35 km towards east.

Note *A vector can be displaced from one position to another. During the displacement if we do not change direction and magnitude then the vector remains unchanged.*

Angle between Two Vectors (θ)

To find angle between two vectors both the vectors are drawn from one point in such a manner that arrows of both the vectors are outwards from that point. Now, the smaller angle is called the angle between two vectors.

For example in Fig. 5.4, angle between **A** and **B** is 60° not 120°.Because in Fig.(a), they are wrongly drawn while in Fig. (b) they are drawn as we desire.

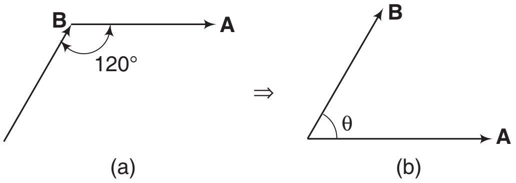


Fig. 5.4

Note $0° \leq \theta \leq 180°$

Kinds of Vectors

Unit Vector

A vector of unit magnitude is called a unit vector and the notation for it in the direction of **A** is $\hat{\mathbf{A}}$ read as 'A cap or A caret'.

Thus,
$$\mathbf{A} = A\hat{\mathbf{A}} \text{ or } \hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$

A unit vector merely indicates a direction. Unit vector along x, y and z-directions are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

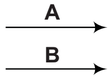
Zero Vector or Null Vector

A vector having zero magnitude is called a null vector or zero vector.

Note *(i) Zero vector has no specific direction.*
(ii) The position vector of origin is a zero vector.
(iii) Zero vectors are only of mathematical importance.

Equal Vectors

Vectors are said to be equal if both vectors have same magnitude and direction.



A = B
Fig. 5.5

Parallel Vectors

Vectors are said to be parallel if they have the same directions.

The vectors **A** and **B** shown in Fig. 5.6 represent parallel vectors.

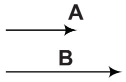


Fig. 5.6

Note *Two equal vectors are always parallel but, two parallel vectors may not be equal vectors.*

Anti-parallel Vectors (Unlike Vectors)

Vectors are said to be anti-parallel if they act in opposite direction.

The vectors **A** and **B** shown in Fig. 5.7 are anti-parallel vectors.

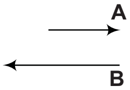
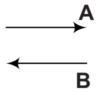


Fig. 5.7

Negative Vector

The negative vector of any vector is a vector having equal magnitude but acts in opposite direction.



A = –B or B = –A
Fig. 5.8

Concurrent Vectors (Co-initial Vectors)

Vectors having the same initial points are called concurrent vectors or co-initial vectors.

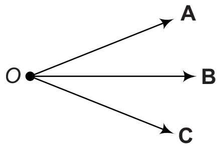
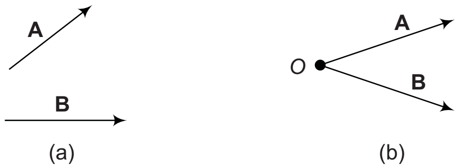


Fig. 5.9

A, B and **C** are concurrent at point *O*.

Coplanar Vectors

The vectors lying in the same plane are called coplanar vectors.



(a) (b)

Fig. 5.10

The vector **A** and **B** are coplanar vectors. The vectors **A** and **B** shown in Fig. 5.10 (b) are concurrent coplanar vectors.

Orthogonal Vectors

Two vectors are said to be orthogonal if the angle between them is 90°.

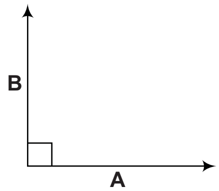


Fig. 5.11

The vector shown in Fig. 5.11, **A** and **B** are orthogonal to one another.

Multiplication and Division of Vectors by Scalars

The product of a vector **A** and a scalar m is a vector $m\mathbf{A}$ whose magnitude is m times the magnitude of **A** and which is in the direction or opposite to **A** according as the scalar m is positive or negative. Thus,

$$|m\mathbf{A}| = mA$$

Further, if m and n are two scalars, then

$$(m+n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$$

and

$$m(n\mathbf{A}) = n(m\mathbf{A}) = (mn)\mathbf{A}$$

The division of vector **A** by a non-zero scalar m is defined as the multiplication of **A** by $\frac{1}{m}$.

Example 5.1

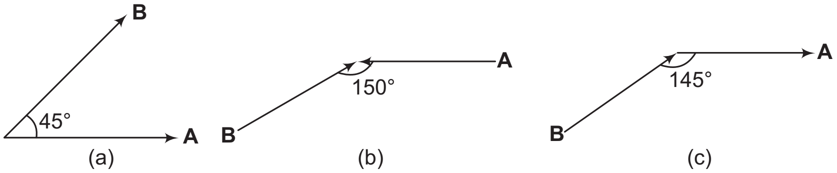


Fig. 5.12

In the shown Fig. 5.12 *(a), (b) and (c), find the angle between* **A** *and* **B**.

Solution If we draw both the vectors from one point with their arrows outwards, then they can be shown as below

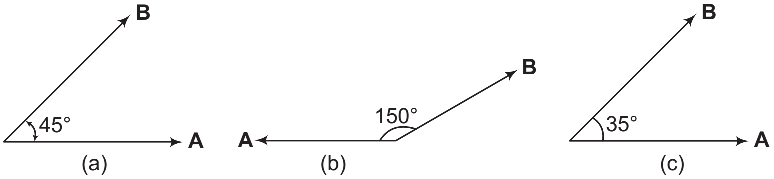


Fig. 5.13

In Fig. (a), $\theta = 45°$

In Fig. (b), $\theta = 150°$ and

In Fig. (c), $\theta = 35°$

Example 5.2 *What is the angle between* **a** *and* $-\frac{3}{2}\mathbf{a}$.

Solution $-\frac{3}{2}\mathbf{a}$ has a magnitude equal to $\frac{3}{2}$ times the magnitude of **a** and its direction is opposite to **a**. Therefore, **a** and $-\frac{3}{2}\mathbf{a}$ are antiparallel to each other or angle between them is 180°.

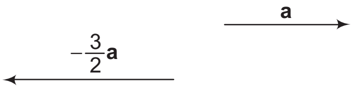


Fig. 5.14

5.3 Addition and Subtraction of Two Vectors

Addition

(i) **The parallelogram law** Let **R** be the resultant of two vectors **A** and **B**. According to parallelogram law of vector addition, the resultant **R** is the diagonal of the parallelogram of which **A** and **B** are the adjacent sides as shown in figure. Magnitude of **R** is given by

$$\boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}} \quad \text{...(i)}$$

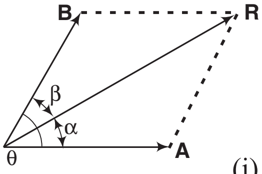


Fig. 5.15

Here, θ = angle between **A** and **B**. The direction of **R** can be found by angle α or β of **R** with **A** or **B**.

Here, $$\boxed{\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}} \quad \text{and} \quad \boxed{\tan\beta = \frac{A\sin\theta}{B + A\cos\theta}} \quad \text{...(ii)}$$

Special cases

If $\theta = 0°$, $R = \text{maximum} = A + B$

$\theta = 180°$, $R = \text{minimum} = A \sim B$

and if $\theta = 90°$, $R = \sqrt{A^2 + B^2}$

In all other cases magnitude and direction of **R** can be calculated by using Eqs. (i) and (ii).

(ii) **The triangle law** According to this law, if the tail of one vector be placed at the head of the other, their sum or resultant **R** is drawn from the tail end of the first to the head end of the other. As is evident from the figure that the resultant **R** is the same irrespective of the order in which the vectors **A** and **B** are taken, Thus, **R** = **A** + **B** = **B** + **A**

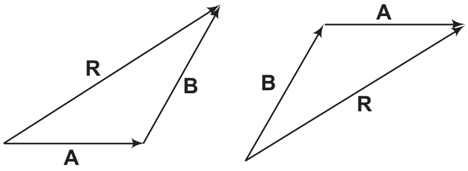


Fig. 5.16

Subtraction

Negative of a vector say **–A** is a vector of the same magnitude as vector **A** but pointing in a direction opposite to that of **A**.

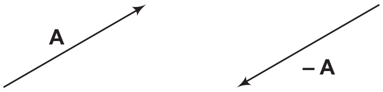


Fig. 5.17

Thus, **A** – **B** can be written as **A** + (–**B**) or **A** – **B** is really the vector addition of **A** and –**B**. Suppose angle between two vectors **A** and **B** is θ. Then, angle between **A** and –**B** will be 180 – θ as shown in Fig. 5.18 (b).

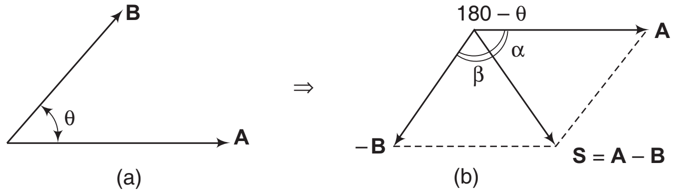


Fig. 5.18

Magnitude of **S** = **A** – **B** will be thus given by

$$S = |\mathbf{A} - \mathbf{B}| = |\mathbf{A} + (-\mathbf{B})|$$

$$= \sqrt{A^2 + B^2 + 2AB\cos(180 - \theta)}$$

or $$\boxed{S = \sqrt{A^2 + B^2 - 2AB\cos\theta}} \qquad \ldots\text{(i)}$$

For direction of **S** we will either find angle α or β, where,

$$\tan\alpha = \frac{B\sin(180-\theta)}{A + B\cos(180-\theta)} = \frac{B\sin\theta}{A - B\cos\theta} \qquad \ldots\text{(ii)}$$

or $$\tan\beta = \frac{A\sin(180-\theta)}{B + A\cos(180-\theta)} = \frac{A\sin\theta}{B - A\cos\theta} \qquad \ldots\text{(iii)}$$

Note **A** –**B** *or* **B** – **A** *can also be found by making triangles as shown in Fig.* 5.19 *(a) and (b).*

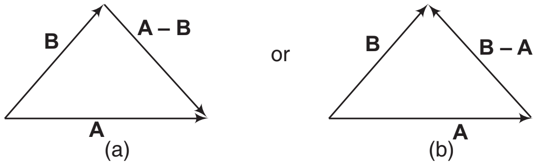


Fig. 5.19

Example 5.3 *Find* **A** + **B** *and* **A** – **B** *in the diagram shown in figure. Given* $A = 4$ *units and* $B = 3$ *units.*

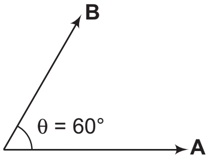


Fig. 5.20

Solution **Addition** $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$= \sqrt{16 + 9 + 2 \times 4 \times 3\cos 60°}$$

$$= \sqrt{37} \text{ units}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

$$= \frac{3\sin 60^\circ}{4 + 3\cos 60^\circ} = 0.472$$

$$\therefore \qquad \alpha = \tan^{-1}(0.472) = 25.3^\circ$$

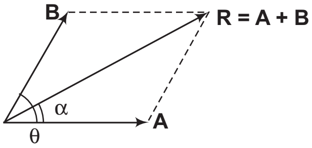


Fig. 5.21

Thus, resultant of **A** and **B** is $\sqrt{37}$ units at angle 25.3° from **A** in the direction shown in figure.

Subtraction $\qquad S = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

$$= \sqrt{16 + 9 - 2\times 4\times 3\cos 60^\circ} = \sqrt{13} \text{ units}$$

and

$$\tan\alpha = \frac{B\sin\theta}{A - B\cos\theta}$$

$$= \frac{3\sin 60^\circ}{4 - 3\cos 60^\circ} = 1.04$$

$$\therefore \qquad \alpha = \tan^{-1}(1.04) = 46.1^\circ$$

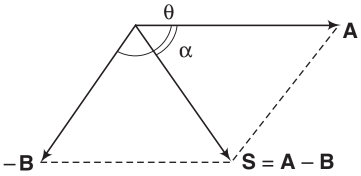


Fig. 5.22

Thus, **A** – **B** is $\sqrt{13}$ units at 46.1° from **A** in the direction shown in figure.

Polygon Law of Vector Addition for more than Two Vectors

This law states that if a vector polygon be drawn, placing the tail end of each succeeding vector at the head or the arrow end of the preceding one their resultant **R** is drawn from the tail end of the first to the head or the arrow end of the last.

Thus, in the figure **R** = **A** + **B** + **C**

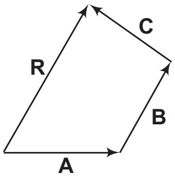


Fig. 5.23

INTRODUCTORY EXERCISE 5.1

1. What is the angle between 2**a** and 4**a**?
2. What is the angle between 3**a** and –5**a**? What is the ratio of magnitude of two vectors?
3. Two vectors have magnitudes 6 units and 8 units respectively. Find magnitude of resultant of two vectors if angle between two vectors is

   (a) 0° (b) 180° (c) 60° (d) 120° (e) 90°
4. Two vectors **A** and **B** have magnitudes 6 units and 8 units respectively. Find |**A** – **B**|, if the angle between two vectors is

   (a) 0° (b) 180° (c) 60° (d) 120° (e) 90°
5. For what angle between **A** and **B**, |**A** + **B**| = |**A** – **B**|.

5.4 Components of a Vector

Two or more vectors which, when compounded in accordance with the parallelogram law of vector **R** are said to be components of vector **R**. The most important components with which we are concerned are mutually perpendicular or rectangular ones along the three co-ordinate axes *ox, oy* and *oz* respectively. Thus, a vector **R** can be written as $\mathbf{R} = R_x\,\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$.

Here, R_x, R_y and R_z are the components of **R** in *x, y* and *z*-axes respectively and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along these directions. The magnitude of **R** is given by

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

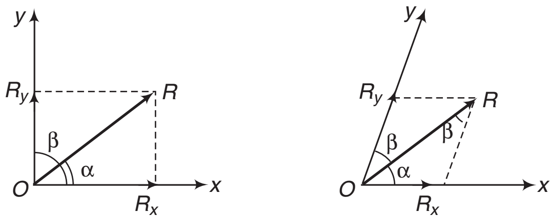
This vector **R** makes an angle of $\alpha = \cos^{-1}\left(\frac{R_x}{R}\right)$ with *x*-axis or $\cos\alpha = \frac{R_x}{R}$

$$\beta = \cos^{-1}\left(\frac{R_y}{R}\right) \text{ with } y\text{-axis or } \cos\beta = \frac{R_y}{R}$$

and
$$\gamma = \cos^{-1}\left(\frac{R_z}{R}\right) \text{ with } z\text{-axis or } \cos\gamma = \frac{R_z}{R}$$

Note *Here* $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$ *are called direction cosines of* **R** *with x, y and z-axes.*

Refer Fig. (a)



(a) (b)

Fig. 5.24

We have resolved a two dimensional vector **R** (in *xy* plane) in mutually perpendicular directions *x* and *y*.

Component along *x*-axis $= R_x = R\cos\alpha$ or $R\sin\beta$ and component along *y*-axis $= R_y = R\cos\beta$ or $R\sin\alpha$.

If $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along *x* and *y*-axes respectively, we can write

$$\mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

Refer Fig. (b)

Vector **R** has been resolved in two axes *x* and *y* not perpendicular to each other. Applying sine law in the triangle shown , we have

$$\frac{R}{\sin[180 - (\alpha + \beta)]} = \frac{R_x}{\sin\beta} = \frac{R_y}{\sin\alpha}$$

or
$$R_x = \frac{R\sin\beta}{\sin(\alpha + \beta)} \quad \text{and} \quad R_y = \frac{R\sin\alpha}{\sin(\alpha + \beta)}$$

If $\alpha + \beta = 90°$, $R_x = R\sin\beta$ and $R_y = R\sin\alpha$

Position Vector

To locate the position of any point P in a plane or space, generally a fixed point of reference called the origin O is taken. The vector $\mathbf{OP}$ is called the position vector of P with respect to O as shown in figure. If coordinates of point P are (x, y) then position vector of point P with respect to point O is

$$\mathbf{OP} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

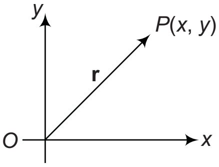


Fig. 5.25

Note *(i) For a point P, there is one and only one position vector with respect to the origin O.*
(ii) Position vector of a point P changes if the position of the origin O is changed.

Displacement Vector

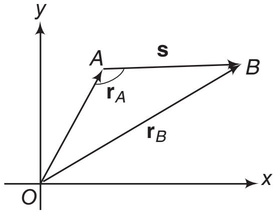


Fig. 5.26

If coordinates of point A are (x_1, y_1, z_1) and B are (x_2, y_2, z_2).
Then, position vector of A

$$= \mathbf{r}_A = \mathbf{OA} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$$

Position vector of $B = \mathbf{r}_B = \mathbf{OB} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$

and $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{r}_B - \mathbf{r}_A =$ displacement vector $(\mathbf{s})$

$$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

Example 5.4 *A force* $\mathbf{F}$ *has magnitude of* 15 N. *Direction of* $\mathbf{F}$ *is at* 37° *from negative x-axis towards positive y-axis. Represent* $\mathbf{F}$ *in terms of* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$.

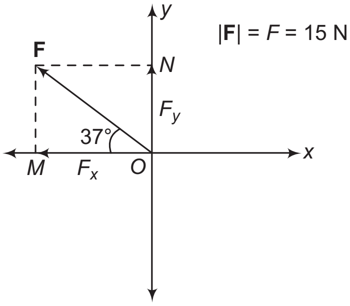


Fig. 5.27

Solution The given force is as shown in figure. Let us find its x and y components.

$$\begin{aligned} F_x &= F\cos 37° \\ &= 15 \times \frac{4}{5} \\ &= 12\text{ N} \quad \text{(along negative } x\text{-axis)} \\ F_y &= F\sin 37° \\ &= 15 \times \frac{3}{5} \\ &= 9\text{ N} \quad \text{(along positive } y\text{-axis)} \end{aligned}$$

From parallelogram law of vector addition, we can see that

$$\begin{aligned} \mathbf{F} &= \mathbf{OM} + \mathbf{ON} \\ &= F_x(-\hat{\mathbf{i}}) + F_y(\hat{\mathbf{j}}) \\ &= (-12\hat{\mathbf{i}} + 9\hat{\mathbf{j}})\text{ N} \end{aligned}$$

Ans.

Example 5.5 *Find magnitude and direction of a vector,* $\mathbf{A} = (6\hat{\mathbf{i}} - 8\hat{\mathbf{j}})$.

Solution **Magnitude of A**

$$|\mathbf{A}| \text{ or } A = \sqrt{(6)^2 + (-8)^2}$$

$$= 10 \text{ units}$$ **Ans.**

Direction of A Vector **A** can be shown as

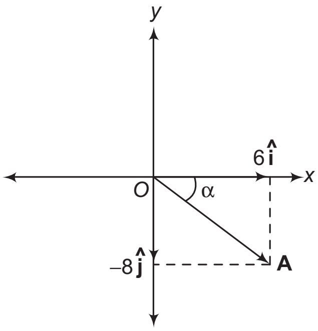


Fig. 5.28

$$\tan\alpha = \frac{8}{6} = \frac{4}{3}$$

$$\therefore \qquad \alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53°$$

Therefore, **A** is making an angle of 53° from positive x-axis towards negative y-axis.

Example 5.6 *Resolve a weight of* 10 *N in two directions which are parallel and perpendicular to a slope inclined at* 30° *to the horizontal.*

Solution Component perpendicular to the plane

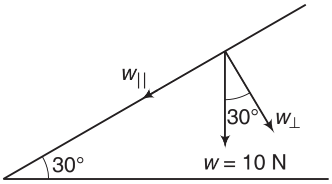


Fig. 5.29

$$w_{\perp} = w\cos 30°$$

$$= (10)\frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ N}$$

and component parallel to the plane

$$w_{||} = w\sin 30° = (10)\left(\frac{1}{2}\right)$$

$$= 5 \text{ N}$$

Example 5.7 *Resolve horizontally and vertically a force $F = 8$ N which makes an angle of* 45° *with the horizontal.*

Solution Horizontal component of **F** is

$$F_H = F\cos 45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

and vertical component of **F** is

$$F_V = F\sin 45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

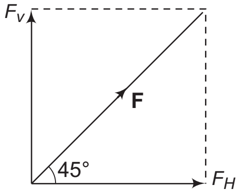


Fig. 5.30

Note *Two vectors given in the form of* $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$ *can be added, subtracted or multiplied (by a scalar) directly as is done in the example* 5.8.

Example 5.8 *Obtain the magnitude of* $2\mathbf{A}-3\mathbf{B}$ *if*

$$\mathbf{A} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}} \quad \textit{and} \quad \mathbf{B} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Solution $2\mathbf{A} - 3\mathbf{B} = 2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) - 3(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = -4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

$\therefore$ Magnitude of $2\mathbf{A} - 3\mathbf{B} = \sqrt{(-4)^2 + (5)^2 + (-7)^2} = \sqrt{16 + 25 + 49} = \sqrt{90}$

INTRODUCTORY EXERCISE 5.2

1. Find magnitude and direction cosines of the vector, $\mathbf{A} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$.
2. Resolve a force $F = 10$ N along x and y-axes. Where this force vector is making an angle of 60° from negative x-axis towards negative y-axis?
3. Find magnitude of $\mathbf{A} - 2\mathbf{B} + 3\mathbf{C}$, where, $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{C} = \hat{\mathbf{k}}$.
4. Find angle between **A** and **B**, where,
   (a) $\mathbf{A} = 2\hat{\mathbf{i}}$ and $\mathbf{B} = -6\hat{\mathbf{i}}$
   (b) $\mathbf{A} = 6\hat{\mathbf{j}}$ and $\mathbf{B} = -2\hat{\mathbf{k}}$
   (c) $\mathbf{A} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ and $\mathbf{B} = 4\hat{\mathbf{k}}$
   (d) $\mathbf{A} = 4\hat{\mathbf{i}}$ and $\mathbf{B} = (-3\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$

5.5 Product of Two Vectors

The product of two vectors is of two kinds

(i) scalar or dot product.
(ii) a vector or a cross product.

Scalar or Dot Product

The scalar or dot product of two vectors **A** and **B** is denoted by $\mathbf{A}\cdot\mathbf{B}$ and is read as **A** dot **B**.

It is defined as the product of the magnitudes of the two vectors **A** and **B** and the cosine of their included angle θ.

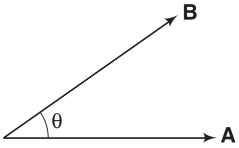


Fig. 5.31

Thus, $$\boxed{\mathbf{A}\cdot\mathbf{B} = AB\cos\theta}$$ (a scalar quantity)

Important Points Regarding Dot Product

The following points should be remembered regarding the dot product.

(i) $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$

(ii) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$

(iii) $\mathbf{A}\cdot\mathbf{A}=A^2$

(iv) $\mathbf{A}\cdot\mathbf{B}=A(B\cos\theta)=A$ (Component of **B** along **A**)
or $\mathbf{A}\cdot\mathbf{B}=B(A\cos\theta)=B$ (Component of **A** along **B**)

(v) $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=(1)(1)\cos 0^\circ=1$

(vi) $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=(1)(1)\cos 90^\circ=0$

(vii) $(a_1\hat{\mathbf{i}}+b_1\hat{\mathbf{j}}+c_1\hat{\mathbf{k}})\cdot(a_2\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+c_2\hat{\mathbf{k}})=a_1a_2+b_1b_2+c_1c_2$

(viii) $\cos\theta=\dfrac{\mathbf{A}\cdot\mathbf{B}}{AB}=$ (cosine of angle between **A** and **B**)

(ix) Two vectors are perpendicular if their dot product is zero. $(\theta=90^\circ)$

Example 5.9 *Work done by a force* **F** *on a body is* $W=\mathbf{F}\bullet\mathbf{s}$, *where* **s** *is the displacement of body. Given that under a force* $\mathbf{F}=(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})$ *N a body is displaced from position vector* $\mathbf{r}_1=(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ *m to the position vector* $\mathbf{r}_2=(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ *m. Find the work done by this force.*

Solution The body is displaced from $\mathbf{r}_1$ to $\mathbf{r}_2$. Therefore, displacement of the body is

$$\mathbf{s}=\mathbf{r}_2-\mathbf{r}_1=(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})-(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}})=(-\hat{\mathbf{i}}-2\hat{\mathbf{j}})\text{ m}$$

Now, work done by the force is $W=\mathbf{F}\cdot\mathbf{s}$

$$=(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}}-2\hat{\mathbf{j}})$$

$$=(2)(-1)+(3)(-2)+(4)(0)=-8\text{ J}$$

Example 5.10 *Find the angle between two vectors* $\mathbf{A}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *and* $\mathbf{B}=\hat{\mathbf{i}}-\hat{\mathbf{k}}$.

Solution

$$A=|\mathbf{A}|=\sqrt{(2)^2+(1)^2\ (-1)^2}=\sqrt{6}$$

$$B=|\mathbf{B}|=\sqrt{(1)^2+(-1)^2}=\sqrt{2}$$

$$\mathbf{A}\cdot\mathbf{B}=(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})\bullet(\hat{\mathbf{i}}-\hat{\mathbf{k}})$$

$$=(2)(1)+(1)(0)+(-1)(-1)=3$$

Now,

$$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{AB}=\frac{3}{\sqrt{6}\cdot\sqrt{2}}$$

$$=\frac{3}{\sqrt{12}}=\frac{\sqrt{3}}{2}$$

∴ $\theta=30^\circ$

Example 5.11 *Prove that the vectors* $\mathbf{A}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *and* $\mathbf{B}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ *are mutually perpendicular.*

Solution
$$\mathbf{A}\cdot\mathbf{B} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$
$$= (2)(1) + (-3)(1) + (1)(1) = 0 = AB\cos\theta$$

∴ $\cos\theta = 0$ (as $A \neq 0, B \neq 0$)

or $\theta = 90^\circ$

or the vectors **A** and **B** are mutually perpendicular.

Vector or Cross Product

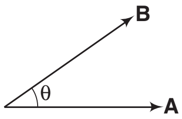
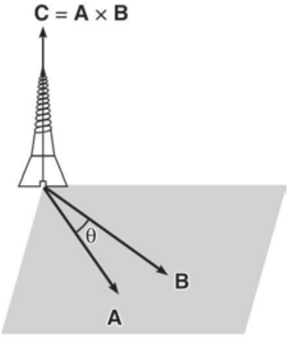


Fig. 5.32

The cross product of two vectors **A** and **B** is denoted by $\mathbf{A}\times\mathbf{B}$ and read as **A** cross **B**. It is defined as a third vector **C** whose magnitude is equal to the product of the magnitudes of the two vectors **A** and **B** and the sine of their included angle θ.

Thus, if $\boxed{\mathbf{C} = \mathbf{A}\times\mathbf{B}, \quad \text{then} \quad C = AB\sin\theta.}$



Plane of **A** and **B**

Fig. 5.33

The vector **C** is normal to the plane of **A** and **B** and points in the direction in which a right handed screw would advance when rotated about an axis perpendicular to the plane of the two vectors in the direction from **A** to **B** through the smaller angle θ between them or alternatively, we might state the rule as below

If the fingers of the right hand be curled in the direction in which vector **A** must be turned through the smaller included angle θ to coincide with the direction of vector **B**, the thumb points in the direction of **C** as shown in Fig. 5.33.

Either of these rules is referred to as the right handed screw rule. Thus, if $\hat{\mathbf{n}}$ be the unit vector in the direction of **C**, we have

$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}}$$

where, $0 \leq \theta \leq \pi$

Important Points About Vector Product

(i) $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

(ii) The cross product of two parallel (or antiparallel) vectors is zero, as $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$ and $\theta = 0^\circ$ or $\sin\theta = 0$ for two parallel vectors. Thus, $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} =$ a null vector.

(iii) If two vectors are perpendicular to each other, we have $\theta = 90^\circ$ and therefore, $\sin\theta = 1$. So that $\mathbf{A}\times\mathbf{B} = AB\,\hat{\mathbf{n}}$. The vectors **A**, **B** and $\mathbf{A}\times\mathbf{B}$ thus form a right handed system of mutually perpendicular vectors. It follows at once from the above that in case of the orthogonal triad of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ (each perpendicular to each other)

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = -\hat{\mathbf{j}}\times\hat{\mathbf{i}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}}\times\hat{\mathbf{k}} = -\hat{\mathbf{k}}\times\hat{\mathbf{j}} = \hat{\mathbf{i}} \quad \text{and} \quad \hat{\mathbf{k}}\times\hat{\mathbf{i}} = -\hat{\mathbf{i}}\times\hat{\mathbf{k}} = \hat{\mathbf{j}}$$

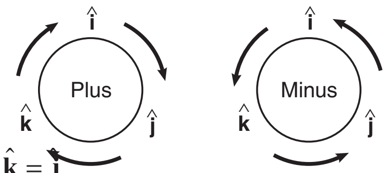


Fig. 5.34

(iv) $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

(v) A vector product can be expressed in terms of rectangular components of the two vectors and put in the determinant form as may be seen from the following:

Let $\mathbf{A} = a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$

and $\mathbf{B} = a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$

Then,
$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}})\times(a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}})\\ &= a_1a_2(\hat{\mathbf{i}}\times\hat{\mathbf{i}}) + a_1b_2(\hat{\mathbf{i}}\times\hat{\mathbf{j}}) + a_1c_2\,(\hat{\mathbf{i}}\times\hat{\mathbf{k}}) + b_1a_2(\hat{\mathbf{j}}\times\hat{\mathbf{i}}) + b_1b_2\,(\hat{\mathbf{j}}\times\hat{\mathbf{j}})\\ &\quad + b_1c_2(\hat{\mathbf{j}}\times\hat{\mathbf{k}}) + c_1a_2(\hat{\mathbf{k}}\times\hat{\mathbf{i}}) + c_1b_2(\hat{\mathbf{k}}\times\hat{\mathbf{j}}) + c_1c_2\,(\hat{\mathbf{k}}\times\hat{\mathbf{k}})\end{aligned}$$

Since, $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}}$ = a null vector and $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$, etc., we have

$$\mathbf{A}\times\mathbf{B} = (b_1c_2 - c_1b_2)\,\hat{\mathbf{i}} + (c_1a_2 - a_1c_2)\,\hat{\mathbf{j}} + (a_1b_2 - b_1a_2)\,\hat{\mathbf{k}}$$

or putting it in determinant form, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

It may be noted that the scalar components of the first vector **A** occupy the middle row of the determinant.

Example 5.12 *Find a unit vector perpendicular to* $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\mathbf{B} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *both.*

Solution As we have read, $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is a vector perpendicular to both **A** and **B**. Hence, a unit vector $\hat{\mathbf{n}}$ perpendicular to **A** and **B** can be written as

$$\hat{\mathbf{n}} = \frac{\mathbf{C}}{C} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

Here,
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 3 & 1\\ 1 & -1 & 1\end{vmatrix}$$

$$= \hat{\mathbf{i}}(3+1) + \hat{\mathbf{j}}(1-2) + \hat{\mathbf{k}}(-2-3) = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Further,
$$|\mathbf{A}\times\mathbf{B}| = \sqrt{(4)^2 + (-1)^2 + (-5)^2} = \sqrt{42}$$

∴ The desired unit vector is

$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} \quad\text{or}\quad \hat{\mathbf{n}} = \frac{1}{\sqrt{42}}(4\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$$

Example 5.13 *Show that the vector* $\mathbf{A} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ *is parallel to a vector* $\mathbf{B} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution A vector **A** is parallel to an another vector **B** if it can be written as

$$\mathbf{A} = m\mathbf{B}$$

Here,
$$\mathbf{A} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = \frac{1}{3}(3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \quad\text{or}\quad \mathbf{A} = \frac{1}{3}\mathbf{B}$$

This implies that **A** is parallel to **B** and magnitude of **A** is $\frac{1}{3}$ times the magnitude of **B**.

Note *Two vectors can be shown parallel (or antiparallel) to one another if :*

(i) The coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$ *of both the vectors bear a constant ratio. For example, a vector* $\mathbf{A} = a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$ *is parallel to an another vector* $\mathbf{B} = a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$ *if :* $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ *= constant. If this constant has positive value, then the vectors are parallel and if the constant has negative value then the vectors are antiparallel.*

(ii) The cross product of both the vectors is a null vector. For instance, **A** *and* **B** *are parallel (or antiparallel) to each other if*

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \text{a null vector}$$

Example 5.14 *Let a force* **F** *be acting on a body free to rotate about a point O and let* **r** *the position vector of any point P on the line of action of the force. Then torque* (τ) *of this force about point O is defined as*

$$\tau = \mathbf{r} \times \mathbf{F}$$

Given, $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})\ N$ *and* $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 6\hat{\mathbf{k}})\ m$

Find the torque of this force.

Solution

$$\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 6 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(1-18) + \hat{\mathbf{j}}(12+1) + \hat{\mathbf{k}}(3+2)$$

or

$$\tau = (-17\hat{\mathbf{i}} + 13\hat{\mathbf{j}} + 5\hat{\mathbf{k}})\ \text{N-m}$$

INTRODUCTORY EXERCISE 5.3

1. Cross product of two parallel or antiparallel vectors is a null vector. Is this statement true or false?
2. Find the values of
   (a) $(4\hat{\mathbf{i}}) \times (-6\hat{\mathbf{k}})$ (b) $(3\hat{\mathbf{j}}) \cdot (-4\hat{\mathbf{j}})$ (c) $(2\hat{\mathbf{i}}) \cdot (-4\hat{\mathbf{k}})$
3. Two vectors **A** and **B** have magnitudes 2 units and 4 units respectively. Find $\mathbf{A} \cdot \mathbf{B}$ if angle between these two vectors is
   (a) 0° (b) 60° (c) 90° (d) 120°
   (e) 180°
4. Find $(2\mathbf{A}) \times (-3\mathbf{B})$, if $\mathbf{A} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{B} = (\hat{\mathbf{j}} + \hat{\mathbf{k}})$

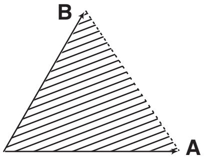
Final Touch Points

1. The moment of inertia has two forms, a scalar form I (used when the axis of rotation is known) and a more general tensor form that does not require knowing the axis of rotation. Although tensor is a generalized term which is characterized by its rank. For example, scalars are tensors of rank zero. Vectors are tensors of rank two.
2. Pressure is a scalar quantity, not a vector quantity. It has magnitude but no direction sense associated with it. Pressure acts in all directions at a point inside a fluid.
3. Surface tension is scalar because it has no specific direction.
4. Stress is neither a scalar nor a vector quantity, it is a tensor.
5. To qualify as a vector, a physical quantity must not only possess magnitude and direction but must also satisfy the parallelogram law of vector addition. For instance, the finite rotation of a rigid body about a given axis has magnitude (the angle of rotation) and also direction (the direction of the axis) but it is not a vector quantity. This is so for the simple reason that the two finite rotations of the body do not add up in accordance with the law of vector addition. However if the rotation be small or infinitesimal, it may be regarded as a vector quantity.
6. Area can behave either as a scalar or a vector and how it behaves depends on circumstances.
7. Area (vector), dipole moment and current density are defined as vectors with specific direction.
8. Vectors associated with a linear or directional effect are called polar vectors or simply as vectors and those associated with rotation about an axis are referred to as axial vectors. Thus, force, linear velocity and linear acceleration are polar vectors and angular velocity, angular acceleration are axial vectors.
9. Examples of dot-product and cross-product

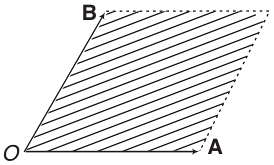
Examples of Dot-product	Examples of Cross-product
$W = \mathbf{F} \cdot \mathbf{s}$	$\tau = \mathbf{r} \times \mathbf{F}$
$P = \mathbf{F} \cdot \mathbf{v}$	$\mathbf{L} = \mathbf{r} \times \mathbf{P}$
$d\phi_e = \mathbf{E} \cdot \mathbf{ds}$	$\mathbf{v} = \omega \times \mathbf{r}$
$d\phi_B = \mathbf{B} \cdot \mathbf{ds}$	$\tau_e = \mathbf{P} \times \mathbf{E}$
$U_e = \mathbf{P} \cdot \mathbf{E}$	$\tau_B = \mathbf{M} \times \mathbf{B}$
$U_B = \mathbf{M} \cdot \mathbf{B}$	$\mathbf{F}_B = q\,(\mathbf{v} \times \mathbf{B})$
	$\mathbf{dB} = \frac{\mu_0}{4\pi} \frac{i\,(\mathbf{dl} \times \mathbf{r})}{r^3}$

10. Students are often confused over the direction of cross product. Let us discuss a simple method. To find direction of **A** × **B** curl your fingers from **A** to **B** through smaller angle. If it is clockwise then **A** × **B** is perpendicular to the plane of **A** and **B** and away from you and if it is anti-clockwise then **A** × **B** is towards you perpendicular to the plane of **A** and **B**.
11. The area of triangle bounded by vectors **A** and **B** is $\frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

Exercise : Prove the above result.



12. Area of parallelogram shown in figure is, Area $= |\mathbf{A} \times \mathbf{B}|$



Exercise : Prove the above relation.

13. **Scalar triple product** : $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is called scalar triple product. It is a scalar quantity. We can show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.
14. The volume of a parallelopiped bounded by vectors **A**, **B** and **C** can be obtained by $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.
15. If three vectors are coplanar then the volume of the parallelopiped bounded by these three vectors should be zero or we can say that their scalar triple product should be zero.
16. If $\mathbf{A} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{B} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{C} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ then $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is also written as $[\mathbf{ABC}]$ and it has the following value :

$$[\mathbf{ABC}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

= Volume of parallelopiped whose adjacent sides are along **A**, **B** and **C**.

17. If $|\mathbf{A}| = |\mathbf{B}| = A$ (say) then,

$$|\mathbf{R}| = |\mathbf{A} + \mathbf{B}| = 2A \cos \frac{\theta}{2}$$

Exercise : Prove the above result.

$$\begin{aligned} \text{For } \theta = 0°, &\quad |\mathbf{R}| = 2A \\ \theta = 60°, &\quad |\mathbf{R}| = \sqrt{3}\,A \\ \theta = 90°, &\quad |\mathbf{R}| = \sqrt{2}\,A \\ \theta = 120°, &\quad |\mathbf{R}| = A \text{ and} \\ \theta = 180°, &\quad |\mathbf{R}| = O \end{aligned}$$

In this case, resultant of **A** and **B** always passes through the bisector line of **A** and **B**.

18. If $|\mathbf{A}| = |\mathbf{B}| = A$ (say) then,

$$|\mathbf{S}| = |\mathbf{A} - \mathbf{B}| = 2A \sin \frac{\theta}{2}$$

Exercise : Prove the above result.

19. Angle between two vectors is obtained by their dot product (not from cross product) i.e.

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB}\right)$$

It is not always, $\sin^{-1}\left\{\frac{|\mathbf{A} \times \mathbf{B}|}{AB}\right\}$

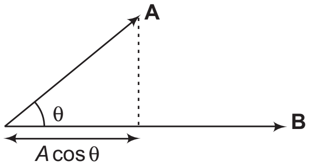
Exercise : Explain the reason why θ is not always given by the following relation ?

$$\theta = \sin^{-1}\left\{\frac{|\mathbf{A} \times \mathbf{B}|}{AB}\right\}$$

20. A unit vector perpendicular to both **A** and **B**

$$\hat{\mathbf{C}} = \pm \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

21. Component of **A** along **B** $= A \cos \theta = \dfrac{\mathbf{A} \cdot \mathbf{B}}{B}$



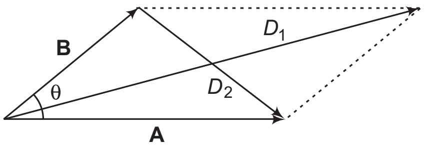
Similarly, component of **B** along **A**

$$= B \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{A}$$

Component of **A** along **B** = component of **B** along **A**

If $|\mathbf{A}| = |\mathbf{B}|$ or $A = B$. Otherwise they are not equal.

22. In the figure shown,



$$\text{diagonal } D_1 = |\mathbf{A} + \mathbf{B} \text{ or } \mathbf{R}| = \sqrt{A^2 + B^2 + 2AB \cos \theta}$$

$$\text{diagonal } D_2 = |\mathbf{A} - \mathbf{B} \text{ or } \mathbf{S}| = \sqrt{A^2 + B^2 - 2AB \cos \theta}$$

$$D_1 = D_2 = \sqrt{A^2 + B^2} \text{ if } \theta = 90^\circ$$

Solved Examples

Example 1 *Find component of vector* $\mathbf{A}+\mathbf{B}$ *along (i) x-axis, (ii)* $\mathbf{C}$.
Given $\mathbf{A}=\hat{\mathbf{i}}-2\hat{\mathbf{j}},\quad \mathbf{B}=2\hat{\mathbf{i}}+3\hat{\mathbf{k}}$ *and* $\mathbf{C}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$.

Solution $\mathbf{A}+\mathbf{B}=(\hat{\mathbf{i}}-2\hat{\mathbf{j}})+(2\hat{\mathbf{i}}+3\hat{\mathbf{k}})=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

(i) Component of $\mathbf{A}+\mathbf{B}$ along x-axis is 3.

(ii) Component of $\mathbf{A}+\mathbf{B}=\mathbf{R}$ (say) along $\mathbf{C}$ is

$$R\cos\theta=\frac{\mathbf{R}\cdot\mathbf{C}}{C}=\frac{(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}})\bullet(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{\sqrt{(1)^2+(1)^2}}=\frac{3-2}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

Example 2 *Find the angle that the vector* $\mathbf{A}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ *makes with y-axis.*

Solution $\cos\theta=\dfrac{A_y}{A}=\dfrac{3}{\sqrt{(2)^2+(3)^2+(-1)^2}}=\dfrac{3}{\sqrt{14}}$

$$\therefore \qquad \theta=\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

Example 3 *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are the vectors* $\mathbf{AB}$ *and* $\mathbf{BC}$ *determined by the adjacent sides of a regular hexagon. What are the vectors determined by the other sides taken in order?*

Solution Given $\mathbf{AB}=\mathbf{a}$ and $\mathbf{BC}=\mathbf{b}$

From the method of vector addition (or subtraction) we can show that,

$$\mathbf{CD}=\mathbf{b}-\mathbf{a}$$

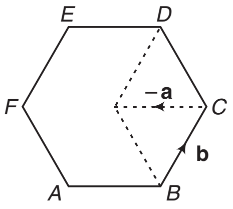
Then

$$\mathbf{DE}=-\mathbf{AB}=-\mathbf{a}$$

$$\mathbf{EF}=-\mathbf{BC}=-\mathbf{b}$$

and

$$\mathbf{FA}=-\mathbf{CD}=\mathbf{a}-\mathbf{b}$$



Example 4 *If* $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}\neq\mathbf{0}$ *with* $\mathbf{a}\neq-\mathbf{c}$ *then show that* $\mathbf{a}+\mathbf{c}=k\mathbf{b}$, *where k is scalar.*

Solution $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$

$$\mathbf{a}\times\mathbf{b}=-\mathbf{c}\times\mathbf{b}$$

$$\therefore \qquad \mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}=0$$

$$(\mathbf{a}+\mathbf{c})\times\mathbf{b}=0$$

$\therefore$ $\mathbf{a}\times\mathbf{b}\neq 0$, $\mathbf{b}\times\mathbf{c}\neq 0$, $\mathbf{a},\mathbf{b},\mathbf{c}$ are non-zero vectors. $(\mathbf{a}+\mathbf{c})\neq\mathbf{0}$

Hence, $\mathbf{a}+\mathbf{c}$ is parallel to $\mathbf{b}$.

$$\therefore \qquad \mathbf{a}+\mathbf{c}=k\mathbf{b}$$

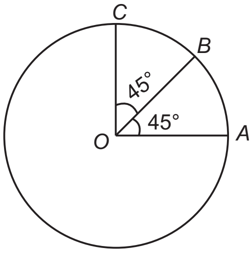
Example 5 *If* $\mathbf{A} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\mathbf{B} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ *and* $\mathbf{C} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$. *Find* $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

Solution $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = [\mathbf{ABC}]$, volume of parallelopiped

$$= \begin{vmatrix} 2 & -3 & 7 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 2(-2-0) + 3(-1-0) + 7(1-0) = -4 - 3 + 7 = 0$$

Therefore **A**, **B** and **C** are coplanar vectors.

Example 6 *Find the resultant of three vectors* **OA**, **OB** *and* **OC** *shown in figure. Radius of circle is 'R'.*



Solution $OA = OC$

$\mathbf{OA} + \mathbf{OC}$ is along **OB**, (bisector) and its magnitude is $2R\cos 45° = R\sqrt{2}$

$(\mathbf{OA} + \mathbf{OC}) + \mathbf{OB}$ is along **OB** and its magnitude is $R\sqrt{2} + R = R(1 + \sqrt{2})$

Example 7 *Prove that* $|\mathbf{a}\times\mathbf{b}|^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$

Solution Let $|\mathbf{a}| = a$, $|\mathbf{b}| = b$ and θ be the angle between them.

$$\begin{aligned} |\mathbf{a}\times\mathbf{b}|^2 &= (ab\sin\theta)^2 = a^2b^2\sin^2\theta \\ &= a^2b^2(1-\cos^2\theta) = a^2b^2 - (a\cdot b\cos\theta)^2 \\ &= a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2 \end{aligned}$$

Hence Proved.

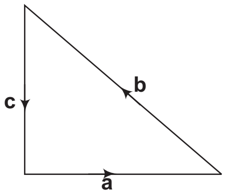
Example 8 *Show that the vectors* $\mathbf{a} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ *and* $\mathbf{c} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ *form a right angled triangle.*

Solution We have $\mathbf{b} + \mathbf{c} = (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}} = \mathbf{a}$

Hence, **a, b, c** are coplanar.

Also, we observe that no two of these vectors are parallel.

Further, $\mathbf{a}\cdot\mathbf{c} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}) = 0$



Dot product of two non-zero vectors is zero. Hence, they are perpendicular so they form a right angled triangle.

$$|\mathbf{a}| = \sqrt{9+4+1} = \sqrt{14},$$

$$|\mathbf{b}| = \sqrt{1+9+25} = \sqrt{35}$$

and $$|\mathbf{c}| = \sqrt{4+1+16} = \sqrt{21}$$

$\Rightarrow$ $$\sqrt{a^2 + c^2} = \sqrt{b^2}$$

Hence Proved.

Example 9 *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be the unit vectors. Suppose that* $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C} = 0$ *and the angle between* $\mathbf{B}$ *and* $\mathbf{C}$ *is* $\frac{\pi}{6}$ *then prove that* $\mathbf{A} = \pm 2(\mathbf{B}\times\mathbf{C})$

Solution Since, $\mathbf{A}\cdot\mathbf{B} = 0, \quad \mathbf{A}\cdot\mathbf{C} = 0$

Hence,
$$(\mathbf{B} + \mathbf{C})\cdot\mathbf{A} = 0$$

So, $\mathbf{A}$ is perpendicular to $(\mathbf{B} + \mathbf{C})$. Further, $\mathbf{A}$ is a unit vector perpendicular to the plane of vectors $\mathbf{B}$ and $\mathbf{C}$.

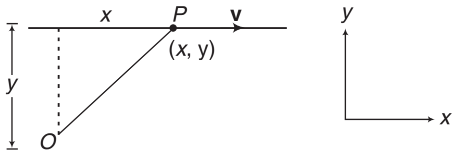
$$\mathbf{A} = \pm \frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}$$

$$|\mathbf{B}\times\mathbf{C}| = |\mathbf{B}||\mathbf{C}|\sin\frac{\pi}{6} = 1\times 1\times\frac{1}{2} = \frac{1}{2}$$

$\therefore$
$$\mathbf{A} = \pm\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|} = \pm 2\,(\mathbf{B}\times\mathbf{C})$$

Example 10 *A particle moves on a given line with a constant speed v. At a certain time, it is at a point P on its straight line path. O is a fixed point. Show that* $(\mathbf{OP}\times\mathbf{v})$ *is independent of the position P.*

Solution Let $\mathbf{v} = v\hat{\mathbf{i}}$



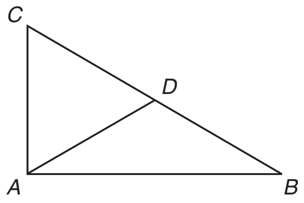
$$\mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Take
$$\mathbf{OP}\times\mathbf{v} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\times v\hat{\mathbf{i}}$$
$$= -\,yv\hat{\mathbf{k}}$$
$$= \text{constant} \qquad \text{(because } y \text{ is constant)}$$

Hence, $\mathbf{OP}\times\mathbf{v}$, which is independent of position of P.

Example 11 *Prove that the mid-point of the hypotenuse of right angled triangle is equidistant from its vertices.*

Solution Here, $\angle CAB = 90°$, let D be the mid-point of hypotenuse, we have



$$\mathbf{BD} = \mathbf{DC}$$
$$\mathbf{AB} = \mathbf{AD} + \mathbf{DB}$$
$$\mathbf{AC} = \mathbf{AD} + \mathbf{DC} = \mathbf{AD} + \mathbf{BD}$$

Since, $\angle BAC = 90°$ $\mathbf{AB}\perp\mathbf{AC}$
$$(\mathbf{AD} + \mathbf{DB})\cdot(\mathbf{AD} + \mathbf{BD}) = 0$$
$$(\mathbf{AD} - \mathbf{BD})\cdot(\mathbf{AD} + \mathbf{BD}) = 0$$
$$AD^2 - BD^2 = 0$$

$\therefore$
$$AD = BD \quad \text{also} \quad BD = DC$$

$\because$ D is mid-point of BC

Thus, $|AD| = |BD| = |DC|$. Hence, the result.

Exercises

Objective Questions

Single Correct Option

1. Which one of the following is a scalar quantity?
(a) Dipole moment (b) Electric field (c) Acceleration (d) Work

2. Which one of the following is not the vector quantity?
(a) Torque (b) Displacement (c) Velocity (d) Speed

3. Which one is a vector quantity?
(a) Time (b) Temperature
(c) Magnetic flux (d) Magnetic field intensity

4. Minimum number of vectors of unequal magnitudes which can give zero resultant are
(a) two (b) three
(c) four (d) more than four

5. Which one of the following statement is **false**?
(a) A vector cannot be displaced from one point to another point
(b) Distance is a scalar quantity but displacement is a vector quantity
(c) Momentum, force and torque are vector quantities
(d) Mass, speed and energy are scalar quantities

6. What is the dot product of two vectors of magnitudes 3 and 5, if angle between them is 60°?
(a) 5.2 (b) 7.5 (c) 8.4 (d) 8.6

7. The forces, which meet at one point but their lines of action do not lie in one plane, are called
(a) non-coplanar non-concurrent forces (b) non-coplanar concurrent forces
(c) coplanar concurrent forces (d) coplanar non-concurrent forces

8. A vector $\mathbf{A}$ points vertically upward and $\mathbf{B}$ points towards north. The vector product $\mathbf{A} \times \mathbf{B}$ is
(a) along west (b) along east (c) zero (d) vertically downward

9. The magnitude of the vector product of two vectors $|\mathbf{A}|$ and $|\mathbf{B}|$ may be
(More than one correct options)
(a) greater than AB (b) equal to AB (c) less than AB (d) equal to zero

10. A force $(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ newton acts on a body and displaces it by $(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ metre. The work done by the force is
(a) 5 J (b) 25 J (c) 10 J (d) 30 J

11. The torque of force $\mathbf{F} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ newton acting at the point $\mathbf{r} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ metre about origin is (in N-m)
(a) $6\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ (b) $17\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 13\hat{\mathbf{k}}$ (c) $-6\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$ (d) $-17\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 13\hat{\mathbf{k}}$

12. If a unit vector is represented by $0.5\,\hat{\mathbf{i}} + 0.8\,\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ the value of c is
(a) 1 (b) $\sqrt{0.11}$ (c) $\sqrt{0.01}$ (d) 0.39

13. Two vectors of equal magnitudes have a resultant equal to either of them, then the angle between them will be

(a) 30° (b) 120° (c) 60° (d) 150°

14. If a vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$ is perpendicular to the vector $4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$, then the value of α is

(a) -1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

15. The angle between the two vectors $\mathbf{A} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\mathbf{B} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ is

(a) 60° (b) 45° (c) 90° (d) 30°

16. Maximum and minimum values of the resultant of two forces acting at a point are 7 N and 3 N respectively. The smaller force will be equal to

(a) 5 N (b) 4 N (c) 2 N (d) 1 N

17. If the vectors $\mathbf{P} = a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{Q} = a\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ are perpendicular to each other, then the positive value of a is

(a) zero (b) 1 (c) 2 (d) 3

18. The (x, y, z) co-ordinates of two points A and B are given respectively as $(0, 3, -1)$ and $(-2, 6, 4)$. The displacement vector from A to B is given by

(a) $-2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ (b) $-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

(c) $-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (d) $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

19. A vector is not changed if

(a) it is rotated through an arbitrary angle

(b) it is multiplied by an arbitrary scalar

(c) it is cross multiplied by a unit vector

(d) it is displaced parallel to itself

20. Which of the sets given below may represent the magnitudes of three vectors adding to zero?

(a) 2, 4, 8 (b) 4, 8, 16

(c) 1, 2, 1 (d) 0.5, 1, 2

21. The resultant of **A** and **B** makes an angle α with **A** and β with **B**, then

(a) α is always less than β (b) $\alpha < \beta$ if $A < B$

(c) $\alpha < \beta$ if $A > B$ (d) $\alpha < \beta$ if $A = B$

22. The angles which the vector $\mathbf{A} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ makes with the co-ordinate axes are

(a) $\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{6}{7}$ and $\cos^{-1}\frac{2}{7}$ (b) $\cos^{-1}\frac{4}{7}, \cos^{-1}\frac{5}{7}$ and $\cos^{-1}\frac{3}{7}$

(c) $\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{4}{7}$ and $\cos^{-1}\frac{1}{7}$ (d) None of these

23. Unit vector parallel to the resultant of vectors $\mathbf{A} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ and $\mathbf{B} = 8\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$ will be

(a) $\frac{24\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{13}$ (b) $\frac{12\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{13}$

(c) $\frac{6\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{13}$ (d) None of these

24. The component of vector $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ along the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is

(a) $\frac{5}{\sqrt{2}}$ (b) $10\sqrt{2}$ (c) $5\sqrt{2}$ (d) 5

25. Two vectors **A** and **B** are such that $\mathbf{A}+\mathbf{B}=\mathbf{C}$ and $A^2+B^2=C^2$. If θ is the angle between positive direction of **A** and **B**, then the correct statement is

(a) $\theta=\pi$ (b) $\theta=\frac{2\pi}{3}$

(c) $\theta=0$ (d) $\theta=\frac{\pi}{2}$

26. If $|\mathbf{A}\times\mathbf{B}|=\sqrt{3}\,\mathbf{A}\cdot\mathbf{B}$, then the value of $|\mathbf{A}+\mathbf{B}|$ is

(a) $(A^2+B^2+AB)^{1/2}$ (b) $\left(A^2+B^2+\frac{AB}{\sqrt{3}}\right)^{1/2}$

(c) $(A+B)$ (d) $(A^2+B^2+\sqrt{3}AB)^{1/2}$

27. If the angle between the vectors **A** and **B** is θ, the value of the product $(\mathbf{B}\times\mathbf{A})\cdot\mathbf{A}$ is equal to

(a) $BA^2\cos\theta$ (b) $BA^2\sin\theta$

(c) $BA^2\sin\theta\cos\theta$ (d) zero

28. Given that $P=12$, $Q=5$ and $R=13$ also $\mathbf{P}+\mathbf{Q}=\mathbf{R}$, then the angle between **P** and **Q** will be

(a) π (b) $\frac{\pi}{2}$

(c) zero (d) $\frac{\pi}{4}$

29. Given that $\mathbf{P}+\mathbf{Q}+\mathbf{R}=0$. Two out of the three vectors are equal in magnitude. The magnitude of the third vector is $\sqrt{2}$ times that of the other two. Which of the following can be the angles between these vectors?

(a) 90°, 135°, 135° (b) 45°, 45°, 90°

(c) 30°, 60°, 90° (d) 45°, 90°, 135°

30. The angle between $\mathbf{P}+\mathbf{Q}$ and $\mathbf{P}-\mathbf{Q}$ will be

(a) 90° (b) between 0° and 180°

(c) 180° only (d) None of these

31. The value of n so that vectors $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-2\hat{\mathbf{k}}$, $5\hat{\mathbf{i}}+n\,\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ may be coplanar, will be

(a) 18 (b) 28 (c) 9 (d) 36

32. If **a** and **b** are two vectors, then the value of $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$ is

(a) $2\,(\mathbf{b}\times\mathbf{a})$ (b) $-2\,(\mathbf{b}\times\mathbf{a})$ (c) $\mathbf{b}\times\mathbf{a}$ (d) $\mathbf{a}\times\mathbf{b}$

33. The resultant of two forces $3P$ and $2P$ is R. If the first force is doubled then the resultant is also doubled. The angle between the two forces is

(a) 60° (b) 120° (c) 30° (d) 135°

34. The resultant of two forces, one double the other in magnitude, is perpendicular to the smaller of the two forces. The angle between the two forces is

(a) 120° (b) 60° (c) 90° (d) 150°

35. Three vectors satisfy the relation $\mathbf{A}\cdot\mathbf{B}=0$ and $\mathbf{A}\cdot\mathbf{C}=0$, then **A** is parallel to

(a) **C** (b) **B** (c) $\mathbf{B}\times\mathbf{C}$ (d) $\mathbf{B}\cdot\mathbf{C}$

36. The sum of two forces at a point is 16 N. If their resultant is normal to the smaller force and has a magnitude of 8 N, then two forces are

(a) 6 N, 10 N (b) 8 N, 8 N

(c) 4 N, 12 N (d) 2 N, 14 N

37. The sum of two vectors **A** and **B** is at right angles to their difference. Then
(a) $A = B$
(b) $A = 2B$
(c) $B = 2A$
(d) **A** and **B** have the same direction

38. Let $\mathbf{C} = \mathbf{A} + \mathbf{B}$.
(a) $|\mathbf{C}|$ is always greater than $|\mathbf{A}|$
(b) It is possible to have $|\mathbf{C}| < |\mathbf{A}|$ and $|\mathbf{C}| < |\mathbf{B}|$
(c) C is always equal to $A + B$
(d) C is never equal to $A + B$

39. Let the angle between two non-zero vectors **A** and **B** be 120° and its resultant be **C**.
(a) C must be equal to $|A - B|$
(b) C must be less than $|A - B|$
(c) C must be greater than $|A - B|$
(d) C may be equal to $|A - B|$

Match the Columns

1. Column I shows some vector equations. Match Column I with the value of angle between **A** and **B** given in Column II.

Column I		Column II	
(a)	$\lvert\mathbf{A} \times \mathbf{B}\rvert = \lvert\mathbf{A} \cdot \mathbf{B}\rvert$	(p)	zero
(b)	$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$	(q)	$\frac{\pi}{2}$
(c)	$\lvert\mathbf{A} + \mathbf{B}\rvert = \lvert\mathbf{A} - \mathbf{B}\rvert$	(r)	$\frac{\pi}{4}$
(d)	$\mathbf{A} + \mathbf{B} = \mathbf{C}$ and $A + B = C$	(s)	$\frac{3\pi}{4}$

Subjective Questions

1. If $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find the angle between **a** and **b**.

2. The vector **A** has a magnitude of 5 unit, **B** has a magnitude of 6 unit and the cross product of **A** and **B** has a magnitude of 15 unit. Find the angle between **A** and **B**.

3. Suppose **a** is a vector of magnitude 4.5 unit due north. What is the vector (a) $3\mathbf{a}$ (b) $-4\mathbf{a}$?

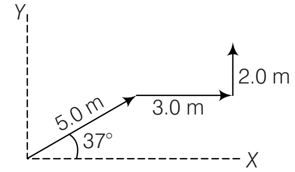
4. Two vectors have magnitudes 3 unit and 4 unit respectively. What should be the angle between them if the magnitude of the resultant is (a) 1 unit, (b) 5 unit and (c) 7 unit.

5. The work done by a force **F** during a displacement **r** is given by $\mathbf{F} \cdot \mathbf{r}$. Suppose a force of 12 N acts on a particle in vertically upward direction and the particle is displaced through 2.0 m in vertically downward direction. Find the work done by the force during this displacement.

6. If **A**, **B**, **C** are mutually perpendicular, then show that $\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$.

7. Prove that $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$.

8. Find the resultant of the three vectors shown in figure.



9. Give an example for which $A \cdot B = C \cdot B$ but $A \neq C$.

10. Obtain the angle between $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$ if $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\mathbf{B} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}}$.

11. Deduce the condition for the vectors $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - a\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ to be parallel.

12. Find the area of the parallelogram whose sides are represented by $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$.

13. If vectors $\mathbf{A}$ and $\mathbf{B}$ be respectively equal to $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$. Find the unit vector parallel to $\mathbf{A} + \mathbf{B}$.

14. If $\mathbf{A} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\mathbf{B} = \hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ and $\mathbf{C} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

15. The x and y-components of vector $\mathbf{A}$ are 4 m and 6 m respectively. The x and y-components of vector $\mathbf{A} + \mathbf{B}$ are 10 m and 9 m respectively. Calculate for the vector $\mathbf{B}$ the following :
(a) its x and y-components
(b) its length
(c) the angle it makes with x-axis

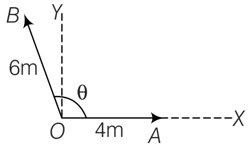
16. Three vectors which are coplanar with respect to a certain rectangular co-ordinate system are given by

$$\mathbf{a} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}},\ \mathbf{b} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} \text{ and } \mathbf{c} = -3\hat{\mathbf{j}}$$

Find
(a) $\mathbf{a} + \mathbf{b} + \mathbf{c}$
(b) $\mathbf{a} + \mathbf{b} - \mathbf{c}$
(c) Find the angle between $\mathbf{a} + \mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{b} - \mathbf{c}$

17. Let $\mathbf{A}$ and $\mathbf{B}$ be the two vectors of magnitude 10 unit each. If they are inclined to the x-axis at angles 30° and 60° respectively, find the resultant.

18. The resultant of vectors $\mathbf{OA}$ and $\mathbf{OB}$ is perpendicular to $\mathbf{OA}$ as shown in figure. Find the angle AOB.



19. Find the components of a vector $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ along the directions of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

20. If two vectors are $\mathbf{A} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{B} = \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$. By calculation, prove that $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

21. The resultant of two vectors $\mathbf{A}$ and $\mathbf{B}$ is at right angles to $\mathbf{A}$ and its magnitude is half of $\mathbf{B}$. Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

22. Four forces of magnitude P, $2P$, $3P$ and $4P$ act along the four sides of a square $ABCD$ in cyclic order. Use the vector method to find the magnitude of resultant force.

23. If $\mathbf{P}+\mathbf{Q}=\mathbf{R}$ and $\mathbf{P}-\mathbf{Q}=\mathbf{S}$, prove that $R^2+S^2=2\,(P^2+Q^2)$

24. Prove by the method of vectors that in a triangle

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Answers

Introductory Exercise 5.1

1. 0°

2. 180° , 0.6

3. (a) 14 units (b) 2 units (c) $2\sqrt{37}$ units (d) $2\sqrt{13}$ units (e) 10 units

4. (a) 2 units (b) 14 units (c) $2\sqrt{13}$ units (d) $2\sqrt{37}$ units (e) 10 units

5. 90°

Introductory Exercise 5.2

1. $A=5\sqrt{2}$ units, $\cos\alpha=\dfrac{3}{5\sqrt{2}}$, $\cos\beta=\dfrac{-4}{5\sqrt{2}}$ and $\cos\gamma=\dfrac{1}{\sqrt{2}}$

2. $F_x=-5$ N, $F_y=-5\sqrt{3}$ N

3. $\sqrt{10}$ units

4. (a) 180° (b) 90° (c) 90° (d) 135°

Introductory Exercise 5.3

1. True

2. (a) $24\,\hat{\mathbf{j}}$ (b) –12 (c) zero

3. (a) 8 units (b) 4 units (c) zero (d) –4 units (e) –8 units

4. $(6\,\hat{\mathbf{i}}+12\,\hat{\mathbf{j}}-12\,\hat{\mathbf{k}})$

Exercises

Single Correct Option

1. (d)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (b,c,d)	**10.** (b)
11. (b)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (d)	**18.** (c)	**19.** (d)	**20.** (c)
21. (c)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (d)	**26.** (a)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (b)
31. (a)	**32.** (a)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (a)	**38.** (b)	**39.** (c)	

Match the Columns

1. (a) $\rightarrow$ r,s (b) $\rightarrow$ p, (c) $\rightarrow$ q (d) $\rightarrow$ p

Subjective Questions

1. $\cos^{-1}\left(\frac{25}{29}\right)$
2. 30° or 150°
3. (a) 13.5 unit due north (b) 18 unit due south
4. (a) 180° (b) 90° (c) 0°
5. –24 J
8. $\sqrt{74}$ m at angle $\tan^{-1}\left(\frac{5}{7}\right)$ from x-axis towards y-axis
9. See the hints
10. $\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)$
11. $a = -4.5$, $b = -6$
12. Area = 13.4 units
13. $\frac{1}{\sqrt{27}}(5\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$
14. zero
15. (a) 6m, 3m (b) $3\sqrt{5}$ m (c) $\theta = \tan^{-1}\left(\frac{1}{2}\right)$
16. (a) $\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ (b) $\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ (c) $\cos^{-1}\left(\frac{-7}{\sqrt{85}}\right)$
17. 20 cos 15° unit at 45° with x-axis.
18. $\cos^{-1}\left(\frac{-2}{3}\right)$
19. $\frac{5}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$
21. 150°
22. $2\sqrt{2}\,P$

06

Kinematics

Chapter Contents

6.1 Introduction to Mechanics and Kinematics
6.2 Few General Points of Motion
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6.4 Basic Definition
6.5 Uniform Motion
6.6 One Dimensional Motion with Uniform Acceleration
6.7 One Dimensional Motion with Non-uniform Acceleration
6.8 Motion in Two and Three Dimensions
6.9 Graphs
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6.1 Introduction to Mechanics and Kinematics

Mechanics is the branch of physics which deals with the motion of particles or bodies in space and time. Position and motion of a body can be determined only with respect to other bodies. Motion of the body involves position and time. For practical purposes a coordinate system, e.g. the cartesian system is fixed to the reference body and position of the body is determined with respect to this reference body. For calculation of time generally clock is used.

Kinematics is the branch of mechanics which deals with the motion regardless of the causes producing it. The study of causes of motion is called **dynamics.**

6.2 Few General Points of Motion

Kinematics is the branch of mechanics which deals with the motion regardless of the causes producing it.

1. Direction of velocity is in the direction of motion. But direction of acceleration is not necessarily in the direction of motion. Direction of acceleration is in the direction of net force acting on the body. For example, if we say that a body is moving due east, it means velocity of the body is towards east. From the above statement, we cannot find the direction of acceleration.
2. Motion in a straight line is called a one-dimensional motion.
3. Motion which is not one dimensional is called a curvilinear motion. Circular motion and projectile motion are the examples of curvilinear motion.

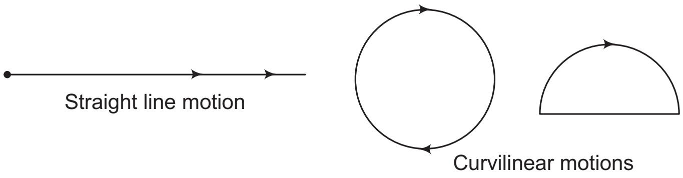


Fig. 6.1

4. Direction of velocity at any point on a curvilinear path is tangential to the path. But with direction of acceleration there is no such condition. As we have stated earlier also, it is in the direction of net force. For example, in the figure shown below a particle is moving on a curvilinear path.

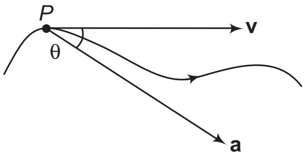


Fig. 6.2

At point P, velocity of the particle is tangential to the path but acceleration is making an angle θ with velocity. If θ is acute $(0° \le \theta < 90°)$, then speed (which is also magnitude of velocity vector) of the particle increases. If θ is obtuse $(90° < \theta \le 180°)$, then speed of the particle decreases.

Note $\theta = 90°$ *is a special case when speed remains constant. This point we shall discuss in our later discussions.*

Example 6.1 *Velocity of a particle at some instant is* $\mathbf{v} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ m/s. *Find speed of the particle at this instant.*

Solution Magnitude of velocity vector at any instant of time is the speed of particle. Hence,

$$\text{Speed} = v \text{ or } |\mathbf{v}| = \sqrt{(3)^2 + (4)^2 + (5)^2} = 5\sqrt{2} \text{ m/s}$$ **Ans.**

Example 6.2 *"A lift is ascending with decreasing speed". What are the directions of velocity and acceleration of the lift at the given instant.*

Solution (i) Direction of motion is the direction of velocity. Lift is ascending (means it is moving upwards). So, direction of velocity is upwards.

(ii) Speed of lift is decreasing. So, direction of acceleration should be in opposite direction or it should be downwards.

INTRODUCTORY EXERCISE 6.1

1. "A lift is descending with increasing speed". What are the directions of velocity and acceleration in the given statement?
2. Velocity and acceleration of a particle at some instant are

$$\mathbf{v} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \text{ m/s} \quad \text{and} \quad \mathbf{a} = (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \text{ m/s}^2$$

   (a) What is the value of dot product of **v** and **a** at the given instant?
   (b) What is the angle between **v** and **a**, acute, obtuse or 90°?
   (c) At the given instant, whether speed of the particle is increasing, decreasing or constant?

6.3 Classification of Motion

A motion can be classified in following two ways :

First According to this way, a motion can be either

(i) One dimensional (1-D)
(ii) Two dimensional (2-D)
(iii) Three dimensional (3-D)

In one dimensional motion, particle (or a body) moves in a straight line, in two dimensional motion, it moves in a plane and in three dimensional motion body moves in space.

Second According to this way, a motion can be either

(i) Uniform motion
(ii) Uniformly accelerated
(iii) Non-uniformly accelerated.

In uniform motion, velocity of the particle is constant and in non-uniformly accelerated motion acceleration of the particle is not constant.

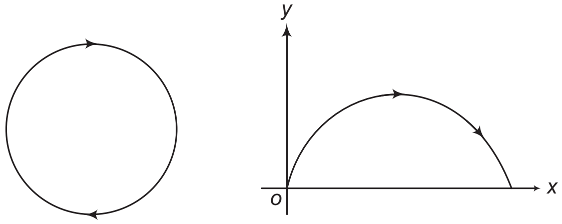
Equations, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ etc. can be applied directly, only for uniformly accelerated motion. If the motion is one dimensional then these equations can be written as, $v = u + at$ etc. For solving a problem of non-uniform acceleration, either integration or differentiation is required.

Extra Points to Remember

In uniform motion, velocity of particle is constant, therefore acceleration is zero. Velocity is constant, means its magnitude (or speed) is constant and direction of velocity is fixed. So, the particle moves in a straight line. Hence, it is always one dimensional motion.

Example 6.3 *Give two examples of two dimensional motion.*

Solution Two dimensional motion takes place in a plane. Its two examples are circular motion and projectile motion.



Circular motion Projectile motion

Fig. 6.3

Normally, the plane of circular motion is either horizontal or vertical and plane of projectile motion is vertical.

Example 6.4 *Velocity of a particle is* $\mathbf{v} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *m/s and its acceleration is zero. State whether it is 1-D, 2-D or 3-D motion?*

Solution Since, acceleration of the particle is zero. Therefore, it is uniform motion or motion in a straight line. So, it is one dimensional motion.

Example 6.5 *Projectile motion is a two dimensional motion with constant acceleration. Is this statement true or false?*

Solution True. Projectile motion takes place in a plane. So, it is two dimensional. For small heights, its acceleration is constant (= acceleration due to gravity). Therefore, it is a two dimensional motion with constant acceleration.

INTRODUCTORY EXERCISE 6.2

1. Velocity and acceleration of a particle are

$$\mathbf{v} = (2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})\ \text{m/s} \quad \text{and} \quad \mathbf{a} = (-2\hat{\mathbf{i}} + 4\hat{\mathbf{j}})\ \text{m/s}^2$$

   Which type of motion is this ?

2. Velocity and acceleration of a particle are

$$\mathbf{v} = (2\hat{\mathbf{i}})\ \text{m/s} \quad \text{and} \quad \mathbf{a} = (4t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}})\ \text{m/s}^2$$

   where, t is the time. Which type of motion is this ?

3. In the above question, can we use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ equation directly?

6.4 Basic Definitions

Position Vector and Displacement Vector

If coordinates of point A are (x_1, y_1, z_1) and B are (x_2, y_2, z_2). Then, position vector of A

$$= \mathbf{r}_A = \mathbf{OA} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$$

Position vector of $B = \mathbf{r}_B = \mathbf{OB} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$

and $$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{r}_B - \mathbf{r}_A$$
$$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

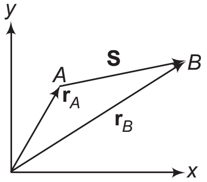


Fig. 6.4

Distance and Displacement

Distance is the actual path length covered by a moving particle or body in a given time interval, while displacement is the change in position vector, i.e. a vector joining initial to final positions. If a particle moves from A to C (Fig. 6.5) through a path ABC. Then, distance travelled is the actual path length ABC, while the displacement is

$$\mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_C - \mathbf{r}_A$$

If a particle moves in a straight line without change in direction, the magnitude of displacement is equal to the distance travelled, otherwise, it is always less than it. Thus,

$$\boxed{|\text{displacement}| \le \text{distance}}$$

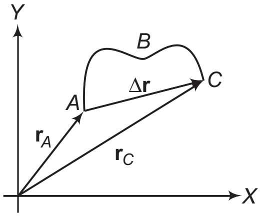


Fig. 6.5

Average Speed and Average Velocity

The average speed of a particle in a given time interval is defined as the ratio of total distance travelled to the total time take.

The average velocity is defined as the ratio of total displacement to the total time taken.

Thus, $$v_{\text{av}} = \text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

and $$\mathbf{v}_{\text{av}} = \text{average velocity} = \frac{\text{total displacement}}{\text{total time}}$$

$$= \frac{\mathbf{S}}{\Delta t} \text{ or } \frac{\mathbf{S}}{t} \text{ or } \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}_f - \mathbf{r}_i}{\Delta t}$$

Here, $\mathbf{r}_f$ = final position vector and $\mathbf{r}_i$ = initial position vector

Instantaneous Velocity and Instantaneous Speed

Instantaneous velocity and instantaneous speed are defined at a particular instant and are given by

$$\boxed{\mathbf{v}_i \text{ or simply } \mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{s}}{\Delta t} \text{ or } \frac{d\mathbf{s}}{dt} \text{ or } \frac{d\mathbf{r}}{dt}}$$

Here, $\mathbf{r}$ is position vector of the body (or particle) at a general time t.

Magnitude of instantaneous velocity at any instant is called its instantaneous speed at that instant. Thus,

$$\boxed{\text{Instantaneous speed} = v = |\mathbf{v}| = \left|\frac{d\mathbf{s}}{dt} \text{ or } \frac{d\mathbf{r}}{dt}\right|}$$

Average and Instantaneous Acceleration

Average acceleration is defined as the ratio of change in velocity, i.e. $\Delta\mathbf{v}$ to the time interval Δt in which this change occurs. Hence,

$$\boxed{\mathbf{a}_{\text{av}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathbf{v}_f - \mathbf{v}_i}{\Delta t}}$$

The instantaneous acceleration is defined at a particular instant and is given by

$$\boxed{\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}}$$

Here, $\mathbf{v}_f =$ final velocity and $\mathbf{v}_i =$ initial velocity vector

Extra Points to Remember

- If motion is one dimensional (let along x-axis) then all vector quantities (displacement, velocity and acceleration) can be treated like scalars by assuming one direction as positive and the other as negative. In this case, all vectors along positive direction are given positive sign and the vectors in negative direction are given negative sign.
- For example, displacement, instantaneous velocity, instantaneous acceleration, average velocity and average acceleration in this case be written as

$$s = \Delta r = r_f - r_i \quad \text{or} \quad x_f - x_i$$

$$v = \frac{ds}{dt} \quad \text{or} \quad \frac{dx}{dt} \quad \text{or} \quad a = \frac{dv}{dt}$$

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{r_f - r_i}{\Delta t} = \frac{x_f - x_i}{\Delta t} \quad \text{and} \quad a_{\text{av}} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t}$$

Example 6.6 *In one second, a particle goes from point A to point B moving in a semicircle (Fig. 6.6). Find the magnitude of the average velocity.*

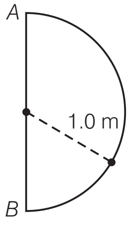


Fig. 6.6

Solution

$$|\mathbf{v}_{\text{av}}| = \frac{AB}{\Delta t} \text{ m/s}$$

$$= \frac{2.0}{1.0} \text{ m/s} = 2 \text{ m/s}$$

Example 6.7 *A table is given below of a particle moving along x-axis. In the table, speed of particle at different time intervals is shown.*

Table 6.1

Time interval (in sec)	**Speed of particle** (in m/s)
0 – 2	2
2 – 5	3
5 – 10	4
10 – 15	2

Find total distance travelled by the particle and its average speed.

Solution Distance = speed × time

∴ Total distance = $(2\times 2)+(3)(3)+(5)(4)+(5)(2)=43$ m

Total time taken is 15 s. Hence,

$$\text{Average speed}=\frac{\text{Total distance}}{\text{Total time}}$$

$$=\frac{43}{15}=2.87\ \text{m/s}$$

Example 6.8 *A particle is moving along x-axis. Its X-coordinate varies with time as,*

$$X=2t^2+4t-6$$

Here, X is in metres and t in seconds. Find average velocity between the time interval $t=0$ to $t=2$ s.

Solution In 1-D motion, average velocity can be written as

$$v_{\text{av}}=\frac{\Delta s}{\Delta t}=\frac{X_f-X_i}{\Delta t}=\frac{X_{2\text{ sec}}-X_{0\text{ sec}}}{2-0}$$

$$=\frac{[2(2)^2+4(2)-6]-[2(0)^2+4(0)-6]}{2}$$

$$=8\ \text{m/s}$$ **Ans.**

Example 6.9 *A particle is moving in x-y plane. Its x and y co-ordinates vary with time as*

$$x=2t^2 \text{ and } y=t^3$$

Here, x and y are in metres and t in seconds. Find average acceleration between a time interval from $t=0$ to $t=2$ s.

Solution The position vector of the particle at any time t can be given as

$$\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}=2t^2\hat{\mathbf{i}}+t^3\hat{\mathbf{j}}$$

The instantaneous velocity is $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=(4t\hat{\mathbf{i}}+3t^2\hat{\mathbf{j}})$

Now,
$$\mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_f - \mathbf{v}_i}{\Delta t} = \frac{\mathbf{v}_{2\,\text{sec}} - \mathbf{v}_{0\,\text{sec}}}{2-0}$$
$$= \frac{[(4)(2)\hat{\mathbf{i}} + (3)(2)^2\,\hat{\mathbf{j}}] - [(4)(0)\hat{\mathbf{i}} + (3)(0)^2\,\hat{\mathbf{j}}]}{2}$$
$$= (4\hat{\mathbf{i}} + 6\hat{\mathbf{j}})\ \text{m/s}^2$$
Ans.

INTRODUCTORY EXERCISE 6.3

1. Average speed is always equal to magnitude of average velocity. Is this statement true or false?
2. When a particle moves with constant velocity its average velocity, its instantaneous velocity and its speed all are equal. Is this statement true or false?
3. A stone is released from an elevator going up with an acceleration of $g/2$. What is the acceleration of the stone just after release?
4. A clock has its second hand 2.0 cm long. Find the average speed and modulus of average velocity of the tip of the second hand in 15 s.
5. (a) Is it possible to be accelerating if you are travelling at constant speed?
   (b) Is it possible to move on a curved path with zero acceleration, constant acceleration, variable acceleration?
6. A particle is moving in a circle of radius 4 cm with constant speed of 1 cm/s. Find
   (a) time period of the particle.
   (b) average speed, average velocity and average acceleration in a time interval from $t = 0$ to $t = T/4$. Here, T is the time period of the particle. Give only their magnitudes.

6.5 Uniform Motion

As we have discussed earlier also, in uniform motion velocity of the particle is constant and acceleration is zero. Velocity is constant means its magnitude (called speed) is constant and direction is fixed. Therefore, motion is 1-D in same direction. If velocity is along positive direction, then displacement is also along positive direction. Therefore, distance travelled (d) is equal to the displacement (s). If velocity is along negative direction then displacement is also negative and distance travelled in this case is the magnitude of displacement. Equations involved in this motion are

(i) Velocity (may be positive or negative) = constant
(ii) Speed, v = constant
(iii) Acceleration = 0
(iv) Displacement (may be positive or negative) = velocity × time
(v) Distance = speed × time or $\boxed{d = vt}$
(vi) Distance and speed are always positive, whereas displacement and velocity may be positive or negative.

Example 6.10 *A particle travels first half of the total distance with constant speed v_1 and second half with constant speed v_2. Find the average speed during the complete journey.*

Solution

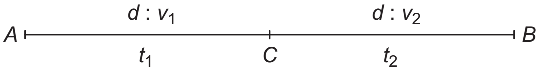


Fig. 6.7

$$t_1 = \frac{d}{v_1} \quad \text{and} \quad t_2 = \frac{d}{v_2}$$

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{d+d}{t_1+t_2} = \frac{2d}{(d/v_1)+(d/v_2)} = \frac{2v_1v_2}{v_1+v_2}$$ **Ans.**

Example 6.11 *A particle travels first half of the total time with speed v_1 and second half time with speed v_2. Find the average speed during the complete journey.*

Solution

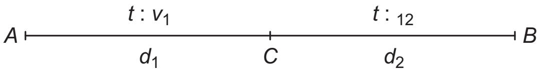


Fig. 6.8

$$d_1 = v_1 t \quad \text{and} \quad d_2 = v_2 t$$

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{d_1+d_2}{t+t} = \frac{v_1t+v_2t}{2t} = \frac{v_1+v_2}{2}$$ **Ans.**

Example 6.12 *A particle travels first half of the total distance with speed v_1. In second half distance, constant speed in $\frac{1}{3}$ rd time is v_2 and in remaining $\frac{2}{3}$ rd time constant speed is v_3. Find average speed during the complete journey.*

Solution

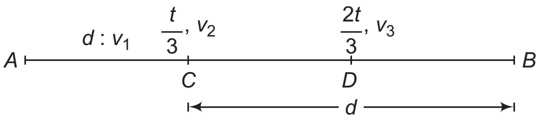


Fig. 6.9

$$CD + DB = d \qquad \Rightarrow \quad v_2\left(\frac{t}{3}\right) + \left(\frac{2t}{3}\right)(v_3) = d$$

or

$$t = \frac{3d}{v_2 + 2v_3} \qquad \text{...(i)}$$

Further,

$$t_{AC} = \frac{d}{v_1}$$

Now,

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{d+d}{t_{AC}+t_{CD}+t_{DB}}$$

$$= \frac{2d}{\frac{d}{v_1}+\frac{t}{3}+\frac{2t}{3}} = \frac{2d}{\left(\frac{d}{v_1}+t\right)}$$

Substituting value of t from Eq. (i), we have

$$\text{average speed} = \frac{2d}{(d/v_1) + (3d/v_2 + 2v_3)}$$

$$= \frac{2v_1(v_2 + 2v_3)}{3v_1 + v_2 + 2v_3}$$

Ans.

INTRODUCTORY EXERCISE 6.4

1. A particle moves in a straight line with constant speed of 4 m/s for 2 s, then with 6 m/s for 3 s. Find the average speed of the particle in the given time interval.
2. A particle travels half of the time with constant speed 2 m/s. In remaining half of the time it travels, $\frac{1}{4}$ th distance with constant speed of 4 m/s and $\frac{3}{4}$ th distance with 6 m/s. Find average speed during the complete journey.

6.6 One Dimensional Motion with Uniform Acceleration

As we have discussed in article 6.3 that equations like $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ etc. can be applied directly with constant (or uniform) acceleration. Further, in one dimensional motion, all vector quantities (displacement, velocity and acceleration) can be treated like scalars by using sign convention method. In this method, one direction is taken as positive and the other as the negative and then all vector quantities are written with paper signs. In most of the cases, we will take following sign convention.

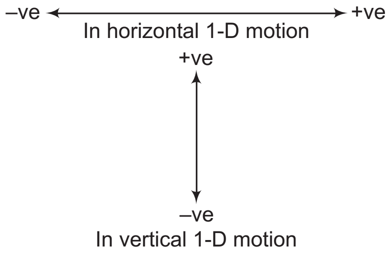


Fig. 6.10

The equations used in 1-D motion with uniform acceleration are

$$v = u + at \qquad \text{...(i)}$$

$$v^2 = u^2 + 2as \qquad \text{...(ii)}$$

$$s = ut + \frac{1}{2}at^2 \qquad \text{...(iii)}$$

$$s_1 = s_0 + ut + \frac{1}{2}at^2 \qquad \text{...(iv)}$$

$$s_t = u + at - \frac{1}{2}a \qquad \text{...(v)}$$

In the above equations, u = initial velocity, v = velocity at time t, a = constant acceleration

s = displacement measured from the starting point

Here, starting point means the point where the particle was at $t = 0$. It is not the point where $u = 0$.

s_1 = displacement measured from any other point, say P, where P is not the starting point.

s_0 = displacement of the starting point from P.

s_t = displacement (not the distance in t^{th} second).

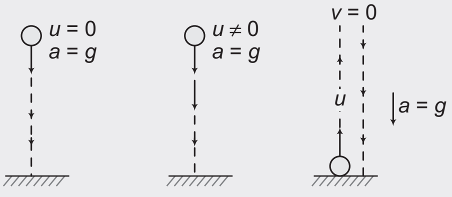
Extra Points to Remember

- In most of the cases, displacement is measured from the starting point, therefore Eq. (iii) or $s = ut + \frac{1}{2}at^2$ is used.
- For small heights, if the motion is taking place under gravity then acceleration is always constant (= acceleration due to gravity). This is 9.8 m/s^2 ($\approx 10 \text{ m/s}^2$) in downward direction. According to our sign convention downward direction is negative. Therefore,

$$a = g = 9.8 \text{ m/s}^2 \approx -10 \text{ m/s}^2$$

- One-dimensional motion (with constant acceleration) can be observed in following three cases:
  Case 1 Initial velocity is zero.
  Case 2 Initial velocity is parallel to constant acceleration.
  Case 3 Initial velocity is antiparallel to constant acceleration.

  In first two cases, motion is only accelerated and direction of motion does not change. In the third case, motion is first retarded (till the velocity becomes zero) and then accelerated in opposite direction.



Case-1 Case-2 Case-3

Fig. 6.11

- In most of the problems of time calculations, $s = ut + \frac{1}{2}at^2$ equation is useful. But s has to be measured from the starting point.
- In case 3 (of point 3), we need not to apply two separate equations, one for retarded motion (when motion is upwards) and other for accelerated motion (when motion is downwards). Problem can be solved by applying the equations only one time, provided s (in $s = ut + \frac{1}{2}at^2$) is measured from the starting point and all vector quantities are substituted with proper signs.

Example 6.13 *A ball is thrown upwards from the top of a tower 40 m high with a velocity of 10 m/s. Find the time when it strikes the ground. Take $g = 10 \text{ m/s}^2$.*

Solution In the problem, $u = +10 \text{ m/s}$, $a = -10 \text{ m/s}^2$

and $s = -40 \text{ m}$ (at the point where stone strikes the ground)

Substituting in $s = ut + \frac{1}{2}at^2$, we have

$$-40 = 10t - 5t^2$$

or $$5t^2 - 10t - 40 = 0$$

or $$t^2 - 2t - 8 = 0$$

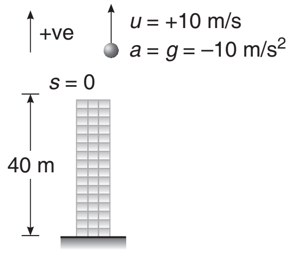


Fig. 6.12

Solving this, we have $t = 4$ s and -2 s.

Taking the positive value $t = 4$ s.

Note *The significance of $t = -2$ s can be understood by following figure*

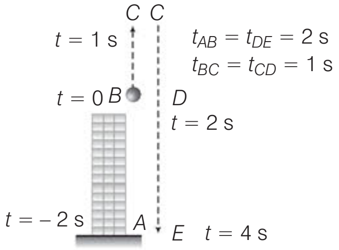


Fig. 6.13

Example 6.14 *A ball is thrown upwards from the ground with an initial speed of u. The ball is at a height of 80 m at two times, the time interval being 6 s. Find u. Take $g = 10\ m/s^2$.*

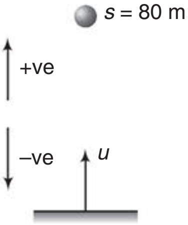


Fig. 6.14

Solution Here, $u = u$ m/s, $a = g = -10\ \text{m/s}^2$ and $s = 80$ m.

Substituting the values in $s = ut + \frac{1}{2}at^2$,

we have $80 = ut - 5t^2$ or $5t^2 - ut + 80 = 0$

or $t = \dfrac{u + \sqrt{u^2 - 1600}}{10}$ and $\dfrac{u - \sqrt{u^2 - 1600}}{10}$

Now, it is given that $\dfrac{u + \sqrt{u^2 - 1600}}{10} - \dfrac{u - \sqrt{u^2 - 1600}}{10} = 6$

or $\dfrac{\sqrt{u^2 - 1600}}{5} = 6$ or $\sqrt{u^2 - 1600} = 30$ or $u^2 - 1600 = 900$

$\therefore$ $u^2 = 2500$ or $u = \pm 50$ m/s

Ignoring the negative sign, we have $u = 50$ m/s

Extra Points to Remember

- In motion under gravity, we can use the following results directly in objective problems:
  (a) If a particle is projected upwards with velocity u , then
    (i) maximum height attained by the particle, $h = \dfrac{u^2}{2g}$
    (ii) time of ascent = time of descent = $\dfrac{u}{g}$ $\Rightarrow$ $\therefore$ Total time of flight = $\dfrac{2u}{g}$
  (b) If a particle is released from rest from a height h (also called free fall), then
    (i) velocity of particle at the time of striking with ground, $v = \sqrt{2gh}$
    (ii) time of descent (also called free fall time) $t = \sqrt{\dfrac{2h}{g}}$

Fig. 6.15

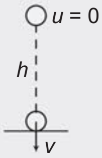


Fig. 6.16

Note *In the above results, air resistance has been neglected and we have already substituted the signs of u, g etc. So, you have to substitute only their magnitudes.*

- **Exercise** Derive the above results.

Difference between Distance (d) and Displacement (s)

The s in equations of motion ($s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$) is really the displacement not the distance. They have different values only when u and a are of opposite sign or $u \uparrow\downarrow a$.

Let us take the following two cases :

Case 1 When u is either zero or parallel to a, then motion is simply accelerated and in this case distance is equal to displacement. So, we can write

$$\boxed{d = s = ut + \frac{1}{2}at^2}$$

Case 2 When u is antiparallel to a, the motion is first retarded then accelerated in opposite direction. So, distance is either greater than or equal to displacement ($d \geq |s|$). In this case, first find the time when velocity becomes zero. Say it is t_0.

$$0 = u - at_0 \quad \Rightarrow \quad \therefore \quad t_0 = \left|\frac{u}{a}\right|$$

Now, if the given time $t \leq t_0$, distance and displacement are equal. So, $d = s = ut + \frac{1}{2}at^2$

For $t \leq t_0$, (with u positive and a negative)

For $t > t_0$, distance is greater than displacement. $d = d_1 + d_2$

Here, d_1 = distance travelled before coming to rest $= \left|\frac{u^2}{2a}\right|$

d_2 = distance travelled in remaining time $t - t_0 = \frac{1}{2}|a(t - t_0)^2|$

$$\therefore \qquad \boxed{d = \left|\frac{u^2}{2a}\right| + \frac{1}{2}|a(t - t_0)^2|}$$

Note *The displacement is still* $s = ut + \frac{1}{2}at^2$ *with u positive and a negative.*

Example 6.15 *A particle is projected vertically upwards with velocity* 40 m/s. *Find the displacement and distance travelled by the particle in*

(a) 2 s *(b)* 4 s *(c)* 6 s

Take $g = 10\,m/s^2$.

Solution Here, u is positive (upwards) and a is negative (downwards). So, first we will find t_0, the time when velocity becomes zero.

$$t_0 = \left|\frac{u}{a}\right| = \frac{40}{10} = 4\text{ s}$$

(a) $t < t_0$. Therefore, distance and displacement are equal.

$$d = s = ut + \frac{1}{2}at^2 = 40 \times 2 - \frac{1}{2} \times 10 \times 4 = 60\text{ m}$$

(b) $t = t_0$. So, again distance and displacement are equal.

$$d = s = 40 \times 4 - \frac{1}{2} \times 10 \times 16 = 80 \text{ m}$$

(c) $t > t_0$. Hence, $d > s$, $\quad s = 40 \times 6 - \frac{1}{2} \times 10 \times 36 = 60 \text{ m}$

While
$$d = \left|\frac{u^2}{2a}\right| + \frac{1}{2} |a\,(t - t_0)^2|$$
$$= \frac{(40)^2}{2 \times 10} + \frac{1}{2} \times 10 \times (6 - 4)^2$$
$$= 100 \text{ m}$$

INTRODUCTORY EXERCISE 6.5

1. Prove the relation, $s_t = u + at - \frac{1}{2}a$.
2. Equation $s_t = u + at - \frac{1}{2}a$ does not seem dimensionally correct, why?
3. A particle is projected vertically upwards. What is the value of acceleration
   (i) during upward journey,
   (ii) during downward journey and
   (iii) at highest point?
4. A ball is thrown vertically upwards. Which quantity remains constant among, speed, kinetic energy, velocity and acceleration?
5. A particle is projected vertically upwards with an initial velocity of 40 m/s. Find the displacement and distance covered by the particle in 6 s. Take $g = 10 \text{ m/s}^2$.
6. A particle moves rectilinearly with initial velocity u and constant acceleration a. Find the average velocity of the particle in a time interval from $t = 0$ to $t = t$ second of its motion.
7. A particle moves in a straight line with uniform acceleration. Its velocity at time $t = 0$ is v_1 and at time $t = t$ is v_2. The average velocity of the particle in this time interval is $\frac{v_1 + v_2}{2}$.
   Is this statement true or false?
8. Find the average velocity of a particle released from rest from a height of 125 m over a time interval till it strikes the ground. Take $g = 10 \text{ m/s}^2$.
9. A particle starts with an initial velocity 2.5 m/s along the positive x-direction and it accelerates uniformly at the rate 0.50 m/s^2.
   (a) Find the distance travelled by it in the first two seconds
   (b) How much time does it take to reach the velocity 7.5 m/s? (c) How much distance will it cover in reaching the velocity 7.5 m/s?
10. A ball is projected vertically upward with a speed of 50 m/s. Find (a) the maximum height, (b) the time to reach the maximum height, (c) the speed at half the maximum height. Take $g = 10 \text{ ms}^2$.

6.7 One Dimensional Motion with Non-uniform Acceleration

When acceleration of a particle is not constant we take help of differentiation or integration.

Equations of Differentiation

(a)
$$\boxed{v=\frac{ds}{dt}} \qquad \text{...(i)}$$

If the motion is taking place along x-axis, then this equation can be written as,

$$v=\frac{dx}{dt}$$

Here, v is the instantaneous velocity and x, the x co-ordinate at a general time t.

(b)
$$\boxed{a=\frac{dv}{dt}} \qquad \text{...(ii)}$$

Here, a is the instantaneous acceleration of the particle. Further, a can also be written as

$$a=\frac{dv}{dt}=\left(\frac{ds}{dt}\right)\left(\frac{dv}{ds}\right)=v\left(\frac{dv}{ds}\right) \qquad \left[\text{as } \frac{ds}{dt}=v\right]$$

∴
$$\boxed{a=v\left(\frac{dv}{ds}\right)} \qquad \text{...(iii)}$$

Equations of Integration

(c)
$$\boxed{\int ds=\int v\,dt} \qquad \text{...(iv)}$$

or
$$\boxed{s=\int v\,dt} \qquad \text{...(v)}$$

In the above equations, v should be either constant or function of t

(d)
$$\boxed{\int dv=\int a\,dt} \qquad \text{...(vi)}$$

or
$$\boxed{\Delta v=v_f-v_i=\int a\,dt} \qquad \text{...(vii)}$$

In the above equations a should be either constant or function of time t.

(e)
$$\boxed{\int v\,dv=\int a\,ds} \qquad \text{...(viii)}$$

In the above equation a should be either constant or function of s.

Note *(i) To convert s-t equation into v-t equation or v-t equation into a-t equation differentiation will be done.*

$$s\text{-}t \rightarrow v\text{-}t \rightarrow a\text{-}t \qquad \textit{(differentiation)}$$

(ii) To convert a-t equation into v-t equation or v-t equation into s-t equation, integration equations (with some limits) are required. By limit we mean the value of physical quantity which we will get after integration should be known at some given time.

For example, after integrating v (w.r.t time) we will get displacement s. Therefore, to get complete s function value of s should be known at some given time. Otherwise constant of integration remains as an unknown.

Thus, $$a\text{-}t \rightarrow v\text{-}t \rightarrow s\text{-}t \qquad \textit{(integration with limits)}$$

Derivation of Equation of Motion ($v = u + at$ etc.)

For one dimensional motion with $a =$ constant.

We can write, $$dv = a\,dt \qquad \left(\text{as } a = \frac{dv}{dt}\right)$$

Integrating both sides, we have $$\int dv = a \int dt \qquad (\text{as } a = \text{constant})$$

At $t = 0$, velocity is u and at $t = t$ velocity is v. Hence,

$$\int_u^v dv = a \int_0^t dt$$

$$\therefore \quad [v]_u^v = a\,[t]_0^t \text{ or } v - u = at$$

$$\therefore \quad \boxed{v = u + at}$$

Hence proved.

Further, we can write $$ds = v\,dt \qquad \left(\text{as } v = \frac{ds}{dt}\right)$$

$$= (u + at)\,dt \qquad (\text{as } v = u + at)$$

At time $t = 0$ suppose $s = 0$ and at $t = t$, displacement is s, then

$$\int_0^s ds = \int_0^t (u + at)\,dt \quad \Rightarrow \quad \therefore \quad [s]_0^s = \left[ut + \frac{1}{2}at^2\right]_0^t$$

or $$\boxed{s = ut + \frac{1}{2}at^2}$$

Hence proved.

We can also write, $$v \cdot dv = a \cdot ds \qquad \left(\text{as } a = v \cdot \frac{dv}{ds}\right)$$

When $s = 0$, v is u and at $s = s$, velocity is v. Therefore,

$$\int_u^v v \cdot dv = a \int_0^s ds \quad \text{or} \quad \left[\frac{v^2}{2}\right]_u^v = a\,[s]_0^s \qquad (\text{as } a = \text{constant})$$

$$\therefore \quad \frac{v^2}{2} - \frac{u^2}{2} = as$$

or $$\boxed{v^2 = u^2 + 2as}$$

Hence proved.

Example 6.16 *Displacement-time equation of a particle moving along x-axis is*

$$x = 20 + t^3 - 12t \text{ (SI units)}$$

(a) Find, position and velocity of particle at time $t = 0$.

(b) State whether the motion is uniformly accelerated or not.

(c) Find position of particle when velocity of particle is zero.

Solution (a) $$x = 20 + t^3 - 12t \qquad \ldots\text{(i)}$$

At $t = 0$, $$x = 20 + 0 - 0 = 20\text{ m}$$

Velocity of particle at time t can be obtained by differentiating Eq. (i) w.r.t. time i.e.

$$v = \frac{dx}{dt} = 3t^2 - 12 \qquad \ldots\text{(ii)}$$

At $t = 0$, $$v = 0 - 12 = -12\text{ m/s}$$

(b) Differentiating Eq. (ii) w.r.t. time t, we get the acceleration $a = \frac{dv}{dt} = 6t$

As acceleration is a function of time, the motion is non-uniformly accelerated.

(c) Substituting $v = 0$ in Eq. (ii), we have $0 = 3t^2 - 12$

Positive value of t comes out to be 2 s from this equation. Substituting $t = 2\,\text{s}$ in Eq. (i), we have $x = 20 + (2)^3 - 12\,(2)$ or $x = 4\ \text{m}$

Example 6.17 *Velocity-time equation of a particle moving in a straight line is,*

$$v = (10 + 2t + 3t^2) \quad (SI\ units)$$

Find

(a) displacement of particle from the mean position at time $t = 1\,\text{s}$, if it is given that displacement is 20 m at time $t = 0$.

(b) acceleration-time equation.

Solution (a) The given equation can be written as,

$$v = \frac{ds}{dt} = (10 + 2t + 3t^2) \quad \text{or} \quad ds = (10 + 2t + 3t^2)\,dt$$

or
$$\int_{20}^{s} ds = \int_0^1 (10 + 2t + 3t^2)\,dt \quad \text{or} \quad s - 20 = [10t + t^2 + t^3]_0^1$$

or
$$s = 20 + 12 = 32\ \text{m}$$

(b) Acceleration-time equation can be obtained by differentiating the given equation w.r.t. time.

Thus,
$$a = \frac{dv}{dt} = \frac{d}{dt}(10 + 2t + 3t^2) \quad \text{or} \quad a = 2 + 6t$$

INTRODUCTORY EXERCISE 6.6

1. Velocity (in m/s) of a particle moving along x-axis varies with time as, $v = (10 + 5t - t^2)$ At time $t = 0$, $x = 0$. Find
   (a) acceleration of particle at $t = 2$ s and
   (b) x-coordinate of particle at $t = 3$ s
2. A particle is moving with a velocity of $v = (3 + 6t + 9t^2)$ cm/s. Find out
   (a) the acceleration of the particle at $t = 3$ s.
   (b) the displacement of the particle in the interval $t = 5$ s to $t = 8$ s.
3. The motion of a particle along a straight line is described by the function $x = (2t - 3)^2$, where x is in metres and t is in seconds. Find
   (a) the position, velocity and acceleration at $t = 2$ s.
   (b) the velocity of the particle at origin.
4. x-coordinate of a particle moving along this axis is $x = (2 + t^2 + 2t^3)$. Here, x is in metres and t in seconds. Find (a) position of particle from where it started its journey, (b) initial velocity of particle and (c) acceleration of particle at $t = 2$ s.
5. The velocity of a particle moving in a straight line is directly proportional to 3/4th power of time elapsed. How does its displacement and acceleration depend on time?

6.8 Motion in Two and Three Dimensions

The motion of a particle thrown in a vertical plane at some angle with horizontal ($\neq 90°$) is an example of two dimensional motion. Similarly, a circular motion is also an example of 2-D motion. A two dimensional motion takes place in a plane. In most of the cases plane of circular motion is horizontal or vertical. According to nature of acceleration we can classify this motion in following two types.

Uniform Acceleration

Equations of motion for uniformly accelerated motion ($\mathbf{a} = \text{constant}$) are as under

$$\mathbf{v} = \mathbf{u} + \mathbf{a}\, t,$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}\, t^2,$$

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + 2\,\mathbf{a} \cdot \mathbf{s}$$

Here, $\mathbf{u}$ = initial velocity of particle, $\mathbf{v}$ = velocity of particle at time t and
$\mathbf{s}$ = displacement of particle in time t

Note *If initial position vector of a particle is* $\mathbf{r}_0$, *then position vector at time t can be written as*

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{s} = \mathbf{r}_0 + \mathbf{u}\,t + \frac{1}{2}\mathbf{a}t^2$$

Non-Uniform Acceleration

When acceleration is not constant then we will have to go for differentiation or integration. The equations in differentiation are

(i) $\mathbf{v} = \dfrac{d\mathbf{s}}{dt}$ or $\dfrac{d\mathbf{r}}{dt}$ (ii) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$

Here, $\mathbf{v}$ and $\mathbf{a}$ are instantaneous velocity, acceleration vectors. The equations of integration are

(iii) $\int d\mathbf{s} = \int \mathbf{v}\,dt$ and (iv) $\int d\mathbf{v} = \int \mathbf{a}\, dt$

Extra Points to Remember

A two or three dimensional motion can also be solved by component method.
For example, in two dimensional motion (in x-y plane) the motion can be resolved along x and y directions. Now, along these two directions we can use sign method, as we used in one-dimensional motion (but separately). By separately we mean, when we are looking the motion along x-axis we need not to bother about the motion along y-axis.

Example 6.18 *A particle of mass 1 kg has a velocity of 2 m/s. A constant force of 2 N acts on the particle for 1 s in a direction perpendicular to its initial velocity. Find the velocity and displacement of the particle at the end of 1 s.*

Solution Force acting on the particle is constant. Hence, acceleration of the particle will also remain constant.

$$a = \frac{F}{m} = \frac{2}{1} = 2\ \text{m/s}^2$$

Since, acceleration is constant. We can apply

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

and $$\mathbf{s} = \mathbf{u}\,t + \frac{1}{2}\,\mathbf{a}\,t^2$$

Refer Fig. 6.17 (a) $$\mathbf{v} = \mathbf{u} + \mathbf{a}\,t$$

Here, **u** and **a** *t* are two mutually perpendicular vectors. So,

$$|\mathbf{v}| = \sqrt{(|\mathbf{u}|)^2 + (|\mathbf{a}\,t\,|)^2}$$

$$= \sqrt{(2)^2 + (2)^2}$$

$$= 2\sqrt{2} \text{ m/s}$$

$$\alpha = \tan^{-1}\frac{|\mathbf{a}t\,|}{|\mathbf{u}\,|} = \tan^{-1}\left(\frac{2}{2}\right)$$

$$= \tan^{-1}(1) = 45°$$

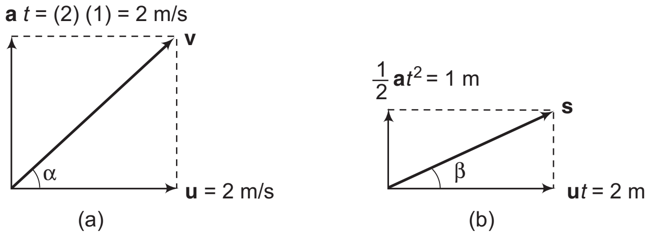


Fig. 6.17

Thus, velocity of the particle at the end of 1s is $2\sqrt{2}$ m/s at an angle of 45° with its initial velocity.

Refer Fig. 6.17 (b), $$\mathbf{s} = \mathbf{u}\,t + \frac{1}{2}\,\mathbf{a}\,t^2$$

Here, **u** *t* and $\frac{1}{2}\,\mathbf{a}\,t^2$ are also two mutually perpendicular vectors. So,

$$|\mathbf{s}\,| = \sqrt{(|\mathbf{u}\,t\,|)^2 + (|\tfrac{1}{2}\,\mathbf{a}\,t^2\,|)^2}$$

$$= \sqrt{(2)^2 + (1)^2}$$

$$= \sqrt{5} \text{ m}$$

and $$\beta = \tan^{-1}\frac{\left|\dfrac{1}{2}\mathbf{a}t^2\right|}{|\mathbf{u}t\,|}$$

$$= \tan^{-1}\left(\frac{1}{2}\right)$$

Thus, displacement of the particle at the end of 1 s is $\sqrt{5}$ m at an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ from its initial velocity.

Example 6.19 *Velocity and acceleration of a particle at time* $t = 0$ *are* $\mathbf{u} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})\ m/s$ *and* $\mathbf{a} = (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ m/s^2$ *respectively. Find the velocity and displacement of particle at* $t = 2$ *s.*

Solution Here, acceleration $\mathbf{a} = (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ \text{m/s}^2$ is constant.

So, we can apply

$$\mathbf{v} = \mathbf{u} + \mathbf{a}\,t$$

and

$$\mathbf{s} = \mathbf{u}\,t + \frac{1}{2}\,\mathbf{a}\,t^2$$

Substituting the proper values, we get

$$\mathbf{v} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) + (2)(4\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$

$$= (10\hat{\mathbf{i}} + 7\hat{\mathbf{j}})\ \text{m/s}$$

and

$$\mathbf{s} = (2)(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) + \frac{1}{2}(2)^2\ (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$$

$$= (12\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\ \text{m}$$

Therefore, velocity and displacement of particle at $t = 2$s are $(10\hat{\mathbf{i}} + 7\hat{\mathbf{j}})$ m/s and $(12\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m respectively.

Example 6.20 *Velocity of a particle in x-y plane at any time t is*

$$\mathbf{v} = (2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})\ m/s$$

At $t = 0$*, particle starts from the co-ordinates* $(2\,m, 4\,m)$*. Find*

(a) acceleration of the particle at $t = 1$ *s.*

(b) position vector and co-ordinates of the particle at $t = 2$ *s.*

Solution (a) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d}{dt}(2t\,\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})$

$$= (2\hat{\mathbf{i}} + 6t\,\hat{\mathbf{j}})\ \text{m/s}^2$$

At $t = 1$s,

$$\mathbf{a} = (2\hat{\mathbf{i}} + 6\hat{\mathbf{j}})\ \text{m/s}^2$$ **Ans.**

(b) $\int d\mathbf{s} = \int \mathbf{v}\,dt$

or

$$\mathbf{s} = \int \mathbf{v}\,dt = \int (2t\,\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})\,dt$$

∴

$$\mathbf{r}_f - \mathbf{r}_i = \int_{\text{initial}}^{\text{final}} (2t\,\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})\,dt$$

or

$$\mathbf{r}_{2\,\text{sec}} - \mathbf{r}_{0\,\text{sec}} = \int_0^2 (2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})\,dt$$

∴

$$\mathbf{r}_{2\,\text{sec}} = \mathbf{r}_{0\,\text{sec}} + [t^2\hat{\mathbf{i}} + t^3\hat{\mathbf{j}}]_0^2$$

$$= (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) + (4\hat{\mathbf{i}} + 8\hat{\mathbf{j}})$$

$$= (6\hat{\mathbf{i}} + 12\hat{\mathbf{j}})\ \text{m}$$ **Ans.**

Therefore, coordinates of the particle at $t = 2\,\text{s}$ are (6 m, 12 m) **Ans.**

INTRODUCTORY EXERCISE 6.7

1. Velocity of a particle at time $t = 0$ is 2 m/s. A constant acceleration of 2 m/s^2 acts on the particle for 2 s at an angle of 60° with its initial velocity. Find the magnitude of velocity and displacement of particle at the end of $t = 2\,\text{s}$.
2. Velocity of a particle at any time t is $\mathbf{v} = (2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}})$ m/s. Find acceleration and displacement of particle at $t = 1\,\text{s}$. Can we apply $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ or not?
3. Acceleration of a particle in x-y plane varies with time as

$$\mathbf{a} = (2t\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}})\ \text{m/s}^2$$

At time $t = 0$, velocity of particle is 2 m/s along positive x-direction and particle starts from origin. Find velocity and coordinates of particle at $t = 1\,\text{s}$.

6.9 Graphs

Before studying graphs of kinematics, let us first discuss some general points :

(i) Mostly a graph is drawn between two variable quantities (say x and y). In kinematics, the frequently asked graphs are s-t, v-t, a-t or v-s.

(ii) Equation between x and y will decide the shape of graph whether it is straight line, circle, parabola or rectangular hyperbola etc. If the equation is linear, graph is a straight line. If equation is quadratic then graph is a parabola. In kinematics, most of the graphs are straight line or parabola.

(iii) By putting $x = 0$ in y-x equation if we get $y = 0$, then graph passes through origin, otherwise not.

(iv) If $z = \frac{dy}{dx}$, then the value of z at any point can be obtained by the slope of the graph at that point.

For example,

instantaneous velocity $v = \frac{ds}{dt}$ = slope of s-t graph.

instantaneous acceleration $a = \frac{dv}{dt}$ = slope of v-t graph.

(v) If $dz = y\,dx$. Then, $\int dz = \int y\,dx$

$\Rightarrow$ z or $z_f - z_i$ or Δz = area under y-x graph, with projection along x-axis.

For example,

$$ds = v dt$$

$\Rightarrow$ Displacement $s = \int v dt$ = area under v-t graph with projection along t-axis.

Further, $dv = a dt$

$\Rightarrow$ $v_f - v_i$ or $\Delta v = \int a dt$ = area under a-t graph with projection along t-axis.

These results have been summarized in following table :

Table 6.2

Name of Graph	Slope	Area
s-t	v	No physical quantity
v-t	a	s
a-t	Rate of change of acceleration	$v_f - v_i$ or Δv

For better understanding of three graphs (s-t, v-t and a-t) of kinematics, we have classified the one dimensional motion in following four types.

Uniform Motion

Equations

The three equations of uniform motion are as under

$a = 0$, $v =$ constant and $s = vt$ or $s = s_0 + vt$

Important Points

(i) s-t equation is linear. Therefore, s-t graph is straight line.

(ii) In $s = vt$, displacement is measured from the starting point ($t = 0$). Corresponding to this equation s-t graph passes through origin, as $s = 0$ when $t = 0$. In $s = s_0 + vt$, displacement is measured from any other point and s_0 is the initial displacement.

(iii) Slope of s-t graph $= v$. Now, since $v =$ constant, therefore slope of s-t graph $=$ constant.

(iv) Slope of v-t graph $= a$. Now, since $a = 0$, therefore slope of v-t graph $= 0$.

The Corresponding Graphs

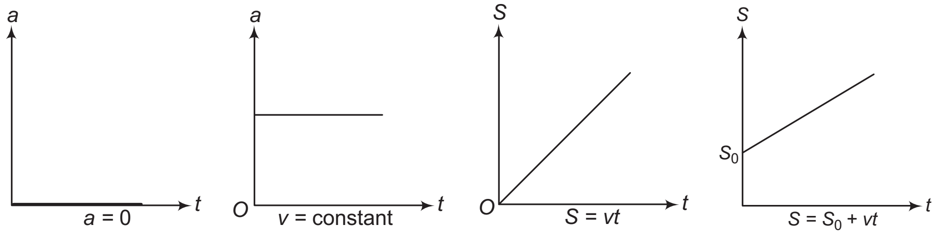


Fig. 6.18

Example 6.21 *s-t graph of a particle in motion is as shown below.*

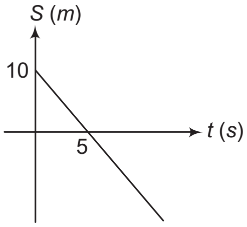


Fig. 6.19

(a) State, whether the given graph represents a uniform motion or not.

(b) Find velocity of the particle.

Solution (a)

$v =$ slope of s-t graph. Since, the given s-t graph is a straight line and slope of a straight line is always constant. Hence, velocity is constant. Therefore, the given graph represents a uniform motion.

(b) $v =$ slope of s-t graph $= -\frac{10}{5} = -2$ m/s **Ans.**

Example 6.22 *A particle is moving along x-axis. Its x-coordinate versus time graph is as shown below.*

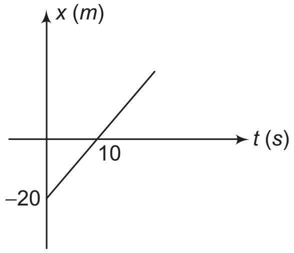


Fig. 6.20

Draw some conclusions from the given graph.

Solution The conclusions drawn from the graph are as under:

(i) x-t graph is a straight line, slope of which $\left(v = \frac{dx}{dt}\right)$ is positive and constant. Therefore, velocity is positive and constant.

(ii) $v = \frac{dx}{dt} =$ slope of x-t graph

$$= +\frac{20}{10} = +2 \text{ m/s}$$

Therefore, velocity is 2 m/s along positive x-direction.

(iii) At $t = 0, x = -20$ m and at $t = 10$ s, $x = 0$

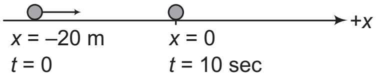


Fig. 6.21

Uniformly Accelerated Motion

Equations

$$a = \text{constant (and positive)}$$

$$v = u + at \text{ or } v = at, \text{ if } u = 0$$

$$s = ut + \frac{1}{2}at^2 \text{ or } s = \frac{1}{2}at^2 \text{ if } u = 0$$

or $$s = s_0 + ut + \frac{1}{2}at^2 \text{ if } s_0 \neq 0$$

or $$s = s_0 + \frac{1}{2}at^2 \text{ if } s_0 \neq 0 \text{ but } u = 0$$

Important Points

(i) v-t equation is linear. Therefore, v-t graph is a straight line. Further, $v = at$ is a straight line passing through origin (as $v = 0$ when $t = 0$)

(ii) All s-t equations are quadratic. Therefore, all s-t graphs should be parabolic.

(iii) Slope of s-t graph gives the instantaneous velocity. Therefore, initial slope of s-t graph gives initial velocity u. In this case, we are considering only accelerated motion (in which speed keeps on increasing in positive direction). Therefore, velocity is positive and continuously increasing. Hence, slope of s-t graph should be positive and should keep on increasing.

(iv) Slope of v-t graph gives instantaneous acceleration. Now, acceleration is positive and constant. Therefore, slope of v-t graph should be positive and constant.

Graphs

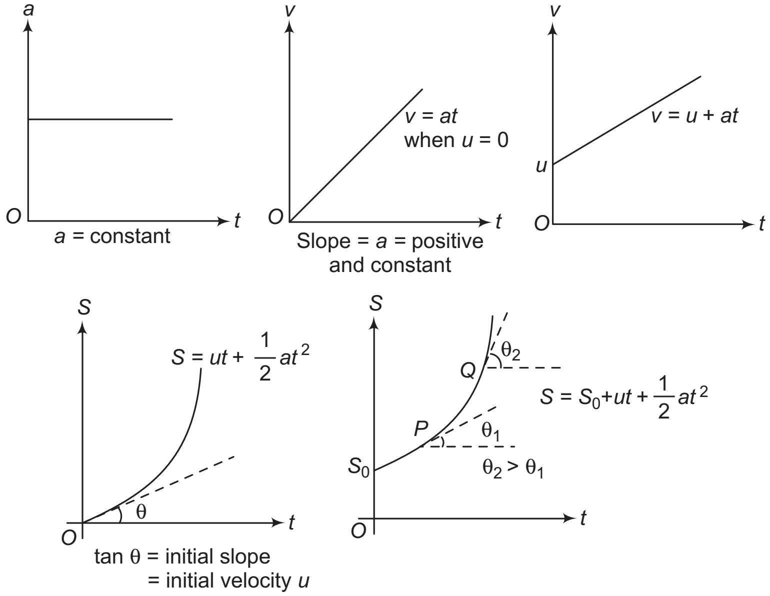


Fig. 6.22

From P to Q slope is increasing (positive at both points). Therefore, velocity is positive and increasing.

Uniformly Retarded Motion (till velocity becomes zero)

We are considering the case when initial velocity is positive and a constant acceleration acts in negative direction (till the velocity becomes zero).

Equations

$$a = \text{constant (and negative)}$$

$$v = u - at$$

$$s = ut - \frac{1}{2}at^2$$

Important Points

(i) In this case, u cannot be zero. Therefore, v-t straight line cannot pass through origin. Further, initial slope of s-t parabolic graph cannot be zero.

(ii) Velocity is positive but keeps on decreasing from u to zero.

(iii) Slope of v-t graph gives instantaneous acceleration. Acceleration is constant and negative. Therefore, slope of v-t graph should be negative and constant.

(iv) Initial slope of s-t graph will give us initial velocity u. Final slope of s-t graph will give us final velocity zero. In between these two times, velocity is positive and decreasing. Therefore, slope of s-t graph (= instantaneous velocity) should be positive and decreasing.

Graphs

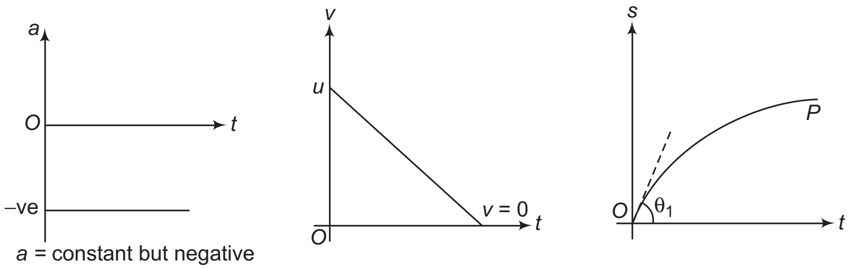


Fig. 6.23

$\tan\theta_1 = \text{initial slope} = u$

From O to P, slope or v is positive but decreasing.

At P, slope $= 0$, therefore $v = 0$

Uniformly Retarded and then Accelerated Motion in Opposite Direction

If a particle is projected upwards then first it is retarded in upward (say positive) direction. At highest point its velocity becomes zero and finally it is retarded in downward (or negative) direction. Throughout the motion, its acceleration is downwards and constant (= acceleration due to gravity). Therefore, it is negative and constant. If air resistance is neglected, then speed of the particle at the time of projection is equal to speed at the time of striking with the ground. But velocities are in opposite directions. So, their signs are different. During retardation, velocity is upwards (therefore positive) but decreasing.

During acceleration, velocity is downwards (therefore negative) and increasing. Upward journey time is equal to the downward journey time. Finally, the particle returns to the ground. Therefore final displacement is zero. In upward journey, displacement increases (parabolically) in positive direction. In downward journey, it decreases. But displacement from the starting point (ground) is still positive. Slope of s-t graphs gives the instantaneous velocity. In upward journey, velocity is positive and decreasing. Therefore, slope is positive and decreasing. At highest point velocity is zero. Therefore, slope is zero. In downward journey, velocity is negative and increasing. Therefore, slope is negative and increasing.

Graphs

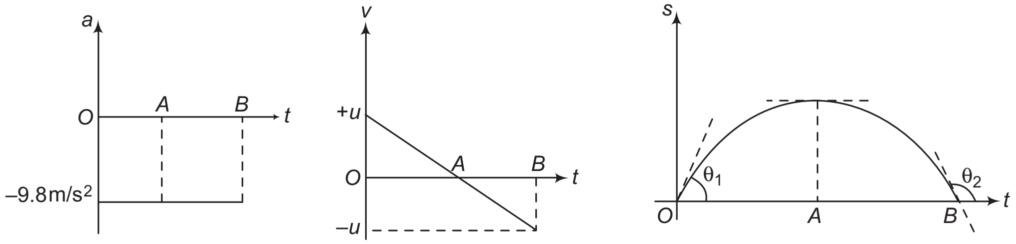


Fig. 6.24

In the above graphs,

(i) $a = -9.8\ \text{m/s}^2$ if the motion is taking place under gravity.

(ii) O is the starting point, where

$$v = +u \Rightarrow \text{slope of } s\text{-}t \text{ graph} = \tan\theta_1 = u$$

(iii) A is the highest point, where

$$v = 0 \Rightarrow \text{slope of } s\text{-}t \text{ graph} = 0$$

(iv) B is the point when particle again strikes the ground. At this point,

$$v = -u \Rightarrow \text{slope of } s\text{-}t \text{ graph} = \tan\theta_2 = -u$$

At this point,

$$s = 0$$

(v) Upwards journey time t_{OA} = downward journey time t_{AB}

(vi) In upward motion (from O to A), velocity is positive and decreasing. Therefore, slope of s-t graph is positive and decreasing.

(vii) In downward motion (from A to B), velocity is negative and increasing. Therefore, slope of s-t graph is negative and increasing.

✓ *Extra Points to Remember*

- Slope of v-t or s-t graph can never be infinite at any point, because infinite slope of v-t graph means infinite acceleration. Similarly, infinite slope of s-t graph means infinite velocity. Hence, the following graphs are not possible :

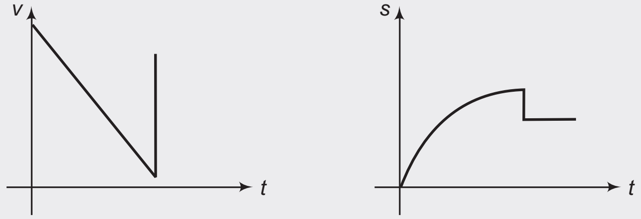


Fig. 6.25

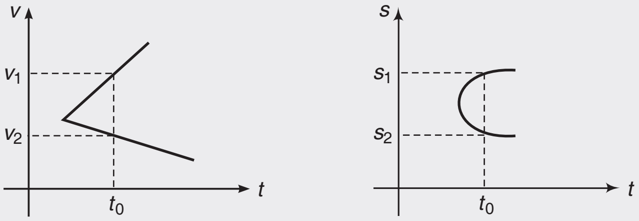


Fig. 6.26

Example 6.23 *Acceleration-time graph of a particle moving in a straight line is as shown in Fig. 6.28. Velocity of particle at time $t = 0$ is 2 m/s. Find the velocity at the end of fourth second.*

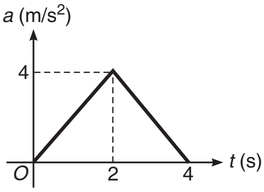


Fig. 6.28

Solution $$\int dv = \int a\,dt$$

or change in velocity = area under a-t graph

Hence,
$$v_f - v_i = \frac{1}{2}(4)(4)$$
$$= 8\,\text{m/s}$$

∴
$$v_f = v_i + 8 = (2 + 8)\,\text{m/s}$$
$$= 10\,\text{m/s}$$

Example 6.24 *A particle is projected upwards with velocity 40 m/s. Taking the value of $g = 10\ m/s^2$ and upward direction as positive, plot a-t, v-t and s-t graphs of the particle from the starting point till it further strikes the ground.*

Solution Upward journey time = downward journey time $= \frac{u}{g} = \frac{40}{10} = 4\ \text{s}$

$\therefore$ Total time of journey = 8 s

Maximum height attained by the particle $= \frac{u^2}{2g} = \frac{(40)^2}{2 \times 10} = 80\ \text{m}$

***a-t* graph** During complete journey $a = g = -10\ \text{m/s}^2$

Corresponding *a-t* graph is as shown below.

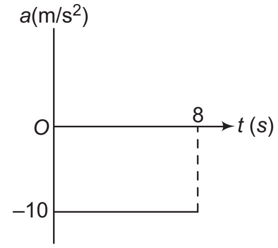


Fig. 6.29

***v-t* graph** In upward journey velocity first decreases from + 40 m/s to 0. Then, in downward journey it increases from 0 to – 40 m/s. Negative sign just signifies its downward direction. Corresponding *v-t* graph is as shown below.

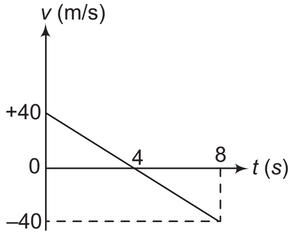


Fig. 6.30

***s-t* graph** In upward journey displacement first increases from 0 to + 80 m. Then, it decreases from + 80 m to 0. Corresponding *s-t* graph is as shown below.

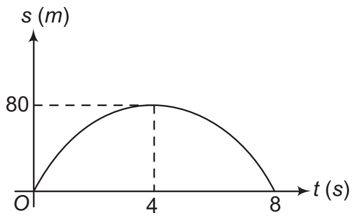


Fig. 6.31

Example 6.25 *A car accelerates from rest at a constant rate* α *for some time, after which it decelerates at a constant rate* β*, to come to rest. If the total time elapsed is t seconds, then evaluate (a) the maximum velocity reached and (b) the total distance travelled.*

Solution (a) Let the car accelerates for time t_1 and decelerates for time t_2. Then,

$$t = t_1 + t_2 \qquad \ldots\text{(i)}$$

and corresponding velocity-time graph will be as shown in Fig. 6.32.

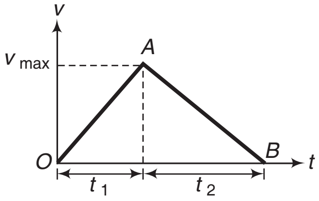


Fig. 6.32

From the graph,

$$\alpha = \text{slope of line } OA = \frac{v_{\max}}{t_1} \quad \text{or} \quad t_1 = \frac{v_{\max}}{\alpha} \qquad \ldots\text{(ii)}$$

and
$$\beta = -\text{ slope of line } AB = \frac{v_{\max}}{t_2}$$

or
$$t_2 = \frac{v_{\max}}{\beta} \qquad \ldots\text{(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{v_{\max}}{\alpha} + \frac{v_{\max}}{\beta} = t$$

or
$$v_{\max}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = t$$

or
$$v_{\max} = \frac{\alpha\beta t}{\alpha+\beta}$$
Ans.

(b) Total distance = total displacement = area under v-t graph

$$= \frac{1}{2} \times t \times v_{\max}$$

$$= \frac{1}{2} \times t \times \frac{\alpha\beta t}{\alpha+\beta}$$

or
$$\text{Distance} = \frac{1}{2}\left(\frac{\alpha\beta t^2}{\alpha+\beta}\right)$$
Ans.

Note *This problem can also be solved by using equations of motion ($v = u + at$ etc.). Try it yourself.*

Example 6.26 *The acceleration versus time graph of a particle moving along a straight line is shown in the figure. Draw the respective velocity-time graph. Given* $v = 0$ *at* $t = 0$.

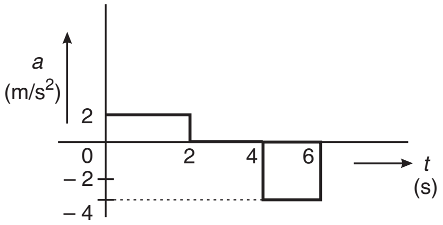


Fig. 6.33

Solution From $t = 0$ to $t = 2\,\text{s}$, $a = +2\ \text{m/s}^2$

$\therefore$ $v = at = 2t$

or v-t graph is a straight line passing through origin with slope $2\ \text{m/s}^2$.

At the end of 2 s,

$$v = 2 \times 2 = 4\ \text{m/s}$$

From $t = 2$ to 4 s, $a = 0$.

Hence, $v = 4$ m/s will remain constant.

From $t = 4$ to 6 s, $a = -4\ \text{m/s}^2$.

Hence, $v = u - at = 4 - 4t$ (with $t = 0$ at 4 s)

$v = 0$ at $t = 1\,\text{s}$ or at 5 s from origin.

At the end of 6 s (or $t = 2\,\text{s}$) $v = -4$ m/s. Corresponding v-t graph is as shown in Fig. 6.34.

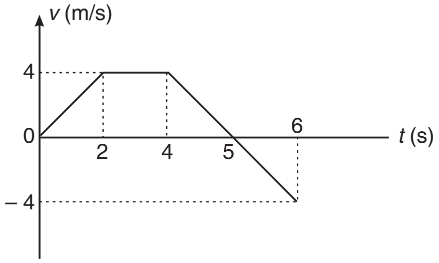


Fig. 6.34

INTRODUCTORY EXERCISE 6.8

1. Two particles A and B are moving along x-axis. Their x-coordinate *versus* time graphs are as shown below

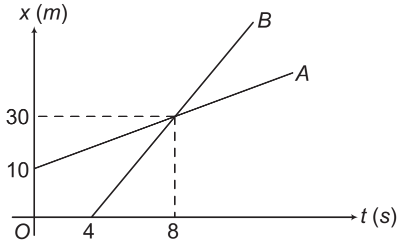


Fig. 6.35

(a) Find the time when the particles start their journey and the x-coordinate at that time.
(b) Find velocities of the two particles.
(c) When and where the particles strike with each other.

2. The velocity of a car as a function of time is shown in Fig. 6.36. Find the distance travelled by the car in 8 s and its acceleration.

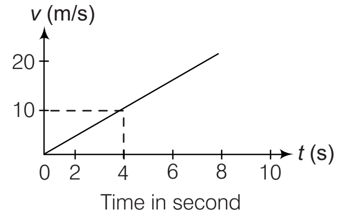


Fig. 6.36

3. Fig. 6.37 shows the graph of velocity versus time for a particle going along the *x*-axis. Find (a) acceleration, (b) the distance travelled in 0 to 10 s and (c) the displacement in 0 to 10 s.

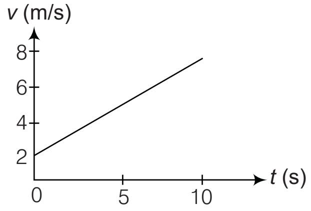


Fig. 6.37

4. Fig. 6.38 shows the graph of the *x*-coordinate of a particle going along the *x*-axis as a function of time. Find (a) the average velocity during 0 to 10 s, (b) instantaneous velocity at 2, 5, 8 and 12 s.

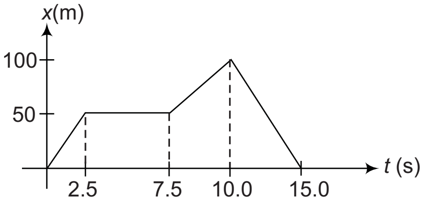


Fig. 6.38

5. From the velocity-time plot shown in Fig. 6.39, find the distance travelled by the particle during the first 40 s. Also find the average velocity during this period.

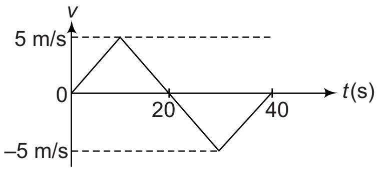


Fig. 6.39

6.10 Relative Motion

The word 'relative' is a very general term, which can be applied to physical, non-physical, scalar or vector quantities. For example, my height is 167 cm while my wife's height is 162 cm. If I ask you what is my height relative to my wife, your answer will be 5 cm. What you did? You simply subtracted my wife's height from my height. The same concept is applied everywhere, whether it is a relative velocity, relative acceleration or anything else. So, from the above discussion we may now conclude that relative velocity of A with respect to B (written as $\mathbf{v}_{AB}$) is

$$\boxed{\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B}$$

Similarly, relative acceleration of A with respect to B is

$$\boxed{\mathbf{a}_{AB} = \mathbf{a}_A - \mathbf{a}_B}$$

If it is a one dimensional motion we can treat the vectors as scalars just by assigning the positive sign to one direction and negative to the other. So, in case of a one dimensional motion the above equations can be written as

$$\boxed{\begin{aligned} v_{AB} &= v_A - v_B \\ \text{and} \quad a_{AB} &= a_A - a_B \end{aligned}}$$

Further, we can see that

$$\boxed{\mathbf{v}_{AB} = -\mathbf{v}_{BA} \text{ or } \mathbf{a}_{BA} = -\mathbf{a}_{AB}}$$

Example 6.27 *Anoop is moving due east with a velocity of 1 m/s and Dhyani is moving due west with a velocity of 2 m/s. What is the velocity of Anoop with respect to Dhyani?*

Solution It is a one dimensional motion. So, let us choose the east direction as positive and the west as negative. Now, given that

$$v_A = \text{velocity of Anoop} = 1\,\text{m/s}$$

and $$v_D = \text{velocity of Dhyani} = -2\,\text{m/s}$$

Thus, $$v_{AD} = \text{velocity of Anoop with respect to Dhyani}$$
$$= v_A - v_D = 1 - (-2) = 3\,\text{m/s}$$

Hence, velocity of Anoop with respect to Dhyani is 3 m/s due east.

Example 6.28 *Car A has an acceleration of $2\ m/s^2$ due east and car B, $4\ m/s^2$ due north. What is the acceleration of car B with respect to car A?*

Solution It is a two dimensional motion. Therefore,

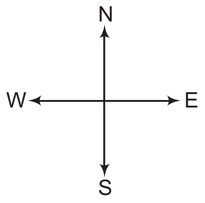


Fig. 6.40

$$\mathbf{a}_{BA} = \text{acceleration of car } B \text{ with respect to car } A$$
$$= \mathbf{a}_B - \mathbf{a}_A$$

Here, $$\mathbf{a}_B = \text{acceleration of car } B$$
$$= 4\ \text{m/s}^2 \text{ (due north)}$$

and $$\mathbf{a}_A = \text{acceleration of car } A$$
$$= 2\ \text{m/s}^2 \text{ (due east)}$$

$$|\mathbf{a}_{BA}| = \sqrt{(4)^2 + (2)^2} = 2\sqrt{5}\ \text{m/s}^2$$

and $$\alpha = \tan^{-1}\left(\frac{4}{2}\right) = \tan^{-1}(2)$$

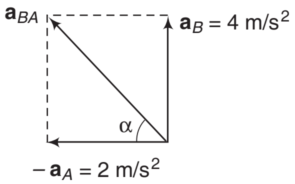


Fig. 6.41

Thus, $\mathbf{a}_{BA}$ is $2\sqrt{5}\ \text{m/s}^2$ at an angle of $\alpha = \tan^{-1}(2)$ from west towards north.

The topic 'relative motion' is very useful in two and three dimensional motion. Questions based on relative motion are usually of following four types :

(a) Minimum distance or collision or overtaking problems
(b) River-boat problems
(c) Aircraft-wind problems
(d) Rain problems

Minimum Distance or Collision or Overtaking Problems

When two bodies are in motion, the questions like, the minimum distance between them or the time when one body overtakes the other can be solved easily by the principle of relative motion. In these type of problems, one body is assumed to be at rest and the relative motion of the other body is considered. By assuming so, two body problem is converted into one body problem and the solution becomes easy. Following example will illustrate the statement:

Example 6.29 *Car A and car B start moving simultaneously in the same direction along the line joining them. Car A moves with a constant acceleration $a = 4\ m/s^2$, while car B moves with a constant velocity $v = 1\ m/s$. At time $t = 0$, car A is 10 m behind car B. Find the time when car A overtakes car B.*

Solution Given, $u_A = 0$, $u_B = 1\,\text{m/s}$, $a_A = 4\,\text{m/s}^2$ and $a_B = 0$

Assuming car B to be at rest, we have

$$u_{AB} = u_A - u_B = 0 - 1 = -1\,\text{m/s}$$
$$a_{AB} = a_A - a_B = 4 - 0 = 4\,\text{m/s}^2$$

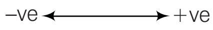


Fig. 6.42

Now, the problem can be assumed in simplified form as shown below.

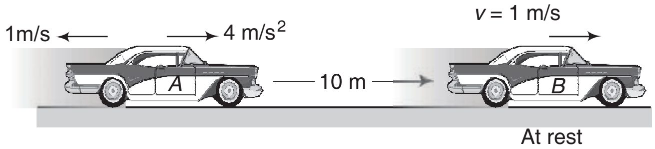


Fig. 6.43

Substituting the proper values in equation $s = ut + \frac{1}{2}at^2$,

we get $$10 = -t + \frac{1}{2}(4)(t^2)$$

or $$2t^2 - t - 10 = 0$$

or $$t = \frac{1 \pm \sqrt{1 + 80}}{4} = \frac{1 \pm \sqrt{81}}{4} = \frac{1 \pm 9}{4}$$

or $$t = 2.5\,\text{s} \quad \text{and} \quad -2\,\text{s}$$

Ignoring the negative value, the desired time is 2.5 s.

Example 6.30 *Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 km/h and ship B is streaming north at 20 km/h. What is their distance of closest approach and how long do they take to reach it?*

Solution Ships A and B are moving with same speed 20 km/h in the directions shown in figure. It is a two dimensional, two body problem with zero acceleration.

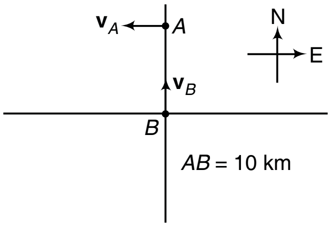


Fig. 6.44

Let us find $\mathbf{v}_{BA}$.

$\therefore$ $$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$

Here, $$|\mathbf{v}_{BA}| = \sqrt{(20)^2 + (20)^2}$$
$$= 20\sqrt{2} \text{ km/h}$$

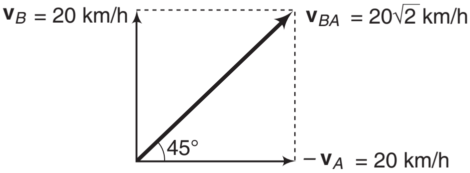


Fig. 6.45

i.e. $\mathbf{v}_{BA}$ is $20\sqrt{2}$ km/h at an angle of 45° from east towards north. Thus, the given problem can be simplified as A is at rest and B is moving with $\mathbf{v}_{BA}$ in the direction shown in Fig. 6.46.

Therefore, the minimum distance between the two is

$$s_{\min} = AC = AB \sin 45^\circ$$
$$= 10\left(\frac{1}{\sqrt{2}}\right) \text{ km}$$
$$= 5\sqrt{2} \text{ km} \qquad \textbf{Ans.}$$

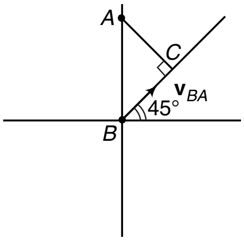


Fig. 6.46

and the desired time is

$$t = \frac{BC}{|\mathbf{v}_{BA}|} = \frac{5\sqrt{2}}{20\sqrt{2}} \qquad (BC = AC = 5\sqrt{2} \text{ km})$$
$$= \frac{1}{4} \text{ h} = 15 \text{ min} \qquad \textbf{Ans.}$$

River-Boat Problems

In river-boat problems, we come across the following three terms :

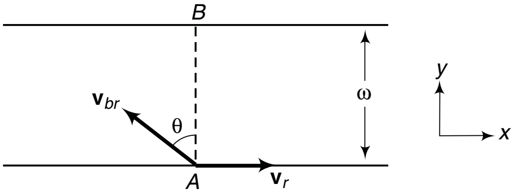


Fig. 6.47

$\mathbf{v}_r =$ absolute velocity of river

$\mathbf{v}_{br} =$ velocity of boatman with respect to river or velocity of boatman in still water

and $\mathbf{v}_b =$ absolute velocity of boatman.

Here, it is important to note that $\mathbf{v}_{br}$ is the velocity of boatman with which he steers and $\mathbf{v}_b$ is the actual velocity of boatman relative to ground.

Further, $$\mathbf{v}_b = \mathbf{v}_{br} + \mathbf{v}_r \qquad (\text{as } \mathbf{v}_{br} = \mathbf{v}_b - \mathbf{v}_r)$$

Now, let us derive some standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\mathbf{v}_{br}$ in the direction shown in Fig. 6.47. River is flowing along positive x-direction with velocity $\mathbf{v}_r$. Width of the river is ω, then

$$\mathbf{v}_b = \mathbf{v}_r + \mathbf{v}_{br}$$

Therefore, $$v_{bx} = v_{rx} + v_{brx} = v_r - v_{br} \sin\theta$$

and $$v_{by} = v_{ry} + v_{bry}$$

$$= 0 + v_{br} \cos\theta = v_{br} \cos\theta$$

Now, time taken by the boatman to cross the river is

$$t = \frac{\omega}{v_{by}} = \frac{\omega}{v_{br}\cos\theta} \quad \text{or} \quad \boxed{t = \frac{\omega}{v_{br}\cos\theta}} \qquad \text{...(i)}$$

Further, displacement along x-axis when he reaches the other bank (also called drift) is

$$x = v_{bx}\, t = (v_r - v_{br}\sin\theta)\frac{\omega}{v_{br}\cos\theta}$$

or $$\boxed{x = (v_r - v_{br}\sin\theta)\frac{\omega}{v_{br}\cos\theta}} \qquad \text{...(ii)}$$

Three special cases are:

(i) **Condition when the boatman crosses the river in shortest interval of time** From Eq. (i) we can see that time (t) will be minimum when $\theta = 0°$, i.e. the boatman should steer his boat perpendicular to the river current.

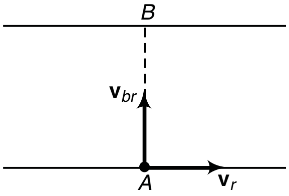


Fig. 6.48

Also, $$t_{\min} = \frac{\omega}{v_{br}}$$

as $$\cos\theta = 1$$

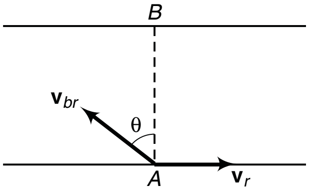


Fig. 6.49

(ii) **Condition when the boatman wants to reach point *B*,** i.e. **at a point just opposite from where he started**

In this case, the drift (x) should be zero.

$$\therefore \qquad x = 0$$

$$\text{or} \qquad (v_r - v_{br}\sin\theta)\frac{\omega}{v_{br}\cos\theta} = 0$$

$$\text{or} \qquad v_r = v_{br}\sin\theta$$

$$\text{or} \qquad \boxed{\sin\theta = \frac{v_r}{v_{br}} \quad \text{or} \quad \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)}$$

Hence, to reach point B the boatman should row at an angle $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$ upstream from AB.

Further, since $\sin\theta \ngtr 1$.

So, if $v_r \geq v_{br}$, the boatman can never reach at point B. Because if $v_r = v_{br}$, $\sin\theta = 1$ or $\theta = 90°$ and it is just impossible to reach at B if $\theta = 90°$. Moreover, it can be seen that $v_b = 0$ if $v_r = v_{br}$ and $\theta = 90°$. Similarly, if $v_r > v_{br}$, $\sin\theta > 1$, i.e. no such angle exists. Practically, it can be realized in this manner that it is not possible to reach at B if river velocity (v_r) is too high.

✓ *Extra Points to Remember*

- In a general case, resolve $\mathbf{v}_{br}$ along the river and perpendicular to river as shown below.

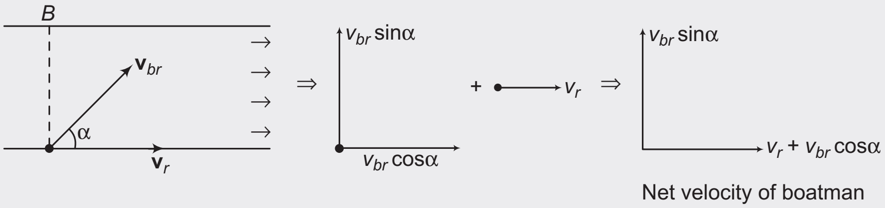


Fig. 6.50

Now, the boatman will cross the river with component of $\mathbf{v}_{br}$ perpendicular to river (= $v_{br}\sin\alpha$ in above case)

$$\therefore \qquad t = \frac{\omega}{v_{br}\sin\alpha}$$

To cross the river in minimum time, why to take help of component of $\mathbf{v}_{br}$ (which is always less that v_{br}), the complete vector $\mathbf{v}_{br}$ should be kept perpendicular to the river current. Due to the other component $v_r + v_{br}\cos\alpha$, boatman will drift along the river by a distance $x = (v_r + v_{br}\cos\alpha)$ (time)

- To reach a point B, which is just opposite to the starting point A, net velocity of boatman $\mathbf{v}_b$ or the vector sum of $\mathbf{v}_r$ and $\mathbf{v}_{br}$ should be along AB. The velocity diagram is as under

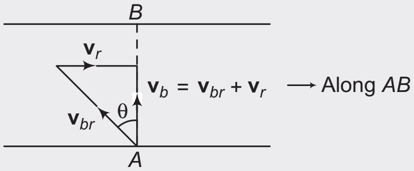


Fig. 6.51

From the diagram we can see that,

$$|\mathbf{v}_b| \text{ or } v_b = \sqrt{v_{br}^2 - v_r^2} \qquad \text{...(i)}$$

$$\text{Time, } t = \frac{\omega}{v_b} = \frac{\omega}{\sqrt{v_{br}^2 - v_r^2}}$$

$$\text{drift } x = 0 \text{ and } \sin\theta = \frac{v_r}{v_{br}} \text{ or } \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$$

From Eq. (i), we can see that this case is possible if,

$$v_{br} > v_r$$

otherwise, v_b is either zero or imaginary.

- If the boatman rows his boat along the river (downstream), then net velocity of boatman will be $v_{br} + v_r$. If he rows along the river upstream then net velocity of boatman will be $v_{br} \sim v_r$.

Example 6.31 *Width of a river is 30 m, river velocity is 2 m/s and rowing velocity is 5 m/s at 37° from the direction of river current (a) find the time taken to cross the river, (b) drift of the boatman while reaching the other shore.*

Solution

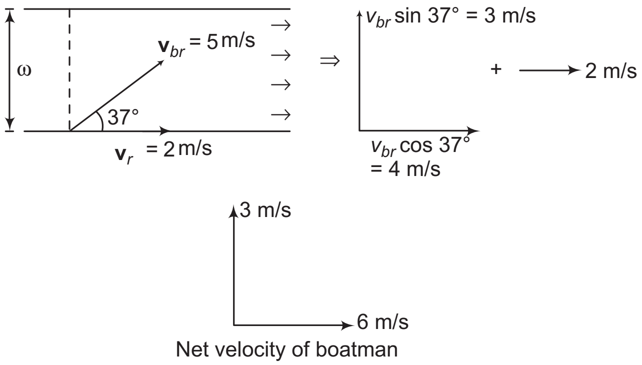


Fig. 6.52

(a) Time taken to cross the river,

$$t = \frac{\omega}{3} = \frac{30}{3} = 10\,\text{s} \qquad \textbf{Ans.}$$

(b) Drift along the river

$$x = (6)(t) = 6 \times 10 = 60\,\text{m} \qquad \textbf{Ans.}$$

Example 6.32 *Width of a river is 30 m, river velocity is 4 m/s and rowing velocity of boatman is 5 m/s*

(a) *Make the velocity diagram for crossing the river in shortest time. Then, find this shortest time, net velocity of boatman and drift along the river.*

(b) *Can the boatman reach a point just opposite on the other shore? If yes then make the velocity diagram, the direction in which he should row his boat and the time taken to cross the river in this case.*

(c) *How long will it take him to row 10 m up the stream and then back to his starting point?*

Solution (a) **Shortest time**

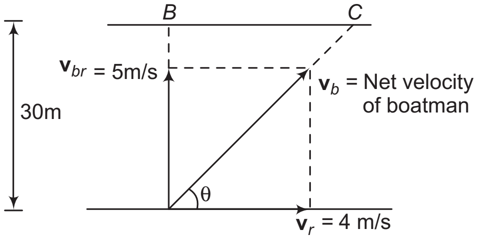


Fig. 6.53

$$t = \frac{30}{5} = 6\,\text{s} = t_{\text{min}}$$ **Ans.**

$$|\mathbf{v}_b| \quad \text{or} \quad v_b = \sqrt{(5)^2 + (4)^2} = \sqrt{41}\ \text{m/s}$$ **Ans.**

$$\tan\theta = \frac{5}{4} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{5}{4}\right)$$ **Ans.**

$$\begin{aligned}\text{Drift} &= BC = (4)(t)\\ &= 4 \times 6 = 24\ \text{m}\end{aligned}$$ **Ans.**

(b) Since, $v_{br} > v_r$, this case is possible. Velocity diagram is as under.

Net velocity $|\mathbf{v}_b|$ or $v_b = \sqrt{(5)^2 - (4)^2} = 3$ m/s along AB

$$\sin\theta = \frac{4}{5} \quad \Rightarrow \quad \theta = \sin^{-1}\frac{4}{5} = 53^\circ$$ **Ans.**

$$t = \frac{AB}{v_b} = \frac{30}{3} = 10\,\text{sec}$$ **Ans.**

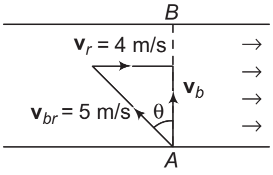


Fig. 6.54

Note *If the boatman wants to return to the same point A, then diagram is as under*

$$t_{BA} = \frac{BA}{3} = \frac{30}{3} = 10\ s$$

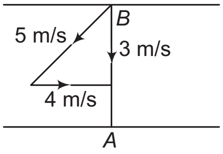


Fig. 6.55

(c)

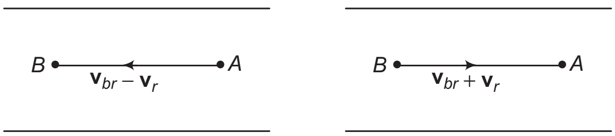

B
A
$\mathbf{v}_{br} - \mathbf{v}_r$
B
A
$\mathbf{v}_{br} + \mathbf{v}_r$

Fig. 6.56

$$t = t_{AB} + t_{BA} = \frac{AB}{v_{br} - v_r} + \frac{BA}{v_{br} + v_r}$$

or $$t = \frac{10}{5-4} + \frac{10}{5+4}$$

$$= \frac{100}{9}\,\text{s}$$ **Ans.**

Aircraft Wind Problems

This is similar to river boat problems. The only difference is that $\mathbf{v}_{br}$ is replaced by $\mathbf{v}_{aw}$ (velocity of aircraft with respect to wind or velocity of aircraft in still air), $\mathbf{v}_r$ is replaced by $\mathbf{v}_w$ (velocity of wind) and $\mathbf{v}_b$ is replaced by $\mathbf{v}_a$ (absolute velocity of aircraft). Further, $\mathbf{v}_a = \mathbf{v}_{aw} + \mathbf{v}_w$.

In this case, problem is slightly different. The given variables are

(i) Complete wind velocity $\mathbf{v}_w$

(ii) Steering speed or $|\mathbf{v}_{aw}|$

(iii) Starting point (say A) and destination point (say B)

We have to find direction of $\mathbf{v}_{aw}$ (or steering velocity) and the time taken in moving from A to B. The concepts is : net velocity of aircraft $\mathbf{v}_a$ or vector sum of $\mathbf{v}_w$ and $\mathbf{v}_{aw}$ should be along AB.

To solve such problems, we can apply the following steps :

(i) Take starting point A as the origin.

(ii) Wind velocity vector is completely given. So, draw $\mathbf{v}_w$ from point A.

(iii) Draw another vector $\mathbf{v}_a$ starting from A in a direction from A to B.

(iv) In above two steps we have already made two sides of a triangle in vector form. Complete the third side. This represents $\mathbf{v}_{aw}$. While completing the triangle for finding direction of $\mathbf{v}_{aw}$, polygon law of vector addition is to be followed, so that,

$$\mathbf{v}_w + \mathbf{v}_{aw} = \mathbf{v}_a$$

(v) Applying, sine law in this triangle, we can find direction of $\mathbf{v}_{aw}$ and the net velocity of aircraft $\mathbf{v}_a$. Now,

time taken, $$t = \frac{AB}{|\mathbf{v}_a|} \quad \text{or} \quad \frac{AB}{v_a}$$

The following example will illustrate the above theory:

Example 6.33 *An aircraft flies at* $400\ km/h$ *in still air. A wind of* $200\sqrt{2}\,km/h$ *is blowing from the south towards north. The pilot wishes to travel from A to a point B north east of A. Find the direction he must steer and time of his journey if* $AB = 1000\ km$.

Solution Given that $v_w = 200\sqrt{2}$ km/h

$v_{aw} = 400$ km/h and $\mathbf{v}_a$ should be along AB or in north-east direction. Thus, the direction of $\mathbf{v}_{aw}$ should be such as the resultant of $\mathbf{v}_w$ and $\mathbf{v}_{aw}$ is along AB or in north-east direction.

Let $\mathbf{v}_{aw}$ makes an angle α with AB as shown in Fig. 6.57. Applying sine law in triangle ABC, we get

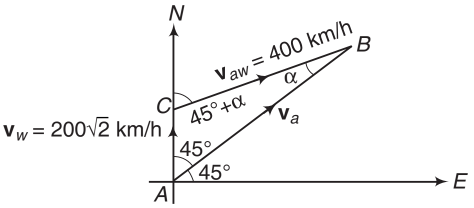

N
$\mathbf{v}_{aw} = 400$ km/h
B
α
C
45°+α
$\mathbf{v}_a$
$\mathbf{v}_w = 200\sqrt{2}$ km/h
45°
45°
A
E

Fig. 6.57

$$\frac{CB}{\sin 45^\circ} = \frac{AC}{\sin \alpha}$$

or $$\sin \alpha = \left(\frac{AC}{CB}\right) \sin 45^\circ$$

$$= \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}} = \frac{1}{2}$$

$\therefore$ $\alpha = 30°$

Therefore, the pilot should steer in a direction at an angle of $(45° + \alpha)$ or 75° from north towards east.

Further, $\dfrac{|\mathbf{v}_a|}{\sin\,(180° - 45° - 30°\,)} = \dfrac{400}{\sin\,45°}$

or
$$|\mathbf{v}_a| = \frac{\sin\,105°}{\sin\,45°} \times (400)\text{ km/h}$$
$$= \left(\frac{\cos\,15°}{\sin\,45°}\right)(400)\text{ km/h}$$
$$= \left(\frac{0.9659}{0.707}\right)(400)\text{ km/h}$$
$$= 546.47\ \text{ km/h}$$

$\therefore$ The time of journey from A to B is

$$t = \frac{AB}{|\mathbf{v}_a|} = \frac{1000}{546.47}\text{ h}$$
$$t = 1.83\text{ h}$$

Ans.

Rain Problems

In these type of problems, we again come across three terms $\mathbf{v}_r$, $\mathbf{v}_m$ and $\mathbf{v}_{rm}$ Here,

$\mathbf{v}_r$ = velocity of rain

$\mathbf{v}_m$ = velocity of man (it may be velocity of cyclist or velocity of motorist also)

and $\mathbf{v}_{rm}$ = velocity of rain with respect to man.

Here, $\mathbf{v}_{rm}$ is the velocity of rain which appears to the man.

So, the man should hold his umbrella in the direction of $\mathbf{v}_{rm}$ or $\mathbf{v}_r - \mathbf{v}_m$, to save him from rain.

Example 6.34 *A man is walking with 3 m/s, due east . Rain is falling vertically downwards with speed 4 m/s. Find the direction in which man should hold his umbrella, so that rain does not wet him.*

Solution As we discussed above, he should hold his umbrella in the direction of $\mathbf{v}_{rm}$ or $\mathbf{v}_r - \mathbf{v}_m$

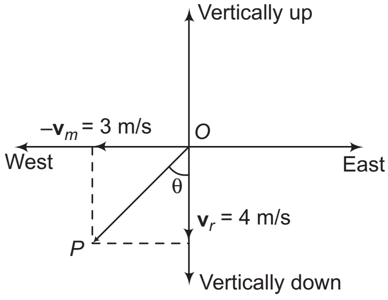


Fig. 6.58

$$\mathbf{OP} = \mathbf{v}_r + (-\mathbf{v}_m) = \mathbf{v}_r - \mathbf{v}_m = \mathbf{v}_{rm}$$

$$\Rightarrow \qquad \tan\theta = \frac{3}{4}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{3}{4}\right) = 37^\circ$$

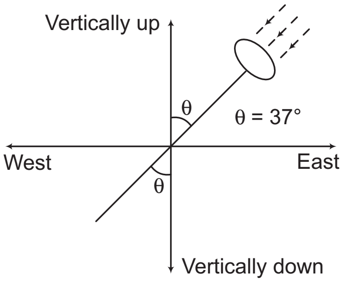


Fig. 6.59

Therefore, man should hold his umbrella at an angle of 37° east of vertical (or 37° from vertical towards east).

Example 6.35 *To a man walking at the rate of* $3\ km/h$ *the rain appears to fall vertically downwards. When he increases his speed to* $6\ km/h$ *it appears to meet him at an angle of* 45° *with vertical. Find the speed of rain.*

Solution Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors in horizontal and vertical directions respectively.

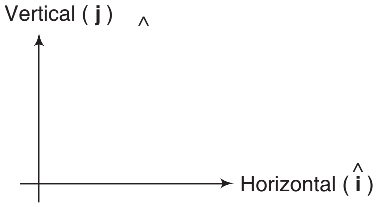


Fig. 6.60

Let velocity of rain $\qquad \mathbf{v}_r = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} \qquad$ ...(i)

Then, speed of rain will be $\qquad |\mathbf{v}_r| = \sqrt{a^2 + b^2} \qquad$ ...(ii)

In the first case,

$$\mathbf{v}_m = \text{ velocity of man} = 3\hat{\mathbf{i}}$$

$$\therefore \qquad \mathbf{v}_{rm} = \mathbf{v}_r - \mathbf{v}_m = (a-3)\,\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

It seems to be in vertical direction.

Hence, $\qquad a - 3 = 0 \quad \text{or} \quad a = 3$

In the second case, $\qquad \mathbf{v}_m = 6\,\hat{\mathbf{i}}$

$$\therefore \qquad \mathbf{v}_{rm} = (a-6)\hat{\mathbf{i}} + b\hat{\mathbf{j}} = -3\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

This seems to be at 45° with vertical. Hence, $|b| = 3$

Therefore, from Eq. (ii) speed of rain is

$$|\mathbf{v}_r| = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2} \text{ km/h}$$ **Ans.**

INTRODUCTORY EXERCISE 6.9

1. Two particles are moving along *x*-axis. Their *x*-coordinate *versus* time graph are as shown below.

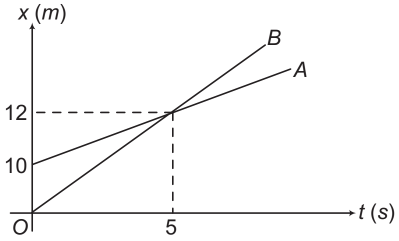


Fig. 6.61

Find velocity of *A* w.r.t. *B*.

2. Two balls *A* and *B* are projected vertically upwards with different velocities. What is the relative acceleration between them?
3. A river 400 m wide is flowing at a rate of 2.0 m/s. A boat is sailing at a velocity of 10.0 m/s with respect to the water in a direction perpendicular to the river.
   (a) Find the time taken by the boat to reach the opposite bank.
   (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?
4. An aeroplane has to go from a point *A* to another point *B*, 500 km away due 30° east of north. Wind is blowing due north at a speed of 20 m/s. The steering-speed of the plane is 150 m/s. (a) Find the direction in which the pilot should head the plane to reach the point *B.* (b) Find the time taken by the plane to go from *A* to *B*.
5. A man crosses a river in a boat. If he cross the river in minimum time he takes 10 min with a drift 120 m. If he crosses the river taking shortest path, he takes 12.5 min, find
   (a) width of the river
   (b) velocity of the boat with respect to water
   (c) speed of the current
6. A river is 20 m wide. River speed is 3 m/s. A boat starts with velocity $2\sqrt{2}$ m/s at angle 45° from the river current (relative to river)
   (a) Find the time taken by the boat to reach the opposite bank.
   (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?

Final Touch Points

1. If a particle is just dropped from a moving body then just after dropping, velocity of the particle (not acceleration) is equal to the velocity of the moving body at that instant.

   For example, if a stone is dropped from a moving train with velocity 20 m/s, then initial velocity of the stone is 20 m/s horizontal in the direction of motion of train. But, after dropping it comes under gravity. Therefore, its acceleration is g downwards.

2. If y (may be velocity, acceleration etc.) is a function of time or $y = f(t)$ and we want to find the average value of y between a time interval of t_1 and t_2. Then,

   $$< y >_{t_1 \text{ to } t_2} = \text{average value of } y \text{ between } t_1 \text{ and } t_2$$

   $$= \frac{\int_{t_1}^{t_2} f(t)\,dt}{t_2 - t_1} \quad \text{or} \quad < y >_{t_1 \text{ to } t_2} = \frac{\int_{t_1}^{t_2} f(t)\,dt}{t_2 - t_1}$$

   If $f(t)$ is a linear function of t, then $y_{\text{av}} = \frac{y_f + y_i}{2}$

   Here, y_f = final value of y and y_i = initial value of y

   At the same time, we should not forget that

   $$v_{\text{av}} = \frac{\text{total displacement}}{\text{total time}} \quad \text{and} \quad a_{\text{av}} = \frac{\text{change in velocity}}{\text{total time}}$$

   Example *In one dimensional uniformly accelerated motion, find average velocity between a time interval from $t = 0$ to $t = t$.*

   Solution We can solve this problem by three methods.

   Method 1. $v = u + at$

   $$\therefore \qquad < v >_{0-t} = \frac{\int_0^t (u + at)\,dt}{t - 0} = u + \frac{1}{2}at$$

   Method 2. Since, v is a linear function of time, we can write

   $$v_{\text{av}} = \frac{v_f + v_i}{2} = \frac{(u + at) + u}{2} = u + \frac{1}{2}at$$

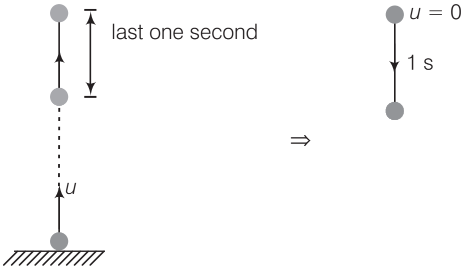
   Method 3. $v_{\text{av}} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{ut + \frac{1}{2}at^2}{t} = u + \frac{1}{2}at$

3. A particle is thrown upwards with velocity u. Suppose it takes time t to reach its highest point, then distance travelled in last second is independent of u.

   This is because this distance is equal to the distance travelled in first second of a freely falling object. Thus,

   $$s = \frac{1}{2}g \times (1)^2 = \frac{1}{2} \times 10 \times 1 = 5 \text{ m}$$

   Exercise: *A particle is thrown upwards with velocity u (> 20 m/s). Prove that distance travelled in last 2 s is 20 m.*



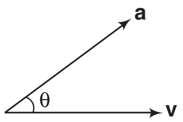
4. Angle between velocity vector **v** and acceleration vector **a** decides whether the speed of particle is increasing, decreasing or constant.

   Speed increases, if $0° \leq \theta < 90°$

Speed decreases, if $90° < \theta \le 180°$

Speed is constant, if $\theta = 90°$

The angle θ between **v** and **a** can be obtained by the relation,



$$\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{a}}{va}\right)$$

Exercise: *Prove that speed of a particle increases if dot product of* **v** *and* **a** *is positive, speed decreases, if the dot product is negative and speed remains constant if dot product is zero.*

5. The magnitude of instantaneous velocity is called the instantaneous speed, i.e.

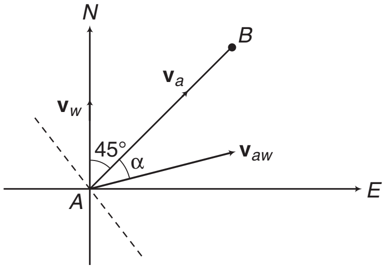
$$v = |\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|$$

Speed is not equal to $\frac{dr}{dt}$, i.e. $v \ne \frac{dr}{dt}$

where, r is the modulus of radius vector **r** because in general $|d\mathbf{r}| \ne dr$. For example, when **r** changes only in direction, i.e. if a point moves in a circle, then $r =$ constant, $dr = 0$ but $|d\mathbf{r}| \ne 0$.

6. Suppose **C** is a vector sum of two vectors **A** and **B** and the direction of **C** is given to us (along PQ), then **A** + **B** should be along PQ or sum of components of **A** and **B** perpendicular to line PQ should be zero.

For instance, in example 6.33, $\mathbf{v}_a$ has to be along AB and we know that $\mathbf{v}_a = \mathbf{v}_{aw} + \mathbf{v}_w$. Therefore, sum of components of $\mathbf{v}_{aw}$ and $\mathbf{v}_w$ perpendicular to line AB (shown as dotted) should be zero.



or $$|\mathbf{v}_{aw}|\sin\alpha = |\mathbf{v}_w|\sin 45°$$

or $$\sin\alpha = \frac{|\mathbf{v}_w|}{|\mathbf{v}_{aw}|}\sin 45°$$

$$= \left(\frac{200\sqrt{2}}{400}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

∴ $$\alpha = 30°$$

Now, $$|\mathbf{v}_a| = |\mathbf{v}_{aw}|\cos\alpha + |\mathbf{v}_w|\cos 45°$$

$$= (400)\cos 30° + (200\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)$$

$$= (400)\frac{\sqrt{3}}{2} + 200 = 346.47 + 200$$

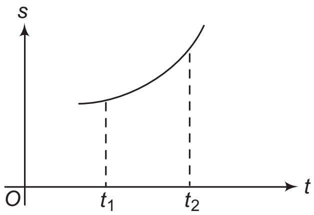
$$= 546.47 \text{ km/h}$$

∴ Time of journey from A to B will be

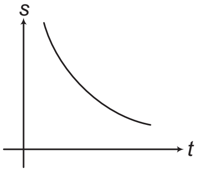
$$t = \frac{AB}{|\mathbf{v}_a|} = \frac{1000}{546.47} = 1.83 \text{ h.}$$

7. From the given *s*-*t* graph, we can find sign of velocity and acceleration.

For example, in the given graph slope at t_1 and t_2 both are positive. Therefore, v_{t_1} and v_{t_2} are positive. Further, slope at $t_2 >$ Slope at t_1. Therefore $v_{t_2} > v_{t_1}$. Hence acceleration of the particle is also positive.



Exercise: *In the given s-t graph, find signs of v and a.*



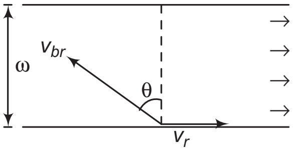
Ans. Negative, positive

8. Shortest path in river boat problems

Path length travelled by the boatman when he reaches the opposite shore is

$$s = \sqrt{\omega^2 + x^2}$$

Here, ω = width of river is constant. So, for s to be minimum modulus of x (drift) should be minimum. Now, two cases are possible.



When $v_r < v_{br}$: In this case $x = 0$, when $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$

or $$\boxed{s_{\min} = \omega \quad \text{at} \quad \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)}$$

When $v_r > v_{br}$: In this case x is minimum, where $\frac{dx}{d\theta} = 0$

or $$\frac{d}{d\theta}\left\{\frac{\omega}{v_{br}\cos\theta}(v_r - v_{br}\sin\theta)\right\} = 0$$

or $$-v_{br}\cos^2\theta - (v_r - v_{br}\sin\theta)(-\sin\theta) = 0$$

or $$-v_{br} + v_r\sin\theta = 0 \quad \text{or} \quad \theta = \sin^{-1}\left(\frac{v_{br}}{v_r}\right)$$

Now, at this angle we can find $x_{\min}$ and then $s_{\min}$ which comes out to be

$$\boxed{s_{\min} = \omega\left(\frac{v_r}{v_{br}}\right) \quad \text{at} \quad \theta = \sin^{-1}\left(\frac{v_{br}}{v_r}\right)}$$

Solved Examples

TYPED PROBLEMS

Type 1. *Collision of two particles or overtaking of one particle by the other particle*

Concept

(i) If two particles start from the same point and they collide, then their displacements are same or

$$S_1 = S_2$$

If they start from different points, then

$$S_1 \neq S_2$$

(ii) If they start their journeys simultaneously, then their time of journeys are same or

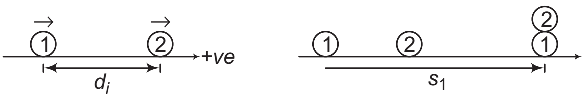
$$t_1 = t_2 = t \quad \text{(say)}$$

otherwise their time of journeys are different

$$t_1 \neq t_2$$

How to Solve? (In 1-D motion)

- Take one direction as positive and the other as negative.
- Without considering, the given directions of their initial velocities and accelerations assume that both particles are moving along positive direction.



Assume this

At the time of collision/overtaking

From the second figure, we can see that particle-1 (which is behind the particle-2) will collide (or overtake) particle-2 if it travels an extra distance d_i (= initial distance between them) or

$$S_1 = S_2 + d_i \qquad \text{...(i)}$$

If motion is uniformly accelerated, then for S we can write

$$S = ut + \frac{1}{2}at^2$$

Now, u and a are vector quantities so, in Eq. (i) we will substitute them with sign.

By putting proper values in Eq. (i) we can find their time of collision. Same method can be applied in vertical motion also.

Note *If two trains of length l_1 and l_2 cross each other or overtake each other (moving on two parallel tracks). Then, the equation will be,*

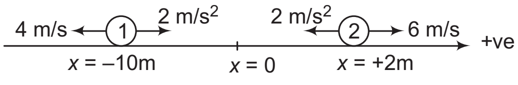
$$S_1 = S_2 + (l_1 + l_2)$$

Example 1 *Two particles are moving along x-axis. Particle-1 starts from* $x = -10$ *m with velocity 4 m/s along negative x-direction and acceleration 2 m/s*^{2} *along positive x-direction. Particle-2 starts from* $x = +2$ *m with velocity 6 m/s along positive x-direction and acceleration 2 m/s*^{2} *along negative x-direction.*

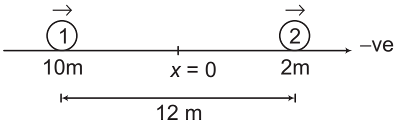
(a) Find the time when they collide.

(b) Find the x-coordinate where they collide. Both start simultaneously.

Solution (a)



Given values



Assume this

Particle-1 is behind the particle-2 at a distance of 12 m. So particle-1 will collide particle-2, if

$$S_1 = S_2 + 12 \quad \Rightarrow \quad \therefore \quad u_1 t + \frac{1}{2} a_1 t^2 = u_2 t + \frac{1}{2} a_2 t^2 + 12$$

But now we will substitute the values of u_1, u_2, a_1 and a_2 with sign

$$\therefore \qquad (-4)t + \frac{1}{2}(+2)t^2 = (+6)t + \frac{1}{2}(-2)t^2 + 12$$

Solving this equation, we get positive value of time,

$$t = 6 \text{ s}$$ **Ans.**

(b) At the time of collision, $S_1 = u_1 t + \frac{1}{2} a_1 t^2 = (-4)(6) + \frac{1}{2}(+2)(6)^2 = +12$ m

At the time of collision, x-coordinate of particle - 1 :

$$x_1 = (\text{Initial } x\text{-coordinate of particle-1}) + S_1$$
$$= -10 + 12 = +2 \text{ m}$$

Since, they collide at the same point. Hence,

$$x_2 = x_1 = +2 \text{ m}$$ **Ans.**

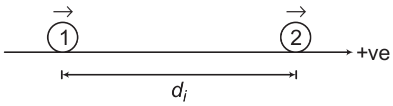
Note *This was also the starting x-coordinate of particle-2.*

Exercise: Find their velocities at the time of collision.

Ans. $v_1 = +8$ m/s, $v_2 = -6$ m/s

Type 2. *To find minimum distance between two particles moving in a straight line*

Concept



If two particles are moving along positive directions as shown in figure.

From the general experience, we can understand that distance between them will increase if $v_2 > v_1$ and distance between them will decrease if $v_1 > v_2$.

Therefore, in most of the cases at minimum distance, $v_1 = v_2$

How to Solve?

- By putting, $v_1 = v_2$ or,

$$u_1 + a_1t = u_2 + a_2t \qquad \text{(if } a = \text{constant)}$$

find the time when they are closest to each other.
- In this time, particle-1 should travel some extra distance and whatever is the extra displacement (of particle-1), that will be subtracted from the initial distance between them to get the minimum distance.

$\therefore$ $$d_{\min} = d_i - \Delta S = d_i - (S_1 - S_2)$$

For S, we can use $ut + \frac{1}{2}at^2$ if acceleration is constant.

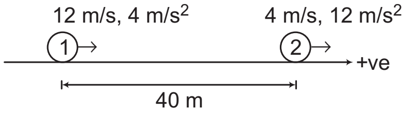
Note *If $S_2 \geq S_1$, then $d_{\min} = d_i$*

Example 2 *Two particles are moving along x-axis. Particle-1 is 40 m behind particle-2. Particle-1 starts with velocity 12 m/s and acceleration 4 m/s^2 both in positive x-direction. Particle-2 starts with velocity 4 m/s and acceleration 12 m/s^2 also in positive x-direction. Find*

(a) the time when distance between them is minimum.

(b) the minimum distance between them.

Solution



(a) As discussed above, distance between them is minimum, when

$$v_1 = v_2$$

or $$u_1 + a_1t = u_2 + a_2t$$

Substituting the values with sign we have,

$$(+12) + (4)t = (+4) + (12)\,t$$

$\therefore$ $$t = 1\text{ s}$$ **Ans.**

(b) **In 1 sec**

$$S_1 = u_1t + \frac{1}{2}a_1t^2$$

$$= 12 \times 1 + \frac{1}{2} \times 4 \times (1)^2 = 14\text{ m}$$

and $$S_2 = u_2t + \frac{1}{2}a_2t^2$$

$$= 4 \times 1 + \frac{1}{2} \times 12 \times (1)^2$$

$$= 10\text{ m}$$

Extra displacement of particle-1 with respect to 2 is

$$\Delta S = S_1 - S_2 = 14 - 10 = 4\text{ m}$$

$\therefore$ Minimum distance between them

$$= d_i - \Delta S = 40 - 4$$

$$= 36\text{ m}$$ **Ans.**

Type 3. *To find trajectory of a particle*

In this type, a particle will be moving in x-y plane. Its x and y co-ordinates as function of time will be given in the question and we have to find trajectory (or x-y relation) of the particle.

How to Solve?

- From the given x and y co-ordinates (as function of time) just eliminate t and find x-y relation. This is a general method which can be applied anywhere in whole physics.

Example 3 *A particle is moving in x-y plane with its x and y co-ordinates varying with time as,* $x = 2t$ *and* $y = 10t - 16\,t^2$. *Find trajectory of the particle.*

Solution Given, $x = 2t$

$$\Rightarrow \qquad t = \frac{x}{2}$$

Now,

$$y = 10\,t - 16\,t^2$$

Substituting value of t in this equation we have,

$$y = 10\left(\frac{x}{2}\right) - 16\left(\frac{x}{2}\right)^2$$

or

$$y = 5x - 4x^2$$

This is the required equation of trajectory of the particle. This is a quadratic equation. Hence, the path of the particle is a parabola.

Type 4. *Two dimensional motion by component method.*

Concept

There are two methods of solving a two (or three) dimensional motion problems. In the first method, we use proper vector method. For example, we will use,

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$ etc. if $\mathbf{a}$ = constant and, $\mathbf{v} = \dfrac{d\mathbf{s}}{dt}$ etc. if $\mathbf{a} \neq$ constant

In the second method, we find the components of all vector quantities along x, y and z-directions. Then, deal different axis separately as one dimension by assigning proper signs to all vector quantities. While dealing x-direction, we don't have to bother about y and z-directions.

Example 4 *A particle is moving in x-y plane. Its initial velocity and acceleration are* $\mathbf{u} = (4\hat{\mathbf{i}} + 8\hat{\mathbf{j}})$ m/s *and* $\mathbf{a} = (2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$ m/s^2. *Find*

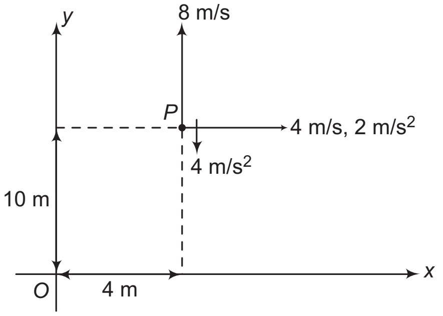
(a) the time when the particle will cross the x-axis.

(b) x-coordinate of particle at this instant.

(c) velocity of the particle at this instant.

Initial coordinates of particle are $(4m, 10m)$.

Solution



Particle starts from point P. Components of its initial velocity and acceleration are as shown in figure.

(a) At the time of crossing the x-axis, its y-coordinate should be zero or its y-displacement (w.r.t initial point P) is –10 m.

Using the equation,
$$s_y = u_y t + \frac{1}{2} a_y t^2$$
$$-10 = 8t - \frac{1}{2} \times 4 \times t^2$$

Solving this equation, we get positive value of time,
$$t = 5 \text{ s}$$

(b) x-coordinate of particle at time t :

x = initial x-coordinate + displacement along x-axis or $x = x_i + s_x$ (at time t)
$$= x_i + u_x t + \frac{1}{2} a_x t^2$$

Substituting the proper values, we have,
$$x = 4 + (4 \times 5) + \frac{1}{2} \times 2 \times (5)^2 = 49 \text{ m}$$ **Ans.**

(c) Since, given acceleration is constant, so we can use,
$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$\therefore \quad \mathbf{v} = (4\hat{\mathbf{i}} + 8\hat{\mathbf{j}}) + (2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})(5)$$
$$= (14\hat{\mathbf{i}} - 12\hat{\mathbf{j}}) \text{ m/s}$$

Type 5. *To convert given v-t graph into s-t graph (For a = 0 or a = constant)*

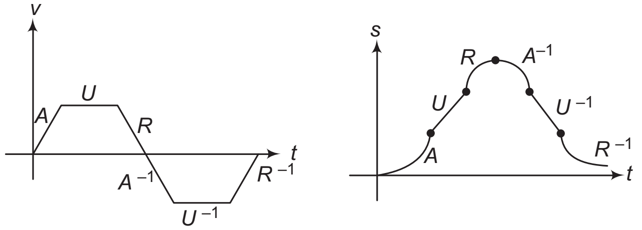
Concept

(i) If we integrate velocity, we get displacement. Therefore, the method discussed in this type is a general method, which can be applied in all those problems where we get the result after integration.

For example
$$v\text{-}t \longrightarrow s\text{-}t$$
$$a\text{-}t \longrightarrow v\text{-}t$$
$$P\text{-}t \longrightarrow F\text{-}t$$

Here, P = linear momentum and F is force ($F = \frac{dP}{dt}$ or $dP = Fdt$).

(ii) For zero or constant acceleration, we can classify the motion into six types. Corresponding v-t and s-t graphs are as shown below.



(iii) The explanation of these six motions is as under

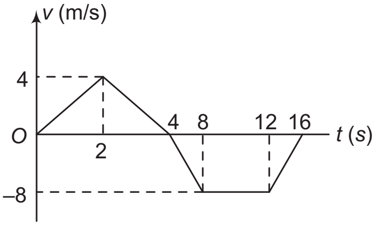
Motion type	About the motion	Velocity or Slope of *s-t* graph ($v = ds/dt$)
A	Accelerated in positive direction	positive and increasing
U	Uniform in positive direction	positive and constant
R	Retarded in positive direction	positive and decreasing
A^{-1}	Accelerated in negative direction	negative and increasing
U^{-1}	Uniform in negative direction	negative and constant
R^{-1}	Retarded in negative direction	negative and decreasing

(iv) In A, U and R motions, velocity is positive (above t-axis). Therefore, body is moving along positive direction. In A^{-1}, U^{-1} and R^{-1} motions, velocity is negative (below t-axis). Therefore, body is moving along negative direction.

How to Solve?

- Mark A, U, R, A^{-1}, U^{-1} or R^{-1} in the given v-t graph for different time intervals.
- Calculate area (= displacement) under v-t graph for different time intervals.
- Plot s-t graph according to their shape of A, U, R etc. motions.
- Keep on adding area for further displacements.

Example 5 *Velocity-time graph* of a particle moving along x-axis is as shown below.



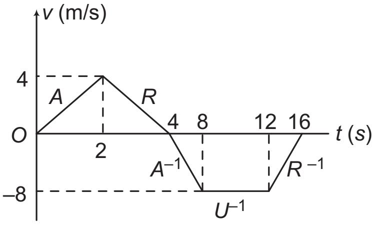
At time $t = 0$, x-coordinate of the particle is $x = 10\,m$.

(a) Plot x-coordinate versus time graph.

(b) Find average velocity and average speed of the particle during the complete journey.

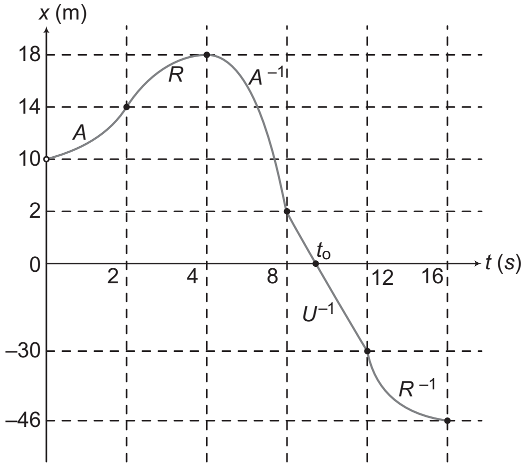
(c) Find average acceleration of the particle between the time interval from $t = 2\,s$ to $t = 8\,s$.

Solution (a) Let us first mark A, U, R etc. in the given v-t diagram and calculate their area (= displacement) in different time intervals.



Time interval	Area or displacement	Final x-coordinate at the end of intervals $x = x_i + s$
0 – 2 s	+ 4 m	10 + 4 = 14 m
2 s – 4 s	+ 4 m	14 + 4 = 18 m
4 s – 8 s	– 16 m	18 – 16 = 2 m
8 s – 12 s	– 32 m	2 – 32 = – 30 m
12 s – 16 s	– 16 m	–30 – 16 = – 46 m

Corresponding x-t graph is as shown below.



Exercise : *Find the time t_0 when x-coordinate of the particle is zero.*

Ans 8.25 s

(b) Total displacement = 4 + 4 – 16 – 32 – 16 = – 56 m

This is also equal to $x_f - x_i = -46 - 10 = -56$ m

Total distance = 4 + 4 + 16 + 32 + 16 = 72 m

Total time = 16 s

Now,

$$\text{average velocity} = \frac{\text{total diaplacement}}{\text{total time}}$$

$$= -\frac{56}{16} = -3.5 \text{ m/s}$$ **Ans.**

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

$$= \frac{72}{16} = 4.5 \text{ m/s} \qquad \textbf{Ans.}$$

(c) Average acceleration $= \dfrac{\Delta v}{\Delta t}$

$$= \frac{v_f - v_i}{\Delta t} = \frac{v_{8\,\text{sec}} - v_{2\text{sec}}}{8-2}$$

$$= \frac{-8-4}{6} = -2 \text{ m/s}^2 \qquad \textbf{Ans.}$$

Type 6. *General method of conversion of graph*

Concept

(i) In some cases, one graph can be converted into the other graph just by finding slope of the given graph. But this method is helpful when different segments of the given graph are straight lines.

For example,

(a) Given s-t graph can be converted into the v-t graph from the slope of s-t graph as

$$v = \frac{ds}{dt} = \text{slope of } s\text{-}t \text{ graph}$$

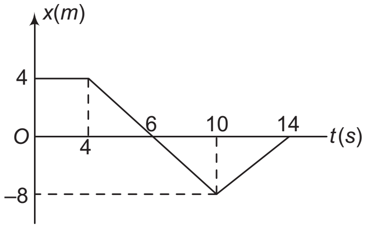
(b) Given v-t graph can be converted into the a-t graph from the slope of v-t graph, as

$$a = \frac{dv}{dt} = \text{slope of } v\text{-}t \text{ graph}$$

(ii) In few cases, we have to convert given y-x graph into z-x graph. For example, suppose we have to convert v-s graph into a-s graph.

In such cases, first you make v-s equation (if it is straight line graph) from the given v-s graph. Then, with the help of this v-s equation and some standard equations (like $a = v \cdot \dfrac{dv}{ds}$) make a-s equation and now draw a-s graph corresponding to this a-s equation.

Example 6 *A particle is moving along x-axis. Its x-coordinate versus time graph is as shown below.*

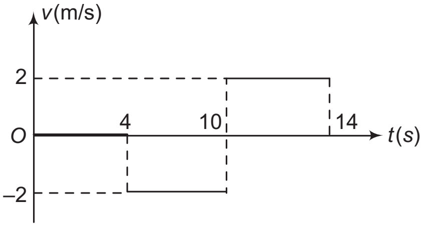


Plot v-t graph corresponding to this.

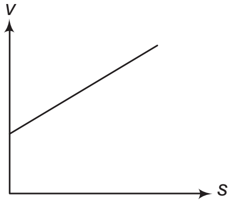
Solution $v = \frac{dx}{dt}$ = slope of x-t graph. Slope for different time intervals is given in following table.

Time interval	Slope of x-t graph (= v)
0 – 4 s	0
4 s – 10 s	– 2 m/s
10 s – 14 s	+ 2 m/s

v-t graph corresponding to this table is as shown below



Example 7 *Corresponding to given v-s graph of a particle moving in a straight line, plot a-s graph.*



Solution The given v-s graph is a straight line with positive slope (say m) and positive intercept (say c). Therefore, v-s equation is

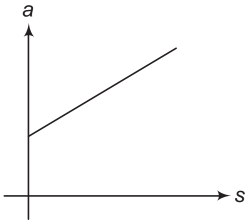
$$v = ms + c$$

$$\Rightarrow \qquad \frac{dv}{ds} = m$$

Now,
$$a = v \cdot \frac{dv}{ds} = (ms + c)(m)$$

$$a = m^2 s + mc$$

a-s equation is a linear equation. Therefore, a-s graph is also a straight line with positive slope ($= m^2$) and positive intercept ($= mc$). a-s graph is as shown below.



Type 7. *Based on difference between distance and displacement*

Concept

There is no direct formula for calculation of distance. In the formula,

$$s = ut + \frac{1}{2}at^2$$

$$s = \text{displacement, not the distance}$$

So, you will have to convert the given distance into proper displacement and then apply the above equation.

Example 8 *A particle is moving along x-axis. At time $t = 0$, its x-coordinate is $x = -4$ m. Its velocity-time equation is $v = 8 - 2t$ where, v is in m/s and t in seconds.*

(a) At how many times, particle is at a distance of 8 m from the origin?

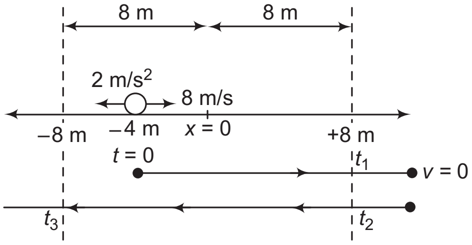
(b) Find those times.

Solution (a) Comparing the given v-t equation with $v = u + at$. We have,

$$u = 8 \text{ m/s and}$$

$$a = -2 \text{ m/s}^2 = \text{constant}$$

Now, motion of the particle is as shown below.



Now, 8 m distance from origin will be at two coordinates $x = 8$ m and $x = -8$ m. From the diagram, we can see that particle will cross these two points three times, t_1, t_2 and t_3.

(b) $\boldsymbol{t_1}$ **and** $\boldsymbol{t_2}$ **:** At $x = 8$ m, displacement from the starting point is

$$s = x_f - x_i = 8 - (-4) = 12 \text{ m}$$

Substituting in $s = ut + \frac{1}{2}at^2$, we have

$$12 = 8t - \frac{1}{2} \times 2 \times t^2$$

Solving this equation, we get

$$\text{smaller time } t_1 = 2 \text{ s}$$ **Ans.**

$$\text{and larger time } t_2 = 6 \text{ s}$$ **Ans.**

$\boldsymbol{t_3}$ **:** At $x = -8$ m, displacement from the starting point is

$$s = x_f - x_i = -8 - (-4) = -4 \text{ m}$$

Substituting in $s = ut + \frac{1}{2}at^2$, we have $\quad -4 = 8t - \frac{1}{2} \times 2 \times t^2$

Solving this equation, we get the positive time,

$$t_3 = 8.47 \text{ s}$$ **Ans.**

Miscellaneous Examples

Example 9 *A rocket is fired vertically upwards with a net acceleration of* $4\ m/s^2$ *and initial velocity zero. After 5 s its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then, it accelerates downwards with acceleration g and return back to ground. Plot velocity-time and displacement-time graphs for the complete journey. Take* $g = 10\ m/s^2$.

Solution In the graphs,

$$v_A = at_{OA} = (4)(5) = 20 \text{ m/s}$$

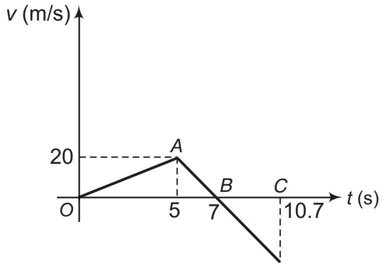
$$v_B = 0 = v_A - gt_{AB}$$

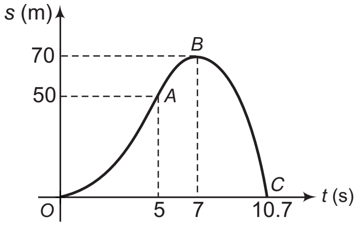
$$\therefore \quad t_{AB} = \frac{v_A}{g} = \frac{20}{10} = 2 \text{ s}$$

$$\therefore \quad t_{OAB} = (5 + 2) \text{ s} = 7 \text{ s}$$

Now, s_{OAB} = area under v-t graph between 0 to 7 s

$$= \frac{1}{2}(7)(20) = 70 \text{ m}$$





Further, $$|s_{OAB}| = |s_{BC}| = \frac{1}{2} gt_{BC}^2$$

$$\therefore \quad 70 = \frac{1}{2}(10)\, t_{BC}^2$$

$$\therefore \quad t_{BC} = \sqrt{14} = 3.7 \text{ s}$$

$$\therefore \quad t_{OABC} = 7 + 3.7 = 10.7 \text{ s}$$

Also, s_{OA} = area under v-t graph between OA

$$= \frac{1}{2}(5)(20) = 50 \text{ m}$$

Example 10 *An open lift is moving upwards with velocity* $10\ m/s$. *It has an upward acceleration of* $2\ m/s^2$. *A ball is projected upwards with velocity* $20\ m/s$ *relative to ground. Find*

(a) time when ball again meets the lift

(b) displacement of lift and ball at that instant.

(c) distance travelled by the ball upto that instant.

Take $g = 10\ m/s^2$

Solution (a) At the time when ball again meets the lift,

$$s_L = s_B$$

$$\therefore \qquad 10\,t + \frac{1}{2} \times 2 \times t^2 = 20\,t - \frac{1}{2} \times 10\,t^2$$

Solving this equation, we get

$$t = 0 \quad \text{and} \quad t = \frac{5}{3}\text{ s}$$

$\therefore$ Ball will again meet the lift after $\frac{5}{3}$ s.

(b) At this instant $s_L = s_B = 10 \times \frac{5}{3} + \frac{1}{2} \times 2 \times \left(\frac{5}{3}\right)^2 = \frac{175}{9}\text{ m} = 19.4\text{ m}$

(c) For the ball u is antiparallel to a. Therefore, we will first find t_0, the time when its velocity becomes zero.

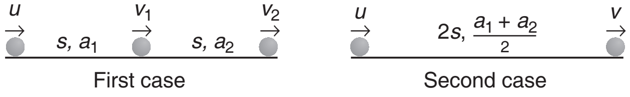
$$t_0 = \left|\frac{u}{a}\right| = \frac{20}{10} = 2\text{ s}$$

As $t\left(= \frac{5}{3}\text{ s}\right) < t_0$, distance and displacement are equal

or $$d = 19.4\text{ m}$$

Example 11 *A particle starts with an initial velocity and passes successively over the two halves of a given distance with constant accelerations* a_1 *and* a_2 *respectively. Show that the final velocity is the same as if the whole distance is covered with a uniform acceleration* $\frac{(a_1 + a_2)}{2}$.

Solution



First case

Second case

In the first case,

$$v_1^2 = u^2 + 2a_1 s \qquad \dots \text{(i)}$$

$$v_2^2 = v_1^2 + 2a_2 s \qquad \dots \text{(ii)}$$

Adding Eqs. (i) and (ii), we have

$$v_2^2 = u^2 + 2\left(\frac{a_1 + a_2}{2}\right)(2s) \qquad \dots \text{(iii)}$$

In the second case,

$$v^2 = u^2 + 2\left(\frac{a_1 + a_2}{2}\right)(2s) \qquad \dots \text{(iv)}$$

From Eqs. (iii) and (iv), we can see that

$$v_2 = v$$

Hence proved.

Example 12 *In a car race, car A takes a time t less than car B at the finish and passes the finishing point with speed v more than that of the car B. Assuming that both the cars start from rest and travel with constant acceleration* a_1 *and* a_2 *respectively. Show that* $v = \sqrt{a_1 a_2}\, t$.

Solution Let A takes t_1 second, then according to the given problem B will take $(t_1 + t)$ seconds. Further, let v_1 be the velocity of B at finishing point, then velocity of A will be $(v_1 + v)$. Writing equations of motion for A and B.

$$v_1 + v = a_1 t_1 \qquad \text{... (i)}$$

and,

$$v_1 = a_2(t_1 + t) \qquad \text{... (ii)}$$

From these two equations, we get

$$v = (a_1 - a_2)\, t_1 - a_2 t \qquad \text{... (iii)}$$

Total distance travelled by both the cars is equal.

or $$s_A = s_B$$

or $$\frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_2 (t_1 + t)^2$$

or $$t_1 = \frac{\sqrt{a_2}\, t}{\sqrt{a_1} - \sqrt{a_2}}$$

Substituting this value of t_1 in Eq. (iii), we get the desired result

$$v = (\sqrt{a_1 a_2})\, t$$

Example 13 *An open elevator is ascending with constant speed* $v = 10$ *m/s. A ball is thrown vertically up by a boy on the lift when he is at a height* $h = 10$ *m from the ground. The velocity of projection is* $v = 30$ *m/s with respect to elevator. Find*

(a) the maximum height attained by the ball.

(b) the time taken by the ball to meet the elevator again.

(c) time taken by the ball to reach the ground after crossing the elevator.

Solution (a) Absolute velocity of ball = 40 m/s (upwards)

$\therefore$ $$h_{\max} = h_i + h_f$$

Here, h_i = initial height = 10 m

and h_f = further height attained by ball

$$= \frac{u^2}{2g} = \frac{(40)^2}{2 \times 10} = 80 \text{ m}$$

$\therefore$ $$h_{\max} = (10 + 80) \text{ m} = 90 \text{ m} \qquad \textbf{Ans.}$$

(b) The ball will meet the elevator again when displacement of lift = displacement of ball

or $$10 \times t = 40 \times t - \frac{1}{2} \times 10 \times t^2 \quad \text{or} \quad t = 6 \text{ s} \qquad \textbf{Ans.}$$

(c) Let t_0 be the total time taken by the ball to reach the ground. Then,

$$-10 = 40 \times t_0 - \frac{1}{2} \times 10 \times t_0^2$$

Solving this equation we get, $t_0 = 8.24$ s

Therefore, time taken by the ball to reach the ground after crossing the elevator,

$$= (t_0 - t) = 2.24 \text{ s}$$

Example 14 *From an elevated point A, a stone is projected vertically upwards. When the stone reaches a distance h below A, its velocity is double of what it was at a height h above A. Show that the greatest height attained by the stone is* $\frac{5}{3}h$.

Solution Let u be the velocity with which the stone is projected vertically upwards.

Given that, $$v_{-h} = 2v_h$$

or $$(v_{-h})^2 = 4v_h^2$$

∴ $$u^2 - 2g(-h) = 4(u^2 - 2gh)$$

∴ $$u^2 = \frac{10gh}{3}$$

Now, $$h_{\max} = \frac{u^2}{2g} = \frac{5h}{3}$$ **Hence proved.**

Example 15 *Velocity of a particle moving in a straight line varies with its displacement as* $v = (\sqrt{4+4s})$ *m/s. Displacement of particle at time* $t = 0$ *is* $s = 0$. *Find displacement of particle at time* $t = 2$ *s.*

Solution Squaring the given equation, we get

$$v^2 = 4 + 4s$$

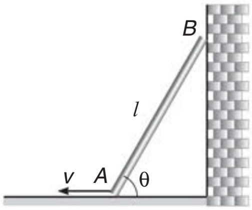
Now, comparing it with $v^2 = u^2 + 2as$, we get

$$u = 2 \text{ m/s} \quad \text{and} \quad a = 2 \text{ m/s}^2$$

∴ Displacement at $t = 2\,\text{s}$ is

$$s = ut + \frac{1}{2}at^2 \quad \text{or} \quad s = (2)(2) + \frac{1}{2}(2)(2)^2 \quad \text{or} \quad s = 8\text{ m}$$ **Ans.**

Example 16 *Figure shows a rod of length l resting on a wall and the floor. Its lower end A is pulled towards left with a constant velocity v. Find the velocity of the other end B downward when the rod makes an angle* θ *with the horizontal.*



Solution In such type of problems, when velocity of one part of a body is given and that of other is required, we first find the relation between the two displacements, then differentiate them with respect to time. Here, if the distance from the corner to the point A is x and that up to B is y. Then,

$$v = \frac{dx}{dt}$$

and $$v_B = -\frac{dy}{dt}$$ (– sign denotes that y is decreasing)

Further, $$x^2 + y^2 = l^2$$

Differentiating with respect to time t

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$xv = yv_B$$

$$v_B = \frac{x}{y}v = v\cot\theta$$ **Ans.**

Example 17 A particle is moving in a straight line with constant acceleration. If x, y and z be the distances described by a particle during the pth, qth and rth second respectively, prove that

$$(q-r)x+(r-p)y+(p-q)z=0$$

Solution As $$s_t = u + at - \frac{1}{2}a = u + \frac{a}{2}(2t-1)$$

$\therefore$ $$x = u + \frac{a}{2}(2p-1) \quad \text{...(i)}$$

$$y = u + \frac{a}{2}(2q-1) \quad \text{...(ii)}$$

$$z = u + \frac{a}{2}(2r-1) \quad \text{...(iii)}$$

Subtracting Eq. (iii) from Eq. (ii), $$y-z=\frac{a}{2}(2q-2r) \quad \text{or} \quad q-r=\frac{y-z}{a}$$

or $$(q-r)x=\frac{1}{a}(yx-zx) \quad \text{...(iv)}$$

Similarly, we can show that

$$(r-p)y=\frac{1}{a}(zy-xy) \quad \text{...(v)}$$

and $$(p-q)z=\frac{1}{a}(xz-yz) \quad \text{...(vi)}$$

Adding Eqs. (iv), (v) and (vi), we get $(q-r)x+(r-p)y+(p-q)z=0$

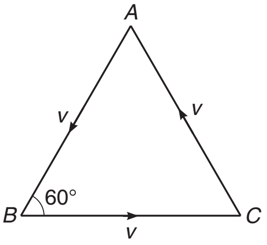
Example 18 *Three particles A, B and C are situated at the vertices of an equilateral triangle ABC of side d at time t = 0. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particles meet each other?*

Solution Velocity of A is v along AB. Velocity of B is along BC. Its component along BA is $v\cos 60° = v/2$. Thus, the separation AB decreases at the rate

$$v+\frac{v}{2}=\frac{3v}{2}$$

Since, this rate is constant, the time taken in reducing the separation AB from d to zero is

$$t=\frac{d}{(3v/2)}=\frac{2d}{3v}$$ **Ans.**



Example 19 *An elevator car whose floor to ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 m/s^2. 2 s after the start, a bolt begins falling from the ceiling of the car. Find*

(a) the time after which bolt hits the floor of the elevator.

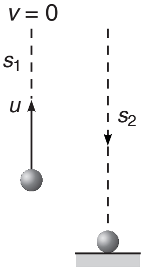
(b) the net displacement and distance travelled by the bolt, with respect to earth. (Take $g = 9.8\ m/s^2$)

Solution (a) If we consider elevator at rest, then relative acceleration of the bolt is

$$a_r = 9.8 + 1.2$$

$$= 11\ \text{m/s}^2 \qquad \text{(downwards)}$$

After 2 s, velocity of lift is $v = at = (1.2)(2) = 2.4$ m/s. Therefore, initial velocity of the bolt is also 2.4 m/s and it gets accelerated with relative acceleration 11 m/s^2. With respect to elevator initial velocity of bolt is zero and it has to travel 2.7 m with 11 m/s^2. Thus, time taken can be directly given as



$$\sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 2.7}{11}}$$

$$= 0.7\ \text{s.} \qquad \textbf{Ans.}$$

(b) Displacement of bolt relative to ground in 0.7 s.

$$s = ut + \frac{1}{2}at^2$$

or $$s = (2.4)(0.7) + \frac{1}{2}(-9.8)(0.7)^2$$

$$s = -0.72\ \text{m} \qquad \textbf{Ans.}$$

Velocity of bolt will become zero after a time

$$t_0 = \frac{u}{g} \qquad (v = u - gt)$$

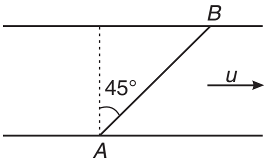
$$= \frac{2.4}{9.8} = 0.245\ \text{s}$$

Therefore, distance travelled by the bolt $= s_1 + s_2 = \dfrac{u^2}{2g} + \dfrac{1}{2}g(t - t_0)^2$

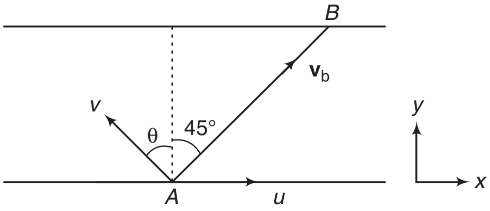
$$= \frac{(2.4)^2}{2 \times 9.8} + \frac{1}{2} \times 9.8\,(0.7 - 0.245)^2$$

$$= 1.3\ \text{m} \qquad \textbf{Ans.}$$

Example 20 *A man wants to reach point B on the opposite bank of a river flowing at a speed as shown in figure. What minimum speed relative to water should the man have so that he can reach point B? In which direction should he swim?*



Solution Let v be the speed of boatman in still water.



Resultant of v and u should be along AB. Components of $\mathbf{v}_b$ (absolute velocity of boatman) along x and y-directions are,

$$v_x = u - v\sin\theta$$

and $$v_y = v\cos\theta$$

Further, $$\tan 45° = \frac{v_y}{v_x}$$

or $$1 = \frac{v\cos\theta}{u - v\sin\theta}$$

$\therefore$ $$v = \frac{u}{\sin\theta + \cos\theta}$$

$$= \frac{u}{\sqrt{2}\sin(\theta + 45°)}$$

v is minimum at, $$\theta + 45° = 90°$$

or $$\theta = 45° \qquad \textbf{Ans.}$$

and $$v_{\min} = \frac{u}{\sqrt{2}} \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

Directions *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Velocity and acceleration of a particle are given as,

$$\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}} \quad \text{and} \quad \mathbf{a} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$$

This is a two dimensional motion with constant acceleration.

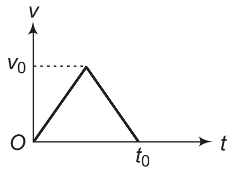
Reason : Velocity and acceleration are two constant vectors.

2. Assertion : Displacement-time graph is a parabola corresponding to straight line velocity-time graph.

Reason : If $v = u + at$ then $s = ut + \frac{1}{2}at^2$

3. Assertion : In v-t graph shown in figure, average velocity in time interval from 0 to t_0 depends only on v_0. It is independent of t_0.

Reason : In the given time interval average velocity is $\frac{v_0}{2}$.



4. Assertion : We know the relation $a = v.\frac{dv}{ds}$. Therefore, if velocity of a particle is zero, then acceleration is also zero.

Reason : In the above equation, a is the instantaneous acceleration.

5. Assertion : Speed of a particle may decrease, even if acceleration is increasing.

Reason : This will happen if acceleration is positive.

6. Assertion : Starting from rest with zero acceleration if acceleration of particle increases at a constant rate of 2 ms^{-3} then velocity should increase at constant rate of $1\,\text{ms}^{-2}$.

Reason : For the given condition.

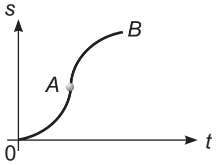
$$\frac{da}{dt} = 2\ \text{ms}^{-3}$$

$$\therefore \qquad a = 2t$$

7. Assertion : Average velocity can't be zero in case of uniform acceleration.

Reason : For average velocity to be zero, a non zero velocity should not remain constant.

8. **Assertion :** In displacement-time graph of a particle as shown in figure, velocity of particle changes its direction at point *A*.



Reason : Sign of slope of *s*-*t* graph decides the direction of velocity.

9. **Assertion :** Displacement-time equation of two particles moving in a straight line are, $s_1 = 2t - 4t^2$ and $s_2 = -2t + 4t^2$. Relative velocity between the two will go on increasing.

Reason : If velocity and acceleration are of same sign then speed will increase.

10. **Assertion :** Acceleration of a moving particle can change its direction without any change in direction of velocity.

Reason : If the direction of change in velocity vector changes, the direction of acceleration vector also changes.

11. **Assertion :** A body is dropped from height h and another body is thrown vertically upwards with a speed $\sqrt{gh}$. They meet at height $h/2$.

Reason : The time taken by both the blocks in reaching the height $h/2$ is same.

12. **Assertion :** Two bodies of unequal masses m_1 and m_2 are dropped from the same height. If the resistance offered by air to the motion of both bodies is the same, the bodies will reach the earth at the same time.

Reason : For equal air resistance, acceleration of fall of masses m_1 and m_2 will be different.

Objective Questions

Single Correct Option

1. A stone is released from a rising balloon accelerating upward with acceleration a. The acceleration of the stone just after the release is

(a) a upward
(b) g downward
(c) $(g - a)$ downward
(d) $(g + a)$ downward

2. A ball is thrown vertically upwards from the ground. If T_1 and T_2 are the respective time taken in going up and coming down, and the air resistance is not ignored, then

(a) $T_1 > T_2$
(b) $T_1 = T_2$
(c) $T_1 < T_2$
(d) nothing can be said

3. The length of a seconds hand in watch is 1 cm. The change in velocity of its tip in 15 s is

(a) zero
(b) $\frac{\pi}{30\sqrt{2}}$ cm/s
(c) $\frac{\pi}{30}$ cm/s
(d) $\frac{\pi\sqrt{2}}{30}$ cm/s

4. When a ball is thrown up vertically with velocity v_0, it reaches a maximum height of h. If one wishes to triple the maximum height then the ball should be thrown with velocity

(a) $\sqrt{3}\, v_0$
(b) $3\, v_0$
(c) $9\, v_0$
(d) $\frac{3}{2}\, v_0$

5. During the first 18 min of a 60 min trip, a car has an average speed of 11 ms^{-1}. What should be the average speed for remaining 42 min so that car is having an average speed of 21 ms^{-1} for the entire trip?

(a) 25.3 ms^{-1}
(b) 29.2 ms^{-1}
(c) 31 ms^{-1}
(d) 35.6 ms^{-1}

6. A particle moves along a straight line. Its position at any instant is given by $x = 32t - \frac{8t^3}{3}$ where x is in metres and t in seconds. Find the acceleration of the particle at the instant when particle is at rest.

(a) – 16 ms^{-2}
(b) – 32 ms^{-2}
(c) 32 ms^{-2}
(d) 16 ms^{-2}

7. The acceleration of a particle is increasing linearly with time t as bt. The particle starts from the origin with an initial velocity v_0. The distance travelled by the particle in time t will be

(a) $v_0 t + \frac{1}{6} bt^3$
(b) $v_0 t + \frac{1}{3} bt^3$
(c) $v_0 t + \frac{1}{3} bt^2$
(d) $v_0 t + \frac{1}{2} bt^2$

8. Water drops fall at regular intervals from a tap 5 m above the ground. The third drop is leaving the tap, the instant the first drop touches the ground. How far above the ground is the second drop at that instant. ($g = 10$ ms^{-2})

(a) 1.25 m
(b) 2.50 m
(c) 3.75 m
(d) 4.00 m

9. A stone is dropped from the top of a tower and one second later, a second stone is thrown vertically downward with a velocity 20 ms^{-1}. The second stone will overtake the first after travelling a distance of ($g = 10$ ms^{-2})

(a) 13 m
(b) 15 m
(c) 11.25 m
(d) 19.5 m

10. A particle moves in the x-y plane with velocity $v_x = 8t - 2$ and $v_y = 2$. If it passes through the point $x = 14$ and $y = 4$ at $t = 2$ s, the equation of the path is

(a) $x = y^2 - y + 2$
(b) $x = y^2 - 2$
(c) $x = y^2 + y - 6$
(d) None of these

11. The horizontal and vertical displacements of a particle moving along a curved line are given by $x = 5t$ and $y = 2t^2 + t$. Time after which its velocity vector makes an angle of 45° with the horizontal is

(a) 0.5 s
(b) 1 s
(c) 2 s
(d) 1.5 s

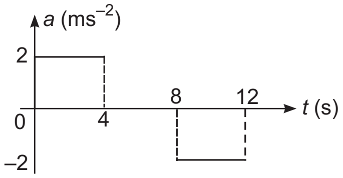
12. A ball is released from the top of a tower of height h metre. It takes T second to reach the ground. What is the position of the ball in $T/3$ second?

(a) $\frac{h}{9}$ metre from the ground
(b) $(7h/9)$ metre from the ground
(c) $(8h/9)$ metre from the ground
(d) $(17h/18)$ metre from the ground

13. An ant is at a corner of a cubical room of side a. The ant can move with a constant speed u. The minimum time taken to reach the farthest corner of the cube is

(a) $\frac{3a}{u}$
(b) $\frac{\sqrt{3}a}{u}$
(c) $\frac{\sqrt{5}a}{u}$
(d) $\frac{(\sqrt{2}+1)a}{u}$

14. A lift starts from rest. Its acceleration is plotted against time. When it comes to rest its height above its starting point is



(a) 20 m (b) 64 m (c) 32 m (d) 36 m

15. A lift performs the first part of its ascent with uniform acceleration a and the remaining with uniform retardation $2a$. If t is the time of ascent, find the depth of the shaft.

(a) $\frac{at^2}{4}$ (b) $\frac{at^2}{3}$ (c) $\frac{at^2}{2}$ (d) $\frac{at^2}{8}$

16. Two objects are moving along the same straight line. They cross a point A with an acceleration a, $2a$ and velocity $2u$, u at time $t = 0$. The distance moved by the object when one overtakes the other is

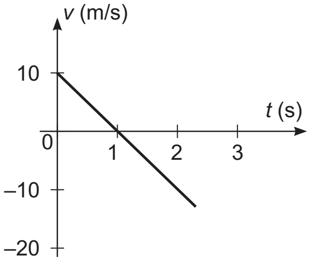
(a) $\frac{6u^2}{a}$ (b) $\frac{2u^2}{a}$ (c) $\frac{4u^2}{a}$ (d) $\frac{8u^2}{a}$

17. A cart is moving horizontally along a straight line with constant speed 30 ms^{-1}. A particle is to be fired vertically upwards from the moving cart in such a way that it returns to the cart at the same point from where it was projected after the cart has moved 80 m. At what speed (relative to the cart) must the projectile be fired? (Take $g = 10$ ms^{-2})

(a) 10 ms^{-1} (b) $10\sqrt{8}$ ms^{-1}
(c) $\frac{40}{3}$ ms^{-1} (d) None of these

18. The figure shows velocity–time graph of a particle moving along a straight line. Identify the correct statement.

(a) The particle starts from the origin
(b) The particle crosses it initial position at $t = 2$ s
(c) The average speed of the particle in the time interval, $0 \le t \le 2$ s is zero
(d) All of the above



19. A ball is thrown vertically upwards from the ground and a student gazing out of the window sees it moving upward past him at 10 ms^{-1}. The window is at 15 m above the ground level. The velocity of ball 3 s after it was projected from the ground is [Take $g = 10$ ms^{-2}]

(a) 10 m/s, up (b) 20 ms^{-1}, up
(c) 20 ms^{-1}, down (d) 10 ms^{-1}, down

20. A body starts moving with a velocity $v_0 = 10$ ms^{-1}. It experiences a retardation equal to $0.2v^2$. Its velocity after 2s is given by

(a) + 2 ms^{-1} (b) + 4 ms^{-1}
(c) – 2 ms^{-1} (d) + 6 ms^{-1}

21. Two trains are moving with velocities $v_1 = 10\ \text{ms}^{-1}$ and $v_2 = 20\ \text{ms}^{-1}$ on the same track in opposite directions. After the application of brakes if their retarding rates are $a_1 = 2\ \text{ms}^{-2}$ and $a_2 = 1\ \text{ms}^{-2}$ respectively, then the minimum distance of separation between the trains to avoid collision is

(a) 150 m	(b) 225 m
(c) 450 m	(d) 300 m

22. Two identical balls are shot upward one after another at an interval of 2s along the same vertical line with same initial velocity of $40\ \text{ms}^{-1}$. The height at which the balls collide is

(a) 50 m	(b) 75 m
(c) 100 m	(d) 125 m

23. A particle is projected vertically upwards and reaches the maximum height H in time T. The height of the particle at any time $t\ (< T)$ will be

(a) $g\,(t - T)^2$	(b) $H - g\,(t - T)^2$
(c) $\frac{1}{2}\,g\,(t - T)^2$	(d) $H - \frac{1}{2}\,g\,(T - t)^2$

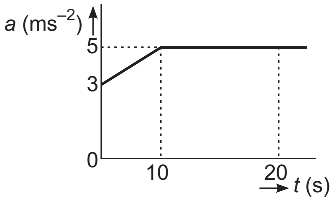
24. A particle moves along the curve $y = \frac{x^2}{2}$. Here x varies with time as $x = \frac{t^2}{2}$. Where x and y are measured in metres and t in seconds. At $t = 2$ s, the velocity of the particle (in ms^{-1}) is

(a) $4\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$	(b) $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$
(c) $4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$	(d) $4\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

25. If the displacement of a particle varies with time as $\sqrt{x} = t + 3$

(a) velocity of the particle is inversely proportional to t
(b) velocity of particle varies linearly with t
(c) velocity of particle is proportional to $\sqrt{t}$
(d) initial velocity of the particle is zero

26. The graph describes an airplane's acceleration during its take-off run. The airplane's velocity when it lifts off at $t = 20$ s is



(a) $40\ \text{ms}^{-1}$	(b) $50\ \text{ms}^{-1}$
(c) $90\ \text{ms}^{-1}$	(d) $180\ \text{ms}^{-1}$

27. A particle moving in a straight line has velocity-displacement equation as $v = 5\sqrt{1 + s}$. Here v is in ms^{-1} and s in metres. Select the correct alternative.

(a) Particle is initially at rest
(b) Initially velocity of the particle is 5 m/s and the particle has a constant acceleration of $12.5\ \text{ms}^{-2}$
(c) Particle moves with a uniform velocity
(d) None of the above

28. A particle is thrown upwards from ground. It experiences a constant resistance force which can produce a retardation of 2 ms^{-2}. The ratio of time of ascent to time of descent is ($g = 10$ ms^{-2})

(a) 1 : 1
(b) $\sqrt{\frac{2}{3}}$
(c) $\frac{2}{3}$
(d) $\sqrt{\frac{3}{2}}$

29. A body of mass 10 kg is being acted upon by a force $3t^2$ and an opposing constant force of 32 N. The initial speed is 10 ms^{-1}. The velocity of body after 5 s is

(a) 14.5 ms^{-1}
(b) 6.5 ms^{-1}
(c) 3.5 ms^{-1}
(d) 4.5 ms^{-1}

30. A stone is thrown vertically upwards. When stone is at a height half of its maximum height, its speed is 10 ms^{-1}; then the maximum height attained by the stone is ($g = 10$ ms^{-2})

(a) 25 m
(b) 10 m
(c) 15 m
(d) 20 m

Subjective Questions

1. (a) What does $\left|\frac{d\mathbf{v}}{dt}\right|$ and $\frac{d|\mathbf{v}|}{dt}$ represent? (b) Can these be equal?

2. The coordinates of a particle moving in x-y plane at any time t are $(2t, t^2)$. Find (a) the trajectory of the particle, (b) velocity of particle at time t and (c) acceleration of particle at any time t.

3. A farmer has to go 500 m due north, 400 m due east and 200 m due south to reach his field. If he takes 20 min to reach the field.

(a) What distance he has to walk to reach the field ?
(b) What is the displacement from his house to the field ?
(c) What is the average speed of farmer during the walk ?
(d) What is the average velocity of farmer during the walk ?

4. A rocket is fired vertically up from the ground with a resultant vertical acceleration of 10 m/s^2. The fuel is finished in 1 min and it continues to move up.(a) What is the maximum height reached? (b) After how much time from then will the maximum height be reached?
(Take $g = 10$ m/s^2)

5. A particle is projected upwards from the roof of a tower 60 m high with velocity 20 m/s. Find

(a) the average speed and
(b) average velocity of the particle upto an instant when it strikes the ground. Take $g = 10$ m/s^2.

6. A block moves in a straight line with velocity v for time t_0. Then, its velocity becomes $2v$ for next t_0 time. Finally, its velocity becomes $3v$ for time T. If average velocity during the complete journey was 2.5 v, then find T in terms of t_0.

7. A particle starting from rest has a constant acceleration of 4 m/s^2 for 4 s. It then retards uniformly for next 8 s and comes to rest. Find during the motion of particle (a) average acceleration (b) average speed and (c) average velocity.

8. A particle moves in a circle of radius $R = \frac{21}{22}$ m with constant speed 1m/s. Find,

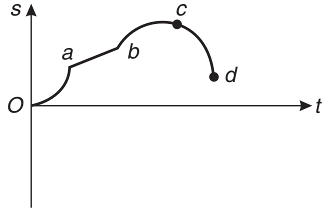
(a) magnitude of average velocity and (b) magnitude of average acceleration in 2 s.

9. Two particles A and B start moving simultaneously along the line joining them in the same direction with acceleration of 1 m/s^2 and 2 m/s^2 and speeds 3 m/s and 1 m/s respectively. Initially, A is 10 m behind B. What is the minimum distance between them?

10. Two diamonds begin a free fall from rest from the same height, 1.0 s apart. How long after the first diamond begins to fall will the two diamonds be 10 m apart? Take $g = 10\ \text{m/s}^2$.

11. Two bodies are projected vertically upwards from one point with the same initial velocity v_0. The second body is projected t_0 s after the first. How long after will the bodies meet?

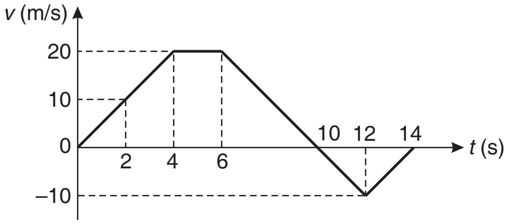
12. Displacement-time graph of a particle moving in a straight line is as shown in figure.



(a) Find the sign of velocity in regions *oa,ab, bc* and *cd.*

(b) Find the sign of acceleration in the above region.

13. Velocity-time graph of a particle moving in a straight line is shown in figure. In the time interval from $t = 0$ to $t = 14$ s, find

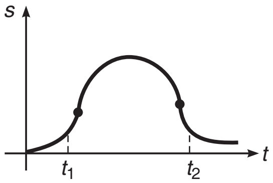


(a) average velocity and

(b) average speed of the particle.

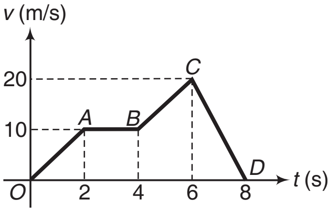
14. A person walks up a stalled 15 m long escalator in 90 s. When standing on the same escalator, now moving, the person is carried up in 60 s. How much time would it take that person to walk up the moving escalator? Does the answer depend on the length of the escalator?

15. Figure shows the displacement-time graph of a particle moving in a straight line. Find the signs of velocity and acceleration of particle at time $t = t_1$ and $t = t_2$.

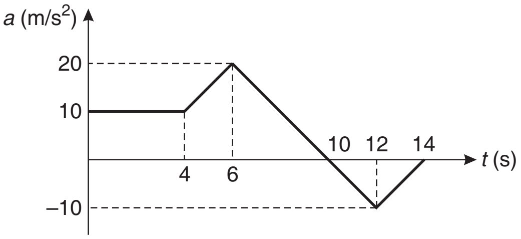


16. Velocity of a particle moving along positive x-direction is $v = (40 - 10\,t)$ m/s. Here, t is in seconds. At time $t = 0$, the x coordinate of particle is zero. Find the time when the particle is at a distance of 60 m from origin.

17. Velocity-time graph of a particle moving in a straight line is shown in figure. Plot the corresponding displacement-time graph of the particle if at time $t = 0$, displacement $s = 0$.



18. Acceleration-time graph of a particle moving in a straight line is as shown in figure. At time $t = 0$, velocity of the particle is zero. Find



(a) average acceleration in a time interval from $t = 6$ s to $t = 12$ s,
(b) velocity of the particle at $t = 14$ s.

19. A particle is moving in x-y plane. At time $t = 0$, particle is at (1m, 2m) and has velocity $(4\hat{\mathbf{i}} + 6\hat{\mathbf{j}})$ m/s. At $t = 4$ s, particle reaches at (6m, 4m) and has velocity $(2\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s. In the given time interval, find
(a) average velocity,
(b) average acceleration and
(c) from the given data, can you find average speed?

20. A stone is dropped from the top of a tower. When it crosses a point 5 m below the top, another stone is let fall from a point 25 m below the top. Both stones reach the bottom of the tower simultaneously. Find the height of the tower. Take $g = 10$ m/s^2.

21. A point mass starts moving in a straight line with constant acceleration. After time t_0 the acceleration changes its sign, remaining the same in magnitude. Determine the time T from the beginning of motion in which the point mass returns to the initial position.

22. A football is kicked vertically upward from the ground and a student gazing out of the window sees it moving upwards past her at 5.00 m/s. The window is 15.0 m above the ground. Air resistance may be ignored. Take $g = 10$ m/s^2.
(a) How high does the football go above ground?
(b) How much time does it take to go from the ground to its highest point?

23. A car moving with constant acceleration covered the distance between two points 60.0 m apart in 6.00 s. Its speed as it passes the second point was 15.0 m/s.
(a) What is the speed at the first point?
(b) What is the acceleration?
(c) At what prior distance from the first was the car at rest?

24. A particle moves along the x-direction with constant acceleration. The displacement, measured from a convenient position, is 2 m at time $t = 0$ and is zero when $t = 10$ s. If the velocity of the particle is momentary zero when $t = 6$ s, determine the acceleration a and the velocity v when $t = 10$ s.

25. At time $t = 0$, a particle is at (2m, 4m). It starts moving towards positive x-axis with constant acceleration 2 m/s^2 (initial velocity = 0). After 2 s, an additional acceleration of 4 m/s^2 starts acting on the particle in negative y-direction also. Find after next 2 s.

(a) velocity and (b) coordinates of particle.

26. A particle starts from the origin at $t = 0$ with a velocity of $8.0\hat{\mathbf{j}}$ m/s and moves in the x-y plane with a constant acceleration of $(4.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})$ m/s^2. At the instant the particle's x-coordinate is 29 m, what are

(a) its y-coordinate and (b) its speed ?

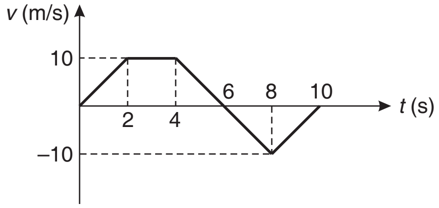
27. The velocity of a particle moving in a straight line is decreasing at the rate of 3 m/s per metre of displacement at an instant when the velocity is 10 m/s. Determine the acceleration of the particle at this instant.

28. A particle moves along a horizontal path, such that its velocity is given by $v = (3t^2 - 6t)$ m/s, where t is the time in seconds. If it is initially located at the origin O, determine the distance travelled by the particle in time interval from $t = 0$ to $t = 3.5$ s and the particle's average velocity and average speed during the same time interval.

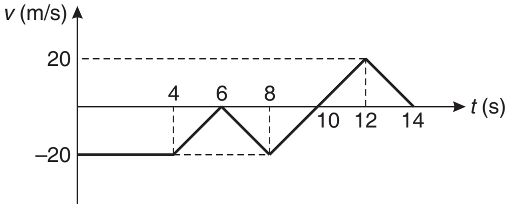
29. A particle travels in a straight line, such that for a short time $2\text{ s} \le t \le 6$ s, its motion is described by $v = (4/a)$ m/s, where a is in m/s^2. If $v = 6$ m/s when $t = 2$ s, determine the particle's acceleration when $t = 3$ s.

30. If the velocity v of a particle moving along a straight line decreases linearly with its displacement from 20 m/s to a value approaching zero at $s = 30$ m, determine the acceleration of the particle when $s = 15$ m.

31. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, $s = -10$ m. Plot corresponding a-t and s-t graphs.



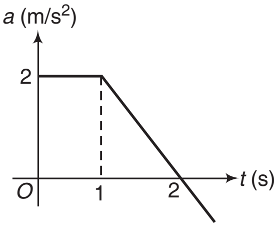
32. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, $s = 20$ m. Plot a-t and s-t graphs of the particle.



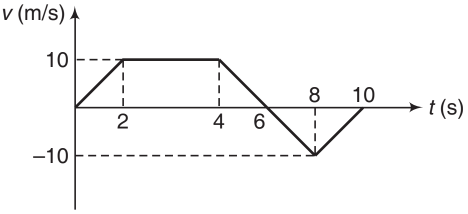
33. A particle of mass m is released from a certain height h with zero initial velocity. It strikes the ground elastically (direction of its velocity is reversed but magnitude remains the same). Plot the graph between its kinetic energy and time till it returns to its initial position.

34. A ball is dropped from a height of 80 m on a floor. At each collision, the ball loses half of its speed. Plot the speed-time graph and velocity-time graph of its motion till two collisions with the floor. [Take $g = 10\ \text{m/s}^2$]

35. Figure shows the acceleration-time graph of a particle moving along a straight line. After what time the particle acquires its initial velocity?

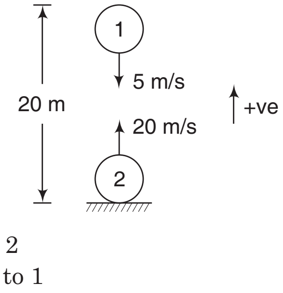


36. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, displacement of the particle from mean position is 10 m. Find



(a) acceleration of particle at $t = 1\ \text{s}, 3\ \text{s}$ and $9\ \text{s}$.
(b) position of particle from mean position at $t = 10\ \text{s}$.
(c) write down s-t equation for time interval
  (i) $0 \le t \le 2\ \text{s}$, (ii) $4\ \text{s} \le t \le 8\ \text{s}$

37. Two particles 1 and 2 are thrown in the directions shown in figure simultaneously with velocities 5 m/s and 20 m/s. Initially, particle 1 is at height 20 m from the ground. Taking upwards as the positive direction, find



(a) acceleration of 1 with respect to 2
(b) initial velocity of 2 with respect to 1
(c) velocity of 1 with respect to 2 after time $t = \frac{1}{2}\ \text{s}$
(d) time when the particles will collide.

38. A ball is thrown vertically upward from the 12 m level with an initial velocity of 18 m/s. At the same instant an open platform elevator passes the 5 m level, moving upward with a constant velocity of 2 m/s. Determine ($g = 9.8 \text{ m/s}^2$)

(a) when and where the ball will meet the elevator,

(b) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.

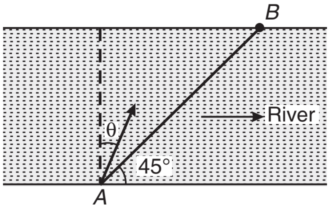
39. An automobile and a truck start from rest at the same instant, with the automobile initially at some distance behind the truck. The truck has a constant acceleration of 2.2 m/s^2 and the automobile has an acceleration of 3.5 m/s^2. The automobile overtakes the truck when it (truck) has moved 60 m.

(a) How much time does it take the automobile to overtake the truck ?

(b) How far was the automobile behind the truck initially ?

(c) What is the speed of each during overtaking ?

40. Given $|\mathbf{v}_{br}| = 4 \text{ m/s}$ = magnitude of velocity of boatman with respect to river, $\mathbf{v}_r = 2 \text{ m/s}$ in the direction shown. Boatman wants to reach from point A to point B. At what angle θ should he row his boat?



41. An aeroplane has to go from a point P to another point Q, 1000 km away due north. Wind is blowing due east at a speed of 200 km/h. The air speed of plane is 500 km/h.

(a) Find the direction in which the pilot should head the plane to reach the point Q.

(b) Find the time taken by the plane to go from P to Q.

42. A train stopping at two stations 4 km apart takes 4 min on the journey from one of the station to the other. Assuming that it first accelerates with a uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x} + \frac{1}{y} = 2$.

LEVEL 2

Objective Questions

Single Correct Option

1. When a man moves down the inclined plane with a constant speed 5 ms^{-1} which makes an angle of 37° with the horizontal, he finds that the rain is falling vertically downward. When he moves up the same inclined plane with the same speed, he finds that the rain makes an angle $\theta = \tan^{-1}\left(\frac{7}{8}\right)$ with the horizontal. The speed of the rain is

(a) $\sqrt{116} \text{ ms}^{-1}$ (b) $\sqrt{32} \text{ ms}^{-1}$

(c) 5 ms^{-1} (d) $\sqrt{73} \text{ ms}^{-1}$

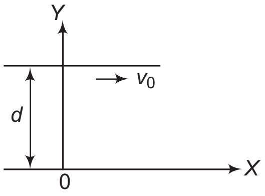
2. Equation of motion of a body is $\frac{dv}{dt} = -4v + 8$, where v is the velocity in ms^{-1} and t is the time in second. Initial velocity of the particle was zero. Then,

(a) the initial rate of change of acceleration of the particle is 8 ms^{-2}
(b) the terminal speed is 2 ms^{-1}
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

3. Two particles A and B are placed in gravity free space at (0, 0, 0) m and (30, 0, 0) m respectively. Particle A is projected with a velocity $(5\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ ms^{-1}, while particle B is projected with a velocity $(10\,\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ ms^{-1} simultaneously. Then,

(a) they will collide at (10, 20, 10) m
(b) they will collide at (10, 10, 10) m
(c) they will never collide
(d) they will collide at 2 s

4. Velocity of the river with respect to ground is given by v_0. Width of the river is d. A swimmer swims (with respect to water) perpendicular to the current with acceleration $a = 2t$ (where t is time) starting from rest from the origin O at $t = 0$. The equation of trajectory of the path followed by the swimmer is



(a) $y = \frac{x^3}{3v_0^3}$
(b) $y = \frac{x^2}{2v_0^2}$
(c) $y = \frac{x}{v_0}$
(d) $y = \sqrt{\frac{x}{v_0}}$

5. The relation between time t and displacement x is $t = \alpha x^2 + \beta x$, where α and β are constants. The retardation is

(a) $2\,\alpha v^3$
(b) $2\,\beta v^3$
(c) $2\,\alpha\beta v^3$
(d) $2\,\beta^2 v^3$

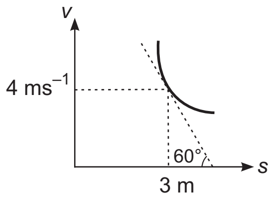
6. A street car moves rectilinearly from station A to the next station B (from rest to rest) with an acceleration varying according to the law $f = a - bx$, where a and b are constants and x is the distance from station A. The distance between the two stations and the maximum velocity are

(a) $x = \frac{2a}{b}$, $v_{\text{max}} = \frac{a}{\sqrt{b}}$
(b) $x = \frac{b}{2a}$, $v_{\text{max}} = \frac{a}{b}$
(c) $x = \frac{a}{2b}$, $v_{\text{max}} = \frac{b}{\sqrt{a}}$
(d) $x = \frac{a}{b}$, $v_{\text{max}} = \frac{\sqrt{a}}{b}$

7. A particle of mass m moves on positive x-axis under the influence of force acting towards the origin given by $-kx^2\hat{\mathbf{i}}$. If the particle starts from rest at $x = a$, the speed it will attain when it crosses the origin is

(a) $\sqrt{\frac{k}{ma}}$
(b) $\sqrt{\frac{2k}{ma}}$
(c) $\sqrt{\frac{ma}{2k}}$
(d) None of these

8. A particle is moving along a straight line whose velocity-displacement graph is as shown in the figure. What is the magnitude of acceleration when displacement is 3 m ?



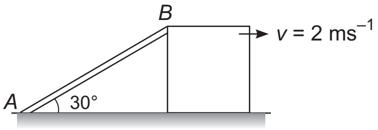
(a) $4\sqrt{3}$ ms^{-2} (b) $3\sqrt{3}$ ms^{-2}

(c) $\sqrt{3}$ ms^{-2} (d) $\frac{4}{\sqrt{3}}$ ms^{-2}

9. A particle is falling freely under gravity. In first t second it covers distance x_1 and in the next t second, it covers distance x_2, then t is given by

(a) $\sqrt{\frac{x_2 - x_1}{g}}$ (b) $\sqrt{\frac{x_2 + x_1}{g}}$ (c) $\sqrt{\frac{2(x_2 - x_1)}{g}}$ (d) $\sqrt{\frac{2(x_2 + x_1)}{g}}$

10. A rod AB is shown in figure. End A of the rod is fixed on the ground. Block is moving with velocity 2 ms^{-1} towards right. The velocity of end B of rod at the instant shown in figure is



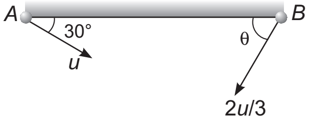
(a) $\sqrt{3}$ ms^{-1} (b) 2 ms^{-1}

(c) $2\sqrt{3}$ ms^{-1} (d) 4 ms^{-1}

11. A thief in a stolen car passes through a police check post at his top speed of 90 kmh^{-1}. A motorcycle cop, reacting after 2 s, accelerates from rest at 5 ms^{-2}. His top speed being 108 kmh^{-1}. Find the maximum separation between policemen and thief.

(a) 112.5 m (b) 115 m (c) 116.5 m (d) None of these

12. Anoop (A) hits a ball along the ground with a speed u in a direction which makes an angle 30° with the line joining him and the fielder Babul (B). Babul runs to intercept the ball with a speed $\frac{2u}{3}$. At what angle θ should he run to intercept the ball ?



(a) $\sin^{-1}\left[\frac{\sqrt{3}}{2}\right]$ (b) $\sin^{-1}\left[\frac{2}{3}\right]$ (c) $\sin^{-1}\left[\frac{3}{4}\right]$ (d) $\sin^{-1}\left[\frac{4}{5}\right]$

13. A car is travelling on a straight road. The maximum velocity the car can attain is 24 ms^{-1}. The maximum acceleration and deceleration it can attain are 1 ms^{-2} and 4 ms^{-2} respectively. The shortest time the car takes from rest to rest in a distance of 200 m is,

(a) 22.4 s (b) 30 s (c) 11.2 s (d) 5.6 s

14. A car is travelling on a road. The maximum velocity the car can attain is 24 ms^{-1} and the maximum deceleration is 4 ms^{-2}. If car starts from rest and comes to rest after travelling 1032 m in the shortest time of 56 s, the maximum acceleration that the car can attain is

(a) 6 ms^{-2} (b) 1.2 ms^{-2}

(c) 12 ms^{-2} (d) 3.6 ms^{-2}

15. Two particles are moving along two long straight lines, in the same plane with same speed equal to 20 cm/s. The angle between the two lines is 60° and their intersection point is O. At a certain moment, the two particles are located at distances 3m and 4m from O and are moving towards O. Subsequently, the shortest distance between them will be

(a) 50 cm (b) $40\sqrt{2}$ cm

(c) $50\sqrt{2}$ cm (d) $50\sqrt{3}$ cm

More than One Correct Options

1. A particle having a velocity $v = v_0$ at $t = 0$ is decelerated at the rate $|a| = \alpha\sqrt{v}$, where α is a positive constant.

(a) The particle comes to rest at $t = \dfrac{2\sqrt{v_0}}{\alpha}$

(b) The particle will come to rest at infinity

(c) The distance travelled by the particle before coming to rest is $\dfrac{2v_0^{3/2}}{\alpha}$

(d) The distance travelled by the particle before coming to rest is $\dfrac{2v_0^{3/2}}{3\alpha}$

2. At time $t = 0$, a car moving along a straight line has a velocity of 16 ms^{-1}. It slows down with an acceleration of $-0.5t$ ms^{-2}, where t is in second. Mark the correct statement (s).

(a) The direction of velocity changes at $t = 8$ s

(b) The distance travelled in 4 s is approximately 58.67 m

(c) The distance travelled by the particle in 10 s is 94 m

(d) The speed of particle at $t = 10$ s is 9 ms^{-1}

3. An object moves with constant acceleration **a**. Which of the following expressions are also constant ?

(a) $\dfrac{d|\mathbf{v}|}{dt}$ (b) $\left|\dfrac{d\mathbf{v}}{dt}\right|$

(c) $\dfrac{d(v^2)}{dt}$ (d) $\dfrac{d\left(\dfrac{\mathbf{v}}{|\mathbf{v}|}\right)}{dt}$

4. Ship A is located 4 km north and 3 km east of ship B. Ship A has a velocity of 20 kmh^{-1} towards the south and ship B is moving at 40 kmh^{-1} in a direction 37° north of east. X and Y-axes are along east and north directions, respectively

(a) Velocity of A relative to B is $(-32\hat{\mathbf{i}} - 44\hat{\mathbf{j}})$ km/h

(b) Position of A relative to B as a function of time is given by $\mathbf{r}_{AB} = [(3 - 32t)\hat{\mathbf{i}} + (4 - 44t)\hat{\mathbf{j}}]$ km

(c) Velocity of A relative to B is $(32\hat{\mathbf{i}} - 44\hat{\mathbf{j}})$ km/h

(d) Position of A relative to B as a function of time is given by $(32t\hat{\mathbf{i}} - 44t\hat{\mathbf{j}})$ km

5. Starting from rest a particle is first accelerated for time t_1 with constant acceleration a_1 and then stops in time t_2 with constant retardation a_2. Let v_1 be the average velocity in this case and s_1 the total displacement. In the second case it is accelerating for the same time t_1 with constant acceleration $2a_1$ and come to rest with constant retardation a_2 in time t_3. If v_2 is the average velocity in this case and s_2 the total displacement, then

(a) $v_2 = 2v_1$ (b) $2v_1 < v_2 < 4v_1$

(c) $s_2 = 2s_1$ (d) $2s_1 < s_2 < 4s_1$

6. A particle is moving along a straight line. The displacement of the particle becomes zero in a certain time ($t > 0$). The particle does not undergo any collision.

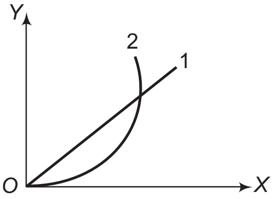
(a) The acceleration of the particle may be zero always

(b) The acceleration of the particle may be uniform

(c) The velocity of the particle must be zero at some instant

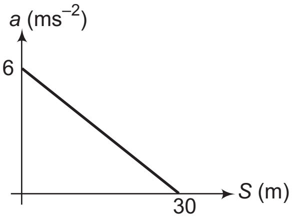
(d) The acceleration of the particle must change its direction

7. A particle is resting over a smooth horizontal floor. At $t = 0$, a horizontal force starts acting on it. Magnitude of the force increases with time according to law $F = \alpha t$, where α is a positive constant. From figure, which of the following statements are correct ?



(a) Curve 1 can be the plot of acceleration against time
(b) Curve 2 can be the plot of velocity against time
(c) Curve 2 can be the plot of velocity against acceleration
(d) Curve 1 can be the plot of displacement against time

8. A train starts from rest at $S = 0$ and is subjected to an acceleration as shown in figure. Then,

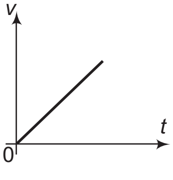


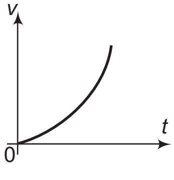
(a) velocity at the end of 10 m displacement is $20\ \text{ms}^{-1}$
(b) velocity of the train at $S = 10$ m is $10\ \text{ms}^{-1}$
(c) The maximum velocity attained by train is $\sqrt{180}\ \text{ms}^{-1}$
(d) The maximum velocity attained by the train is $15\ \text{ms}^{-1}$

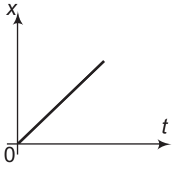
9. For a moving particle, which of the following options may be correct?
(a) $|\mathbf{V}_{\text{av}}| < v_{\text{av}}$ (b) $|\mathbf{V}_{\text{av}}| > v_{\text{av}}$ (c) $\mathbf{V}_{\text{av}} = 0$ but $v_{\text{av}} \neq 0$ (d) $\mathbf{V}_{\text{av}} \neq 0$ but $v_{\text{av}} = 0$

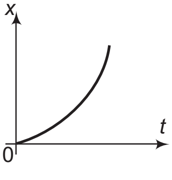
Here, $\mathbf{V}_{\text{av}}$ is average velocity and v_{av} the average speed.

10. Identify the correct graph representing the motion of a particle along a straight line with constant acceleration with zero initial velocity.

(a)


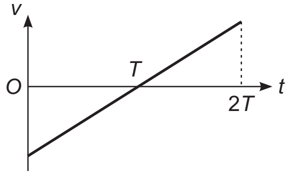
(b)


(c)


(d)


11. A man who can swim at a velocity v relative to water wants to cross a river of width b, flowing with a speed u.

(a) The minimum time in which he can cross the river is $\frac{b}{v}$
(b) He can reach a point exactly opposite on the bank in time $t = \frac{b}{\sqrt{v^2 - u^2}}$ if $v > u$
(c) He cannot reach the point exactly opposite on the bank if $u > v$
(d) He cannot reach the point exactly opposite on the bank if $v > u$



12. The figure shows the velocity (v) of a particle plotted against time (t).

(a) The particle changes its direction of motion at some point
(b) The acceleration of the particle remains constant
(c) The displacement of the particle is zero
(d) The initial and final speeds of the particle are the same

13. The speed of a train increases at a constant rate α from zero to v and then remains constant for an interval and finally decreases to zero at a constant rate β. The total distance travelled by the train is l. The time taken to complete the journey is t. Then,

(a) $t = \dfrac{l(\alpha + \beta)}{\alpha\beta}$
(b) $t = \dfrac{l}{v} + \dfrac{v}{2}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$
(c) t is minimum when $v = \sqrt{\dfrac{2l\alpha\beta}{(\alpha - \beta)}}$
(d) t is minimum when $v = \sqrt{\dfrac{2l\alpha\beta}{(\alpha + \beta)}}$

14. A particle moves in x-y plane and at time t is at the point $(t^2, t^3 - 2t)$, then which of the following is/are correct?

(a) At $t = 0$, particle is moving parallel to y-axis
(b) At $t = 0$, direction of velocity and acceleration are perpendicular
(c) At $t = \sqrt{\dfrac{2}{3}}$, particle is moving parallel to x-axis
(d) At $t = 0$, particle is at rest

15. A car is moving with uniform acceleration along a straight line between two stops X and Y. Its speed at X and Y are $2\ \text{ms}^{-1}$ and $14\ \text{ms}^{-1}$, Then

(a) its speed at mid-point of XY is $10\ \text{ms}^{-1}$
(b) its speed at a point A such that $XA : AY = 1:3$ is $5\ \text{ms}^{-1}$
(c) the time to go from X to the mid-point of XY is double of that to go from mid-point to Y
(d) the distance travelled in first half of the total time is half of the distance travelled in the second half of the time

Comprehension Based Questions

Passage 1 (Q.Nos. 1 to 4)

An elevator without a ceiling is ascending up with an acceleration of $5\ ms^{-2}$. *A boy on the elevator shoots a ball in vertical upward direction from a height of* $2\ m$ *above the floor of elevator. At this instant the elevator is moving up with a velocity of* $10\ ms^{-1}$ *and floor of the elevator is at a height of* $50\ m$ *from the ground. The initial speed of the ball is* $15\ ms^{-1}$ *with respect to the elevator. Consider the duration for which the ball strikes the floor of elevator in answering following questions.* $(g = 10\ ms^{-2})$

1. The time in which the ball strikes the floor of elevator is given by

(a) 2.13 s (b) 2.0 s (c) 1.0 s (d) 3.12 s

2. The maximum height reached by ball, as measured from the ground would be

(a) 73.65 m (b) 116.25 m (c) 82.56 m (d) 63.25 m

3. Displacement of ball with respect to ground during its flight would be

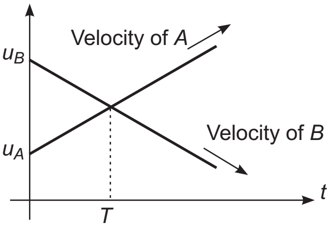
(a) 16.25 m (b) 8.76 m (c) 20.24 m (d) 30.56 m

4. The maximum separation between the floor of elevator and the ball during its flight would be

(a) 12 m (b) 15 m (c) 9.5 m (d) 7.5 m

Passage 2 (Q.Nos. 5 to 7)

A situation is shown in which two objects A and B start their motion from same point in same direction. The graph of their velocities against time is drawn. u_A and u_B are the initial velocities of A and B respectively. T is the time at which their velocities become equal after start of motion. You cannot use the data of one question while solving another question of the same set. So all the questions are independent of each other.



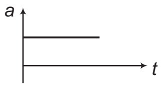
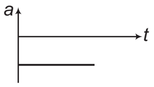
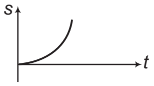
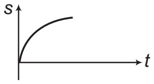
5. If the value of T is 4 s, then the time after which A will meet B is
(a) 12 s (b) 6 s
(c) 8 s (d) data insufficient

6. Let v_A and v_B be the velocities of the particles A and B respectively at the moment A and B meet after start of the motion. If $u_A = 5\ \text{ms}^{-1}$ and $u_B = 15\ \text{ms}^{-1}$, then the magnitude of the difference of velocities v_A and v_B is
(a) $5\ \text{ms}^{-1}$ (b) $10\ \text{ms}^{-1}$
(c) $15\ \text{ms}^{-1}$ (d) data insufficient

7. After 10 s of the start of motion of both objects A and B, find the value of velocity of A if $u_A = 6\ \text{ms}^{-1}$, $u_B = 12\ \text{ms}^{-1}$ and at T velocity of A is $8\ \text{ms}^{-1}$ and $T = 4$ s
(a) $12\ \text{ms}^{-1}$ (b) $10\ \text{ms}^{-1}$
(c) $15\ \text{ms}^{-1}$ (d) None of these

Match the Columns

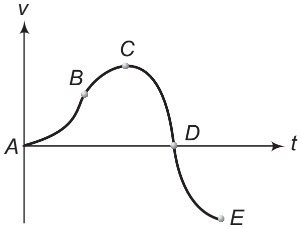
1. Match the following two columns :

Column I	Column II
(a) 	(p) speed must be increasing
(b) 	(q) speed must be decreasing
(c) 	(r) speed may be increasing
(d) 	(s) speed may be decreasing

2. Match the following two columns :

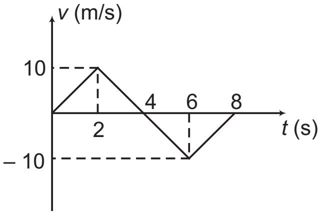
Column I	Column II
(a) $\mathbf{v} = -2\hat{\mathbf{i}}, \mathbf{a} = -4\hat{\mathbf{j}}$	(p) speed increasing
(b) $\mathbf{v} = 2\hat{\mathbf{i}}, \mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$	(q) speed decreasing
(c) $\mathbf{v} = -2\hat{\mathbf{i}}, \mathbf{a} = +2\hat{\mathbf{i}}$	(r) speed constant
(d) $\mathbf{v} = 2\hat{\mathbf{i}}, \mathbf{a} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$	(s) Nothing can be said

3. The velocity-time graph of a particle moving along X-axis is shown in figure. Match the entries of Column I with the entries of Column II.



Column I	Column II
(a) For AB, particle is	(p) Moving in +ve X-direction with increasing speed
(b) For BC, particle is	(q) Moving in +ve X-direction with decreasing speed
(c) For CD, particle is	(r) Moving in –ve X-direction with increasing speed
(d) For DE, particle is	(s) Moving in –ve X-direction with decreasing speed

4. Corresponding to velocity-time graph in one dimensional motion of a particle as shown in figure, match the following two columns.



Column I	Column II
(a) Average velocity between zero second and 4 s	(p) 10 SI units
(b) Average acceleration between 1 s and 4 s	(q) 2.5 SI units
(c) Average speed between zero seccond and 6 s	(r) 5 SI units
(d) Rate of change of speed at 4 s	(s) None of the above

5. A particle is moving along x-axis. Its x-coordinate varies with time as :

$$x = -20 + 5t^2$$

For the given equation match the following two columns :

Column I	Column II
(a) Particle will cross the origin at	(p) zero second
(b) At what time velocity and acceleration are equal	(q) 1 s
(c) At what time particle changes its direction of motion	(r) 2 s
(d) At what time velocity is zero	(s) None of the above

6. x and y-coordinates of a particle moving in x - y plane are,

$$x = 1 - 2t + t^2 \text{ and } y = 4 - 4t + t^2$$

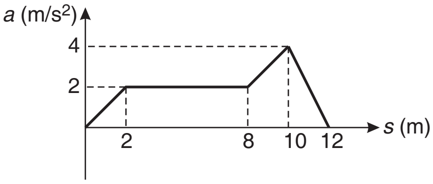
For the given situation match the following two columns :

Column I	Column II
(a) y-component of velocity when it crosses the y-axis	(p) + 2 SI unit
(b) x-component of velocity when it crosses the x-axis	(q) − 2 SI units
(c) Initial velocity of particle	(r) + 4 SI units
(d) Initial acceleration of particle	(s) None of the above

Subjective Questions

1. To test the quality of a tennis ball, you drop it onto the floor from a height of 4.00 m. It rebounds to a height of 2.00 m. If the ball is in contact with the floor for 12.0 ms, what is its average acceleration during that contact? Take $g = 98$ m/s^2.

2. The acceleration-displacement graph of a particle moving in a straight line is as shown in figure, initial velocity of particle is zero. Find the velocity of the particle when displacement of the particle is $s = 12$ m.



3. At the initial moment three points A, B and C are on a horizontal straight line at equal distances from one another. Point A begins to move vertically upward with a constant velocity v and point C vertically downward without any initial velocity but with a constant acceleration a. How should point B move vertically for all the three points to be constantly on one straight line. The points begin to move simultaneously.

4. A particle moves in a straight line with constant acceleration a. The displacements of particle from origin in times t_1, t_2 and t_3 are s_1, s_2 and s_3 respectively. If times are in AP with common difference d and displacements are in GP, then prove that $a = \dfrac{(\sqrt{s_1} - \sqrt{s_3})^2}{d^2}$.

5. A car is to be hoisted by elevator to the fourth floor of a parking garage, which is 14 m above the ground. If the elevator can have maximum acceleration of 0.2 m/s^2 and maximum deceleration of 0.1 m/s^{2}and can reach a maximum speed of 2.5 m/s, determine the shortest time to make the lift, starting from rest and ending at rest.

6. To stop a car, first you require a certain reaction time to begin braking; then the car slows under the constant braking deceleration. Suppose that the total distance moved by your car during these two phases is 56.7 m when its initial speed is 80.5 km/h and 24.4 m when its initial speed is 48.3 km/h. What are

(a) your reaction time and

(b) the magnitude of the deceleration?

7. An elevator without a ceiling is ascending with a constant speed of 10 m/s. A boy on the elevator shoots a ball directly upward, from a height of 2.0 m above the elevator floor. At this time the elevator floor is 28 m above the ground. The initial speed of the ball with respect to the elevator is 20 m/s. (Take $g = 9.8\ \text{m/s}^2$)

(a) What maximum height above the ground does the ball reach?

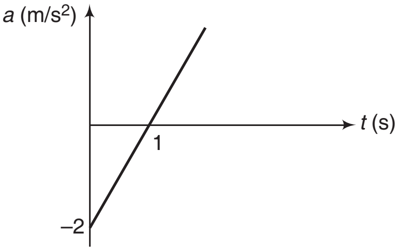
(b) How long does the ball take to return to the elevator floor?

8. A particle moves along a straight line and its velocity depends on time as $v = 3t - t^2$. Here, v is in m/s and t in second. Find

(a) average velocity and

(b) average speed for first five seconds.

9. The acceleration of particle varies with time as shown.



(a) Find an expression for velocity in terms of t.

(b) Calculate the displacement of the particle in the interval from $t = 2$ s to $t = 4$ s. Assume that $v = 0$ at $t = 0$.

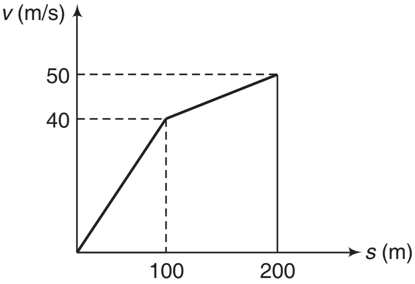
10. A man wishes to cross a river of width 120 m by a motorboat. His rowing speed in still water is 3 m/s and his maximum walking speed is 1 m/s. The river flows with velocity of 4 m/s.

(a) Find the path which he should take to get to the point directly opposite to his starting point in the shortest time.

(b) Also, find the time which he takes to reach his destination.

11. The current velocity of river grows in proportion to the distance from its bank and reaches the maximum value v_0 in the middle. Near the banks the velocity is zero. A boat is moving along the river in such a manner that the boatman rows his boat always perpendicular to the current. The speed of the boat in still water is u. Find the distance through which the boat crossing the river will be carried away by the current, if the width of the river is c. Also determine the trajectory of the boat.

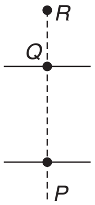
12. The v-s graph for an airplane travelling on a straight runway is shown. Determine the acceleration of the plane at $s = 50$ m and $s = 150$ m. Draw the a-s graph.



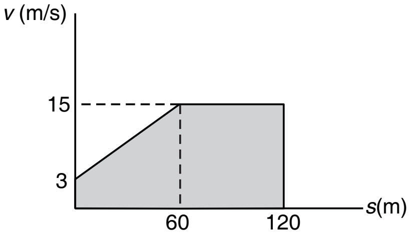
13. A river of width a with straight parallel banks flows due north with speed u. The points O and A are on opposite banks and A is due east of O. Coordinate axes Ox and Oy are taken in the east and north directions respectively. A boat, whose speed is v relative to water, starts from O and crosses the river. If the boat is steered due east and u varies with x as : $u = x(a - x)\dfrac{v}{a^2}$. Find

(a) equation of trajectory of the boat,
(b) time taken to cross the river,
(c) absolute velocity of boatman when he reaches the opposite bank,
(d) the displacement of boatman when he reaches the opposite bank from the initial position.

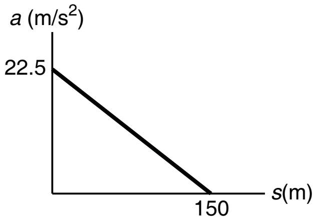
14. A river of width ω is flowing with a uniform velocity v. A boat starts moving from point P also with velocity v relative to the river. The direction of resultant velocity is always perpendicular to the line joining boat and the fixed point R. Point Q is on the opposite side of the river. P, Q and R are in a straight line. If $PQ = QR = \omega$, find (a) the trajectory of the boat, (b) the drifting of the boat and (c) the time taken by the boat to cross the river.



15. The v-s graph describing the motion of a motorcycle is shown in figure. Construct the a-s graph of the motion and determine the time needed for the motorcycle to reach the position $s = 120$ m. Given ln 5 = 1.6.



16. The jet plane starts from rest at $s = 0$ and is subjected to the acceleration shown. Determine the speed of the plane when it has travelled 60 m.



17. A particle leaves the origin with an initial velocity $\mathbf{v} = (3.00\,\hat{\mathbf{i}})$ m/s and a constant acceleration $\mathbf{a} = (-1.00\,\hat{\mathbf{i}} - 0.500\,\hat{\mathbf{j}})\,\mathrm{m/s^2}$. When the particle reaches its maximum x coordinate, what are

(a) its velocity and (b) its position vector?

18. The speed of a particle moving in a plane is equal to the magnitude of its instantaneous velocity,

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}.$$

(a) Show that the rate of change of the speed is $\dfrac{dv}{dt} = (v_x a_x + v_y a_y) / \sqrt{v_x^2 + v_y^2}.$

(b) Show that the rate of change of speed can be expressed as $\dfrac{dv}{dt} = \mathbf{v} \cdot \mathbf{a} / v$, and use this result to explain why $\dfrac{dv}{dt}$ is equal to a_t the component of **a** that is parallel to **v**.

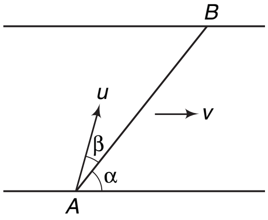
19. A man with some passengers in his boat, starts perpendicular to flow of river 200 m wide and flowing with 2 m/s. Speed of boat in still water is 4 m/s. When he reaches half the width of river the passengers asked him that they want to reach the just opposite end from where they have started.

(a) Find the direction due which he must row to reach the required end.

(b) How many times more time, it would take to that if he would have denied the passengers?

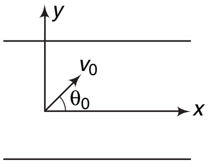
20. A child in danger of drowning in a river is being carried downstream by a current that flows uniformly at a speed of 2.5 km/h. The child is 0.6 km from shore and 0.8 km upstream of a boat landing when a rescue boat sets out. If the boat proceeds at its maximum speed of 20 km/h with respect to the water, what angle does the boat velocity v make with the shore? How long will it take boat to reach the child?

21. A launch plies between two points A and B on the opposite banks of a river always following the line AB. The distance S between points A and B is 1200 m. The velocity of the river current $v = 1.9$ m/s is constant over the entire width of the river. The line AB makes an angle $\alpha = 60°$ with the direction of the current. With what velocity u and at what angle β to the line AB should the launch move to cover the distance AB and back in a time $t = 5$ min? The angle β remains the same during the passage from A to B and from B to A.

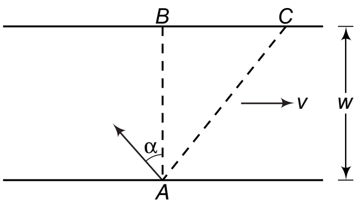


22. The slopes of wind screen of two cars are $\alpha_1 = 30°$ and $\alpha_2 = 15°$ respectively. At what ratio v_1/v_2 of the velocities of the cars will their drivers see the hail stones bounced back by the wind screen on their cars in vertical direction? Assume hail stones fall vertically downwards and collisions to be elastic.

23. A projectile of mass m is fired into a liquid at an angle θ_0 with an initial velocity v_0 as shown. If the liquid develops a frictional or drag resistance on the projectile which is proportional to its velocity, i.e. $F = -kv$ where k is a positive constant, determine the x and y components of its velocity at any instant. Also find the maximum distance $x_{\max}$ that it travels?



24. A man in a boat crosses a river from point A. If he rows perpendicular to the banks he reaches point C ($BC = 120$ m) in 10 min. If the man heads at a certain angle α to the straight line AB (AB is perpendicular to the banks) against the current he reaches point B in 12.5 min. Find the width of the river w, the rowing velocity u, the speed of the river current v and the angle α. Assume the velocity of the boat relative to water to be constant and the same magnitude in both cases.



Answers

Introductory Exercise 6.1

1. Both downwards **2.** (a) $-2\ \text{m}^2/\text{s}^3$, (b) obtuse, (c) decreasing

Introductory Exercise 6.2

1. One dimensional with constant acceleration **2.** Two dimensional with non-uniform acceleration
3. No

Introductory Exercise 6.3

1. False **2.** True **3.** g (downwards) **4.** $\frac{\pi}{15}$ cm/s, $\frac{2\sqrt{2}}{15}$ cm/s
5. (a) Yes, in uniform circular motion (b) No, yes (projectile motion), yes
6. (a) 25.13 s (b) 1 cm/s, 0.9 cm/s, $0.23\ \text{cm/s}^2$

Introductory Exercise 6.4

1. 5.2 m/s **2.** $\frac{11}{3}$ m/s

Introductory Exercise 6.5

2. See the hints
3. Always g **4.** Acceleration **5.** 60 m, 100 m
6. $u + \frac{1}{2}at$ **7.** True **8.** 25 m/s (downwards)
9. (a) 6.0 m, (b) 10 s, (c) 50 m
10. 125 m, (b) 5 s, (c) approximately 35 m/s

Introductory Exercise 6.6

1. (a) $1\ \text{m/s}^2$ (b) 43.5m **2.** (a) $60\ \text{cm/s}^2$, (b) 1287 cm
3. (a) $x = 1.0$m, $v = 4$ m/s, $a = 8\ \text{m/s}^2$, (b) zero **4.** (a) $x = 2.0$ m (b) zero (c) $26\ \text{ms}^{-2}$
5. $s \propto t^{7/4}$ and $a \propto t^{-1/4}$

Introductory Exercise 6.7

1. $2\sqrt{7}$ m/s, $4\sqrt{3}$ m **2.** $(2\hat{\mathbf{j}})\ \text{m/s}^2$, $(2\hat{\mathbf{i}} + \hat{\mathbf{j}})$ m, yes
3. $\mathbf{v} = (3\hat{\mathbf{i}} + \hat{\mathbf{j}})$ m/s, co-ordinates $= \left(\frac{7}{3}\ \text{m}, \frac{1}{4}\ \text{m}\right)$

Introductory Exercise 6.8

1. (a) Particle A starts at $t = 0$ from $x = 10$ m. Particle B starts at $t = 4$ s from $x = 0$.
(b) $v_A = +\ 2.5$ m/s, $v_B = +\ 7.5$ m/s, (c) They strike at $x = 30$ m and $t = 8$ s
2. 80 m, $2.5\ \text{m/s}^2$
3. (a) $0.6\ \text{m/s}^2$, (b) 50 m, (c) 50 m
4. (a) 10 m/s, (b) 20 m/s, zero, 20 m/s, –20 m/s
5. 100 m, zero

Introductory Exercise 6.9

1. –2 m/s

2. zero

3. (a) 40 s (b) 80 m

4. (a) $\sin^{-1}\left(\frac{1}{15}\right)$ east of the line AB (b) 50 min

5. (a) 200 m, (b) 20 m/min, (c) 12 m/min

6. (a) 10 s, (b) 50 m

Exercises

LEVEL 1

Assertion and Reason

1.(d) **2.**(d) **3.**(a) **4.**(d) **5.**(c) **6.**(d) **7.**(d) **8.**(d) **9.**(d) **10.**(a or b)
11.(a) **12.**(d)

Single Correct Option

1.(b) **2.**(c) **3.**(d) **4.**(a) **5.**(a) **6.**(b) **7.**(a) **8.**(c) **9.**(c) **10.**(a)
11.(b) **12.**(c) **13.**(c) **14.**(b) **15.**(b) **16.**(a) **17.**(c) **18.**(b) **19.**(d) **20.**(a)
21.(b) **22.**(b) **23.**(d) **24.**(b) **25.**(b) **26.**(c) **27.**(b) **28.**(b) **29.**(b) **30.**(b)

Subjective Questions

1. (a) Magnitude of total acceleration and tangential acceleration, (b) equal in 1-D motion

2. (a) $x^2 = 4y$ (b) $(2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}})$ units (c) $(2\hat{\mathbf{j}})$ units

3. (a) 1100 m, (b) 500 m, (c) 55 m/min, (d) 25 m/min,

4. (a) 36 km (b) 1 min

5. (a) 16.67 ms^{-1} (b) 10 ms^{-1} (downwards)

6. $T = 4\,t_0$

7. (a) zero (b) 8 ms^{-1} (c) 8 ms^{-1}

8. (a) $\frac{21\sqrt{3}}{44}$ ms^{-1} (b) $\frac{\sqrt{3}}{2}$ ms^{-2}

9. 8 m

10. 1.5 s

11. $\frac{v_0}{g} + \frac{t_0}{2}$

12. (a) positive, positive, positive, negative (b) positive, zero, negative, negative

13. (a) $\frac{50}{7}$ ms^{-1} (b) 10 ms^{-1}

14. 36 s, No

15. v_{t_1}, a_{t_1} and a_{t_2} are positive while v_{t_2} is negative

16. 2 s, 6 s, $2(2 + \sqrt{7})$ s

17. See the hints

18. (a) –5 ms^{-2} (b) 90 ms^{-1}

19. (a) $(1.25\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}})$ ms^{-1}(b) $(-0.5\hat{\mathbf{i}} + \hat{\mathbf{j}})$ ms^{-2} (c) No

20. 45 m

21. $(3.414)\,t_0$

22. (a) 16.25 m (b) 1.8 s

23. (a) 5 ms^{-1} (b)1.67 ms^{-2} (c) 7.5 m

24. 0.2 ms^{-2}, 0.8 ms^{-1}

25. (a) $(8\hat{\mathbf{i}} - 8\hat{\mathbf{j}})$ ms^{-1} (b) (18 m , – 4m)

26. (a) 45 m (b) 22 ms^{-1}

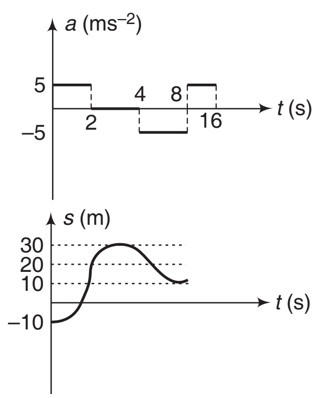
27. – 30 ms^{-2}

28. 14.125 m, 1.75 ms^{-1}, 4.03 ms^{-1}

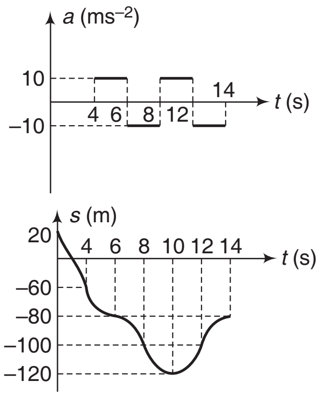
29. 0.603 ms^{-2}

30. $\left(-\frac{20}{3}\right)$ ms^{-2}

31.



32.



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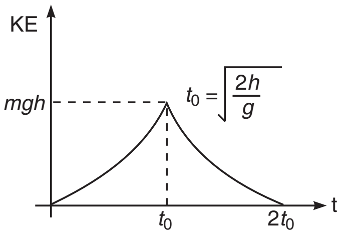
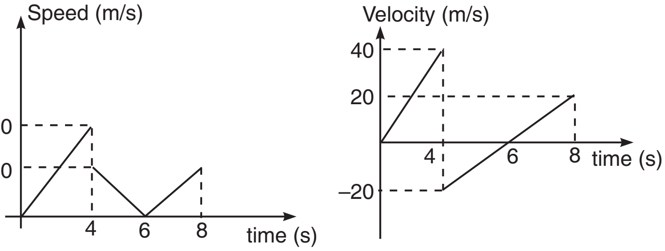


34.



35. $(2 + \sqrt{3})$ s

36. (a) 5 ms^{-2}, zero, 5 ms^{-2} (b) $s = 30$ m (c) (i) $s = 10 + 2.5t^2$ (ii) $s = 40 + 10(t - 4) - 2.5(t - 4)^2$

37. (a) zero (b) 25 ms^{-1} (c) – 25 ms^{-1} (d) 0.8 s

38. (a) 3.65 s, at 12.30 m level (b) 19.8 ms^{-1} (downwards)

39. (a) 7.39 s (b) 35.5 m (c) automobile 25.9 ms^{-1}, truck 16.2 ms^{-1}

40. $45° - \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 24.3°$

41. (a) at an angle $\theta = \sin^{-1}(0.4)$ west of north (b) $\frac{10}{\sqrt{21}}h$

LEVEL 2

Single Correct Option

1.(b) **2.**(b) **3.**(c) **4.**(a) **5.**(a) **6.**(a) **7.**(d) **8.**(a) **9.**(a) **10.**(d)
11.(a) **12.**(c) **13.**(a) **14.**(b) **15.**(d)

More than One Correct Options

1.(a,d) **2.**(all) **3.**(b) **4.**(a,b) **5.**(a,d) **6.**(b,c) **7.**(a,b) **8.**(b,c) **9.**(a,c) **10.**(a,d)
11.(a,b,c) **12.**(all) **13.**(b,d) **14.**(a,b,c) **15.**(a,c)

Comprehension Based Questions

1.(a) **2.**(c) **3.**(d) **4.**(c) **5.**(c) **6.**(b) **7.**(d)

Match the Columns

1. (a) → r,s (b) → r,s (c) → p (d) → q
2. (a) → r (b) → p (c) → q (d) → q
3. (a) → p (b) → p (c) → q (d) → r
4. (a) → r (b) → s (c) → r (d) → r
5. (a) → r (b) → q (c) → s (d) → p
6. (a) → q (b) → p (c) → s (d) → s

Subjective Questions

1. 1.26×10^3 ms^{-2} (upward) **2.** $4\sqrt{3}$ ms^{-1}

3. B moves up with initial velocity $\frac{v}{2}$ and downward acceleration $-\frac{a}{2}$ **5.** 20.5 s

6. (a) 0.74 s (b) 6.2 ms^{-2} **7.** (a) 76 m (b) 4.2 s **8.** (a) – 0.833 ms^{-1} (b) 2.63 ms^{-1}

9. (a) $v = t^2 - 2t$ (b) 6.67 m **10.** (a) 90° + $\sin^{-1}(3/5)$ from river current (b) 2 min 40 s

11. $\frac{cv_0}{2u}$, $y^2 = \frac{ucx}{v_0}$ **12.** 8 ms^{-2}, 4.5 ms^{-2}, For the graph see the hints

13. (a) $y = \frac{x^2}{2a} - \frac{x^3}{3a^2}$ (b) $\frac{a}{v}$ (c) v (due east) (d) $a\hat{\mathbf{i}} + \frac{a\hat{\mathbf{j}}}{6}$

14. (a) circle (b) $\sqrt{3}\,\omega$ (c) $\frac{1.317\,\omega}{v}$ **15.** 12.0 s, For the graph see the hints **16.** 46.47 ms^{-1}

17. (a) $(-1.5\hat{\mathbf{j}})$ ms^{-1} (b) $(4.5\hat{\mathbf{i}} - 2.25\hat{\mathbf{j}})$ m

19. (a) At an angle $(90° + 2\theta)$ from river current (upstream). Here : $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ (b) $\frac{4}{3}$

20. 37°, 3 min **21.** $u = 8$ ms^{-1}, $\beta = 12°$ **22.** $\frac{v_1}{v_2} = 3$

23. $v_x = v_0 \cos\theta_0\, e^{-kt/m}$, $v_y = \frac{m}{k}\left[\left(\frac{k}{m} v_0 \sin\theta_0 + g\right) e^{-\frac{kt}{m}} - g\right]$, $x_m = \frac{mv \cos\theta_0}{k}$

24. 200 m, 20 m /min, 12 m /min, 36°50.

07

Projectile Motion

Chapter Contents

7.1 Introduction
7.2 Projectile Motion
7.3 Two Methods of Solving a Projectile Motion
7.4 Time of Flight, Maximum Height and Horizontal Range of a Projectile
7.5 Projectile Motion along an Inclined Plane
7.6 Relative Motion between Two Projectiles

7.1 Introduction

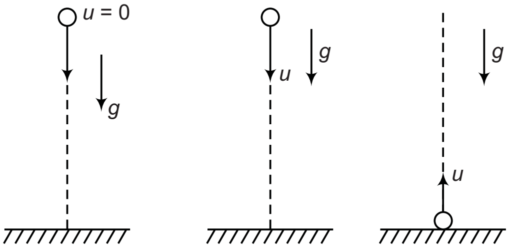
Motion of a particle under constant acceleration is either a straight line (one-dimensional) or parabolic (two-dimensional). Motion is one dimensional under following three conditions :

(i) Initial velocity of the particle is zero.

(ii) Initial velocity of the particle is in the direction of constant acceleration (or parallel to it).

(iii) Initial velocity of the particle is in the opposite direction of constant acceleration (or antiparallel to it).

For small heights acceleration due to gravity (g) is almost constant. The three cases discussed about are as shown in the Fig. 7.1.



Case-(i) Case-(ii) Case-(iii)

Fig. 7.1

In all other cases when initial velocity is at some angle (≠ 0° or 180°) with constant acceleration, motion is parabolic as shown below.

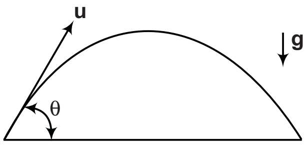


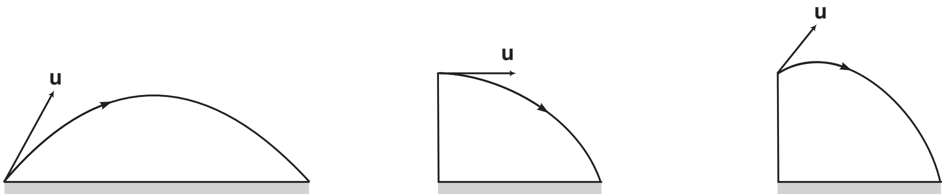
Fig. 7.2

This motion under acceleration due to gravity is called projectile motion.

7.2 Projectile Motion

As we have seen above, projectile motion is a two-dimensional motion (or motion in a plane) with constant acceleration (or acceleration due to gravity for small heights).

The different types of projectile motion are as shown below.



(a) (b) (c)

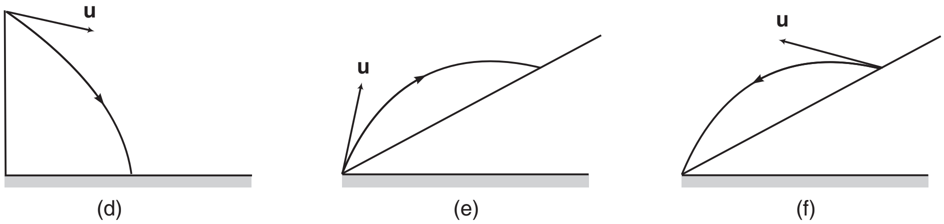


Fig. 7.3

The plane of the projectile motion is a vertical plane.

7.3 Two Methods of Solving a Projectile Motion

Every projectile motion can be solved by either of the following two methods:

Method 1 Projectile motion is a two dimensional motion with constant acceleration. Therefore, we can use the equations

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t \text{ and } \mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

For example, in the shown figure

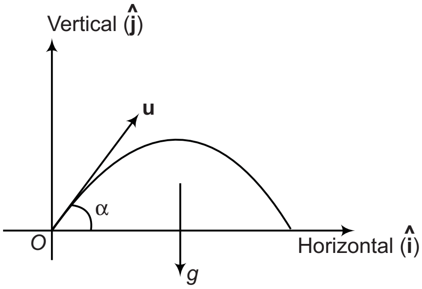

Vertical ($\hat{\mathbf{j}}$)
u
α
O
Horizontal ($\hat{\mathbf{i}}$)
g

Fig. 7.4

$$\mathbf{u} = u\cos\alpha\hat{\mathbf{i}} + u\sin\alpha\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{a} = -g\hat{\mathbf{j}}$$

Now, suppose we want to find velocity at time t.

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t = (u\cos\alpha\hat{\mathbf{i}} + u\sin\alpha\hat{\mathbf{j}}) - gt\hat{\mathbf{j}}$$

or

$$\mathbf{v} = u\cos\alpha\hat{\mathbf{i}} + (u\sin\alpha - gt)\,\hat{\mathbf{j}}$$

Similarly, displacement at time t will be

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\,\mathbf{a}t^2 = (u\cos\alpha\hat{\mathbf{i}} + u\sin\alpha\hat{\mathbf{j}})\,t - \frac{1}{2}gt^2\,\hat{\mathbf{j}}$$

$$= ut\cos\alpha\hat{\mathbf{i}} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

Note *In all problems, value of* $\mathbf{a}$ $(=\mathbf{g})$ *will be same only* $\mathbf{u}$ *will be different.*

Example 7.1 *A particle is projected with a velocity of* 50 m/s *at* $37°$ *with horizontal. Find velocity, displacement and co-ordinates of the particle (w.r.t. the starting point) after* 2 *s.*

Given, $g = 10 \, m/s^2$, $\sin 37° = 0.6$ *and* $\cos 37° = 0.8$

Solution In the given problem,

$$\mathbf{u} = (50\cos 37°)\hat{\mathbf{i}} + (50\sin 37°)\hat{\mathbf{j}}$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}}) \text{ m/s}$$
$$\mathbf{a} = (-10\hat{\mathbf{j}}) \text{ m/s}^2$$
$$t = 2\text{s}$$
$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}}) + (-10\hat{\mathbf{j}})(2)$$
$$= (40\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) \text{ m/s} \qquad \textbf{Ans.}$$

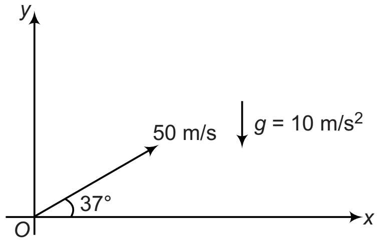


Fig. 7.5

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}})(2) + \frac{1}{2}(-10\hat{\mathbf{j}})(2)^2$$
$$= (80\hat{\mathbf{i}} + 40\hat{\mathbf{j}}) \text{m}$$

Coordinates of the particle are

$$x = 80 \text{ m} \quad \text{and} \quad y = 40 \text{ m} \qquad \textbf{Ans.}$$

Example 7.2 *A particle is projected with velocity u at angle* θ *with horizontal. Find the time when velocity vector is perpendicular to initial velocity vector.*

Solution

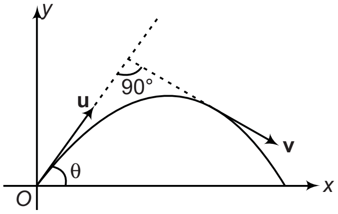


Fig. 7.6

Given, $\mathbf{v} \perp \mathbf{u}$

$$\Rightarrow \quad \mathbf{v} \cdot \mathbf{u} = 0$$
$$\Rightarrow \quad (\mathbf{u} + \mathbf{a}t) \cdot \mathbf{u} = 0 \qquad \text{......(i)}$$

Substituting the proper values in Eq. (i), we have

$$[\{(u\cos\theta)\hat{\mathbf{i}} + (u\sin\theta)\hat{\mathbf{j}}\} + (-g\hat{\mathbf{j}})t] \cdot [(u\cos\theta)\hat{\mathbf{i}} + (u\sin\theta)\hat{\mathbf{j}}] = 0$$
$$\Rightarrow \quad u^2\cos^2\theta + u^2\sin^2\theta - (ug\sin\theta)t = 0$$
$$\Rightarrow \quad u^2(\sin^2\theta + \cos^2\theta) = (ug\sin\theta)t$$

Solving this equation, we get

$$t=\frac{u}{g\sin\theta}=\frac{u\operatorname{cosec}\theta}{g}$$ **Ans.**

Alternate method

$$\mathbf{a}=\mathbf{g}$$

Angle between **u** and **a** is $\alpha=90^\circ+\theta$

Now, Eq. (i) can be written as

$$\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{a}t=0$$

or $$u^2+(ug\cos\alpha)t=0$$

or $$u^2+[ug\cos(90^\circ+\theta)]t=0$$

Solving this equation, we get

$$t=\frac{u}{g\sin\theta}=\frac{u\operatorname{cosec}\theta}{g}$$ **Ans.**

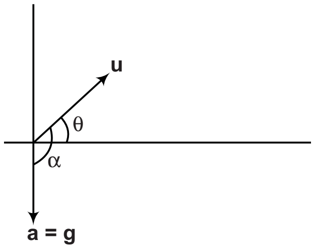


Fig. 7.7

Method 2 In this method, select two mutually perpendicular directions x and y and find the two components of initial velocity and acceleration along these two directions, i.e. find u_x, u_y, a_x and a_y. Now apply the appropriate equation (s) of the following six equations :

$$\left.\begin{aligned}v_x&=u_x+a_xt\\ s_x&=u_xt+\frac{1}{2}a_xt^2\\ v_x^2&=u_x^2+2a_xs_x\end{aligned}\right]\rightarrow\text{Along } x\text{- axis}$$

and

$$\left.\begin{aligned}v_y&=u_y+a_yt\\ s_y&=u_yt+\frac{1}{2}a_yt^2\\ v_y^2&=u_y^2+2a_ys_y\end{aligned}\right]\rightarrow\text{Along } y\text{- axis}$$

Substitute $v_x, u_x, a_x, s_x, v_y, u_y, a_y$ and s_y with proper signs but choosing one direction as positive and other as the negative along both axes. In most of the problems $s=ut+\frac{1}{2}at^2$ equation is useful for time calculation. Under normal projectile motion, x-axis is taken along horizontal direction and y-axis along vertical direction. In projectile motion along an inclined plane, x-axis is normally taken along the plane and y-axis perpendicular to the plane. Two simple cases are shown below.

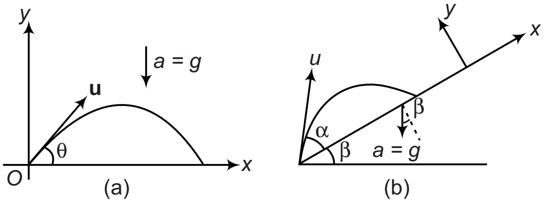


Fig. 7.8

In Fig. 7.8 (a)

$$u_x = u\cos\theta,\ u_y = u\sin\theta,\ a_x = 0,\ a_y = -g$$

In Fig. 7.8 (b)

$$u_x = u\cos\alpha,\ u_y = u\sin\alpha,\ a_x = -g\sin\beta,\ a_y = -g\cos\beta$$

Example 7.3 *A projectile is fired horizontally with velocity of 98 m/s from the top of a hill 490 m high. Find*

(a) the time taken by the projectile to reach the ground,

(b) the distance of the point where the particle hits the ground from foot of the hill and

(c) the velocity with which the projectile hits the ground. ($g = 9.8\ m/s^2$)

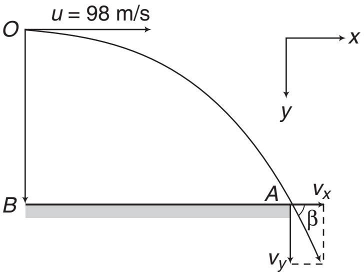


Fig. 7.9

Solution Here, it will be more convenient to choose x and y directions as shown in figure.

Here, $u_x = 98$ m/s, $a_x = 0$, $u_y = 0$ and $a_y = g$

(a) At A, $s_y = 490\ \text{m}$. So, applying

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$\therefore \quad 490 = 0 + \frac{1}{2}(9.8)\,t^2$$

$$\therefore \quad t = 10\,\text{s}$$ **Ans.**

(b) $BA = s_x = u_x t + \frac{1}{2} a_x t^2$

$$\text{or} \quad BA = (98)(10) + 0$$

$$\text{or} \quad BA = 980\ \text{m}$$ **Ans.**

(c) $v_x = u_x + a_x t = 98 + 0 = 98$ m/s

$$v_y = u_y + a_y t = 0 + (9.8)(10) = 98\ \text{m/s}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2}\ \text{m/s}$$

$$\text{and} \quad \tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$$

$$\therefore \quad \beta = 45^\circ$$

Thus, the projectile hits the ground with velocity $98\sqrt{2}$ m/s at an angle of $\beta = 45^\circ$ with horizontal as shown in Fig. 7.9.

Example 7.4 *A body is thrown horizontally from the top of a tower and strikes the ground after three seconds at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take* $g = 9.8\ m/s^2$.

Solution As shown in the figure of Example 7.3.

$$u_y = 0 \text{ and } a_y = g = 9.8 \text{ m/s}^2 ,$$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$s_y = 0 \times 3 + \frac{1}{2} \times 9.8 \times (3)^2$$

$$= 44.1 \text{ m}$$

Thus, height of the tower is 44.1 m.

Further, $v_y = u_y + a_y t = 0 + (9.8)(3) = 29.4$ m/s

As the resultant velocity v makes an angle of 45° with the horizontal, so

$$\tan 45° = \frac{v_y}{v_x} \quad \text{or} \quad 1 = \frac{29.4}{v_x}$$

$$v_x = 29.4 \text{ m/s}$$

$$v_x = u_x + a_x t$$

$$\Rightarrow \qquad 29.4 = u_x + 0$$

$$\text{or} \qquad u_x = 29.4 \text{ m/s}$$

Therefore, the speed with which the body was projected (horizontally) is 29.4 m/s.

INTRODUCTORY EXERCISE 7.1

1. Two particles are projected from a tower horizontally in opposite directions with velocities 10 m/s and 20 m/s. Find the time when their velocity vectors are mutually perpendicular. Take $g = 10 \text{ m/s}^2$.
2. Projectile motion is a 3-dimensional motion. Is this statement true or false?
3. Projectile motion (at low speed) is uniformly accelerated motion. Is this statement true or false?
4. A particle is projected from ground with velocity 50 m/s at 37° from horizontal. Find velocity and displacement after 2 s. $\sin 37° = \frac{3}{5}$.
5. A particle is projected from a tower of height 25 m with velocity $20\sqrt{2}$ m/s at 45°. Find the time when particle strikes with ground. The horizontal distance from the foot of tower where it strikes. Also find the velocity at the time of collision.

Note *In question numbers 4 and 5, $\hat{\mathbf{i}}$ is in horizontal direction and $\hat{\mathbf{j}}$ is vertically upwards.*

7.4 Time of Flight, Maximum Height and Horizontal Range of a Projectile

Fig. 7.10 shows a particle projected from the point O with an initial velocity u at an angle α with the horizontal. It goes through the highest point A and falls at B on the horizontal surface through O. The point O is called the **point of projection**, the angle α is called the **angle of projection**, the distance OB is called the **horizontal range (R)** or simply range and the vertical height AC is called the **maximum height (H).** The total time taken by the particle in describing the path OAB is called the **time of flight (T).**

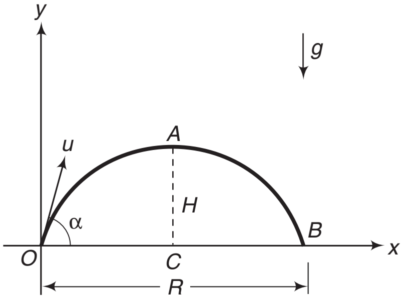


Fig. 7.10

Time of Flight (T)

Refer Fig. 7.10. Here, x and y-axes are in the directions shown in figure. Axis x is along horizontal direction and axis y is vertically upwards. Thus,

$$u_x = u\cos\alpha,$$

$$u_y = u\sin\alpha,\ a_x = 0$$

and $$a_y = -g$$

At point B, $s_y = 0$. So, applying

$$s_y = u_y t + \frac{1}{2} a_y t^2, \text{ we have}$$

$$0 = (u\sin\alpha)t - \frac{1}{2} g t^2$$

$$\therefore \quad t = 0, \frac{2u\sin\alpha}{g}$$

Both $t = 0$ and $t = \frac{2u\sin\alpha}{g}$ correspond to the situation where $s_y = 0$. The time $t = 0$ corresponds to point O and time $t = \frac{2u\sin\alpha}{g}$ corresponds to point B. Thus, time of flight of the projectile is

$$T = t_{OAB} \quad \text{or} \quad \boxed{T = \frac{2u\sin\alpha}{g}}$$

Maximum Height (H)

At point A vertical component of velocity becomes zero, i.e. $v_y = 0$. Substituting the proper values in

$$v_y^2 = u_y^2 + 2a_y s_y$$

we have, $$0 = (u\sin\alpha)^2 + 2(-g)(H)$$

$$\therefore \quad \boxed{H = \frac{u^2\sin^2\alpha}{2g}}$$

Horizontal Range (R)

Distance OB is the range R. This is also equal to the displacement of particle along x-axis in time $t = T$. Thus, applying $s_x = u_x t + \frac{1}{2} a_x t^2$, we get

$$R = (u \cos \alpha)\left(\frac{2u \sin \alpha}{g}\right) + 0$$

as $\quad a_x = 0 \quad$ and $\quad t = T = \dfrac{2u \sin \alpha}{g}$

$$\therefore \quad R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g} \quad \text{or} \quad \boxed{R = \frac{u^2 \sin 2\alpha}{g}}$$

Following are given two important points regarding the range of a projectile

(i) Range is maximum where $\sin 2\alpha = 1$ or $\alpha = 45°$ and this maximum range is

$$R_{\max} = \frac{u^2}{g} \qquad \text{(at } \alpha = 45°\text{)}$$

(ii) For given value of u range at α and range at $90° - \alpha$ are equal, although times of flight and maximum heights may be different. Because

$$R_{90° - \alpha} = \frac{u^2 \sin 2(90° - \alpha)}{g} = \frac{u^2 \sin (180° - 2\alpha)}{g} = \frac{u^2 \sin 2\alpha}{g} = R_\alpha$$

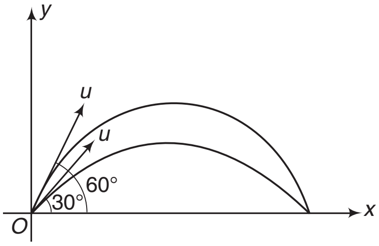


Fig. 7.11

So, $\quad R_{30°} = R_{60°}$

or $\quad R_{20°} = R_{70°}$

This is shown in Fig. 7.11.

Extra Points to Remember

- Formulae of T, H and R can be applied directly between two points lying on same horizontal line.

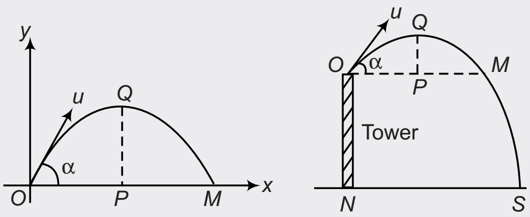


Fig. 7.12

For example, in the two projectile motions shown in figure,

$$t_{OQM} = T = \frac{2u \sin\alpha}{g}, \; PQ = H = \frac{u^2 \sin^2 \alpha}{2g} \text{ and } OM = R = \frac{u^2 \sin 2\alpha}{g}$$

For finding t_{OQMS} or distance NS method-2 discussed in article 7.3 is more useful.

- As we have seen in the above derivations that $a_x = 0$, i.e. motion of the projectile in horizontal direction is uniform. Hence, horizontal component of velocity $u \cos \alpha$ does not change during its motion.
- Motion in vertical direction is first retarded then accelerated in opposite direction. Because u_y is upwards and a_y is downwards. Hence, vertical component of its velocity first decreases from O to A and then increases from A to B. This can be shown as in Fig. 7.13.

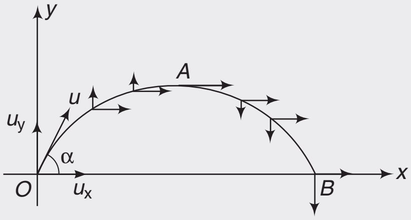


Fig. 7.13

- The coordinates and velocity components of the projectile at time t are

$$x = s_x = u_x t = (u \cos \alpha) t$$

$$y = s_y = u_y t + \frac{1}{2} a_y t^2$$

$$= (u \sin \alpha) t - \frac{1}{2} g t^2$$

$$v_x = u_x = u \cos \alpha$$

and $$v_y = u_y + a_y t = u \sin \alpha - gt$$

Therefore, speed of projectile at time t is $v = \sqrt{v_x^2 + v_y^2}$ and the angle made by its velocity vector with positive x-axis is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

- **Equation of trajectory of projectile**

$$x = (u \cos \alpha) t$$

∴ $$t = \frac{x}{u \cos \alpha}$$

Substituting this value of t in, $y = (u \sin \alpha) t - \frac{1}{2} g t^2$, we get

$$\boxed{y = x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha}}$$

or $$y = x \tan \alpha - \frac{g x^2}{2 u^2} \sec^2 \alpha$$

$$\boxed{y = x \tan \alpha - \frac{g x^2}{2 u^2} (1 + \tan^2 \alpha)}$$

These are the standard equations of trajectory of a projectile. The equation is quadratic in x. This is why the path of a projectile is a parabola. The above equation can also be written in terms of range (R) of projectile as:

$$\boxed{y = x\left(1 - \frac{x}{R}\right) \tan \alpha}$$

Example 7.5 *Find the angle of projection of a projectile for which the horizontal range and maximum height are equal.*

Solution Given, $R = H$

$$\therefore \quad \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin^2 \alpha}{2g} \quad \text{or} \quad 2\sin\alpha\cos\alpha = \frac{\sin^2\alpha}{2}$$

$$\text{or} \quad \frac{\sin\alpha}{\cos\alpha} = 4 \quad \text{or} \quad \tan\alpha = 4$$

$$\therefore \quad \alpha = \tan^{-1}(4)$$ **Ans.**

Example 7.6 *Prove that the maximum horizontal range is four times the maximum height attained by the projectile; when fired at an inclination so as to have maximum horizontal range.*

Solution For $\theta = 45°$, the horizontal range is maximum and is given by

$$R_{\max} = \frac{u^2}{g}$$

Maximum height attained

$$H_{\max} = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g} = \frac{R_{\max}}{4}$$

or

$$R_{\max} = 4\,H_{\max}$$ **Proved.**

Example 7.7 *For given value of u, there are two angles of projection for which the horizontal range is the same. Show that the sum of the maximum heights for these two angles is independent of the angle of projection.*

Solution There are two angles of projection α and $90° - \alpha$ for which the horizontal range R is same.

Now,

$$H_1 = \frac{u^2 \sin^2\alpha}{2g} \quad \text{and} \quad H_2 = \frac{u^2 \sin^2(90° - \alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$$

Therefore,

$$H_1 + H_2 = \frac{u^2}{2g}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{u^2}{2g}$$

Clearly the sum of the heights for the two angles of projection is independent of the angles of projection.

Example 7.8 *Show that there are two values of time for which a projectile is at the same height. Also show mathematically that the sum of these two times is equal to the time of flight.*

Solution For vertically upward motion of a projectile,

$$y = (u\sin\alpha)\,t - \frac{1}{2}gt^2 \quad \text{or} \quad \frac{1}{2}gt^2 - (u\sin\alpha)\,t + y = 0$$

This is a quadratic equation in t. Its roots are

$$t_1 = \frac{u \sin \alpha - \sqrt{u^2 \sin^2 \alpha - 2gy}}{g}$$

and
$$t_2 = \frac{u \sin \alpha + \sqrt{u^2 \sin^2 \alpha - 2gy}}{g}$$

$\therefore$
$$t_1 + t_2 = \frac{2u \sin \alpha}{g} = T$$
(time of flight of the projectile)

INTRODUCTORY EXERCISE 7.2

1. A particle is projected from ground with velocity $40\sqrt{2}$ m/s at $45°$. Find

(a) velocity and

(b) displacement of the particle after 2 s. ($g = 10\ \text{m/s}^2$)

2. Under what conditions the formulae of range, time of flight and maximum height can be applied directly in case of a projectile motion?

3. What is the average velocity of a particle projected from the ground with speed u at an angle α with the horizontal over a time interval from beginning till it strikes the ground again?

4. What is the change in velocity in the above question?

5. A particle is projected from ground with initial velocity $u = 20\sqrt{2}$ m/s at $\theta = 45°$. Find

(a) R, H and T,

(b) velocity of particle after 1 s

(c) velocity of particle at the time of collision with the ground (x-axis).

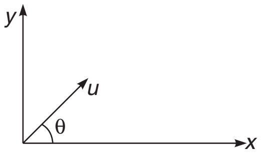


Fig. 7.14

6. A particle is projected from ground at angle 45° with initial velocity $20\sqrt{2}$ m/s. Find

(a) change in velocity,

(b) magnitude of average velocity in a time interval from $t = 0$ to $t = 3$ s.

7. The coach throws a baseball to a player with an initial speed of 20 m/s at an angle of 45° with the horizontal. At the moment the ball is thrown, the player is 50 m from the coach. At what speed and in what direction must the player run to catch the ball at the same height at which it was released? ($g = 10\ \text{m/s}^2$)

8. A ball is thrown horizontally from a point 100 m above the ground with a speed of 20 m/s. Find (a) the time it takes to reach the ground, (b) the horizontal distance it travels before reaching the ground, (c) the velocity (direction and magnitude) with which it strikes the ground.

9. A bullet fired at an angle of $30°$ with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 away? Assume the muzzle speed to be fixed and neglect air resistance.

10. A particle moves in the xy-plane with constant acceleration a directed along the negative y-axis. The equation of path of the particle has the form $y = bx - cx^2$, where b and c are positive constants. Find the velocity of the particle at the origin of coordinates.

7.5 Projectile Motion along an Inclined Plane

Here, two cases arise. One is up the plane and the other is down the plane. Let us discuss both the cases separately.

Up the Plane

In this case direction x is chosen up the plane and direction y is chosen perpendicular to the plane. Hence,

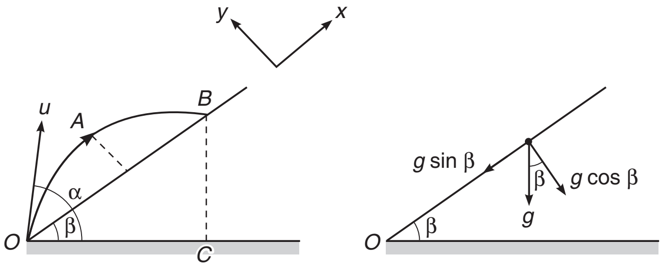


Fig. 7.15

$$u_x = u\cos(\alpha-\beta), \qquad a_x = -g\sin\beta$$
$$u_y = u\sin(\alpha-\beta) \quad \text{and} \quad a_y = -g\cos\beta$$

Now, let us derive the expressions for time of flight (T) and range (R) along the plane.

Time of Flight

At point B displacement along y-direction is zero. So, substituting the proper values in $s_y = u_y t + \frac{1}{2} a_y t^2$, we get

$$0 = ut\sin(\alpha-\beta) + \frac{1}{2}(-g\cos\beta)\,t^2 \quad \Rightarrow \quad \therefore \quad t = 0 \quad \text{and} \quad \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$t = 0$, corresponds to point O and $t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$ corresponds to point B. Thus,

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

Note *Substituting* $\beta = 0$*, in the above expression, we get* $T = \frac{2u\sin\alpha}{g}$ *which is quite obvious because* $\beta = 0$ *is the situation shown in Fig. 7.16.*

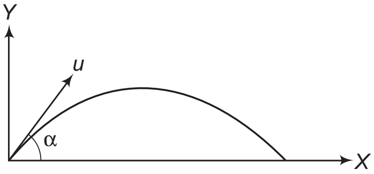


Fig. 7.16

Range

Range (R) or the distance OB can be found by following two methods:

Method 1 Horizontal component of initial velocity is

$$u_H = u\cos\alpha$$

$$\therefore \quad OC = u_H T \qquad (\text{as } a_H = 0)$$

$$= \frac{(u\cos\alpha)\,2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$= \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos\beta}$$

$$\therefore \quad R = OB = \frac{OC}{\cos\beta}$$

$$= \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

Using, $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$,

Range can also be written as,

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta]$$

This range will be maximum when

$$2\alpha - \beta = \frac{\pi}{2} \qquad \text{or} \qquad \alpha = \frac{\pi}{4} + \frac{\beta}{2}$$

and $$R_{\max} = \frac{u^2}{g\cos^2\beta}[1 - \sin\beta]$$

Here, also we can see that for $\beta = 0$, range is maximum at $\alpha = \frac{\pi}{4}$ or $\alpha = 45°$

and $$R_{\max} = \frac{u^2}{g\cos^2 0°}(1 - \sin 0°) = \frac{u^2}{g}$$

Method 2 Range (R) or the distance OB is also equal to the displacement of projectile along x-direction in time $t = T$. Therefore,

$$R = s_x = u_x T + \frac{1}{2}a_x T^2$$

Substituting the values of u_x, a_x and T, we get the same result.

(ii) **Down the Plane** Here, x and y-directions are down the plane and perpendicular to plane respectively as shown in Fig. 7.17. Hence,

$$u_x = u\cos(\alpha+\beta), \quad a_x = g\sin\beta$$

$$u_y = u\sin(\alpha+\beta), \quad a_y = -g\cos\beta$$

Proceeding in the similar manner, we get the following results :

$$\boxed{T = \frac{2u\sin(\alpha+\beta)}{g\cos\beta}, \quad R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta)+\sin\beta]}$$

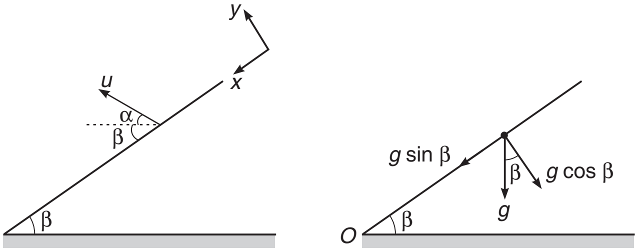


Fig. 7.17

From the above expressions, we can see that if we replace β by –β, the equations of T and R for up the plane and down the plane are interchanged provided α (angle of projection) in both the cases is measured from the horizontal not from the plane.

Example 7.9 *A man standing on a hill top projects a stone horizontally with speed v_0 as shown in figure. Taking the co-ordinate system as given in the figure. Find the co-ordinates of the point where the stone will hit the hill surface.*

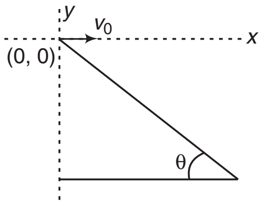


Fig. 7.18

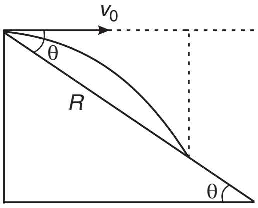


Fig. 7.19

Solution Range of the projectile on an inclined plane (down the plane) is,

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta)+\sin\beta]$$

Here, $u = v_0, \quad \alpha = 0 \quad \text{and} \quad \beta = \theta$

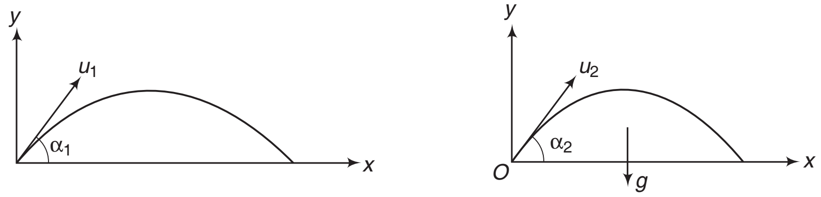
$\therefore$ $$R = \frac{2v_0^2\sin\theta}{g\cos^2\theta}$$

Now, $$x = R\cos\theta = \frac{2v_0^2\tan\theta}{g}$$

and $$y = -R\sin\theta = -\frac{2v_0^2\tan^2\theta}{g}$$

7.6 Relative Motion between Two Projectiles

Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in Fig. 7.20(a) and (b). Acceleration of both the particles is g downwards. So, relative acceleration between them is zero because



(a) (b)

Fig. 7.20

$$\mathbf{a}_{12} = \mathbf{a}_1 - \mathbf{a}_2 = \mathbf{g} - \mathbf{g} = 0$$

i.e. the relative motion between the two particles is uniform.

Now, $$u_{1x} = u_1 \cos\alpha_1, \quad u_{2x} = u_2 \cos\alpha_2$$

$$u_{1y} = u_1 \sin\alpha_1 \quad \text{and} \quad u_{2y} = u_2 \sin\alpha_2$$

Therefore, $$u_{12x} = u_{1x} - u_{2x} = u_1\cos\alpha_1 - u_2\cos\alpha_2$$

and $$u_{12y} = u_{1y} - u_{2y} = u_1\sin\alpha_1 - u_2\sin\alpha_2$$

u_{12x} and u_{12y} are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straight line at an angle $\theta = \tan^{-1}\left(\dfrac{u_{12y}}{u_{12x}}\right)$ with positive x-axis.

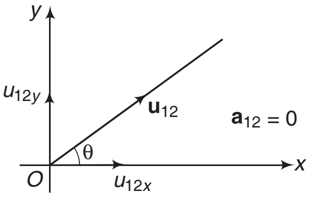


Fig. 7.21

Now, if $u_{12x} = 0$ or $u_1\cos\alpha_1 = u_2\cos\alpha_2$, the relative motion is along y-axis or in vertical direction (as $\theta = 90°$). Similarly, if $u_{12y} = 0$ or $u_1\sin\alpha_1 = u_2\sin\alpha_2$, the relative motion is along x-axis or in horizontal direction (as $\theta = 0°$).

Note *Relative acceleration between two projectiles is zero. Relative motion between them is uniform. Therefore, condition of collision of two particles in air is that relative velocity of one with respect to the other should be along line joining them, i.e., if two projectiles A and B collide in mid air, then* $\mathbf{v}_{AB}$ *should be along AB or* $\mathbf{v}_{BA}$ *along BA.*

Example 7.10 *A particle A is projected with an initial velocity of 60 m/s at an angle 30° to the horizontal. At the same time a second particle B is projected in opposite direction with initial speed of 50 m/s from a point at a distance of 100 m from A. If the particles collide in air, find (a) the angle of projection α of particle B, (b) time when the collision takes place and (c) the distance of P from A, where collision occurs.* ($g = 10\ m/s^2$)

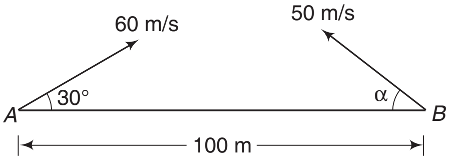


Fig. 7.22

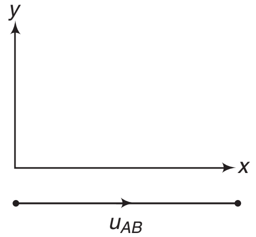


Fig. 7.23

Solution (a) Taking x and y-directions as shown in figure.

Here,
$$\mathbf{a}_A = -g\hat{\mathbf{j}}$$
$$\mathbf{a}_B = -g\hat{\mathbf{j}}$$
$$u_{Ax} = 60\cos 30° = 30\sqrt{3}\text{ m/s}$$
$$u_{Ay} = 60\sin 30° = 30\text{ m/s}$$
$$u_{Bx} = -50\cos\alpha$$
and
$$u_{By} = 50\sin\alpha$$

Relative acceleration between the two is zero as $\mathbf{a}_A = \mathbf{a}_B$. Hence, the relative motion between the two is uniform. It can be assumed that B is at rest and A is moving with $\mathbf{u}_{AB}$. Hence, the two particles will collide, if $\mathbf{u}_{AB}$ is along AB. This is possible only when

$$u_{Ay} = u_{By}$$

i.e. component of relative velocity along y-axis should be zero.

or
$$30 = 50\sin\alpha$$
$\therefore$
$$\alpha = \sin^{-1}(3/5) = 37°$$ **Ans.**

(b) Now,
$$|\mathbf{u}_{AB}| = u_{Ax} - u_{Bx}$$
$$= (30\sqrt{3} + 50\cos\alpha)\text{ m/s}$$
$$= \left(30\sqrt{3} + 50\times\frac{4}{5}\right)\text{ m/s}$$
$$= (30\sqrt{3} + 40)\text{ m/s}$$

Therefore, time of collision is
$$t = \frac{AB}{|\mathbf{u}_{AB}|} = \frac{100}{30\sqrt{3}+40}$$
or
$$t = 1.09\text{ s}$$ **Ans.**

(c) Distance of point P from A where collision takes place is
$$d = \sqrt{(u_{Ax}t)^2 + \left(u_{Ay}t - \frac{1}{2}gt^2\right)^2}$$
$$= \sqrt{(30\sqrt{3}\times 1.09)^2 + \left(30\times 1.09 - \frac{1}{2}\times 10\times 1.09\times 1.09\right)^2}$$
or
$$d = 62.64\text{ m}$$ **Ans.**

INTRODUCTORY EXERCISE 7.3

1. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10\ \text{m/s}^2$)

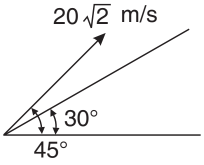


Fig. 7.24

2. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10\ \text{m/s}^2$)

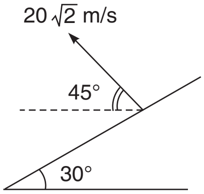


Fig. 7.25

3. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10\ \text{m/s}^2$)

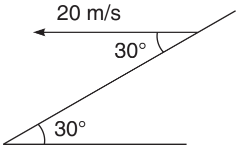


Fig. 7.26

4. Passenger of a train just drops a stone from it. The train was moving with constant velocity. What is path of the stone as observed by (a) the passenger itself, (b) a man standing on ground?
5. A particle is projected upwards with velocity 20 m/s. Simultaneously another particle is projected with velocity $20\sqrt{2}$ m/s at 45°. ($g = 10\ \text{m/s}^2$)
   - (a) What is acceleration of first particle relative to the second?
   - (b) What is initial velocity of first particle relative to the other?
   - (c) What is distance between two particles after 2 s?
6. A particle is projected from the bottom of an inclined plane of inclination 30°. At what angle α (from the horizontal) should the particle be projected to get the maximum range on the inclined plane.

Final Touch Points

1. In projectile motion speed (and hence kinetic energy) is minimum at highest point.

Speed $= (\cos\theta)$ times the speed of projection

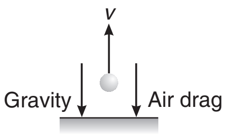
and kinetic energy $= (\cos^2\theta)$ times the initial kinetic energy

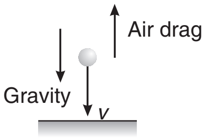
Here, $\theta =$ angle of projection

2. In projectile motion it is sometimes better to write the equations of H, R and T in terms of u_x and u_y as

$$T = \frac{2u_y}{g}, \quad H = \frac{u_y^2}{2g} \quad \text{and} \quad R = \frac{2u_x u_y}{g}$$

3. If a particle is projected vertically upwards, then during upward journey gravity forces (weight) and air drag both are acting downwards. Hence, $|\text{retardation}| > |g|$. During its downward journey air drag is upwards while gravity is downwards. Hence, acceleration $< g$. Therefore we may conclude that,





time of ascent < time of descent

Exercise : In projectile motion, if air drag is taken into consideration than state whether the H, R and T will increase, decrease or remain same.

Ans. T will increase, H will decrease and R may increase, decrease or remain same.

4. At the time of collision coordinates of particles should be same, i.e.

$$x_1 = x_2, \quad \text{and} \quad y_1 = y_2 \qquad \text{(for a 2-D motion)}$$

Similarly

$$x_1 = x_2, \quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2 \qquad \text{(for a 3-D motion)}$$

Two particles collide at the same moment. Of course their time of journeys may be different, i.e. they may start at different times (t_1 and t_2 may be different). If they start together then $t_1 = t_2$.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on the concept that horizontal component of velocity remains unchanged*

This type can be better understood by the following example.

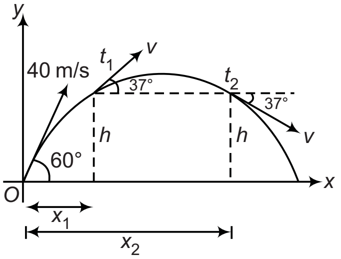
Example 1 *A particle is projected from ground with velocity 40m/s at 60° from horizontal.*

(a) Find the speed when velocity of the particle makes an angle of 37° from horizontal.

(b) Find the time for the above situation.

(c) Find the vertical height and horizontal distance of the particle from the starting point in the above position. Take $g = 10\ m/s^2$.

Solution In the figure shown,



$$u_x = 40\cos 60° = 20\text{ m/s}$$

$$u_y = 40\sin 60° = 20\sqrt{3}\text{ m/s}$$

$$a_x = 0 \text{ and } a_y = -10\text{ m/s}^2$$

(a) Horizontal component of velocity remains unchanged.

$$v_x = u_x$$

or $\quad v\cos 37° = 20$

$\Rightarrow \quad v(0.8) = 20$

$\therefore \quad v = 25\text{ m/s}.$

(b) Using, $v_y = u_y + a_y t \quad \Rightarrow \quad t = \dfrac{v_y - u_y}{a_y}$

For t_1, $\quad t_1 = \dfrac{v\sin 37° - 20\sqrt{3}}{-10} = \dfrac{(25)(0.6) - 20\sqrt{3}}{-10} = 1.96\text{ s}$.

For t_2, $\quad t_2 = \dfrac{-v\sin 37° - 20\sqrt{3}}{-10} = \dfrac{-(25)(0.6) - 20\sqrt{3}}{-10} = 4.96\text{ s}$.

(c) Vertical height $\quad h = s_y \quad$ (at t_1 or t_2)

Let us calculate at t_1

$$\therefore \quad h = u_y\, t_1 + \frac{1}{2} a_y\, t_1^2 = (20\sqrt{3})(1.96) + \frac{1}{2}(-10)(1.96)^2$$

$$= 48.7\text{ m}.$$

Horizontal distances

$$x_1 = u_x t_1 \qquad \text{(as } a_x = 0\text{)}$$

$$= (20)(1.96) = 39.2\text{ m}$$

Similarly, $\quad x_2 = u_x t_2 = (20)(4.96)$

$$= 99.2\text{ m}.$$

Type 2. *Situations where the formulae, H, R and T cannot be applied directly.*

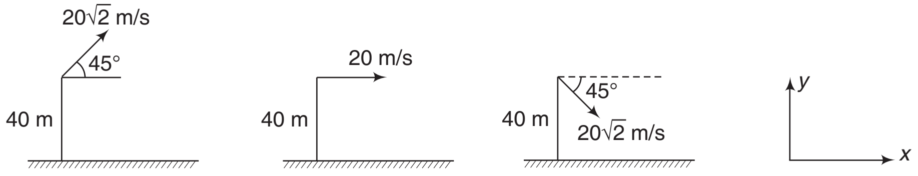
Concept

As discussed earlier also, formulae of H, R and T can be applied directly between two points lying on the same horizontal line.

How to Solve?

- In any other situation apply component method (along x and y-axes). In most of the problems it is advisable to first find the time, using the equation, $s = ut + \frac{1}{2}at^2$ in vertical (or y) direction.

Example 2 *In the figures shown, three particles are thrown from a tower of height* 40 m *as shown in figure. In each case find the time when the particles strike the ground and the distance of this point from foot of the tower.*



(i) (ii) (iii)

Solution For time calculation, apply

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

in vertical direction.

In all figures, $s = -40 \text{ m}, a_y = -10 \text{ m/s}^2$

In first figure $u_y = +20\sqrt{2} \cos 45° = +20 \text{ m/s}$

In second figure, $u_y = 0$

In third figure, $u_y = -20\sqrt{2} \cos 45° = -20 \text{ m/s}$

For horizontal distance,

$$x = u_x t$$

$$u_x = 20 \text{ m/s} \quad \text{in all cases}$$

Substituting the proper values and then solving we get,

In first figure

Time $t_1 = 5.46$ s and horizontal distance $x_1 = 109.2$ m

In second figure

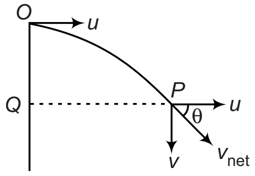
Time $t_2 = 2.83$ s and the horizontal distance $x_2 = 56.6$ m

In third figure

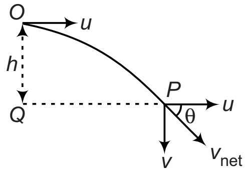
Time $t_3 = 1.46$ s and the horizontal distance $x_3 = 29.2$ m

Type 3. *Horizontal projection of a projectile from some height*

Concept



After time *t*
Fig. (i)



After falling *h*
Fig. (ii)

Suppose a particle is projected from point O with a horizontal velocity 'u' as shown in two figures. Then,

In Fig. (i) or after time *t*

Suppose the particle is at point P, then

$$\text{Horizontal component of velocity} = u$$

$$\text{Vertical component of velocity, } v = gt$$

If $g = 10\,\text{m/s}$, then

$$v = 10t \quad \text{(downwards)}$$

Horizontal distance $QP = ut$ and vertical height fallen $OQ = \frac{1}{2}gt^2$

If $g = 10\,\text{m/s}^2$, then

$$OQ = 5t^2$$

If Fig. (ii) or after falling a height '*h*'

Suppose the particle is at point P, then

Horizontal component of velocity $= u$

Vertical component of velocity $v = \sqrt{2gh}$ (downwards)

Time taken in falling a height h is

$$t = \sqrt{\frac{2h}{g}} \qquad \left(\text{as } h = \frac{1}{2}gt^2\right)$$

The horizontal distance $QP = ut$

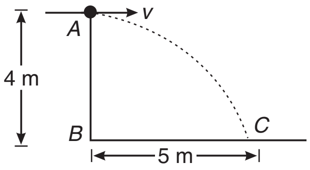
Note *In both figures, net velocity of the particle at point P is,*

$$v_{net} = \sqrt{v^2 + u^2}$$

and the angle θ of this net velocity with horizontal is

$$\tan\theta = \frac{v}{u} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{v}{u}\right)$$

Example 3 *A ball rolls off the edge of a horizontal table top 4 m high. If it strikes the floor at a point 5 m horizontally away from the edge of the table, what was its speed at the instant it left the table?*



Solution Using $h = \frac{1}{2} gt^2$, we have

$$h_{AB} = \frac{1}{2} gt_{AC}^2$$

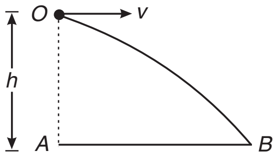
or
$$t_{AC} = \sqrt{\frac{2h_{AB}}{g}}$$

$$= \sqrt{\frac{2 \times 4}{9.8}} = 0.9 \text{ s}$$

Further,
$$BC = vt_{AC}$$

or
$$v = \frac{BC}{t_{AC}} = \frac{5.0}{0.9} = 5.55 \text{ m/s}$$ **Ans.**

Example 4 *An aeroplane is flying in a horizontal direction with a velocity 600 km/h at a height of 1960 m. When it is vertically above the point A on the ground, a body is dropped from it. The body strikes the ground at point B. Calculate the distance AB.*



Solution From $h = \frac{1}{2} gt^2$

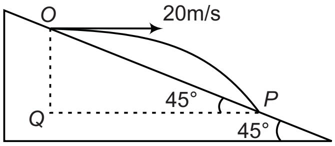
we have,
$$t_{OB} = \sqrt{\frac{2h_{OA}}{g}} = \sqrt{\frac{2 \times 1960}{9.8}} = 20 \text{ s}$$

Horizontal distance
$$AB = vt_{OB}$$

$$= \left(600 \times \frac{5}{18} \text{ m/s}\right)(20 \text{ s})$$

$$= 3333.33 \text{ m} = 3.33 \text{ km}$$ **Ans.**

Example 5 *In the figure shown, find*



(a) the time of flight of the projectile over the inclined plane

(b) range OP

Solution (a) Let the particle strikes the plane at point P at time t, then

$$OQ = \frac{1}{2} g t^2 = 5t^2$$

$$QP = 20\,t$$

In ΔOPQ, angle OPQ is 45°.

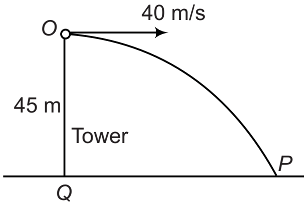
$\therefore$
$$OQ = QP \quad \text{or} \quad 5t^2 = 20t$$

$\therefore$
$$t = 4 \text{ s}$$ **Ans.**

(b)
$$OP = QP \text{ s } 45° = (20t)(\sqrt{2})$$

Substituting $t = 4$ s, we have

$$OP = 80\sqrt{2} \text{ m}$$ **Ans.**



Example 6 *In the figure shown, find*

(a) the time when the particle strikes the ground at P

(b) the horizontal distance QP

(c) velocity of the particle at P

Take $g = 10\ m/s^2$

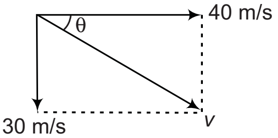
Solution (a) $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 45}{10}} = 3\,\text{s}$

(b) Horizontal distance $QP = 40\,t$

$$= 40 \times 3 = 120\,\text{m}$$ **Ans.**

(c) Horizontal component of velocity at $P = 40$ m/s

Vertical compound of velocity $= gt = 10t = 10 \times 3 = 30\,\text{m/s}$ (downwards)



Net velocity $v = \sqrt{(40)^2 + (30)^2} = 50$ m/s **Ans.**

$$\tan\theta = \frac{30}{40} \quad \text{or} \quad \frac{3}{4}$$

$\therefore$ $$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 37°$$ **Ans.**

Type 4. *Based on trajectory of a projectile*

Concept

We have seen that equation of trajectory of projectile is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

In some problems, the given equation of a projectile is compared with this standard equation to find the unknowns.

Example 7 *A particle moves in the plane xy with constant acceleration 'a' directed along the negative y-axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the particle at the origin of co-ordinates.*

Solution Comparing the given equation with the equation of a projectile motion,

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

We find that $$g = a, \tan\theta = p \text{ and } \frac{a}{2u^2}(1 + \tan^2\theta) = q$$

$\therefore$ u = velocity of particle at origin

$$= \sqrt{\frac{a(1 + \tan^2\theta)}{2q}} = \sqrt{\frac{a(1 + p^2)}{2q}}$$

Type 5. *Based on basic concepts of projectile motion.*

Concept

Following are given some basic concepts of any projectile motion :

(i) Horizontal component of velocity always remains constant.

(ii) Vertical component of velocity changes by 10 m/s or 9.8 m/s in every second in downward direction. For example, if vertical component of velocity at $t=0$ is 30 m/s then change in vertical component in first five seconds is as given in following table:

Time (in sec)	**Vertical component** (in m/s)	**Direction**
0	30	upwards
1	20	upwards
2	10	upwards
3	0	-
4	10	downwards
5	20	downwards

In general we can use, $v_y = u_y + a_y t$

(iii) At a height difference 'h' between two points 1 and 2, the vertical components v_1 and v_2 are related as,

$$v_2 = \pm\sqrt{v_1^2 \pm 2gh}$$

In moving upwards, vertical component decreases. So, take $-2gh$ in the above equation if point 2 is higher than point 1.

(iv) Horizontal displacement is simply,

$$s_x = u_x t \qquad \text{(in the direction of } u_x\text{)}$$

(v) Vertical displacement has two components (say s_1 and s_2), one due to initial velocity u_y and the other due to gravity.

$s_1 = u_y t$ = displacement due to initial component of velocity u_y.

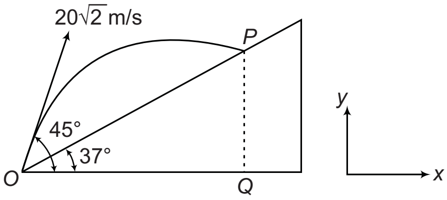
This s_1 is in the direction of u_y (upwards or downwards)

$$s_2 = \frac{1}{2}gt^2 = 5t^2 \qquad (\text{if } g = 10\,\text{m/s}^2)$$

This s_2 is always downwards.

Net vertical displacement is the resultant of s_1 and s_2.

Example 8 *In the figure shown, find*



(a) time of flight of the projectile along the inclined plane.

(b) range OP

Solution (a) Horizontal component of initial velocity,

$$u_x = 20\sqrt{2}\cos 45° = 20 \text{ m/s}$$

Vertical component of initial velocity

$$u_y = 20\sqrt{2}\sin 45° = 20 \text{ m/s}$$

Let the particle strikes at P after time t, then

horizontal displacement $\qquad OQ = u_x t = 20t$

In vertical displacement, $u_y t$ or $20t$ is upwards and $\frac{1}{2}gt^2$ or $5t^2$ is downwards. But net displacement is upwards, therefore $20\,t$ should be greater than $5t^2$ and

$$QP = 20t - 5t^2$$

In ΔOPQ,

$$\tan 37° = \frac{QP}{OQ}$$

or,

$$\frac{3}{4} = \frac{20t - 5t^2}{20t}$$

Solving this equation, we get

$$t = 1 \text{ s}$$ **Ans.**

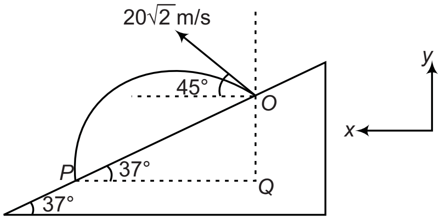
(b) Range $OP = OQ \sec 37°$

$$= (20t)\left(\frac{5}{4}\right)$$

Substituting $t = 1$ s, we have

$$OP = 25 \text{ m}$$ **Ans.**

Example 9 *In the shown figure, find*



(a) time of flight of the projectile along the inclined plane

(b) range OP

Solution (a) Horizontal component of initial velocity, $u_x = 20\sqrt{2}\cos 45° = 20$ m/s

Vertical component of initial velocity,

$$u_y = 20\sqrt{2}\sin 45° = 20 \text{ m/s}$$

Let the particle, strikes the inclined plane at P after time t, then horizontal displacement

$$QP = u_x t = 20t$$

In vertical displacement, $u_y t$ or $20t$ is upwards and $\frac{1}{2}gt^2$ or $5t^2$ is downwards. But net vertical displacement is downwards. Hence $5t^2$ should be greater than $20t$ and therefore,

$$OQ = 5t^2 - 20t$$

In ΔOQP,

$$\tan 37° = \frac{OQ}{QP}$$

or $$\frac{3}{4} = \frac{5t^2 - 20t}{20t}$$

Solving this equation, we get

$$t = 7\,\text{s}$$ **Ans.**

(b) Range, $OP = (PQ)\sec 37°$

$$= (20t)\left(\frac{5}{4}\right)$$

Substituting the value of t, we get

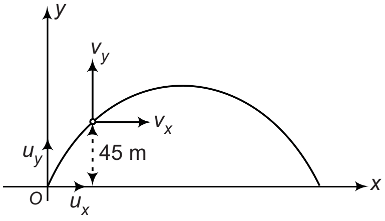
$$OP = 175\,\text{m}$$ **Ans.**

Example 10 *At a height of 45 m from ground velocity of a projectile is,*

$$\mathbf{v} = (30\hat{\mathbf{i}} + 40\hat{\mathbf{j}})\ \text{m/s}$$

Find initial velocity, time of flight, maximum height and horizontal range of this projectile. Here $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *are the unit vectors in horizontal and vertical directions.*

Solution Given, $v_x = 30\,\text{m/s}$ and $v_y = 40\,\text{m/s}$



Horizontal component of velocity remains unchanged.

∴ $$u_x = v_x = 30\,\text{m/s}$$

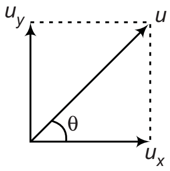
Vertical component of velocity is more at lesser heights. Therefore

$$u_y > v_y$$

or $$u_y = \sqrt{v_y^2 + 2\,gh}$$

$$= \sqrt{(40)^2 + (2)(10)(45)}$$

$$= 50\,\text{m/s}$$



Initial velocity of projectile,

$$u = \sqrt{u_x^2 + u_y^2}$$

$$= \sqrt{(30)^2 + (50)^2}$$

$$= 10\sqrt{34}\ \text{m/s}$$ **Ans.**

$$\tan\theta = \frac{u_y}{u_x} = \frac{50}{30} = \frac{5}{3}$$

∴ $$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$ **Ans.**

Time of flight, $$T = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}$$

$$= \frac{2 \times 50}{10}$$

$$= 10\,\text{s}$$ **Ans.**

Maximum height, $$H=\frac{u^2\sin^2\theta}{2g}=\frac{u_y^2}{2g}$$

$$=\frac{(50)^2}{2\times 10}=125\,\text{m}$$ **Ans.**

Horizontal range $R=u_xT$

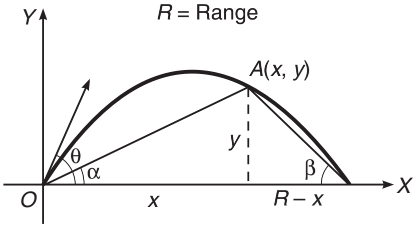
$$=30\times 10$$

$$=300\,\text{m}$$ **Ans.**

Miscellaneous Examples

Example 11 *A particle is thrown over a triangle from one end of a horizontal base and after grazing the vertex falls on the other end of the base. If* α *and* β *be the base angles and* θ *the angle of projection, prove that* tan θ = tan α + tan β.

Solution The situation is shown in figure.



From figure, we have

$$\tan\alpha+\tan\beta=\frac{y}{x}+\frac{y}{R-x}$$

$$\tan\alpha+\tan\beta=\frac{yR}{x(R-x)} \qquad \text{...(i)}$$

Equation of trajectory is

$$y=x\tan\theta\left[1-\frac{x}{R}\right]$$

or, $$\tan\theta=\frac{yR}{x(R-x)} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\tan\theta=\tan\alpha+\tan\beta$$ **Hence Proved.**

Example 12 *The velocity of a projectile when it is at the greatest height is* $\sqrt{2/5}$ *times its velocity when it is at half of its greatest height. Determine its angle of projection.*

Solution Suppose the particle is projected with velocity u at an angle θ with the horizontal. Horizontal component of its velocity at all height will be $u\cos\theta$.

At the greatest height, the vertical component of velocity is zero, so the resultant velocity is

$$v_1 = u\cos\theta$$

At half the greatest height during upward motion,

$$y = h/2,\ a_y = -g,\ u_y = u\sin\theta$$

Using $$v_y^2 - u_y^2 = 2a_y y$$

we get, $$v_y^2 - u^2\sin^2\theta = 2(-g)\frac{h}{2}$$

or $$v_y^2 = u^2\sin^2\theta - g\times\frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2\theta}{2} \qquad \left[\because\ h = \frac{u^2\sin^2\theta}{2g}\right]$$

or $$v_y = \frac{u\sin\theta}{\sqrt{2}}$$

Hence, resultant velocity at half of the greatest height is

$$v_2 = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{u^2\cos^2\theta + \frac{u^2\sin^2\theta}{2}}$$

Given, $$\frac{v_1}{v_2} = \sqrt{\frac{2}{5}}$$

$\therefore$ $$\frac{v_1^2}{v_2^2} = \frac{u^2\cos^2\theta}{u^2\cos^2\theta + \dfrac{u^2\sin^2\theta}{2}} = \frac{2}{5}$$

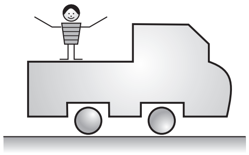
or $$\frac{1}{1 + \frac{1}{2}\tan^2\theta} = \frac{2}{5}$$

or $$2 + \tan^2\theta = 5 \quad\text{or}\quad \tan^2\theta = 3$$

or $$\tan\theta = \sqrt{3}$$

$\therefore$ $$\theta = 60°$$ **Ans.**

Example 13 *A car accelerating at the rate of $2\ m/s^2$ from rest from origin is carrying a man at the rear end who has a gun in his hand. The car is always moving along positive x-axis. At $t = 4$ s, the man fires a bullet from the gun and the bullet hits a bird at $t = 8$ s. The bird has a position vector $40\hat{\mathbf{i}} + 80\hat{\mathbf{j}} + 40\hat{\mathbf{k}}$. Find velocity of projection of the bullet. Take the y-axis in the horizontal plane. ($g = 10\ m/s^2$)*



Solution Let velocity of bullet be,

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

At $t = 4$ s, x-coordinate of car is $$x_c = \frac{1}{2}at^2 = \frac{1}{2}\times 2\times 16 = 16\text{ m}$$

x-coordinate of bird is $x_b = 40$ m

$\therefore$ $$x_b = x_c + v_x(8-4)$$

or $$40 = 16 + 4v_x$$

$\therefore$ $$v_x = 6\text{ m/s}$$

Similarly, $y_b = y_c + v_y\ (8-4)$

or $80 = 0 + 4v_y$

or $v_y = 20$ m/s

and $z_b = z_c + v_z\ (8-4) - \frac{1}{2} g\ (8-4)^2$

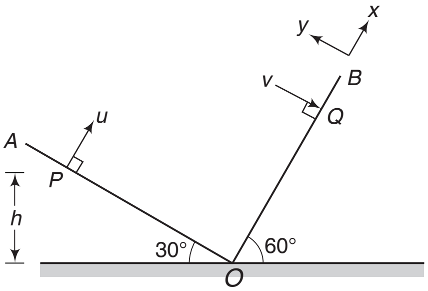
or $40 = 0 + 4v_z - \frac{1}{2} \times 10 \times 16$

or $v_z = 30$ m/s

∴ Velocity of projection of bullet

$$\mathbf{v} = (6\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 30\hat{\mathbf{k}})\ \text{m/s}$$

Example 14 *Two inclined planes OA and OB having inclinations 30° and 60° with the horizontal respectively intersect each other at O, as shown in figure. A particle is projected from point P with velocity $u = 10\sqrt{3}$ m/s along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicular at Q. Calculate*



(a) time of flight,

(b) velocity with which the particle strikes the plane OB,

(c) height h of point P from point O,

(d) distance PQ. (Take $g = 10\ m/s^2$)

Solution Let us choose the x and y directions along OB and OA respectively. Then,

$$u_x = u = 10\sqrt{3}\ \text{m/s},\ u_y = 0$$

$$a_x = -g \sin 60° = -5\sqrt{3}\ \text{m/s}^2$$

and $$a_y = -g \cos 60° = -5\ \text{m/s}^2$$

(a) At point Q, x-component of velocity is zero. Hence, substituting in

$$v_x = u_x + a_x t$$

$$0 = 10\sqrt{3} - 5\sqrt{3}t$$

or $$t = \frac{10\sqrt{3}}{5\sqrt{3}} = 2\ \text{s}$$ **Ans.**

(b) At point Q, $v = v_y = u_y + a_y t$

∴ $v = 0 - (5)(2) = -10$ m/s **Ans.**

Here, negative sign implies that velocity of particle at Q is along negative y-direction.

(c) Distance $PO = |$ displacement of particle along y-direction $| = |s_y|$

Here, $$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$= 0 - \frac{1}{2}(5)(2)^2 = -10\,\text{m}$$

∴ $$PO = 10\text{ m}$$

Therefore, $$h = PO \sin 30° = (10)\left(\frac{1}{2}\right)$$

or $$h = 5\text{ m}$$ **Ans.**

(d) Distance $OQ =$ displacement of particle along x-direction $= s_x$

Here, $$s_x = u_x t + \frac{1}{2} a_x\, t^2$$

$$= (10\sqrt{3})(2) - \frac{1}{2}(5\sqrt{3})(2)^2 = 10\sqrt{3}\text{ m}$$

or $$OQ = 10\sqrt{3}\text{ m}$$

∴ $$PQ = \sqrt{(PO)^2 + (OQ)^2}$$

$$= \sqrt{(10)^2 + (10\sqrt{3})^2}$$

$$= \sqrt{100 + 300} = \sqrt{400}$$

∴ $$PQ = 20\text{ m}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : A particle follows only a parabolic path if acceleration is constant.

Reason : In projectile motion path is parabolic, as acceleration is assumed to be constant at low heights.

2. Assertion : Projectile motion is called a two dimensional motion, although it takes place in space.

Reason : In space it takes place in a plane.

3. Assertion : If time of flight in a projectile motion is made two times, its maximum height will become four times.

Reason : In projectile motion $H \propto T^2$, where H is maximum height and T the time of flight.

4. Assertion : A particle is projected with velocity $\mathbf{u}$ at angle 45° with ground. Let $\mathbf{v}$ be the velocity of particle at time $t (\neq 0)$, then value of $\mathbf{u} \cdot \mathbf{v}$ can be zero.

Reason : Value of dot product is zero when angle between two vectors is 90°.

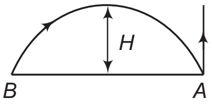
5. Assertion : A particle has constant acceleration is x-y plane. But neither of its acceleration components (a_x and a_y) is zero. Under this condition particle cannot have parabolic path.

Reason : In projectile motion, horizontal component of acceleration is zero.

6. Assertion : In projectile motion at any two positions $\frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$ always remains constant.

Reason : The given quantity is average acceleration, which should remain constant as acceleration is constant.

7. Assertion : Particle A is projected upwards. Simultaneously particle B is projected as projectile as shown. Particle A returns to ground in 4 s. At the same time particle B collides with A. Maximum height H attained by B would be 20 m. ($g = 10 \text{ ms}^{-2}$)



Reason : Speed of projection of both the particles should be same under the given condition.

8. Assertion : Two projectiles have maximum heights $4H$ and H respectively. The ratio of their horizontal components of velocities should be 1 : 2 for their horizontal ranges to be same.

Reason : Horizontal range = horizontal component of velocity × time of flight.

9. **Assertion :** If $g = 10\ \text{m/s}^2$ then in projectile motion speed of particle in every second will change by $10\ \text{ms}^{-1}$.

Reason : Acceleration is nothing but rate of change of velocity.

10. **Assertion :** In projectile motion if particle is projected with speed u, then speed of particle at height h would be $\sqrt{u^2 - 2gh}$.

Reason : If particle is projected with vertical component of velocity u_y. Then vertical component at the height h would be $\pm\sqrt{u_y^2 - 2gh}$

Objective Questions

Single Correct Option

1. Identify the correct statement related to the projectile motion.

(a) It is uniformly accelerated everywhere

(b) It is uniformly accelerated everywhere except at the highest position where it is moving with constant velocity

(c) Acceleration is never perpendicular to velocity

(d) None of the above

2. Two bodies are thrown with the same initial velocity at angles θ and $(90° - \theta)$ respectively with the horizontal, then their maximum heights are in the ratio

(a) 1 : 1 (b) $\sin\theta : \cos\theta$ (c) $\sin^2\theta : \cos^2\theta$ (d) $\cos\theta : \sin\theta$

3. The range of a projectile at an angle θ is equal to half of the maximum range if thrown at the same speed. The angle of projection θ is given by

(a) 15° (b) 30° (c) 60° (d) data insufficient

4. A ball is projected with a velocity $20\ \text{ms}^{-1}$ at an angle to the horizontal. In order to have the maximum range. Its velocity at the highest position must be

(a) $10\ \text{ms}^{-1}$ (b) $14\ \text{ms}^{-1}$ (c) $18\ \text{ms}^{-1}$ (d) $16\ \text{ms}^{-1}$

5. A particle has initial velocity, $\mathbf{v} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ and a constant force $\mathbf{F} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ acts on it. The path of the particle is

(a) straight line (b) parabolic (c) circular (d) elliptical

6. A body is projected at an angle 60° with the horizontal with kinetic energy K. When the velocity makes an angle 30° with the horizontal, the kinetic energy of the body will be

(a) $K/2$ (b) $K/3$ (c) $2K/3$ (d) $3K/4$

7. If T_1 and T_2 are the times of flight for two complementary angles, then the range of projectile R is given by

(a) $R = 4gT_1T_2$ (b) $R = 2gT_1T_2$ (c) $R = \frac{1}{4}gT_1T_2$ (d) $R = \frac{1}{2}gT_1T_2$

8. A gun is firing bullets with velocity v_0 by rotating it through 360° in the horizontal plane. The maximum area covered by the bullets is

(a) $\frac{\pi v_0^2}{g}$ (b) $\frac{\pi^2 v_0^2}{g}$ (c) $\frac{\pi v_0^4}{g^2}$ (d) $\frac{\pi^2 v_0^4}{g}$

9. A grass hopper can jump maximum distance 1.6 m. It spends negligible time on ground. How far can it go in $10\sqrt{2}$ s ?

(a) 45 m (b) 30 m (c) 20 m (d) 40 m

10. Two stones are projected with the same speed but making different angles with the horizontal. Their horizontal ranges are equal. The angle of projection of one is $\frac{\pi}{3}$ and the maximum height reached by it is 102 m. Then the maximum height reached by the other in metres is

(a) 76	(b) 84
(c) 56	(d) 34

11. A ball is projected upwards from the top of a tower with a velocity 50 ms^{-1} making an angle 30° with the horizontal. The height of tower is 70 m. After how many seconds from the instant of throwing, will the ball reach the ground. ($g = 10\ ms^{-2}$)

(a) 2 s	(b) 5 s	(c) 7 s	(d) 9 s

12. Average velocity of a particle in projectile motion between its starting point and the highest point of its trajectory is (projection speed = u, angle of projection from horizontal = θ)

(a) $u\cos\theta$	(b) $\frac{u}{2}\sqrt{1 + 3\cos^2\theta}$
(c) $\frac{u}{2}\sqrt{2 + \cos^2\theta}$	(d) $\frac{u}{2}\sqrt{1 + \cos^2\theta}$

13. A train is moving on a track at 30 ms^{-1}. A ball is thrown from it perpendicular to the direction of motion with 30 ms^{-1} at 45° from horizontal. Find the distance of ball from the point of projection on train to the point where it strikes the ground.

(a) 90 m	(b) $90\sqrt{3}$ m	(c) 60 m	(d) $60\sqrt{3}$ m

14. A body is projected at time $t = 0$ from a certain point on a planet's surface with a certain velocity at a certain angle with the planet's surface (assumed horizontal). The horizontal and vertical displacements x and y (in metre) respectively vary with time t in second as, $x = (10\sqrt{3})\,t$ and $y = 10t - t^2$. The maximum height attained by the body is

(a) 75 m	(b) 100 m
(c) 50 m	(d) 25 m

15. A particle is fired horizontally from an inclined plane of inclination 30° with horizontal with speed 50 ms^{-1}. If $g = 10\ ms^{-2}$, the range measured along the incline is

(a) 500 m	(b) $\frac{1000}{3}$ m	(c) $200\sqrt{2}$ m	(d) $100\sqrt{3}$ m

16. A fixed mortar fires a bomb at an angle of 53° above the horizontal with a muzzle velocity of 80 ms^{-1}. A tank is advancing directly towards the mortar on level ground at a constant speed of 5 m/s. The initial separation (at the instant mortar is fired) between the mortar and tank, so that the tank would be hit is [Take $g = 10\ ms^{-2}$]

(a) 662.4 m	(b) 526.3 m
(c) 486.6 m	(d) None of these

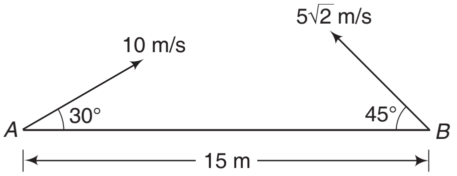
Subjective Questions

1. At time $t = 0$, a small ball is projected from point A with a velocity of 60 m/s at 60° angle with horizontal. Neglect atmospheric resistance and determine the two times t_1 and t_2 when the velocity of the ball makes an angle of 45° with the horizontal x-axis.

2. A particle is projected from ground with velocity $20\sqrt{2}$ m/s at 45°. At what time particle is at height 15 m from ground? ($g = 10\ m/s^2$)

3. A particle is projected at an angle 60° with horizontal with a speed $v = 20$ m/s. Taking $g = 10\ \text{m/s}^2$. Find the time after which the speed of the particle remains half of its initial speed.

4. Two particles A and B are projected from ground towards each other with speeds 10 m/s and $5\sqrt{2}$ m/s at angles 30° and 45° with horizontal from two points separated by a distance of 15 m. Will they collide or not?

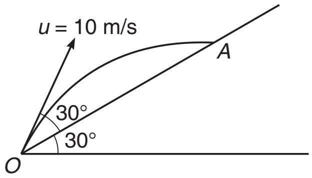


5. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located over a tower at one point and moved with velocities $v_1 = 3$ m/s and $v_2 = 4$ m/s horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

6. A ball is thrown from the ground to clear a wall 3 m high at a distance of 6 m and falls 18 m away from the wall. Find the angle of projection of ball.

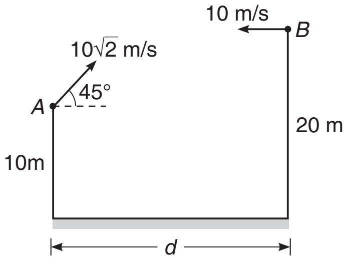
7. A body is projected up such that its position vector varies with time as $\mathbf{r} = \{3t\,\hat{\mathbf{i}} + (4t - 5t^2)\hat{\mathbf{j}}\}$ m. Here, t is in seconds. Find the time and x-coordinate of particle when its y-coordinate is zero.

8. A particle is projected along an inclined plane as shown in figure. What is the speed of the particle when it collides at point A ? ($g = 10\ \text{m/s}^2$)



9. In the above problem, what is the component of its velocity perpendicular to the plane when it strikes at A ?

10. Two particles A and B are projected simultaneously from two towers of heights 10 m and 20 m respectively. Particle A is projected with an initial speed of $10\sqrt{2}$ m/s at an angle of 45° with horizontal, while particle B is projected horizontally with speed 10 m/s. If they collide in air, what is the distance d between the towers?

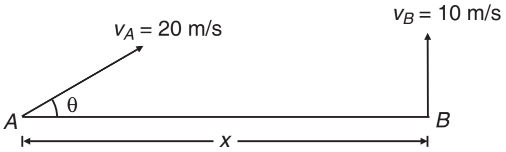


11. A particle is projected from the bottom of an inclined plane of inclination 30° with velocity of 40 m/s at an angle of 60° with horizontal. Find the speed of the particle when its velocity vector is parallel to the plane. Take $g = 10\,\text{m/s}^2$.

12. Two particles A and B are projected simultaneously in the directions shown in figure with velocities $v_A = 20$ m/s and $v_B = 10$ m/s respectively. They collide in air after $\frac{1}{2}$ s. Find

(a) the angle θ

(b) the distance x.



13. A ball is shot from the ground into the air. At a height of 9.1 m, its velocity is observed to be $\mathbf{v} = 7.6\hat{\mathbf{i}} + 6.1\hat{\mathbf{j}}$ in metre per second ($\hat{\mathbf{i}}$ is horizontal, $\hat{\mathbf{j}}$ is upward). Give the approximate answers.

(a) To what maximum height does the ball rise?

(b) What total horizontal distance does the ball travel?

(c) What are the magnitude and

(d) What are the direction of the ball's velocity just before it hits the ground?

14. A particle is projected with velocity $2\sqrt{gh}$, so that it just clears two walls of equal height h which are at a distance of $2h$ from each other. Show that the time of passing between the walls is $2\sqrt{\frac{h}{g}}$.

[**Hint :** First find velocity at height h. Treat it as initial velocity and 2h as the range.]

15. A particle is projected at an angle of elevation α and after t second it appears to have an elevation of β as seen from the point of projection. Find the initial velocity of projection.

16. A projectile aimed at a mark, which is in the horizontal plane through the point of projection, falls a cm short of it when the elevation is α and goes b cm far when the elevation is β. Show that, if the speed of projection is same in all the cases the proper elevation is

$$\frac{1}{2}\sin^{-1}\left[\frac{b\sin 2\alpha + a\sin 2\beta}{a+b}\right]$$

17. Two particles are simultaneously thrown in horizontal direction from two points on a riverbank, which are at certain height above the water surface. The initial velocities of the particles are $v_1 = 5$ m/s and $v_2 = 7.5$ m/s respectively. Both particles fall into the water at the same time. First particle enters the water at a point $s = 10$ m from the bank. Determine

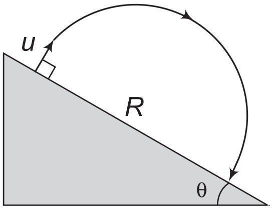
(a) the time of flight of the two particles,

(b) the height from which they are thrown,

(c) the point where the second particle falls in water.

18. A balloon is ascending at the rate $v = 12$ km/h and is being carried horizontally by the wind at $v_w = 20$ km/h. If a ballast bag is dropped from the balloon at the instant $h = 50$ m, determine the time needed for it to strike the ground. Assume that the bag was released from the balloon with the same velocity as the balloon. Also, find the speed with which the bag strikes the ground?

19. A projectile is fired with a velocity u at right angles to the slope, which is inclined at an angle θ with the horizontal. Derive an expression for the distance R to the point of impact.



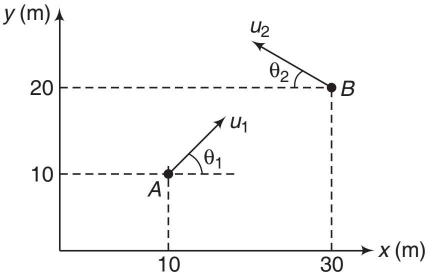
20. An elevator is going up with an upward acceleration of $1\ \mathrm{m/s^2}$. At the instant when its velocity is 2 m/s, a stone is projected upward from its floor with a speed of 2 m/s relative to the elevator, at an elevation of 30°.

(a) Calculate the time taken by the stone to return to the floor.

(b) Sketch the path of the projectile as observed by an observer outside the elevator.

(c) If the elevator was moving with a downward acceleration equal to g, how would the motion be altered?

21. Two particles A and B are projected simultaneously in a vertical plane as shown in figure. They collide at time t in air. Write down two necessary equations for collision to take place.



LEVEL 2

Objective Questions

Single Correct Option

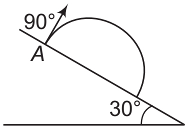
1. Two bodies were thrown simultaneously from the same point, one straight up, and the other, at an angle of $\theta = 30^\circ$ to the horizontal. The initial velocity of each body is $20\ \mathrm{ms^{-1}}$. Neglecting air resistance, the distance between the bodies at $t = 1.2$ later is

(a) 20 m
(b) 30 m
(c) 24 m
(d) 50 m

2. A particle is dropped from a height h. Another particle which is initially at a horizontal distance d from the first is simultaneously projected with a horizontal velocity u and the two particles just collide on the ground. Then

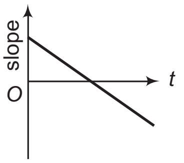
(a) $d^2 = \dfrac{u^2h}{2h}$
(b) $d^2 = \dfrac{2u^2h}{g}$
(c) $d = h$
(d) $gd^2 = u^2h$

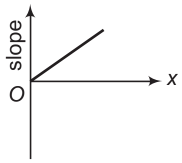
3. A ball is projected from point A with velocity $10\ \text{ms}^{-1}$ perpendicular to the inclined plane as shown in figure. Range of the ball on the inclined plane is

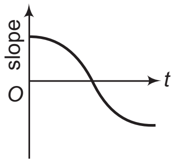


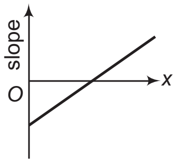
(a) $\frac{40}{3}$ m (b) $\frac{20}{3}$ m (c) $\frac{12}{3}$ m (d) $\frac{60}{3}$ m

4. A heavy particle is projected with a velocity at an angle with the horizontal into the uniform gravitational field. The slope of the trajectory of the particle varies as

(a)


(b)


(c)


(d)


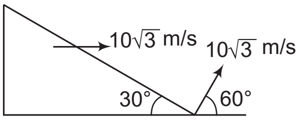
5. A particle starts from the origin of coordinates at time $t = 0$ and moves in the xy plane with a constant acceleration α in the y-direction. Its equation of motion is $y = \beta x^2$. Its velocity component in the x-direction is

(a) variable (b) $\sqrt{\frac{2\alpha}{\beta}}$ (c) $\frac{\alpha}{2\beta}$ (d) $\sqrt{\frac{\alpha}{2\beta}}$

6. A projectile is projected with speed u at an angle of 60° with horizontal from the foot of an inclined plane. If the projectile hits the inclined plane horizontally, the range on inclined plane will be

(a) $\frac{u^2\sqrt{21}}{2g}$ (b) $\frac{3u^2}{4g}$ (c) $\frac{u^2}{2\beta}$ (d) $\frac{\sqrt{21}\,u^2}{8g}$

7. A particle is projected at an angle 60° with speed $10\sqrt{3}$ m/s, from the point A, as shown in the figure. At the same time the wedge is made to move with speed $10\sqrt{3}$ m/s towards right as shown in the figure. Then the time after which particle will strike with wedge is

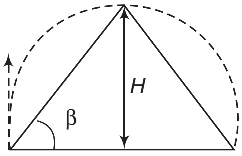


(a) 2 s (b) $2\sqrt{3}$ s (c) $\frac{4}{\sqrt{3}}$ s (d) None of these

8. A particle moves along the parabolic path $x = y^2 + 2y + 2$ in such a way that Y-component of velocity vector remains $5\ \text{ms}^{-1}$ during the motion. The magnitude of the acceleration of the particle is

(a) $50\ \text{ms}^{-2}$ (b) $100\ \text{ms}^{-2}$ (c) $10\sqrt{2}\ \text{ms}^{-2}$ (d) $0.1\ \text{ms}^{-2}$

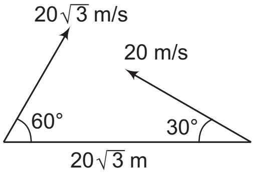
9. A shell fired from the base of a mountain just clears it. If α is the angle of projection, then the angular elevation of the summit β is



(a) $\frac{\alpha}{2}$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}\left(\frac{\tan\alpha}{2}\right)$ (d) $\tan^{-1}(2\tan\alpha)$

10. In the figure shown, the two projectiles are fired simultaneously. The minimum distance between them during their flight is



(a) 20 m
(b) $10\sqrt{3}$ m
(c) 10 m
(d) None of the above

More than One Correct Options

1. Two particles projected from the same point with same speed u at angles of projection α and β strike the horizontal ground at the same point. If h_1 and h_2 are the maximum heights attained by the projectile, R is the range for both and t_1 and t_2 are their times of flights, respectively , then

(a) $\alpha + \beta = \frac{\pi}{2}$
(b) $R = 4\sqrt{h_1 h_2}$
(c) $\frac{t_1}{t_2} = \tan\alpha$
(d) $\tan\alpha = \sqrt{\frac{h_1}{h_2}}$

2. A ball is dropped from a height of 49 m. The wind is blowing horizontally. Due to wind a constant horizontal acceleration is provided to the ball. Choose the correct statement (s). [Take $g = 9.8\ \text{ms}^{-2}$]

(a) Path of the ball is a straight line
(b) Path of the ball is a curved one
(c) The time taken by the ball to reach the ground is 3.16 s
(d) Actual distance travelled by the ball is more then 49 m

3. A particle is projected from a point P with a velocity v at an angle θ with horizontal. At a certain point Q it moves at right angles to its initial direction. Then

(a) velocity of particle at Q is $v\sin\theta$
(b) velocity of particle at Q is $v\cot\theta$
(c) time of flight from P to Q is $(v/g)\operatorname{cosec}\theta$
(d) time of flight from P to Q is $(v/g)\sec\theta$

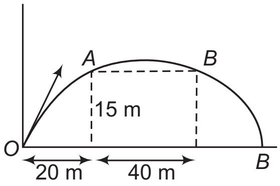
4. At a height of 15 m from ground velocity of a projectile is $\mathbf{v} = (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$. Here, $\hat{\mathbf{j}}$ is vertically upwards and $\hat{\mathbf{i}}$ is along horizontal direction then ($g = 10\ \text{ms}^{-2}$)

(a) particle was projected at an angle of 45° with horizontal
(b) time of flight of projectile is 4 s
(c) horizontal range of projectile is 100 m
(d) maximum height of projectile from ground is 20 m

5. Which of the following quantities remain constant during projectile motion?

(a) Average velocity between two points
(b) Average speed between two points
(c) $\frac{d\mathbf{v}}{dt}$
(d) $\frac{d^2\mathbf{v}}{dt^2}$

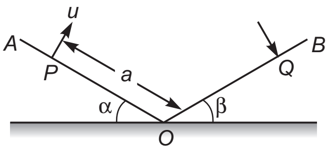
6. In the projectile motion shown is figure, given $t_{AB} = 2$ s then ($g = 10\ \text{ms}^{-2}$)



(a) particle is at point B at 3 s
(b) maximum height of projectile is 20 m
(c) initial vertical component of velocity is $20\ \text{ms}^{-1}$
(d) horizontal component of velocity is $20\ \text{ms}^{-1}$

Comprehension Based Questions

Passage (Q. Nos. 1 to 2)



Two inclined planes OA and OB intersect in a horizontal plane having their inclinations α and β *with the horizontal as shown in figure. A particle is projected from point P with velocity u along a direction perpendicular to plane OA. The particle strikes plane OB perpendicularly at Q.*

1. If $\alpha = 30^\circ$, $\beta = 30^\circ$, the time of flight from P to Q is

(a) $\frac{u}{g}$ (b) $\frac{\sqrt{3}\,u}{g}$ (c) $\frac{\sqrt{2}\,u}{g}$ (d) $\frac{2\,u}{g}$

2. If $\alpha = 30^\circ$, $\beta = 30^\circ$ and $a = 4.9$ m, the initial velocity of projection is

(a) 9.8 ms^{-1} (b) 4.9 ms^{-1} (c) $4.9\sqrt{2}$ ms^{-1} (d) 19.6 ms^{-1}

Match the Columns

1. Particle-1 is just dropped from a tower. 1 s later particle-2 is thrown from the same tower horizontally with velocity 10 ms^{-1}. Taking $g = 10$ ms^{-2}, match the following two columns at $t = 2$ s.

Column I	Column II
(a) Horizontal displacement between two	(p) 10 SI units
(b) Vertical displacement between two	(q) 20 SI units
(c) Magnitude of relative horizontal component of velocity	(r) $10\sqrt{2}$ SI units
(d) Magnitude of relative vertical component of velocity	(s) None of the above

2. In a projectile motion, given $H = \frac{R}{2} = 20$ m. Here, H is maximum height and R the horizontal range. For the given condition match the following two columns.

Column I	Column II
(a) Time of flight	(p) 1
(b) Ratio of vertical component of velocity and horizontal component of velocity	(q) 2
(c) Horizontal component of velocity (in m/s)	(r) 10
(d) Vertical component of velocity (in m/s)	(s) None of the above

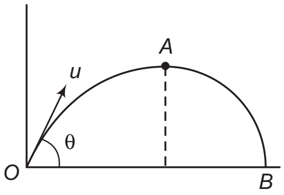
3. A particle can be thrown at a constant speed at different angles. When it is thrown at 15° with horizontal, it falls at a distance of 10 m from point of projection. For this speed of particle match following two columns.

Column I	Column II
(a) Maximum horizontal range which can be taken with this speed	(p) 10 m
(b) Maximum height which can be taken with this speed	(q) 20 m
(c) Range at 75°	(r) 15 m
(d) Height at 30°	(s) None of the above

4. In projectile motion, if vertical component of velocity is increased to two times, keeping horizontal component unchanged, then

Column I	Column II
(a) Time of flight	(p) will remain same
(b) Maximum height	(q) will become two times
(c) Horizontal range	(r) will become four times
(d) Angle of projection with horizontal	(s) None of the above

5. In projectile motion shown in figure.

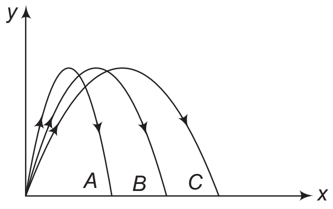


Column I	Column II
(a) Change in velocity between O and A	(p) $u \cos\theta$
(b) Average velocity between O and A	(q) $u \sin\theta$
(c) Change in velocity between O and B	(r) $2u\sin\theta$
(d) Average velocity between O and B	(s) None of the above

6. Particle-1 is projected from ground (take it origin) at time $t = 0$, with velocity $(30\hat{\mathbf{i}} + 30\hat{\mathbf{j}})$ ms^{-1}. Particle-2 is projected from (130 m, 75 m) at time $t = 1$ s with velocity $(-20\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}})$ ms^{-1}. Assuming $\hat{\mathbf{j}}$ to be vertically upward and $\hat{\mathbf{i}}$ to be in horizontal direction, match the following two columns at $t = 2$ s.

Column I	Column II
(a) horizontal distance between two	(p) 30 SI units
(b) vertical distance between two	(q) 40 SI units
(c) relative horizontal component of velocity between two	(r) 50 SI units
(d) relative vertical component of velocity between two	(s) None of the above

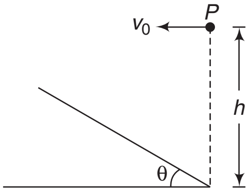
7. The trajectories of the motion of three particles are shown in the figure. Match the entries of Column I with the entries of Column II. Neglect air resistance.



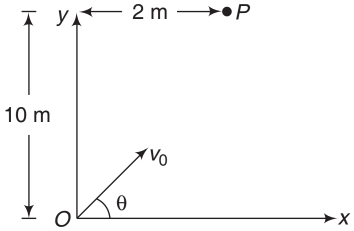
Column I	Column II
(a) Time of flight is least for	(p) A
(b) Vertical component of velocity is greatest for	(q) B
(c) Horizontal component of velocity is greatest for	(r) C
(d) Launch speed is least for	(s) same for all

Subjective Questions

1. Determine the horizontal velocity v_0 with which a stone must be projected horizontally from a point P, so that it may hit the inclined plane perpendicularly. The inclination of the plane with the horizontal is θ and point P is at a height h above the foot of the incline, as shown in the figure.



2. A particle is dropped from point P at time $t = 0$. At the same time another particle is thrown from point O as shown in the figure and it collides with the particle P. Acceleration due to gravity is along the negative y-axis. If the two particles collide 2 s after they start, find the initial velocity v_0 of the particle which was projected from O. Point O is not necessarily on ground.



3. Two particles are simultaneously projected in the same vertical plane from the same point with velocities u and v at angles α and β with horizontal. Find the time that elapses when their velocities are parallel.

4. A projectile takes off with an initial velocity of 10 m/s at an angle of elevation of 45°. It is just able to clear two hurdles of height 2 m each, separated from each other by a distance d. Calculate d. At what distance from the point of projection is the first hurdle placed? Take $g = 10 \text{ m/s}^2$.

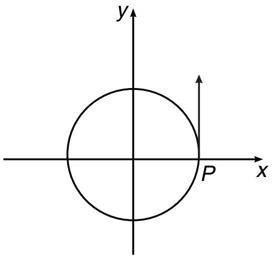
5. A stone is projected from the ground in such a direction so as to hit a bird on the top of a telegraph post of height h and attains the maximum height of $2h$ above the ground. If at the instant of projection, the bird were to fly away horizontally with a uniform speed, find the ratio between the horizontal velocity of bird and the horizontal component of velocity of stone, if the stone hits the bird while descending.

6. A particle is released from a certain height $H = 400$ m. Due to the wind, the particle gathers the horizontal velocity component $v_x = ay$ where $a = \sqrt{5}\ \text{s}^{-1}$ and y is the vertical displacement of the particle from the point of release, then find

(a) the horizontal drift of the particle when it strikes the ground,

(b) the speed with which particle strikes the ground.
(Take $g = 10\ \text{m/s}^2$)

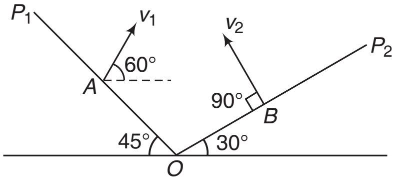
7. A train is moving with a constant speed of 10 m/s in a circle of radius $\dfrac{16}{\pi}$ m. The plane of the circle lies in horizontal x-y plane. At time $t = 0$, train is at point P and moving in counter-clockwise direction. At this instant, a stone is thrown from the train with speed 10 m/s relative to train towards negative x-axis at an angle of 37° with vertical z-axis. Find



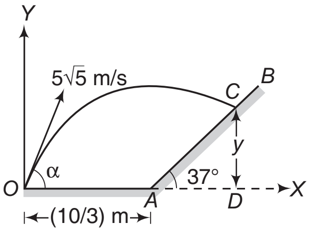
(a) the velocity of particle relative to train at the highest point of its trajectory.

(b) the co-ordinates of points on the ground where it finally falls and that of the highest point of its trajectory.
Take $g = 10\ \text{m/s}^2, \sin 37° = \dfrac{3}{5}$

8. A particle is projected from an inclined plane OP_1 from A with velocity $v_1 = 8\ \text{ms}^{-1}$ at an angle 60° with horizontal. An another particle is projected at the same instant from B with velocity $v_2 = 16\ \text{ms}^{-1}$ and perpendicular to the plane OP_2 as shown in figure. After time $10\sqrt{3}$ s there separation was minimum and found to be 70 m. Then find distance AB.



9. A particle is projected from point O on the ground with velocity $u = 5\sqrt{5}$ m/s at angle $\alpha = \tan^{-1}(0.5)$. It strikes at a point C on a fixed smooth plane AB having inclination of 37° with horizontal as shown in figure. If the particle does not rebound, calculate



(a) coordinates of point C in reference to coordinate system as shown in the figure.

(b) maximum height from the ground to which the particle rises.
($g = 10\ \text{m/s}^2$).

10. A plank fitted with a gun is moving on a horizontal surface with speed of 4 m/s along the positive x-axis. The z-axis is in vertically upward direction. The mass of the plank including the mass of the gun is 50 kg. When the plank reaches the origin, a shell of mass 10 kg is fired at an angle of 60° with the positive x-axis with a speed of $v = 20$ m/s with respect to the gun in x-z plane. Find the position vector of the shell at $t = 2$ s after firing it. Take $g = 9.8\ \text{m/s}^2$.

Answers

Introductory Exercise 7.1

1. $\sqrt{2}$ s
2. False
3. True
4. $\mathbf{v} = (40\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s, $\mathbf{s} = (80\hat{\mathbf{i}} + 40\hat{\mathbf{j}})$ m
5. $t = 5$ s, $d = 100$ m, $\mathbf{v} = (20\hat{\mathbf{i}} - 30\hat{\mathbf{j}})$ ms^{-1}

Introductory Exercise 7.2

1. (a) $20\sqrt{2}$ m/s at angle $\tan^{-1}\left(\frac{1}{2}\right)$ with horizontal, (b) 100 m.
2. Between two points lying on the same horizontal line
3. $u \cos \alpha$
4. $2u \sin \alpha$, downwards
5. (a) 80 m, 20 m, 4s (b) $(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ ms^{-1} (c) $(20\hat{\mathbf{i}} - 20\hat{\mathbf{j}})$ ms^{-1}
6. (a) 30 ms^{-1} (vertically downwards) (b) 20.62 ms^{-1}
7. $\frac{5}{\sqrt{2}}$ ms^{-1}
8. (a) $\sqrt{20}$ s (b) $20\sqrt{20}$ m (c) 49 m/s, $\theta = \tan^{-1}(\sqrt{5})$ with horizontal
9. No
10. $\sqrt{\frac{a(1+b^2)}{2c}}$

Introductory Exercise 7.3

1. 1.69 s, 39 m
2. 6.31 s, 145.71 m
3. 2.31 s, 53.33 mm
4. (a) A vertical straight line (b) A parabola
5. (a) zero (b) 20 ms^{-1} in horizontal direction (c) 40 m
6. 60°

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (a) **4.** (b) **5.** (d) **6.** (a) **7.** (c) **8.** (a or b) **9.** (d) **10.** (b)

Single Correct Option

1. (a) **2.** (c) **3.** (a) **4.** (b) **5.** (b) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (d)
11. (c) **12.** (b) **13.** (a) **14.** (d) **15.** (b) **16.** (d)

Subjective Questions

1 $t_1 = 2.19\text{s},\ t_2 = 8.20\text{ s}$ **2.** 3 s and 1 s **3.** $\sqrt{3}$ s

4. No **5.** 2.5 m **6.** $\tan^{-1}\left(\frac{2}{3}\right)$

7. time = zero, 0.8 s, x-coordinate = 0, 2.4 m **8.** $\frac{10}{\sqrt{3}}$ m/s **9.** 5 m/s

10. 20 m **11.** $\frac{40}{\sqrt{3}}$ m/s **12.** (a) 30° (b) $5\sqrt{3}$ m

13. (a) 11 m, (b) 23 m (c) 16.6 m/s (d) $\tan^{-1}(2)$, below horizontal

15. $u = \dfrac{gt\cos\beta}{\sin(\alpha - \beta)}$ **17.** (a) 2s (b) 19.6 m (c) 15 m

18. 3.55 s, 32.7 m/s **19.** $R = \dfrac{2u^2}{g}\tan\theta\sec\theta$

20. (a) 0.18 s (c) a straight line with respect to elevator and projectile with respect to ground

21. $(u_1\cos\theta_1 + u_2\cos\theta_2)\,t = 20$ ...(i) $(u_1\sin\theta_1 - u_2\sin\theta_2)\,t = 10$ ...(ii)

LEVEL 2

Single Correct Option

1. (c) **2.** (b) **3.** (a) **4.** (a) **5.** (d) **6.** (d) **7.** (a) **8.** (a) **9.** (c) **10.** (b)

More than One Correct Options

1. (all) **2.** (a,c,d) **3.** (b,c) **4.** (b,d) **5.** (c,d) **6.** (all)

Comprehension Based Questions

1. (b) **2.** (a)

Match the Columns

1. (a) → (p), (b) → (s), (c) → (p), (d) → (p)
2. (a) → (s), (b) → (q), (c) → (r), (d) → (s)
3. (a) → (q), (b) → (p), (c) → (p), (d) → (s)
4. (a) → (q), (b) → (r), (c) → (q), (d) → (s)
5. (a) → (q), (b) → (s), (c) → (r), (d) → (p)
6. (a) → (r), (b) → (r), (c) → (r), (d) → (s)
7. (a) → (s), (b) → (s), (c) → (r), (d) → (p)

Subjective Questions

1. $v_0 = \sqrt{\dfrac{2gh}{2 + \cot^2\theta}}$ **2.** $\sqrt{26}$ ms^{-1} at angle $\theta = \tan^{-1}(5)$ with x-axis

3. $t = \dfrac{uv\sin(\alpha - \beta)}{g(v\cos\beta - u\cos\alpha)}$ **4.** 4.47 m, 2.75 m

5. $\dfrac{2}{\sqrt{2} + 1}$ **6.** (a) 2.67 km (b) 0.9 km/s

7. (a) $(-6\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}})$ ms^{-1} (b) (–4.5 m, 16 m, 0), (0.3 m, 8.0 m, 3.2 m)

8. 250 m **9.** (a) (5 m, 1.25 m) (b) 4.45 m

10. $[24\,\hat{\mathbf{i}} + 15\,\hat{\mathbf{k}}]$ m

08

Laws of Motion

Chapter Contents

8.1 Types of Forces
8.2 Free Body Diagram
8.3 Equilibrium
8.4 Newton's Laws of Motion
8.5 Constraint Equations
8.6 Pseudo Force
8.7 Friction

8.1 Types of Forces

There are basically three forces which are commonly encountered in mechanics.

Field Forces

These are the forces in which contact between two objects is not necessary. Gravitational force between two bodies and electrostatic force between two charges are two examples of field forces. Weight ($w = mg$) of a body comes in this category.

Contact Forces

Two bodies in contact exert equal and opposite forces on each other. If the contact is frictionless, the contact force is perpendicular to the common surface and known as **normal reaction.**

If, however the objects are in rough contact and move (or have a tendency to move) relative to each other without losing contact then **frictional force** arise which oppose such motion. Again each object exerts a frictional force on the other and the two forces are equal and opposite. This force is perpendicular to normal reaction. Thus, the contact force (F) between two objects is made up of two forces.

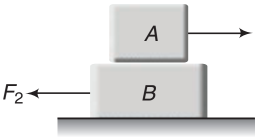


Fig. 8.1

(i) Normal reaction (N) (ii) Force of friction (f)

and since these two forces are mutually perpendicular.

$$\boxed{F = \sqrt{N^2 + f^2}}$$

Note *In this book normal reaction at most of the places has been represented by N. But at some places, it is also represented by R. This is because N is confused with the SI unit of force newton.*

Consider two wooden blocks A and B being rubbed against each other.

In Fig. 8.1, A is being moved to the right while B is being moved leftward. In order to see more clearly which forces act on A and which on B, a second diagram is drawn showing a space between the blocks but they are still supposed to be in contact.

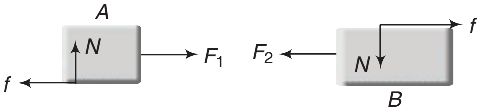


Fig. 8.2

In Fig. 8.2, the two normal reactions each of magnitude N are perpendicular to the surface of contact between the blocks and the two frictional forces each of magnitude f act along that surface, each in a direction opposing the motion of the block upon which it acts.

Note *Forces on block B from the ground are not shown in the figure.*

Attachment to Another Body

Tension (T) in a string and **spring force** ($F = kx$) come in this group. Regarding the tension and string, the following three points are important to remember:

1. If a string is inextensible the magnitude of acceleration of any number of masses connected through the string is always same.

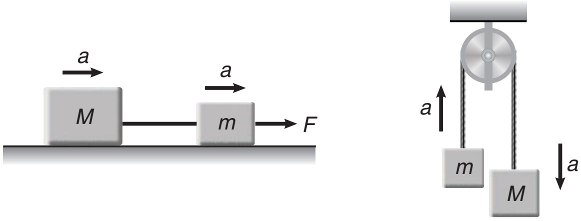
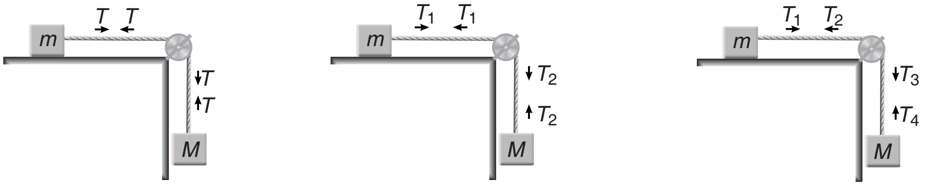


Fig. 8.3

2. If a string is massless, the tension in it is same everywhere. However, if a string has a mass and it is accelerated, tension at different points will be different.
3. If pulley is massless and frictionless, tension will be same on both sides of the pulley.



String and pulley are massless and there is no friction between pulley and string

String is massless but pulley is not massless and frictionless

String and pulley are not massless and there is a friction between pulley and string

Fig. 8.4

Spring force ($F = kx$) has been discussed in detail in the chapter of work, energy and power.

Hinge Force

In the figure shown there is a hinge force on the rod (from the hinge). There are two methods of finding a hinge force :

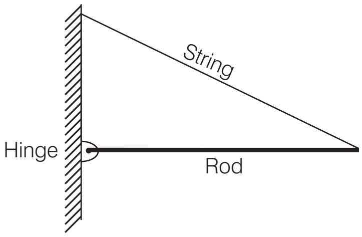


Fig. 8.5

(i) either you find its horizontal (H) and vertical (V) components

(ii) or you find its magnitude and direction.

✓ *Extra Points to Remember*

- Normal reaction is perpendicular to the common tangent direction and always acts towards the body. It is just like a pressure force ($F = PA$) which is also perpendicular to a surface and acts towards it.

  For example

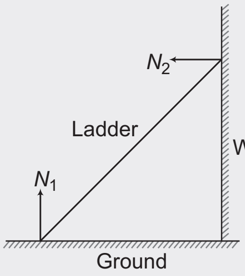


Fig. 8.6

Normal reaction on ladder from ground is N_1 and from wall is N_2.

- Tension in a string is as shown in Fig. 8.7.

  In the figure :

  T_1 goes to block A (force applied by string on block A).

  T_2 and T_3 to pulley P_1

  T_4, T_5 and T_7 to pulley P_2

  T_8 to block B and

  T_6 to roof

  If string and pullies are massless and there is no friction in the pullies, then

$$T_1 = T_2 = T_3 = T_4 = T_5 = T_6 \quad \text{and} \quad T_7 = T_8$$

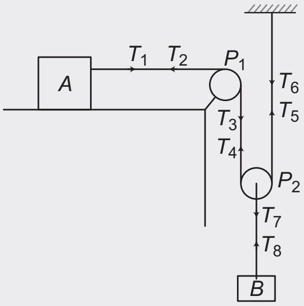
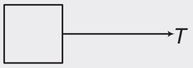


Fig. 8.7

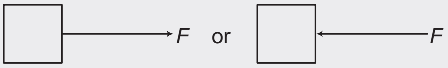
- If a string is attached with a block then it can apply force on the block only in a direction away from the block (in the form of tension).



String attached with a block

Fig. 8.8

If the block is attached with a rod, then it can apply force on the block in both directions, towards the block (may be called push) or away from the block (called pull)



Rod attached with a block

Fig. 8.9

- All forces discussed above make a pair of equal and opposite forces acting on two different bodies (Newton's third law).

8.2 Free Body Diagram

No system, natural or man made, consists of a single body alone or is complete in itself. A single body or a part of the system can, however be isolated from the rest by appropriately accounting for its effect on the remaining system.

A free body diagram (FBD) consists of a diagrammatic representation of a single body or a sub-system of bodies isolated from its surroundings showing all the forces acting on it.

Consider, for example, a book lying on a horizontal surface.

A free body diagram of the book alone would consist of its weight ($w = mg$), acting through the centre of gravity and the reaction (N) exerted on the book by the surface.

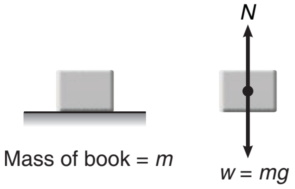


Fig. 8.10

Example 8.1 *A cylinder of weight W is resting on a V-groove as shown in figure. Draw its free body diagram.*

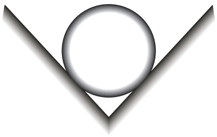


Fig. 8.11

Solution The free body diagram of the cylinder is as shown in Fig. 8.12. Here, w = weight of cylinder and N_1 and N_2 are the normal reactions between the cylinder and the two inclined walls.

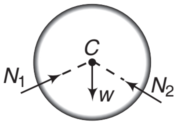


Fig. 8.12

Example 8.2 *Three blocks A, B and C are placed one over the other as shown in figure. Draw free body diagrams of all the three blocks.*

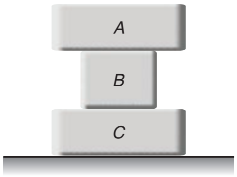
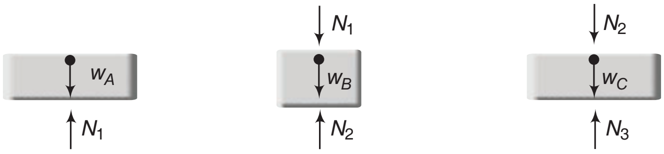


Fig. 8.13

Solution Free body diagrams of A, B and C are shown below.



FBD of A FBD of B FBD of C

Fig. 8.14

Here, N_1 = normal reaction between A and B

N_2 = normal reaction between B and C

and N_3 = normal reaction between C and ground.

Example 8.3 *A block of mass m is attached with two strings as shown in figure. Draw the free body diagram of the block.*

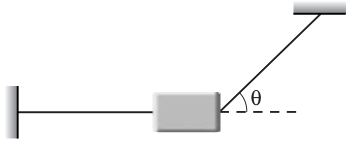


Fig. 8.15

Solution The free body diagram of the block is as shown in Fig. 8.16.

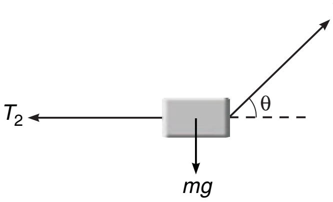


Fig. 8.16

8.3 Equilibrium

Forces which have zero resultant and zero turning effect will not cause any change in the motion of the object to which they are applied. Such forces (and the object) are said to be in equilibrium. For understanding the equilibrium of an object under two or more concurrent or coplanar forces let us first discuss the resolution of force and moment of a force about some point.

Resolution of a Force

When a force is replaced by an equivalent set of components, it is said to be resolved. One of the most useful ways in which to resolve a force is to choose only two components (although a force may be resolved in three or more components also) which are at right angles also. The magnitude of these components can be very easily found using trigonometry.

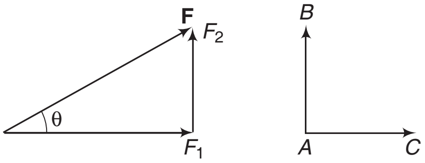


Fig. 8.17

In Fig. 8.17, $F_1 = F \cos\theta =$ component of $\mathbf{F}$ along AC

$F_2 = F \sin\theta =$ component of $\mathbf{F}$ perpendicular to AC or along AB

Finding such components is referred to as resolving a force in a pair of perpendicular directions. Note that the component of a force in a direction perpendicular to itself is zero. For example, if a force of 10 N is applied on an object in horizontal direction then its component along vertical is zero. Similarly, the component of a force in a direction parallel to the force is equal to the magnitude of the force. For example component of the above force in the direction of force (horizontal) will be 10 N.

In the opposite direction the component is −10 N.

Example 8.4 *Resolve a weight of 10 N in two directions which are parallel and perpendicular to a slope inclined at 30° to the horizontal.*

Solution Component perpendicular to the plane

$$w_{\perp} = w\cos 30^\circ = (10)\frac{\sqrt{3}}{2} = 5\sqrt{3}\text{ N}$$

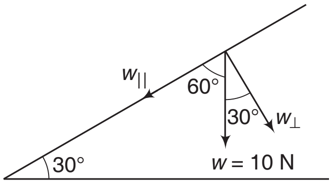


Fig. 8.18

and component parallel to the plane

$$w_{||} = w\sin 30^\circ = (10)\left(\frac{1}{2}\right) = 5\text{ N}$$

Example 8.5 *Resolve horizontally and vertically a force F = 8 N which makes an angle of 45° with the horizontal.*

Solution Horizontal component of **F** is

$$F_H = F\cos 45^\circ = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

and vertical component of **F** is $F_V = F\sin 45^\circ = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$

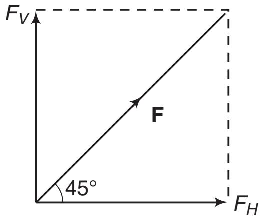


Fig. 8.19

Example 8.6 *A body is supported on a rough plane inclined at 30° to the horizontal by a string attached to the body and held at an angle of 30° to the plane. Draw a diagram showing the forces acting on the body and resolve each of these forces*

(a) horizontally and vertically,

(b) parallel and perpendicular to the plane.

Solution The forces are

The tension in the string T

The normal reaction with the plane N

The weight of the body w and the friction f

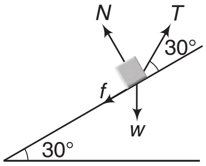


Fig. 8.20

(a) Resolving horizontally and vertically

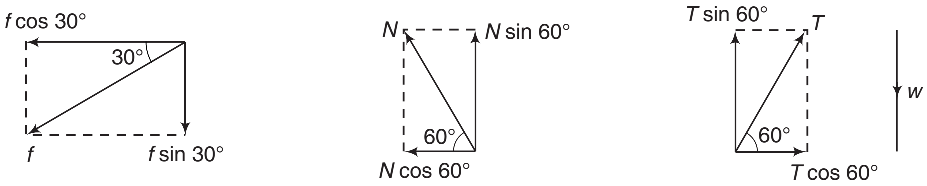


Fig. 8.21

Resolving horizontally and vertically in the senses OX and OY as shown, the components are

Force	Components	
	Parallel to *OX* (horizontal)	Parallel to *OY* (vertical)
f	$-f\cos 30°$	$-f\sin 30°$
N	$-N\cos 60°$	$N\sin 60°$
T	$T\cos 60°$	$T\sin 60°$
w	0	$-w$

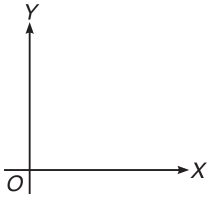


Fig. 8.22

(b) Resolving parallel and perpendicular to the plane

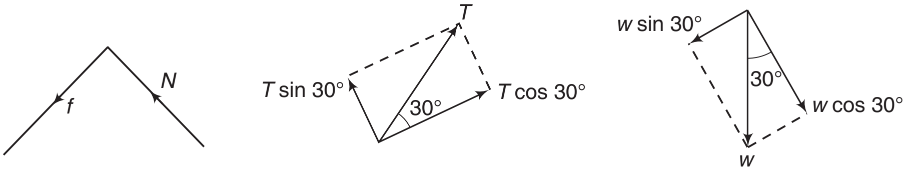


Fig. 8.23

Resolving parallel and perpendicular to the plane in the senses OX' and OY' as shown, the components are :

Force	Components	
	Parallel to *OX′* (parallel to plane)	Parallel to *OY′* (perpendicular to plane)
f	$-f$	0
N	0	N
T	$T\cos 30°$	$T\sin 30°$
w	$-w\sin 30°$	$-w\cos 30°$

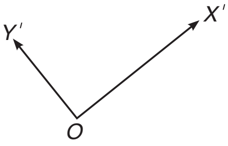


Fig. 8.24

Moment of a Force

The general name given to any turning effect is **torque.** The magnitude of torque, also known as the moment of a force F is calculated by multiplying together the magnitude of the force and its perpendicular distance $r_\perp$ from the axis of rotation. This is denoted by C or τ (tau).

i.e.
$$\boxed{C = Fr_\perp \quad \text{or} \quad \tau = Fr_\perp}$$

Direction of Torque

The angular direction of a torque is the sense of the rotation it would cause.

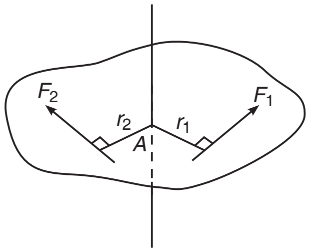


Fig. 8.25

Consider a lamina that is free to rotate in its own plane about an axis perpendicular to the lamina and passing through a point A on the lamina. In the diagram the moment about the axis of rotation of the force F_1 is F_1r_1 anticlock-wise and the moment of the force F_2 is F_2r_2 clockwise. A convenient way to differentiate between clockwise and anticlock-wise torques is to allocate a positive sign to one sense (usually, but not invariably, this is anticlockwise) and negative sign to the other. With this convention, the moments of F_1 and F_2 are $+F_1r_1$ and $-F_2r_2$ (when using a sign convention in any problem it is advisable to specify the chosen positive sense).

Zero Moment

If the line of action of a force passes through the axis of rotation, its perpendicular distance from the axis is zero. Therefore, its moment about that axis is also zero.

Note *Later in the chapter of rotation we will see that torque is a vector quantity.*

Example 8.7 *ABCD is a square of side 2 m and O is its centre. Forces act along the sides as shown in the diagram. Calculate the moment of each force about*

(a) an axis through A and perpendicular to the plane of square.

(b) an axis through O and perpendicular to the plane of square.

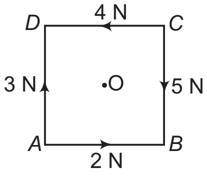


Fig. 8.26

Solution Taking anticlockwise moments as positive we have:

(a) Magnitude of force	2 N	5 N	4 N	3 N
Perpendicular distance from *A*	0	2 m	2 m	0
Moment about *A*	0	–10 N-m	+8 N-m	0

(b) Magnitude of force	2 N	5 N	4 N	3 N
Perpendicular distance from *O*	1 m	1 m	1 m	1 m
Moment about *O*	+2 N-m	–5 N-m	+4 N-m	–3 N-m

Example 8.8 *Forces act as indicated on a rod AB which is pivoted at A. Find the anticlockwise moment of each force about the pivot.*

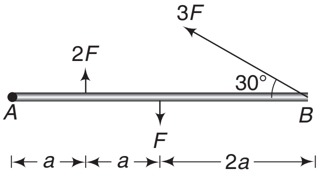


Fig. 8.27

Solution

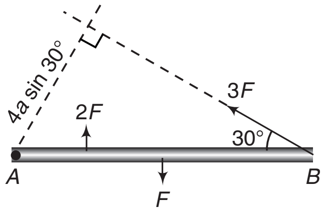


Fig. 8.28

Magnitude of force	2*F*	*F*	3*F*
Perpendicular distance from *A*	*a*	2*a*	4*a* sin 30° = 2*a*
Anticlockwise moment about *A*	+2 *Fa*	–2 *Fa*	+6 *Fa*

Coplanar Forces in Equilibrium

When an object is in equilibrium under the action of a set of two or more coplanar forces, each of three factors which comprise the possible movement of the object must be zero, i.e. the object has

(i) no linear movement along any two mutually perpendicular directions OX and OY.

(ii) no rotation about any axis.

The set of forces must, therefore, be such that

(a) the algebraic sum of the components parallel to OX is zero or $\Sigma F_x = 0$

(b) the algebraic sum of the components parallel to OY is zero or $\Sigma F_y = 0$

(c) the resultant moment about any specified axis is zero or $\Sigma \tau_{\text{any axis}} = 0$

Thus, for the equilibrium of a set of two or more coplanar forces

$$\boxed{\begin{gathered}\Sigma F_x = 0\\ \Sigma F_y = 0 \quad \text{and} \quad \Sigma \tau_{\text{any axis}} = 0\end{gathered}}$$

Using the above three conditions, we get only three set of equations. So, in a problem number of unknowns should not be more than three.

Example 8.9 *A rod AB rests with the end A on rough horizontal ground and the end B against a smooth vertical wall. The rod is uniform and of weight w. If the rod is in equilibrium in the position shown in figure. Find*

(a) frictional force at A

(b) normal reaction at A

(c) normal reaction at B.

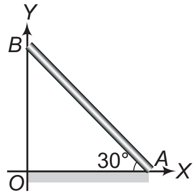


Fig. 8.29

Solution Let length of the rod be $2l$. Using the three conditions of equilibrium. Anticlockwise moment is taken as positive.

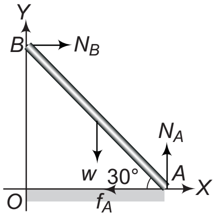


Fig. 8.30

(i) $\Sigma F_X = 0 \quad \Rightarrow \quad \therefore \quad N_B - f_A = 0$

or $\quad N_B = f_A \quad$ ...(i)

(ii) $\Sigma F_Y = 0 \quad \Rightarrow \quad \therefore \quad N_A - w = 0$

or $\quad N_A = w \quad$ ...(ii)

(iii) $\Sigma \tau_O = 0$

$$\therefore \quad N_A (2l \cos 30^\circ) - N_B (2l \sin 30^\circ) - w(l \cos 30^\circ) = 0$$

$$\text{or} \quad \sqrt{3} N_A - N_B - \frac{\sqrt{3}}{2} w = 0 \quad \text{...(iii)}$$

Solving these three equations, we get

(a) $f_A = \frac{\sqrt{3}}{2} w$ (b) $N_A = w$ (c) $N_B = \frac{\sqrt{3}}{2} w$

Exercise : What happens to N_A, N_B and f_A if (a) Angle $\theta = 30^\circ$ is slightly increased,

(b) A child starts moving on the ladder from A to B without changing the angle θ.

Ans (a) Unchanged, decreases, decrease, (b) Increases, increase, increase

Equilibrium of Concurrent Coplanar Forces

If an object is in equilibrium under two or more concurrent coplanar forces the algebraic sum of the components of forces in any two mutually perpendicular directions OX and OY should be zero, i.e. the set of forces must be such that

(i) the algebraic sum of the components parallel to OX is zero, i.e. $\Sigma F_x = 0$.

(ii) the algebraic sum of the components parallel to OY is zero, i.e. $\Sigma F_y = 0$.

Thus, for the equilibrium of two or more concurrent coplanar forces

$$\boxed{\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\end{aligned}}$$

The third condition of zero moment about any specified axis is automatically satisfied if the moment is taken about the point of intersection of the forces. So, here we get only two equations. Thus, number of unknown in any problem should not be more than two.

Example 8.10 *An object is in equilibrium under four concurrent forces in the directions shown in figure. Find the magnitudes of* $\mathbf{F}_1$ *and* $\mathbf{F}_2$.

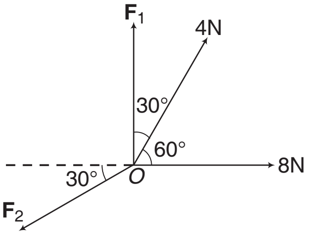


Fig. 8.31

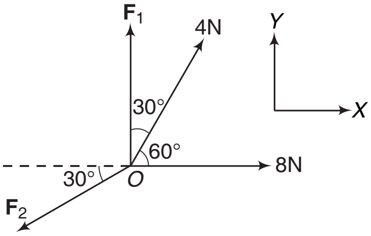


Fig. 8.32

Solution The object is in equilibrium. Hence,

(i) $\Sigma F_x = 0$

$$\therefore \quad 8 + 4\cos 60° - F_2 \cos 30° = 0$$

$$\text{or} \quad 8 + 2 - F_2 \frac{\sqrt{3}}{2} = 0$$

$$\text{or} \quad F_2 = \frac{20}{\sqrt{3}}\ \text{N}$$

(ii) $\Sigma F_y = 0$

$$\therefore \quad F_1 + 4\sin 60° - F_2 \sin 30° = 0$$

$$\text{or} \quad F_1 + \frac{4\sqrt{3}}{2} - \frac{F_2}{2} = 0$$

$$\text{or} \quad F_1 = \frac{F_2}{2} - 2\sqrt{3} = \frac{10}{\sqrt{3}} - 2\sqrt{3}$$

$$\text{or} \quad F_1 = \frac{4}{\sqrt{3}}\ \text{N}$$

Lami's Theorem

If an object O is in equilibrium under three concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ as shown in figure. Then,

$$\frac{F_1}{\sin\alpha}=\frac{F_2}{\sin\beta}=\frac{F_3}{\sin\gamma}$$

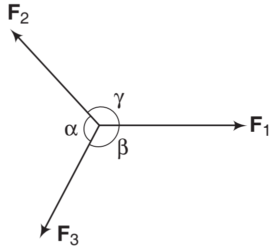


Fig. 8.33

This property of three concurrent forces in equilibrium is known as Lami's theorem and is very useful method of solving problems related to three concurrent forces in equilibrium.

Example 8.11 *One end of a string* 0.5 *m long is fixed to a point A and the other end is fastened to a small object of weight* 8 *N. The object is pulled aside by a horizontal force F, until it is* 0.3 *m from the vertical through A. Find the magnitudes of the tension T in the string and the force F.*

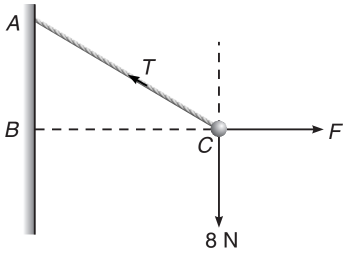


Fig. 8.34

Solution $AC = 0.5\text{ m}$, $BC = 0.3\text{ m}$

$\therefore$ $AB = 0.4\text{ m}$

and if $\angle BAC = \theta.$

Then $$\cos\theta=\frac{AB}{AC}=\frac{0.4}{0.5}=\frac{4}{5}$$

and $$\sin\theta=\frac{BC}{AC}=\frac{0.3}{0.5}=\frac{3}{5}$$

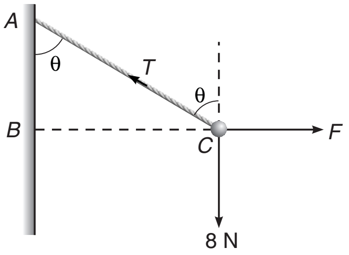


Fig. 8.35

Here, the object is in equilibrium under three concurrent forces. So, we can apply Lami's theorem.

or $$\frac{F}{\sin(180^\circ-\theta)}=\frac{8}{\sin(90^\circ+\theta)}=\frac{T}{\sin 90^\circ}$$

or $$\frac{F}{\sin\theta}=\frac{8}{\cos\theta}=T$$

$\therefore$ $$T=\frac{8}{\cos\theta}=\frac{8}{4/5}=10\text{ N}$$

and $$F=\frac{8\sin\theta}{\cos\theta}=\frac{(8)(3/5)}{(4/5)}=6\text{ N}$$ **Ans.**

Example 8.12 *The rod shown in figure has a mass of* 2 *kg and length* 4 *m. In equilibrium, find the hinge force (or its two components) acting on the rod and tension in the string. Take* $g = 10\ m/s^2$, $\sin 37° = \frac{3}{5}$ and $\cos 37° = \frac{4}{5}$.

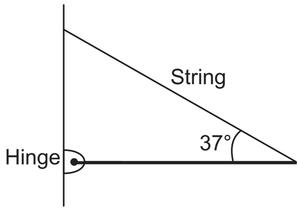


Fig. 8.36

Solution

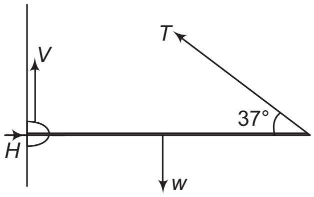


Fig. 8.37

In the figure, only those forces which are acting on the rod has been shown. Here H and V are horizontal and vertical components of the hinge force.

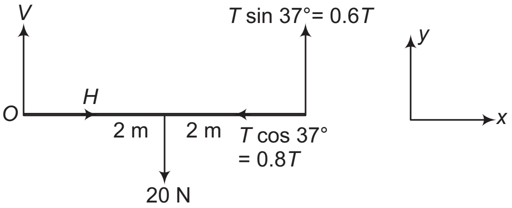


Fig. 8.38

$$\Sigma F_x = 0 \quad \Rightarrow \quad H - 0.8T = 0 \qquad \text{...(i)}$$

$$\Sigma F_y = 0 \quad \Rightarrow \quad V + 0.6T - 20 = 0 \qquad \text{...(ii)}$$

$$\Sigma \tau_O = 0$$

⇒ Clockwise torque of 20 N = anticlock-wise torque of 0.6 T.

All other forces (H, V and 0.8 T pass through O, hence their torques are zero).

$$\therefore \qquad 20 \times 2 = 0.6T \times 4 \qquad \text{...(iii)}$$

Solving these three equations, we get

$$T = 16.67 \text{ N},$$

$$H = 13.33 \text{ N}$$

and $$V = 10 \text{ N}$$ **Ans.**

Hinge force (F)

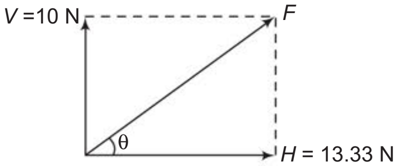


Fig. 8.39

$$F = \sqrt{(13.33)^2 + (10)^2} = 16.67 \text{ N}$$ **Ans.**

$$\tan\theta = \frac{10}{13.33}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{10}{13.33}\right) = 37°$$ **Ans.**

INTRODUCTORY EXERCISE 8.1

1. The diagram shows a rough plank resting on a cylinder with one end of the plank on rough ground. Neglect friction between plank and cylinder. Draw diagrams to show
   (a) the forces acting on the plank,
   (b) the forces acting on the cylinder.

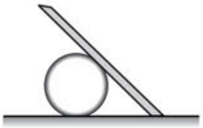


Fig. 8.40

2. Two spheres A and B are placed between two vertical walls as shown in figure. Friction is absent everywhere. Draw the free body diagrams of both the spheres.

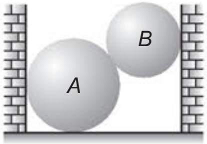


Fig. 8.41

3. A point A on a sphere of weight w rests in contact with a smooth vertical wall and is supported by a string joining a point B on the sphere to a point C on the wall. Draw free body diagram of the sphere.

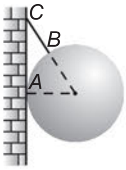


Fig. 8.42

4. A rod AB of weight w_1 is placed over a sphere of weight w_2 as shown in figure. Ground is rough and there is no friction between rod and sphere and sphere and wall. Draw free body diagrams of sphere and rod separately.

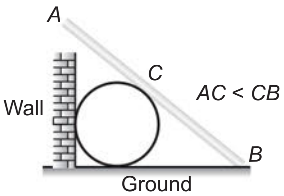


Fig. 8.43

Note *No friction will act between sphere and ground, think why ?*

5. A rod *OA* is suspended with the help of a massless string *AB* as shown in Fig. 8.44. Rod is hinged at point *O*. Draw free body diagram of the rod.

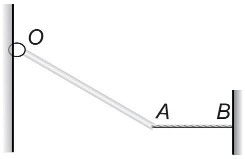


Fig. 8.44

6. A rod *AB* is placed inside a rough spherical shell as shown in Fig.8.45. Draw the free body diagram of the rod.

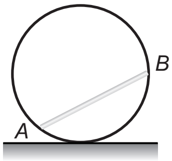


Fig. 8.45

7. Write down the components of four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ along *ox* and *oy* directions as shown in Fig. 8.46.

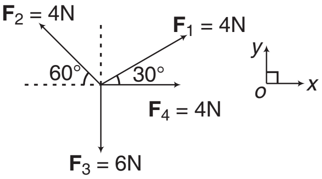

$\mathbf{F}_2 = 4\text{N}$
$\mathbf{F}_1 = 4\text{N}$
60°
30°
y
o
x
$\mathbf{F}_4 = 4\text{N}$
$\mathbf{F}_3 = 6\text{N}$

Fig. 8.46

8. All the strings shown in figure are massless. Tension in the horizontal string is 30 N. Find *W*.

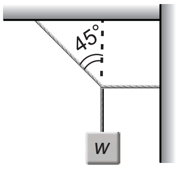


Fig. 8.47

9. The 50 kg homogeneous smooth sphere rests on the 30° incline *A* and against the smooth vertical wall *B*. Calculate the contact forces at *A* and *B*.

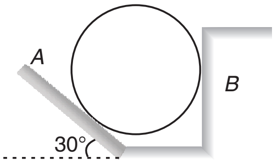


Fig. 8.48

10. In question 3 of the same exercise, the radius of the sphere is *a*. The length of the string is also *a*. Find tension in the string.

11. A sphere of weight $w = 100$ N is kept stationary on a rough inclined plane by a horizontal string *AB* as shown in figure. Find
    (a) tension in the string,
    (b) force of friction on the sphere and
    (c) normal reaction on the sphere by the plane.

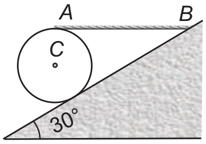


Fig. 8.49

8.4 Newton's Laws of Motion

It is interesting to read Newton's original version of the laws of motion.

Law I Every body continues in its state of rest or in uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Law II The change of motion is proportional to the magnitude of force impressed and is made in the direction of the straight line in which that force is impressed.

Law III To every action there is always an equal and opposite reaction or the mutual actions of two bodies upon each other are always directed to contrary parts.

The modern versions of these laws are:

1. A body continues in its initial state of rest or motion with uniform velocity unless acted on by an unbalanced external force.
2. The acceleration of a body is inversely proportional of its mass and directly proportional to the resultant external force acting on it, i.e.

$$\Sigma \mathbf{F} = \mathbf{F}_{\text{net}} = m\,\mathbf{a} \quad \text{or} \quad \mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$$

3. Forces always occur in pairs. If body A exerts a force on body B, an equal but opposite force is exerted by body B on body A.

Working with Newton's First and Second Laws

Normally any problem relating to Newton's laws is solved in following four steps:

1. First of all we decide the system on which the laws of motion are to be applied. The system may be a single particle, a block or a combination of two or more blocks, two blocks connected by a string, etc. The only restriction is that all parts of the system should have the same acceleration.
2. Once the system is decided, we make the list of all the forces acting on the system. Any force applied by the system on other bodies is not included in the list of the forces.
3. Then we make a free body diagram of the system and indicate the magnitude and directions of all the forces listed in step 2 in this diagram.
4. In the last step we choose any two mutually perpendicular axes say x and y in the plane of the forces in case of coplanar forces. Choose the x-axis along the direction in which the system is known to have or is likely to have the acceleration. A direction perpendicular to it may be chosen as the y-axis. If the system is in equilibrium any mutually perpendicular directions may be chosen. Write the components of all the forces along the x-axis and equate their sum to the product of the mass of the system and its acceleration, i.e.

$$\Sigma F_x = ma \qquad \ldots\text{(i)}$$

This gives us one equation. Now, we write the components of the forces along the y-axis and equate the sum to zero. This gives us another equation, i.e.

$$\Sigma F_y = 0 \qquad \ldots\text{(ii)}$$

Note *(i) If the system is in equilibrium we will write the two equations as*

$$\Sigma F_x = 0 \quad \textit{and} \quad \Sigma F_y = 0$$

(ii) If the forces are collinear, the second equation, i.e. $\Sigma F_y = 0$ is not needed.

Extra Points to Remember

- If $\mathbf{a}$ is the acceleration of a body, then $m\mathbf{a}$ force does not act on the body but this much force is required to provide $\mathbf{a}$ acceleration to the body. The different available forces acting on the body provide this $m\mathbf{a}$ force or, we can say that vector sum of all forces acting on the body is equal to $m\mathbf{a}$. The available forces may be weight, tension, normal reaction, friction or any externally applied force etc.
- If all bodies of a system has a common acceleration then that common acceleration can be given by

$$a = \frac{\text{Net pulling/pusing force}}{\text{Total mass}} = \frac{\text{NPF}}{\text{TM}}$$

Net pulling/pushing force (NPF) is actually the net force.

Example Suppose two unequal masses m and $2m$ are attached to the ends of a light inextensible string which passes over a smooth massless pulley. We have to find the acceleration of the system. We can assume that the mass $2m$ is pulled downwards by a force equal to its weight, i.e. $2mg$. Similarly, the mass m is being pulled by a force of mg downwards. Therefore, net pulling force on the system is $2mg - mg = mg$ and total mass being pulled is $2m + m = 3m$.

∴ Acceleration of the system is

$$a = \frac{\text{Net pulling force}}{\text{Total mass to be pulled}} = \frac{mg}{3m} = \frac{g}{3}$$

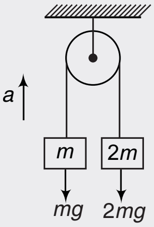


Fig. 8.50

Note *While finding net pulling force, take the forces (or their components) which are in the direction of motion (or opposite to it) and are single (i.e. they are not forming pair of equal and opposite forces). For example weight (mg) or some applied force F. Tension makes an equal and opposite pair. So, they are not to be included, unless the system in broken at some place and only one tension is considered on the system under consideration.*

- After finding that common acceleration, we will have to draw free body diagrams of different blocks to find normal reaction or tension etc.

Example 8.13 *Two blocks of masses 4 kg and 2 kg are placed side by side on a smooth horizontal surface as shown in the figure. A horizontal force of 20 N is applied on 4 kg block. Find*

(a) the acceleration of each block.

(b) the normal reaction between two blocks.

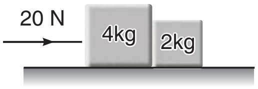


Fig. 8.51

Solution (a) Both the blocks will move with same acceleration (say a) in horizontal direction.

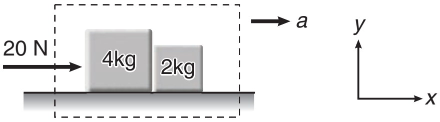


Fig. 8.52

Let us take both the blocks as a system. Net external force on the system is 20 N in horizontal direction.

Using $\Sigma F_x = ma_x$

$$20 = (4+2)a = 6a$$

or $a = \frac{10}{3}\text{ m/s}^2$ **Ans.**

Alternate Method

$$a = \frac{\text{Net pushing force}}{\text{Total mass}} = \frac{20}{4+2} = \frac{10}{3}\ \text{m/s}^2$$

(b) The free body diagram of both the blocks are as shown in Fig. 8.53.

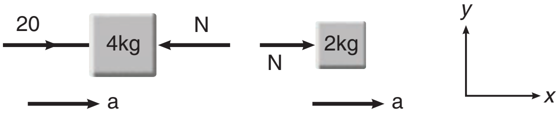


Fig. 8.53

Using $\Sigma F_x = ma_x$

For 4 kg block, $20 - N = 4a = 4 \times \frac{10}{3}$

$$N = 20 - \frac{40}{3} = \frac{20}{3}\ \text{newton}$$ **Ans.**

This can also be solved as under

For 2 kg block, $N = 2a = 2 \times \frac{10}{3} = \frac{20}{3}$ newton

Here, N is the normal reaction between the two blocks.

Note *In free body diagram of the blocks we have not shown the forces acting on the blocks in vertical direction, because normal reaction between the blocks and acceleration of the system can be obtained without using* $\Sigma F_y = 0$.

Example 8.14 *Three blocks of masses 3 kg, 2 kg and 1 kg are placed side by side on a smooth surface as shown in figure. A horizontal force of 12 N is applied on 3 kg block. Find the net force on 2 kg block.*

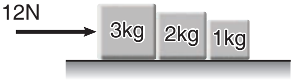


Fig. 8.54

Solution Since, all the blocks will move with same acceleration (say a) in horizontal direction. Let us take all the blocks as a single system.

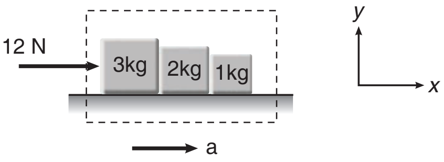


Fig. 8.55

Net external force on the system is 12 N in horizontal direction.

Using $\Sigma F_x = ma_x$,

we get, $12 = (3+2+1)a = 6a$

or $a = \frac{12}{6} = 2\ \text{m/s}^2$

Alternate Method

$$a = \frac{\text{Net pushing force}}{\text{Total mass}} = \frac{12}{3+2+1} = 2\ \text{m/s}^2$$

Now, let F be the net force on 2 kg block in x-direction, then using $\Sigma F_x = ma_x$ for 2 kg block, we get

$$F = (2)(2) = 4\ \text{N}$$ **Ans.**

Note *Here, net force F on 2 kg block is the resultant of N_1 and N_2 $(N_1 > N_2)$*
where, N_1 = normal reaction between 3 kg and 2 kg block,
and N_2 = normal reaction between 2 kg and 1 kg block.
Thus, $F = N_1 - N_2$

Example 8.15 *In the arrangement shown in figure. The strings are light and inextensible. The surface over which blocks are placed is smooth. Find*
(a) the acceleration of each block,
(b) the tension in each string.

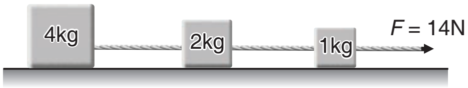


Fig. 5.56

Solution (a) Let a be the acceleration of each block and T_1 and T_2 be the tensions, in the two strings as shown in figure.

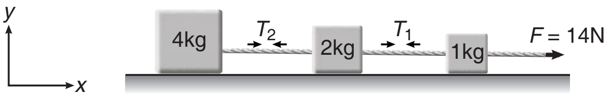


Fig. 8.57

Taking the three blocks and the two strings as the system.

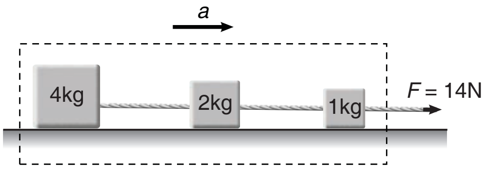


Fig. 8.58

Using $\Sigma F_x = ma_x$ or $14 = (4+2+1)a$ or $a = \frac{14}{7} = 2\ \text{m/s}^2$ **Ans.**

Alternate Method

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{14}{4+2+1} = 2\ \text{m/s}^2$$

(b) Free body diagram (showing the forces in x-direction only) of 4 kg block and 1kg block are shown in Fig. 8.59.

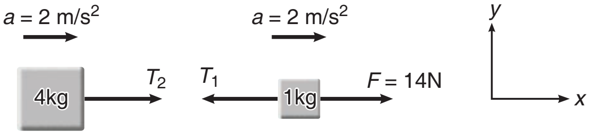


Fig. 8.59

Using $\Sigma F_x = ma_x$

For 1 kg block, $F - T_1 = (1)(a)$

or $14 - T_1 = (1)(2) = 2$

$\therefore$ $T_1 = 14 - 2 = 12\,\text{N}$ **Ans.**

For 4 kg block, $T_2 = (4)(a)$

$\therefore$ $T_2 = (4)(2) = 8\,\text{N}$ **Ans.**

Example 8.16 *Two blocks of masses 4 kg and 2 kg are attached by an inextensible light string as shown in figure. Both the blocks are pulled vertically upwards by a force F = 120 N. Find*

(a) the acceleration of the blocks,

(b) tension in the string. (Take $g = 10\ m/s^2$).

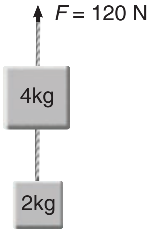


Fig. 8.60

Solution (a) Let a be the acceleration of the blocks and T the tension in the string as shown in figure.

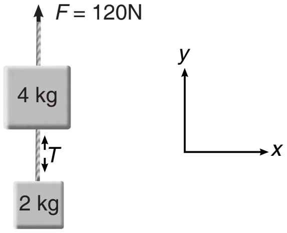


Fig. 8.61

Taking the two blocks and the string as the system shown in figure Fig. 8.62.

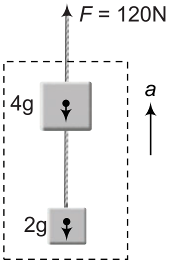


Fig. 8.62

Using $\Sigma F_y = ma_y$, we get

$$F - 4g - 2g = (4+2)a$$

or $120 - 40 - 20 = 6a$ or $60 = 6a$

$\therefore$ $a = 10\ \text{m/s}^2$ **Ans.**

Alternate Method

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{120-60}{4+2} = 10\,\text{m/s}^2$$

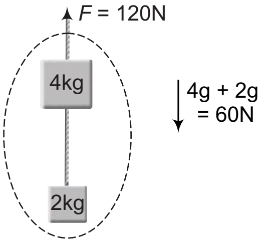


Fig. 8.63

(b) Free body diagram of 2 kg block is as shown in Fig. 8.64.

Using $$\Sigma F_y = ma_y$$

we get, $$T - 2g = 2a$$

or $$T - 20 = (2)(10)$$

$$\therefore \quad T = 40\ \text{N} \qquad \textbf{Ans.}$$

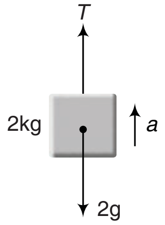


Fig. 8.64

Example 8.17 *In the system shown in figure pulley is smooth. String is massless and inextensible. Find acceleration of the system a, tensions T_1 and T_2.* $(g = 10\ m/s^2)$

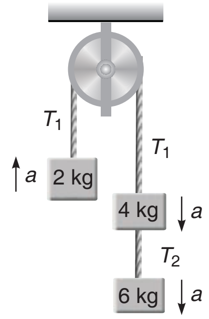


Fig. 8.65

Solution Here, net pulling force will be

Weight of 4 kg and 6 kg blocks on one side – weight of 2 kg block on the other side. Therefore,

$$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

$$= \frac{(6 \times 10) + (4 \times 10) - (2)(10)}{6 + 4 + 2}$$

$$= \frac{20}{3}\ \text{m/s}^2 \qquad \textbf{Ans.}$$

For T_1, let us consider FBD of 2 kg block. Writing equation of motion, we get

$$T_1 - 20 = 2a \quad \text{or} \quad T_1 = 20 + 2 \times \frac{20}{3} = \frac{100}{3}\ \text{N} \qquad \textbf{Ans.}$$

For T_2, we may consider FBD of 6 kg block. Writing equation of motion, we get

$$60 - T_2 = 6a$$

$$\therefore \quad T_2 = 60 - 6a = 60 - 6\left(\frac{20}{3}\right)$$

$$= \frac{60}{3}\ \text{N} \qquad \textbf{Ans.}$$

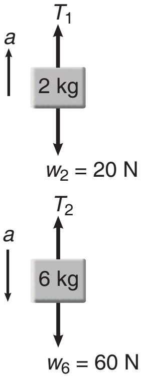


Fig. 8.66

Exercise: *Draw FBD of 4 kg block. Write down the equation of motion for it and check whether the values calculated above are correct or not.*

Example 8.18 *In the system shown in figure all surfaces are smooth. String is massless and inextensible. Find acceleration a of the system and tension T in the string.* ($g = 10\ m/s^2$)

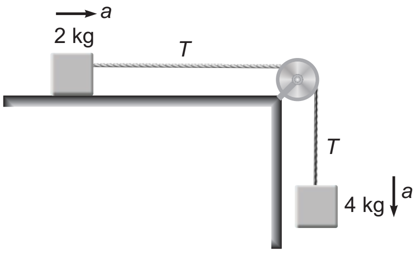


Fig. 8.67

Solution Here, weight of 2 kg is perpendicular to motion (or a). Hence, it will not contribute in net pulling force. Only weight of 4 kg block will be included.

$$\therefore \qquad a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

$$= \frac{(4)(10)}{(4+2)} = \frac{20}{3}\ \text{m/s}^2 \qquad \textbf{Ans.}$$

For T, consider FBD of 4 kg block. Writing equation of motion.

$$40 - T = 4a$$

$$\therefore \qquad T = 40 - 4a$$

$$= 40 - 4\left(\frac{20}{3}\right) = \frac{40}{3}\ \text{N} \qquad \textbf{Ans.}$$

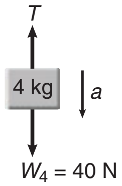


Fig. 8.68

Exercise: *Draw FBD of 2 kg block and write down equation of motion for it. Check whether the values calculated above are correct or not.*

Example 8.19 *In the adjacent figure, masses of A, B and C are 1 kg, 3 kg and 2 kg respectively. Find*

(a) the acceleration of the system and

(b) tensions in the strings.

Neglect friction. ($g = 10\ m/s^2$)

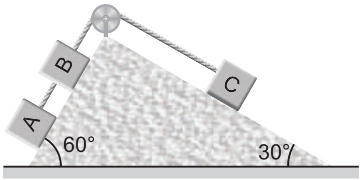


Fig. 8.69

Solution (a) In this case net pulling force

$$= m_A g \sin 60° + m_B g \sin 60° - m_C g \sin 30°$$

$$= (1)(10)\frac{\sqrt{3}}{2} + (3)(10)\left(\frac{\sqrt{3}}{2}\right) - (2)(10)\left(\frac{1}{2}\right)$$

$$= 24.64\ \text{N}$$

Total mass being pulled = 1 + 3 + 2 = 6 kg

$$\therefore \quad \text{Acceleration of the system } a = \frac{21.17}{6} = 4.1\ \text{m/s}^2 \qquad \textbf{Ans.}$$

(b) For the tension in the string between *A* and *B*.

FBD of *A*

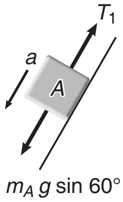


Fig. 8.70

$$m_A g \sin 60° - T_1 = (m_A)(a)$$

$$\therefore \quad T_1 = m_A g \sin 60° - m_A a$$

$$= m_A (g \sin 60° - a)$$

$$\therefore \quad T_1 = (1)\left(10 \times \frac{\sqrt{3}}{2} - 4.1\right)$$

$$= 4.56 \text{ N}$$

Ans.

For the tension in the string between *B* and *C*.

FBD of *C*

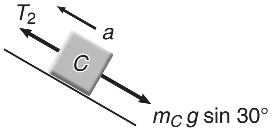


Fig. 8.71

$$T_2 - m_C g \sin 30° = m_C a$$

$$\therefore \quad T_2 = m_C (a + g \sin 30°)$$

$$\therefore \quad T_2 = 2\left[4.1 + 10\left(\frac{1}{2}\right)\right]$$

$$= 18.2 \text{ N}$$

Ans.

INTRODUCTORY EXERCISE 8.2

1. Three blocks of masses 1 kg, 4 kg and 2 kg are placed on a smooth horizontal plane as shown in figure. Find
   (a) the acceleration of the system,
   (b) the normal force between 1 kg block and 4 kg block,
   (c) the net force on 2 kg block.

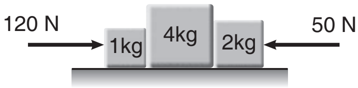


Fig. 8.72

2. In the arrangement shown in figure, find the ratio of tensions in the strings attached with 4 kg block and that with 1 kg block.

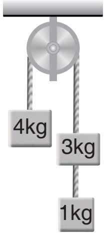


Fig. 8.73

3. Two unequal masses of 1 kg and 2 kg are connected by an inextensible light string passing over a smooth pulley as shown in figure. A force $F = 20$ N is applied on 1 kg block. Find the acceleration of either block. $(g = 10 \text{ m/s}^2)$.

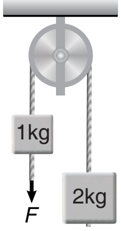


Fig. 8.74

4. In the arrangement shown in figure what should be the mass of block A, so that the system remains at rest? Neglect friction and mass of strings.

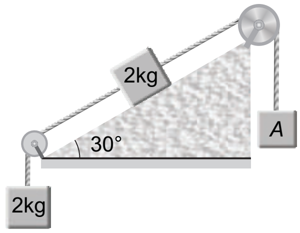


Fig. 8.75

5. Two blocks of masses 2 kg and 4 kg are released from rest over a smooth inclined plane of inclination 30° as shown in figure. What is the normal force between the two blocks?

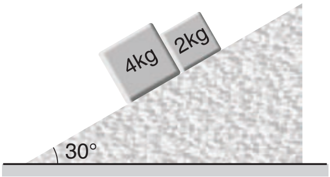


Fig. 8.76

6. What should be the acceleration a of the box shown in Fig. 8.77 so that the block of mass m exerts a force $\frac{mg}{4}$ on the floor of the box?

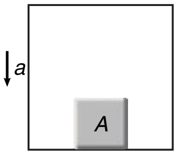


Fig. 8.77

7. In the figure shown, find acceleration of the system and tensions T_1, T_2 and T_3. (Take $g = 10\,\text{m/s}^2$)

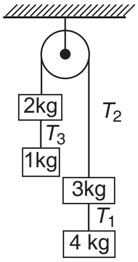


Fig. 8.78

8. In the figure shown, all surfaces are smooth. Find
   (a) acceleration of all the three blocks,
   (b) net force on 6 kg, 4 kg and 10 kg blocks and
   (c) force acting between 4 kg and 10 kg blocks.

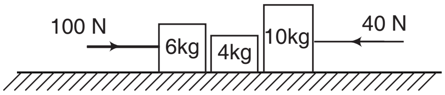


Fig. 8.79

9. Three blocks of masses $m_1 = 10$ kg, $m_2 = 20$ kg and $m_3 = 30$ kg are on a smooth horizontal table, connected to each other by light horizontal strings. A horizontal force $F = 60$ N is applied to m_3, towards right. Find
   (a) tensions T_1 and T_2 and
   (b) tension T_2 if all of a sudden the string between m_1 and m_2 snaps.

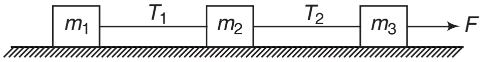


Fig. 8.80

8.5 Constraint Equations

In the above article, we have discussed the cases where different blocks of the system had a common acceleration and that common acceleration was given by

$$a = \frac{\text{Net pulling / pushing force}}{\text{Total mass}}$$

Now, the question is, if different blocks have different accelerations then what? In those cases, we take help of constraint equations. These equations establish the relation between accelerations (or velocities) of different blocks of a system. Depending upon different kinds of problems we have divided the constraint equations in following two types. Most of them are directly explained with the help of some example (s) in their support.

Type 1

Example 8.20 *Using constraint method find the relation between accelerations of* 1 *and* 2.

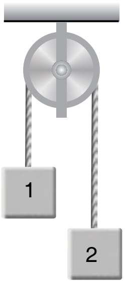


Fig. 8.81

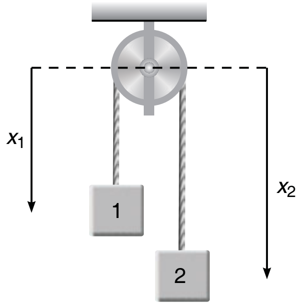


Fig. 8.82

Solution At any instant of time let x_1 and x_2 be the displacements of 1 and 2 from a fixed line (shown dotted). Here x_1 and x_2 are variables but,

$$x_1 + x_2 = \text{constant}$$

or $$x_1 + x_2 = l \qquad \text{(length of string)}$$

Differentiating with respect to time, we have

$$v_1 + v_2 = 0 \quad \text{or} \quad v_1 = -v_2$$

Again differentiating with respect to time, we get

$$a_1 + a_2 = 0 \quad \text{or} \quad a_1 = -a_2$$

This is the required relation between a_1 and a_2, i.e. accelerations of 1 and 2 are equal but in opposite directions.

Note *(i) In the equation* $x_1 + x_2 = l$*, we have neglected the length of string over the pulley. But that length is also constant.*

(ii) In constraint equation if we get $a_1 = -a_2$*, then negative sign does not always represent opposite directions of* a_1 *and* a_2*. The real significance of this sign is,* x_2 *decreases if* x_1 *increases and vice-versa.*

Example 8.21 *Using constraint equations find the relation between* a_1 *and* a_2.

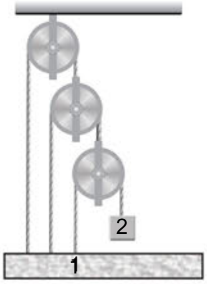


Fig. 8.83

Solution In Fig. 8.84, points 1, 2, 3 and 4 are movable. Let their displacements from a fixed dotted line be x_1, x_2, x_3 and x_4

$$x_1 + x_3 = l_1$$
$$(x_1 - x_3) + (x_4 - x_3) = l_2$$
$$(x_1 - x_4) + (x_2 - x_4) = l_3$$

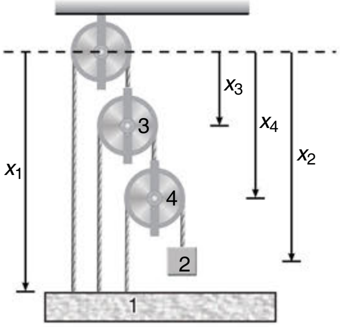


Fig. 8.84

On double differentiating with respect to time, we will get following three constraint relations

$$a_1 + a_3 = 0 \quad \ldots\text{(i)}$$
$$a_1 + a_4 - 2a_3 = 0 \quad \ldots\text{(ii)}$$
$$a_1 + a_2 - 2a_4 = 0 \quad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$a_2 = -7a_1$$

Which is the desired relation between a_1 and a_2.

Example 8.22 *In the above example, if two blocks have masses* 1 *kg and* 2 *kg respectively then find their accelerations and tensions in different strings.*

Solution Pulleys 3 and 4 are massless. Hence net force on them should be zero. Therefore, if we take T tension in the shortest string, then tension in other two strings will be $2T$ and $4T$.

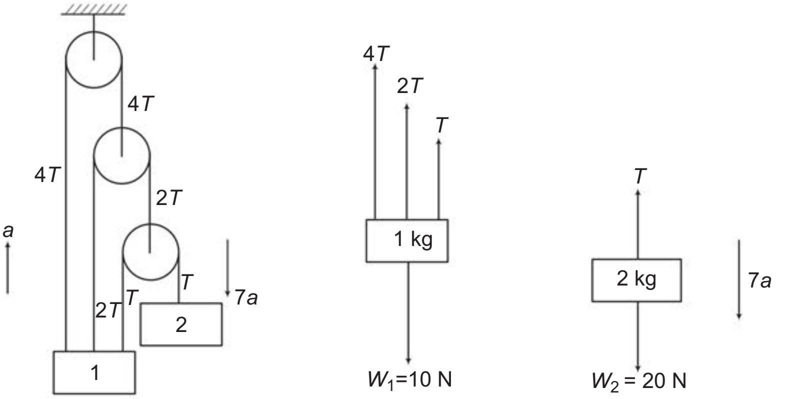


Fig. 8.85

Further, if a is the acceleration of 1 in upward direction, then from the constraint equation $a_2 = -7a_1$, acceleration of 2 will be $7a$ downwards.

Writing the equation, $F_{\text{net}} = ma$ for the two blocks we have

$$4T + 2T + T - 10 = 1 \times a$$

or $$7T - 10 = a \qquad \ldots\text{(i)}$$

$$20 - T = 2 \times (7a)$$

or $$20 - T = 14a \qquad \ldots\text{(ii)}$$

Solving these two equations we get,

$$T = 1.62\ \text{N} \qquad \textbf{Ans.}$$

and $$a = 1.31\ \text{m/s}^2 \qquad \textbf{Ans.}$$

Note *In a problem if 'a' comes out to be negative after calculations then we will change the initially assumed directions of accelerations.*

Type 2

Example 8.23 *The system shown in figure is released from rest. Find acceleration of different blocks and tension in different strings.*

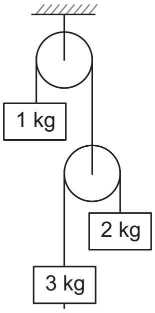


Fig. 8.86

Solution

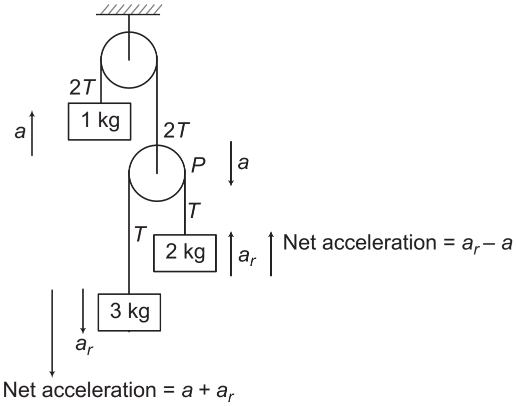


Fig. 8.87

(i) Pulley P and 1 kg mass are attached with the same string. Therefore, if 1 kg mass has an acceleration 'a' in upward direction, then pulley P will have an acceleration 'a' downwards.

(ii) 2 kg and 3 kg blocks are attached with the same string passing over a moveable pulley P. Therefore their relative acceleration, a_r (relative to pulley) will be same. Their net accelerations (relative to ground) are as shown in figure.

(iii) Pulley P is massless. Hence net force on this pulley should be zero. If T is the tension in the string connecting 2 kg and 3 kg mass, then tension in the upper string will be $2T$.

Now writing the equation, $F_{\text{net}} = ma$ for three blocks, we have:

1 kg block:

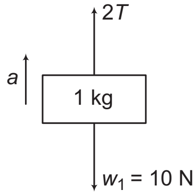


Fig. 8.88

$$2T - 10 = 1 \times a \qquad \ldots\text{(i)}$$

2 kg block:

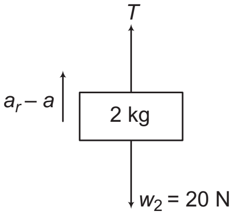


Fig. 8.89

$$T - 20 = 2(a_r - a) \qquad \ldots\text{(ii)}$$

3 kg block:

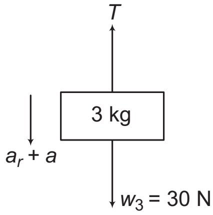


Fig. 8.90

$$30 - T = 3(a_r + a) \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii) we get,

$$T = 8.28\text{ N},\ a = 6.55\text{ m/s}^2 \quad \text{and} \quad a_r = 0.7\text{ m/s}^2.$$

Now, acceleration of 3kg block is $(a + a_r)$ or $7.25\,\text{m/s}^2$ downwards and acceleration of 2kg is $(a_r - a)$ or $-5.85\,\text{m/s}^2$ upwards. Since, this comes out to be negative, hence acceleration of 2 kg block is $5.85\,\text{m/s}^2$ downwards.

Extra Points to Remember

- In some cases, acceleration of a block is inversely proportional to tension force acting on the block (or its component in the direction of motion or acceleration).If tension is double (as compared to other block), then acceleration will be half.

  In Fig. (a): Tension force on block-1 is double (=2*T*) than the tension force on block-2 (=*T*). Therefore, acceleration of block -1 will be half. If block-1 has an acceleration '*a*' in downward direction, then block -2 will have an acceleration '2*a*' towards right.

  In Fig. (b): Tension force on block-1 is three times ($2T+T=3T$) than the tension force on block-2 (=*T*). Therefore acceleration of block-2 will be three times. If block-1 has an acceleration '*a*' in upwards direction, then acceleration of block-2 will be '3*a*' downwards.

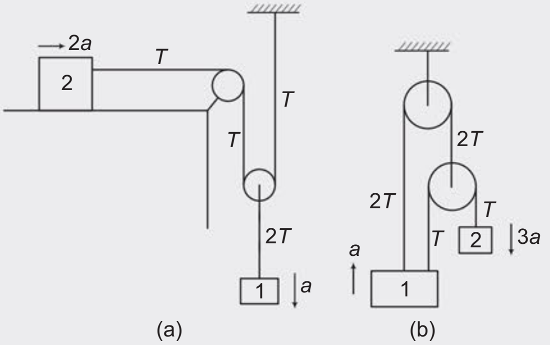


Fig. 8.91

INTRODUCTORY EXERCISE 8.3

1. Make the constraint relation between a_1, a_2 and a_3.

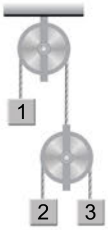


Fig. 8.92

2. At certain moment of time, velocities of 1 and 2 both are 1 m/s upwards. Find the velocity of 3 at that moment.

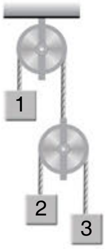


Fig. 8.93

3. Consider the situation shown in figure. Both the pulleys and the string are light and all the surfaces are smooth.
   (a) Find the acceleration of 1 kg block.
   (b) Find the tension in the string. ($g=10\,\text{m/s}^2$).

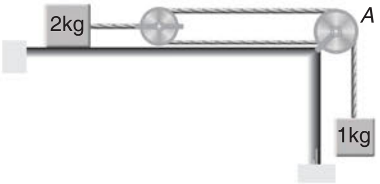


Fig. 8.94

4. Calculate the acceleration of either blocks and tension in the string shown in figure. The pulley and the string are light and all surfaces are smooth.

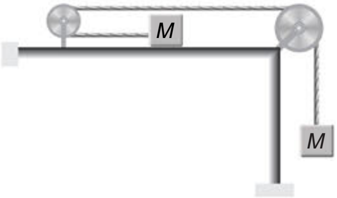


Fig. 8.95

5. Find the mass M so that it remains at rest in the adjoining figure. Both the pulley and string are light and friction is absent everywhere. ($g = 10 \text{ m/s}^2$).

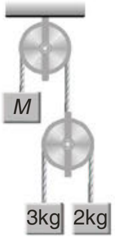


Fig. 8.96

6. In Fig. 8.97 assume that there is negligible friction between the blocks and table. Compute the tension in the cord connecting m_2 and the pulley and acceleration of m_2 if $m_1 = 300 \text{ g}$, $m_2 = 200 \text{ g}$ and $F = 0.40 \text{ N}$.

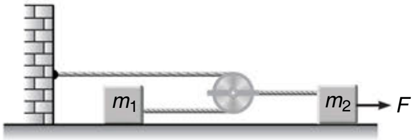


Fig. 8.97

7. In the figure shown, $a_3 = 6 \text{ m/s}^2$ (downwards) and $a_2 = 4 \text{ m/s}^2$ (upwards). Find acceleration of 1.

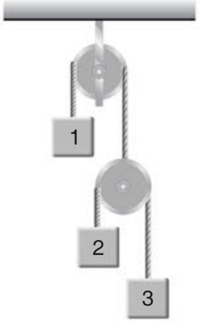


Fig. 8.98

8. Find the acceleration of the block of mass M in the situation shown in the figure. All the surfaces are frictionless.

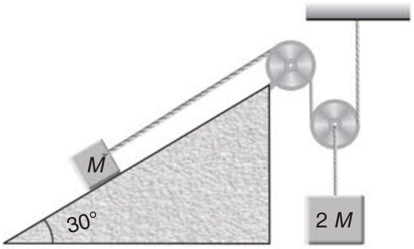


Fig. 8.99

8.6 Pseudo Force

Before studying the concept of pseudo force let us first discuss frame of reference.

Frame of reference is the way of observation the things.

Inertial Frame of Reference

A non-accelerating frame of reference is called an inertial frame of reference. A frame of reference moving with a constant velocity is an inertial frame of reference.

Non-inertial Frame of Reference

An accelerating frame of reference is called a non-inertial frame of reference.

Note *(i) A rotating frame of reference is a non-inertial frame of reference, because it is also an accelerating one.*

(ii) Earth is rotating about its axis of rotation and it is revolving around the centre of sun also. So, it is non-intertial frame of reference. But for most of the cases, we consider its as an inertial frame of reference.

Now let us come to the pseudo force. Instead of ground (or inertial frame of reference) when we start watching the objects from a non-inertial (accelerating) frame of reference its motion conditions are felt differently.

For example Suppose a child is standing inside an accelerating lift. From ground frame of reference this child appears to be accelerating but from lift (non-inertial) frame of reference child appears to be at rest. To justify this changed condition of motion, from equations point of view we have to apply a pseudo force. This pseudo force is given by

$$\boxed{\mathbf{F}_p = -m\mathbf{a}}$$

Here, 'm' is the mass of that body/object which is being observed from non-inertial frame of reference and $\mathbf{a}$ is the acceleration of frame of reference. Negative sign implies that direction of pseudo force $\mathbf{F}_p$ is opposite to $\mathbf{a}$. Hence whenever you make free body diagram of a body from a non-inertial frame, apply all real forces (actually acting) on the body plus one pseudo force. Magnitude of this pseudo force is 'ma' and the direction is opposite to $\mathbf{a}$.

Example Suppose a block A of mass m is placed on a lift ascending with an acceleration a_0. Let N be the normal reaction between the block and the floor of the lift. Free body diagram of A in ground frame of reference (inertial) is shown in Fig. 8.100.

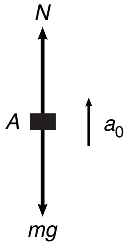


Fig. 8.100

$$\therefore \qquad N - mg = ma_0$$

$$\text{or} \qquad N = m(g + a_0) \qquad \ldots\text{(i)}$$

But if we draw the free body diagram of A with respect to the elevator (a non-inertial frame of reference) without applying the pseudo force, as shown in Fig. 8.101, we get

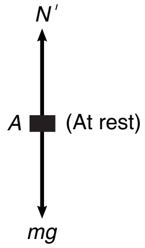


Fig. 8.101

$$N' - mg = 0$$

or $$N' = mg \qquad \ldots\text{(ii)}$$

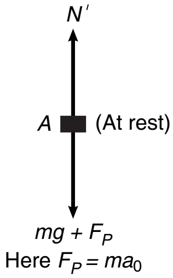


Fig. 8.102

Since, $N' \neq N$, either of the equations is wrong. If we apply a pseudo force in non-inertial frame of reference, N' becomes equal to N as shown in Fig. 8.102. Acceleration of block with respect to elevator is zero.

$\therefore$ $$N' - mg - ma_0 = 0$$

or $$N' = m(g + a_0) \qquad \ldots\text{(iii)}$$

$\therefore$ $$N' = N$$

Example 8.24 *All surfaces are smooth in following figure. Find F, such that block remains stationary with respect to wedge.*

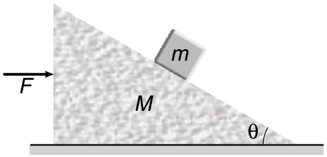


Fig. 8.103

Solution Acceleration of (block + wedge) $a = \dfrac{F}{(M+m)}$

Let us solve the problem by both the methods. Such problems can be solved with or without using the concept of pseudo force.

From Inertial Frame of Reference (Ground)

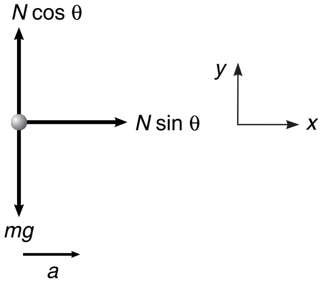


Fig. 8.104

FBD of block w.r.t. ground (Apply real forces):

With respect to ground block is moving with an acceleration a.

$\therefore$ $$\Sigma F_y = 0$$

$\Rightarrow$ $$N\cos\theta = mg \qquad \ldots\text{(i)}$$

and $$\Sigma F_x = ma$$

$\Rightarrow$ $$N\sin\theta = ma \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$a = g \tan\theta$$

$$\therefore \quad F = (M+m)a$$

$$= (M+m)\, g \tan\theta$$

From Non-inertial Frame of Reference (Wedge)

FBD of block w.r.t. wedge (real forces + pseudo force)

w.r.t. wedge, block is stationary

$$\therefore \quad \Sigma F_y = 0 \Rightarrow N\cos\theta = mg \quad \text{...(iii)}$$

$$\Sigma F_x = 0 \Rightarrow N\sin\theta = ma \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we will get the same result

i.e. $F = (M+m)\, g \tan\theta.$

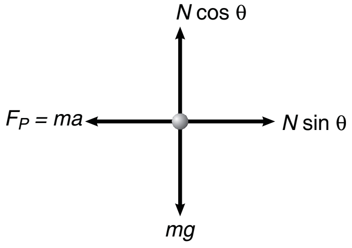


Fig. 8.105

Example 8.25 *A bob of mass m is suspended from the ceiling of a train moving with an acceleration a as shown in figure. Find the angle θ in equilibrium position.*

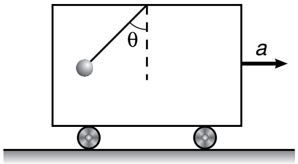


Fig. 8.106

Solution This problem can also be solved by both the methods.

Inertial Frame of Reference (Ground)

FBD of bob w.r.t. ground (only real forces)

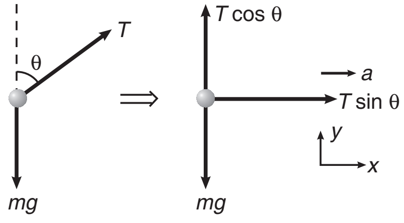


Fig. 8.107

With respect to ground, bob is also moving with an acceleration a.

$$\therefore \quad \Sigma F_x = 0 \Rightarrow T\sin\theta = ma \quad \text{...(i)}$$

$$\text{and} \quad \Sigma F_y = 0 \Rightarrow T\cos\theta = mg \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\tan\theta = \frac{a}{g} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Non-inertial Frame of Reference (Train)

FBD of bob w.r.t. train (real forces + pseudo force):

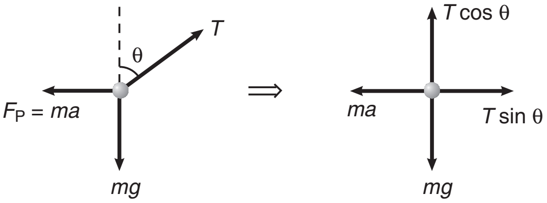


Fig. 8.108

with respect to train, bob is in equilibrium

$$\therefore \quad \Sigma F_x = 0$$

$$\Rightarrow \quad T\sin\theta = ma \qquad \text{...(iii)}$$

$$\therefore \quad \Sigma F_y = 0$$

$$\Rightarrow \quad T\cos\theta = mg \qquad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get the same result, i.e.

$$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

INTRODUCTORY EXERCISE 8.4

1. Two blocks *A* and *B* of masses 1kg and 2 kg have accelerations $(2\hat{\mathbf{i}})\,\text{m/s}^2$ and $(-4\hat{\mathbf{j}})\,\text{m/s}^2$. Find
   (a) Pseudo force on block *A* as applied with respect to the block *B*.
   (b) Pseudo force on block *B* as applied with respect to the block *A*.
2. Pseudo force with respect to a frame moving with constant velocity is zero. Is this statement true or false?
3. Problems of non-intertial frames can be solved only with the concept of pseudo force. Is this statement true or false?

8.7 Friction

Regarding the frictional force (f) following points are worthnoting :

1. It is the tangential component of net contact force (F) acting between two bodies in contact.
2. It starts acting when there is tendency of relative motion (different velocities) between two bodies in contact or actual relative motion takes place. So, friction has a tendency to stop relative motion between two bodies in contact.
3. If there is only tendency of relative motion then static friction acts and if actual relative motion takes place, then kinetic friction acts.

4. Like any other force of nature friction force also makes a pair of equal and opposite forces acting on two different bodies.
5. Direction of friction force on a given body is opposite to the direction of relative motion (or its tendency) of this body.
6.

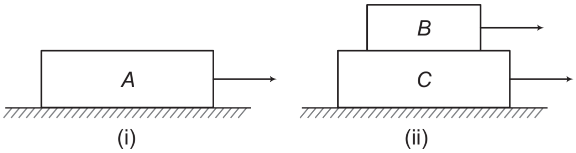


Fig. 8.109

In Fig. (i), motion of block *A* means its relative motion with respect to ground. So, in this case friction between block and ground has a tendency to stop its motion.

In Fig. (ii), relative motion between two blocks *B* and *C* means their different velocities. So, friction between these two blocks has a tendency to make their velocities same.

7. Static friction is self adjusting in nature. This varies from zero to a limiting value f_L. Only that much amount of friction will act which can stop the relative motion.
8. Kinetic friction is constant and it can be denoted by f_k.
9. It is found experimentally that limiting value of static friction f_L and constant value of kinetic friction f_k both are directly proportional to normal reaction *N* acting between the two bodies.

$$\therefore \qquad f_L \text{ or } f_k \propto N$$

$$\Rightarrow \qquad f_L = \mu_s N$$

$$\text{and} \qquad f_k = \mu_k N$$

Here, μ_s = coefficient of static friction

and μ_k = coefficient of kinetic friction.

Both μ_s and μ_k are dimensionless constants which depend on the nature of surfaces in contact. Value of μ_k is usually less than the value of μ_s i.e. constant value of kinetic friction is less than the limiting value of static friction.

Note *(i) In problems, if μ_s and μ_k are not given separately but only μ is given. Then use*

$$f_L = f_k = \mu N$$

(ii) If more than two blocks are placed one over the other on a horizontal ground then normal reaction between two blocks will be equal to the weight of the blocks over the common surface.

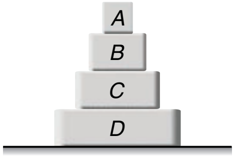


Fig. 8.110

For example, N_1 = *normal reaction between A and B*

$= m_A g$

N_2 = *normal reaction between B and C*

$= (m_A + m_B) g$ *and so on.*

Extra Points to Remember

- **Friction force is electromagnetic in nature.**

The surfaces in contact, however smooth they may appear, actually have imperfections called asperities. When one surface rests on the other the actual area of contact is very less than the surface area of the face of contact.

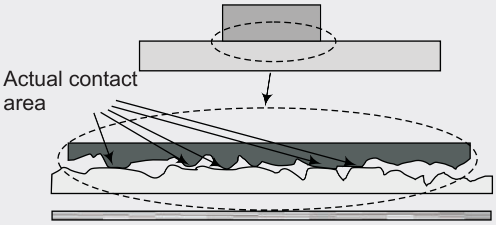


Fig. 8.111

The pressure due to the reaction force between the surfaces is very high as the true contact area is very small. Hence, these contact points deform a little and cold welds are formed at these points.
So, in order to start the relative sliding between these surfaces, enough force has to be applied to break these welds. But, once the welds break and the surfaces start sliding over each other, the further formation of these welds is relatively slow and weak and hence a smaller force is enough to keep the block moving with uniform velocity.
This is the reason why limiting value of static friction is greater than the kinetic friction.

Note *By making the surfaces extra smooth, frictional force increases as actual area of contact increases and the two bodies in contact act like a single body.*

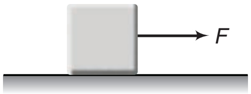


Fig. 8.112

Example 8.26 *Suppose a block of mass 1 kg is placed over a rough surface and a horizontal force F is applied on the block as shown in figure. Now, let us see what are the values of force of friction f and acceleration of the block a if the force F is gradually increased. Given that* $\mu_s = 0.5, \mu_k = 0.4$ *and* $g = 10\ m/s^2$.

Solution Free body diagram of the block is

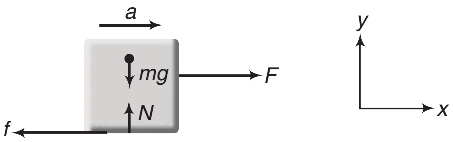


Fig. 8.113

$$\Sigma F_y = 0$$

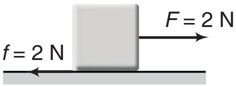
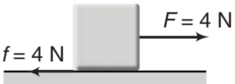
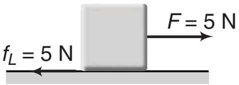
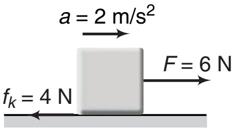
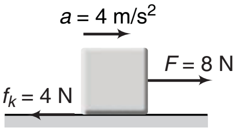
$\therefore$ $$N - mg = 0$$

or $$N = mg = (1)(10) = 10\,\text{N}$$

$$f_L = \mu_s N = (0.5)(10) = 5\,\text{N}$$

and $$f_k = \mu_k N = (0.4)(10) = 4\,\text{N}$$

Below is explained in tabular form, how the force of friction f depends on the applied force F.

F	f	$F_{net} = F - f$	Static or kinetic friction	Relative motion or tendency of relative motion	Acceleration of block $a = \frac{F_{net}}{m}$	Diagram
0	0	0	static	Neither tendency nor actual relative motion	0	
2 N	2 N	0	static	Tendency	0	
4 N	4 N	0	static	Tendency	0	
5 N	5 N	0	static	Tendency	0	
6 N	4 N	2 N	kinetic	Actual relative motion	2 m/s^2	
8 N	4 N	4 N	kinetic	Actual relative motion	4 m/s^2	

Graphically, this can be understood as under:

Note that $f = F$ till $F \le f_L$. Therefore, slope of line OA will be 1 ($y = mx$) or angle of line OA with F-axis is 45°.

Here, $a = 0$ for $F \le 5$ N

and $$a = \frac{F - f_K}{m} = \frac{F - 4}{1} = F - 4 \text{ for } F > 5 \text{ N}$$

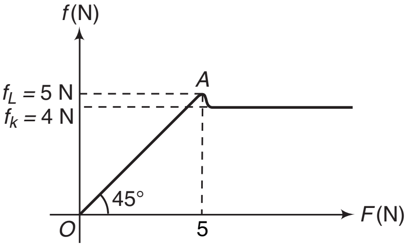


Fig. 8.114

a-F graph is as shown in Fig. 8.115. When F is slightly increased from 5 N, acceleration of block increases from 0 to 1 m/s^2. Think why?

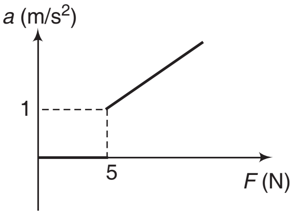


Fig. 8.115

Note *Henceforth, we will take coefficient of friction as μ unless and until specially mentioned in the question μ_s and μ_k separately.*

Angle of Friction (λ)

At a point of rough contact, where slipping is about to occur, the two forces acting on each object are the normal reaction N and frictional force μN.

The resultant of these two forces is F and it makes an angle λ with the normal reaction, where

$$\tan\lambda = \frac{\mu N}{N} = \mu \quad \text{or} \quad \lambda = \tan^{-1}(\mu) \qquad \ldots\text{(i)}$$

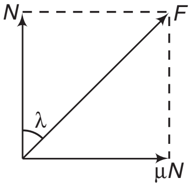


Fig. 8.116

This angle λ is called the angle of friction.

Angle of Repose (α)

Suppose a block of mass m is placed on an inclined plane whose inclination θ can be increased or decreased. Let, μ be the coefficient of friction between the block and the plane. At a general angle θ,

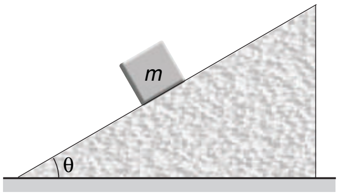


Fig. 8.117

Normal reaction $N = mg\cos\theta$

Limiting friction $f_L = \mu N = \mu mg\cos\theta$

and the driving force (or pulling force)

$$F = mg\sin\theta \qquad \text{(Down the plane)}$$

From these three equations we see that, when θ is increased from 0° to 90°, normal reaction N and hence, the limiting friction f_L is decreased while the driving force F is increased. There is a critical angle called angle of repose (α) at which these two forces are equal. Now, if θ is further increased, then the driving force F becomes more than the limiting friction f_L and the block starts sliding.

Thus, $$f_L = F \quad \text{at} \quad \theta = \alpha$$

or $$\mu\, mg\cos\alpha = mg\sin\alpha$$

or $$\tan\alpha = \mu$$

or $$\alpha = \tan^{-1}(\mu) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we see that angle of friction (λ) is numerically equal to the angle of repose.

or $$\lambda = \alpha$$

From the above discussion we can conclude that

If $\theta < \alpha$, $F < f_L$ the block is stationary.

If $\theta = \alpha$, $F = f_L$ the block is on the verge of sliding.

and if $\theta > \alpha$, $F > f_L$ the block slides down with acceleration

$$a = \frac{F - f_L}{m} = g(\sin\theta - \mu\cos\theta)$$

Variation of N, f_L and F with θ, is shown graphically in Fig. 8.118.

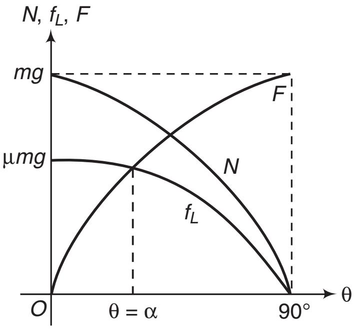


Fig. 8.118

$$N = mg\cos\theta$$

or $$N \propto \cos\theta$$

$$f_L = \mu mg\cos\theta$$

or $$f_L \propto \cos\theta$$

$$F = mg\sin\theta$$

or $$F \propto \sin\theta$$

Normally $$\mu < 1,$$

So, $$f_L < N.$$

Example 8.27 *A particle of mass 1 kg rests on rough contact with a plane inclined at 30° to the horizontal and is just about to slip. Find the coefficient of friction between the plane and the particle.*

Solution

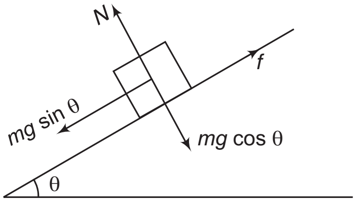


Fig. 8.119

Weight mg has two components $mg\sin\theta$ and $mg\cos\theta$. Block is at rest

$$\therefore \quad N = mg\cos\theta \qquad \text{...(i)}$$

$$f = mg\sin\theta \qquad \text{...(ii)}$$

Block is about to slip.

$$\therefore \quad f = f_L = \mu N \qquad \text{...(iii)}$$

Here $$\mu_s = \mu$$

Solving these three equations, we get

$$\mu = \tan\theta = \tan 30^\circ$$

$$= \frac{1}{\sqrt{3}}$$ **Ans.**

Note *The given angle is also angle of repose* α.

$$\therefore \quad \mu = \tan\theta = \tan\alpha = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

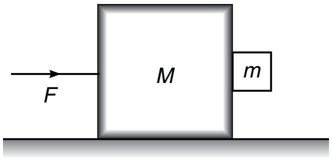


Fig. 8.120

Example 8.28 *In the adjoining figure, the coefficient of friction between wedge (of mass M) and block (of mass m) is μ.*

Find the minimum horizontal force F required to keep the block stationary with respect to wedge.

Solution This problem can be solved with or without using the concept of pseudo force. Let us solve the problem by both the methods.

$$a = \text{acceleration of (wedge + block) in horizontal direction}$$

$$= \frac{F}{M+m}$$

Inertial Frame of Reference (Ground)

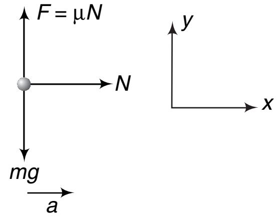


Fig. 8.121

FBD of block with respect to ground (only real forces have to applied) is as shown in Fig. 8.121. With respect to ground block is moving with an acceleration a. Therefore,

$$\Sigma F_y = 0 \quad \text{and} \quad \Sigma F_x = ma$$

$$mg = \mu N \quad \text{and} \quad N = ma$$

$$\therefore \qquad a = \frac{g}{\mu}$$

$$\therefore \qquad F = (M+m)a = (M+m)\frac{g}{\mu}$$

Non-inertial Frame of Reference (Wedge)

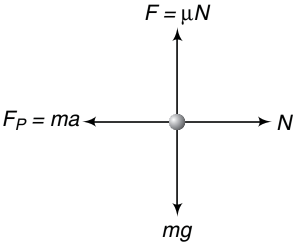


Fig. 8.122

FBD of m with respect to wedge (real + one pseudo force) is as shown in Fig. 8.122. With respect to wedge block is stationary.

$$\therefore \qquad \Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

$$\therefore \qquad mg = \mu N \quad \text{and} \quad N = ma$$

$$\therefore \qquad a = \frac{g}{\mu} \quad \text{and} \quad F = (M+m)a$$

$$= (M+m)\frac{g}{\mu}$$

Example 8.29 *A 6 kg block is kept on an inclined rough surface as shown in figure. Find the force F required to*

(a) keep the block stationary,

(b) move the block downwards with constant velocity and

(c) move the block upwards with an acceleration of 4 m/s^2. (Take $g = 10\,m/s^2$)

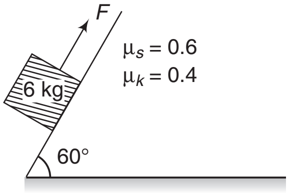


Fig. 8.123

Solution $N = mg\cos 60° = (6)(10)\left(\frac{1}{2}\right) = 30$ newton

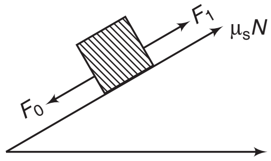


Fig. 8.124

$$\mu_S N = 18 \text{ newton}$$

$$\mu_K N = 12 \text{ newton}$$

Driving force $F_0 = mg\sin 60° = (6)(10)\left(\frac{\sqrt{3}}{2}\right) = 52$ N

(a) Force needed to keep the block stationary is

$$F_1 = F_0 - \mu_S N \quad \text{(upwards)}$$
$$= 52 - 18$$
$$= 34 \text{ N} \quad \text{(upwards)}$$

Ans.

(b) If the block moves downwards with constant velocity ($a = 0, F_{\text{net}} = 0$), then kinetic friction will act in upward direction.

∴ Force needed,

$$F_2 = F_0 - \mu_K N \quad \text{(upwards)}$$
$$= 52 - 12$$
$$= 40 \text{ N} \quad \text{(upwards)}$$

Ans.

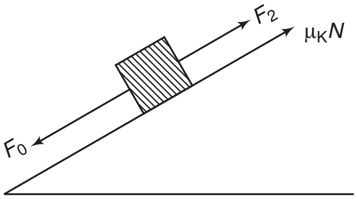


Fig. 8.125

(c) In this case, kinetic friction will act in downward direction

$$F_3 - F_0 - \mu_K N = ma$$

or $$F_3 - 52 - 12 = ma = (6)(4)$$

∴ $$F_3 = 88 \text{ N} \quad \text{(upwards)}$$

Ans.

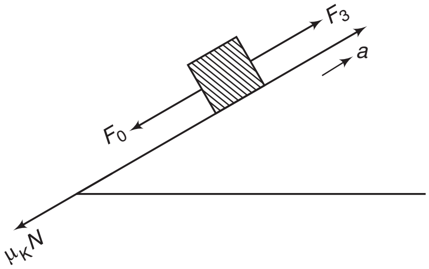


Fig. 8.126

Example 8.30 *A block of mass m is at rest on a rough wedge as shown in figure. What is the force exerted by the wedge on the block?*

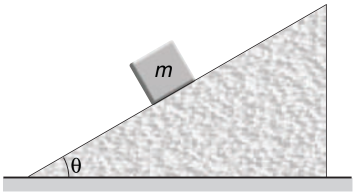


Fig. 8.127

Solution Since, the block is permanently at rest, it is in equilibrium. Net force on it should be zero. In this case, only two forces are acting on the block

(1) Weight = mg (downwards).

(2) Contact force (resultant of normal reaction and friction force) applied by the wedge on the block.

For the block to be in equilibrium, these two forces should be equal and opposite.

Therefore, force exerted by the wedge on the block is mg (upwards).

Note (i) *From Newton's third law of motion, force exerted by the block on the wedge is also mg but downwards.*

(ii) *This result can also be obtained in a different manner. The normal force on the block is* $N = mg\cos\theta$ *and the friction force on the block is* $f = mg\sin\theta$ *(not* $\mu\, mg\cos\theta$*)*

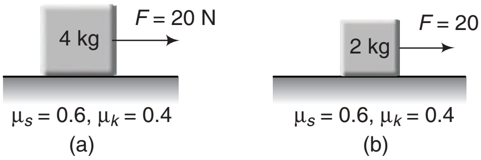
These two forces are mutually perpendicular.

∴ *Net contact force would be* $\sqrt{N^2 + f^2}$

or $\sqrt{(mg\cos\theta)^2 + (mg\sin\theta)^2}$ *which is equal to mg.*

INTRODUCTORY EXERCISE 8.5

1. In the three figures shown, find acceleration of block and force of friction on it in each case.



$\mu_s = 0.6, \mu_k = 0.4$
(a)

$\mu_s = 0.6, \mu_k = 0.4$
(b)

Fig. 8.128

2. In the figure shown, angle of repose is 45°. Find force of friction, net force and acceleration of the block when

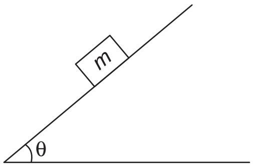
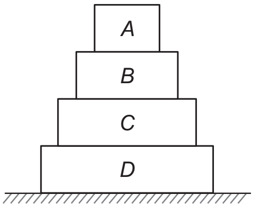


Fig. 8.129

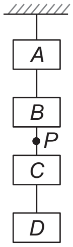
(a) $\theta = 30°$ (b) $\theta = 45°$ and (c) $\theta = 60°$

Final Touch Points

1. In Fig. (i), normal reaction at point P (between blocks C and D) is given by,



(i)



(ii)

$$N = [\Sigma(\text{mass above } P)] \times g_{\text{eff}} = (m_A + m_B + m_C)\, g_{\text{eff}}$$

In Fig. (ii), tension at point P is given by,

$$T = \Sigma\,[(\text{mass below } P)] \times g_{\text{eff}}$$

If the strings are massless then, $T = (m_C + m_D) g_{\text{eff}}$

Here, $g_{\text{eff}} = g$ if acceleration of system is zero

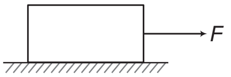
$= (g + a)$ if acceleration a is upwards

$= (g - a)$ if acceleration a is downwards

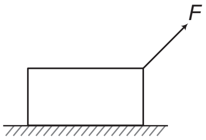
2. Feeling of weight to a person is due to the normal reaction. Under normal conditions, $N = mg$. Therefore feeling of weight is the actual weight mg. If we are standing on a lift and the lift has an acceleration 'a' upwards then $N = m(g + a)$. Therefore feeling of weight is more than the actual weight mg. Similarly if 'a' is downwards then $N = m(g - a)$ and feeling of weight is less than the actual weight mg.

3. If $\mu_s = \mu_N = \mu$ then, limiting value of static friction = constant value of kinetic friction = μN.

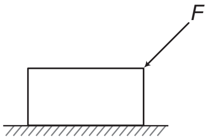
Here, $N = mg$ on horizontal ground or $N = mg \cos\theta$ on inclined ground as long as the external forces (other than weight and normal reaction) are either zero or tangential to the surface. If the external force is inclined to the horizontal surface (or inclined plane), then normal reaction either increase or decrease depending on the direction of F.



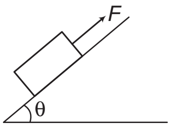
$N = mg$



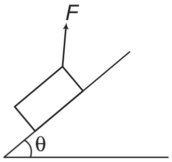
$N < mg$



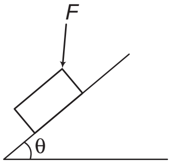
$N > mg$



$N = mg\cos\theta$



$N < mg\cos\theta$



$N > mg\cos\theta$

Solved Examples

TYPED PROBLEMS

Type 1. *Resolution of forces*

Concept

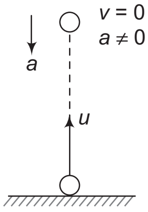
Different situations of this type can be classified in following two types:

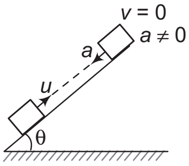
(i) Permanent rest, body in equilibrium, net force equal to zero, net acceleration equal to zero or moving with constant velocity.

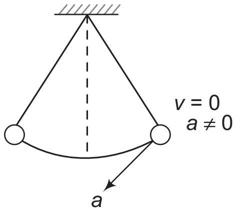
(ii) Accelerated and temporary rest.

How to Solve?

- In the first situation, forces can be resolved in any direction. Net force (or summation of components of different forces acting on the body) in any direction should be zero.
- In the second situation forces are normally resolved along acceleration and perpendicular to it. In a direction perpendicular to acceleration net force is zero and along acceleration net force is *ma*.
- In temporary rest situation velocity of the body is zero but acceleration is not zero. The direction of acceleration in this case is the direction in which the body is supposed to move just after few seconds. Three situations of temporary rest are shown below.

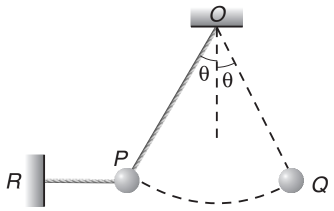






Note *In the second situation also, we can resolve the forces in any direction. In that case, net force along this direction = (mass) (component of acceleration in this direction)*

Example 1 *A ball of mass 1 kg is at rest in position P by means of two light strings OP and RP. The string RP is now cut and the ball swings to position Q. If* $\theta = 45°$. *Find the tensions in the strings in positions OP (when RP was not cut) and OQ (when RP was cut). (Take* $g = 10\ m/s^2$*).*

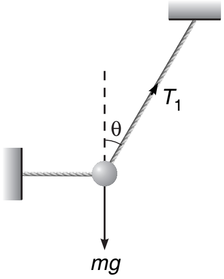


Solution In the first case, ball is in equilibrium (permanent rest). Therefore, net force on the ball in any direction should be zero.

$\therefore$ $(\Sigma \mathbf{F})$ in vertical direction $= 0$

or $T_1 \cos\theta = mg$

or $$T_1 = \frac{mg}{\cos\theta}$$

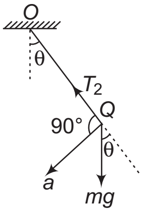


Substituting $m_1 = 1$ kg, $g = 10$ m/s^2 and $\theta = 45°$

we get, $T_1 = 10\sqrt{2}$ N

Note *Here, we deliberately resolved all the forces in vertical direction because component of the tension in RP in vertical direction is zero. In a direction other than vertical we will also have to consider component of tension in RP, which will unnecessarily increase our calculation.*

In the second case ball is not in equilibrium (temporary rest). After few seconds it will move in a direction perpendicular to OQ. Therefore, net force on the ball at Q is perpendicular to OQ or net force along $OQ = 0$.



$\therefore$ $T_2 = mg\cos\theta$

Substituting the values, we get $T_2 = 5\sqrt{2}$ N

Here, we can see that $T_1 \neq T_2$

Type 2. *To find tension at some point (say at P) if it is variable*

How to Solve?

- Find acceleration (a common acceleration) of the system by using the equation

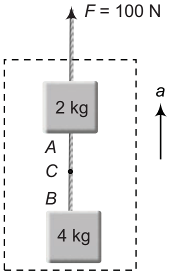
$$a = \frac{\text{net pulling or pushing force}}{\text{total mass}}$$

  In some cases, '*a*' will be given in the question.
- Cut the string at *P* and divide the system in two parts.
- Make free body diagram of any one part (preferably of the smaller one).
- In its FBD make one tension at point *P* in a direction away from the block with which this part of the string is attached.
- Write the equation,

$$F_{\text{net}} = ma$$

  for this part. You will get tension at *P*. In this equation *m* is not the total mass. It is mass of this part only.

Example 2 *In the given figure mass of string AB is 2 kg. Find tensions at A, B and C, where C is the mid point of string.*



Solution $$a = \frac{F - \text{weight of 2 kg} - \text{weight of 4 kg} - \text{weight of string}}{\text{mass of 2 kg} + \text{mass of 4 kg} + \text{mass of string}}$$

$$= \frac{100 - 20 - 40 - 20}{2 + 4 + 2} \qquad (g = 10 \text{ m/s}^2)$$

$$= \frac{20}{8} = 2.5 \text{ m/s}^2$$

Refer Fig. (a) $T_A - m_{AB}g - 40 = (m_{AB} + 4)a$

or $T_A - 20 - 40 = (2 + 4)(2.5)$

$T_A = 75$ N **Ans.**

Refer Fig. (b) $T_C - m_{BC}g - 40 = (m_{BC} + 4)a$

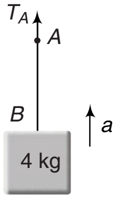
or $T_C - 10 - 40 = (1 + 4)(2.5)$

or $T_C = 62.5$ N **Ans.**

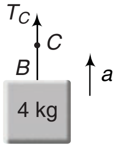
Refer Fig. (c) $T_B - 40 = 4a$

or $T_B = 40 + 4 \times 2.5$

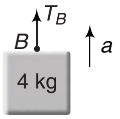
or $T_B = 50$ N **Ans.**



(a)



(b)



(c)

Note *Tension at a general point P can be given by :*

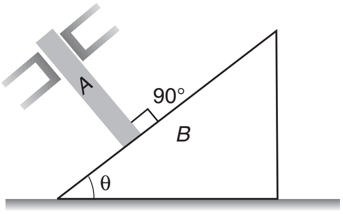
$$T_P = [(\Sigma \text{ mass below } P) \times g_{eff}]$$

Here $$g_{eff} = g + a = 12.5 \text{ m/s}^2$$

Type 3. *Based on constraint relation between a block (or a plank) and a wedge.*

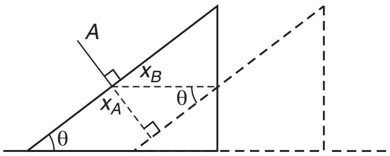
These type of problems can be understood by following two examples:

Example 3 *In the figure shown find relation between magnitudes of* $\mathbf{a}_A$ *and* $\mathbf{a}_B$.



Solution $x_A = x_B \sin\theta$ ...(i)

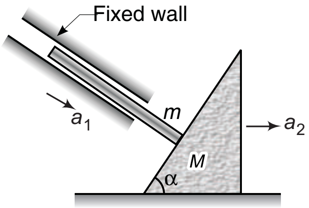
Here, θ = constant



Double differentiating Eq. (i) with respect to time, we get

$$a_A = a_B \sin\theta$$ **Ans.**

Example 4 *In the arrangement shown in the figure, the rod of mass m held by two smooth walls, remains always perpendicular to the surface of the wedge of mass M. Assuming all the surfaces to be frictionless, find the acceleration of the rod and that of the wedge.*



Solution Let acceleration of m be a_1 (absolute) and that of M be a_2 (absolute).

Writing equations of motion only in the directions of a_1 or a_2.

For *m* $$mg\cos\alpha - N = ma_1 \quad \text{...(i)}$$

For *M*, $$N\sin\alpha = Ma_2 \quad \text{...(ii)}$$

Here, N = normal reaction between m and M

As discussed above, constraint equation can be written as,

$$a_1 = a_2 \sin\alpha \quad \text{...(iii)}$$

Solving above three equations, we get

acceleration of rod, $$a_1 = \frac{mg\cos\alpha\sin\alpha}{\left(m\sin\alpha + \dfrac{M}{\sin\alpha}\right)}$$ **Ans.**

and acceleration of wedge $$a_2 = \frac{mg\cos\alpha}{m\sin\alpha + \dfrac{M}{\sin\alpha}}$$ **Ans.**

Type 4. *Based on constraint relation which keeps on changing.*

Concept

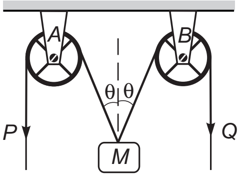
(i) In the constraint relations discussed so far the relation between different accelerations was fixed.

For example: In the two illustrations discussed above $a_A = a_B \sin\theta$ or $a_1 = a_2 \sin\alpha$ but these relations were fixed, as θ or α was constant.

(ii) In some cases, θ or α keeps on changing. Therefore, the constraint relation also keeps on changing.

(iii) In this case, constraint relation between different accelerations becomes very complex. So, normally constraint relation between velocities is only asked.

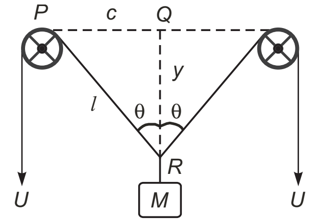
Example 5 *In the arrangement shown in the figure, the ends P and Q of an unstretchable string move downwards with uniform speed U. Pulleys A and B are fixed. Mass M moves upwards with a speed* **(JEE 1982)**



(a) $2U\cos\theta$ (b) $\frac{U}{\cos\theta}$ (c) $\frac{2U}{\cos\theta}$ (d) $U\cos\theta$

Solution In the right angle ΔPQR

$$l^2 = c^2 + y^2$$



c = constant, l and y are variables

Differentiating this equation with respect to time, we get

$$2l\frac{dl}{dt} = 0 + 2y\frac{dy}{dt} \quad \text{or} \quad \left(-\frac{dy}{dt}\right) = \frac{l}{y}\left(-\frac{dl}{dt}\right)$$

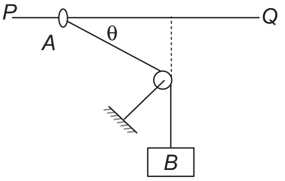
Here, $-\frac{dy}{dt} = v_M, \frac{l}{y} = \frac{1}{\cos\theta}$ and $-dl/dt = U$

Hence, $$v_M = \frac{U}{\cos\theta}$$

∴ The correct option is (b).

Note *Here θ is variable. Therefore the constraint relation* $v_M = \frac{U}{\cos\theta}$ *is also variable.*

Example 6 *In the adjoining figure, wire PQ is smooth, ring A has a mass 1 kg and block B, 2 kg. If system is released from rest with θ = 60°, find*



(a) constraint relation between their velocities as a function of θ.

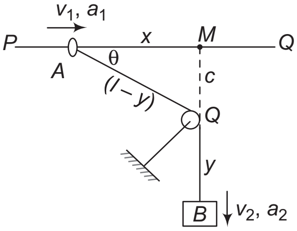
(b) constraint relation between their accelerations just after the release at θ = 60°.

(c) tension in the string and the values of these accelerations at this instant.

Solution M and Q are two fixed fixed points. Therefore,

$$MQ = \text{constant} = c$$

$$l = \text{length of string} = \text{constant}.$$



(a) In triangle MQA, $(l-y)^2 = x^2 + c^2$

Differentiating w.r.t time, we get

$$2(l-y)\left(-\frac{dy}{dt}\right) = 2x\left(+\frac{dx}{dt}\right) + 0$$

or $$(l-y)\left(+\frac{dy}{dt}\right)=x\left(-\frac{dx}{dt}\right) \quad \text{...(i)}$$

$\therefore$ $$\frac{dy}{dt}=\left(\frac{x}{l-y}\right)\left(-\frac{dx}{dt}\right) \quad \text{...(ii)}$$

y is increasing with time,

$\therefore$ $$+\frac{dy}{dt}=v_2$$

x is decreasing with time

$\therefore$ $$-\frac{dx}{dt}=v_1 \quad \text{and} \quad \frac{x}{l-y}=\cos\theta$$

Substituting these values in Eq. (ii), we have

$$v_2=v_1\cos\theta$$ **Ans.**

(b) Further differentiating Eq. (i) we have,

$$\frac{d^2y}{dt^2}(l-y)-\left(\frac{dy}{dt}\right)^2=-\left[x\cdot\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2\right] \quad \text{...(iii)}$$

Just after the release, $v_1, v_2, \frac{dx}{dt}$ and $\frac{dy}{dt}$ all are zero. Substituting in Eq. (iii), we have,

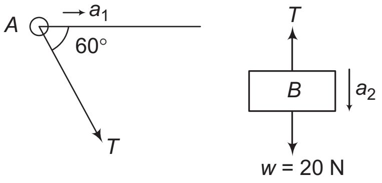
$$\frac{d^2y}{dt^2}=\left(\frac{x}{l-y}\right)\left(-\frac{d^2x}{dt^2}\right) \quad \text{...(iv)}$$

Here, $$\frac{d^2y}{dt^2}=a_2 \text{ and } -\frac{d^2x}{dt^2}=a_1$$

$$\frac{x}{l-y}=\cos\theta=\cos 60°=\frac{1}{2}$$

Substituting in Eq. (iv), we have $$a_2=\frac{a_1}{2} \quad \text{...(v)}$$

(c) **For *A*** Equation is



$$T\cos 60°=m_A a_1 \quad \text{or} \quad \frac{T}{2}=(1)a_1=a_1 \quad \text{...(vi)}$$

For *B* $$20-T=m_Ba_2$$
$$20-T=2a_2 \quad \text{...(vii)}$$

Solving Eqs. (v), (vi) and (vii), we get

$$T=\frac{40}{3}\text{ N} \Rightarrow a_1=\frac{20}{3}\text{ m/s}^2$$

and $$a_2=\frac{10}{3}\text{ m/s}^2$$

Note *Eq. (iii) converts into a simple Eq. (iv), just after the release when* $v_1, v_2, \frac{dx}{dt}$ *and* $\frac{dy}{dt}$ *all are zero.*

Type 5. *To find whether the block will move or not under different forces kept over a rough surface*

How to Solve?

- The rough surface may be horizontal, inclined or vertical.
- Resolve the forces along the surface and perpendicular to the surface.
- In most of the cases, acceleration perpendicular to the surface is zero.
- So net force perpendicular to the surface should be zero. By putting net force perpendicular to the surface equal to zero we will get normal reaction N.
- After finding, N, calculate $\mu_s N$, $\mu_k N$ or μN.
- Calculate net force along the plane and call it the driving force F.

Now, if $F \le \mu_s N$

Then $f = F$ (f = Force of friction)

and $F_{\text{net}} = 0$ or $a = 0$,

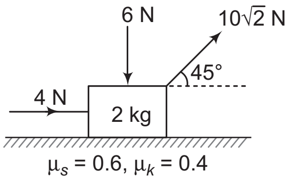
If $F > \mu_s N$

Then, $f = \mu_k N$ and $F_{\text{net}} = F - f$ or $a = \dfrac{F_{\text{net}}}{m}$ (in the direction of F)

Example 7 *In the figure shown,*

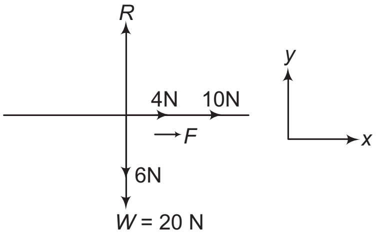
(a) find the force of friction acting on the block.

(b) state whether the block will move or not. If yes then with what acceleration?



Solution Resolving the force in horizontal (along the plane) and in vertical (perpendicular to the plane) directions (except friction)

Here, R is the normal reaction.



$$\Sigma F_y = 0 \Rightarrow R = 26\text{ N}$$
$$\mu_s R = 0.6 \times 26 = 16.6\text{ N}$$
$$\mu_k R = 0.4 \times 26 = 10.4\text{ N}$$
$$\Sigma F_x = \text{net driving force } F = 14\text{ N}$$

(a) Since, $F \le \mu_s R$

$\therefore$ Force of friction $f = F = 14$ N

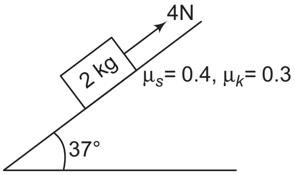
This friction will act in the opposite direction of F.

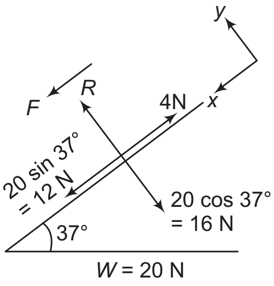
(b) Since, $F \le \mu_s R$, the block will not move.

Example 8 *In the figure shown,*

(a) find the force of friction acting on the block.

(b) state whether the block will move or not. If yes then with what acceleration?





Solution Resolving the forces along the plane and perpendicular to the plane. (except friction)

Here, R is the normal reaction. $\Sigma F_y = 0 \Rightarrow R = 16\text{ N}$

$$\mu_s R = 0.4 \times 16 = 6.4\text{ N}$$

$$\mu_k R = 0.3 \times 16 = 4.8\text{ N}$$

$$\Sigma F_x = \text{net driving force } F = (12 - 4)\text{ N} = 8\text{ N}$$

(a) Since, $F > \mu_s R$, therefore kinetic friction or 4.8 N will act in opposite direction of F.

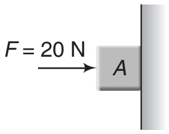
(b) Since, $F > \mu_s R$, the block will move in the direction of F (or downwards) with an acceleration,

$$a = \frac{F_{\text{net}}}{m} = \frac{F - f}{m} = \frac{8 - 4.8}{2} = 1.6\text{ m/s}^2$$

This acceleration is in the direction of F (or downwards).

Example 9 *A block of mass 1 kg is pushed against a rough vertical wall with a force of 20 N, coefficient of static friction being* $\frac{1}{4}\cdot$ *Another horizontal force of 10 N is applied on the block in a direction parallel to the wall. Will the block move? If yes, in which direction? If no, find the frictional force exerted by the wall on the block.* ($g = 10\ m/s^2$)

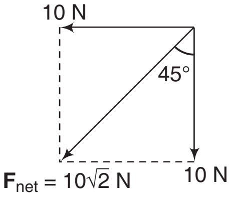
Solution Normal reaction on the block from the wall will be ($F_{\text{net}} = 0$, perpendicular to the wall)



$$R = F = 20\text{ N}$$

Therefore, limiting friction $f_L = \mu R = \left(\frac{1}{4}\right)(20) = 5\text{ N}$

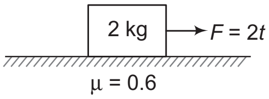
Weight of the block is $w = mg = (1)(10) = 10\text{ N}$



A horizontal force of 10 N is applied to the block. Both weight and this force are along the wall. The resultant of these two forces will be $10\sqrt{2}$ N in the direction shown in figure. Since, this resultant is greater than the limiting friction. The block will move in the direction of $\mathbf{F}_{\text{net}}$ with acceleration

$$a = \frac{\mathbf{F}_{\text{net}} - f_L}{m} = \frac{10\sqrt{2} - 5}{1} = 9.14\text{ m/s}^2$$

Type 6. *To draw acceleration versus time graph. Following three examples will illustrate this type.*



Example 10 *In the figure shown, F is in newton and t in seconds. Take* $g = 10\ m/s^2$.

(a) Plot acceleration of the block versus time graph.

(b) Find force of friction at, $t = 2\text{ s}$ *and* $t = 8\text{ s}$.

Solution (a) Normal reaction, $R = mg = 20\text{N}$. Limiting value of friction, $f_1 = \mu R = 0.6 \times 20 = 12\text{N}$

The applied force F $(=2t)$ crosses this limiting value of friction at 6s. Therefore, upto 6 s block remains stationary and after 6s it starts moving. After 6 s, friction becomes constant at 12 N but the applied force keeps on increasing. Therefore, acceleration keeps on increasing.

For $t \le 6$ s

$$f = F = 2t \qquad \text{...(i)}$$

$$F_{\text{net}} = F - f = 0$$

$$\therefore \quad a = \frac{F_{\text{net}}}{m} = 0$$

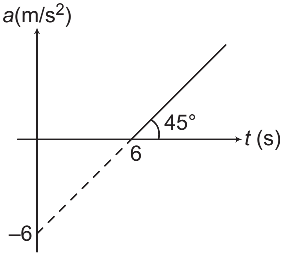
For **$t > 6$ s**

$$F = 2t$$

$$f = 12\text{ N} = f_1 \qquad \text{...(ii)}$$

$$F_{\text{net}} = F - f = 2t - 12$$

$$\therefore \quad a = \frac{F_{\text{net}}}{m} = \frac{2t-12}{2} = (t-6)$$



$\therefore$ a-t graph is a straight line with slope =1 and intercept $= -6$.

Corresponding a-t graph is as shown.

(b) At $t = 2\text{s}$, $f = 4\text{ N}$ [from Eq. (i)]

At $t = 8\text{s}$, $f = 12\text{ N}$ [from Eq. (ii)]

Example 11 *Repeat the above problem, if instead of μ we are given μ_s and μ_k, where $\mu_s = 0.6$ and $\mu_k = 0.4$.*

Solution (a) $R = mg = 20\text{ N}$

$$\mu_s R = 0.6 \times 20 = 12\text{ N}$$

$$\mu_k R = 0.4 \times 20 = 8\text{ N}$$

Upto 6 s, situation is same but after 6 s, a constant kinetic friction of 8N will act. At 6 s, friction will suddenly change from 12 N $(=\mu_s R)$ to 8N $(=\mu_k R)$ and direction of friction is opposite to its motion. Therefore, at 6 s it will start with an initial acceleration.

$$a_i = \frac{\text{decrease in friction}}{\text{mass}} = \frac{12-8}{2} = 2\text{ m/s}^2$$

For $t \le 6$ s

$$f = F = 2t \qquad \text{...(i)}$$

$$F_{\text{net}} = F - f = 0$$

$$\therefore \quad a = \frac{F_{\text{net}}}{m} = 0$$

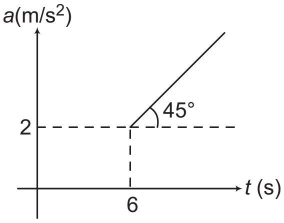
For $t \ge 6$ s

$$F = 2t$$

$$f = \mu_k R = 8\text{N}$$

$$F_{\text{net}} = F - f = 2t - 8$$

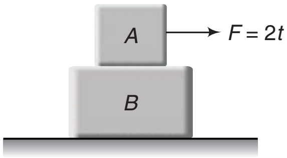
$$\therefore \quad a = \frac{F_{\text{net}}}{m} = \frac{2t-8}{2} = (t-4)$$



At $t = 6$ s, we can see that, $a_i = 2\text{ m/s}^2$

Further, a - t graph is a straight line of slope = 1 and intercept $= -4$. Corresponding a - t graph is as shown in figure.

Example 12 *Two blocks A and B of masses 2 kg and 4 kg are placed one over the other as shown in figure. A time varying horizontal force $F = 2t$ is applied on the upper block as shown in figure. Here t is in second and F is in newton. Draw a graph showing accelerations of A and B on y-axis and time on x-axis. Coefficient of friction between A and B is $\mu = \frac{1}{2}$ and the horizontal surface over which B is placed is smooth. ($g = 10\ m/s^2$)*



Concept

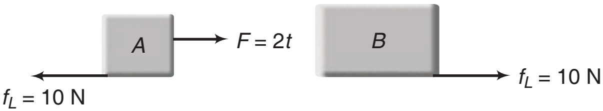
In the given example, block A will move due to the applied force but block B moves due to friction (between A and B). But there is a limiting value of friction between them. Therefore, there is a limiting value of acceleration (of block B). Up to this acceleration they move as a single block with a common acceleration, but after that acceleration of B will become constant (as friction acting on this block will become constant). But acceleration of A will keep on increasing as a time increasing force is acting on it.

Solution Limiting friction between A and B is

$$f_L = \mu m_A g = \left(\frac{1}{2}\right)(2)\ (10) = 10\ \text{N}$$

Block B moves due to friction only. Therefore, maximum acceleration of B can be

$$a_{\max} = \frac{f_L}{m_B} = \frac{10}{4} = 2.5\ \text{m/s}^2$$



For $t \geq 7.5$ s

Thus, both the blocks move together with same acceleration till the common acceleration becomes $2.5\ \text{m/s}^2$, after that acceleration of B will become constant while that of A will go on increasing. To find the time when the acceleration of both the blocks becomes $2.5\ \text{m/s}^2$ (or when slipping will start between A and B) we will write

$$2.5 = \frac{F}{(m_A + m_B)} = \frac{2t}{6}$$

$\therefore$ $t = 7.5$ s

Hence, for $t \leq 7.5$ s

$$a_A = a_B = \frac{F}{m_A + m_B} = \frac{2t}{6} = \frac{t}{3}$$

Thus, a_A *versus* t or a_B *versus* t graph is a straight line passing through origin of slope $\frac{1}{3}$.

For, $t \geq 7.5$ s

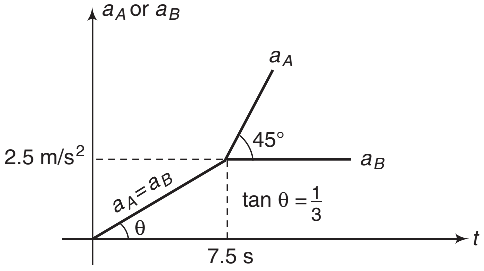
$$a_B = 2.5\ \text{m/s}^2 = \text{constant}$$

and

$$a_A = \frac{F - f_L}{m_A}$$

or

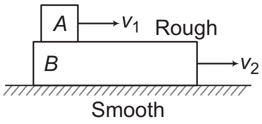
$$a_A = \frac{2t - 10}{2} \quad \text{or} \quad a_A = t - 5$$



Thus, a_A *versus* t graph is a straight line of slope 1 and intercept –5. While a_B *versus* t graph is a straight line parallel to t-axis. The corresponding graph is as shown in above figure.

Type 7. *When two blocks in contact are given different velocities and after some time, due to friction their velocities become equal.*

Concept



In the figure shown, if $v_1 > v_2$ (or $v_1 \neq v_2$) then there is a relative motion between the two blocks.

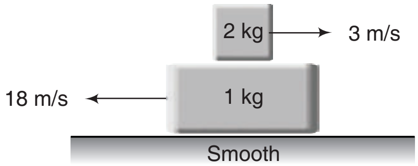
As $v_1 > v_2$, relative motion of A is towards right and relation motion of B is towards left. Since, relative motion is there, so kinetic friction (or limiting value of friction) will act in the opposite direction of relative motion. This friction (and acceleration due to this force) with decrease the velocity of A and increase the velocity of B. After some time when their velocities become equal, frictional force between them becomes zero and they continue to be moving with that common velocity (as the ground is smooth).

How to Solve?

- Find value of kinetic friction or limiting value of friction ($f = \mu_k N$ or μN) between the two blocks and then accelerations of these blocks $\left(= \frac{f}{m}\right)$. Then write $v_1 = v_2$, as their velocities become same when relatative motion is stopped.

  or $$u_1 + a_1 t = u_2 + a_2 t \qquad \ldots(\text{i})$$

  Substituting the proper values of u_1, a_1, u_2 and a_2 in Eq. (i), we can find the time when the velocities become equal.



Example 13 *Coefficient of friction between two blocks shown in figure is* $\mu = 0.6$ *. The blocks are given velocities in the directions shown in figure. Find*

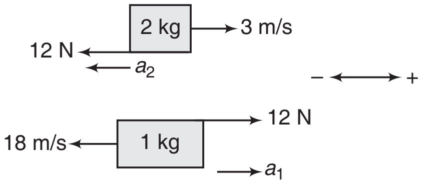
(a) the time when relative motion between them is stopped.

(b) the common velocity of the two blocks.

(c) the displacements of 1 kg and 2 kg blocks upto that instant. (Take $g = 10\,\text{m/s}^2$*)*

Note *Assume that lower block is sufficiently long and upper block does not fall from it.*

Solution Relative motion of 2 kg block is towards right. Therefore, maximum friction on this block will act towards left



$$f = \mu N = (0.6)(2)(10) = 12 \text{ N}$$

$$a_2 = -\frac{12}{2} = -6 \text{ m/s}^2$$

$$a_1 = \frac{12}{1} = 12 \text{ m/s}^2$$

(a) Relative motion between them will stop when,

$$v_1 = v_2 \quad \text{or} \quad u_1 + a_1 t = u_2 + a_2 t$$

or $$-18 + 12t = 3 - 6t \qquad \text{...(i)}$$

Solving we get, $$t = \frac{7}{6} \text{ s}$$ **Ans.**

(b) Substituting value of 't' in Eq. (i) either on RHS or on LHS we have,

common velocity $= -4$ m/s **Ans.**

(c) $$s_1 = u_1 t + \frac{1}{2} a_1 t^2$$

$$= (-18)\left(\frac{7}{6}\right) + \frac{1}{2}(12)\left(\frac{7}{6}\right)^2$$

$$= -12.83 \text{ m}$$ **Ans.**

$$s_2 = u_2 t + \frac{1}{2} a_2 t^2$$

$$= (3)\left(\frac{7}{6}\right) + \frac{1}{2}(-6)\left(\frac{7}{6}\right)^2$$

$$= -0.58 \text{ m}$$ **Ans.**

Type 8. *Acceleration or retardation of a car*

Concept

A car accelerates or retards due to friction. On a horizontal road maximum available friction is μN or $\mu\, mg$ (as $N = mg$). Therefore, maximum acceleration or retardation of a car on a horizontal road is

$$a_{\max} = \frac{f_{\max}}{m} = \frac{\mu\, mg}{m} = \mu\, g$$

On an inclined plane maximum value of friction is μN or $\mu\, mg\cos\theta$ (as $N = mg\cos\theta$). Now $mg\sin\theta$ is a force which is always downwards but the frictional force varying from 0 to $\mu\, mg\cos\theta$ can be applied in upward or downward direction by the application of brakes or accelerator.

Example 14 *On a horizontal rough road, value of coefficient of friction* $\mu = 0.4$. *Find the minimum time in which a distance of* $400 m$ *can be covered. The car starts from rest and finally comes to rest.*

Solution Maximum friction on horizontal rough road, $f_{\max} = \mu mg$

$\therefore$ Maximum acceleration or retardation of the car may be

$$a_{\max} \text{ or } a = \frac{f_{\max}}{m} = \frac{\mu mg}{m} = \mu g$$

$$= 0.4 \times 10 = 4 \text{ m/s}^2$$

Let, the car accelerates and retards for time 't' with 4 m/s^2.

Then, $$\frac{1}{2}at^2 + \frac{1}{2}at^2 = 400 \text{ m}$$

or $$at^2 = 400 \text{ m} \quad \text{or} \quad 4t^2 = 400$$

or $$t = 10 \text{ s}$$

Therefore, the minimum time is 20 s (10 s of acceleration and 10 s of retardation). **Ans.**

Example 15 *A car is moving up the plane. Angle of inclination is* θ *and coefficient of friction is* μ.

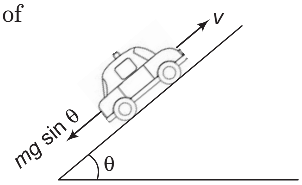
(a) What is the condition in which car can be accelerated?
If this condition is satisfied then find

(b) maximum acceleration of the car.

(c) minimum retardation of the car.

(d) maximum retardation of the car.

Solution (a) $mg \sin\theta$ in all conditions is downwards but direction of friction may be upwards or downwards. We will have to press accelerator for upward friction and brakes for downward friction.



To accelerate the car friction should be upwards. Therefore, car can be accelerated if maximum upward friction $> mg \sin\theta$

or $$\mu mg \cos\theta > mg \sin\theta \quad \text{or} \quad \mu > \tan\theta$$

(b) Maximum acceleration

$$= \frac{\text{maximum upwards force}}{\text{mass}}$$

$$= \frac{\mu mg \cos\theta - mg \sin\theta}{m} = (\mu g \cos\theta - g \sin\theta)$$

(c) Minimum retardation will be zero, when upward friction $= mg \sin\theta$

(d) Maximum retardation

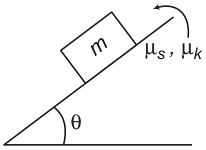
$$= \frac{\text{maximum downward force}}{m}$$

$$= \frac{mg \sin\theta + \mu mg \cos\theta}{m}$$

$$= (g \sin\theta + \mu g \cos\theta)$$

This is the case, when maximum friction force acts in downward direction.

Miscellaneous Examples



Example 16 *In the adjoining figure, angle of plane* θ *is increased from* $0°$ *to* $90°$. *Plot force of friction 'f' versus* θ *graph.*

Solution Normal reaction $N = mg\cos\theta$. Limiting value of static friction,

$$f_L = \mu_s N = \mu_s\, mg\cos\theta$$

Constant value of kinetic friction

$$f_K = \mu_k N = \mu_k mg\cos\theta$$

Driving force down the plane,

$$F = mg\sin\theta$$

Now block remains stationary and $f = F$ until F becomes equal to f_L

or $$mg\sin\theta = \mu_s\, mg\cos\theta$$

or $$\tan\theta = \mu_s \quad \text{or} \quad \theta = \tan^{-1}(\mu_s) = \theta_r \text{ (say)}$$

After this, block starts moving and constant value of kinetic friction will act. Thus,

For $\theta \le \tan^{-1}(\mu_s)$ or θ_r

$$f = F = mg\sin\theta \quad \text{or} \quad f \propto \sin\theta$$

At, $\theta = 0°$, $f = 0$ and at $\theta = \tan^{-1}(\mu_s)$ or θ_r

or $$f = mg\sin\theta_r \quad \text{or} \quad \mu_s mg\cos\theta_r$$

For $\theta > \tan^{-1}(\mu_s)$ or θ_r

$$f = f_k = \mu_k mg\cos\theta \quad \text{or} \quad f \propto \cos\theta$$

At $\theta = \tan^{-1}(\mu_s)$ or θ_r

$$f = \mu_k mg\cos\theta_r \text{ and at } \theta = 90°$$

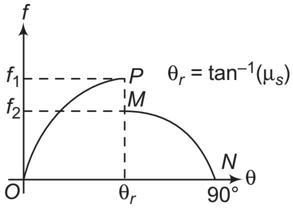
$$f = 0$$

Corresponding f *versus* θ graph is as shown in figure

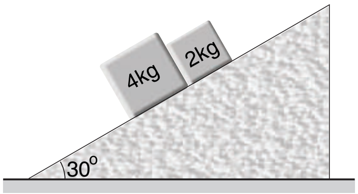
In the figure, OP is sine graph and MN is cos graph,

$$f_1 = mg\sin\theta_r = \mu_s\, mg\cos\theta_r$$

$$f_2 = \mu_k\, mg\cos\theta_r$$



Example 17 *Figure shows two blocks in contact sliding down an inclined surface of inclination* $30°$. *The friction coefficient between the block of mass 2.0 kg and the incline is* $\mu_1 = 0.20$ *and that between the block of mass 4.0 kg and the incline is* $\mu_2 = 0.30$. *Find the acceleration of 2.0 kg block.* ($g = 10\ m/s^2$).



Solution Since, $\mu_1 < \mu_2$, acceleration of 2 kg block down the plane will be more than the acceleration of 4 kg block, if allowed to move separately. But, as the 2.0 kg block is behind the 4.0 kg block both of them will move with same acceleration say a. Taking both the blocks as a single system:

Force down the plane on the system $= (4 + 2)\, g \sin 30°$

$$= (6)(10)\left(\frac{1}{2}\right) = 30 \text{ N}$$

Force up the plane on the system

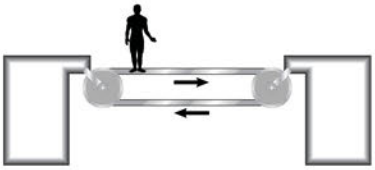
$$= \mu_1 (2)(g) \cos 30° + \mu_2 (4)(g) \cos 30°$$
$$= (2\,\mu_1 + 4\,\mu_2)\, g \cos 30°$$
$$= (2 \times 0.2 + 4 \times 0.3)\,(10)\,(0.86)$$
$$\approx 13.76 \text{ N}$$

$\therefore$ Net force down the plane is $F = 30 - 13.76 = 16.24$ N

$\therefore$ Acceleration of both the blocks down the plane will be a.

$$a = \frac{F}{4+2} = \frac{16.24}{6} = 2.7 \text{ m/s}^2$$ **Ans.**

Example 18 *Figure shows a man standing stationary with respect to a horizontal conveyor belt that is accelerating with 1 ms^{-2}. What is the net force on the man? If the coefficient of static friction between the man's shoes and the belt is 0.2, upto what maximum acceleration of the belt can the man continue to be stationary relative to the belt? Mass of the man = 65 kg. ($g = 9.8\ m/s^2$)*



Solution As the man is standing stationary w.r.t. the belt,

$\therefore$ Acceleration of the man = Acceleration of the belt

$$= a = 1 \text{ ms}^{-2}$$

Mass of the man, $m = 65$ kg

Net force on the man $= ma = 65 \times 1 = 65$ N **Ans.**

Given coefficient of friction, $\mu = 0.2$

$\therefore$ Limiting friction, $f_L = \mu mg$

If the man remains stationary with respect to the maximum acceleration a_0 of the belt, then

$$ma_0 = f_L = \mu mg$$

$\therefore$ $a_0 = \mu g = 0.2 \times 9.8 = 1.96 \text{ ms}^{-2}$ **Ans.**

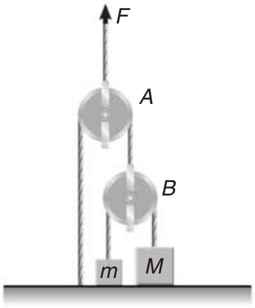
Example 19 *Two blocks of masses m = 5 kg and M = 10 kg are connected by a string passing over a pulley B as shown. Another string connects the centre of pulley B to the floor and passes over another pulley A as shown. An upward force F is applied at the centre of pulley A. Both the pulleys are massless.*

Find the acceleration of blocks m and M, if F is

(a) 100 *N*

(b) 300 *N*

(c) 500 *N (Take* $g = 10\ m/s^2$*)*

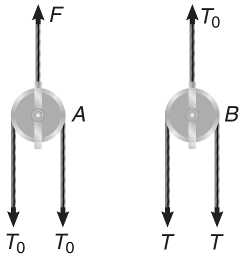


Solution Let $T_0 =$ tension in the string passing over A

$T =$ tension in the string passing over B

$$2T_0 = F \quad \text{and} \quad 2T = T_0$$

$$\therefore \qquad T = F/4$$



(a) $T = F/4 = 25$ N

weights of blocks are $mg = 50$ N

$Mg = 100$ N

As $T < mg$ and Mg both, the blocks will remain stationary on the floor.

(b) $T = F/4 = 75$ N

As $T < Mg$ and $T > mg$, M will remain stationary on the floor, whereas m will move.

Acceleration of m,

$$a = \frac{T - mg}{m} = \frac{75 - 50}{5}$$

$$= 5 \text{ m/s}^2$$ **Ans.**

(c) $T = F/4 = 125$ N

As $T > mg$ and Mg both the blocks will accelerate upwards.

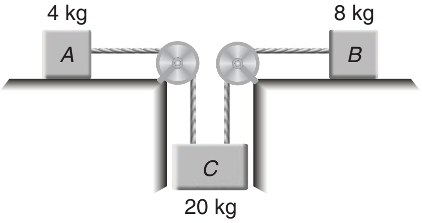
Acceleration of m,

$$a_1 = \frac{T - mg}{m} = \frac{125 - 50}{5} = 15 \text{ m/s}^2$$

Acceleration of M,

$$a_2 = \frac{T - Mg}{M} = \frac{125 - 100}{10} = 2.5 \text{ m/s}^2$$

Example 20 *Consider the situation shown in figure. The block B moves on a frictionless surface, while the coefficient of friction between A and the surface on which it moves is 0.2. Find the acceleration with which the masses move and also the tension in the strings. (Take $g = 10$ m/s^2).*



Solution Let a be the acceleration with which the masses move and T_1 and T_2 be the tensions in left and right strings. Friction on mass A is $\mu mg = 8$N. Then equations of motion of masses A, B and C are

For mass A $\quad T_1 - 8 = 4a$ ...(i)

For mass B $\quad T_2 = 8a$ ...(ii)

For mass C $\quad 200 - T_1 - T_2 = 20a$ ...(iii)

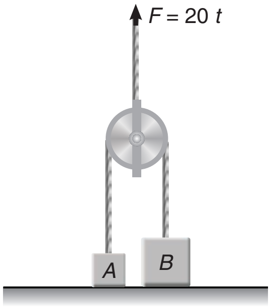
Adding the above three equations, we get $32a = 192$

or $\quad a = 6 \text{ m/s}^2$

From Eqs. (i) and (ii), we have $\quad T_2 = 48$ N

and $\quad T_1 = 32$ N

Example 21 *Two blocks A and B of masses 1 kg and 2 kg respectively are connected by a string, passing over a light frictionless pulley. Both the blocks are resting on a horizontal floor and the pulley is held such that string remains just taut.At moment* $t = 0$, *a force* $F = 20t$ *newton starts acting on the pulley along vertically upward direction as shown in figure. Calculate*



(a) *velocity of A when B loses contact with the floor.*

(b) *height raised by the pulley upto that instant.*

(Take, $g = 10\ m/s^2$*)*

Solution (a) Let T be the tension in the string. Then,

$$2T = 20t$$

or $$T = 10t \text{ newton}$$

Let the block A loses its contact with the floor at time $t = t_1$. This happens when the tension in string becomes equal to the weight of A. Thus,

$$T = mg$$

or $$10t_1 = 1 \times 10$$

or $$t_1 = 1 \text{ s} \qquad \text{...(i)}$$

Similarly, for block B, we have

$$10t_2 = 2 \times 10$$

or $$t_2 = 2 \text{ s} \qquad \text{...(ii)}$$

i.e. the block B loses contact at 2 s. For block A, at time t such that $t \geq t_1$ let a be its acceleration in upward direction. Then,

$$10t - 1 \times 10 = 1 \times a = (dv/dt)$$

or $$dv = 10(t - 1)\,dt \qquad \text{...(iii)}$$

Integrating this expression, we get

$$\int_0^v dv = 10 \int_1^t (t - 1)\,dt$$

or $$v = 5t^2 - 10t + 5 \qquad \text{...(iv)}$$

Substituting $t = t_2 = 2$ s

$$v = 20 - 20 + 5 = 5 \text{ m/s} \qquad \text{...(v)}$$

(b) From Eq. (iv), $$dy = (5t^2 - 10t + 5)\,dt \qquad \text{...(vi)}$$

where, y is the vertical displacement of block A at time t ($\geq t_1$).

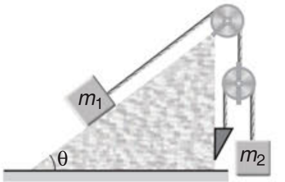
Integrating, we have

$$\int_{y=0}^{y=h} dy = \int_{t=1}^{t=2} (5t^2 - 10t + 5)\, dt$$

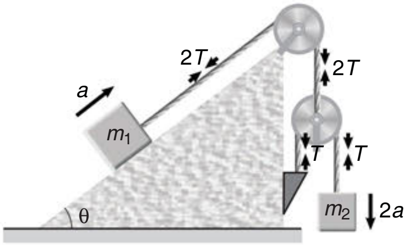
$$h = 5\left[\frac{t^3}{3}\right]_1^2 - 10\left[\frac{t^2}{2}\right]_1^2 + 5\,[t]_1^2 = \frac{5}{3} \text{ m}$$

$\therefore$ Height raised by pulley upto that instant $= \dfrac{h}{2} = \dfrac{5}{6}$ m **Ans.**

Example 22 *Find the acceleration of the body of mass m_2 in the arrangement shown in figure. If the mass m_2 is η times great as the mass m_1, and the angle that the inclined plane forms with the horizontal is equal to θ. The masses of the pulleys and threads, as well as the friction, are assumed to be negligible.*



Solution Here, by constraint relation we can see that the acceleration of m_2 is two times that of m_1. So, we assume if m_1 is moving up the inclined plane with an acceleration a, the acceleration of mass m_2 going down is $2a$. The tensions in different strings are shown in figure.



The dynamic equations can be written as

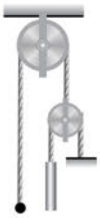
For mass m_1: $2T - m_1 g \sin\theta = m_1 a$ ...(i)

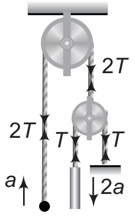
For mass m_2: $m_2 g - T = m_2(2a)$ ...(ii)

Substituting $m_2 = \eta m_1$ and solving Eqs. (i) and (ii), we get

$$\text{Acceleration of } m_2 = 2a = \frac{2g(2\eta - \sin\theta)}{4\eta + 1}$$ **Ans.**

Example 23 *In the arrangement shown in figure the mass of the ball is η times as great as that of the rod. The length of the rod is l, the masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod ?*





Solution From constraint relation we can see that the acceleration of the rod is double than that of the acceleration of the ball. If ball is going up with an acceleration a, rod will be coming down with the acceleration $2a$, thus, the relative acceleration of the ball with respect to rod is $3a$ in upward direction. If it takes time t seconds to reach the upper end of the rod, we have

$$t = \sqrt{\frac{2l}{3a}} \qquad \text{...(i)}$$

Let mass of ball be m and that of rod is M, the dynamic equations of these are

For rod $$Mg - T = M(2a) \qquad \text{...(ii)}$$

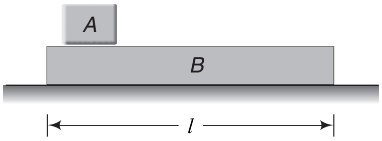
For ball $$2T - mg = ma \qquad \text{...(iii)}$$

Substituting $m = \eta M$ and solving Eqs. (ii) and (iii), we get

$$a = \left(\frac{2-\eta}{\eta+4}\right) g$$

From Eq. (i), we have $$t = \sqrt{\frac{2l(\eta+4)}{3g(2-\eta)}}$$ **Ans.**

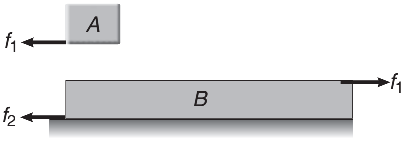
Example 24 *Figure shows a small block A of mass m kept at the left end of a plank B of mass $M = 2m$ and length l. The system can slide on a horizontal road. The system is started towards right with the initial velocity v. The friction coefficients between the road and the plank is 1/2 and that between the plank and the block is 1/4. Find*



(a) the time elapsed before the block separates from the plank.

(b) displacement of block and plank relative to ground till that moment.

Solution There will be relative motion between block and plank and plank and road. So at each surface limiting friction will act. The direction of friction forces at different surfaces are as shown in figure.



Here, $$f_1 = \left(\frac{1}{4}\right)(mg)$$

and $$f_2 = \left(\frac{1}{2}\right)(m + 2m)g = \left(\frac{3}{2}\right) mg$$

Retardation of A is $$a_1 = \frac{f_1}{m} = \frac{g}{4}$$

and retardation of B is $a_2 = \dfrac{f_2 - f_1}{2m} = \dfrac{5}{8}g$

Since, $a_2 > a_1$

Relative acceleration of A with respect to B is

$$a_r = a_2 - a_1 = \frac{3}{8}g$$

Initial velocity of both A and B is v. So, there is no relative initial velocity. Hence,

(a) Applying $s = \dfrac{1}{2}at^2$

or $l = \dfrac{1}{2}a_r t^2 = \dfrac{3}{16}gt^2$

$\therefore$ $t = 4\sqrt{\dfrac{l}{3g}}$ **Ans.**

(b) Displacement of block $s_A = u_A t - \dfrac{1}{2}a_A t^2$

or $s_A = 4v\sqrt{\dfrac{l}{3g}} - \dfrac{1}{2}\cdot\dfrac{g}{4}\cdot\left(\dfrac{16l}{3g}\right)$ $\left(a_A = a_1 = \dfrac{g}{4}\right)$

or $s_A = 4v\sqrt{\dfrac{l}{3g}} - \dfrac{2}{3}l$

Displacement of plank $s_B = u_B t - \dfrac{1}{2}a_B t^2$

or $s_B = 4v\sqrt{\dfrac{l}{3g}} - \dfrac{1}{2}\left(\dfrac{5}{8}g\right)\left(\dfrac{16l}{3g}\right)$ $\left(a_B = a_2 = \dfrac{5}{8}g\right)$

or $s_B = 4v\sqrt{\dfrac{l}{3g}} - \dfrac{5}{3}l$ **Ans.**

Note *We can see that $s_A - s_B = l$. Which is quite obvious because block A has moved a distance l relative to plank.*

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : If net force on a rigid body in zero, it is either at rest or moving with a constant linear velocity. Nothing else can happen.

Reason : Constant velocity means linear acceleration is zero.

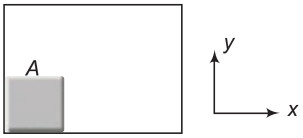
2. Assertion : Three concurrent forces are $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$. Angle between $\mathbf{F}_1$ and $\mathbf{F}_2$ is 30° and between $\mathbf{F}_1$ and $\mathbf{F}_3$ is 120°. Under these conditions, forces cannot remain in equilibrium.

Reason : At least one angle should be greater than 180°.

3. Assertion : Two identical blocks are placed over a rough inclined plane. One block is given an upward velocity to the block and the other in downward direction. If $\mu = \frac{1}{3}$ and $\theta = 45°$ the ratio of magnitudes of accelerations of two is 2 : 1.

Reason : The desired ratio is $\frac{1+\mu}{1-\mu}$.

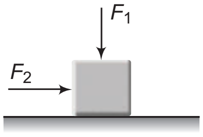
4. Assertion : A block A is just placed inside a smooth box B as shown in figure. Now, the box is given an acceleration $\mathbf{a} = (3\hat{\mathbf{j}} - 2\hat{\mathbf{i}})\ \text{ms}^{-2}$. Under this acceleration block A cannot remain in the position shown.



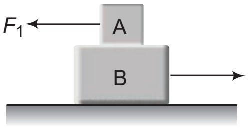
Reason : Block will require $m\mathbf{a}$ force for moving with acceleration $\mathbf{a}$.

5. Assertion : A block is kept at rest on a rough ground as shown. Two forces F_1 and F_2 are acting on it. If we increase either of the two forces F_1 or F_2, force of friction acting on the block will increase.

Reason : By increasing F_1, normal reaction from ground will increase.

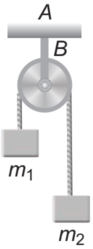


6. **Assertion :** In the figure shown, force of friction on A from B will always be right wards.
   Reason : Friction always opposes the relative motion between two bodies in contact.



Smooth

7. **Assertion :** In the figure shown tension in string AB always lies between m_1g and m_2g. $(m_1 \neq m_2)$

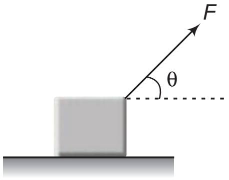


   Reason : Tension in massless string is uniform throughout.

8. **Assertion :** Two frames S_1 and S_2 are non-inertial. Then frame S_2 when observed from S_1 is inertial.
   Reason : A frame in motion is not necessarily a non-inertial frame.

9. **Assertion :** Moment of concurrent forces about any point is constant.
   Reason : If vector sum of all the concurrent forces is zero, then moment of all the forces about any point is also zero.

10. **Assertion :** Minimum force is needed to move a block on rough surface, if θ = angle of friction.
    Reason : Angle of friction and angle of repose are numerically same.



Rough

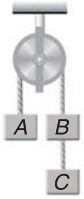
11. **Assertion :** When a person walks on a rough surface, the frictional force exerted by surface on the person is opposite to the direction of his motion.

    Reason : It is the force exerted by the road on the person that causes the motion.

Objective Questions

Single Correct Option

1. Three equal weights A, B and C of mass 2 kg each are hanging on a string passing over a fixed frictionless pulley as shown in the figure. The tension in the string connecting weights B and C is approximately



(a) zero
(b) 13 N
(c) 3.3 N
(d) 19.6 N

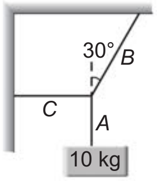
2. Two balls A and B of same size are dropped from the same point under gravity. The mass of A is greater than that of B. If the air resistance acting on each ball is same, then

(a) both the balls reach the ground simultaneously
(b) the ball A reaches earlier
(c) the ball B reaches earlier
(d) nothing can be said

3. A block of mass m is placed at rest on an inclined plane of inclination θ to the horizontal. If the coefficient of friction between the block and the plane is μ, then the total force the inclined plane exerts on the block is

(a) mg (b) $\mu \, mg \cos\theta$ (c) $mg \sin\theta$ (d) $\mu \, mg \tan\theta$

4. In the figure a block of mass 10 kg is in equilibrium. Identify the string in which the tension is zero.

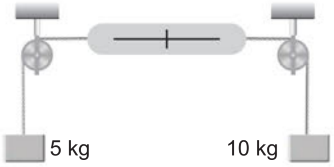


(a) B
(b) C
(c) A
(d) None of the above

5. At what minimum acceleration should a monkey slide a rope whose breaking strength is $\frac{2}{3}$ rd of its weight ?

(a) $\frac{2g}{3}$ (b) g (c) $\frac{g}{3}$ (d) zero

6. For the arrangement shown in the figure, the reading of spring balance is



(a) 50 N
(b) 100 N
(c) 150 N
(d) None of the above

7. The time taken by a body to slide down a rough 45° inclined plane is twice that required to slide down a smooth 45° inclined plane. The coefficient of kinetic friction between the object and rough plane is given by

(a) $\frac{1}{3}$ (b) $\frac{3}{4}$
(c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{2}{3}}$

8. The force required to just move a body up the inclined plane is double the force required to just prevent the body from sliding down the plane. The coefficient of friction is μ. If θ is the angle of inclination of the plane than $\tan\theta$ is equal to

(a) μ (b) 3μ (c) 2μ (d) $0.5\,\mu$

9. A force F_1 accelerates a particle from rest to a velocity v. Another force F_2 decelerates the same particle from v to rest, then

(a) F_1 is always equal to F_2

(b) F_2 is greater than F_1

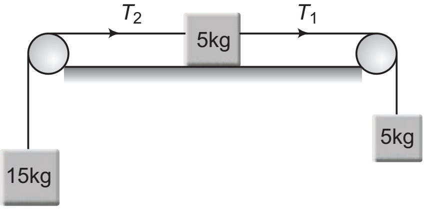
(c) F_2 may be smaller than, greater than or equal to F_1

(d) F_2 cannot be equal to F_1

10. A particle is placed at rest inside a hollow hemisphere of radius R. The coefficient of friction between the particle and the hemisphere is $\mu = \dfrac{1}{\sqrt{3}}$. The maximum height up to which the particle can remain stationary is

(a) $\dfrac{R}{2}$ (b) $\left(1 - \dfrac{\sqrt{3}}{2}\right)R$ (c) $\dfrac{\sqrt{3}}{2}R$ (d) $\dfrac{3R}{8}$

11. In the figure shown, the frictional coefficient between table and block is 0.2. Find the ratio of tensions in the right and left strings.



(a) 17 : 24 (b) 34 : 12 (c) 2 : 3 (d) 3 : 2

12. A smooth inclined plane of length L having inclination θ with the horizontal is inside a lift which is moving down with a retardation a. The time taken by a body to slide down the inclined plane from rest will be

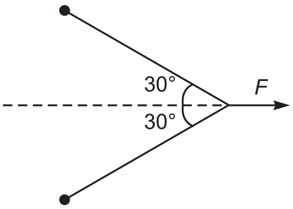
(a) $\sqrt{\dfrac{2L}{(g+a)\sin\theta}}$ (b) $\sqrt{\dfrac{2L}{(g-a)\sin\theta}}$ (c) $\sqrt{\dfrac{2L}{a\sin\theta}}$ (d) $\sqrt{\dfrac{2L}{g\sin\theta}}$

13. A block rests on a rough inclined plane making an angle of 30° with horizontal. The coefficient of static friction between the block and inclined plane is 0.8. If the frictional force on the block is 10 N, the mass of the block in kg is ($g = 10\ \text{m/s}^2$)

(a) 2.0 (b) 4.0

(c) 1.6 (d) 2.5

14. In figure, two identical particles each of mass m are tied together with an inextensible string. This is pulled at its centre with a constant force F. If the whole system lies on a smooth horizontal plane, then the acceleration of each particle towards each other is



(a) $\dfrac{\sqrt{3}}{2}\dfrac{F}{m}$ (b) $\dfrac{1}{2\sqrt{3}}\dfrac{F}{m}$

(c) $\dfrac{2}{\sqrt{3}}\dfrac{F}{m}$ (d) $\sqrt{3}\dfrac{F}{m}$

15. A block of mass m is placed at rest on a horizontal rough surface with angle of friction ϕ. The block is pulled with a force F at an angle θ with the horizontal. The minimum value of F required to move the block is

(a) $\dfrac{mg\sin\phi}{\cos(\theta-\phi)}$ (b) $\dfrac{mg\cos\phi}{\cos(\theta-\phi)}$ (c) $mg\tan\phi$ (d) $mg\sin\phi$

16. A block of mass 4 kg is placed on a rough horizontal plane. A time dependent horizontal force $F = kt$ acts on the block. Here $k = 2\ \text{Ns}^{-1}$. The frictional force between the block and plane at time $t = 2$ s is ($\mu = 0.2$)

(a) 4 N (b) 8 N
(c) 12 N (d) 10 N

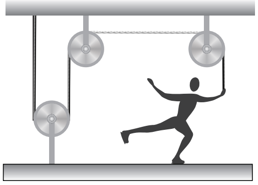
17. A body takes time t to reach the bottom of a smooth inclined plane of angle θ with the horizontal. If the plane is made rough, time taken now is $2t$. The coefficient of friction of the rough surface is

(a) $\dfrac{3}{4}\tan\theta$ (b) $\dfrac{2}{3}\tan\theta$
(c) $\dfrac{1}{4}\tan\theta$ (d) $\dfrac{1}{2}\tan\theta$

18. A man of mass m slides down along a rope which is connected to the ceiling of an elevator with deceleration a relative to the rope. If the elevator is going upward with an acceleration a relative to the ground, then tension in the rope is

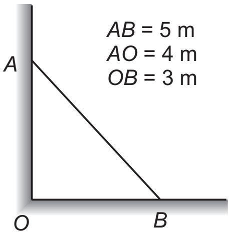
(a) mg (b) $m(g+2a)$
(c) $m(g+a)$ (d) zero

19. A 50 kg person stands on a 25 kg platform. He pulls on the rope which is attached to the platform *via* the frictionless pulleys as shown in the figure. The platform moves upwards at a steady rate if the force with which the person pulls the rope is



(a) 500 N (b) 250 N (c) 25 N (d) None of these

20. A ladder of length 5 m is placed against a smooth wall as shown in figure. The coefficient of friction is μ between ladder and ground. What is the minimum value of μ, if the ladder is not to slip?



(a) $\mu = \dfrac{1}{2}$ (b) $\mu = \dfrac{1}{4}$ (c) $\mu = \dfrac{3}{8}$ (d) $\mu = \dfrac{5}{8}$

21. If a ladder weighing 250 N is placed against a smooth vertical wall having coefficient of friction between it and floor 0.3, then what is the maximum force of friction available at the point of contact between the ladder and the floor?

(a) 75 N (b) 50 N (c) 35 N (d) 25 N

22. A rope of length L and mass M is being pulled on a rough horizontal floor by a constant horizontal force $F = Mg$. The force is acting at one end of the rope in the same direction as the length of the rope. The coefficient of kinetic friction between rope and floor is 1/2. Then, the tension at the midpoint of the rope is

(a) $\frac{Mg}{4}$ (b) $\frac{2Mg}{5}$ (c) $\frac{Mg}{8}$ (d) $\frac{Mg}{2}$

23. A heavy body of mass 25 kg is to be dragged along a horizontal plane $\left(\mu = \frac{1}{\sqrt{3}}\right)$. The least force required is (1 kgf = 9.8 N)

(a) 25 kgf (b) 2.5 kgf (c) 12.5 kgf (d) 6.25 kgf

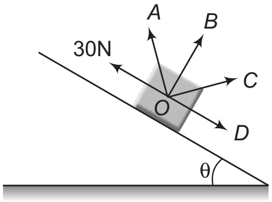
24. A block A of mass 4 kg is kept on ground. The coefficient of friction between the block and the ground is 0.8. The external force of magnitude 30 N is applied parallel to the ground. The resultant force exerted by the ground on the block is ($g = 10\ \text{m/s}^2$)

(a) 40 N (b) 30 N
(c) zero (d) 50 N

25. A block A of mass 2 kg rests on another block B of mass 8 kg which rests on a horizontal floor. The coefficient of friction between A and B is 0.2 while that between B and floor is 0.5. When a horizontal force F of 25 N is applied on the block B, the force of friction between A and B is

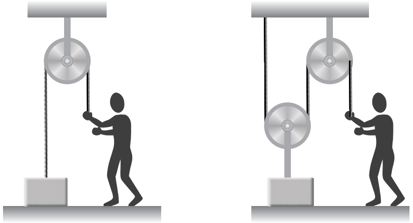
(a) 3 N (b) 4 N
(c) 2 N (d) zero

26. A body of mass 10 kg lies on a rough inclined plane of inclination $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ with the horizontal. When the force of 30 N is applied on the block parallel to and upward the plane, the total force by the plane on the block is nearly along



(a) OA (b) OB
(c) OC (d) OD

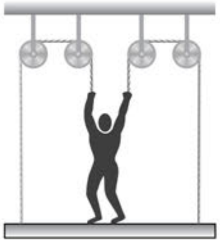
27. In the figure shown, a person wants to raise a block lying on the ground to a height h. In which case he has to exert more force. Assume pulleys and strings are light



(i) (ii)

(a) Fig. (i) (b) Fig. (ii)
(c) Same in both (d) Cannot be determined

28. A man of mass m stands on a platform of equal mass m and pulls himself by two ropes passing over pulleys as shown in figure. If he pulls each rope with a force equal to half his weight, his upward acceleration would be



(a) $\frac{g}{2}$

(b) $\frac{g}{4}$

(c) g

(d) zero

29. A varying horizontal force $F = at$ acts on a block of mass m kept on a smooth horizontal surface. An identical block is kept on the first block. The coefficient of friction between the blocks is μ. The time after which the relative sliding between the blocks prevails is

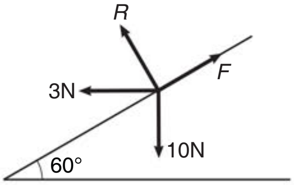
(a) $\frac{2mg}{a}$ (b) $\frac{2\mu mg}{a}$ (c) $\frac{\mu mg}{a}$ (d) $2\mu mga$

30. Two particles start together from a point O and slide down along straight smooth wires inclined at 30° and 60° to the vertical plane and on the same side of vertical through O. The relative acceleration of second with respect to first will be of magnitude

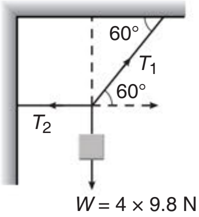
(a) $\frac{g}{2}$ (b) $\frac{\sqrt{3}g}{2}$ (c) $\frac{g}{\sqrt{3}}$ (d) g

Subjective Questions

1. Find the values of the unknown forces if the given set of forces shown in figure are in equilibrium.



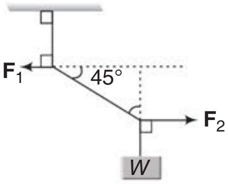
2. Determine the tensions T_1 and T_2 in the strings as shown in figure.



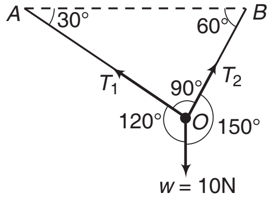
3. In figure the tension in the diagonal string is 60 N.

(a) Find the magnitude of the horizontal forces $\mathbf{F}_1$ and $\mathbf{F}_2$ that must be applied to hold the system in the position shown.

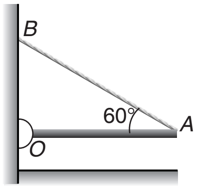
(b) What is the weight of the suspended block ?



4. A ball of mass 1 kg hangs in equilibrium from two strings *OA* and *OB* as shown in figure. What are the tensions in strings *OA* and *OB*? (Take $g = 10\ \text{m/s}^2$).

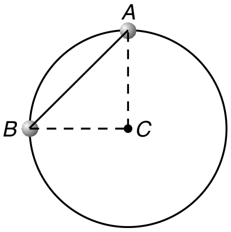


5. A rod *OA* of mass 4 kg is held in horizontal position by a massless string *AB* as shown in figure. Length of the rod is 2 m. Find

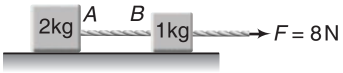


(a) tension in the string,
(b) net force exerted by hinge on the rod. ($g = 10\ \text{m/s}^2$)

6. Two beads of equal masses m are attached by a string of length $\sqrt{2}a$ and are free to move in a smooth circular ring lying in a vertical plane as shown in figure. Here, a is the radius of the ring. Find the tension and acceleration of *B* just after the beads are released to move.

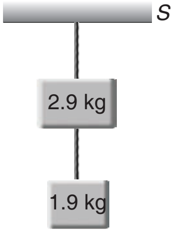


7. Two blocks of masses 1 kg and 2 kg are connected by a string *AB* of mass 1 kg. The blocks are placed on a smooth horizontal surface. Block of mass 1 kg is pulled by a horizontal force *F* of magnitude 8 N. Find the tension in the string at points *A* and *B*.



8. Two blocks of masses 2.9 kg and 1.9 kg are suspended from a rigid support *S* by two inextensible wires each of length 1 m, as shown in the figure. The upper wire has negligible mass and the lower wire has a uniformly distributed mass of 0.2 kg. The whole system of blocks, wires and support have an upward acceleration of $0.2\ \text{m/s}^2$. Acceleration due to gravity is $9.8\ \text{m/s}^2$.

(a) Find the tension at the mid-point of the lower wire.
(b) Find the tension at the mid-point of the upper wire.



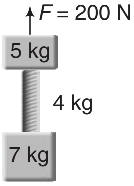
9. Two blocks shown in figure are connected by a heavy uniform rope of mass 4 kg. An upward force of 200 N is applied as shown.

(a) What is the acceleration of the system ?

(b) What is the tension at the top of the rope ?

(c) What is the tension at the mid-point of the rope ?

(Take $g = 9.8\ \text{m/s}^2$)

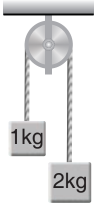


10. A 4 m long ladder weighing 25 kg rests with its upper end against a smooth wall and lower end on rough ground. What should be the minimum coefficient of friction between the ground and the ladder for it to be inclined at 60° with the horizontal without slipping? (Take $g = 10\ \text{m/s}^2$).

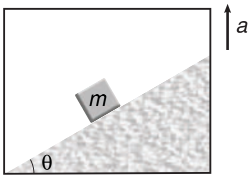
11. A plumb bob of mass 1 kg is hung from the ceiling of a train compartment. The train moves on an inclined plane with constant velocity. If the angle of incline is 30°. Find the angle made by the string with the normal to the ceiling. Also, find the tension in the string. ($g = 10\ \text{m/s}^2$)

12. Repeat both parts of the above question, if the train moves with an acceleration $a = g/2$ up the plane.

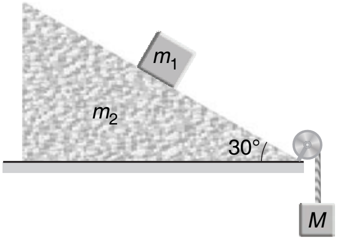
13. Two unequal masses of 1 kg and 2 kg are connected by a string going over a clamped light smooth pulley as shown in figure. The system is released from rest. The larger mass is stopped for a moment 1.0 s after the system is set in motion. Find the time elapsed before the string is tight again.



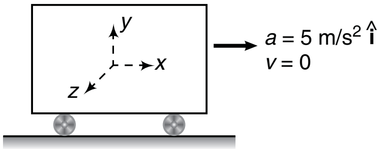
14. In the adjoining figure, a wedge is fixed to an elevator moving upwards with an acceleration a. A block of mass m is placed over the wedge. Find the acceleration of the block with respect to wedge. Neglect friction.



15. In figure $m_1 = 1$ kg and $m_2 = 4$ kg. Find the mass M of the hanging block which will prevent the smaller block from slipping over the triangular block. All the surfaces are frictionless and the strings and the pulleys are light.



Note *In exercises 16 to 18 the situations described take place in a box car which has initial velocity* $v = 0$ *but acceleration* $\mathbf{a} = (5\ m/s^2)\,\hat{\mathbf{i}}$. *(Take* $g = 10\ m/s^2$*)*

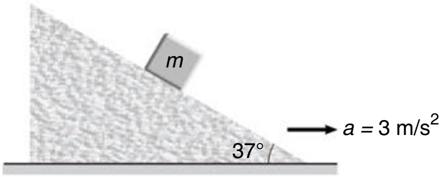


16. A 2 kg object is slid along the frictionless floor with initial velocity (10 m/s) $\hat{\mathbf{i}}$ (a) Describe the motion of the object relative to car (b) when does the object reach its original position relative to the box car.

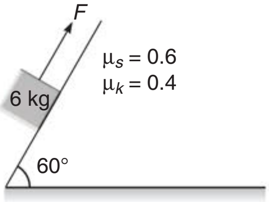
17. A 2 kg object is slid along the frictionless floor with initial transverse velocity (10 m/s) $\hat{\mathbf{k}}$. Describe the motion (a) in car's frame and (b) in ground frame.

18. A 2 kg object is slid along a rough floor (coefficient of sliding friction = 0.3) with initial velocity (10 m/s) $\hat{\mathbf{i}}$. Describe the motion of the object relative to car assuming that the coefficient of static friction is greater than 0.5.

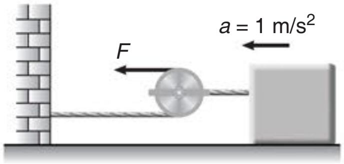
19. A block is placed on an inclined plane as shown in figure. What must be the frictional force between block and incline if the block is not to slide along the incline when the incline is accelerating to the right at 3 m/s^2 $\left(\sin 37° = \frac{3}{5}\right)$? (Take $g = 10$ m/s^2)



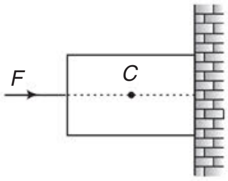
20. A 6 kg block is kept on an inclined rough surface as shown in figure. Find the force F required to
(a) keep the block stationary,
(b) move the block downwards with constant velocity and
(c) move the block upwards with an acceleration of 4 m/s^2.
(Take $g = 10$ m/s^2)



21. A block of mass 200 kg is set into motion on a frictionless horizontal surface with the help of frictionless pulley and a rope system as shown in figure. What horizontal force F should be applied to produce in the block an acceleration of 1 m/s^2 ?



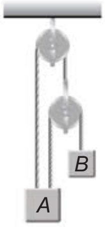
22. A cube of mass 2 kg is held stationary against a rough wall by a force $F = 40$ N passing through centre C. Find perpendicular distance of normal reaction between wall and cube from point C. Side of the cube is 20 cm. Take $g = 10$ m/s^2.



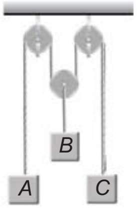
23. A 20 kg monkey has a firm hold on a light rope that passes over a frictionless pulley and is attached to a 20 kg bunch of bananas. The monkey looks upward, sees the bananas and starts to climb the rope to get them.

(a) As the monkey climbs, do the bananas move up, move down or remain at rest ?
(b) As the monkey climbs, does the distance between the monkey and the bananas decrease, increase or remain constant ?
(c) The monkey releases her hold on the rope. What happens to the distance between the monkey and the bananas while she is falling ?
(d) Before reaching the ground, the monkey grabs the rope to stop her fall. What do the bananas do ?

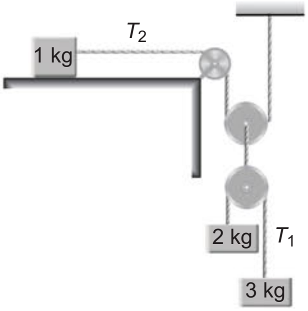
24. In the pulley-block arrangement shown in figure, find the relation between acceleration of blocks A and B.



25. In the pulley-block arrangement shown in figure, find relation between a_A, a_B and a_C.

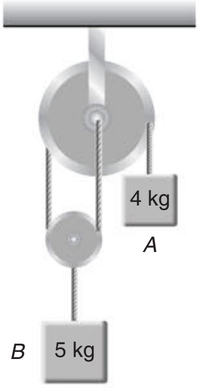


26. In the figure shown, find : ($g = 10\ \text{m/s}^2$)

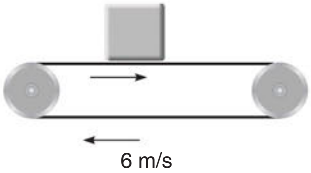


(a) acceleration of 1 kg, 2 kg and 3 kg blocks and
(b) tensions T_1 and T_2.

27. Find the acceleration of the blocks A and B in the situation shown in the figure.

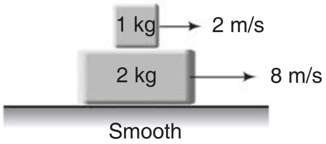


28. A conveyor belt is moving with constant speed of 6 m/s. A small block is just dropped on it. Coefficient of friction between the two is $\mu = 0.3$. Find



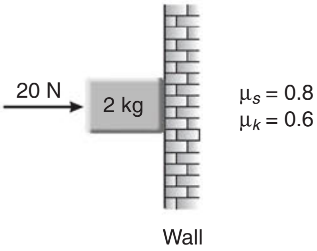
(a) The time when relative motion between them will stop.
(b) Displacement of block upto that instant. ($g = 10\,\text{m/s}^2$).

29. Coefficient of friction between two blocks shown in figure is $\mu = 0.4$. The blocks are given velocities of 2 m/s and 8 m/s in the directions shown in figure. Find

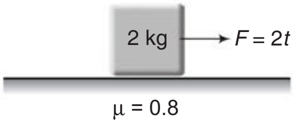


(a) the time when relative motion between them will stop.
(b) the common velocities of blocks upto that instant.
(c) displacements of 1 kg and 2 kg blocks upto that instant. ($g = 10\,\text{m/s}^2$)

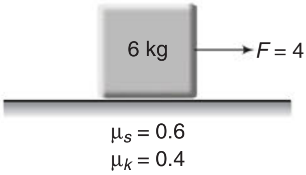
30. A 2 kg block is pressed against a rough wall by a force $F = 20$ N as shown in figure. Find acceleration of the block and force of friction acting on it. (Take $g = 10\,\text{m/s}^2$)



31. A 2 kg block is kept over a rough ground with coefficient of friction $\mu = 0.8$ as shown in figure. A time varying force $F = 2t$ (F in newton and t in second) is applied on the block. Plot a graph between acceleration of block *versus* time. ($g = 10\,\text{m/s}^2$)



32. A 6 kg block is kept over a rough surface with coefficients of friction $\mu_s = 0.6$ and $\mu_k = 0.4$ as shown in figure. A time varying force $F = 4t$ (F in newton and t in second) is applied on the block as shown. Plot a graph between acceleration of block and time. (Take $g = 10\,\text{m/s}^2$)

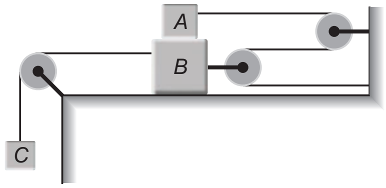


LEVEL 2

Objective Questions

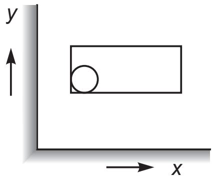
Single Correct Option

1. What is the largest mass of C in kg that can be suspended without moving blocks A and B? The static coefficient of friction for all plane surface of contact is 0.3. Mass of block A is 50kg and block B is 70kg. Neglect friction in the pulleys.



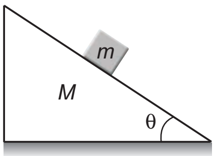
(a) 120 kg
(b) 92 kg
(c) 81 kg
(d) None of the above

2. A sphere of mass 1 kg rests at one corner of a cube. The cube is moved with a velocity $\mathbf{v} = (8t\hat{\mathbf{i}} - 2t^2)\hat{\mathbf{j}}$, where t is time in second. The force by sphere on the cube at $t = 1$ s is ($g = 10\ \text{ms}^{-2}$) [Figure shows vertical plane of the cube]



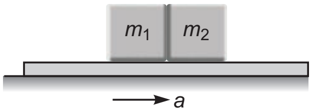
(a) 8 N
(b) 10 N
(c) 20 N
(d) 6 N

3. A smooth block of mass m is held stationary on a smooth wedge of mass M and inclination θ as shown in figure. If the system is released from rest, then the normal reaction between the block and the wedge is



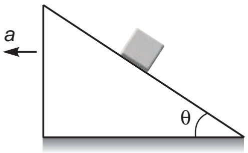
(a) $mg\cos\theta$
(b) less than $mg\cos\theta$
(c) greater than $mg\cos\theta$
(d) may be less or greater than $mg\cos\theta$ depending upon whether M is less or greater than m

4. Two blocks of masses m_1 and m_2 are placed in contact with each other on a horizontal platform as shown in figure. The coefficient of friction between m_1 and platform is 2μ and that between block m_2 and platform is μ. The platform moves with an acceleration a. The normal reaction between the blocks is

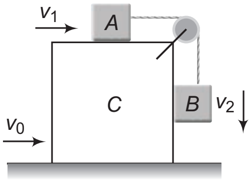


(a) zero in all cases
(b) zero only if $m_1 = m_2$
(c) non zero only if $a > 2\mu g$
(d) non zero only if $a > \mu g$

5. A block of mass m is resting on a wedge of angle θ as shown in the figure. With what minimum acceleration a should the wedge move so that the mass m falls freely?



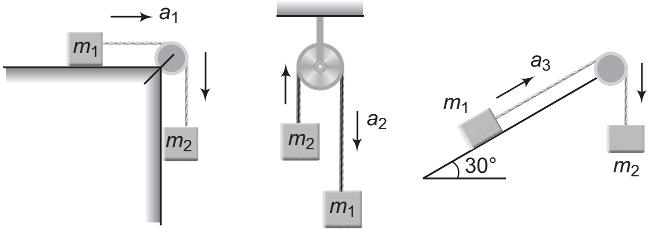
(a) g
(b) $g\cos\theta$
(c) $g\cot\theta$
(d) $g\tan\theta$



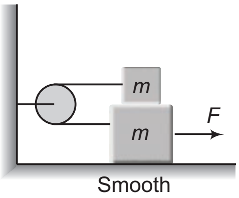
6. To a ground observer the block C is moving with v_0 and the block A with v_1 and B is moving with v_2 relative to C as shown in the figure. Identify the correct statement.

(a) $v_1 - v_2 = v_0$
(b) $v_1 = v_2$
(c) $v_1 + v_0 = v_2$
(d) None of the above

7. In each case $m_1 = 4\,\text{kg}$ and $m_2 = 3\,\text{kg}$. If a_1, a_2 and a_3 are the respective accelerations of the block m_1 in given situations, then

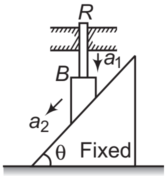


(a) $a_1 > a_2 > a_3$
(b) $a_1 > a_2 = a_3$
(c) $a_1 = a_2 = a_3$
(d) $a_1 > a_3 > a_2$



8. For the arrangement shown in figure the coefficient of friction between the two blocks is μ. If both the blocks are identical and moving, then the acceleration of each block is

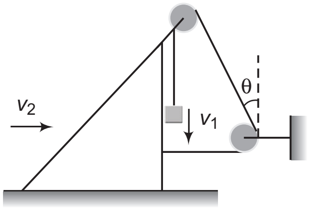
(a) $\dfrac{F}{2m} - 2\mu g$
(b) $\dfrac{F}{2m}$
(c) $\dfrac{F}{2m} - \mu g$
(d) zero



9. In the arrangement shown in the figure the rod R is restricted to move in the vertical direction with acceleration a_1, and the block B can slide down the fixed wedge with acceleration a_2. The correct relation between a_1 and a_2 is given by

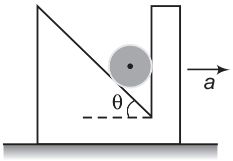
(a) $a_2 = a_1 \sin\theta$
(b) $a_2 \sin\theta = a_1$
(c) $a_2 \cos\theta = a_1$
(d) $a_2 = a_1 \cos\theta$

10. In the figure block moves downwards with velocity v_1, the wedge moves rightwards with velocity v_2. The correct relation between v_1 and v_2 is



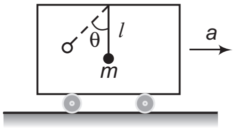
(a) $v_2 = v_1$
(b) $v_2 = v_1 \sin\theta$
(c) $2v_2 \sin\theta = v_1$
(d) $v_2 (1 + \sin\theta) = v_1$

11. In the figure, the minimum value of a at which the cylinder starts rising up the inclined surface is



(a) $g \tan\theta$ (b) $g \cot\theta$ (c) $g \sin\theta$ (d) $g \cos\theta$

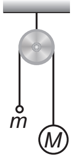
12. When the trolley shown in figure is given a horizontal acceleration a, the pendulum bob of mass m gets deflected to a maximum angle θ with the vertical. At the position of maximum deflection, the net acceleration of the bob with respect to trolley is



(a) $\sqrt{g^2 + a^2}$ (b) $a \cos\theta$

(c) $g \sin\theta - a \cos\theta$ (d) $a \sin\theta$

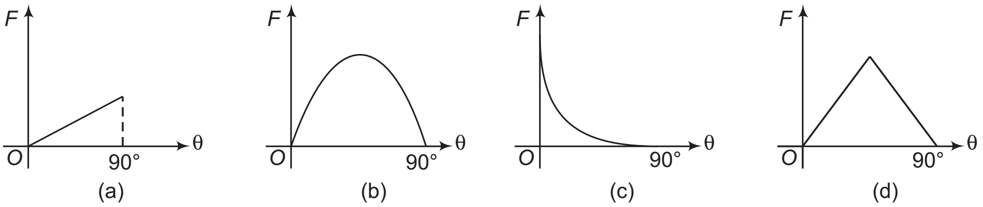
13. In the arrangement shown in figure the mass M is very heavy compared to m ($M >> m$). The tension T in the string suspended from the ceiling is



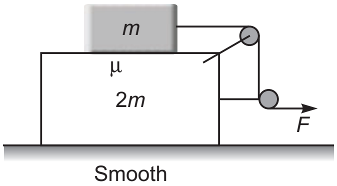
(a) 4 mg (b) 2 mg

(c) zero (d) None of these

14. A block rests on a rough plane whose inclination θ to the horizontal can be varied. Which of the following graphs indicates how the frictional force F between the block and the plane varies as θ is increased ?

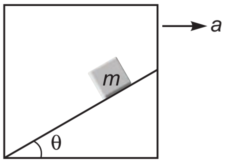


15. The minimum value of μ between the two blocks for no slipping is



(a) $\dfrac{F}{mg}$ (b) $\dfrac{F}{3mg}$ (c) $\dfrac{2F}{3mg}$ (d) $\dfrac{4F}{3mg}$

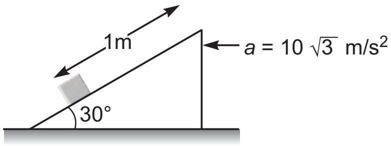
16. A block is sliding along an inclined plane as shown in figure. If the acceleration of chamber is a as shown in the figure. The time required to cover a distance L along incline is



(a) $\sqrt{\dfrac{2L}{g \sin\theta - a \cos\theta}}$ (b) $\sqrt{\dfrac{2L}{g \sin\theta + a \sin\theta}}$

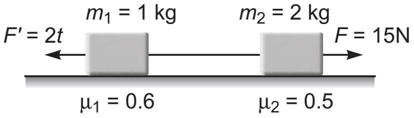
(c) $\sqrt{\dfrac{2L}{g \sin\theta + a \cos\theta}}$ (d) $\sqrt{\dfrac{2L}{g \sin\theta}}$

17. In the figure, the wedge is pushed with an acceleration of $10\sqrt{3}\ \text{m/s}^2$. It is seen that the block starts climbing up on the smooth inclined face of wedge. What will be the time taken by the block to reach the top?



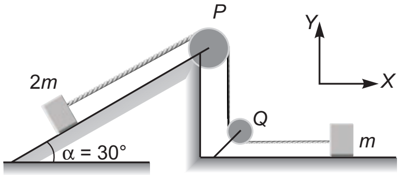
(a) $\dfrac{2}{\sqrt{5}}$ s
(b) $\dfrac{1}{\sqrt{5}}$ s
(c) $\sqrt{5}$ s
(d) $\dfrac{\sqrt{5}}{2}$ s

18. Two blocks A and B are separated by some distance and tied by a string as shown in the figure. The force of friction in both the blocks at $t = 2$ s is



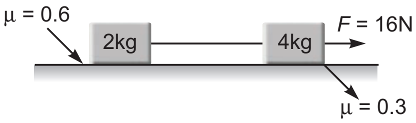
(a) 4 N (→), 5 N(←)
(b) 2 N(→), 5 N(←)
(c) 0 N(→), 10 N(←)
(d) 1 N(←), 10 N(←)

19. All the surfaces and pulleys are frictionless in the shown arrangement. Pulleys P and Q are massless. The force applied by clamp on pulley P is



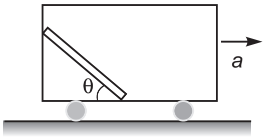
(a) $\dfrac{mg}{6}(-\sqrt{3}\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}})$
(b) $\dfrac{mg}{6}(\sqrt{3}\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}})$
(c) $\dfrac{mg}{6}\sqrt{2}$
(d) None of these

20. Two blocks of masses 2 kg and 4 kg are connected by a light string and kept on horizontal surface. A force of 16 N is acted on 4 kg block horizontally as shown in figure. Besides, it is given that coefficient of friction between 4 kg and ground is 0.3 and between 2 kg block and ground is 0.6. Then frictional force between 2 kg block and ground is

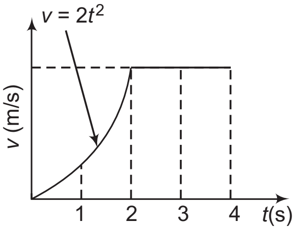


(a) 12 N
(b) 6 N
(c) 4 N
(d) zero

21. A smooth rod of length l is kept inside a trolley at an angle θ as shown in the figure. What should be the acceleration a of the trolley so that the rod remains in equilibrium with respect to it?



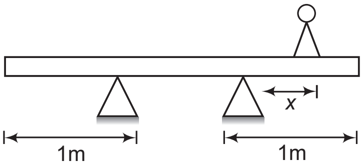
(a) $g\tan\theta$
(b) $g\cos\theta$
(c) $g\sin\theta$
(d) $g\cot\theta$



22. A car begins from rest at time $t = 0$, and then accelerates along a straight track during the interval $0 < t \leq 2$ s and thereafter with constant velocity as shown in the graph. A coin is initially at rest on the floor of the car. At $t = 1$ s, the coin begins to slip and its stops slipping at $t = 3$ s. The coefficient of static friction between the floor and the coin is ($g = 10$ m/s^2)

(a) 0.2
(b) 0.3
(c) 0.4
(d) 0.5

23. A horizontal plank is 10.0 m long with uniform density and mass 10 kg. It rests on two supports which are placed 1.0 m from each end as shown in the figure. A man of mass 80 kg can stand upto distance x on the plank without causing it to tip. The value of x is



(a) $\frac{1}{2}$ m
(b) $\frac{1}{4}$ m
(c) $\frac{3}{4}$ m
(d) $\frac{1}{8}$ m

24. A block is kept on a smooth inclined plane of angle of inclination θ that moves with a constant acceleration so that the block does not slide relative to the inclined plane. If the inclined plane stops, the normal contact force offered by the plane on the block changes by a factor

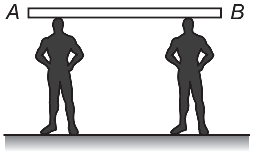
(a) $\tan\theta$
(b) $\tan^2\theta$
(c) $\cos^2\theta$
(d) $\cot\theta$

25. A uniform cube of mass m and side a is resting in equilibrium on a rough 45° inclined surface. The distance of the point of application of normal reaction measured from the lower edge of the cube is

(a) zero
(b) $\frac{a}{3}$
(c) $\frac{a}{\sqrt{2}}$
(d) $\frac{a}{4}$

26. A horizontal force $F = \frac{mg}{3}$ is applied on the upper surface of a uniform cube of mass m and side a which is resting on a rough horizontal surface having $\mu = \frac{1}{2}$. The distance between lines of action of mg and normal reaction is

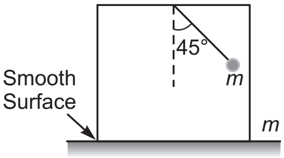
(a) $\frac{a}{2}$
(b) $\frac{a}{3}$
(c) $\frac{a}{4}$
(d) None of these



27. Two persons of equal heights are carrying a long uniform wooden plank of length l. They are at distance $\frac{l}{4}$ and $\frac{l}{6}$ from nearest end of the rod. The ratio of normal reaction at their heads is

(a) 2 : 3
(b) 1 : 3
(c) 4 : 3
(d) 1 : 2

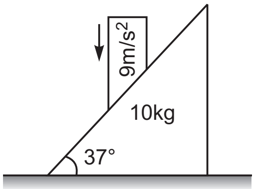
28. A ball connected with string is released at an angle 45° with the vertical as shown in the figure. Then the acceleration of the box at this instant will be (mass of the box is equal to mass of ball)



(a) $\frac{g}{4}$ (b) $\frac{g}{3}$

(c) $\frac{g}{2}$ (d) g

29. In the system shown in figure all surfaces are smooth. Rod is moved by external agent with acceleration 9 ms^{-2} vertically downwards. Force exerted on the rod by the wedge will be



(a) 120 N
(b) 200 N
(c) 160 N
(d) 180 N

30. A thin rod of length 1 m is fixed in a vertical position inside a train, which is moving horizontally with constant acceleration 4 ms^{-2}. A bead can slide on the rod and friction coefficient between them is 0.5. If the bead is released from rest at the top of the rod, it will reach the bottom in

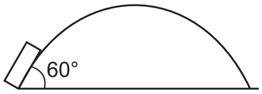
(a) $\sqrt{2}$ s (b) 1 s (c) 2 s (d) 0.5 s

31. Mr. X of mass 80 kg enters a lift and selects the floor he wants. The lift now accelerates upwards at 2 ms^{-2} for 2 s and then moves with constant velocity. As the lift approaches his floor, it decelerates at the same rate as it previously accelerates. If the lift cables can safely withstand a tension of 2×10^4 N and the lift itself has a mass of 500 kg, how many Mr. X's could it safely carry at one time?

(a) 22 (b) 14

(c) 18 (d) 12

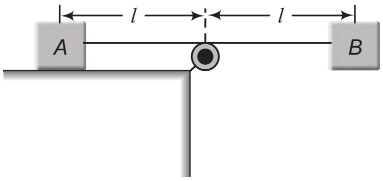
32. A particle when projected in vertical plane moves along smooth surface with initial velocity 20 ms^{-1} at an angle of 60°, so that its normal reaction on the surface remains zero throughout the motion. Then the slope of the surface at height 5m from the point of projection will be



(a) $\sqrt{3}$ (b) 1

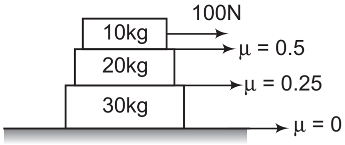
(c) 2 (d) None of these

33. Two blocks A and B, each of same mass are attached by a thin inextensible string through an ideal pulley. Initially block B is held in position as shown in figure. Now, the block B is released. Block A will slide to right and hit the pulley in time t_A. Block B will swing and hit the surface in time t_B. Assume the surface as frictionless, then



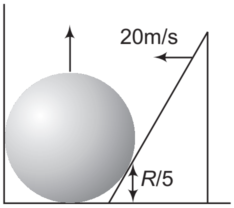
(a) $t_A > t_B$ (b) $t_A < t_B$ (c) $t_A = t_B$ (d) data insufficient

34. Three blocks are kept as shown in figure. Acceleration of 20 kg block with respect to ground is



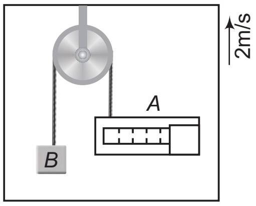
(a) 5 ms^{-2} (b) 2 ms^{-2} (c) 1 ms^{-2} (d) None of these

35. A sphere of radius R is in contact with a wedge. The point of contact is $\frac{R}{5}$ from the ground as shown in the figure. Wedge is moving with velocity 20 ms^{-1} towards left then the velocity of the sphere at this instant will be



(a) 20 ms^{-1} (b) 15 ms^{-1} (c) 16 ms^{-1} (d) 12 ms^{-1}

36. In the figure it is shown that the velocity of lift is 2 ms^{-1} while string is winding on the motor shaft with velocity 2 ms^{-1} and shaft A is moving downward with velocity 2 ms^{-1} with respect to lift, then find out the velocity of block B

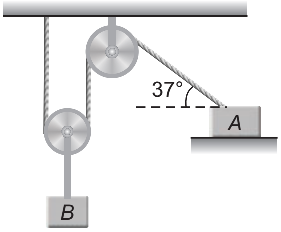


(a) 2 ms^{-1}↑ (b) 2 ms^{-1}↓ (c) 4 ms^{-1}↑ (d) None of these

37. A monkey pulls the midpoint of a 10 cm long light inextensible string connecting two identical objects A and B lying on smooth table of masses 0.3 kg continuously along the perpendicular bisector of line joining the masses. The masses are found to approach each other at a relative acceleration of 5 ms^{-2} when they are 6 cm apart. The constant force applied by monkey is

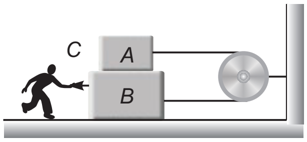
(a) 4 N (b) 2 N (c) 3 N (d) None of these

38. In the figure shown the block B moves with velocity 10 ms^{-1}. The velocity of A in the position shown is



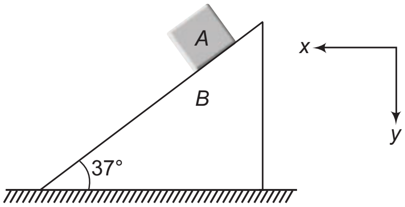
(a) 12.5 ms^{-1} (b) 25 ms^{-1} (c) 8 ms^{-1} (d) 16 ms^{-1}

39. In the figure $m_A = m_B = m_c = 60$ kg. The coefficient of friction between C and ground is 0.5, B and ground is 0.3, A and B is 0.4. C is pulling the string with the maximum possible force without moving. Then the tension in the string connected to A will be



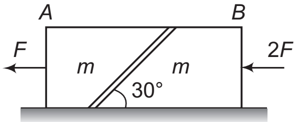
(a) 120 N
(b) 60 N
(c) 100 N
(d) zero

40. In the figure shown the acceleration of A is $\mathbf{a}_A = (15\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$. Then the acceleration of B is (A remains in contact with B)



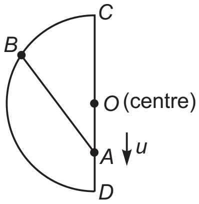
(a) $5\,\hat{\mathbf{i}}$
(b) $-15\,\hat{\mathbf{i}}$
(c) $-10\,\hat{\mathbf{i}}$
(d) $-5\,\hat{\mathbf{i}}$

41. Two blocks A and B each of mass m are placed on a smooth horizontal surface. Two horizontal forces F and $2F$ are applied on the blocks A and B respectively as shown in figure. The block A does not slide on block B. Then the normal reaction acting between the two blocks is



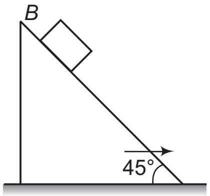
(a) F
(b) $\dfrac{F}{2}$
(c) $\dfrac{F}{\sqrt{3}}$
(d) $3F$

42. Two beads A and B move along a semicircular wire frame as shown in figure. The beads are connected by an inelastic string which always remains tight. At an instant the speed of A is u, $\angle BAC = 45°$ and $BOC = 75°$, where O is the centre of the semicircular arc. The speed of bead B at that instant is

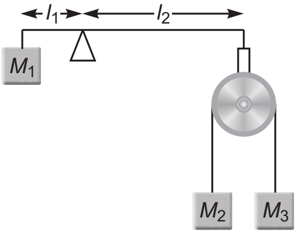


(a) $\sqrt{2}u$
(b) u
(c) $\dfrac{u}{2\sqrt{2}}$
(d) $\sqrt{\dfrac{2}{3}}u$

43. If the coefficient of friction between A and B is μ, the maximum acceleration of the wedge A for which B will remain at rest with respect to the wedge is

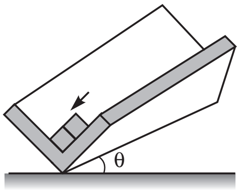


(a) μg
(b) $g\left(\dfrac{1+\mu}{1-\mu}\right)$
(c) $g\left(\dfrac{1-\mu}{1+\mu}\right)$
(d) $\dfrac{g}{\mu}$



44. A pivoted beam of negligible mass has a mass suspended from one end and an Atwood's machine suspended from the other. The frictionless pulley has negligible mass and dimension. Gravity is directed downward and $M_2 = 3M_3,\ l_2 = 3l_1$. Find the ratio M_1/M_2 which will ensure that the beam has no tendency to rotate just after the masses are released.

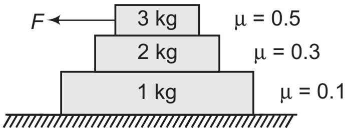
(a) $\frac{M_1}{M_2} = 2$
(b) $\frac{M_1}{M_2} = 3$
(c) $\frac{M_1}{M_2} = 4$
(d) None of these



45. A block of mass m slides down an inclined right angled trough. If the coefficient of friction between block and the trough is μ_k, acceleration of the block down the plane is

(a) $g(\sin\theta + \sqrt{2}\mu_k \cos\theta)$
(b) $g(\sin\theta + \mu_k \cos\theta)$
(c) $g(\sin\theta - \sqrt{2}\mu_k \cos\theta)$
(d) $g(\sin\theta - \mu_k \cos\theta)$

46. If force F is increasing with time and at $t = 0$, $F = 0$, where will slipping first start?



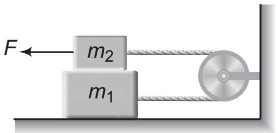
(a) between 3 kg and 2 kg
(b) between 2 kg and 1 kg
(c) between 1 kg and ground
(d) Both (a) and (b)

47. A plank of mass 2 kg and length 1 m is placed on horizontal floor. A small block of mass 1 kg is placed on top of the plank, at its right extreme end. The coefficient of friction between plank and floor is 0.5 and that between plank and block is 0.2. If a horizontal force = 30 N starts acting on the plank to the right, the time after which the block will fall off the plank is ($g = 10\ \text{ms}^{-2}$)

(a) $\left(\frac{2}{3}\right)$ s
(b) 1.5 s
(c) 0.75 s
(d) $\left(\frac{4}{3}\right)$ s

More than One Correct Options

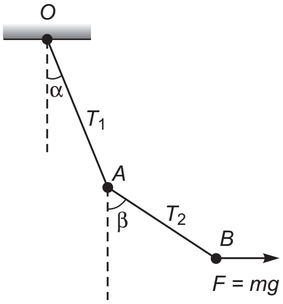
1. Two blocks each of mass 1 kg are placed as shown. They are connected by a string which passes over a smooth (massless) pulley.



There is no friction between m_1 and the ground. The coefficient of friction between m_1 and m_2 is 0.2. A force F is applied to m_2. Which of the following statements is/are correct?

(a) The system will be in equilibrium if $F \le 4$ N
(b) If $F > 4$ N tension in the string will be 4 N
(c) If $F > 4$ N the frictional force between the blocks will be 2 N
(d) If $F = 6$ N tension in the string will be 3 N

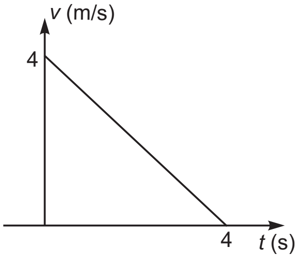
2. Two particles A and B, each of mass m are kept stationary by applying a horizontal force $F = mg$ on particle B as shown in figure. Then



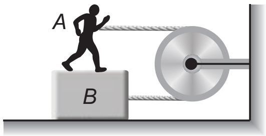
(a) $\tan\beta = 2\tan\alpha$
(b) $2T_1 = 5T_2$
(c) $\sqrt{2}\,T_1 = \sqrt{5}T_2$
(d) $\alpha = \beta$

3. The velocity-time graph of the figure shows the motion of a wooden block of mass 1 kg which is given an initial push at $t = 0$ along a horizontal table.

(a) The coefficient of friction between the block and the table is 0.1
(b) The coefficient of friction between the block and the table is 0.2
(c) If the table was half of its present roughness, the time taken by the block to complete the journey is 4 s
(d) If the table was half of its present roughness, the time taken by the block to complete the journey is 8 s

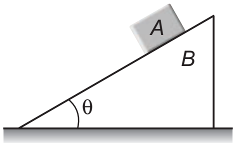


4. As shown in the figure, A is a man of mass 60kg standing on a block B of mass 40 kg kept on ground. The coefficient of friction between the feet of the man and the block is 0.3 and that between B and the ground is 0.2. If the person pulls the string with 125 N force, then



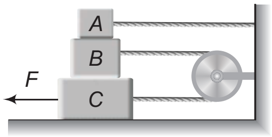
(a) B will slide on ground
(b) A and B will move with acceleration 0.5 ms^{-2}
(c) the force of friction acting between A and B will be 125 N
(d) the force of friction acting between B and ground will be 250 N

5. In the figure shown A and B are free to move. All the surfaces are smooth. Mass of A is m. Then



(a) the acceleration of A will be more than $g\sin\theta$
(b) the acceleration of A will be less than $g\sin\theta$
(c) normal reaction on A due to B will be more than $mg\cos\theta$
(d) normal reaction on A due to B will be less than $mg\cos\theta$

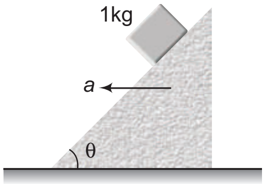
6. $M_A = 3$ kg, $M_B = 4$ kg, and $M_C = 8$ kg. Coefficient of friction between any two surfaces is 0.25. Pulley is frictionless and string is massless. A is connected to wall through a massless rigid rod.



(a) value of F to keep C moving with constant speed is 80 N
(b) value of F to keep C moving with constant speed is 120 N
(c) if F is 200 N then acceleration of B is 10 ms^{-2}
(d) to slide C towards left, F should be at least 50 N (Take $g = 10 \text{ ms}^{-2}$)

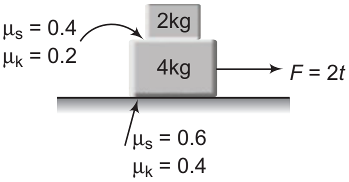
7. A man pulls a block of mass equal to himself with a light string. The coefficient of friction between the man and the floor is greater than that between the block and the floor
(a) if the block does not move, then the man also does not move
(b) the block can move even when the man is stationary
(c) if both move then the acceleration of the block is greater than the acceleration of man
(d) if both move then the acceleration of man is greater than the acceleration of block

8. A block of mass 1 kg is at rest relative to a smooth wedge moving leftwards with constant acceleration $a = 5 \text{ ms}^{-2}$. Let N be the normal reaction between the block and the wedge. Then ($g = 10 \text{ ms}^{-2}$)



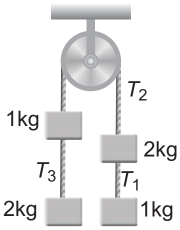
(a) $N = 5\sqrt{5}$ newton
(b) $N = 15$ newton
(c) $\tan\theta = \frac{1}{2}$
(d) $\tan\theta = 2$

9. For the given situation shown in figure, choose the correct options ($g = 10 \text{ ms}^{-2}$)



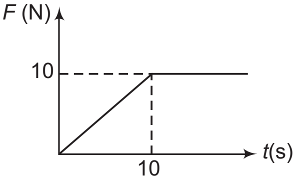
(a) At $t = 1$ s, force of friction between 2 kg and 4 kg is 2 N
(b) At $t = 1$ s, force of friction between 2 kg and 4 kg is zero
(c) At $t = 4$ s, force of friction between 4 kg and ground is 8 N
(d) At $t = 15$ s, acceleration of 2kg is 1 ms^{-2}

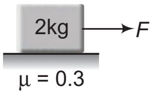
10. In the figure shown, all the strings are massless and friction is absent everywhere. Choose the correct options.



(a) $T_1 > T_3$
(b) $T_3 > T_1$
(c) $T_2 > T_1$
(d) $T_2 > T_3$

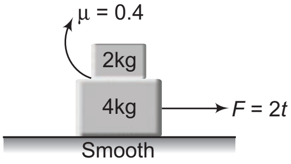
11. Force acting on a block *versus* time graph is as shown in figure. Choose the correct options. ($g = 10 \text{ ms}^{-2}$)





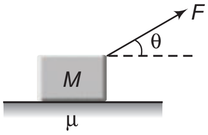
(a) At $t = 2$ s, force of friction is 2 N
(b) At $t = 8$ s, force of friction is 6 N
(c) At $t = 10$ s, acceleration of block is 2 ms^{-2}
(d) At $t = 12$ s, velocity of block is 8 ms^{-1}

12. For the situation shown in figure, mark the correct options.



(a) At $t = 3$ s, pseudo force on 4 kg block applied from 2 kg is 4 N in forward direction
(b) At $t = 3$ s, pseudo force on 2 kg block applied from 4 kg is 2 N in backward direction
(c) Pseudo force does not make an equal and opposite pairs
(d) Pseudo force also makes a pair of equal and opposite forces

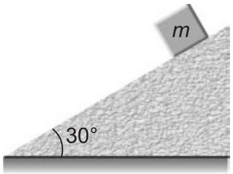
13. For the situation shown in figure, mark the correct options.



(a) Angle of friction is $\tan^{-1}(\mu)$
(b) Angle of repose is $\tan^{-1}(\mu)$
(c) At $\theta = \tan^{-1}(\mu)$, minimum force will be required to move the block
(d) Minimum force required to move the block is $\dfrac{\mu Mg}{\sqrt{1+\mu^2}}$.

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 5)



A man wants to slide down a block of mass m which is kept on a fixed inclined plane of inclination 30° as shown in the figure. Initially the block is not sliding.

To just start sliding the man pushes the block down the incline with a force F. Now, the block starts accelerating. To move it downwards with constant speed the man starts pulling the block with same force. Surfaces are such that ratio of maximum static friction to kinetic friction is 2. Now, answer the following questions.

1. What is the value of F?
(a) $\dfrac{mg}{4}$ (b) $\dfrac{mg}{6}$ (c) $\dfrac{mg\sqrt{3}}{4}$ (d) $\dfrac{mg}{2\sqrt{3}}$

2. What is the value of μ_s, the coefficient of static friction?
(a) $\dfrac{4}{3\sqrt{3}}$ (b) $\dfrac{2}{3\sqrt{3}}$ (c) $\dfrac{3}{3\sqrt{3}}$ (d) $\dfrac{1}{2\sqrt{3}}$

3. If the man continues pushing the block by force F, its acceleration would be
(a) $\dfrac{g}{6}$ (b) $\dfrac{g}{4}$ (c) $\dfrac{g}{2}$ (d) $\dfrac{g}{3}$

4. If the man wants to move the block up the incline, what minimum force is required to start the motion?
(a) $\dfrac{2}{3}mg$ (b) $\dfrac{mg}{2}$ (c) $\dfrac{7mg}{6}$ (d) $\dfrac{5mg}{6}$

5. What minimum force is required to move it up the incline with constant speed?
(a) $\dfrac{2}{3}mg$ (b) $\dfrac{mg}{2}$ (c) $\dfrac{7mg}{6}$ (d) $\dfrac{5mg}{6}$

Passage 2 (Q. Nos. 6 to 7)

A lift with a mass 1200 *kg is raised from rest by a cable with a tension* 1350 *kg-f. After some time the tension drops to* 1000 *kg-f and the lift comes to rest at a height of* 25 *m above its initial point.* (1 *kg-f* = 9.8 *N)*

6. What is the height at which the tension changes?

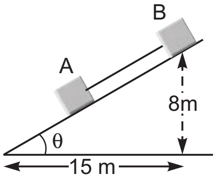
(a) 10.8 m (b) 12.5 m
(c) 14.3 m (d) 16 m

7. What is the greatest speed of lift?

(a) $9.8\ \text{ms}^{-1}$ (b) $7.5\ \text{ms}^{-1}$
(c) $5.92\ \text{ms}^{-1}$ (d) None of these

Passage 3 (Q. Nos. 8 to 9)

Blocks A and B shown in the figure are connected with a bar of negligible weight. A and B each has mass 170 *kg, the coefficient of friction between A and the plane is* 0.2 *and that between B and the plane is* 0.4 ($g = 10\ ms^{-2}$)



8. What is the total force of friction between the blocks and the plane?

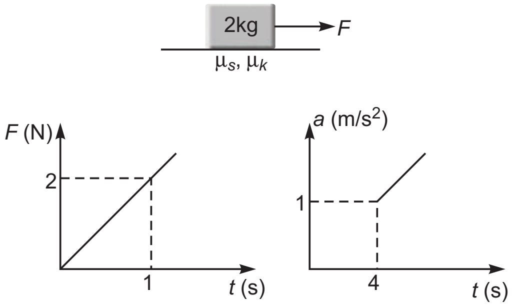
(a) 900 N (b) 700 N
(c) 600 N (d) 300 N

9. What is the force acting on the connecting bar?

(a) 140 N (b) 100 N
(c) 75 N (d) 125 N

Match the Columns

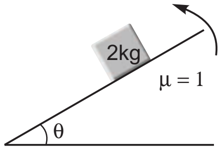
1.



Force acting on a block *versus* time and acceleration *versus* time graph are as shown in figure. Taking value of $g = 10\ \text{ms}^{-2}$, match the following two columns.

Column I	Column II
(a) Coefficient of static friction	(p) 0.2
(b) Coefficient of kinetic friction	(q) 0.3
(c) Force of friction (in N) at $t = 0.1$ s	(r) 0.4
(d) Value of $\frac{a}{10}$, where a is acceleration of block (in m/s^2) at $t = 8$ s	(s) 0.5

2. Angle θ is gradually increased as shown in figure. For the given situation match the following two columns. ($g = 10\ \text{ms}^{-2}$)

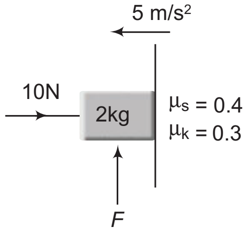


Column I	Column II
(a) Force of friction when $\theta = 0°$	(p) 10 N
(b) Force of friction when $\theta = 90°$	(q) $10\sqrt{3}$ N
(c) Force of friction when $\theta = 30°$	(r) $\frac{10}{\sqrt{3}}$ N
(d) Force of friction when $\theta = 60°$	(s) None of the above

3. Match the following two columns regarding fundamental forces of nature.

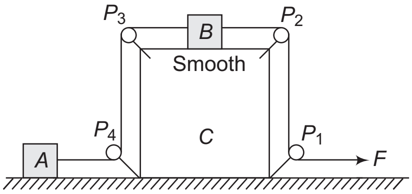
Column I	Column II
(a) Force of friction	(p) field force
(b) Normal reaction	(q) contact force
(c) Force between two neutrons	(r) electromagnetic force
(d) Force between two protons	(s) nuclear force

4. In the figure shown, match the following two columns. ($g = 10\ \text{ms}^{-2}$)



Column I	Column II
(a) Normal reaction	(p) 5 N
(b) Force of friction when $F = 15$ N	(q) 10 N
(c) Minimum value of F for stopping the block moving down	(r) 15 N
(d) Minimum value of F for stopping the block moving up	(s) None of the above

5. There is no friction between blocks B and C. But ground is rough. Pulleys are smooth and massless and strings are light. For $F = 10$ N, whole system remains stationary. Match the following two columns.($m_B = m_C = 1$ kg and $g = 10\ \text{ms}^{-2}$)



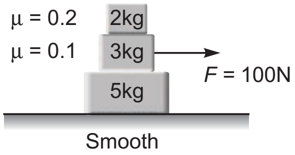
Column I	Column II
(a) Force of friction between A and ground	(p) 10 N
(b) Force of friction between C and ground	(q) 20 N
(c) Normal reaction on C from ground	(r) 5 N
(d) Tension in string between P_3 and P_4	(s) None of the above

6. Match Column I with Column II.

Note *Applied force is parallel to plane.*

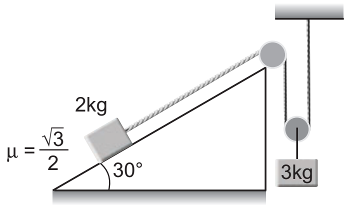
Column I	Column II
(a) If friction force is less than applied force then friction may be	(p) Static
(b) If friction force is equal to the force applied, then friction may be	(q) Kinetic
(c) If a block is moving on ground, then friction is	(r) Limiting
(d) If a block kept on ground is at rest, then friction may be	(s) No conclusion can be drawn

7. For the situation shown in figure, in Column I, the statements regarding friction forces are mentioned, while in Column II some information related to friction forces are given. Match the entries of Column I with the entries of Column II (Take $g = 10\ \text{ms}^{-2}$)



Column I	Column II
(a) Total friction force on 3 kg block is	(p) Towards right
(b) Total friction force on 5 kg block is	(q) Towards left
(c) Friction force on 2 kg block due to 3 kg block is	(r) Zero
(d) Friction force on 3 kg block due to 5 kg block is	(s) Non-zero

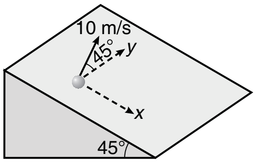
8. If the system is released from rest, then match the following two columns.



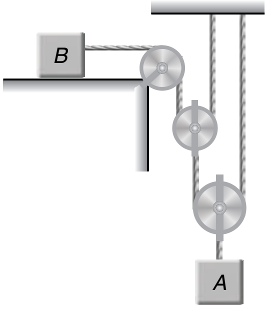
Column I	Column II
(a) Acceleration of 2 kg mass	(p) 2 SI unit
(b) Acceleration of 3 kg mass	(q) 5 SI unit
(c) Tension in the string connecting 2 kg mass	(r) Zero
(d) Frictional force on 2 kg mass	(s) None of these

Subjective Questions

1. A small marble is projected with a velocity of 10 m/s in a direction 45° from the y-direction on the smooth inclined plane. Calculate the magnitude v of its velocity after 2s. (Take $g = 10\,\text{m/s}^2$)



2. Determine the acceleration of the 5 kg block A. Neglect the mass of the pulley and cords. The block B has a mass of 10 kg. The coefficient of kinetic friction between block B and the surface is $\mu_k = 0.1$. (Take $g = 10\,\text{m/s}^2$)

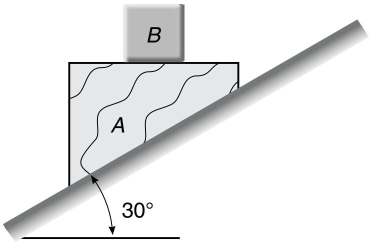


3. A 30 kg mass is initially at rest on the floor of a truck. The coefficient of static friction between the mass and the floor of truck in 0.3 and coefficient of kinetic friction is 0.2. Initially the truck is travelling due east at constant speed. Find the magnitude and direction of the friction force acting on the mass, if : (Take $g = 10\,\text{m/s}^2$)

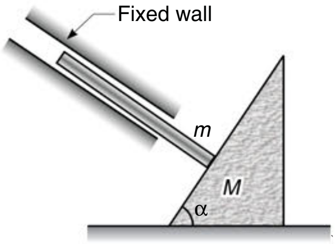
(a) The truck accelerates at $1.8\,\text{m/s}^2$ eastward

(b) The truck accelerates at $3.8\,\text{m/s}^2$ westward.

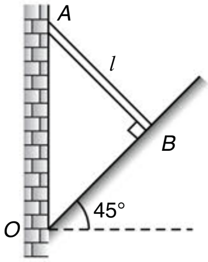
4. A 6 kg block B rests as shown on the upper surface of a 15 kg wedge A. Neglecting friction, determine immediately after the system is released from rest (a) the acceleration of A (b) the acceleration of B relative to A. (Take $g = 10\,\text{m/s}^2$)



5. In the arrangement shown in the figure, the rod of mass m held by two smooth walls, remains always perpendicular to the surface of the wedge of mass M. Assuming all the surfaces are frictionless, find the acceleration of the rod and that of the wedge.



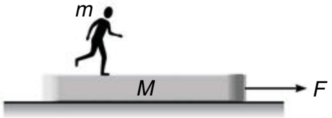
6. At the bottom edge of a smooth vertical wall, an inclined plane is kept at an angle of 45°. A uniform ladder of length l and mass M rests on the inclined plane against the wall such that it is perpendicular to the incline.



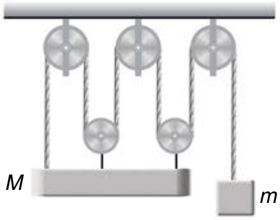
(a) If the plane is also smooth, which way will the ladder slide.

(b) What is the minimum coefficient of friction necessary so that the ladder does not slip on the incline?

7. A plank of mass M is placed on a rough horizontal surface and a constant horizontal force F is applied on it. A man of mass m runs on the plank. Find the range of acceleration of the man so that the plank does not move on the surface. Coefficient of friction between the plank and the surface is μ. Assume that the man does not slip on the plank.



8. Find the acceleration of two masses as shown in figure. The pulleys are light and frictionless and strings are light and inextensible.

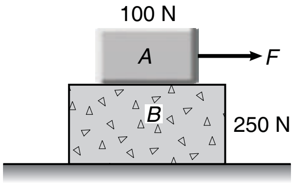


9. The upper portion of an inclined plane of inclination α is smooth and the lower portion is rough. A particle slides down from rest from the top and just comes to rest at the foot. If the ratio of smooth length to rough length is $m : n$, find the coefficient of friction.

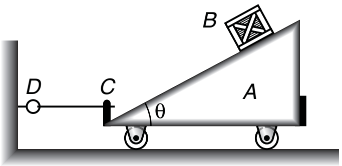
10. Block B rests on a smooth surface. If the coefficient of static friction between A and B is $\mu = 0.4$. Determine the acceleration of each, if

(a) $F = 30$ N and

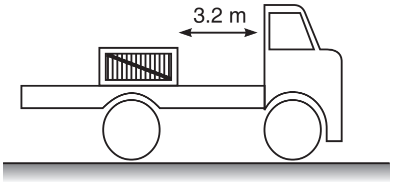
(b) $F = 250$ N ($g = 10\,\text{m/s}^2$)



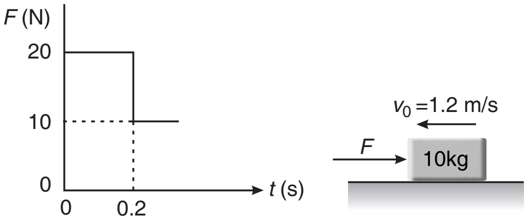
11. Block B has a mass m and is released from rest when it is on top of wedge A, which has a mass $3m$. Determine the tension in cord CD while B is sliding down A. Neglect friction.



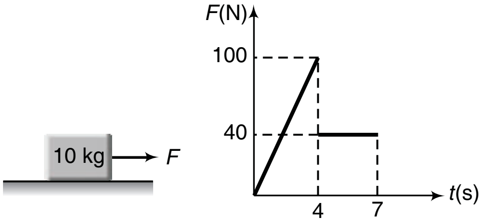
12. Coefficients of friction between the flat bed of the truck and crate are $\mu_s = 0.8$ and $\mu_k = 0.7$. The coefficient of kinetic friction between the truck tires and the road surface is 0.9. If the truck stops from an initial speed of 15 m/s with maximum braking (wheels skidding). Determine where on the bed the crate finally comes to rest. (Take $g = 10\,\text{m/s}^2$)



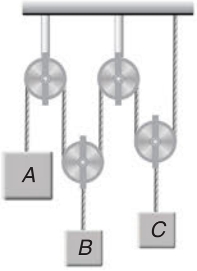
13. The 10 kg block is moving to the left with a speed of 1.2 m/s at time $t = 0$. A force F is applied as shown in the graph. After 0.2 s, the force continues at the 10 N level. If the coefficient of kinetic friction is $\mu_k = 0.2$. Determine the time t at which the block comes to a stop. ($g = 10\,\text{m/s}^2$)



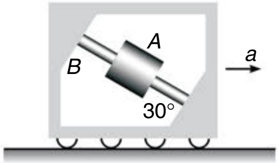
14. The 10 kg block is resting on the horizontal surface when the force F is applied to it for 7 s. The variation of F with time is shown. Calculate the maximum velocity reached by the block and the total time t during which the block is in motion. The coefficients of static and kinetic friction are both 0.50. ($g = 9.8\,\text{m/s}^2$)



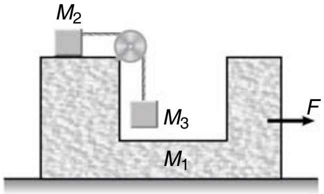
15. If block A of the pulley system is moving downward with a speed of 1 m/s while block C is moving up at 0.5 m/s, determine the speed of block B.



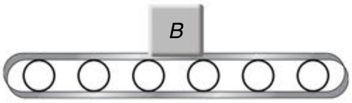
16. The collar A is free to slide along the smooth shaft B mounted in the frame. The plane of the frame is vertical. Determine the horizontal acceleration a of the frame necessary to maintain the collar in a fixed position on the shaft. ($g = 9.8 \text{ m/s}^2$)



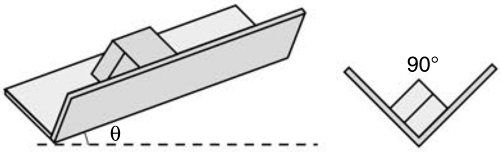
17. In the adjoining figure all surfaces are frictionless. What force F must by applied to M_1 to keep M_3 free from rising or falling?



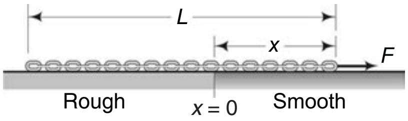
18. The conveyor belt is designed to transport packages of various weights. Each 10 kg package has a coefficient of kinetic friction $\mu_k = 0.15$. If the speed of the conveyor belt is 5 m/s, and then it suddenly stops, determine the distance the package will slide before coming to rest. ($g = 9.8 \text{ m/s}^2$)



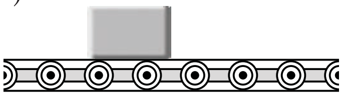
19. In figure, a crate slides down an inclined right-angled trough. The coefficient of kinetic friction between the crate and the trough is μ_k. What is the acceleration of the crate in terms of μ_k, θ and g?



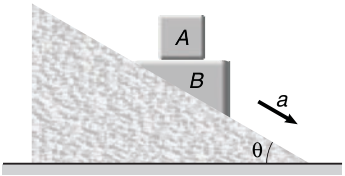
20. A heavy chain with a mass per unit length ρ is pulled by the constant force F along a horizontal surface consisting of a smooth section and a rough section. The chain is initially at rest on the rough surface with $x = 0$. If the coefficient of kinetic friction between the chain and the rough surface is μ_k, determine the velocity v of the chain when $x = L$. The force F is greater than $\mu_k \rho g L$ in order to initiate the motion.



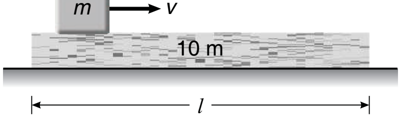
21. A package is at rest on a conveyor belt which is initially at rest. The belt is started and moves to the right for 1.3 s with a constant acceleration of 2 m/s^2. The belt then moves with a constant deceleration a_2 and comes to a stop after a total displacement of 2.2 m. Knowing that the coefficients of friction between the package and the belt are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (a) the deceleration a_2 of the belt, (b) the displacement of the package relative to the belt as the belt comes to a stop. ($g = 9.8$ m/s^2)



22. Determine the normal force the 10 kg crate A exerts on the smooth cart B, if the cart is given an acceleration of $a = 2$ m/s^2 down the plane. Also, find the acceleration of the crate. Set $\theta = 30°$. ($g = 10$ m/s^2).



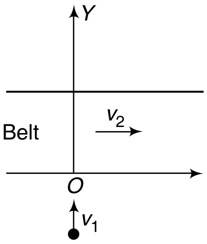
23. A small block of mass m is projected on a larger block of mass 10 m and length l with a velocity v as shown in the figure. The coefficient of friction between the two blocks is μ_2 while that between the lower block and the ground is μ_1. Given that $\mu_2 > 11\mu_1$.



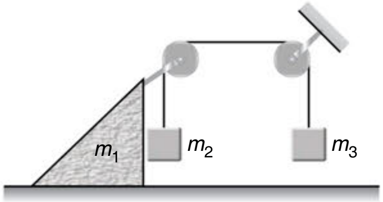
(a) Find the minimum value of v, such that the mass m falls off the block of mass 10 m.

(b) If v has this minimum value, find the time taken by block m to do so.

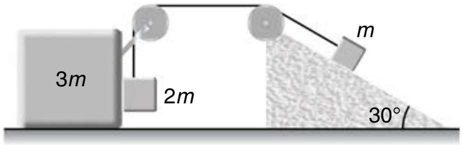
24. A particle of mass m and velocity v_1 in positive y direction is projected on to a belt that is moving with uniform velocity v_2 in x-direction as shown in figure. Coefficient of friction between particle and belt is μ. Assuming that the particle first touches the belt at the origin of fixed x-y coordinate system and remains on the belt, find the co-ordinates (x, y) of the point where sliding stops.



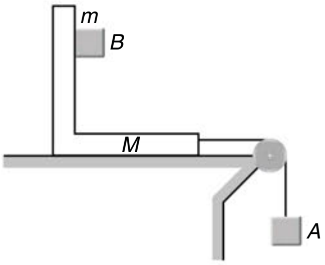
25. In the shown arrangement, both pulleys and the string are massless and all the surfaces are frictionless. Find the acceleration of the wedge.



26. Neglect friction. Find accelerations of m, $2m$ and $3m$ as shown in the figure. The wedge is fixed.



27. The figure shows an L shaped body of mass M placed on smooth horizontal surface. The block A is connected to the body by means of an inextensible string, which is passing over a smooth pulley of negligible mass. Another block B of mass m is placed against a vertical wall of the body. Find the minimum value of the mass of block A so that block B remains stationary relative to the wall. Coefficient of friction between the block B and the vertical wall is μ.



Answers

Introductory Exercise 8.1

1. See the hints **2.** See the hints **3.** See the hints

4. See the hints **5.** See the hints **6.** See the hints

7. $F_{1x} = 2\sqrt{3}$ N, $F_2x = -2$ N, $F_3x = 0$, $F_4x = 4$ N, $F_1y = 2$ N, $F_{2y} = 2\sqrt{3}$ N, $F_{3y} = -6$ N, $F_{4y} = 0$

8. 30 N **9.** $N_A = \dfrac{1000}{\sqrt{3}}$ N, $N_B = \dfrac{500}{\sqrt{3}}$ N

10. $\dfrac{2}{\sqrt{3}}W$ **11.** (a) 26.8 N (b) 26.8 N (c) 100 N

Introductory Exercise 8.2

1. (a) 10 ms^{-2} (b) 110 N (c) 20 N **2.** 4 **3.** $\dfrac{10}{3}$ ms^{-2} **4.** 3 kg

5. zero **6.** $\dfrac{3g}{4}$ **7.** 4 ms^{-2}, 24 N, 42 N, 14 N

8. (a) 3 ms^{-2} (b) 18 N, 12 N, 30 N, (c) 70 N **9.** (a) 10 N, 30 N (b) 24 N

Introductory Exercise 8.3

1. $2a_1 + a_2 + a_3 = 0$ **2.** 3 m/s downwards **3.** (a) $\dfrac{2g}{3}$ (b) $\dfrac{10}{3}$ N **4.** (a) $\dfrac{g}{2}, \dfrac{Mg}{2}$

5. 4.8 kg **6.** $\dfrac{12}{35}$ N, $\dfrac{2}{7}$ ms^{-2} **7.** 1 ms^{-2} (upwards) **8.** $\dfrac{g}{3}$ (up the plane)

Introductory Exercise 8.4

1. (a) $(4\,\hat{\mathbf{j}})$ N (b) $(-4\hat{\mathbf{i}})$ N **2.** True **3.** False

Introductory Exercise 8.5

1. (a) zero, 20 N (b) 6 ms^{-2}, 8 N **2.** (a) $\dfrac{mg}{2}$, 0, 0 (b) $\dfrac{mg}{\sqrt{2}}$, 0, 0 (c) $\dfrac{mg}{2}, \left(\dfrac{\sqrt{3}-1}{2}\right)mg, \left(\dfrac{\sqrt{3}-1}{2}\right)g$

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (a) **4.** (b) **5.** (d) **6.** (a) **7.** (b) **8.** (d) **9.** (d) **10.** (b)
11. (d)

Single Correct Option

1. (b) **2.** (b) **3.** (a) **4.** (d) **5.** (c) **6.** (d) **7.** (b) **8.** (b) **9.** (c) **10.** (b)
11. (a) **12.** (a) **13.** (a) **14.** (b) **15.** (a) **16.** (a) **17.** (a) **18.** (b) **19.** (b) **20.** (c)
21. (a) **22.** (d) **23.** (c) **24.** (d) **25.** (d) **26.** (a) **27.** (a) **28.** (d) **29.** (b) **30.** (a)

Subjective Questions

1. $F = 10.16$ N, $R = 2.4$ N

2. 45.26 N, 22.63 N

3. $F_1 = F_2 = W = 30\sqrt{2}$ N

4. 5N, $5\sqrt{3}$ N

5. (a) $\frac{40}{\sqrt{3}}$ N (b) $\frac{40}{\sqrt{3}}$ N

6. $\frac{mg}{\sqrt{2}}, \frac{g}{2}$

7. 4N, 6N

8. (a) 20 N (b) 50 N

9. (a) 2.7 ms^{-2} (b) 136.5 N (c) 112.5 N

10. 0.288

11. 30°, 10 N

12. $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$, $5\sqrt{7}$ N

13. $\frac{1}{3}$ s

14. $(g + a)\sin\theta$, down the plane

15. 6.83 kg

16. (a) $x = x_0 + 10t - 2.5\,t^2$ $v_x = 10 - 5t$ (b) $t = 4$ s

17. (a) $x = x_0 - 2.5\,t^2$, $z = z_0 + 10t$, $v_x = -5t$, $v_z = 10\ \text{ms}^{-1}$

(b) $x = x_0$, $z = z_0 + 10t$, $v_x = 0$, $v_z = 10\ \text{ms}^{-1}$

18. **For $t \le 1.25$ s :**

$$x = x_0 + 10t - 4t^2$$
$$v_x = 10 - 8t$$

After 1.25 s : Block remains stationary

19. $\frac{9}{25}$ mg

20. (a) 34 N (up) (b) 40 N (up) (c) 88 N (up)

21. $F = 100$ N

22. 5 cm

23. (a) move up (b) constant (c) constant (d) stop

24. $a_B = -3a_A$

25. $a_A + 2a_B + a_C = 0$

26. (a) $a_1 = \frac{120}{11}\ \text{ms}^{-2}$, $a_2 = \frac{50}{11}\ \text{ms}^{-2}$ (downwards) $a_3 = \frac{70}{11}\ \text{ms}^{-2}$ (downwards) (b) $T_1 = T_2 = \frac{120}{11}$ N

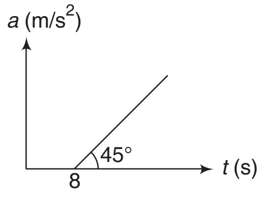
27. $\frac{2}{7}g$ (downwards), $\frac{g}{7}$ (upwards)

28. (a) 2 s (b) 6 m

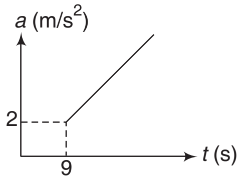
29. (a) 1 s (b) 6 ms^{-1} (c) 4m, 7m (both towards right)

30. 4 ms^{-2} (downwards), 12 N (upwards)

31. $a = 0$ for $t \le 8$ s, $a = t - 8$ for $t \ge 8$ s



32. $a = 0$ for $t \le 9$s, $a = \left(\frac{2}{3}t - 4\right)$ for $t \ge 9$ s



LEVEL 2

Single Correct Option

1. (c) **2.** (b) **3.** (b) **4.** (d) **5.** (c) **6.** (a) **7.** (b) **8.** (c) **9.** (b) **10.** (d)
11. (a) **12.** (c) **13.** (b) **14.** (b) **15.** (c) **16.** (c) **17.** (b) **18.** (d) **19.** (b) **20.** (c)
21. (d) **22.** (c) **23.** (a) **24.** (c) **25.** (a) **26.** (b) **27.** (c) **28.** (b) **29.** (b) **30.** (d)
31. (b) **32.** (d) **33.** (b) **34.** (c) **35.** (b) **36.** (d) **37.** (b) **38.** (d) **39.** (d) **40.** (d)
41. (d) **42.** (a) **43.** (b) **44.** (b) **45.** (c) **46.** (c) **47.** (a)

More than One Correct Options

1. (a,c,d) **2.** (a,c) **3.** (a,d) **4.** (c,d) **5.** (a,d) **6.** (a,c) **7.** (a,b, c)
8. (a,c) **9.** (b,c) **10.** (b,c, d) **11.** (all) **12.** (b, c) **13.** (all)

Comprehension Based Questions

1. (b) **2.** (a) **3.** (d) **4.** (c) **5.** (d) **6.** (c) **7.** (c) **8.** (a) **9.** (a)

Match the Columns

1. (a) → (r) (b) → (q) (c) → (p) (d) → (s)
2. (a) → (s) (b) → (s) (c) → (p) (d) → (p)
3. (a) → (q,r) (b) → (q,r) (c) → (s) (d) → (p, s)
4. (a) → (s) (b) → (p) (c) → (s) (d) → (s)
5. (a) → (p) (b) → (s) (c) → (q) (d) → (p)
6. (a) → (q) (b) → (p,r) (c) → (q) (d) → (p,r)
7. (a) → (q, s) (b) → (p,s) (c) → (p,s) (d) → (q,s)
8. (a) → (r) (b) → (r) (c) → (s) (d) → (q)

Subjective Questions

1. 10 ms^{-1} **2.** $\frac{2}{33}\ \text{ms}^{-2}$ **3.** (a) 54 N (due east) (b) 60 N (due west) **4.** (a) 6.36 ms^{-2} (b) 5.5 ms^{-2}

5. $\dfrac{mg\cos\alpha\sin\alpha}{m\sin\alpha + \dfrac{M}{\sin\alpha}}, \dfrac{mg\cos\alpha}{m\sin\alpha + \dfrac{M}{\sin\alpha}}$ **6.** (a) Clockwise (b) $\frac{1}{3}$ **7.** $\dfrac{F}{m} - \dfrac{\mu(M+m)g}{m} \le a \le \dfrac{F}{m} + \dfrac{\mu(M+m)g}{m}$

8. $a_M = \left(\dfrac{5m - M}{25m + M}\right)g$. (upwards) $a_m = 5a_M$ **9.** $\mu = \left(\dfrac{m+n}{m}\right)\tan\alpha$

10. (a) $a_A = a_B = 0.857\ \text{m/s}^2$ (b) $a_A = 21\ \text{m/s}^2$, $a_B = 1.6\ \text{m/s}^2$ **11.** $\dfrac{mg}{2}\sin 2\theta$ **12.** 2.77 m

13. $t = 0.33$ s **14.** 5.2 m/s , 5.55 s **15.** zero **16.** 5.66 m/s^2 **17.** $\dfrac{M_3}{M_2}(M_1 + M_2 + M_3)g$ **18.** 8.5 m

19. $g(\sin\theta - \sqrt{2}\,\mu_k\cos\theta)$ **20.** $\sqrt{\dfrac{2F}{\rho} - \mu_k gL}$ **21.** (a) 6.63 m/s^2 (b) 0.33 m **22.** 90 N, 1 ms^{-2}

23. (a) $v_{\min} = \sqrt{\dfrac{22(\mu_2 - \mu_1)gl}{10}}$ (b) $t = \sqrt{\dfrac{20l}{11g(\mu_2 - \mu_1)}}$ **24.** $x = v_2\dfrac{\sqrt{v_1^2 + v_2^2}}{2\mu g}$, $y = \dfrac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$

25. $\dfrac{2m_1m_3g}{(m_2 + m_3)(m_1 + m_2) + m_2m_3}$ **26.** $a_m = \dfrac{13}{34}g$, $a_{2m} = \dfrac{\sqrt{397}}{34}g$, $a_{3m} = \dfrac{3}{17}g$ **27.** $m_A = \dfrac{M + m}{\mu - 1}$ but $\mu > 1$

09

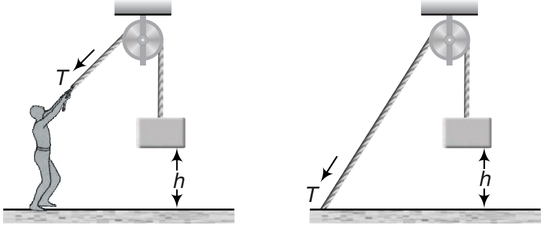
Work, Enengy and Power

Chapter Contents

9.1 Introduction to Work
9.2 Work Done
9.3 Conservative and Non-conservative Forces
9.4 Kinetic Energy
9.5 Work-Energy Theorem
9.6 Potential Energy
9.7 Three Types of Equilibrium
9.8 Power of a Force
9.9 Law of Conservation of Mechanical Energy

9.1 Introduction to Work

In our daily life 'work' has many different meanings. For example, Ram is working in a factory. The machine is in working order. Let us work out a plan for the next year, etc. In physics however, the term 'work' has a special meaning. In physics, work is always associated with a force and a displacement. We note that for work to be done, the force must act through a distance. Consider a person holding a weight at a distance '*h*' off the floor as shown in figure.



No work is done by the man holding the weight at a fixed position. The same task could be accomplished by tying the rope to a fixed point.

Fig. 9.1

In everyday usage, we might say that the man is doing a work, but in our scientific definition, no work is done by a force acting on a stationary object. We could eliminate the effort of holding the weight by merely tying the string to some object and the weight could be supported with no help from us.

Let us now see what does 'work' mean in the language of physics.

9.2 Work Done

There are mainly three methods of finding work done by a force:

(i) Work done by a constant force.
(ii) Work done by a variable force.
(iii) Work done by the area under force and displacement graph.

Work done by a Constant Force

Work done by a constant force is given by

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{S} \qquad (\mathbf{F} = \text{force}, \mathbf{S} = \text{displacement}) \\ &= FS\cos\theta \\ &= (\text{magnitude of force})\ (\text{component of displacement in the direction of force}) \\ &= (\text{magnitude of displacement})\ (\text{component of force in the direction of displacement}) \end{aligned}$$

Here, θ is the angle between **F** and **S**.

Thus, work done is the dot product of **F** and **S**.

Special Cases

(i) If $\theta = 0°$, $W = FS\cos 0° = FS$
(ii) If $\theta = 90°$, $W = FS\cos 90° = 0$
(iii) If $\theta = 180°$, $W = FS\cos 180° = -FS$

Extra Points to Remember

- Work done by a force may be positive, negative or even zero also, depending on the angle (θ) between the force vector **F** and displacement vector **S**. Work done by a force is zero when $\theta = 90°$, it is positive when $0° \le \theta < 90°$ and negative when $90° < \theta \le 180°$. For example, when a person lifts a body, the work done by the lifting force is positive (as $\theta = 0°$) but work done by the force of gravity is negative (as $\theta = 180°$).
- Work depends on frame of reference. With change of frame of reference, inertial force does not change while displacement may change. So, the work done by a force will be different in different frames. For example, if a person is pushing a box inside a moving train, then work done as seen from the frame of reference of train is $\mathbf{F}\cdot\mathbf{S}$ while as seen from the ground it is $\mathbf{F}\cdot(\mathbf{S}+\mathbf{S}_0)$. Here $\mathbf{S}_0$, is the displacement of train relative to ground.
- Suppose a body is displaced from point A to point B, then

$$\mathbf{S} = \mathbf{r}_B - \mathbf{r}_A = (x_B - x_A)\,\hat{\mathbf{i}} + (y_B - y_A)\,\hat{\mathbf{j}} + (z_B - z_A)\,\hat{\mathbf{k}}$$

Here, (x_A, y_A, z_A) and (x_B, y_B, z_B) are the co-ordinates of points A and B.

Example 9.1 *A body is displaced from* $A = (2\ m, 4\ m, -6\ m)$ *to* $\mathbf{r}_B = (6\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\ m$ *under a constant force* $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})\ N$. *Find the work done.*

Solution $\mathbf{r}_A = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}})\ \text{m}$

$\therefore$

$$\mathbf{S} = \mathbf{r}_B - \mathbf{r}_A$$

$$= (6\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

$$= 4\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$$

$$W = \mathbf{F}\cdot\mathbf{S} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 8\hat{\mathbf{k}}) = 8 - 24 - 8 = -24\ \text{J}$$ **Ans.**

Note *Work done is negative. Therefore angle between* **F** *and* **S** *is obtuse.*

Example 9.2 *A block of mass* $m = 2$ *kg is pulled by a force* $F = 40$ *N upwards through a height* $h = 2$ *m. Find the work done on the block by the applied force* F *and its weight* mg. $(g = 10\ m/s^2)$

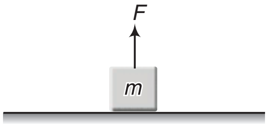


Fig. 9.2

Solution Weight $mg = (2)(10) = 20\ \text{N}$

Work done by the applied force $W_F = Fh\cos 0°$.

As the angle between force and displacement is 0°

or $W_F = (40)(2)(1) = 80\ \text{J}$ **Ans.**

Similarly, work done by its weight

$$W_{mg} = (mg)(h)\cos 180°$$

or $W_{mg} = (20)(2)(-1) = -40\ \text{J}$ **Ans.**

Example 9.3 *Two unequal masses of 1 kg and 2 kg are attached at the two ends of a light inextensible string passing over a smooth pulley as shown in Fig. 9.3. If the system is released from rest, find the work done by string on both the blocks in 1 s.*
(Take $g = 10\ m/s^2$).

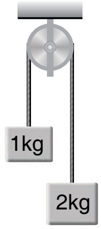


Fig. 9.3

Solution Net pulling force on the system is

$$F_{\text{net}} = 2g - 1g = 20 - 10 = 10\ \text{N}$$

Total mass being pulled $\quad m = (1 + 2) = 3\ \text{kg}$

Therefore, acceleration of the system will be

$$a = \frac{F_{\text{net}}}{m} = \frac{10}{3}\ \text{m/s}^2$$

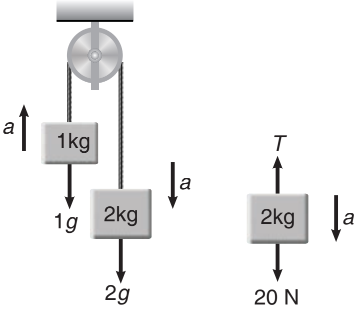
Displacement of both the blocks in 1 s is

$$S = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{10}{3}\right)(1)^2 = \frac{5}{3}\ \text{m}$$

Free body diagram of 2 kg block is shown in Fig. 9.4 (b).

Using $\Sigma F = ma$, we get

$$20 - T = 2a = 2\left(\frac{10}{3}\right) \quad \text{or} \quad T = 20 - \frac{20}{3} = \frac{40}{3}\ \text{N}$$



(a) (b)

Fig. 9.4

$\therefore$ Work done by string (tension) on 1 kg block in 1 s is

$$W_1 = (T)(S)\cos 0^\circ$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(1) = \frac{200}{9}\ \text{J}$$ **Ans.**

Similarly, work done by string on 2 kg block in 1 s will be

$$W_2 = (T)(S)(\cos 180^\circ)$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(-1) = -\frac{200}{9}\ \text{J}$$ **Ans.**

Work Done by a Variable Force

So far we have considered the work done by a force which is constant both in magnitude and direction. Let us now consider a force which acts always in one direction but whose magnitude may keep on varying. We can choose the direction of the force as x-axis. Further, let us assume that the magnitude of the force is also a function of x or say $F(x)$ is known to us. Now, we are interested in finding the work done by this force in moving a body from x_1 to x_2.

Work done in a small displacement from x to $x + dx$ will be

$$dW = F \cdot dx$$

Now, the total work can be obtained by integration of the above elemental work from x_1 to x_2 or

$$\boxed{W = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} F \cdot dx}$$

Note *In this method of finding work done, you need not to worry for the sign of work done. If we put proper limits in integration then sign of work done automatically comes.*

It is important to note that $\int_{x_1}^{x_2} F\ dx$ is also the area under F-x graph between $x = x_1$ to $x = x_2$.

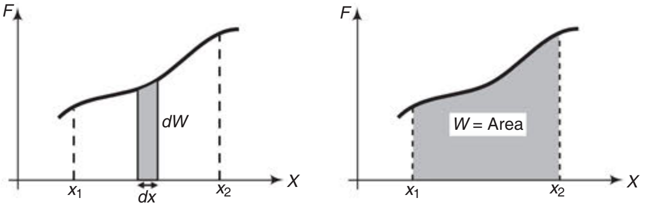


Fig. 9.5

Spring Force

An important example of the above idea is a spring that obeys Hooke's law. Consider the situation shown in figure. One end of a spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x = 0$ denote the position of the block when the spring is in its natural length. When the block is displaced by an amount x (either compressed or elongated) a restoring force (F) is applied by the spring on the block. The direction of this force F is always towards its mean position ($x = 0$) and the magnitude is directly proportional to x or

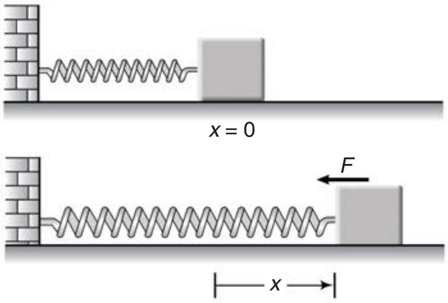


Fig. 9.6

$$F \propto x \qquad \text{(Hooke's law)}$$

$$\therefore \qquad F = -kx \qquad \ldots\text{(i)}$$

Here, k is a constant called force constant of spring and depends on the nature of spring. From Eq. (i) we see that F is a variable force and F-x graph is a straight line passing through origin with slope $= -k$. Negative sign in Eq. (i) implies that the spring force F is directed in a direction opposite to the displacement x of the block.

Let us now find the work done by this force F when the block is displaced from $x = 0$ to $x = x$. This can be obtained either by integration or the area under F-x graph.

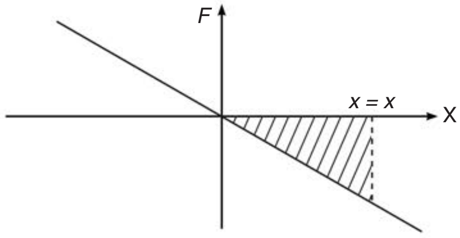


Fig. 9.7

Thus, $$W = \int dW = \int_0^x F dx = \int_0^x -kx\ dx = -\frac{1}{2}kx^2$$

Here, work done is negative because force is in opposite direction of displacement.

Similarly, if the block moves from $x = x_1$ to $x = x_2$. The limits of integration are x_1 and x_2 and the work done is

$$W = \int_{x_1}^{x_2} -kx\ dx = \frac{1}{2}k\ (x_1^2 - x_2^2)$$

Example 9.4 *A force $F = (2 + x)$ acts on a particle in x-direction where F is in newton and x in metre. Find the work done by this force during a displacement from $x = 1.0$ m to $x = 2.0$ m.*

Solution As the force is variable, we shall find the work done in a small displacement from x to $x + dx$ and then integrate it to find the total work. The work done in this small displacement is

$$dW = F\,dx = (2+x)\,dx$$

Thus,

$$W = \int_{1.0}^{2.0} dW = \int_{1.0}^{2.0} (2+x)\,dx$$

$$= \left[2x + \frac{x^2}{2}\right]_{1.0}^{2.0} = 3.5 \text{ J}$$ **Ans.**

Example 9.5 *A force $F = -\dfrac{k}{x^2}$ $(x \neq 0)$ acts on a particle in x-direction. Find the work done by this force in displacing the particle from. $x = +a$ to $x = +2a$. Here, k is a positive constant.*

Solution

$$W = \int F\,dx = \int_{+a}^{+2a} \left(\frac{-k}{x^2}\right) dx = \left[\frac{k}{x}\right]_{+a}^{+2a}$$

$$= -\frac{k}{2a}$$ **Ans.**

Note *It is important to note that work comes out to be negative which is quite obvious as the force acting on the particle is in negative x-direction $\left(F = -\dfrac{k}{x^2}\right)$ while displacement is along positive x-direction. (from $x = a$ to $x = 2a$)*

Work Done by Area Under *F-S* or *F-x* Graph

This method is normally used when force and displacement are either parallel or antiparallel (or one dimensional). As we have discussed above

$$W = \int F dx = \text{area under } F\text{-}x \text{ graph}$$

So, work done by a force can be obtained from the area under F-x graph. Unlike the integration method of finding work done in which sign of work done automatically comes after integration, in this method area of the graph will only give us the magnitude of work done. If force and displacement have same sign, work done will be positive and if both have opposite signs, work done is negative.

Let us take an example.

Example 9.6 *A force F acting on a particle varies with the position x as shown in figure. Find the work done by this force in displacing the particle from*

(a) $x = -2$ m to $x = 0$

(b) $x = 0$ to $x = 2$ m.

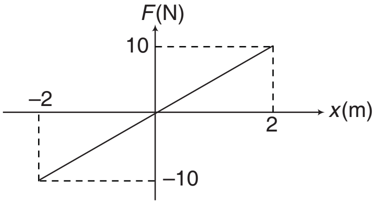


Fig. 9.8

Solution (a) From $x = -2\,\text{m}$ to $x = 0$, displacement of the particle is along positive x-direction while force acting on the particle is along negative x-direction. Therefore, work done is negative and given by the area under F-x graph with projection along x-axis.

$$\therefore \qquad W = -\frac{1}{2}(2)(10) = -10\ \text{J} \qquad \textbf{Ans.}$$

(b) From $x = 0$ to $x = 2\,\text{m}$, displacement of particle and force acting on the particle both are along positive x-direction. Therefore, work done is positive and given by the area under F-x graph,

$$\text{or} \qquad W = \frac{1}{2}(2)(10) = 10\ \text{J} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 9.1

1. A block is displaced from (1m, 4m, 6m) to $(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ m under a constant force $\mathbf{F} = (6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ N. Find the work done by this force.
2. A block of mass 2.5 kg is pushed 2.20 m along a frictionless horizontal table by a constant force 16 N directed 45° above the horizontal. Determine the work done by
   (a) the applied force,
   (b) the normal force exerted by the table,
   (c) the force of gravity and
   (d) determine the total work done on the block
3. A block is pulled a distance x along a rough horizontal table by a horizontal string. If the tension in the string is T, the weight of the block is w, the normal reaction is N and frictional force is F. Write down expressions for the work done by each of these forces.
4. A bucket tied to a string is lowered at a constant acceleration of $g/4$. If mass of the bucket is m and it is lowered by a distance l then find the work done by the string on the bucket.
5. A 1.8 kg block is moved at constant speed over a surface for which coefficient of friction $\mu = \frac{1}{4}$. It is pulled by a force F acting at 45° with horizontal as shown in Fig. 9.9. The block is displaced by 2 m. Find the work done on the block by (a) the force F (b) friction (c) gravity.

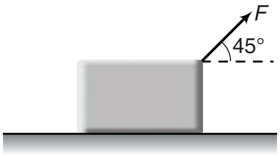


Fig. 9.9

6. A block is constrained to move along x-axis under a force $F = -2x$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 2$ m to $x = -4$ m.
7. A block is constrained to move along x-axis under a force $F = \frac{4}{x^2}$ $(x \neq 0)$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 4$ m to $x = 2$ m.

8. Force acting on a particle varies with displacement as shown in Fig. 9.10. Find the work done by this force on the particle from $x = -4$ m to $x = +4$ m.

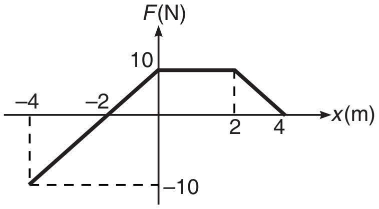


Fig. 9.10

9. A particle is subjected to a force F_x that varies with position as shown in figure. Find the work done by the force on the body as it moves
   (a) from $x = 10.0$ m to $x = 5.0$ m,
   (b) from $x = 5.0$ m to $x = 10.0$ m,
   (c) from $x = 10.0$ m to $x = 15.0$ m,
   (d) what is the total work done by the force over the distance $x = 0$ to $x = 15.0$ m ?

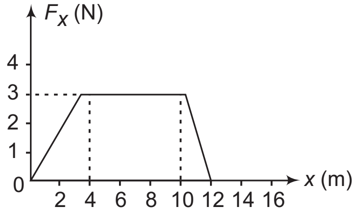


Fig. 9.11

10. A child applies a force **F** parallel to the x-axis to a block moving on a horizontal surface. As the child controls the speed of the block, the x-component of the force varies with the x-coordinate of the block as shown in figure. Calculate the work done by the force **F** when the block moves

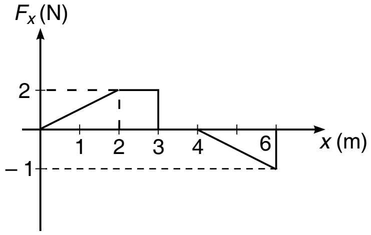


Fig. 9.12

(a) from $x = 0$ to $x = 3.0$ m
(b) from $x = 3.0$ m to $x = 4.0$ m
(c) from $x = 4.0$ m to $x = 7.0$ m
(d) from $x = 0$ to $x = 7.0$ m

9.3 Conservative and Non-Conservative Forces

In the above article, we considered the forces which were although variable but always directed in one direction. However, the most general expression for work done is

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

and

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} dW = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r}$$

Here,

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

$\mathbf{r}_i =$ initial position vector and $\mathbf{r}_f =$ final position vector

Conservative and non-conservative forces can be better understood after going through the following two examples.

Example 9.7 *An object is displaced from point $A(2\text{ m}, 3\text{ m}, 4\text{ m})$ to a point $B(1\text{ m}, 2\text{ m}, 3\text{ m})$ under a constant force $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ N. Find the work done by this force in this process.*

Solution

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r} = \int_{(2\text{ m},\, 3\text{ m},\, 4\text{ m})}^{(1\text{ m},\, 2\text{ m},\, 3\text{ m})} (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$= [2x + 3y + 4z]_{(2\text{ m},\, 3\text{ m},\, 4\text{ m})}^{(1\text{ m},\, 2\text{ m},\, 3\text{ m})} = -9\text{ J}$$ **Ans.**

Alternate Solution

Since, $\mathbf{F}$ = constant, we can also use.

$$W = \mathbf{F}\cdot\mathbf{S}$$

Here, $$\mathbf{S} = \mathbf{r}_f - \mathbf{r}_i = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$= (-\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$$

∴ $$W = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$$= -2 - 3 - 4 = -9\text{ J}$$ **Ans.**

Example 9.8 *An object is displaced from position vector $\mathbf{r}_1 = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ m to $\mathbf{r}_2 = (4\hat{\mathbf{i}} + 6\hat{\mathbf{j}})$ m under a force $\mathbf{F} = (3x^2\hat{\mathbf{i}} + 2y\hat{\mathbf{j}})$ N. Find the work done by this force.*

Solution

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} (3x^2\hat{\mathbf{i}} + 2y\hat{\mathbf{j}})\cdot(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$= \int_{\mathbf{r}_1}^{\mathbf{r}_2} (3x^2\,dx + 2y\,dy) = [x^3 + y^2]_{(2,\,3)}^{(4,\,6)}$$

$$= 83\text{ J}$$ **Ans.**

In the above two examples, we saw that while calculating the work done we did not mention the path along which the object was displaced. Only initial and final coordinates were required. It shows that in both the examples, the work done is path independent or work done will be same along all paths. The forces in which work is path independent are known as **conservative forces.**

Thus, if a particle or an object is displaced from position A to position B through three different paths under a conservative force field. Then,

$$W_1 = W_2 = W_3$$

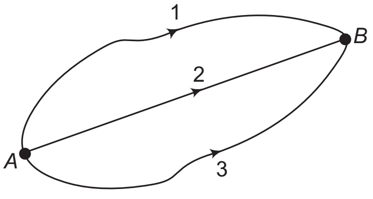


Fig. 9.13

Further, it can be shown that work done in a closed path is zero under a conservative force field. ($W_{AB} = -W_{BA}$ or $W_{AB} + W_{BA} = 0$). Gravitational force, Coulomb's force and spring force are few examples of conservative forces. On the other hand, if the work is path dependent or $W_1 \neq W_2 \neq W_3$, the force is called a **non-conservative.** Frictional forces, viscous forces are non-conservative in nature. Work done in a closed path is not zero in a non-conservative force field.

Note *The word potential energy is defined only for conservative forces like gravitational force, electrostatic force and spring force etc.*

We can differentiate the conservative and non-conservative forces in a better way by making a table as given below.

S.No	Conservative Forces	Non-conservative Forces
1.	Work done is path independent	Work done is path dependent
2.	Work done in a closed path is zero	Work done in a closed path is not zero
3.	The word potential energy is defined for conservative forces	The word potential energy is not defined for non-conservative forces.
4.	Examples are: gravitational force, electrostatic force, spring force etc.	Examples are: frictional force, viscous force etc.

✓ Extra Points to Remember

- Gravitational force is a conservative force. Its work done is path independent. For small heights it only depends on the height difference 'h' between two points. Work done by gravitational force is ($\pm mgh$), in moving the mass 'm' from one point to another point. This is ($+mgh$) if the mass is moving downwards (as the force mg and displacement both are downwards, in the same direction) and ($-mgh$) if the mass is moving upwards.

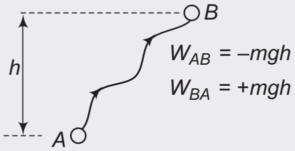


Fig. 9.14

- The magnetic field (and therefore the magnetic force) is neither conservative nor non-conservative.
- Electric field (and therefore electric force) is produced either by static charge or by time varying magnetic field. First is conservative and the other non-conservative.

9.4 Kinetic Energy

Kinetic energy (KE) is the capacity of a body to do work by virtue of its motion. If a body of mass m has a velocity v, its kinetic energy is equivalent to the work which an external force would have to do to bring the body from rest upto its velocity v. The numerical value of the kinetic energy can be calculated from the formula.

$$\boxed{\text{KE} = \frac{1}{2} mv^2}$$

This can be derived as follows:

Consider a constant force F which acting on a mass m initially at rest. This force provides the mass m a velocity v.

If in reaching this velocity, the particle has been moving with an acceleration a and has been given a displacement s, then

$$F = ma \qquad \text{(Newton's law)}$$

$$v^2 = 2as$$

Work done by the constant force $= Fs$

or $$W = (ma)\left(\frac{v^2}{2a}\right) = \frac{1}{2} mv^2$$

But the kinetic energy of the body is equivalent to the work done in giving the body this velocity.

Hence, $$\text{KE} = \frac{1}{2} mv^2$$

Regarding the kinetic energy the following two points are important to note:

1. Since, both m and v^2 are always positive. KE is always positive and does not depend on the direction of motion of the body.
2. Kinetic energy depends on the frame of reference. For example, the kinetic energy of a person of mass m sitting in a train moving with speed v is zero in the frame of train but $\frac{1}{2} mv^2$ in the frame of earth.

9.5 Work Energy Theorem

This theorem is a very important tool that relates the works to kinetic energy. According to this theorem:

Work done by all the forces (conservative or non-conservative, external or internal) acting on a particle or an object is equal to the change in kinetic energy of it.

$$\therefore \qquad W_{\text{net}} = \Delta\text{KE} = K_f - K_i$$

Let, $\mathbf{F}_1, \mathbf{F}_2 \ldots$ be the individual forces acting on a particle. The resultant force is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots$ and the work done by the resultant force is

$$W = \int \mathbf{F} \cdot \mathbf{dr} = \int (\mathbf{F}_1 + \mathbf{F}_2 + \ldots) \cdot \mathbf{dr}$$
$$= \int \mathbf{F}_1 \cdot \mathbf{dr} + \int \mathbf{F}_2 \cdot \mathbf{dr} + \ldots$$

where, $\int \mathbf{F}_1 \cdot \mathbf{dr}$ is the work done on the particle by $\mathbf{F}_1$ and so on. Thus, work energy theorem can also be written as: work done by the resultant force which is also equal to the sum of the work done by the individual forces is equal to change in kinetic energy.

Regarding the work-energy theorem it is worthnoting that :

(i) If W_{net} is positive then $K_f - K_i$ = positive,

i.e. $K_f > K_i$ or kinetic energy will increase and if W_{net} is negative then kinetic energy will decrease.

(ii) This theorem can be applied to non-inertial frames also. In a non-inertial frame it can be written as work done by all the forces (including the pseudo forces) = change in kinetic energy in non-inertial frame. Let us take an example.

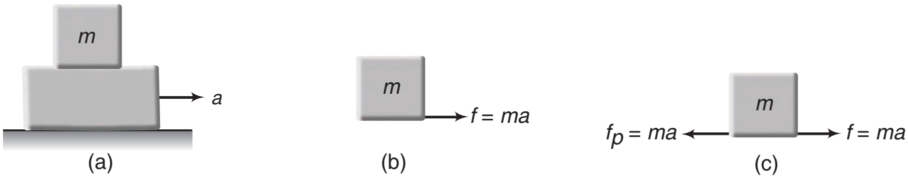


Fig. 9.15

Refer Figure (a)

A block of mass m is kept on a rough plank moving with an acceleration a. There is no relative motion between block and plank. Hence, force of friction on block is $f = ma$ in forward direction.

Refer Figure (b)

Horizontal force on the block has been shown from ground (inertial) frame of reference.

If the plank moves a distance s on the ground, the block will also move the same distance s (as there is no slipping between the two). Hence, work done by friction on the block (w.r.t. ground) is

$$W_f = fs = mas$$

From work-energy principle if v is the speed of block (w.r.t. ground), then

$$\text{KE} = W_f \quad \text{or} \quad \frac{1}{2}mv^2 = mas \quad \text{or} \quad v = \sqrt{2as}$$

Thus, velocity of block relative to ground is $\sqrt{2as}$.

Refer Figure (c)

Free body diagram of the block has been shown from accelerating frame (plank).

Here, $$f_p = \text{pseudo force} = ma$$

Work done by all the forces,

$$W = W_f + W_p = mas - mas = 0$$

From work-energy theorem,

$$\frac{1}{2}mv_r^2 = W = 0 \qquad \text{or} \qquad v_r = 0$$

Thus, velocity of block relative to plank is zero.

Note *Work-energy theorem is very useful in finding the work-done by a force whose exact nature is not known to us or to find the work done of a variable force whose exact variation is not known to us.*

Example 9.9 *An object of mass 5 kg falls from rest through a vertical distance of 20 m and attains a velocity of 10 m/s. How much work is done by the resistance of the air on the object ? ($g = 10\ m/s^2$)*

Solution Applying work-energy theorem,

work done by all the forces = change in kinetic energy

or $$W_{mg} + W_{\text{air}} = \frac{1}{2}mv^2$$

$\therefore$ $$W_{\text{air}} = \frac{1}{2}mv^2 - W_{mg} = \frac{1}{2}mv^2 - mgh$$

$$= \frac{1}{2} \times 5 \times (10)^2 - (5) \times (10) \times (20)$$

$$= -750 \text{ J}$$ **Ans.**

Example 9.10 *An object of mass m is tied to a string of length l and a variable force F is applied on it which brings the string gradually at angle θ with the vertical. Find the work done by the force F.*

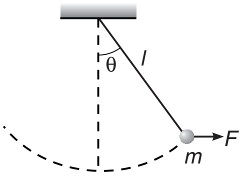


Fig. 9.16

Solution In this case, three forces are acting on the object:

1. tension (T)
2. weight (mg) and
3. applied force (F)

Using work-energy theorem

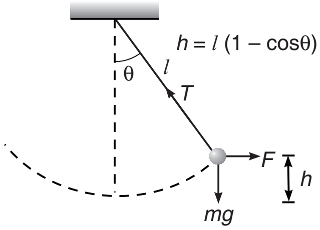


Fig. 9.17

$$W_{\text{net}} = \Delta\text{KE}$$

or $$W_T + W_{mg} + W_F = 0 \qquad \ldots\text{(i)}$$

as $$\Delta\text{KE} = 0$$

because $$K_i = K_f = 0$$

Further, $W_T = 0$, as tension is always perpendicular to displacement.

$$W_{mg} = -mgh$$

or $$W_{mg} = -mgl(1-\cos\theta)$$

Substituting these values in Eq. (i), we get

$$W_F = mgl(1-\cos\theta)$$ **Ans.**

Note *Here, the applied force F is variable. So, if we do not apply the work energy theorem we will first find the magnitude of F at different locations and then integrate dW ($= \mathbf{F}\cdot\mathbf{dr}$) with proper limits.*

Example 9.11 *A body of mass m was slowly hauled up the hill as shown in the Fig. 9.18 by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.*

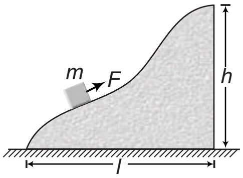


Fig. 9.18

Solution Four forces are acting on the body:

1. weight (mg)
2. normal reaction (N)
3. friction (f) and
4. the applied force (F)

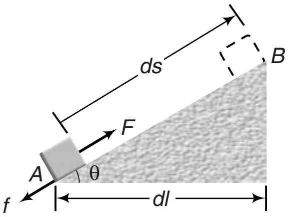


Fig. 9.19

Using work-energy theorem

$$W_{\text{net}} = \Delta\text{KE}$$

or $$W_{mg} + W_N + W_f + W_F = 0 \qquad \text{...(i)}$$

Here, $\Delta\text{KE} = 0$, because $K_i = 0 = K_f$

$$W_{mg} = -\,mgh$$

$$W_N = 0$$

(as normal reaction is perpendicular to displacement at all points)

W_f can be calculated as under

$$f = \mu\, mg \cos\theta$$

$$\therefore \qquad (dW_{AB})_f = -\,f\,ds$$

$$= -\,(\mu\, mg \cos\theta)\,ds$$

$$= -\,\mu\, mg\,(dl) \qquad \text{(as } ds \cos\theta = dl\text{)}$$

$$\therefore \qquad f = -\,\mu\, mg\, \Sigma\, dl = -\,\mu\, mgl$$

Substituting these values in Eq. (i), we get

$$W_F = mgh + \mu mgl$$ **Ans.**

Note *Here again, if we want to solve this problem without using work-energy theorem we will first find magnitude of applied force* **F** *at different locations and then integrate* $dW\ (= \mathbf{F} \cdot \mathbf{dr})$ *with proper limits.*

Example 9.12 *The displacement x of a particle moving in one dimension, under the action of a constant force is related to time t by the equation*

$$t = \sqrt{x} + 3$$

where, x is in metre and t in second. Calculate: (a) the displacement of the particle when its velocity is zero, (b) the work done by the force in the first 6 *s.*

Solution As $t = \sqrt{x} + 3$

i.e. $$x = (t-3)^2 \qquad \text{...(i)}$$

So, $$v = (dx/dt) = 2(t-3) \qquad \text{...(ii)}$$

(a) v will be zero when $2(t-3) = 0$ i.e. $t = 3$

Substituting this value of t in Eq. (i),

$$x = (3-3)^2 = 0$$

i.e. when velocity is zero, displacement is also zero. **Ans.**

(b) From Eq. (ii), $(v)_{t=0} = 2(0-3) = -6$ m/s

and $(v)_{t=6} = 2(6-3) = 6$ m/s

So, from work-energy theorem

$$w = \Delta\text{KE} = \frac{1}{2}m[v_f^2 - v_i^2] = \frac{1}{2}m[6^2 - (-6)^2] = 0$$

i.e. work done by the force in the first 6 s is zero. **Ans.**

INTRODUCTORY EXERCISE 9.2

1. A ball of mass 100 gm is projected upwards with velocity 10 m/s. It returns back with 6 m/s. Find work done by air resistance.
2. Velocity-time graph of a particle of mass 2 kg moving in a straight line is as shown in Fig. 9.20. Find the work done by all the forces acting on the particle.

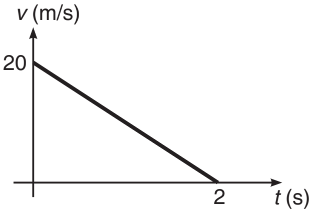


Fig. 9.20

3. Is work-energy theorem valid in a non-inertial frame?
4. A particle of mass m moves on a straight line with its velocity varying with the distance travelled according to the equation $v = \alpha\sqrt{x}$, where α is a constant. Find the total work done by all the forces during a displacement from $x = 0$ to $x = b$.
5. A 5 kg mass is raised a distance of 4 m by a vertical force of 80 N. Find the final kinetic energy of the mass if it was originally at rest. $g = 10\ \text{m/s}^2$.
6. An object of mass m has a speed v_0 as it passes through the origin. It is subjected to a retarding force given by $F_x = -Ax$. Here, A is a positive constant. Find its x-coordinate when it stops.
7. A block of mass M is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force F. The kinetic energy of the block increases by 40J in 1s. State whether the following statements are true or false:
   (a) The tension in the string is Mg
   (b) The work done by the tension on the block is 40 J
   (c) The tension in the string is F
   (d) The work done by the force of gravity is 40J in the above 1s
8. Displacement of a particle of mass 2 kg varies with time as $s = (2t^2 - 2t + 10)$ m. Find total work done on the particle in a time interval from $t = 0$ to $t = 2$ s.
9. A block of mass 30 kg is being brought down by a chain. If the block acquires a speed of 40 cm/s in dropping down 2 m. Find the work done by the chain during the process. $(g = 10\ \text{m/s}^2)$

9.6 Potential Energy

The energy possessed by a body or system by virtue of its position or configuration is known as the potential energy. For example, a block attached to a compressed or elongated spring possesses some energy called elastic potential energy. This block has a capacity to do work. Similarly, a stone when released from a certain height also has energy in the form of gravitational potential energy. Two charged particles kept at certain distance has electric potential energy.

Regarding the potential energy it is important to note that it is defined for a conservative force field only. For non-conservative forces it has no meaning. The change in potential energy (dU) of a system corresponding to a conservative force is given by

$$dU = -\mathbf{F}\cdot\mathbf{dr} = -dW$$

or

$$\int_i^f dU = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot\mathbf{dr}$$

or

$$\Delta U = U_f - U_i = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot\mathbf{dr}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, i.e. if we take $r_i = \infty$ (infinite) and $U_i = 0$ then we can write

$$\boxed{U = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot\mathbf{dr} = -W}$$

or potential energy of a body or system is the negative of work done by the conservative forces in bringing it from infinity to the present position.

Regarding the potential energy it is worth noting that:

1. Potential energy can be defined only for conservative forces and it should be considered to be a property of the entire system rather than assigning it to any specific particle.
2. Potential energy depends on frame of reference.
3. If conservative force F and potential energy associated with this force U are functions of single variable r or x then :

$$F = -\frac{dU}{dr} \quad \text{or} \quad -\frac{dU}{dx}$$

Now, let us discuss three types of potential energies which we usually come across.

Elastic Potential Energy

In Article 9.2, we have discussed the spring forces. We have seen there that the work done by the spring force (of course conservative for an ideal spring) is $-\frac{1}{2}kx^2$ when the spring is stretched or compressed by an amount x from its unstretched position. Thus,

$$U = -W = -\left(-\frac{1}{2}kx^2\right) \quad \text{or} \quad \boxed{U = \frac{1}{2}kx^2} \qquad (k = \text{spring constant})$$

Note that elastic potential energy is always positive.

Gravitational Potential Energy

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by

$$\boxed{U = -G\frac{m_1 m_2}{r}}$$

Here, G = universal gravitation constant $= 6.67 \times 10^{-11} \dfrac{\text{N-m}^2}{\text{kg}^2}$

If a body of mass m is raised to a height h from the surface of earth, the change in potential energy of the system (earth + body) comes out to be

$$\boxed{\Delta U = \frac{mgh}{\left(1+\frac{h}{R}\right)}} \qquad (R = \text{radius of earth})$$

or $$\Delta U \approx mgh \quad \text{if} \quad h << R$$

Thus, the potential energy of a body at height h, i.e. mgh is really the change in potential energy of the system for $h << R$. So, be careful while using $U = mgh$, that h should not be too large. This we will discuss in detail in the chapter of Gravitation.

Electric Potential Energy

The electric potential energy of two point charges q_1 and q_2 separated by a distance r in vacuum is given by

$$\boxed{U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r}}$$

Here, $$\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \ \frac{\text{N-m}^2}{\text{C}^2} = \text{constant}$$

Extra Points to Remember

- Elastic potential energy is either zero or positive, but gravitational and electric potential energy may be zero, positive or negative.
- For increase or decrease in gravitational potential energy of a particle (for small heights) we write,

$$\Delta U = mgh$$

Here, h is the change in height of particle. In case of a rigid body, h of centre of mass of the rigid body is seen.

- Change in potential energy is equal to the negative of work done by the conservative force ($\Delta U = -\Delta W$). If work done by the conservative force is negative, change in potential energy will be positive or potential energy of the system will increase and *vice-versa.*

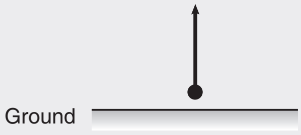


Fig. 9.21

This can be understood by a simple example. Suppose a ball is taken from the ground to some height, work done by gravity is negative, i.e. change in potential energy should increase or potential energy of the ball will increase.

$$\Delta W_{\text{gravity}} = -\text{ve}$$

$$\therefore \quad \Delta U = +\text{ ve} \qquad (\Delta U = -\Delta W)$$

or $$U_f - U_i = +\text{ ve}$$

- $F = -\frac{dU}{dr}$, i.e. conservative forces always act in a direction where potential energy of the system is decreased. This can also be shown as in Fig 9.22.

If a ball is dropped from a certain height. The force on it (its weight) acts in a direction in which its potential energy decreases.

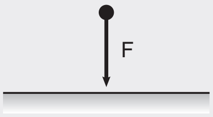


Fig. 9.22

Example 9.13 *A chain of mass 'm' and length 'l' is kept in three positions as shown below. Assuming $h = 0$ on the ground find potential energy of the chain in all three cases.*

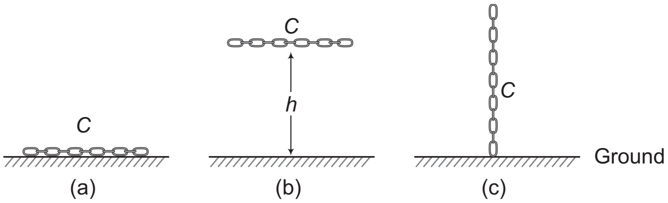


Fig. 9.23

Solution For finding potential energy of chain we will have to see its centre of mass height 'h' from ground.

In figure (a): $h_c = 0 \Rightarrow$ Potential energy, $U = 0$

In figure (b): $h_c = h \Rightarrow$ Potential energy, $U = mgh$

In figure (c): $h_c = \frac{l}{2} \Rightarrow$ Potential energy, $U = mg\frac{l}{2}$

Example 9.14 *Potential energy of a body in position A is $-40\,J$. Work done by conservative force in moving the body from A to B is $-20\,J$. Find potential energy of the body in position B.*

Solution Work done by a conservative force is given by

$$W_{A\to B} = -\Delta U = -(U_B - U_A) = U_A - U_B$$

$$\Rightarrow \qquad U_B = U_A - W_{A\to B} = (-40) - (-20) = -20\,\text{J}$$ **Ans.**

INTRODUCTORY EXERCISE 9.3

1. If work done by a conservative force is positive then select the correct option(s).
   (a) potential energy will decrease
   (b) potential energy may increase or decrease
   (c) kinetic energy will increase
   (d) kinetic energy may increase or decrease.
2. Work done by a conservative force in bringing a body from infinity to A is 60 J and to B is 20 J. What is the difference in potential energy between points A and B, i.e. $U_B - U_A$.

9.7 Three Types of Equilibrium

A body is said to be in translatory equilibrium, if net force acting on the body is zero, i.e.

$$\mathbf{F}_{\text{net}} = 0$$

If the forces are conservative

$$F = -\frac{dU}{dr}$$

and for equilibrium $F = 0$.

So,

$$-\frac{dU}{dr} = 0, \quad \text{or} \quad \frac{dU}{dr} = 0$$

i.e. at equilibrium position slope of U-r graph is zero or the potential energy is optimum (maximum, minimum or constant). Equilibrium are of three types, i.e. the situation where $F = 0$ and $\frac{dU}{dr} = 0$ can be obtained under three conditions. These are stable equilibrium, unstable equilibrium and neutral equilibrium. These three types of equilibrium can be better understood from the given three figures.

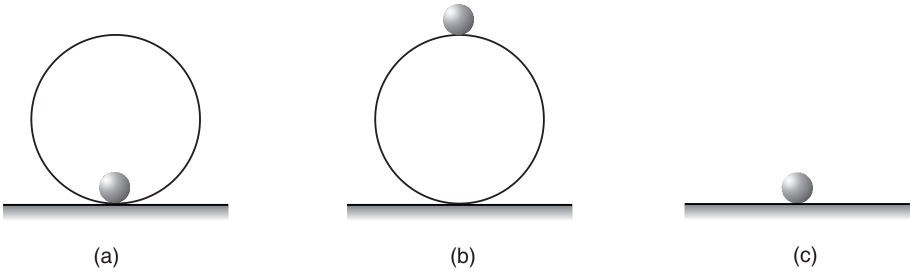


Fig. 9.24

Three identical balls are placed in equilibrium in positions as shown in figures (a), (b) and (c) respectively.

In Fig. (a), ball is placed inside a smooth spherical shell. This ball is in stable equilibrium position. In Fig. (b), the ball is placed over a smooth sphere. This is in unstable equilibrium position. In Fig. (c), the ball is placed on a smooth horizontal ground. This ball is in neutral equilibrium position.

The table given below explains what is the difference and what are the similarities between these three equilibrium positions in the language of physics.

Table 9.1

S. No.	Stable Equilibrium	Unstable Equilibrium	Neutral Equilibrium
1.	Net force is zero.	Net force is zero.	Net force is zero.
2.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.
3.	When displaced from its equilibrium position a net restoring force starts acting on the body which has a tendency to bring the body back to its equilibrium position.	When displaced from its equilibrium position, a net force starts acting on the body which moves the body in the direction of displacement or away from the equilibrium position.	When displaced from its equilibrium position the body has neither the tendency to come back nor to move away from the original position. It is again in equilibrium.
4.	Potential energy in equilibrium position is minimum as compared to its neighbouring points. or $\frac{d^2U}{dr^2}$ = positive	Potential energy in equilibrium position is maximum as compared to its neighbouring points. or $\frac{d^2U}{dr^2}$ = negative	Potential energy remains constant even if the body is displaced from its equilibrium position. or $\frac{d^2U}{dr^2} = 0$
5.	When displaced from equilibrium position the centre of gravity of the body goes up.	When displaced from equilibrium position the centre of gravity of the body comes down.	When displaced from equilibrium position the centre of gravity of the body remains at the same level.

✓ Extra Points to Remember

- If we plot graphs between F and r or U and r, F will be zero at equilibrium while U will be maximum, minimum or constant depending on the type of equilibrium. This all is shown in Fig. 9.24

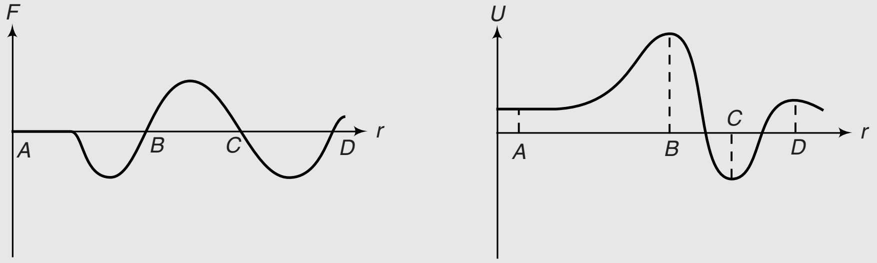


Fig. 9.25

At point A, $F = 0$, $\frac{dU}{dr} = 0$, but U is constant. Hence, A is neutral equilibrium position.

At points B and D, $F = 0$, $\frac{dU}{dr} = 0$ but U is maximum. Thus, these are the points of unstable equilibrium.

At point C, $F = 0$, $\frac{dU}{dr} = 0$, but U is minimum. Hence, point C is in stable equilibrium position.

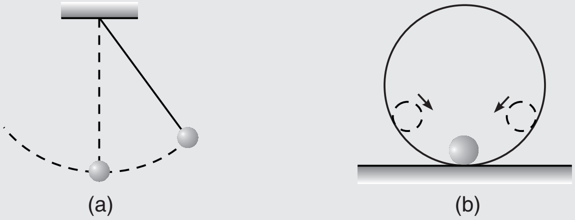

(a) (b)

Fig. 9.26

- Oscillations of a body take place about stable equilibrium position. For example, bob of a pendulum oscillates about its lowest point which is also the stable equilibrium position of the bob. Similarly, in Fig. 9.26 (b), the ball will oscillate about its stable equilibrium position.
- If a graph between F and r is as shown in figure, then $F = 0$, at $r = r_1$, $r = r_2$ and $r = r_3$. Therefore, at these three points, body is in equilibrium. But these three positions are three different types of equilibriums. For example :

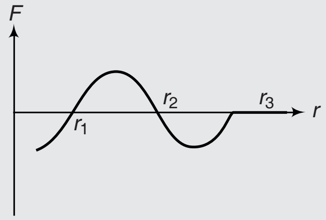


Fig. 9.27

at $r = r_1$, body is in unstable equilibrium. This is because, if we displace the body slightly rightwards (positive direction), force acting on the body is also positive, i.e. away from $r = r_1$ position.

At $r = r_2$, body is in stable equilibrium. Because if we displace the body rightwards (positive direction) force acting on the body is negative (or leftwards) or the force acting is restoring in nature.

At $r = r_3$, equilibrium is neutral in nature. Because if we displace the body rightwards or leftwards force is again zero.

Example 9.15 *For the potential energy curve shown in figure.*

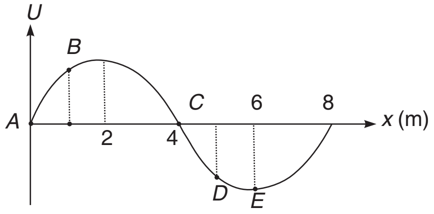


Fig. 9.28

(a) Find directions of force at points A, B, C, D and E.

(b) Find positions of stable, unstable and neutral equilibriums.

Solution (a) $F = -\dfrac{dU}{dr}$ or $-\dfrac{dU}{dx} = -$ (slope of U - x graph)

Point	Slope of U - x graph	F = (Slope of U - x graph)
A	Positive	Negative
B	Positive	Negative
C	Negative	Positive
D	Negative	Positive
E	Zero	Zero

(b) At point $x = 6$ m, potential energy is minimum. So. it is stable equilibrium position.

At $x = 2$ m, potential energy is maximum. So, it is unstable equilibrium position.

There is no point, where potential energy is constant. So, we don't have any point of unstable equilibrium position.

Example 9.16 *The potential energy of a conservative force field is given by*

$$U = ax^2 - bx$$

where, a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

Solution In a conservative field $F = -\dfrac{dU}{dx}$

$\therefore$ $$F = -\frac{d}{dx}(ax^2 - bx) = b - 2ax$$

For equilibrium $F = 0$

or $$b - 2ax = 0 \quad \therefore \quad x = \frac{b}{2a}$$

From the given equation we can see that $\dfrac{d^2U}{dx^2} = 2a$ (positive), i.e. U is minimum.

Therefore, $x = \dfrac{b}{2a}$ is the stable equilibrium position. **Ans.**

INTRODUCTORY EXERCISE 9.4

1. Potential energy of a particle moving along x-axis is given by

$$U = \left(\frac{x^3}{3} - 4x + 6\right)$$

Here, U is in joule and x in metre. Find position of stable and unstable equilibrium.

2. Force acting on a particle moving along x-axis is as shown in figure. Find points of stable and unstable equilibrium.

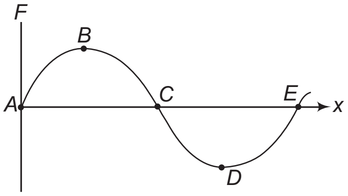


Fig. 9.29

3. Two point charges $+q$ and $+q$ are fixed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third point charge $-q$ is at origin. State whether its equilibrium is stable, unstable or neutral if it is slightly displaced :
(a) along x-axis. (b) along y-axis.

4. Potential energy of a particle along x-axis, varies as, $U = -20 + (x - 2)^2$, where U is in joule and x in meter. Find the equilibrium position and state whether it is stable or unstable equilibrium.

5. Force acting on a particle constrained to move along x-axis is $F = (x - 4)$. Here, F is in newton and x in metre. Find the equilibrium position and state whether it is stable or unstable equilibrium.

9.8 Power of a Force

Power of a force is the rate of work done by this force. Now, power may be of two types:

(i) Instantaneous power (P_i or P) (ii) Average power (P_{av})

Instantaneous Power

The rate of doing work done by a force at a given instant is called instantaneous power of this force. Thus,

$$P = \frac{dW}{dt} = \frac{\mathbf{F}\cdot d\mathbf{r}}{dt} \qquad (\text{as } dW = \mathbf{F}\cdot d\mathbf{r})$$

$$= \mathbf{F}\cdot\mathbf{v} \qquad \left(\text{as } \frac{d\mathbf{r}}{dt} = \mathbf{v}\right)$$

$$= Fv\cos\theta$$

$$\therefore \qquad \boxed{P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = Fv\cos\theta}$$

Here, θ is the angle between **F** and **v**. Hence power of a force is the dot product of this force and instantaneous velocity. If angle between **F** and **v** is acute then dot product is positive (or $\cos\theta$ is positive). So, power is positive. If angle is 90°, then dot product, $\cos\theta$ and hence power are zero. If θ is obtuse, then dot product, $\cos\theta$ and hence power are negative.

Average Power

The ratio of total work done and total time is defined as the average power.

Thus: $$\boxed{P_{av} = \frac{W_{\text{Total}}}{t_{\text{total}}}}$$

Example 9.17 *A ball of mass 1 kg is dropped from a tower. Find power of gravitational force at time $t = 2$ s. Take $g = 10\ m/s^2$.*

Solution At $t = 2$s, velocity of the ball,

$$v = gt = 10 \times 2 = 20\ \text{m/s} \qquad \text{(downwards)}$$

Gravitational force on the ball,

$$F = mg = 1 \times 10 = 10\ \text{N} \qquad \text{(downwards)}$$

∴ Instantaneous power,

$$P = \mathbf{F} \cdot \mathbf{v} = Fv\cos\theta$$
$$= (10)(20)\cos 0°$$
$$= 200\ \text{W}$$

Ans.

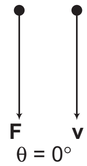


Fig. 9.30

Example 9.18 *A particle of mass m is lying on smooth horizontal table. A constant force F tangential to the surface is applied on it. Find*

(a) average power over a time interval from $t = 0$ to $t = t$,

(b) instantaneous power as function of time t.

Solution (a) $a = \dfrac{F}{m} = \text{constant}$

$$v = at = \frac{F}{m}t \quad \Rightarrow \quad P_{\text{av}} = \frac{W}{t} = \frac{\frac{1}{2}mv^2}{t} = \frac{\left(\frac{1}{2}\right)(m)\left(\frac{Ft}{m}\right)^2}{t}$$

$$= \frac{F^2 t}{2m}$$

Ans.

(b) $P_i = Fv\cos 0° = Fv = (F)\left(\dfrac{Ft}{m}\right) = \dfrac{F^2 t}{m}$ **Ans.**

INTRODUCTORY EXERCISE 9.5

1. A block of mass 1 kg starts moving with constant acceleration $a = 4\ \text{m/s}^2$. Find
   (a) average power of the net force in a time interval from $t = 0$ to $t = 2$ s,
   (b) instantaneous power of the net force at $t = 4$ s.
2. A constant power P is applied on a particle of mass m. Find kinetic energy, velocity and displacement of particle as function of time t.
3. A time varying power $P = 2t$ is applied on a particle of mass m. Find
   (a) kinetic energy and velocity of particle as function of time,
   (b) average power over a time interval from $t = 0$ to $t = t$.

9.9 Law of Conservation of Mechanical Energy

Suppose, only conservative forces are acting on a system of particles and potential energy U is defined corresponding to these forces. There are either no other forces or the work done by them is zero. We have

$$U_f - U_i = -W$$

and $$W = K_f - K_i \quad \text{(from work energy theorem)}$$

then $$U_f - U_i = -(K_f - K_i)$$

or $$U_f + K_f = U_i + K_i \quad \ldots\text{(i)}$$

The sum of the potential energy and the kinetic energy is called the total mechanical energy. We see from Eq. (i), that the **total mechanical energy of a system remains constant, if only conservative forces are acting on a system of particles and the work done by all other forces is zero.** This is called the conservation of mechanical energy.

The total mechanical energy is not constant, if non-conservative forces such as friction is also acting on the system. However, the work energy theorem, is still valid. Thus, we can apply

$$W_c + W_{nc} + W_{\text{ext}} = K_f - K_i$$

Here, $$W_c = -(U_f - U_i)$$

So, we get $$W_{nc} + W_{\text{ext}} = (K_f + U_f) - (K_i + U_i)$$

or $$W_{nc} + W_{\text{ext}} = E_f - E_i = \Delta E$$

Here, $E = K + U$ is the total mechanical energy.

Extra Points to Remember

- Work done by conservative forces is equal to minus of change in potential energy

$$W_c = -\Delta U = -(U_f - U_i) = U_i - U_f$$

- Work done by all the forces is equal to change in kinetic energy.

$$W_{\text{all}} = \Delta K = K_f - K_i$$

- Work done by the forces other than the conservative forces (non-conservative + external forces) is equal to change in mechanical energy

$$W_{nc} + W_{\text{ext}} = \Delta E = E_f - E_i = (K_f + U_f) - (K_i + U_i)$$

- If there are no non-conservative forces, then

$$W_{\text{ext}} = \Delta E = E_f - E_i$$

Further, in this case if no information is given regarding the change in kinetic energy then we can take it zero. In that case,

$$W_{\text{ext}} = \Delta U = U_f - U_i$$

Example 9.19 *A body is displaced from position A to position B. Kinetic and potential energies of the body at positions A and B are*

$$K_A = 50\,J,\ U_A = -30\,J,\ K_B = 10\,J \text{ and } U_B = 20\,J.$$

Find work done by

(a) conservative forces (b) all forces (c) forces other than conservative forces.

Solution (a) $W_c = -\Delta U = -(U_f - U_i)$

$$= U_i - U_f = U_A - U_B$$

$$= -30 - 20 = -50\,\text{J}$$ **Ans.**

(b) $W_{all} = \Delta K = K_f - K_i$

$$= K_B - K_A$$

$$= 10 - 50 = -40\,\text{J}$$ **Ans.**

(c) Work done by the forces other than conservative

$$= \Delta E = E_f - E_i$$

$$= (K_f + U_f) - (K_i + U_i)$$

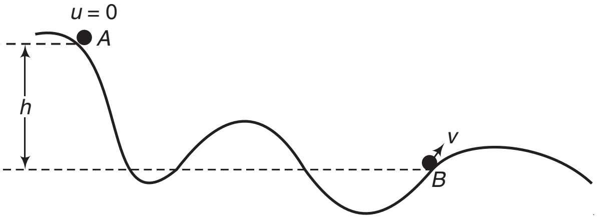
$$= (K_B + U_B) - (K_A + U_A)$$

$$= (10 + 20) - (50 - 30) = 10\,\text{J}$$ **Ans.**

Note *Work done by conservative force is negative* $(= -50\ J)$. *Therefore potential energy should increase and we can see that,* $U_f > U_i$ *as* $U_B > U_A$

Final Touch Points

1. Suppose a particle is released from point A with $u = 0$.



Friction is absent everywhere. Then velocity at B will be

$$v = \sqrt{2gh} \qquad \text{(irrespective of the track it follows from } A \text{ to } B\text{)}$$

Here, $h = h_A - h_B$

2. In circular motion, centripetal force acts towards the centre. This force is perpendicular to small displacement **dS** and velocity **v**. Hence, work done by it is zero and power of this force is also zero.

Solved Examples

TYPED PROBLEMS

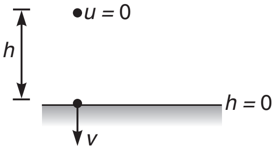
Type 1. *Based on conversation of mechanical energy.*

Concept

If only conservative forces are acting on a system then its mechanical energy remains conserved. Otherwise, if all surfaces are given smooth, then also, mechanical energy will be conserved.

How to Solve?

- If mechanical energy is conserved then it is possible that some part of the energy may be decreasing while the other part may be increasing. Now the energy conservation equation can be written in following two ways :
- **First method** Magnitude of decrease of energy = magnitude of increase of energy.



- **Second method** $E_i = E_f$ ($i \rightarrow$ initial and $f \rightarrow$ final)

  i.e. write down total initial mechanical energy on one side and total final mechanical energy on the other side. While writing gravitational potential energy we choose some reference point (where $h = 0$), but throughout the question this reference point should not change. Let us take a simple example.

  A ball of mass 'm' is released from a height h as shown in figure. The velocity of particle at the instant when it strikes the ground can be found using energy conservation principle by following two methods.
- **Method-1** Decrease in gravitational potential energy = increase in kinetic energy

$\therefore$
$$mgh = \frac{1}{2}mv^2$$

or
$$v = \sqrt{2gh}$$

- **Method-2**

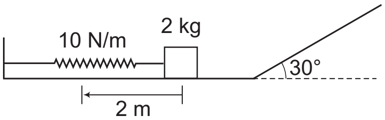
$$E_i = E_f$$

or
$$K_i + U_i = K_f + U_f$$

or
$$0 + mgh = \frac{1}{2}mv^2 + 0$$

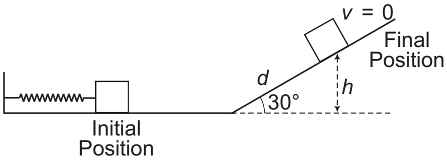
$\Rightarrow$
$$v = \sqrt{2gh}$$

Example 1 *In the figure shown, all surfaces are smooth and force constant of spring is* $10\,N/m$. *Block of mass 2 kg is not attached with the spring. The spring is compressed by 2m and then released. Find the maximum distance 'd' travelled by the block over the inclined plane. Take* $g = 10\,m/s^2$.



Solution In the final position, block will stop for a moment and then it will return back.

In the initial position system has only spring potential energy $\frac{1}{2}kx^2$ and in the final position it has only gravitational potential energy.



Since, all surfaces are smooth, therefore mechanical energy will remain conserved.

$$\Rightarrow \qquad E_i = E_f \quad \text{or} \quad \frac{1}{2}kx^2 = mgh = mg\left(\frac{d}{2}\right)$$

where $$h = d \sin 30° = \frac{d}{2}$$

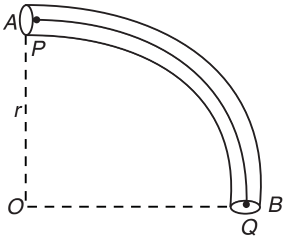
$$\Rightarrow \qquad d = \frac{kx^2}{mg}$$

Substituting the values we have,

$$d = \frac{(10)(2)^2}{(2)(10)}$$

$$= 2\,\text{m}$$ **Ans.**

Example 2 *A smooth narrow tube in the form of an arc AB of a circle of centre O and radius r is fixed so that A is vertically above O and OB is horizontal. Particles P of mass m and Q of mass 2 m with a light inextensible string of length* $(\pi r/2)$ *connecting them are placed inside the tube with P at A and Q at B and released from rest. Assuming the string remains taut during motion, find the speed of particles when P reaches B.*



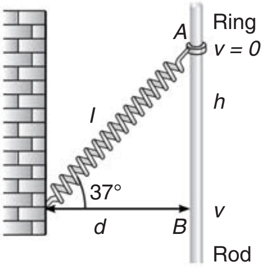
Solution All surfaces are smooth. Therefore, mechanical energy of the system will remain conserved.

$\therefore$ Decrease in PE of both the blocks = increase in KE of both the blocks

$$\therefore \qquad (mgr) + (2mg)\left(\frac{\pi r}{2}\right) = \frac{1}{2}(m + 2m)v^2$$

or $$v = \sqrt{\frac{2}{3}(1 + \pi)gr}$$ **Ans.**

Example 3 *One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially the spring makes an angle of 37° with the horizontal as shown in figure. When the system is released from rest, find the speed of the ring when the spring becomes horizontal.* ($\sin 37° = 3/5$)



Solution If l is the stretched length of the spring, then from figure

$$\frac{d}{l} = \cos 37° = \frac{4}{5}, \quad \text{i.e.} \quad l = \frac{5}{4}d$$

So, the stretch

$$x = l - d = \frac{5}{4}d - d = \frac{d}{4}$$

and

$$h = l \sin 37° = \frac{5}{4}d \times \frac{3}{5} = \frac{3}{4}d$$

Now, taking point B as reference level and applying law of conservation of mechanical energy between A and B,

$$E_A = E_B$$

or

$$mgh + \frac{1}{2}kx^2 = \frac{1}{2}mv^2 \qquad [\text{At } B,\ h = 0 \text{ and } x = 0]$$

or

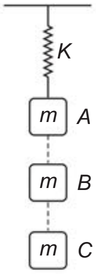
$$\frac{3}{4}mgd + \frac{1}{2}k\left(\frac{d}{4}\right)^2 = \frac{1}{2}mv^2 \qquad \left[\text{as for } A,\ h = \frac{3}{4}d \text{ and } y = \frac{1}{4}d\right]$$

or

$$v = d\sqrt{\frac{3g}{2d} + \frac{k}{16\,m}} \qquad \textbf{Ans.}$$

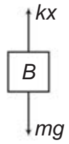
Type 2. *Based on the position of equilibrium and momentary rest.*

Concept



In the figure shown, block attached with the spring is released from rest at natural length of the spring at A.

At this position a constant force mg is acting on the block in downward direction. So, block starts moving downwards and spring is stretched. So, a variable spring force kx starts acting on the block in upward direction which keeps on increasing with extension 'x'. In position B, also called equilibrium position,



$$kx = mg$$

$$\therefore \qquad F_{net} = 0$$

But the block does not stop here. Rather, it has maximum velocity in this position. After crossing B the block retards, as $kx > mg$ and net force is upwards. At point C (the maximum extension) block stops for a moment ($v = 0$) and then it returns back. The block starts oscillating between A and C.

Thus,

From	kx and mg	Direction of net force	Direction of velocity	Speed
A to B	$mg > kx$	Downwards	Downwards	Increasing
at B	$mg = kx$	$F_{net} = 0$	-	Maximum
B to C	$kx > mg$	Upwards	Downwards	Decreasing
C to B	$kx > mg$	Upwards	Upwards	Increasing
B to A	$mg > kx$	Downwards	Upwards	Decreasing

Note (i) *Points A and C are the points of momentary rest, where $v = 0$ but $\mathbf{F}_{net} \neq 0$. So the maximum extension (at point C) can be obtained by energy conservation principle but not by putting $\mathbf{F}_{net} = 0$.*

(ii) *Point B is the point of equilibrium where $\mathbf{F}_{net} = 0$ and speed is maximum. This point can be obtained by putting $\mathbf{F}_{net} = 0$ and after that, maximum speed can be obtained by energy conservation principle.*

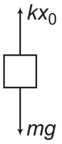
Example 4 *In the figure shown in the concept, find*

(a) Equilibrium extension $x_0 (= AB)$

(b) Maximum extension $x_m (= AC)$

(c) Maximum speed at point B.

Solution (a) At point B,



$$\mathbf{F}_{\text{net}} = 0$$

$$\Rightarrow \quad kx_0 = mg$$

$$\Rightarrow \quad x_0 = \frac{mg}{K}$$ **Ans.**

(b) **From A to C** $(v_A = v_C = 0)$

Decreasing in gravitational potential energy = increasing in spring potential energy.

$$\therefore \quad mgx_m = \frac{1}{2} Kx_m^2 \qquad (AC = x_m)$$

$$\Rightarrow \quad x_m = \frac{2mg}{K}$$ **Ans.**

(c) **From A to B**

Decreasing in gravitational potential energy = increasing in (spring potential energy + kinetic energy)

$$\Rightarrow \quad mg\,x_0 = \frac{1}{2} Kx_0^2 + \frac{1}{2} mv_{\text{max}}^2$$

Substituting the value of $x_0 = \frac{mg}{K}$ in the above equation, we get

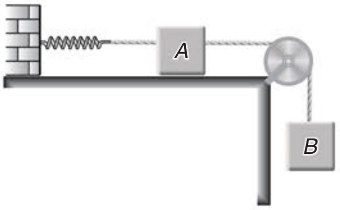
$$v_{\text{max}} = \left(\sqrt{\frac{m}{K}}\right) g$$ **Ans.**

Example 5 *Consider the situation shown in figure. Mass of block A is m and that of block B is 2 m. The force constant of spring is K. Friction is absent everywhere. System is released from rest with the spring unstretched. Find*

(a) the maximum extension of the spring x_m

(b) the speed of block A when the extension in the spring is $x = \frac{x_m}{2}$

(c) net acceleration of block B when extension in the spring is $x = \frac{x_m}{4}$



Solution (a) At maximum extension in the spring

$$v_A = v_B = 0 \qquad \text{(momentarily)}$$

Therefore, applying conservation of mechanical energy:

decreasing in gravitational potential energy of block B = increasing in elastic potential energy of spring.

or
$$m_B g x_m = \frac{1}{2} K x_m^2$$

or
$$2\,mg x_m = \frac{1}{2} K x_m^2$$

∴
$$x_m = \frac{4\,mg}{K} \qquad \textbf{Ans.}$$

(b) At $x = \frac{x_m}{2} = \frac{2\,mg}{K}$

Let
$$v_A = v_B = v \text{ (say)}$$

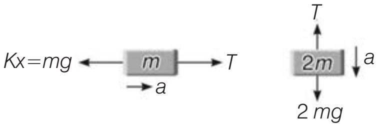
Then, decrease in gravitational potential energy of block B = increase in elastic potential energy of spring + increase in kinetic energy of both the blocks.

∴
$$m_B g x = \frac{1}{2} K x^2 + \frac{1}{2}(m_A + m_B) v^2$$

or
$$(2m)\,(g)\left(\frac{2\,mg}{K}\right) = \frac{1}{2} K \left(\frac{2\,mg}{K}\right)^2 + \frac{1}{2}(m + 2m) v^2$$

∴
$$v = 2g\sqrt{\frac{m}{3K}} \qquad \textbf{Ans.}$$

(c) At $x = \frac{x_m}{4} = \frac{mg}{K}$



or
$$Kx = mg$$

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{2\,mg - mg}{3\,m}$$

$$= \frac{g}{3} \text{ (downwards)} \qquad \textbf{Ans.}$$

Type 3. *Problems with friction*

Concept

Mechanical energy does not remain constant if friction is there and work done by friction is not zero. So, initial mechanical energy is more than the final mechanical energy and the difference goes in the work done against friction.

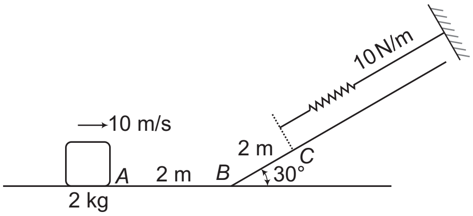
How to Solve?

- The problem can be solved by the following simple equation:

  initial mechanical energy – final mechanical energy if work done against friction.

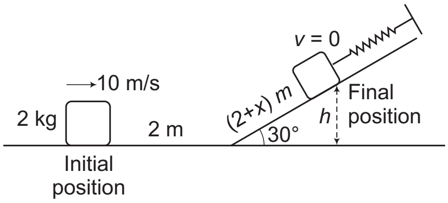
  Here, work done against friction is equal to (μmgd) if the block is moving on horizontal ground and this is equal to ($\mu d\, mg\cos\theta$) if the block is moving on an inclined plane. In these expressions 'd' is the distance travelled over the rough ground (not the displacement). If μ_s and μ_k two coefficients of friction are given, then we will have to take μ_k.

Example 6



In the figure shown, $AB = BC = 2\,\text{m}$. Friction coefficient everywhere is $\mu = 0.2$. Find the maximum compression of the spring.

Solution Let x be the maximum extension.



$$h = (2 + x)\sin 30° = (1 + 0.5\,x)\,m$$

The block has travelled $d_1 = 2\,\text{m}$ on rough horizontal ground and $d_2 = (2 + x)\,\text{m}$ on rough inclined ground. In the initial position block has only kinetic energy and in the final position spring and gravitational potential energy. So, applying the equation.

$$E_i - E_f = \text{work done against friction}$$

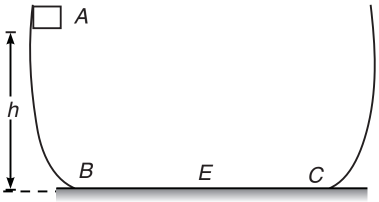
$$\Rightarrow \quad \left(\frac{1}{2}mv^2\right) - \left(\frac{1}{2}kx^2 + mgh\right) = \mu\, mg\, d_1 + (\mu\, mg\cos\theta)\, d_2$$

$$\Rightarrow \frac{1}{2}\times 2\times(10)^2 - \frac{1}{2}\times 10\times x^2 - 2\times 10\times(1 + 0.5\,x) = 0.2\times 2\times 10\times 2 + (0.2\times 2\times 10\times\cos 30°)\,(2 + x)$$

Solving this equation we get,

$$x = 2.45\,\text{m}$$ **Ans.**

Example 7 *A small block slides along a track with elevated ends and a flat central part as shown in figure. The flat portion BC has a length* $l = 3.0$ *m. The curved portions of the track are frictionless. For the flat part, the coefficient of kinetic friction is* $\mu_k = 0.20$, *the particle is released at point A which is at height* $h = 1.5$ *m above the flat part of the track. Where does the block finally comes to rest?*



Solution As initial mechanical energy of the block is mgh and final is zero, so loss in mechanical energy = mgh. This mechanical energy is lost in doing work against friction in the flat part,

So, loss in mechanical energy = work done against friction

or $$mgh = \mu mgd \quad \text{i.e.} \quad d = \frac{h}{\mu} = \frac{1.5}{0.2} = 7.5 \text{ m}$$

After starting from B, the block will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend and at C will have the same value as it had when ascending, but now it will move from C to B. The same will be repeated and finally the block will come to rest at E such that

$$BC + CB + BE = 7.5$$

or $$3 + 3 + BE = 7.5$$

i.e. $$BE = 1.5$$

So, the block comes to rest at the centre of the flat part. **Ans.**

Example 8 *A* 0.5 *kg block slides from the point A on a horizontal track with an initial speed* 3 *m/s towards a weightless horizontal spring of length* 1 *m and force constant* 2 *N/m. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as* 0.22 *and* 0.20 *respectively. If the distances AB and BD are* 2 *m and* 2.14 *m respectively, find the total distance through which the block moves before it comes to rest completely.* ($g = 10$ m/s^2)

(JEE 1983)

Solution As the track AB is frictionless, the block moves this distance without loss in its initial $\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 3^2 = 2.25$ J. In the path BD as friction is present, so work done against friction

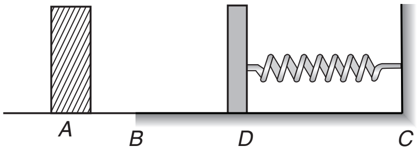
$$= \mu_k mgd = 0.2 \times 0.5 \times 10 \times 2.14 = 2.14 \text{ J}$$

So, at D the KE of the block is $= 2.25 - 2.14 = 0.11$ J

Now, if the spring is compressed by x

$$0.11 = \tfrac{1}{2} \times k \times x^2 + \mu_k mgx$$

i.e. $$0.11 = \tfrac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x$$



or $$x^2 + x - 0.11 = 0$$

which on solving gives positive value of $x = 0.1$ m

After moving the distance $x = 0.1$ m the block comes to rest.

Now the compressed spring exerts a force:

$$F = kx = 2 \times 0.1 = 0.2 \text{ N}$$

on the block while limiting frictional force between block and track is $f_L = \mu_s mg = 0.22 \times 0.5 \times 10 = 1.1$ N. Since, $F < f_L$. The block will not move back. So, the total distance moved by the block

$$= AB + BD + 0.1 = 2 + 2.14 + 0.1$$
$$= 4.24 \text{ m}$$ **Ans.**

Type 4. *Dependent and path independent works*

Concept

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

Here, $d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$

In the following three cases work done is path independent.

(i) **F** is a constant force.

(ii) **F** is of the type $\mathbf{F} = f_1(x)\,\hat{\mathbf{i}} + f_2(y)\,\hat{\mathbf{j}} + f_3(z)\,\hat{\mathbf{k}}$

(iii) $\mathbf{F} \cdot d\mathbf{r}$ is in the form d (function of x, y and z) e.g. $d(xy)$, so that,

$$W_{A \to B} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B d(xy) = [xy]_A^B$$

In all other cases, we will have to mention the path. Along different paths work done will be different.

Example 9 *A body is displaced from origin to (2m, 4m) under the following two forces :*

(a) $\mathbf{F} = (2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}})\,N$, *a constant force*

(b) $\mathbf{F} = (2x\,\hat{\mathbf{i}} + 3y^2\,\hat{\mathbf{j}})\,N$

Find work done by the given forces in both cases.

Solution (a) $\mathbf{F} = (2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}})$ N

$$d\mathbf{r} = (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}})$$

$$\therefore \quad \mathbf{F} \cdot d\mathbf{r} = 2\,dx + 6\,dy$$

$$W = \int_{(0,0)}^{(2\text{m},\,4\text{m})} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(2\text{m},\,4\text{m})} (2\,dx + 6\,dy)$$

$$= [2x + 6y]_{(0,0)}^{(2\text{m},\,4\text{m})} = (2 \times 2 + 6 \times 4)$$

$$= 28 \text{ J}$$ **Ans.**

Note *Here,* **F** *is constant, so the work done is path independent.*

(b) $\mathbf{F} = (2\,x\,\hat{\mathbf{i}} + 3\,y^2\,\hat{\mathbf{j}})$ N

$$d\mathbf{r} = (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}})$$

$$\therefore \quad \mathbf{F} \cdot d\mathbf{r} = (2x\,dx + 3y^2 dy)$$

$$W = \int_{(0,0)}^{(2\text{m},\,4\text{m})} \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(2\text{m},\,4\text{m})} (2x\,dx + 3y^2 dy)$$

$$= [x^2 + y^3]_{(0,0)}^{(2\text{m},\,4\text{m})} = (2)^2 + (4)^3$$

$$= 68\text{ J}$$ **Ans.**

Note *Here the given force is of type* $\mathbf{F} = f_1(x)\,\hat{\mathbf{i}} + f_2(y)\,\hat{\mathbf{j}}$

So, the work done is path independent.

Example 10 *A force* $\mathbf{F} = -k(y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$ *(where k is a positive constant) acts on a particle moving in the x-y plane. Starting from the origin, the particle is taken along the positive x-axis to the point* $(a,0)$ *and then parallel to the y-axis to the point* (a,a). *The total work done by the force* $\mathbf{F}$ *on the particle is* **(JEE 1998)**

(a) $-2ka^2$ (b) $2ka^2$ (c) $-ka^2$ (d) ka^2

Solution $dW = \mathbf{F}\cdot d\mathbf{r}$, where $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$

and $\mathbf{F} = -k(y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$

$\therefore$ $dW = -k(ydx + xdy) = -kd(xy)$

$\therefore$ $W = \int_{(0,0)}^{(a,a)} dW = -k\int_{(0,0)}^{(a,a)} d(xy)$

$= -k[xy]_{(0,0)}^{(a,a)}$

$W = -ka^2$

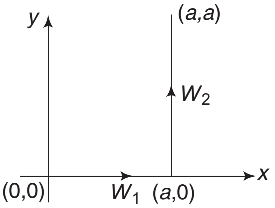
$\therefore$ The correct option is (c).

Alternate Method

While moving from (0, 0) to $(a, 0)$ along positive x-axis, $y = 0$

$\therefore \mathbf{F} = -kx\hat{\mathbf{j}}$ i.e. force is in negative y-direction while the displacement is in positive x-direction.

Therefore, $W_1 = 0$ (Force $\perp$ displacement).



Then, it moves from $(a, 0)$ to (a, a) along a line parallel to y-axis $(x = +a)$. During this

$$\mathbf{F} = -k(y\hat{\mathbf{i}} + a\hat{\mathbf{j}})$$

The first component of force, $-ky\hat{\mathbf{i}}$ will not contribute any work, because this component is along negative x-direction $(-\hat{\mathbf{i}})$ while displacement is in positive y-direction $(a,0)$ to (a, a).

The second component of force i.e. $-ka\hat{\mathbf{j}}$ will perform negative work as :

$$\mathbf{F} = -ka\,\hat{\mathbf{j}} \text{ and } \mathbf{S} = a\,\hat{\mathbf{j}}$$

$\therefore$ $W_2 = \mathbf{F}\cdot\mathbf{S}$

or $W_2 = (-ka)(a) = -ka^2$

$\therefore$ $W = W_1 + W_2 = -ka^2$

Note *For the given force, work done is path independent. It depends only on initial and final positions.*

Example 11 *A body is displaced from origin to* (1m,1m) *by a force* $\mathbf{F} = (2y\,\hat{\mathbf{i}} + 3x^2\,\hat{\mathbf{j}})$ *along two paths*

(a) $x = y$ *(b)* $y = x^2$

Find the work done along both paths.

Solution $\mathbf{F} = (2y\,\hat{\mathbf{i}} + 3x^2\,\hat{\mathbf{j}})$

$$d\mathbf{r} = (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}})$$

$$\mathbf{F}\cdot d\mathbf{r} = (2y\,dx + 3x^2\,dy)$$

We cannot integrate $\mathbf{F}\cdot d\mathbf{r}$ or $(2y\,dx + 3x^2\,dy)$ as such to find the work done. But along the given paths we can change this expression.

(a) Along the path $x = y$,

$$(2y\,dx + 3x^2 dy) = (2x\,dx + 3y^2 dy)$$

$$\therefore \quad W_1 = \int_{(0,0)}^{(1\text{m},1\text{m})} \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(1\text{m},1\text{m})} (2x\,dx + 3y^2 dy)$$

$$= [x^2 + y^3]_{(0,0)}^{(1\text{m},1\text{m})}$$

$$= (1)^2 + (1)^3 = 2\text{ J}$$ **Ans.**

(b) Along the path $y = x^2$

$$(2y\,dx + 3x^2 dy) = (2x^2\,dx + 3y\,dy)$$

$$\therefore \quad W_2 = \int_{(0,0)}^{(1\text{m},1\text{m})} \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(1\text{m},1\text{m})} (2x^2 dx + 3y\,dy)$$

$$= \left[\frac{2}{3}x^3 + \frac{3}{2}y^2\right]_{(0,0)}^{(1\text{m},1\text{m})}$$

$$= \frac{2}{3}(1)^3 + \frac{3}{2}(1)^2$$

$$= \frac{13}{6}\text{ J}$$ **Ans.**

Note *We can see that* $W_1 \neq W_2$ *or work done is path dependent in this case.*

Type 5. *Based on relation between conservative force (F) and potential energy (U) associated with this force.*

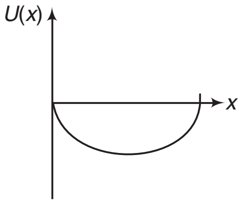
Concept

$$F = -\frac{dU}{dx} \qquad \text{...(i)}$$

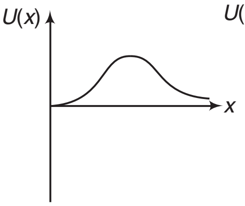
$$\therefore \quad \int dU = -\int F dx \qquad \text{...(ii)}$$

If U-x function is given, we can make F-x function by simple differentiation, using Eq. (i). If F-x function is given, then U-x function can be made by integration, using Eq. (ii). In this case, some limit of U (or value of U at some given value of x) should be known to us to make complete U-x function. Otherwise an unknown constant of integration will be there in U-x equation. If no limit is given in the question and we have to select the most appropriate answer then we can take $U = 0$ at $x = 0$.

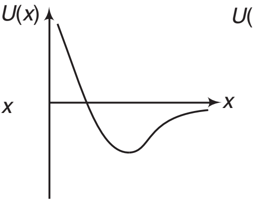
Example 12 *A particle, which is constrained to move along x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here, k and a are positive constants. For $x \geq 0$, the functional form of the potential energy $U(x)$ of the particle is* **(JEE 2002)**



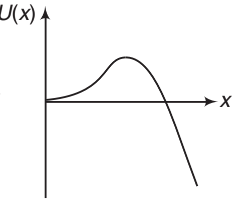
(a)



(b)



(c)



(d)

Solution $F = -\dfrac{dU}{dx}$

$$\therefore \qquad dU = -F \cdot dx \quad \text{or} \quad U(x) = -\int_0^x (-kx + ax^3)\,dx$$

$$U(x) = \frac{kx^2}{2} - \frac{ax^4}{4}$$

$$U(x) = 0 \ \text{ at } \ x = 0 \text{ and } x = \sqrt{\frac{2k}{a}}$$

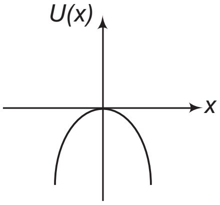
$$U(x) = \text{negative for } x > \sqrt{\frac{2k}{a}}$$

Further, $F = 0$ at $x = 0$. Therefore slope of U-x graph should be zero at $x = 0$.

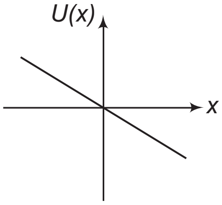
Hence, the correct answer is (d).

Note *In this example, we have assumed a limit : $U = 0$ at $x = 0$.*

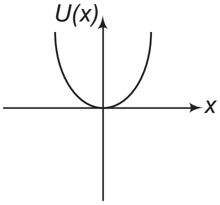
Example 13 *A particle is placed at the origin and a force $F = kx$ is acting on it (where k is a positive constant). If $U(0) = 0$, the graph of $U(x)$ versus x will be (where, U is the potential energy function)* **(JEE 2004)**



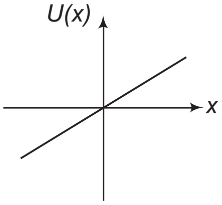
(a)



(b)



(c)



(d)

Solution From $F = -\dfrac{dU}{dx}$

$$\int_0^{U(x)} dU = -\int_0^x F\,dx = -\int_0^x (kx)\,dx$$

$$\therefore \qquad U(x) = -\frac{kx^2}{2} \quad \text{as } U(0) = 0$$

Therefore, the correct option is (a).

Miscellaneous Examples

Example 14 *A small mass m starts from rest and slides down the smooth spherical surface of R. Assume zero potential energy at the top. Find*

(a) the change in potential energy,

(b) the kinetic energy,

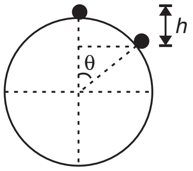
(c) the speed of the mass as a function of the angle θ made by the radius through the mass with the vertical.

Solution In the figure, $h = R(1 - \cos\theta)$

(a) As the mass comes down, potential energy will decrease. Hence,

$$\Delta U = -mgh = -mgR(1 - \cos\theta)$$

(b) Magnitude of decrease in potential energy = increase in kinetic energy



$\therefore$ Kinetic energy $= mgh$

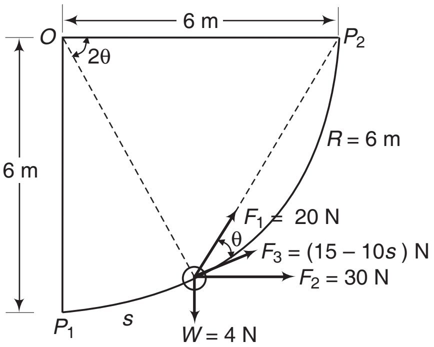
$= mgR(1 - \cos\theta)$ **Ans.**

(c) $$\frac{1}{2}mv^2 = mgR(1 - \cos\theta)$$

$\therefore$ $v = \sqrt{2gR(1 - \cos\theta)}$ **Ans.**

Example 15 *A smooth track in the form of a quarter-circle of radius 6 m lies in the vertical plane. A ring of weight 4 N moves from P_1 and P_2 under the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Force $\mathbf{F}_1$ is always towards P_2 and is always 20 N in magnitude; force $\mathbf{F}_2$ always acts horizontally and is always 30 N in magnitude; force $\mathbf{F}_3$ always acts tangentially to the track and is of magnitude $(15 - 10\,s)$ N, where s is in metre. If the particle has speed 4 m/s at P_1, what will its speed be at P_2?*

Solution The work done by $\mathbf{F}_1$ is



$$W_1 = \int_{P_1}^{P_2} F_1 \cos\theta \, ds$$

From figure, $s = R\left(\frac{\pi}{2} - 2\theta\right)$

or $ds = (6\text{ m})\, d(-2\theta) = -12\, d\theta$

and $F_1 = 20.$

At P_1, $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

At P_2, $2\theta = 0 \Rightarrow \theta = 0$

Hence, $$W_1 = -240 \int_{\pi/4}^{0} \cos\theta \, d\theta$$

$$= 240 \sin\frac{\pi}{4} = 120\sqrt{2}\text{ J}$$

The work done by $\mathbf{F}_3$ is

$$W_3 = \int F_3 \, ds = \int_0^{6(\pi/2)} (15 - 10s)\, ds \qquad \left(P_1P_2 = R\frac{\pi}{2} = \frac{6\pi}{2}\right)$$

$$= [15s - 5s^2]_0^{3\pi} = -302.8\text{ J}$$

To calculate the work done by $\mathbf{F}_2$ and by w, it is convenient to take the projection of the path in the direction of the force. Thus,

$$W_2 = F_2(OP_2) = 30(6) = 180 \text{ J}$$

$$W_4 = (-w)(P_1O) = (-4)(6) = -24 \text{ J} \qquad (w = \text{weight})$$

The total work done is $W_1 + W_3 + W_2 + W_4 = 23$ J

Then, by the work-energy principle.

$$K_{P_2} - K_{P_1} = 23 \text{ J}$$

$$= \frac{1}{2}\left(\frac{4}{9.8}\right)v_2^2 - \frac{1}{2}\left(\frac{4}{9.8}\right)(4)^2 = 23$$

$$v_2 = 11.3 \text{ m/s}$$ **Ans.**

Example 16 *A single conservative force $F(x)$ acts on a 1.0 kg particle that moves along the x-axis. The potential energy $U(x)$ is given by:*

$$U(x) = 20 + (x-2)^2$$

where, x is in meters. At $x = 5.0$ m the particle has a kinetic energy of 20 J.

(a) What is the mechanical energy of the system?

(b) Make a plot of $U(x)$ as a function of x for $-10 \text{ m} \le x \le 10$ m, and on the same graph draw the line that represents the mechanical energy of the system.
Use part (b) to determine.

(c) The least value of x and

(d) The greatest value of x between which the particle can move.

(e) The maximum kinetic energy of the particle and

(f) The value of x at which it occurs.

(g) Determine the equation for $F(x)$ as a function of x.

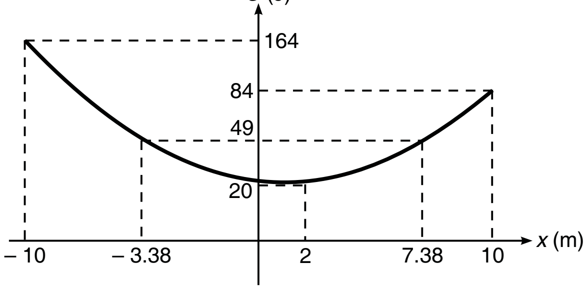
(h) For what (finite) value of x does $F(x) = 0$?

Solution (a) Potential energy at $x = 5.0$ m is

$$U = 20 + (5-2)^2 = 29 \text{ J}$$

∴ Mechanical energy

$$E = K + U = 20 + 29 = 49 \text{ J}$$



(b) At $x = 10$ m, $U = 84$ J at $x = -10$ m, $U = 164$ J
and at $x = 2$ m, $U = \text{minimum} = 20$ J

(c) and (d) Particle will move between the points where its kinetic energy becomes zero or its potential energy is equal to its mechanical energy.

Thus, $49 = 20 + (x-2)^2$

or $(x-2)^2 = 29$

or $x - 2 = \pm\sqrt{29} = \pm 5.38$ m

$\therefore$ $x = 7.38$ m and -3.38 m

or the particle will move between $x = -3.38$ m and $x = 7.38$ m

(e) and (f) Maximum kinetic energy is at $x = 2$ m, where the potential energy is minimum and this maximum kinetic energy is,

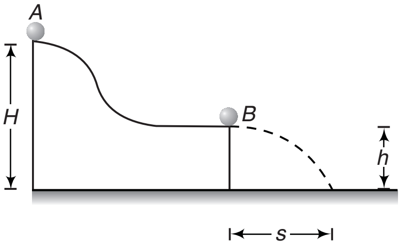
$$K_{\max} = E - U_{\min} = 49 - 20$$
$$= 29 \text{ J}$$

(g) $F = -\dfrac{dU}{dx} = -2(x-2) = 2(2-x)$

(h) $F(x) = 0$, at $x = 2.0$ m

where potential energy is minimum (the position of stable equilibrium).

Example 17 *A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion. What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?*



Solution In order to obtain the velocity at point *B*, we apply the law of conservation of energy. So,

$$\text{Loss in PE} = \text{Gain in KE}$$
$$m\,g\,(H-h) = \frac{1}{2}mv^2$$

$\therefore$ $v = \sqrt{[2g\,(H-h)]}$

Further $h = \dfrac{1}{2}gt^2$

$\therefore$ $t = \sqrt{(2h/g)}$

Now, $s = v \times t = \sqrt{[2g(H-h)]} \times \sqrt{(2\,h/g)}$

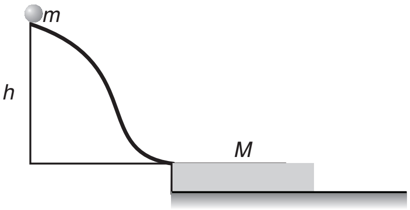
or $s = \sqrt{[4\,h(H-h)]}$ ...(i)

For maximum value of s, $\dfrac{ds}{dh} = 0$

$\therefore$ $\dfrac{1}{2\sqrt{[4\,h(H-h)]}} \times 4(H-2h) = 0$ or $h = \dfrac{H}{2}$ **Ans.**

Substituting $h = H/2$, in Eq. (i), we get

$s = \sqrt{[4(H/2)\,(H - H/2)]} = \sqrt{H^2} = H$ **Ans.**



Example 18 *A small disc of mass m slides down a smooth hill of height h without initial velocity and gets onto a plank of mass M lying on a smooth horizontal plane at the base of hill figure. Due to friction between the disc and the plank, disc slows down and finally moves as one piece with the plank. (a) Find out total work performed by the friction forces in this process. (b) can it be stated that the result obtained does not depend on the choice of the reference frame.*

Solution (a) When the disc slides down and comes onto the plank, then

$$mgh = \frac{1}{2} mv^2$$

$$\therefore \qquad v = \sqrt{(2\,gh)} \qquad \text{...(i)}$$

Let v_1 be the common velocity of both, the disc and plank when they move together. From law of conservation of linear momentum,

$$mv = (M + m)\,v_1$$

$$\therefore \qquad v_1 = \frac{mv}{(M + m)} \qquad \text{...(ii)}$$

Now, $\quad$ change in KE $= (K)_f - (K)_i =$ (work done)$_{\text{friction}}$

$$\therefore \qquad \frac{1}{2}(M + m)v_1^2 - \frac{1}{2} mv^2 = (\text{work done})_{\text{friction}}$$

or

$$W_{\text{fr}} = \frac{1}{2}(M + m)\left[\frac{m\,v}{M + m}\right]^2 - \frac{1}{2} m\,v^2$$

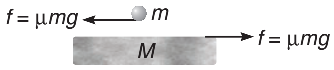
$$= \frac{1}{2} mv^2 \left[\frac{m}{M + m} - 1\right]$$

as

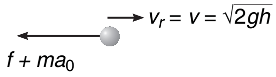
$$\frac{1}{2} mv^2 = mgh$$

$$\therefore \qquad W_{fr} = -mgh\left[\frac{M}{M + m}\right] \qquad \textbf{Ans.}$$

(b) In part (a), we have calculated work done from the ground frame of reference. Now, let us take plank as the reference frame.



Acceleration of plank $a_0 = \dfrac{f}{M} = \dfrac{\mu\, mg}{M}$



Free body diagram of disc with respect to plank is shown in figure.

Here, ma_0 = pseudo force.

$\therefore$ Retardation of disc w.r.t. plank.

$$a_r = \frac{f + ma_0}{m} = \frac{\mu mg + \dfrac{\mu m^2 g}{M}}{m} = \mu g + \frac{\mu mg}{M}$$

$$= \left(\frac{M + m}{M}\right)\mu g$$

The disc will stop after travelling a distance S_r relative to plank, where

$$S_r = \frac{v_r^2}{2a_r} = \frac{Mgh}{(M+m)\mu g} \qquad (0 = v_r^2 - 2a_r S_r)$$

$\therefore$ Work done by friction in this frame of reference

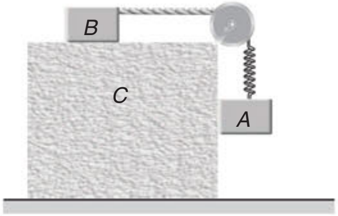
$$W_{fr} = -fS_r = -(\mu mg)\left[\frac{Mgh}{(M+m)\mu g}\right]$$

$$= -\frac{M\,mgh}{(M+m)}$$

which is same as part (a).

Note *Work done by friction in this problem does not depend upon the frame of reference, otherwise in general work depends upon reference frame.*

Example 19 *Two blocks A and B are connected to each other by a string and a spring. The string passes over a frictionless pulley as shown in figure. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2. The force constant of the spring is 1960 Nm^{-1}. If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring. ($g = 9.8\ m/s^2$)*

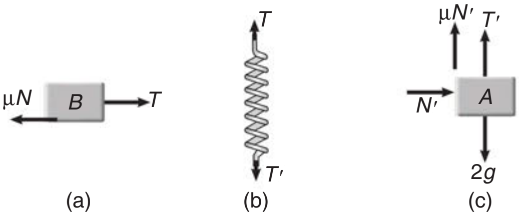


Solution Let m be the mass of B. From its free-body diagram

$$T - \mu N = m \times 0 = 0$$

where, T = tension of the string and $N = mg$

$$\therefore \qquad T = \mu mg$$



(a) (b) (c)

From the free-body diagram of the spring

$$T - T' = 0$$

where, T' is the force exerted by A on the spring or $T = T' = \mu mg$

From the free-body diagram of A $\qquad 2g - (T' + \mu N') = 2 \times 0 = 0$

where, N' is the normal reaction of the vertical wall of C on A and $N' = 2 \times 0$ (as there is no horizontal acceleration of A)

$$\therefore \qquad 2g = T' = \mu mg \quad \text{or} \quad m = \frac{2g}{\mu g} = \frac{2}{0.2} = 10 \text{ kg}$$ **Ans.**

Tensile force on the spring $= T$ or $T' = \mu mg = 0.2 \times 10 \times 9.8 = 19.6$ N

Now, in a spring tensile force = force constant × extension

$$\therefore \qquad 19.6 = 1960\,x \quad \text{or} \quad x = \frac{1}{100}\text{ m} \quad \text{or} \quad U \text{ (energy of a spring)} = \frac{1}{2}kx^2$$

$$= \frac{1}{2} \times 1960 \times \left(\frac{1}{100}\right)^2 = 0.098 \text{ J}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Power of a constant force is also constant.

Reason : Net constant force will always produce a constant acceleration.

2. Assertion : A body is moved from $x = 2$ to $x = 1$, under a force $F = 4x$, the work done by this force is negative.

Reason : Force and displacement are in opposite directions.

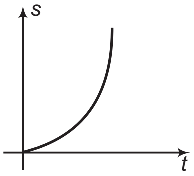
3. Assertion : If work done by conservative forces is positive, kinetic energy will increase.

Reason : Because potential energy will decrease.

4. Assertion : In circular motion work done by all the forces acting on the body is zero.

Reason : Centripetal force and velocity are mutually perpendicular.

5. Assertion : Corresponding to displacement- time graph of a particle moving in a straight line we can say that total work done by all the forces acting on the body is positive.



Reason : Speed of particle is increasing.

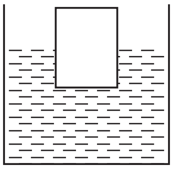
6. Assertion : Work done by a constant force is path independent.

Reason : All constant forces are conservative in nature.

7. Assertion : Work-energy theorem can be applied for non-inertial frames also.

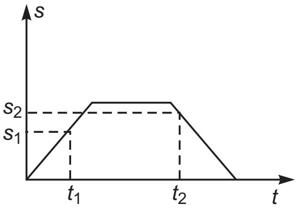
Reason : Earth is a non-inertial frame.

8. Assertion : A wooden block is floating in a liquid as shown in figure. In vertical direction equilibrium of block is stable.



Reason : When depressed in downward direction is starts oscillating.

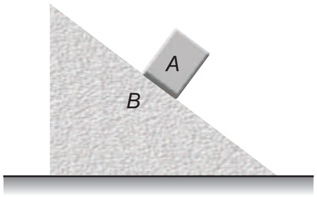
9. Assertion : Displacement-time graph of a particle moving in a straight line is shown in figure. Work done by all the forces between time interval t_1 and t_2 is definitely zero.



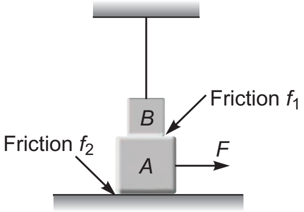
Reason : Work done by all the forces is equal to change in kinetic energy.

10. Assertion : All surfaces shown in figure are smooth. Block *A* comes down along the wedge *B*. Work done by normal reaction (between *A* and *B*) on *B* is positive while on *A* it is negative.

Reason : Angle between normal reaction and net displacement of *A* is greater than 90° while between normal reaction and net displacement of *B* is less than 90°.



11. Assertion : A plank *A* is placed on a rough surface over which a block *B* is placed. In the shown situation, elastic cord is unstretched. Now a gradually increasing force *F* is applied slowly on *A* until the relative motion between the block and plank starts.



At this moment cord is making an angle θ with the vertical. Work done by force *F* is equal to energy lost against friction f_2, plus potential energy stored in the cord.

Reason : Work done by static friction f_1 on the system as a whole is zero.

12. Assertion : A block of mass *m* starts moving on a rough horizontal surface with a velocity *v*. It stops due to friction between the block and the surface after moving through a ceratin distance. The surface is now tilted to an angle of 30° with the horizontal and the same block is made to go up on the surface with the same initial velocity *v*. The decrease in the mechanical energy in the second situation is smaller than that in the first situation.

Reason : The coefficient of friction between the block and the surface decreases with the increase in the angle of inclination.

Objective Questions

Single Correct Option

1. Identify, which of the following energies can be positive (or zero) only?

(a) Kinetic energy

(b) Potential energy

(c) Mechanical energy

(d) Both kinetic and mechanical energies

2. The total work done on a particle is equal to the change in its kinetic energy

(a) always

(b) only if the forces acting on the body are conservative

(c) only in the inertial frame

(d) only if no external force is acting

3. Work done by force of static friction

(a) can be positive (b) can be negative (c) can be zero (d) All of these

4. Work done when a force $\mathbf{F} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ N acting on a particle takes it from the point $\mathbf{r}_1 = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ to the point $\mathbf{r}_2 = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ is

(a) −3 J (b) − 1 J (c) zero (d) 2 J

5. A particle moves along the x-axis from $x = 0$ to $x = 5$ m under the influence of a force given by $F = 7 - 2x + 3x^2$. The work done in the process is

(a) 360 J (b) 85 J (c) 185 J (d) 135 J

6. A particle moves with a velocity $\mathbf{v} = (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})\ \text{ms}^{-1}$ under the influence of a constant force $\mathbf{F} = (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 20\hat{\mathbf{k}})$ N. The instantaneous power applied to the particle is

(a) 200 W (b) 320 W (c) 140 W (d) 170 W

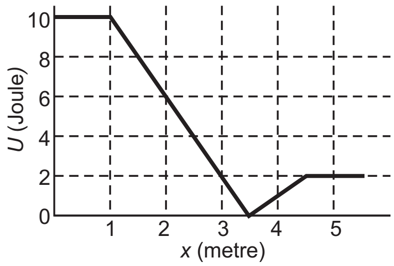
7. A pump is required to lift 800 kg of water per minute from a 10 m deep well and eject it with speed of 20 m/s. The required power in watts of the pump will be

(a) 6000 (b) 4000 (c) 5000 (d) 8000

8. A ball is dropped onto a floor from a height of 10 m. If 20% of its initial energy is lost, then the height of bounce is

(a) 2 m (b) 4 m (c) 8 m (d) 6.4 m

9. A body with mass 1 kg moves in one direction in the presence of a force which is described by the potential energy graph. If the body is released from rest at $x = 2$ m, than its speed when it crosses $x = 5$ m is (Neglect dissipative forces)



(a) $2\sqrt{2}\ \text{ms}^{-1}$ (b) $1\ \text{ms}^{-1}$ (c) $2\ \text{ms}^{-1}$ (d) $3\ \text{ms}^{-1}$

10. A body has kinetic energy E when projected at angle of projection for maximum range. Its kinetic energy at the highest point of its path will be

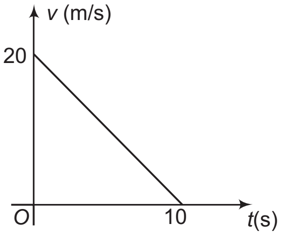
(a) E (b) $\frac{E}{2}$ (c) $\frac{E}{\sqrt{2}}$ (d) zero

11. A person pulls a bucket of water from a well of depth h. If the mass of uniform rope is m and that of the bucket full of water is M, then work done by the person is

(a) $\left(M+\frac{m}{2}\right)gh$ (b) $\frac{1}{2}(M+m)gh$

(c) $(M+m)\,gh$ (d) $\left(\frac{M}{2}+m\right)gh$

12. The velocity of a particle decreases uniformly from 20 ms^{-1} to zero in 10 s as shown in figure. If the mass of the particle is 2 kg, then identify the correct statement.



(a) The net force acting on the particle is opposite to the direction of motion
(b) The work done by friction force is –400 J
(c) The magnitude of friction force acting on the particle is 4 N
(d) All of the above

13. The minimum stopping distance of a car moving with velocity v is x. If the car is moving with velocity $2v$, then the minimum stopping distance will be

(a) $2x$ (b) $4x$
(c) $3x$ (d) $8x$

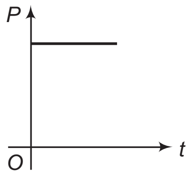
14. A projectile is fired from the origin with a velocity v_0 at an angle θ with the x-axis. The speed of the projectile at an altitude h is

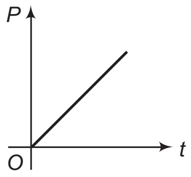
(a) $v_0 \cos\theta$ (b) $\sqrt{v_0^2-2gh}$ (c) $\sqrt{v_0^2\sin^2\theta-2gh}$ (d) None of these

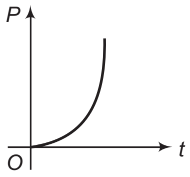
15. A particle of mass m moves from rest under the action of a constant force F which acts for two seconds. The maximum power attained is

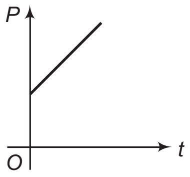
(a) $2Fm$ (b) $\frac{F^2}{m}$ (c) $\frac{2F}{m}$ (d) $\frac{2F^2}{m}$

16. A body moves under the action of a constant force along a straight line. The instantaneous power developed by this force with time t is correctly represented by

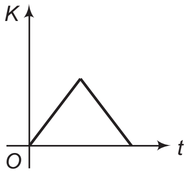
(a)


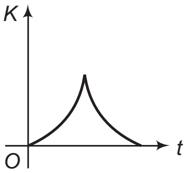
(b)


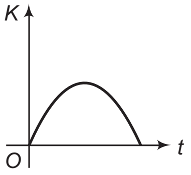
(c)


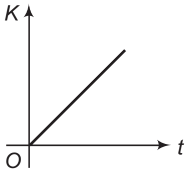
(d)


17. A ball is dropped at $t = 0$ from a height on a smooth elastic surface. Identify the graph which correctly represents the variation of kinetic energy K with time t.

(a)


(b)


(c)


(d)


18. A block of mass 5 kg is raised from the bottom of the lake to a height of 3 m without change in kinetic energy. If the density of the block is 3000 kg m^{-3}, then the work done is equal to

(a) 100 J (b) 150 J (c) 50 J (d) 75 J

19. A body of mass m is projected at an angle θ with the horizontal with an initial velocity u. The average power of gravitational force over the whole time of flight is

(a) $mgu \cos\theta$ (b) $\frac{1}{2} mg \sqrt{u \cos\theta}$ (c) $\frac{1}{2} mgu \sin\theta$ (d) zero

20. A spring of force constant k is cut in two parts at its one-third length. When both the parts are stretched by same amount. The work done in the two parts will be

(a) equal in both (b) greater for the longer part

(c) greater for the shorter part (d) data insufficient

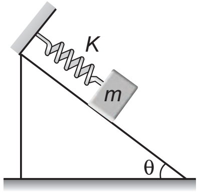
Note *Spring constant of a spring is inversely proportional to length of spring.*

21. A particle moves under the action of a force $\mathbf{F} = 20\hat{\mathbf{i}} + 15\hat{\mathbf{j}}$ along a straight line $3y + \alpha x = 5$, where, α is a constant. If the work done by the force F is zero, then the value of α is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$

(c) 3 (d) 4

22. A system of wedge and block as shown in figure, is released with the spring in its natural length. All surfaces are frictionless. Maximum elongation in the spring will be



(a) $\frac{2\,mg \sin\theta}{K}$ (b) $\frac{mg \sin\theta}{K}$

(c) $\frac{4\,mg \sin\theta}{K}$ (d) $\frac{mg \sin\theta}{2K}$

23. A force $\mathbf{F} = (3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}})$N acts on a body due to which its displacement varies as $\mathbf{S} = (2t^2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}})$ m. Work done by this force in 2 second is

(a) 32 J (b) 24 J

(c) 46 J (d) 20 J

24. An open knife of mass m is dropped from a height h on a wooden floor. If the blade penetrates up to the depth d into the wood, the average resistance offered by the wood to the knife edge is

(a) $mg\left(1 + \frac{h}{d}\right)$ (b) $mg\left(1 + \frac{h}{d}\right)^2$

(c) $mg\left(1 - \frac{h}{d}\right)$ (d) $mg\left(1 + \frac{d}{h}\right)$

25. Two springs have force constants k_A and k_B such that $k_B = 2k_A$. The four ends of the springs are stretched by the same force. If energy stored in spring A is E, then energy stored in spring B is

(a) $\frac{E}{2}$ (b) $2E$ (c) E (d) $4E$

26. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a nearly weightless spring of force constant $k = 50$ N/m. The maximum compression of the spring would be

(a) 0.15 m (b) 0.12 m (c) 0.5 m (d) 0.25 m

27. A bullet moving with a speed of 100 ms^{-1} can just penetrate into two planks of equal thickness. Then the number of such planks, if speed is doubled will be

(a) 6 (b) 10 (c) 4 (d) 8

28. A body of mass 100 g is attached to a hanging spring whose force constant is 10 N/m. The body is lifted until the spring is in its unstretched state and then released. Calculate the speed of the body when it strikes the table 15 cm below the release point

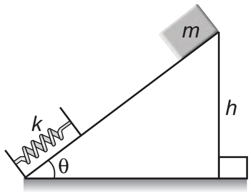
(a) 1 m/s (b) 0.866 m/s (c) 0.225 m/s (d) 1.5 m/s

29. An ideal massless spring S can be compressed 1.0 m in equilibrium by a force of 100 N. This same spring is placed at the bottom of a friction less inclined plane which makes an angle $\theta = 30°$ with the horizontal. A 10 kg mass m is released from the rest at the top of the inclined plane and is brought to rest momentarily after compressing the spring by 2.0 m. The distance through which the mass moved before coming to rest is

(a) 8 m (b) 6 m

(c) 4 m (d) 5 m

30. A body of mass m is released from a height h on a smooth inclined plane that is shown in the figure. The following can be true about the velocity of the block knowing that the wedge is fixed



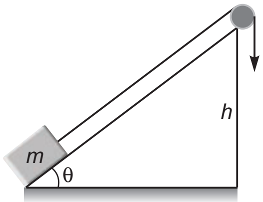
(a) v is highest when it just touches the spring

(b) v is highest when it compresses the spring by some amount

(c) v is highest when the spring comes back to natural position

(d) v is highest at the maximum compression

31. A block of mass m is directly pulled up slowly on a smooth inclined plane of height h and inclination θ with the help of a string parallel to the incline. Which of the following statement is **incorrect** for the block when it moves up from the bottom to the top of the incline?



(a) Work done by the normal reaction force is zero

(b) Work done by the string is mgh

(c) Work done by gravity is mgh

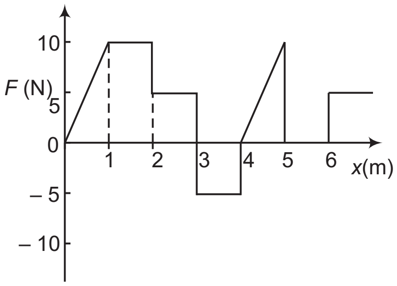
(d) Net work done on the block is zero

32. A spring of natural length l is compressed vertically downward against the floor so that its compressed length becomes $\frac{l}{2}$. On releasing, the spring attains its natural length. If k is the stiffness constant of spring, then the work done by the spring on the floor is

(a) zero (b) $\frac{1}{2}kl^2$

(c) $\frac{1}{2}k\left(\frac{l}{2}\right)^2$ (d) kl^2

33. The relationship between the force F and position x of a body is as shown in figure. The work done in displacing the body from $x = 1$ m to $x = 5$ m will be



(a) 30 J (b) 15 J
(c) 25 J (d) 20 J

34. Under the action of a force, a 2 kg body moves such that its position x as a function of time is given by $x = \frac{t^3}{3}$, where x is in metre and t in second. The work done by the force in the first two seconds is

(a) 1600 J (b) 160 J
(c) 16 J (d) 1.6 J

35. The kinetic energy of a projectile at its highest position is K. If the range of the projectile is four times the height of the projectile, then the initial kinetic energy of the projectile is

(a) $\sqrt{2}K$ (b) $2K$
(c) $4K$ (d) $2\sqrt{2}K$

36. Power applied to a particle varies with time as $P = (3t^2 - 2t + 1)$ watt, where t is in second. Find the change in its kinetic energy between time $t = 2$ s and $t = 4$ s

(a) 32 J (b) 46 J
(c) 61 J (d) 102 J

37. A block of mass 10 kg is moving in x-direction with a constant speed of 10 m/s. It is subjected to a retarding force $F = -0.1x$ J/m during its travel from $x = 20$ m to $x = 30$ m. Its final kinetic energy will be

(a) 475 J (b) 450 J
(c) 275 J (d) 250 J

38. A ball of mass 12 kg and another of mass 6 kg are dropped from a 60 feet tall building. After a fall of 30 feet each, towards earth, their kinetic energies will be in the ratio of

(a) $\sqrt{2} : 1$ (b) $1 : 4$
(c) $2 : 1$ (d) $1 : \sqrt{2}$

39. A spring of spring constant 5×10^3 N/m is stretched initially by 5 cm from the unstretched position. The work required to further stretch the spring by another 5 cm is

(a) 6.25 N-m (b) 12.50 N-m
(c) 18.75 N-m (d) 25.00 N-m

Subjective Questions

1. Momentum of a particle is increased by 50%. By how much percentage kinetic energy of particle will increase ?

2. Kinetic energy of a particle is increased by 1%. By how much percentage momentum of the particle will increase ?

3. Two equal masses are attached to the two ends of a spring of force constant k. The masses are pulled out symmetrically to stretch the spring by a length $2x_0$ over its natural length. Find the work done by the spring on each mass.

4. A rod of length 1.0 m and mass 0.5 kg fixed at one end is initially hanging vertical. The other end is now raised until it makes an angle 60° with the vertical. How much work is required ?

5. A particle is pulled a distance l up a rough plane inclined at an angle α to the horizontal by a string inclined at an angle β to the plane $(\alpha + \beta < 90°)$. If the tension in the string is T, the normal reaction between the particle and the plane is N, the frictional force is F and the weight of the particle is w. Write down expressions for the work done by each of these forces.

6. A chain of mass m and length l lies on a horizontal table. The chain is allowed to slide down gently from the side of the table. Find the speed of the chain at the instant when last link of the chain slides from the table. Neglect friction everywhere.

7. A helicopter lifts a 72 kg astronaut 15 m vertically from the ocean by means of a cable. The acceleration of the astronaut is $\frac{g}{10}$. How much work is done on the astronaut by $(g = 9.8 \text{ m/s}^2)$

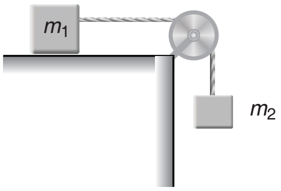
(a) the force from the helicopter and
(b) the gravitational force on her ?
(c) What are the kinetic energy and
(d) the speed of the astronaut just before she reaches the helicopter ?

8. A 1.5 kg block is initially at rest on a horizontal frictionless surface when a horizontal force in the positive direction of x-axis is applied to the block. The force is given by $\mathbf{F}(x) = (2.5 - x^2)\hat{\mathbf{i}}$ N, where, x is in metre and the initial position of the block is $x = 0$.

(a) What is the kinetic energy of the block as it passes through $x = 2.0$ m ?
(b) What is the maximum kinetic energy of the block between $x = 0$ and $x = 2.0$ m ?

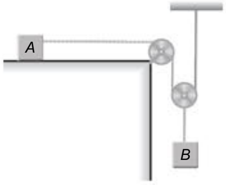
9. A small block of mass 1 kg is kept on a rough inclined wedge of inclination 45° fixed in an elevator. The elevator goes up with a uniform velocity $v = 2$ m/s and the block does not slide on the wedge. Find the work done by the force of friction on the block in 1 s. $(g = 10 \text{ m/s}^2)$

10. Two masses $m_1 = 10$ kg and $m_2 = 5$ kg are connected by an ideal string as shown in the figure. The coefficient of friction between m_1 and the surface is $\mu = 0.2$. Assuming that the system is released from rest. Calculate the velocity of blocks when m_2 has descended by 4 m. $(g = 10 \text{ m/s}^2)$

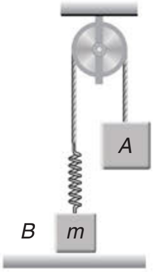


11. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration $a = g$. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of angle θ as it slides down.

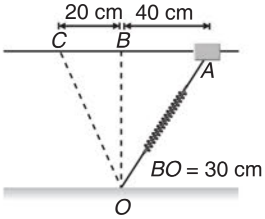
12. In the arrangement shown in figure $m_A = 4.0$ kg and $m_B = 1.0$ kg. The system is released from rest and block B is found to have a speed 0.3 m/s after it has descended through a distance of 1 m. Find the coefficient of friction between the block and the table. Neglect friction elsewhere. (Take $g = 10 \text{ m/s}^2$).



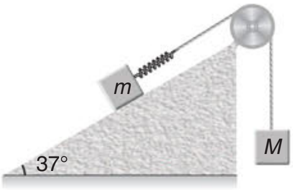
13. In the figure, block A is released from rest when the spring is in its natural length. For the block B of mass m to leave contact with the ground at some stage what should be the minimum mass of block A?



14. As shown in figure a smooth rod is mounted just above a table top. A 10 kg collar, which is able to slide on the rod with negligible friction is fastened to a spring whose other end is attached to a pivot at O. The spring has negligible mass, a relaxed length of 10 cm and a spring constant of 500 N/m. The collar is released from rest at point A. *(a)* What is its velocity as it passes point B? (*b*) Repeat for point C.



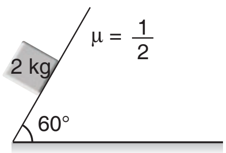
15. A block of mass m is attached with a massless spring of force constant K. The block is placed over a rough inclined surface for which the coefficient of friction is $\mu = \frac{3}{4}$. Find the minimum value of M required to move the block up the plane. (Neglect mass of string and pulley. Ignore friction in pulley).



16. A block of mass 2 kg is released from rest on a rough inclined ground as shown in figure. Find the work done on the block by

(a) gravity, (b) force of friction

when the block is displaced downwards along the plane by 2 m. (Take $g = 10\ \text{m/s}^2$)



17. The potential energy of a two particle system separated by a distance r is given by $U(r) = \frac{A}{r}$, where A is a constant. Find the radial force F_r, that each particle exerts on the other.

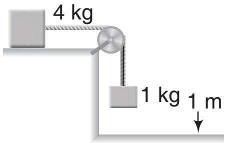
18. A single conservative force F_x acts on a 2 kg particle that moves along the x-axis. The potential energy is given by

$$U = (x - 4)^2 - 16$$

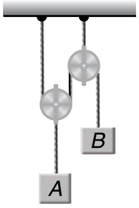
Here, x is in metre and U in joule. At $x = 6.0$ m kinetic energy of particle is 8 J. Find

(a) total mechanical energy
(b) maximum kinetic energy
(c) values of x between which particle moves
(d) the equation of F_x as a function of x
(e) the value of x at which F_x is zero

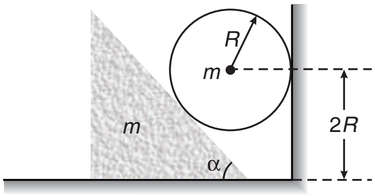
19. A 4 kg block is on a smooth horizontal table. The block is connected to a second block of mass 1 kg by a massless flexible taut cord that passes over a frictionless pulley. The 1 kg block is 1 m above the floor. The two blocks are released from rest. With what speed does the 1 kg block hit the ground ?



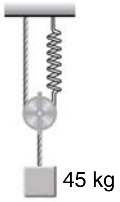
20. Block A has a weight of 300 N and block B has a weight of 50 N. Determine the distance that A must descend from rest before it obtains a speed of 2.5 m/s. Neglect the mass of the cord and pulleys.



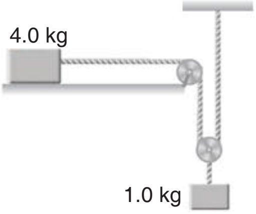
21. A sphere of mass m held at a height $2R$ between a wedge of same mass m and a rigid wall, is released from rest. Assuming that all the surfaces are frictionless. Find the speed of both the bodies when the sphere hits the ground.



22. The system is released from rest with the spring initially stretched 75 mm. Calculate the velocity v of the block after it has dropped 12 mm. The spring has a stiffness of 1050 N/m. Neglect the mass of the small pulley.

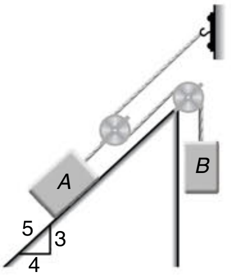


23. Consider the situation shown in figure. The system is released from rest and the block of mass 1 kg is found to have a speed 0.3 m/s after it has descended through a distance of 1 m. Find the coefficient of kinetic friction between the block and the table.

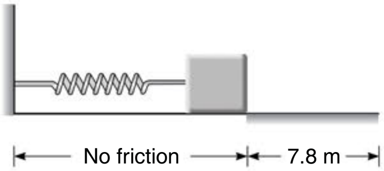


24. A disc of mass 50 g slides with zero initial velocity down an inclined plane set at an angle 30° to the horizontal. Having traversed a distance of 50 cm along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient 0.15 for both inclined and horizontal planes. ($g = 10\,\text{m/s}^2$)

25. Block A has a weight of 300 N and block B has a weight of 50 N. If the coefficient of kinetic friction between the incline and block A is $\mu_k = 0.2$. Determine the speed of block A after it moves 1 m down the plane, starting from rest. Neglect the mass of the cord and pulleys.



26. Figure shows, a 3.5 kg block accelerated by a compressed spring whose spring constant is 640 N/m. After leaving the spring at the spring's relaxed length, the block travels over a horizontal surface, with a coefficient of kinetic friction of 0.25, for a distance of 7.8 m before stopping. ($g = 9.8\,\text{m/s}^2$)



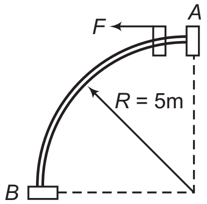
(a) What is the increase in the thermal energy of the block-floor system ?
(b) What is the maximum kinetic energy of the block ?
(c) Through what distance is the spring compressed before the block begins to move ?

LEVEL 2

Objective Questions

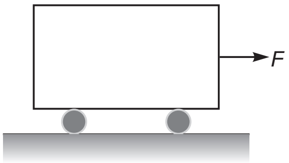
Single Correct Option

1. A bead of mass $\frac{1}{2}$ kg starts from rest from A to move in a vertical plane along a smooth fixed quarter ring of radius 5 m, under the action of a constant horizontal force $F = 5\,\text{N}$ as shown. The speed of bead as it reaches the point B is [Take $g = 10\,\text{ms}^{-2}$]



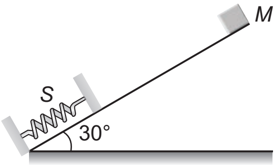
(a) $14.14\,\text{ms}^{-1}$
(b) $7.07\,\text{ms}^{-1}$
(c) $4\,\text{ms}^{-1}$
(d) $25\,\text{ms}^{-1}$

2. A car of mass m is accelerating on a level smooth road under the action of a single force F. The power delivered to the car is constant and equal to P. If the velocity of the car at an instant is v, then after travelling how much distance it becomes double?



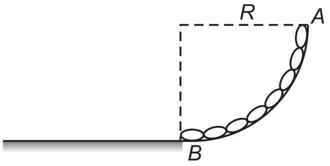
(a) $\frac{7mv^3}{3P}$
(b) $\frac{4mv^3}{3P}$
(c) $\frac{mv^3}{P}$
(d) $\frac{18mv^3}{7P}$

3. An ideal massless spring S can be compressed 1 m by a force of 100 N in equilibrium. The same spring is placed at the bottom of a frictionless inclined plane inclined at 30° to the horizontal. A 10 kg block M is released from rest at the top of the incline and is brought to rest momentarily after compressing the spring by 2 m. If $g = 10\,\text{ms}^{-2}$, what is the speed of mass just before it touches the spring?



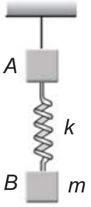
(a) $\sqrt{20}\,\text{ms}^{-1}$
(b) $\sqrt{30}\,\text{ms}^{-1}$
(c) $\sqrt{10}\,\text{ms}^{-1}$
(d) $\sqrt{40}\,\text{ms}^{-1}$

4. A smooth chain AB of mass m rests against a surface in the form of a quarter of a circle of radius R. If it is released from rest, the velocity of the chain after it comes over the horizontal part of the surface is



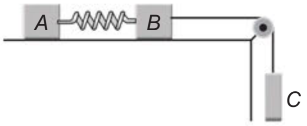
(a) $\sqrt{2gR}$
(b) $\sqrt{gR}$
(c) $\sqrt{2gR\left(1-\frac{2}{\pi}\right)}$
(d) $\sqrt{2gR\,(2-\pi)}$

5. Initially the system shown in figure is in equilibrium. At the moment, the string is cut the downward acceleration of blocks A and B are respectively a_1 and a_2. The magnitudes of a_1 and a_2 are



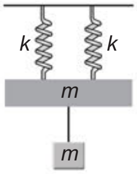
(a) zero and zero
(b) $2g$ and zero
(c) g and zero
(d) None of the above

6. In the diagram shown, the blocks A and B are of the same mass M and the mass of the block C is M_1. Friction is present only under the block A. The whole system is suddenly released from the state of rest. The minimum coefficient of friction to keep the block A in the state of rest is equal to



(a) $\frac{M_1}{M}$ (b) $\frac{2M_1}{M}$ (c) $\frac{M_1}{2M}$ (d) None of these

7. System shown in figure is in equilibrium. Find the magnitude of net change in the string tension between two masses just after, when one of the springs is cut. Mass of both the blocks is same and equal to m and spring constant of both the springs is k

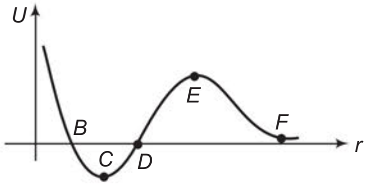


(a) $\frac{mg}{2}$ (b) $\frac{mg}{4}$
(c) $\frac{mg}{3}$ (d) $\frac{3mg}{2}$

8. A body is moving down an inclined plane of slope 37°. The coefficient of friction between the body and the plane varies as $\mu = 0.3\,x$, where x is the distance traveled down the plane by the body. The body will have maximum speed. $\left(\sin 37° = \frac{3}{5}\right)$

(a) at $x = 1.16$ m (b) at $x = 2$ m
(c) at bottommost point of the plane (d) at $x = 2.5$ m

9. The given plot shows the variation of U, the potential energy of interaction between two particles with the distance separating them r.

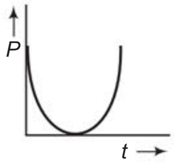


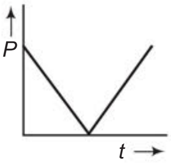
1. B and D are equilibrium points
2. C is a point of stable equilibrium
3. The force of interaction between the two particles is attractive between points C and D and repulsive between D and E
4. The force of interaction between particles is repulsive between points E and F.

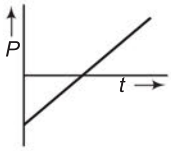
Which of the above statements are correct?

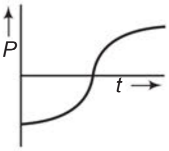
(a) 1 and 2 (b) 1 and 4 (c) 2 and 4 (d) 2 and 3

10. A particle is projected at $t = 0$ from a point on the ground with certain velocity at an angle with the horizontal. The power of gravitation force is plotted against time. Which of the following is the best representation?

(a)


(b)


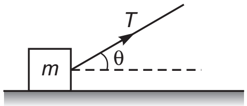
(c)


(d)


11. A block of mass m is attached to one end of a mass less spring of spring constant k. The other end of spring is fixed to a wall. The block can move on a horizontal rough surface. The coefficient of friction between the block and the surface is μ. Then the compression of the spring for which maximum extension of the spring becomes half of maximum compression is

(a) $\frac{2mg\mu}{k}$ (b) $\frac{mg\mu}{k}$ (c) $\frac{4\,mg\mu}{k}$ (d) None of these

12. A block of mass m slides along the track with kinetic friction μ. A man pulls the block through a rope which makes an angle θ with the horizontal as shown in the figure. The block moves with constant speed v. Power delivered by man is



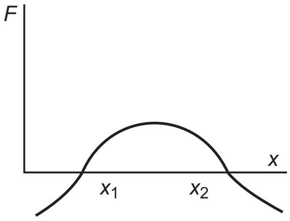
(a) Tv (b) $Tv\cos\theta$

(c) $(T\cos\theta - \mu mg)\,v$ (d) zero

13. The potential energy ϕ in joule of a particle of mass 1 kg moving in x-y plane obeys the law, $\phi = 3x + 4y$. Here, x and y are in metres. If the particle is at rest at (6m, 8m) at time 0, then the work done by conservative force on the particle from the initial position to the instant when it crosses the x-axis is

(a) 25 J (b) – 25 J (c) 50 J (d) – 50 J

14. The force acting on a body moving along x-axis varies with the position of the particle shown in the figure. The body is in stable equilibrium at



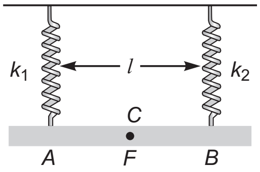
(a) $x = x_1$ (b) $x = x_2$

(c) both x_1 and x_2 (d) neither x_1 nor x_2

15. A small mass slides down an inclined plane of inclination θ with the horizontal. The coefficient of friction is $\mu = \mu_0 x$, where x is the distance through which the mass slides down and μ_0 a positive constant. Then the distance covered by the mass before it stops is

(a) $\frac{2}{\mu_0}\tan\theta$ (b) $\frac{4}{\mu_0}\tan\theta$ (c) $\frac{1}{2\mu_0}\tan\theta$ (d) $\frac{1}{\mu_0}\tan\theta$

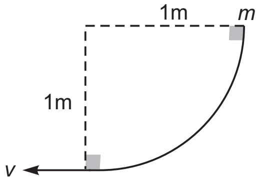
16. Two light vertical springs with spring constants k_1 and k_2 are separated by a distance l. Their upper ends are fixed to the ceiling and their lower ends to the ends A and B of a light horizontal rod AB. A vertical downward force F is applied at point C on the rod. AB will remain horizontal in equilibrium if the distance AC is



(a) $\frac{lk_1}{k_2}$ (b) $\frac{lk_1}{k_2 + k_1}$

(c) $\frac{lk_2}{k_1}$ (d) $\frac{lk_2}{k_1 + k_2}$

17. A block of mass 1 kg slides down a curved track which forms one quadrant of a circle of radius 1 m as shown in figure. The speed of block at the bottom of the track is $v = 2\ \text{ms}^{-1}$. The work done by the force of friction is



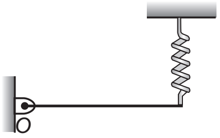
(a) + 4 J (b) − 4 J (c) − 8 J (d) + 8 J

18. The potential energy function for a diatomic molecule is $U(x) = \dfrac{a}{x^{12}} - \dfrac{b}{x^6}$. In stable equilibrium, the distance between the particles is

(a) $\left(\dfrac{2a}{b}\right)^{1/6}$ (b) $\left(\dfrac{a}{b}\right)^{1/6}$

(c) $\left(\dfrac{b}{2a}\right)^{1/6}$ (d) $\left(\dfrac{b}{a}\right)^{1/6}$

19. A rod of mass M hinged at O is kept in equilibrium with a spring of stiffness k as shown in figure. The potential energy stored in the spring is

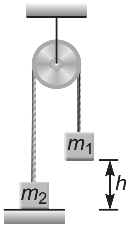


(a) $\dfrac{(mg)^2}{4k}$ (b) $\dfrac{(mg)^2}{2k}$ (c) $\dfrac{(mg)^2}{8k}$ (d) $\dfrac{(mg)^2}{k}$

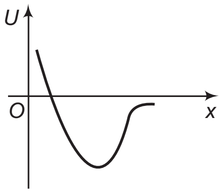
20. In the figure, m_1 and m_2 $(< m_1)$ are joined together by a pulley. When the mass m_1 is released from the height h above the floor, it strikes the floor with a speed

(a) $\sqrt{2gh\left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)}$ (b) $\sqrt{2gh}$

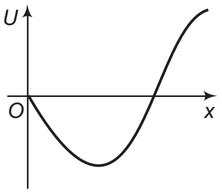
(c) $\sqrt{\dfrac{2m_2gh}{m_1 + m_2}}$ (d) $\sqrt{\dfrac{2m_1gh}{m_1 + m_2}}$



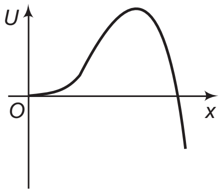
21. A particle free to move along x- axis is acted upon by a force $F = -ax + bx^2$ where a and b are positive constants. For $x \geq 0$, the correct variation of potential energy function $U(x)$ is best represented by



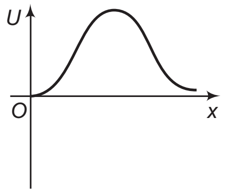
(a)



(b)



(c)



(d)

22. Equal net forces act on two different blocks A and B of masses m and $4m$ respectively. For same displacement, identify the correct statement.

(a) Their kinetic energies are in the ratio $\frac{K_A}{K_B} = \frac{1}{4}$

(b) Their speeds are in the ratio $\frac{v_A}{v_B} = \frac{1}{1}$

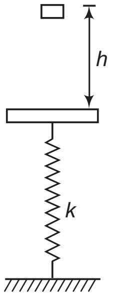
(c) Work done on the blocks are in the ratio $\frac{W_A}{W_B} = \frac{1}{1}$

(d) All of the above

23. The potential energy function of a particle in the x-y plane is given by $U = k(x + y)$, where k is a constant. The work done by the conservative force in moving a particle from (1, 1) to (2, 3) is

(a) $-3k$ (b) $+3k$ (c) k (d) None of these

24. A vertical spring is fixed to one of its end and a massless plank fitted to the other end. A block is released from a height h as shown. Spring is in relaxed position. Then choose the correct statement.



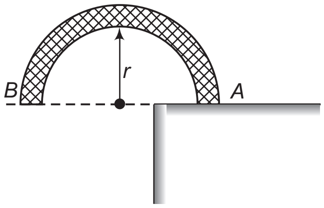
(a) The maximum compression of the spring does not depend on h

(b) The maximum kinetic energy of the block does not depend on h

(c) The compression of the spring at maximum KE of the block does not depend on h

(d) The maximum compression of the spring does not depend on k

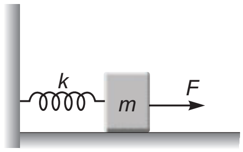
25. A uniform chain of length πr lies inside a smooth semicircular tube AB of radius r. Assuming a slight disturbance to start the chain in motion, the velocity with which it will emerge from the end B of the tube will be



(a) $\sqrt{gr\left(1 + \frac{2}{\pi}\right)}$ (b) $\sqrt{2gr\left(\frac{2}{\pi} + \frac{\pi}{2}\right)}$

(c) $\sqrt{gr\,(\pi + 2)}$ (d) $\sqrt{\pi gr}$

26. A block of mass m is connected to a spring of force constant k. Initially the block is at rest and the spring has natural length. A constant force F is applied horizontally towards right. The maximum speed of the block will be (there is no friction between block and the surface)

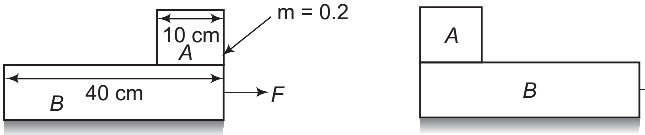


(a) $\frac{F}{\sqrt{2mk}}$ (b) $\frac{F}{\sqrt{mk}}$ (c) $\frac{\sqrt{2}F}{\sqrt{mk}}$ (d) $\frac{2F}{\sqrt{mk}}$

27. Two blocks are connected to an ideal spring of stiffness 200 N/m.At a certain moment, the two blocks are moving in opposite directions with speeds 4 ms^{-1} and 6 ms^{-1}, and the instantaneous elongation of the spring is 10 cm. The rate at which the spring energy $\left(\dfrac{kx^2}{2}\right)$ is increasing is

(a) 500 J/s (b) 400 J/s (c) 200 J/s (d) 100 J/s

28. A block *A* of mass 45 kg is placed on another block *B* of mass 123 kg. Now block *B* is displaced by external agent by 50 cm horizontally towards right. During the same time block *A* just reaches to the left end of block *B*. Initial and final positions are shown in figures. The work done on block *A* in ground frame is



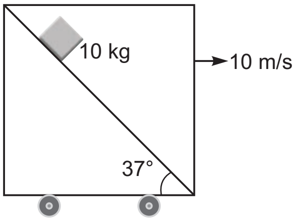
Initial position

Final position

(a) – 18 J (b) 18 J

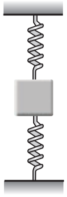
(c) 36 J (d) – 36 J

29. A block of mass 10 kg is released on a fixed wedge inside a cart which is moving with constant velocity 10 ms^{-1} towards right. There is no relative motion between block and cart. Then work done by normal reaction on block in two seconds from ground frame will be ($g = 10\ \text{ms}^{-2}$)



(a) 1320 J (b) 960 J (c) 1200 J (d) 240 J

30. A block tied between two identical springs is in equilibrium. If upper spring is cut, then the acceleration of the block just after cut is 5 ms^{-2}. Now if instead of upper string lower spring is cut, then the acceleration of the block just after the cut will be
(Take $g = 10\ \text{m/s}^2$)



(a) 1.25 ms^{-2} (b) 5 ms^{-2}

(c) 10 ms^{-2} (d) 2.5 ms^{-2}

More than One Correct Options

1. The potential energy of a particle of mass 5 kg moving in *xy*-plane is given as $U = (7x + 24y)$ joule, *x* and *y* being in metre. Initially at $t = 0$, the particle is at the origin (0, 0) moving with a velocity of $(8.6\,\hat{\mathbf{i}} + 23.2\,\hat{\mathbf{j}})\ \text{ms}^{-1}$. Then

(a) The velocity of the particle at $t = 4$ s, is 5 ms^{-1}

(b) The acceleration of the particle is 5 ms^{-2}

(c) The direction of motion of the particle initially (at $t = 0$) is at right angles to the direction of acceleration

(d) The path of the particle is circle

2. The potential energy of a particle is given by formula $U = 100 - 5x + 100x^2$, where U and x are in SI units. If mass of the particle is 0.1 kg then magnitude of it's acceleration

(a) At 0.05 m from the origin is 50 ms^{-2}

(b) At 0.05 m from the mean position is $100\ \text{ms}^{-2}$

(c) At 0.05 m from the origin is $150\ \text{ms}^{-2}$

(d) At 0.05 m from the mean position is $200\ \text{ms}^{-2}$

3. One end of a light spring of spring constant k is fixed to a wall and the other end is tied to a block placed on a smooth horizontal surface. In a displacement, the work done by the spring is $+\left(\frac{1}{2}\right)kx^2$. The possible cases are

(a) The spring was initially compressed by a distance x and was finally in its natural length

(b) It was initially stretched by a distance x and finally was in its natural length

(c) It was initially in its natural length and finally in a compressed position

(d) It was initially in its natural length and finally in a stretched position

4. Identify the correct statement about work energy theorem.

(a) Work done by all the conservative forces is equal to the decrease in potential energy

(b) Work done by all the forces except the conservative forces is equal to the change in mechanical energy

(c) Work done by all the forces is equal to the change in kinetic energy

(d) Work done by all the forces is equal to the change in potential energy

5. A disc of mass $3m$ and a disc of mass m are connected by a massless spring of stiffness k. The heavier disc is placed on the ground with the spring vertical and lighter disc on top. From its equilibrium position the upper disc is pushed down by a distance δ and released. Then

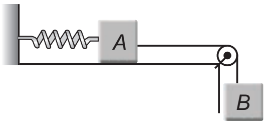
(a) if $\delta > \frac{3\,mg}{k}$, the lower disc will bounce up

(b) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = $6\,mg$

(c) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = $4\,mg$

(d) if $\delta > \frac{4\,mg}{k}$, the lower disc will bounce up

6. In the adjoining figure, block A is of mass m and block B is of mass $2\,m$. The spring has force constant k. All the surfaces are smooth and the system is released from rest with spring unstretched.



(a) The maximum extension of the spring is $\frac{4mg}{k}$

(b) The speed of block A when extension in spring is $\frac{2mg}{k}$, is $2g\sqrt{\frac{2m}{3k}}$

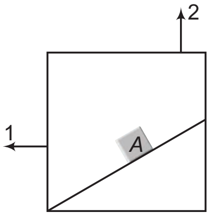
(c) Net acceleration of block B when the extension in the spring is maximum, is $\frac{2}{3}g$

(d) Tension in the thread for extension of $\frac{2mg}{k}$ in spring is mg

7. If kinetic energy of a body is increasing then
(a) work done by conservative forces must be positive
(b) work done by conservative forces may be positive
(c) work done by conservative forces may be zero
(d) work done by non-conservative forces may be zero

8. At two positions kinetic energy and potential energy of a particle are $K_1 = 10\text{ J} : U_1 = -20\text{ J}, K_2 = 20\text{ J}, U_2 = -10\text{ J}$. In moving from 1 to 2
(a) work done by conservative forces is positive
(b) work done by conservative forces is negative
(c) work done by all the forces is positive
(d) work done by all the forces is negative

9. Block A has no relative motion with respect to wedge fixed to the lift as shown in figure during motion-1 or motion-2. Then,
(a) work done by gravity on block A in motion-2 is less than in motion-1
(b) work done by normal reaction on block A in both the motions will be positive
(c) work done by force of friction in motion-1 may be positive
(d) work done by force of friction in motion-1 may be negative

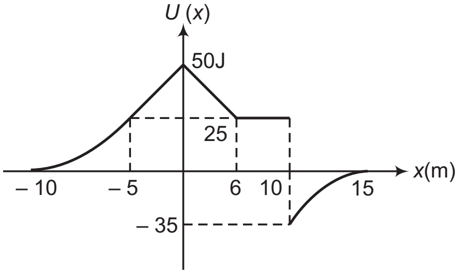


Comprehension Based Questions

Passage (Q. Nos. 1 to 2)

The figure shows the variation of potential energy of a particle as a function of x, the x-coordinate of the region. It has been assumed that potential energy depends only on x. For all other values of x, U is zero, i.e. for $x < -10$ and $x > 15, U = 0$.

Based on above information answer the following questions:



1. If total mechanical energy of the particle is 25 J, then it can be found in the region
(a) $-10 < x < -5$ and $6 < x < 15$
(b) $-10 < x < 0$ and $6 < x < 10$
(c) $-5 < x < 6$
(d) $-10 < x < 10$

2. If total mechanical energy of the particle is -40 J, then it can be found in region
(a) $x < -10$ and $x > 15$
(b) $-10 < x < -5$ and $6 < x < 15$
(c) $10 < x < 15$
(d) It is not possible

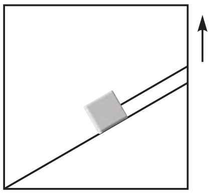
Match the Columns

1. A body is displaced from $x = 4$ m to $x = 2$ m along the x-axis. For the forces mentioned in Column I, match the corresponding work done is Column II.

Column I	Column II
(a) $\mathbf{F} = 4\hat{\mathbf{i}}$	(p) positive
(b) $\mathbf{F} = (4\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$	(q) negative
(c) $\mathbf{F} = -4\hat{\mathbf{i}}$	(r) zero
(d) $\mathbf{F} = (-4\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$	(s) $\lvert W \rvert = 8$ units

2. A block is placed on a rough wedge fixed on a lift as shown in figure. A string is also attached with the block. The whole system moves upwards. Block does not lose contact with wedge on the block. Match the following two columns regarding the work done (on the block).

Column I	Column II
(a) Work done by normal reaction	(p) positive
(b) Work done by gravity	(q) negative
(c) Work done by friction	(r) zero
(d) Work done by tension	(s) Can't say anything

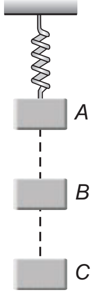


3. Two positive charges $+q$ each are fixed at points $(-a, 0)$ and $(a, 0)$. A third charge $+Q$ is placed at origin. Corresponding to small displacement of $+Q$ in the direction mentioned in Column I, match the corresponding equilibrium of Column II.

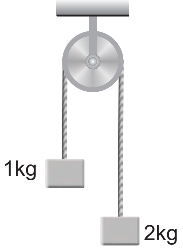
Column I	Column II
(a) Along positive x-axis	(p) stable equilibrium
(b) Along positive y-axis	(q) unstable equilibrium
(c) Along positive z-axis	(r) neutral equilibrium
(d) Along the line $x = y$	(s) no equilibrium

4. A block attached with a spring is released from A. Position-B is the mean position and the block moves to point C. Match the following two columns.

Column I	Column II
(a) From A to B decrease in gravitational potential energy is........ the increase in spring potential energy.	(p) less than
(b) From A to B increase in kinetic energy of block is......... the decrease in gravitational potential energy.	(q) more than
(c) From B to C decrease in kinetic energy of block is..... the increase in spring potential energy.	(r) equal to
(d) From B to C decrease in gravitational potential energy is .... the increase in spring potential energy.	



5. System shown in figure is released from rest. Friction is absent and string is massless. In time $t = 0.3$ s.



Column I	Column II
(a) Work done by gravity on 2 kg block	(p) -1.5 J
(b) Work done by gravity on 1 kg block	(q) 2 J
(c) Work done by string on 2 kg block	(r) 3 J
(d) Work done by string on 1 kg block	(s) -2 J

Take $g = 10 \text{ ms}^{-2}$

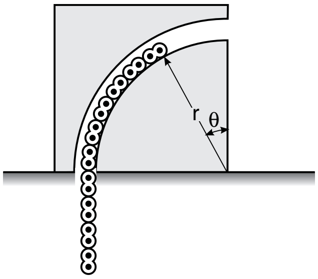
6. In Column I, some statements are given related to work done by a force on an object while in Column II the sign and information about value of work done is given. Match the entries of Column I with the entries of Column II.

Column I	Column II
(a) Work done by friction force on the block as it slides down a rigid fixed incline with respect to ground.	(p) Positive
(b) In above case work done by friction force on incline with respect to ground.	(q) Negative
(c) Work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket with respect to ground.	(r) Zero
(d) Total work done by friction force in (a) with respect to ground.	(s) may be positive, negative or zero.

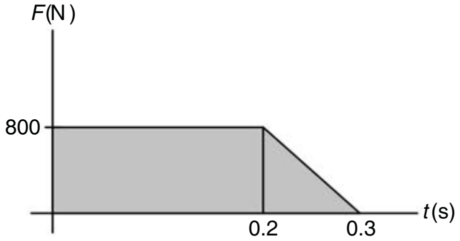
Subjective Questions

1. Two blocks of masses m_1 and m_2 connected by a light spring rest on a horizontal plane. The coefficient of friction between the blocks and the surface is equal to μ. What minimum constant force has to be applied in the horizontal direction to the block of mass m_1 in order to shift the other block?

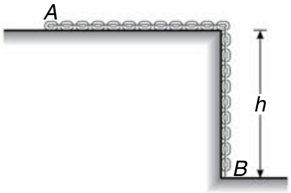
2. The flexible bicycle type chain of length $\frac{\pi r}{2}$ and mass per unit length ρ is released from rest with $\theta = 0°$ in the smooth circular channel and falls through the hole in the supporting surface. Determine the velocity v of the chain as the last link leaves the slot.



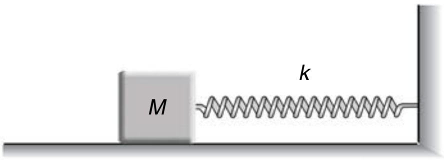
3. A baseball having a mass of 0.4 kg is thrown such that the force acting on it varies with time as shown in the first graph. The corresponding velocity time graph is shown in the second graph. Determine the power applied as a function of time and the work done till $t = 0.3$ s.



4. A chain AB of length l is loaded in a smooth horizontal table so that its fraction of length h hangs freely and touches the surface of the table with its end B. At a certain moment, the end A of the chain is set free. With what velocity will this end of the chain slip out of the table?



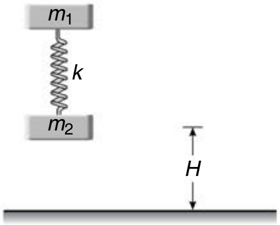
5. The block shown in the figure is acted on by a spring with spring constant k and a weak frictional force of constant magnitude f. The block is pulled a distance x_0 from equilibrium position and then released. It oscillates many times and ultimately comes to rest.



(a) Show that the decrease of amplitude is the same for each cycle of oscillation.

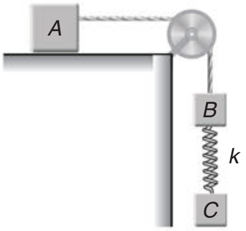
(b) Find the number of cycles the mass oscillates before coming to rest.

6. A spring mass system is held at rest with the spring relaxed at a height H above the ground. Determine the minimum value of H so that the system has a tendency to rebound after hitting the ground. Given that the coefficient of restitution between m_2 and ground is zero.



7. A block of mass m moving at a speed v compresses a spring through a distance x before its speed is halved. Find the spring constant of the spring.

8. In the figure shown masses of the blocks A, B and C are 6 kg, 2 kg and 1 kg respectively. Mass of the spring is negligibly small and its stiffness is 1000 N/m. The coefficient of friction between the block A and the table is $\mu = 0.8$. Initially block C is held such that spring is in relaxed position. The block is released from rest. Find ($g = 10\ \text{m/s}^2$)



(a) the maximum distance moved by the block C.

(b) the acceleration of each block, when elongation in the spring is maximum.

9. A body of mass m slides down a plane inclined at an angle α. The coefficient of friction is μ. Find the rate at which kinetic plus gravitational potential energy is dissipated at any time t.

10. A particle moving in a straight line is acted upon by a force which works at a constant rate and changes its velocity from u and v over a distance x. Prove that the time taken in it is

$$\frac{3}{2}\frac{(u+v)\,x}{u^2+v^2+uv}$$

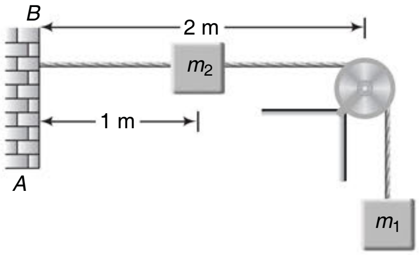
11. A chain of length l and mass m lies on the surface of a smooth sphere of radius $R > l$ with one end tied to the top of the sphere.

(a) Find the gravitational potential energy of the chain with reference level at the centre of the sphere.

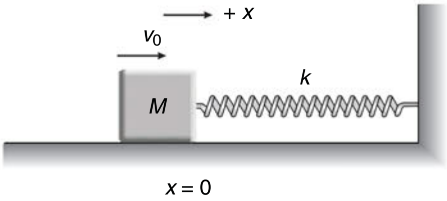
(b) Suppose the chain is released and slides down the sphere. Find the kinetic energy of the chain, when it has slid through an angle θ.

(c) Find the tangential acceleration $\frac{dv}{dt}$ of the chain when the chain starts sliding down.

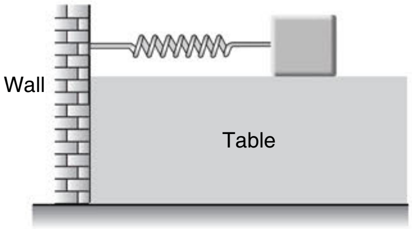
12. Find the speed of both the blocks at the moment the block m_2 hits the wall AB, after the blocks are released from rest. Given that $m_1 = 0.5$ kg and $m_2 = 2$ kg, ($g = 10\ \text{m/s}^2$)



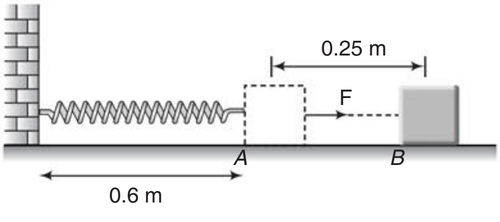
13. A block of mass M slides along a horizontal table with speed v_0. At $x = 0$, it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where b is a positive constant. Find the loss in mechanical energy when the block has first come momentarily to rest.



14. A small block of ice with mass 0.120 kg is placed against a horizontal compressed spring mounted on a horizontal table top that is 1.90 m above the floor. The spring has a force constant $k = 2300$ N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released and the block slides along the table, goes off the edge and travels to the floor. If there is negligible friction between the ice and the table, what is the speed of the block of ice when it reaches the floor. ($g = 9.8$ m/s^2)



15. A 0.500 kg block is attached to a spring with length 0.60 m and force constant $k = 40.0$ N/m. The mass of the spring is negligible. You pull the block to the right along the surface with a constant horizontal force $F = 20.0$ N. (a) What is the block's speed when the block reaches point B, which is 0.25 m to the right of point A? (b) When the block reaches point B, you let go off the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached? Neglect size of block and friction.



Answers

Introductory Exercise 9.1

1. –2 J **2.** (a) 24.9 J (b) zero (c) zero (d) 24.9 J **3.** $Tx, 0, 0, -Fx$ **4.** $-\frac{3}{4}\,mgl$

5. (a) 7.2 J (b) – 7.2 J (c) zero **6.** –12 J **7.** –1 J **8.** 30 J

9. (a) –15 J (b) +15 J (c) 3 J (d) 27 J **10.** (a) 4.0 J (b) zero (c) –1.0 J (d) 3.0 J

Introductory Exercise 9.2

1. – 3.2 J **2.** –400 J **3.** Yes **4.** $\frac{1}{2}\,m\alpha^2 b$

5. 120 J **6.** $v_0\sqrt{\frac{m}{A}}$ **7.** (a) False (b) False (c) True (d) False

8. 32 J **9.** –597.6 J

Introductory Exercise 9.3

1. (a, d) **2.** 40 J

Introductory Exercise 9.4

1. $x = 2$ m is position of stable equilibrium. $x = -2$ m is position of unstable equilibrium.

2. Points A and E are unstable equilibrium positions. Point C is stable equilibrium position.

3. (a) unstable (b) stable **4.** $x = 2$ m, stable **5.** $x = 4$ m, unstable

Introductory Exercise 9.5

1. (a) 16 W (b) 64 W **2.** $K = Pt$, $v = \sqrt{\frac{2Pt}{m}}$, $S = \sqrt{\frac{8P}{9m}}\,t^{3/2}$

3. (a) $K = t^2$, $v = \sqrt{\frac{2}{m}}t$ (b) $P_{av} = t$

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (d) **4.** (d) **5.** (a) **6.** (c) **7.** (b) **8.** (a) **9.** (d) **10.** (a)
11. (a) **12.** (c)

Single Correct Option

1. (a) **2.** (a) **3.** (d) **4.** (b) **5.** (d) **6.** (c) **7.** (b) **8.** (c) **9.** (a) **10.** (b)
11. (a) **12.** (a) **13.** (b) **14.** (b) **15.** (d) **16.** (b) **17.** (b) **18.** (a) **19.** (d) **20.** (c)
21. (d) **22.** (a) **23.** (b) **24.** (a) **25.** (a) **26.** (a) **27.** (d) **28.** (b) **29.** (c) **30.** (b)
31. (c) **32.** (a) **33.** (d) **34.** (c) **35.** (b) **36.** (b) **37.** (a) **38.** (c) **39.** (c)

Subjective Questions

1. 125 % **2.** 0.5% **3.** $-Kx_0^2$ **4.** $W = 1.225$ J **5.** $Tl\cos\beta, 0, -Fl, -Wl\sin\alpha$

6. $\sqrt{gl}$ **7.** (a) 116425 J (b) –10584 J (c) 1058 J (d) 5.42 m/s **8.** (a) 2.33 J (b) 2.635 J

9. 10 J **10.** 4 m/s **11.** $\sqrt{2Rg\,(1+\sin\theta-\cos\theta)}$ **12.** 0.115 **13.** $\frac{m}{2}$

14. (a) 2.45 m/s (b) 2.15 m/s **15.** $\frac{3}{5}$ m **16.** (a) 34.6 J (b) –10 J **17.** $F_r=\frac{A}{r^2}$

18. (a) –4 J (b) 12 J (c) $x=(4-2\sqrt{3})$ m to $x=(4+2\sqrt{3})$ m (d) $F_x=8-2x$ (e) $x=4$ m **19.** 2 m / s

20. 0.796 m **21.** $v_w=\sqrt{2gR}\cos\alpha$, $v_s=\sqrt{2gR}\sin\alpha$

22. $v=0.37\text{ ms}^{-1}$ **23.** $\mu_k=0.12$ **24.** – 0.05 J **25.** 1.12 ms^{-1}

26. (a) 66.88 J (b) 66.88 J (c) 45.7 cm

LEVEL 2

Single Correct Option

1. (a) **2.** (a) **3.** (a) **4.** (c) **5.** (b) **6.** (b) **7.** (a) **8.** (d) **9.** (c) **10.** (c)
11. (c) **12.** (b) **13.** (c) **14.** (b) **15.** (a) **16.** (d) **17.** (c) **18.** (a) **19.** (c) **20.** (a)
21. (c) **22.** (c) **23.** (a) **24.** (c) **25.** (b) **26.** (b) **27.** (c) **28.** (b) **29.** (b) **30.** (b)

More than One Correct Options

1. (a,b) **2.** (a,b,c) **3.** (a,b) **4.** (b,c) **5.** (b,d) **6.** (a) **7.** (b,c, d)
8. (b,c) **9.** (all)

Comprehension Based Questions

1. (a) **2.** (d)

Match the Columns

1. (a) → (q,s) (b) → (q,s) (c) → (p,s) (d) → (p,s)

2. (a) → (p) (b) → (q) (c) → (r) (d) → (p)

3. (a) → (p) (b) → (q) (c) → (q) (d) → (s)

4. (a) → (q) (b) → (p) (c) → (p) (d) → (p)

5. (a) → (r) (b) → (p) (c) → (s) (d) → (q)

6. (a) → (q) (b) → (r) (c) → (p) (d) → (q)

Subjective Questions

1. $\left(m_1+\frac{m_2}{2}\right)\mu g$ **2.** $\sqrt{gr\left(\frac{\pi}{2}+\frac{4}{\pi}\right)}$

3. For $t\le 0.2$ s, $P=(53.3\,t)$ kW, for $t>0.2$ s, $P=(160t-533t^2)$ kW, 1.69 kJ. **4.** $\sqrt{2gh\ln\left(\frac{l}{h}\right)}$

5. (b) $\frac{1}{4}\left[\frac{kx_0}{f}-1\right]$ **6.** $H_{\min}=\frac{m_2 g}{k}\left[\frac{m_2+2m_1}{2m_1}\right]$ **7.** $\frac{3mv^2}{4x^2}$

8. (a) 2×10^{-2}m (b) $a_A=a_B=0$, $a_C=10\text{ m/s}^2$ **9.** $\mu mg^2\cos\alpha(\sin\alpha-\mu\cos\alpha)t$

11. (a) $\frac{mR^2g}{l}\sin\left(\frac{l}{R}\right)$ (b) $\frac{mR^2g}{l}\left[\sin\left(\frac{l}{R}\right)+\sin\theta-\sin\left(\theta+\frac{l}{R}\right)\right]$ (c) $\frac{Rg}{l}\left[1-\cos\left(\frac{l}{R}\right)\right]$

12. $v_1=3.03\text{ ms}^{-1}$; $v_2=3.39\text{ ms}^{-1}$ **13.** $\frac{bgV_0^2M^2}{2(k+bMg)}$

14. 8.72 ms^{-1} **15.** (a) 3.87 ms^{-1} (b) 0.10 m

10

Circular Motion

Chapter Contents

10.1 Introduction
10.2 Kinematics of Circular Motion
10.3 Dynamics of Circular Motion
10.4 Centrifugal Force
10.5 Motion in a Vertical Circle

10.1 Introduction

Circular motion is a two dimensional motion or motion in a plane. This plane may be horizontal, inclined or vertical. But in most of the cases, this plane is horizontal. In circular motion, direction of velocity continuously keeps on changing. Therefore, even though speed is constant and velocity keeps on changing. So body is accelerated. Later we will see that this is a variable acceleration. So, we cannot apply the equations $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ etc. directly.

10.2 Kinematics of Circular Motion

Velocity

In circular motion, a particle has two velocities :

(i) Angular velocity

(ii) Linear velocity

Angular Velocity

Suppose a particle P is moving in a circle of radius r and centre O.

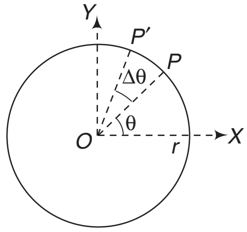


Fig. 10.1

The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle. As the particle moves on the circle its angular position θ changes. Suppose the point rotates an angle $\Delta\theta$ in time Δt. The rate of change of angular position is known as the **angular velocity (ω).** Thus,

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

Here, ω is the angular speed or magnitude of angular velocity. Angular velocity is a vector quantity. Direction of ω is perpendicular to plane of circle and given by screw law.

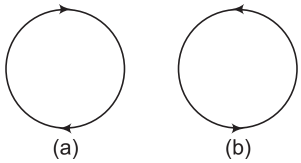

(a) (b)

Fig. 10.2

In the Fig. 10.2 (a), when the particle is rotating clockwise, direction of ω is perpendicular to paper inwards or in $\otimes$ direction.

In Fig. 10.2 (b), when the particle is rotating in anticlockwise direction, direction of ω is perpendicular to paper outwards or in Θ direction.

Linear velocity is as usual, $$\boxed{\mathbf{v} = \frac{d\mathbf{s}}{dt} \quad \text{or} \quad \frac{d\mathbf{r}}{dt}}$$

Magnitude of linear velocity is called linear speed v. Thus,

$$\boxed{v = |\mathbf{v}| = \left|\frac{d\mathbf{s}}{dt} \quad \text{or} \quad \frac{d\mathbf{r}}{dt}\right|}$$

Relation between Linear Speed and Angular Speed

In the Fig. 10.1, linear distance PP' travelled by the particle in time Δt is

$$\Delta s = r\Delta\theta$$

or
$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

or
$$\frac{ds}{dt} = r\frac{d\theta}{dt} \quad \text{or} \quad \boxed{v = r\omega}$$

Acceleration

Like the velocity, a particle in circular motion has two accelerations:

(i) Angular acceleration

(ii) Linear acceleration

The rate of change of angular velocity is called the **angular acceleration (α)**.

Thus,
$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

Angular acceleration is also a vector quantity. Direction of α is also perpendicular to plane of circle, either parallel or antiparallel to ω. If angular speed of the particle is increasing, then α is parallel to ω and if angular speed is decreasing, then α is antiparallel to ω. Angular acceleration is zero if angular speed (or angular velocity) is constant.

In circular motion, linear speed of the particle may or may not be constant but direction of linear velocity continuously keeps on changing. So, velocity is continuously changing. Therefore, acceleration cannot be zero. But of course we can resolve the linear acceleration into two components:

(i) tangential acceleration (a_t)

(ii) radial or centripetal acceleration (a_r)

Component of linear acceleration in tangential direction is called tangential acceleration (a_t). This component is responsible for change in linear speed. This is the rate of change of speed. Thus,

$$\boxed{a_t = \frac{dv}{dt} = \frac{d|\mathbf{v}|}{dt}}$$

If speed of the particle is constant, then a_t is zero. If speed is increasing, then this is positive and in the direction of linear velocity. If speed is decreasing, then this component is negative and in the opposite direction of linear velocity.

Tangential component of the linear acceleration and angular acceleration have following relation:

$$a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt}$$

$$= r\alpha \qquad \left(\text{as } \frac{d\omega}{dt} = \alpha\right)$$

$$\therefore \quad \boxed{a_t = r\alpha}$$

Component of linear acceleration in radial direction (towards centre) is called radial or centripetal acceleration. This component is responsible for change in direction of linear velocity. So, this component can never be zero, as the direction continuously keeps on changing. Value of this component is

$$\boxed{a_r = \frac{v^2}{r} = r\omega^2} \qquad \text{(as } v = r\omega\text{)}$$

These two components are mutually perpendicular. So, the net linear acceleration is the vector sum of these two, as shown in figure.

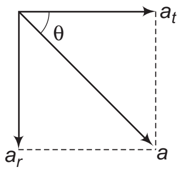


Fig. 10.3

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{r}\right)^2}$$

or

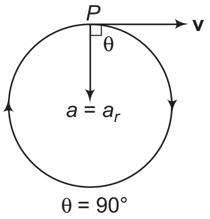
$$a = \sqrt{(r\alpha)^2 + (r\omega^2)^2}$$

and

$$\boxed{\tan\theta = \frac{a_r}{a_t} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a_r}{a_t}\right)}$$

Three Types of Circular Motion

For better understanding, we can classify the circular motion in following three types:



θ = 90°

Fig. 10.4

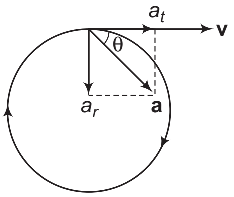
(i) **Uniform circular motion** in which v and ω are constant.

In this motion,

$$v \text{ or } |\mathbf{v}| = \text{ constant } \Rightarrow a_t = 0 \Rightarrow \omega = \text{constant} \Rightarrow \alpha = 0$$

$$a = a_r = \frac{v^2}{r} = r\omega^2$$

a is towards centre, **v** is tangential and according to the shown figure, ω is perpendicular to paper inwards or in ⊗ direction.



θ is acute

Fig. 10.5

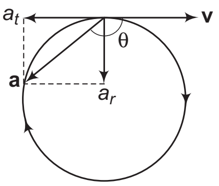
(ii) **Accelerated circular motion** in which v and ω are increasing. So, a_t is in the direction of **v** and α is in the direction of ω.

In the figure shown, α and ω both are perpendicular to paper inwards. Further,

$$a = \sqrt{a_t^2 + a_r^2}$$

and

$$\tan\theta = \frac{a_r}{a_t}$$



θ is obtuse

Fig. 10.6

(iii) **Retarded circular motion** in which v and ω are decreasing. So, a_t is in the opposite direction of **v** and α is in the opposite direction of ω.

In the figure shown, ω is perpendicular to paper inwards in ⊗ direction and α is perpendicular to paper outwards in ⊙ direction.

Note *In the above figures,* θ *is the angle between* **v** *and* **a**.

✓ Extra Points to Remember

- Relation between angular velocity vector $\boldsymbol{\omega}$, velocity vector $\mathbf{v}$ and position vector of the particle with respect to centre $\mathbf{r}$ is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

- In circular motion, if angular acceleration α is constant then we can apply the following equations directly:

$$\omega = \omega_0 + \alpha t \quad \Rightarrow \quad \omega^2 = \omega_0^2 + 2\alpha\theta \text{ and } \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

  Here, ω_0 is the initial angular velocity and ω, the angular velocity at time t. Similarly, θ is the angle rotated by position vector of the particle (with respect to centre).

- If angular acceleration is not constant, then we will have to take help of differentiation or integration. The basic equations are

$$\omega = \frac{d\theta}{dt} \quad \text{and} \quad \alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta}$$

$$\therefore \quad \int d\theta = \int \omega\, dt,\ \int d\omega = \int \alpha\, dt \text{ and } \int \omega\, d\omega = \int \alpha\, d\theta$$

Example 10.1 *A particle moves in a circle of radius* 0.5 *m at a speed that uniformly increases. Find the angular acceleration of particle if its speed changes from* 2.0 *m/s to* 4.0 *m/s in* 4.0 *s.*

Solution The tangential acceleration of the particle is

$$a_t = \frac{dv}{dt} = \frac{4.0 - 2.0}{4.0} = 0.5 \text{ m/s}^2$$

The angular acceleration is $\alpha = \dfrac{a_t}{r} = \dfrac{0.5}{0.5} = 1 \text{ rad/s}^2$ **Ans.**

Example 10.2 *The speed of a particle moving in a circle of radius* $r = 2$ *m varies with time t as* $v = t^2$, *where t is in second and v in m/s. Find the radial, tangential and net acceleration at* $t = 2$ *s.*

Solution Linear speed of particle at $t = 2$ s is

$$v = (2)^2 = 4 \text{ m/s}$$

$\therefore$ Radial acceleration $a_r = \dfrac{v^2}{r} = \dfrac{(4)^2}{2} = 8 \text{ m/s}^2$

The tangential acceleration is $a_t = \dfrac{dv}{dt} = 2t$

$\therefore$ Tangential acceleration at $t = 2$ s is

$$a_t = (2)(2) = 4 \text{ m/s}^2$$

$\therefore$ Net acceleration of particle at $t = 2$s is

$$a = \sqrt{(a_r)^2 + (a_t)^2} = \sqrt{(8)^2 + (4)^2} \quad \text{or} \quad a = \sqrt{80} \text{ m/s}^2$$

Note *On any curved path (not necessarily a circular one) the acceleration of the particle has two components* a_t *and* a_n *in two mutually perpendicular directions. Component of* $\mathbf{a}$ *along* $\mathbf{v}$ *is* a_t *and perpendicular to* $\mathbf{v}$ *is* a_n. *Thus,*

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2}$$

Example 10.3 *In circular motion, what are the possible values (zero, positive or negative) of the following :*

(a) $\boldsymbol{\omega}\cdot\mathbf{v}$ (b) $\mathbf{v}\cdot\mathbf{a}$ (c) $\boldsymbol{\omega}\cdot\boldsymbol{\alpha}$

Solution (a) $\mathbf{v}$ lies in the plane of circle and $\boldsymbol{\omega}$ is always perpendicular to this plane.

$$\therefore \qquad \mathbf{v}\perp\boldsymbol{\omega} \qquad \text{(always)}$$

Hence, $\boldsymbol{\omega}\cdot\mathbf{v}$ is always zero.

(b) $\mathbf{v}$ and $\mathbf{a}$ both lie in the plane of circle and the angle between these two vectors may be acute (when speed is increasing) obtuse (when speed is decreasing) or 90° (when speed is constant).

Hence, $\mathbf{v}\cdot\mathbf{a}$ may be positive, negative or zero.

(c) $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are either parallel ($\theta = 0°$ between $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$) or antiparallel ($\theta = 180°$). In uniform circular motion, $\boldsymbol{\alpha}$ has zero magnitude. Hence, $\boldsymbol{\omega}\cdot\boldsymbol{\alpha}$ may be positive, negative or zero.

INTRODUCTORY EXERCISE 10.1

1. Is the acceleration of a particle in uniform circular motion constant or variable?
2. Which of the following quantities may remain constant during the motion of an object along a curved path?
   (i) Velocity (ii) Speed (iii) Acceleration (iv) Magnitude of acceleration
3. A particle moves in a circle of radius 1.0 cm with a speed given by $v = 2t$, where v is in cm/s and t in seconds.
   (a) Find the radial acceleration of the particle at $t = 1\text{s}$.
   (b) Find the tangential acceleration at $t = 1\text{s}$.
   (c) Find the magnitude of net acceleration at $t = 1\text{s}$.
4. A particle is moving with a constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle rotates an angle $\theta = \left(\frac{\pi}{2}\right)$.
5. A particle is moving with a constant angular acceleration of 4 rad/s^2 in a circular path. At time $t = 0$, particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.
6. A particle rotates in a circular path of radius 54 m with varying speed $v = 4t^2$. Here v is in m/s and t in second. Find angle between velocity and acceleration at $t = 3$ s.
7. Figure shows the total acceleration and velocity of a particle moving clockwise in a circle of radius 2.5 m at a given instant of time. At this instant, find :
   (a) the radial acceleration,
   (b) the speed of the particle and
   (c) its tangential acceleration.

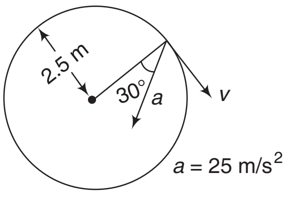


Fig. 10.7

10.3 Dynamics of Circular Motion

In the above article, we have learnt that linear acceleration of a particle in circular motion has two components, tangential and radial (or centripetal). So, normally we resolve the forces acting on the particle in two directions:

(i) tangential
(ii) radial

In tangential direction, net force on the particle is ma_t and in radial direction net force is ma_r.

In uniform circular motion, tangential acceleration is zero. Hence, net force in tangential direction is zero and in radial direction

$$\boxed{F_{\text{net}} = ma_r = \frac{mv^2}{r} = mr\omega^2}$$

as, $$a_r = \frac{v^2}{r} = r\omega^2$$

This net force (towards centre) is also called **centripetal force**.

In most of the cases plane of our uniform circular motion will be horizontal and one of the tangent is in vertical direction also. So, in this case we resolve the forces in:

(i) horizontal radial direction
(ii) vertical tangential direction

In vertical tangential direction net force is zero ($a_t = 0$) and in horizontal radial direction (towards centre) net force is $\frac{mv^2}{r}$ or $mr\omega^2$.

Note *Centripetal force* $\frac{mv^2}{r}$ *or* $mr\omega^2$ *(towards centre) does not act on the particle but this much force is required to the particle for rotating in a circle (as it is accelerated due to change in direction of velocity). The real forces acting on the particle provide this centripetal force or we can say that vector sum of all the forces acting on the particle is equal to* $\frac{mv^2}{r}$ *or* $mr\omega^2$ *(in case of uniform circular motion). The real forces acting on the particle may be, friction force, weight, normal reaction, tension etc.*

Conical Pendulum

If a small particle of mass m tied to a string is whirled in a horizontal circle, as shown in Fig.10.8. The arrangement is called the 'conical pendulum'. In case of conical pendulum, the vertical component of tension balances the weight in tangential direction, while its horizontal component provides the necessary centripetal force in radial direction (towards centre). Thus,

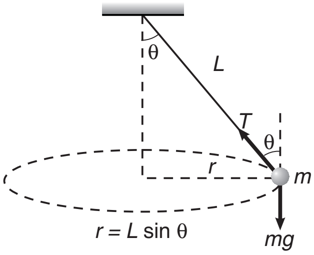


Fig. 10.8

$$T\sin\theta = \frac{mv^2}{r} \qquad \ldots\text{(i)}$$

and $$T\cos\theta = mg \qquad \ldots\text{(ii)}$$

From these two equations, we can find

$$v = \sqrt{rg \tan \theta}$$

$\therefore$ Angular speed $$\omega = \frac{v}{r} = \sqrt{\frac{g \tan \theta}{r}}$$

So, the time period of pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r}{g \tan \theta}} = 2\pi \sqrt{\frac{L \cos \theta}{g}} \qquad (\text{as } r = L \sin \theta)$$

or $$T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

Motion of a Particle Inside a Smooth Cone

A particle of mass 'm' is rotating inside a smooth cone in horizontal circle of radius 'r' as shown in figure. constant speed of the particle is suppose 'v'.

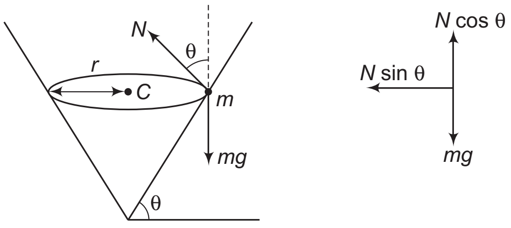


Fig. 10.9

Only two forces are acting on the particle in the shown directions:

(i) normal reaction N

(ii) weight mg

We have resolved these two forces in vertical tangential direction and horizontal radial direction. In vertical tangential direction, net force is zero.

$$\therefore \qquad N \cos \theta = mg$$

In horizontal radial direction (towards centre), net force is $\dfrac{mv^2}{r}$.

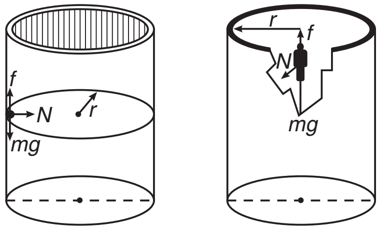
$$\therefore \qquad N \sin \theta = \frac{mv^2}{r}$$

'Death Well' or Rotor

In case of 'death well' a person drives a bicycle on a vertical surface of a large wooden well while in case of a rotor, at a certain angular speed of rotor a person hangs resting against the wall without any support from the bottom. In death well walls are at rest and person revolves while in case of rotor person is at rest and the walls rotate.

In both cases, friction force balances the weight of person while reaction provides the centripetal force for circular motion, i.e.

$$f = mg \quad \text{and} \quad N = \frac{mv^2}{r} = mr\omega^2 \qquad (v = r\omega)$$



(A)
Death well

(B)
Rotor

Fig. 10.10

A Cyclist Bends Towards Centre on a Circular Path

In the figure, F is the resultant of N and f.

$$\therefore \qquad F = \sqrt{N^2 + f^2}$$

When the cyclist is inclined to the centre of the rounding of its path, the resultant of N, f and mg is directed horizontally to the centre of the circular path of the cycle. This resultant force imparts a centripetal acceleration to the cyclist.

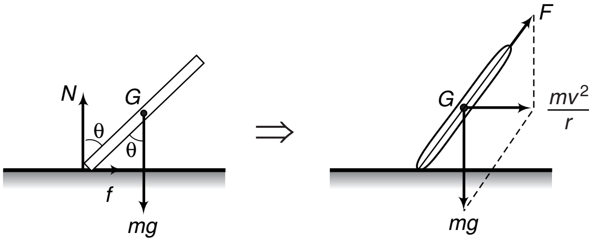


Fig. 10.11

Resultant of N and f, i.e. F should pass through G, the centre of gravity of cyclist (for complete equilibrium, rotational as well as translational). Hence,

$$\tan\theta = \frac{f}{N}, \text{ where } f = \frac{mv^2}{r} \text{ and } N = mg$$

$$\therefore \qquad \tan\theta = \frac{v^2}{rg}$$

Circular Turning of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will provide the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways:

1. By friction only.
2. By banking of roads only.
3. By friction and banking of roads both.

In real life, the necessary centripetal force is provided by friction and banking of roads both. Now, let us write equations of motion in each of the three cases separately and see what are the constraints in each case.

1. By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards centre.

Thus,
$$f = \frac{mv^2}{r}$$

Further, limiting value of f is μN.

or
$$f_L = \mu N = \mu mg \qquad (N = mg)$$

Therefore, for a safe turn without sliding

$$\frac{mv^2}{r} \le f_L \quad \text{or} \quad \frac{mv^2}{r} \le \mu mg$$

or
$$\boxed{\mu \ge \frac{v^2}{rg} \quad \text{or} \quad v \le \sqrt{\mu rg}}$$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

Note *You might have seen that if the speed of the car is too high, car starts skidding outwards. With this, radius of the circle increases or the necessary centripetal force is reduced (centripetal force* $\propto \frac{1}{r}$*).*

2. By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

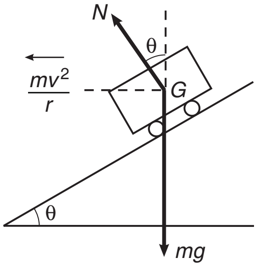


Fig. 10.12

Applying Newton's second law along the radius and the first law in the vertical direction.

$$N \sin\theta = \frac{mv^2}{r} \qquad \text{...(i)}$$

and

$$N \cos\theta = mg \qquad \text{...(ii)}$$

From these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad \boxed{v = \sqrt{rg \tan\theta}}$$

Note *This is the speed at which car does not slide down even if track is smooth. If track is smooth and speed is less than* $\sqrt{rg \tan\theta}$*, vehicle will move down so that r gets decreased and if speed is more than this vehicle will move up.*

3. By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle of these the first force, i.e. weight (mg) is fixed both in magnitude and direction. The direction of second force, i.e. normal reaction N is also fixed (perpendicular to the road) while the direction of the third force, i.e. friction f can be either inwards or outwards, while its magnitude can be varied from zero to a maximum limit ($f_L = \mu N$). So, the magnitude of normal reaction N and direction plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre. Of these m and r are also constant. Therefore, magnitude of N and direction plus magnitude of friction mainly depend on the speed of the vehicle v. Although situation varies from problem to problem yet, we can see that

(i) Friction f is upwards if the vehicle is at rest or $v = 0$.

Because in this case the component of weight $mg \sin\theta$ is balanced by f.

(ii) Friction f is downwards if $v > \sqrt{rg \tan\theta}$

(iii) Friction f is upwards if $v < \sqrt{rg \tan\theta}$

(iv) Friction f is zero if $v = \sqrt{rg \tan\theta}$

Let us now see how the force of friction and normal reaction change as speed is gradually increased.

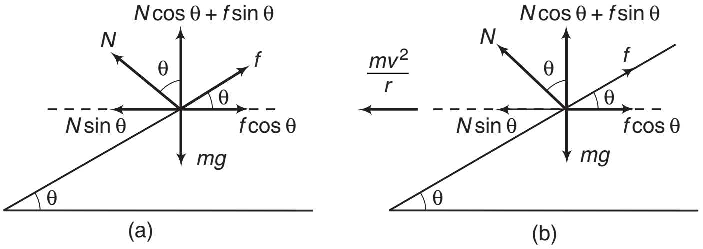


Fig. 10.13

In Fig. (a) When the car is at rest force of friction is upwards. We can resolve the forces in any two mutually perpendicular directions. Let us resolve them in horizontal and vertical directions.

$$\Sigma F_H = 0 \quad \therefore \quad N\sin\theta - f\cos\theta = 0 \qquad \ldots\text{(i)}$$

$$\Sigma F_V = 0 \quad \therefore \quad N\cos\theta + f\sin\theta = mg \qquad \ldots\text{(ii)}$$

In Fig. (b) Now, the car is given a small speed v, so that a centripetal force $\frac{mv^2}{r}$ is now required in horizontal direction towards centre. So, Eq. (i) will now become,

$$N\sin\theta - f\cos\theta = \frac{mv^2}{r}$$

or we can say that in first case $N\sin\theta$ and $f\cos\theta$ are equal while in second case their difference is $\frac{mv^2}{r}$.

This can occur in following three ways:

(i) N increases while f remains same.

(ii) N remains same while f decreases or

(iii) N increases and f decreases.

But only third case is possible, i.e. N will increase and f will decrease. This is because Eq. (ii), $N\cos\theta + f\sin\theta = mg =$ constant is still has to be valid.

So, to keep $N\cos\theta + f\sin\theta$ to be constant N should increase and f should decrease (as $\theta =$ constant).

Now, as speed goes on increasing, force of friction first decreases. Becomes zero at $v = \sqrt{rg\tan\theta}$ and then starts acting in downward direction, so that its horizontal component $f\cos\theta$ with $N\sin\theta$ now provides the required centripetal force.

Example 10.4 *A small block of mass* 100 g *moves with uniform speed in a horizontal circular groove, with vertical side walls of radius* 25 cm. *If the block takes* 2.0 s *to complete one round, find the normal constant force by the side wall of the groove.*

Solution The speed of the block is

$$v = \frac{2\pi \times (25\,\text{cm})}{2.0\text{ s}} = 0.785\text{ m/s}$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0.785\text{ m/s})^2}{0.25\text{m}} = 2.464\text{ m/s}^2$$

towards the centre. The only force in this direction is the normal contact force due to the side walls. Thus, from Newton's second law, this force is

$$N = ma = (0.100\ \text{kg})(2.464\ \text{m/s}^2) = 0.246\ \text{N} \qquad \textbf{Ans.}$$

Example 10.5 *A fighter plane is pulling out for a dive at a speed of* 900 *km/h. Assuming its path to be a vertical circle of radius* 2000 *m and its mass to be* 16000 *kg, find the force exerted by the air on it at the lowest point. Take,* $g = 9.8\ m/s^2$.

Solution At the lowest point in the path, the acceleration is vertically upward (towards the centre) and its magnitude is v^2/r.

The forces on the plane are :

(a) weight Mg downward and

(b) force F by the air upward.

Hence, Newton's second law of motion gives

$$F - Mg = Mv^2/r \quad \text{or} \quad F = M\,(g + v^2/r)$$

Here,
$$v = 900\ \text{km/h} = \frac{9 \times 10^5}{3600}\ \text{m/s} = 250\ \text{m/s}$$

$\therefore$
$$F = 16000\left(9.8 + \frac{62500}{2000}\right)\text{N} = 6.56 \times 10^5\ \text{N} \qquad \text{(upward).}$$

Example 10.6 *Three particles, each of mass m are situated at the vertices of an equilateral triangle of side a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original mutual separation a. Find the initial velocity that should be given to each particle and also the time period of the circular motion.* $\left(F = \dfrac{Gm_1m_2}{r^2}\right)$.

Solution

$$r = \frac{a}{2}\sec 30° = \frac{a}{\sqrt{3}}$$

$$F = \frac{Gmm}{a^2}$$

$$F_{\text{net}} = \sqrt{3}F = \left(\frac{Gmm}{a^2}\right)(\sqrt{3})$$

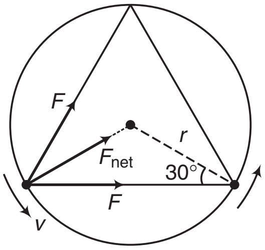


Fig. 10.14

This will provide the necessary centripetal force.

$\therefore$
$$\frac{mv^2}{r} = \frac{\sqrt{3}\,Gm^2}{a^2} \quad \text{or} \quad \frac{mv^2}{(a/\sqrt{3})} = \frac{\sqrt{3}\,Gm^2}{a^2}$$

$\Rightarrow$
$$v = \sqrt{\frac{Gm}{a}} \qquad \textbf{Ans.}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi\,(a/\sqrt{3})}{\sqrt{Gm/a}} = 2\pi\sqrt{\frac{a^3}{3Gm}} \qquad \textbf{Ans.}$$

Example 10.7 *(a) How many revolutions per minute must the apparatus shown in figure make about a vertical axis so that the cord makes an angle of 45° with the vertical?*

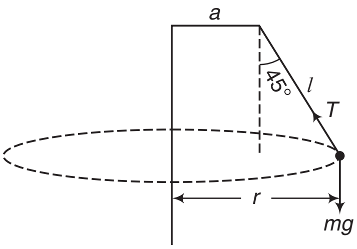


Fig. 10.15

(b) What is the tension in the cord then? Given, $l = \sqrt{2}\ m, a = 20\ cm$ *and* $m = 5.0\ kg$ *?*

Solution (a) $r = a + l \sin 45° = (0.2) + (\sqrt{2})\left(\dfrac{1}{\sqrt{2}}\right) = 1.2 \text{ m}$

Now, $$T \cos 45° = mg \qquad \text{...(i)}$$

and $$T \sin 45° = mr\omega^2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have $\omega = 2n\pi = \sqrt{\dfrac{g}{r}}$

$$\therefore \qquad n = \frac{1}{2\pi}\sqrt{\frac{g}{r}} = \frac{60}{2\pi}\sqrt{\frac{9.8}{1.2}} \text{ rpm } = 27.3 \text{ rpm}$$ **Ans.**

(b) From Eq. (i), we have $T = \sqrt{2}\, mg = (\sqrt{2})(5.0)(9.8)$

$= 69.3$ N **Ans.**

Example 10.8 *A turn of radius* 20 *m is banked for the vehicle of mass* 200 *kg going at a speed of* 10 *m/s. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed*

(a) 5 *m/s*

(b) 15 *m/s.*

Assume that friction is sufficient to prevent slipping. ($g = 10\ m/s^2$)

Solution (a) The turn is banked for speed $v = 10$ m/s

Therefore, $$\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$$

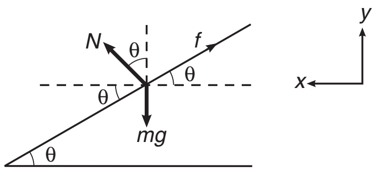


Fig. 10.16

Now, as the speed is decreased, force of friction f acts upwards.

Using the equations $$\Sigma F_x = \frac{mv^2}{r}$$

and $\Sigma F_y = 0$, we get

$$N \sin\theta - f \cos\theta = \frac{mv^2}{r} \qquad \text{...(i)}$$

$$N\cos\theta + f\sin\theta = mg \qquad \text{...(ii)}$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 5$ m/s, $m = 200$ kg and $r = 20$ m, in the above equations,

we get $$f = 300\sqrt{5}\text{ N} \qquad \text{(upwards)}$$

(b) In the second case force of friction f will act downwards.

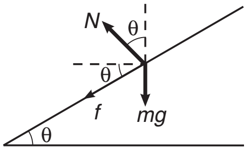


Fig. 10.17

Using $$\Sigma F_x = \frac{mv^2}{r}$$

and $$\Sigma F_y = 0, \quad \text{we get}$$

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{...(iii)}$$

$$N\cos\theta - f\sin\theta = mg \qquad \text{...(iv)}$$

Substituting $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 15$ m/s, $m = 200$ kg

and $r = 20$ m in the above equations, we get

$$f = 500\sqrt{5}\text{ N} \qquad \text{(downwards)}$$

INTRODUCTORY EXERCISE 10.2

1. A turn has a radius of 10 m. If a vehicle goes round it at an average speed of 18 km/h, what should be the proper angle of banking?
2. If the road of the previous problem is horizontal (no banking), what should be the minimum friction coefficient so that a scooter going at 18 km/h does not skid?
3. A circular road of radius 50 m has the angle of banking equal to 30°. At what speed should a vehicle go on this road so that the friction is not used?
4. Is a body in uniform circular motion in equilibrium?
5. A car driver going at speed v suddenly finds a wide wall at a distance r. Should he apply brakes or turn the car in a circle of radius r to avoid hitting the wall.
6. A 4 kg block is attached to a vertical rod by means of two strings of equal length. When the system rotates about the axis of the rod, the strings are extended as shown in figure.
   (a) How many revolutions per minute must the system make in order for the tension in the upper string to be 200 N?
   (b) What is the tension in the lower string then?

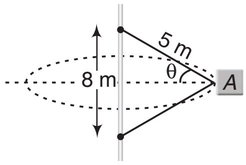


Fig. 10.18

7. A car moves at a constant speed on a straight but hilly road. One section has a crest and dip of the same 250 m radius.
   (a) As the car passes over the crest the normal force on the car is one half the 16 kN weight of the car. What will be the normal force on the car as its passes through the bottom of the dip?
   (b) What is the greatest speed at which the car can move without leaving the road at the top of the hill?
   (c) Moving at a speed found in part (b) what will be the normal force on the car as it moves through the bottom of the dip? (Take, $g = 10\text{ m/s}^2$)

10.4 Centrifugal Force

Newton's laws are valid only in inertial frames. In non-inertial frames a pseudo force $-m\mathbf{a}$ has to be applied. ($\mathbf{a}$ = acceleration of frame of reference). After applying the pseudo force one can apply Newton's laws in their usual form. Now, suppose a frame of reference is rotating with constant angular velocity ω in a circle of radius 'r'. Then, it will become a non-inertial frame of acceleration $r\omega^2$ towards the centre. Now, if we observe an object of mass 'm' from this frame then a pseudo force of magnitude $mr\omega^2$ will have to be applied to this object in a direction away from the centre. This pseudo force is called the centrifugal force. After applying this force we can now apply Newton's laws in their usual form. Following examples will illustrate the concept more clearly:

Example 10.9 *A particle of mass m is placed over a horizontal circular table rotating with an angular velocity ω about a vertical axis passing through its centre. The distance of the object from the axis is r. Find the force of friction f between the particle and the table.*

Solution Let us solve this problem from both frames. The one is a frame fixed on ground and the other is a frame fixed on table itself.

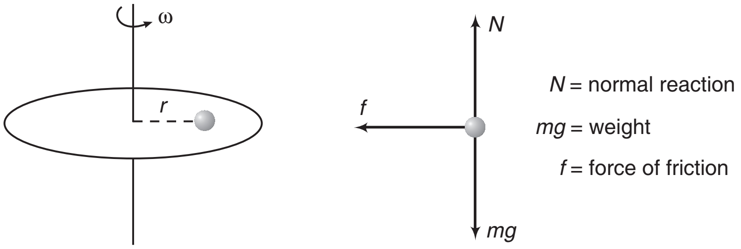


Fig. 10.19

From Frame of Reference Fixed on Ground (Inertial)

Here, N will balance its weight and the force of friction f will provide the necessary centripetal force.

Thus, $$f = mr\omega^2$$ **Ans.**

From Frame of Reference Fixed on Table Itself (Non-inertial)

In the free body diagram of particle with respect to table, in addition to above three forces (N, mg and f) a pseudo force of magnitude $mr\omega^2$ will have to be applied in a direction away from the centre. But one thing should be clear that in this frame the particle is in equilibrium, i.e. N will balance its weight in vertical direction while f will balance the pseudo force in horizontal direction.

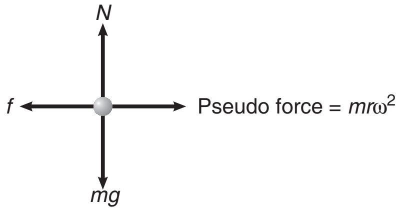


Fig. 10.20

or $$f = mr\omega^2$$ **Ans.**

Thus, we see that f comes out to be $mr\omega^2$ from both the frames.

Example 10.10 *Two blocks A and B of masses 1 kg and 3 kg are attached with two massless strings as shown in figure. The system is kept over a smooth table and it is rotated about the axis shown in figure with constant angular speed* $\omega = 2\,rad/s$.

Find direction and magnitude of centrifugal force on

(a) A as observed by B

(b) B as observed by A

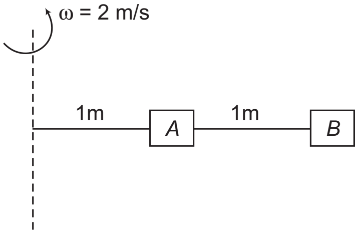


Fig. 10.21

Solution (a) Acceleration of B,

$$a_B = r_B\omega^2 = (2)(2)^2 = 8\text{ m/s}^2 \qquad \text{(towards centre)}$$

Mass of A $\qquad m_A = 1\text{ kg}$.

$\therefore$ Centrifugal force (or pseudo force) on A,

$$F_A = m_A a_B = (1)(8) = 8\text{N}$$ **Ans.**

Direction of this force is in the opposite direction of $\mathbf{a}_B$. Therefore, direction of $\mathbf{F}_A$ is radially outwards.

(b) Acceleration of A, $\qquad a_A = r_A\omega^2 = (1)(2)^2 = 4\text{ m/s}^2 \qquad$ (towards centre)

Mass of B, $\qquad m_B = 3\text{ kg}$

$\therefore$ Centrifugal force on B,

$$F_B = m_B a_A = (3)(4) = 12\text{N}$$ **Ans.**

Direction of $\mathbf{F}_B$ is also radially outwards.

10.5 Motion in a Vertical Circle

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point B of the circle as shown in figure. Here,

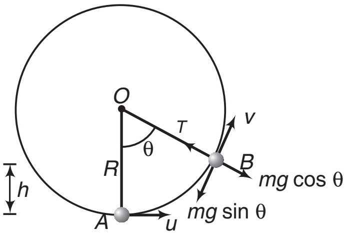


Fig. 10.22

$$h = R(1 - \cos\theta) \qquad \text{...(i)}$$

From conservation of mechanical energy

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

or

$$v^2 = u^2 - 2gh \qquad \text{...(ii)}$$

The necessary centripetal force is provided by the resultant of tension T and $mg\cos\theta$

$$\therefore \qquad T - mg\cos\theta = \frac{mv^2}{R} \qquad \text{...(iii)}$$

Now, following three conditions arise depending on the value of u.

Condition of Looping the Loop ($u \geq \sqrt{5gR}$)

The particle will complete the circle if the string does not slack even at the highest point ($\theta = \pi$). Thus, tension in the string should be greater than or equal to zero ($T \geq 0$) at $\theta = \pi$. In critical case substituting $T = 0$ and $\theta = \pi$ in Eq. (iii), we get

$$mg = \frac{mv_{\min}^2}{R} \quad \text{or} \quad v_{\min}^2 = gR \quad \text{or} \quad v_{\min} = \sqrt{gR} \qquad \text{(at highest point)}$$

Substituting $\theta = \pi$ in Eq. (i), $\qquad h = 2R$

Therefore, from Eq. (ii), we have

$$u_{\min}^2 = v_{\min}^2 + 2gh$$

or
$$u_{\min}^2 = gR + 2g(2R) = 5\,gR$$

or
$$u_{\min} = \sqrt{5\,gR}$$

Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At $u = \sqrt{5gR}$, velocity at highest point is $v = \sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0°$ and $v = \sqrt{5gR}$ in Eq. (iii), we get $T = 6\,mg$ or in the critical condition tension in the string at lowest position is 6 mg. This is shown in Fig. 10.23.

If $u < \sqrt{5gR}$, following two cases are possible

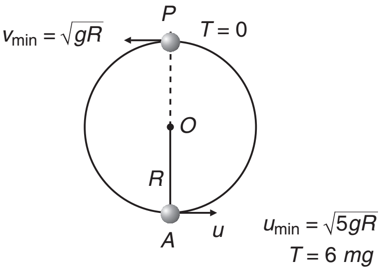


Fig. 10.23

Condition of Leaving the Circle ($\sqrt{2gR} < u < \sqrt{5gR}$)

If $u < \sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From Eq. (iii), tension in the string becomes zero ($T = 0$)

where, $$\cos\theta = \frac{-v^2}{Rg} \quad \text{or} \quad \cos\theta = \frac{2gh - u^2}{Rg}$$

Substituting this value of $\cos\theta$ in Eq. (i), we get

$$\frac{2gh - u^2}{Rg} = 1 - \frac{h}{R} \quad \text{or} \quad h = \frac{u^2 + Rg}{3g} = h_1 \text{ (say)} \qquad \text{...(iv)}$$

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle becomes zero when

$$0 = u^2 - 2gh \quad \text{or} \quad h = \frac{u^2}{2g} = h_2 \text{ (say)} \qquad \text{...(v)}$$

i.e. at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero or $T = 0$ but $v \neq 0$. This is possible only when

$$h_1 < h_2 \quad \text{or} \quad \frac{u^2 + Rg}{3g} < \frac{u^2}{2g}$$

or $$2u^2 + 2Rg < 3u^2 \quad \text{or} \quad u^2 > 2Rg \quad \text{or} \quad u > \sqrt{2Rg}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

From Eq. (iv), we can see that $h > R$ if $u^2 > 2gR$. Thus, the particle will leave the circle when $h > R$ or $90° < \theta < 180°$. This situation is shown in the Fig. 10.24.

$$\sqrt{2gR} < u < \sqrt{5gR} \quad \text{or} \quad 90° < \theta < 180°$$

Note *After leaving the circle, the particle will follow a parabolic path as the particle comes under gravity.*

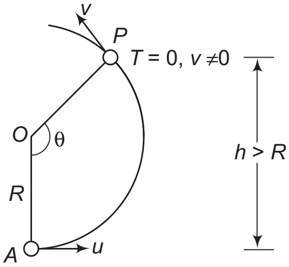


Fig. 10.24

Condition of Oscillation $(0 < u \leq \sqrt{2gR})$

The particle will oscillate, if velocity of the particle becomes zero but tension in the string is not zero. or $v = 0$, but $T \neq 0$. This is possible when

$$h_2 < h_1$$

or $$\frac{u^2}{2g} < \frac{u^2 + Rg}{3g}$$

or $$3u^2 < 2u^2 + 2Rg$$

or $$u^2 < 2Rg$$

or $$u < \sqrt{2Rg}$$

Moreover, if $h_1 = h_2$, $u = \sqrt{2Rg}$ and tension and velocity both becomes zero simultaneously.

Further, from Eq. (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$. Thus, for $0 < u \leq \sqrt{2gR}$, particle oscillates in lower half of the circle $(0° < \theta \leq 90°)$.

This situation is shown in the figure.

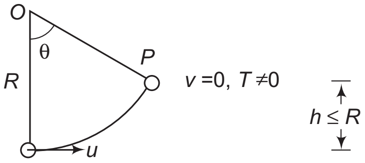


Fig. 10.25

$$0 < u \leq \sqrt{2gR} \quad \text{or} \quad 0° < \theta \leq 90°$$

Note *The above three conditions have been derived for a particle moving in a vertical circle attached to a string. The same conditions apply, if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.*

Condition of Looping the Loop is $u \geq \sqrt{5gR}$

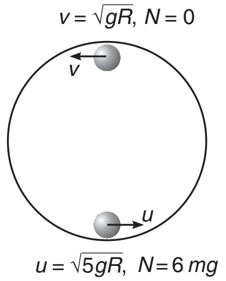


Fig. 10.26

Condition of Leaving the Circle is $\sqrt{2gR} < u < \sqrt{5gR}$

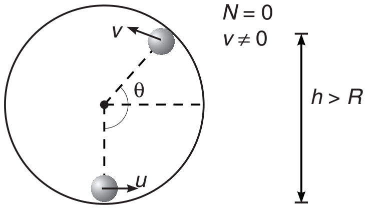


Fig. 10.27

Condition of Oscillation is $0 < u \leq \sqrt{2gR}$

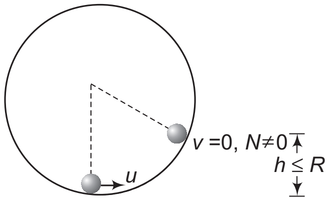


Fig. 10.28

Example 10.11 *A stone tied to a string of length L is whirled in a vertical circle with the other end of the string at the centre. At a certain instant of time the stone is at it lowest position and has a speed u. Find the magnitude of the change in its velocity as it reaches a position, where the string is horizontal.*

Solution $v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gL}$

$$|\Delta \mathbf{v}| = |\mathbf{v}_f - \mathbf{v}_i|$$

$$= \sqrt{v^2 + u^2 - 2v \cdot u \cos 90^\circ}$$

$$= \sqrt{(u^2 - 2gL) + u^2}$$

$$= \sqrt{2(u^2 - gL)}$$ **Ans.**

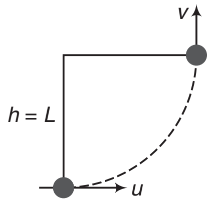


Fig. 10.29

Example 10.12 *With what minimum speed v must a small ball should be pushed inside a smooth vertical tube from a height h so that it may reach the top of the tube? Radius of the tube is R.*

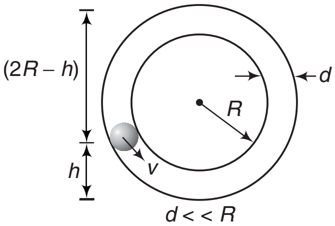


Fig. 10.30

Solution $v_{\text{top}} = \sqrt{v^2 - 2g\,(2R - h)}$

To just complete the vertical circle v_{top} may be zero.

$\therefore$ $$0 = \sqrt{v^2 - 2g\,(2R - h)}$$

or $$v = \sqrt{2g\,(2R - h)}$$ **Ans.**

Example 10.13 *A particle is suspended from a fixed point by a string of length 5 m. It is projected from the equilibrium position with such a velocity that the string slackens after the particle has reached a height 8 m above the lowest point. Find the velocity of the particle, just before the string slackens. Find also, to what height the particle can rise further?*

Solution At P,

$$T = 0$$

$\therefore$ $$mg\cos\theta = \frac{mv^2}{R}$$

or $$g\cos\theta = \frac{v^2}{R}$$

or $$(9.8)\left(\frac{3}{5}\right) = \frac{v^2}{5}$$

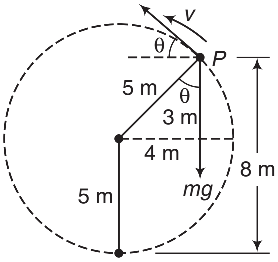


Fig. 10.31

$\therefore$ $$v = 5.42 \text{ m/s}$$ **Ans.**

After point P motion is projectile

$$h = \frac{v^2\sin^2\theta}{2g} = \frac{(5.42)^2\,(4/5)^2}{2 \times 9.8}$$

$$= 0.96 \text{ m}$$ **Ans.**

Example 10.14 *A heavy particle hanging from a fixed point by a light inextensible string of length l is projected horizontally with speed $\sqrt{gl}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.*

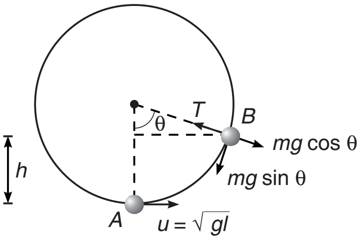


Fig. 10.32

Solution Let $T = mg$ at angle θ as shown in figure.

$$h = l(1 - \cos\theta) \qquad \ldots\text{(i)}$$

Applying conservation of mechanical energy between points *A* and *B,* we get

$$\frac{1}{2} m(u^2 - v^2) = mgh$$

Here, $$u^2 = gl \qquad \ldots\text{(ii)}$$

and v = speed of particle in position *B*

$$\therefore \qquad v^2 = u^2 - 2gh \qquad \ldots\text{(iii)}$$

Further, $$T - mg\cos\theta = \frac{mv^2}{l} \quad \text{or} \quad mg - mg\cos\theta = \frac{mv^2}{l} \qquad (T = mg)$$

or $$v^2 = gl(1 - \cos\theta) \qquad \ldots\text{(iv)}$$

Substituting values of v^2, u^2 and h from Eqs. (iv), (ii) and (i) in Eq. (iii), we get

$$gl(1 - \cos\theta) = gl - 2gl(1 - \cos\theta) \quad \text{or} \quad \cos\theta = \frac{2}{3} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Substituting $\cos\theta = \frac{2}{3}$ in Eq. (iv), we get

$$v = \sqrt{\frac{gl}{3}}$$ **Ans.**

INTRODUCTORY EXERCISE 10.3

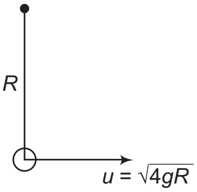


Fig. 10.33

1. In the figure shown in Fig. 10.33, a bob attached with a light string of radius R is given an initial velocity $u = \sqrt{4gR}$ at the bottommost point.
   (a) At what height string will slack.
   (b) What is velocity of the bob just before slacking of string.
2. In the above question, if $u = \sqrt{gR}$ then
   (a) after rotating an angle θ, velocity of the bob becomes zero. Find the value of θ.
   (b) If mass of the bob is '*m*' then what is the tension in the string when velocity becomes zero?
3. In question number-1, if $u = \sqrt{7gR}$ then
   (a) What is the velocity at topmost point ?
   (b) What is tension at the topmost point ?
   (c) What is tension at the bottommost point ?
4. A bob is suspended from a crane by a cable of length $l = 5$ m. The crane and load are moving at a constant speed v_0. The crane is stopped by a bumper and the bob on the cable swings out an angle of 60°. Find the initial speed v_0. ($g = 9.8 \text{ m/s}^2$)

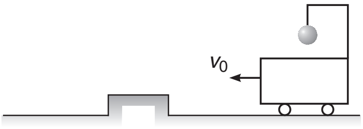
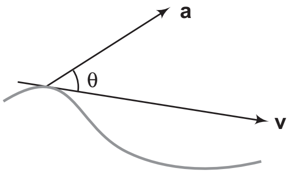


Fig. 10.34

Final Touch Points

1. In general, in any curvilinear motion direction of instantaneous velocity is tangential to the path, while acceleration may have any direction. If we resolve the acceleration in two normal directions, one parallel to velocity and another perpendicular to velocity, the first component is a_t while the other is a_n



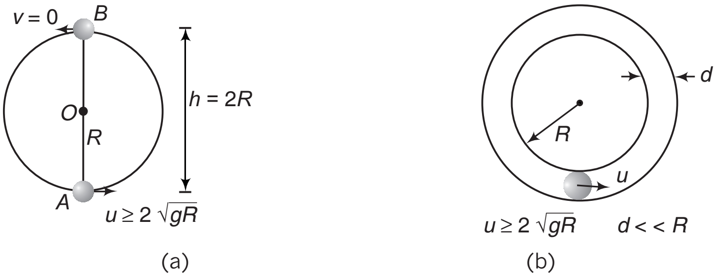
Thus, $a_t =$ component of **a** along **v** $= a\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{v}}{v}$

$$= \frac{dv}{dt} = \frac{d|\mathbf{v}|}{dt} = \text{rate of change of speed}$$

and $a_n =$ component of **a** perpendicular to **v** $= a\sin\theta = \sqrt{a^2 - a_t^2} = v^2/R$

Here, v is the speed of particle at that instant and R is called the radius of curvature to the curvilinear path at that point.

2. In $a_t = a\cos\theta$, if θ is acute, a_t will be positive and speed will increase. If θ is obtuse a_t will be negative and speed will decrease. If θ is 90°, a_t is zero and speed will remain constant.
3. If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottommost point is not $\sqrt{5\,gR}$. Because in this case, velocity of the particle at the topmost point can be zero also. Using conservation of mechanical energy between points A and B as shown in figure (a), we get



(a) (b)

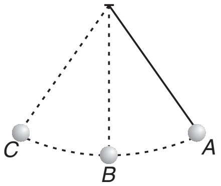
$$\frac{1}{2}m(u^2 - v^2) = mgh \qquad \text{or} \qquad \frac{1}{2}mu^2 = mg\,(2R) \qquad (\text{as } v = 0)$$

$$\therefore \qquad u = 2\sqrt{gR}$$

Therefore, the minimum value of u in this case is $2\sqrt{gR}$.

Same is the case when a particle is compelled to move inside a smooth vertical tube as shown in figure (b).

4. Oscillation of a pendulum is the part of a vertical circular motion. At point A and C since velocity is zero, net centripetal force will be zero. Only tangential force is present. From A and B or C to B speed of the bob increases. Therefore, tangential force is parallel to velocity. From B to A or B to C speed of the bob decreases. Hence, tangential force is antiparallel to velocity.



5. In circular motion, acceleration of the particle has two components

(i) tangential acceleration $a_t = \frac{dv}{dt} = R\alpha$

(ii) normal or radial acceleration $a_n = \frac{v^2}{R} = R\omega^2$

a_t and a_n are two perpendicular components of **a**. Hence, we can write $a = \sqrt{a_t^2 + a_n^2}$

Since, circular motion, is a 2-D motion we can write

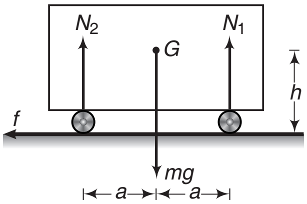
$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{r}\right)^2}$$

Here, $$v = \sqrt{v_x^2 + v_y^2} \quad \text{or} \quad v^2 = v_x^2 + v_y^2$$

6. Condition of toppling of a vehicle on circular tracks

While moving in a circular track normal reaction on the outer wheels (N_1) is more than the normal reaction on inner wheels (N_2).

or $$N_1 > N_2$$



This can be proved as below.

Distance between two wheels = $2a$

Height of centre of gravity of car from road = h

For translational equilibrium of car

$$N_1 + N_2 = mg \qquad \text{...(i)}$$

and $$f = \frac{mv^2}{r} \qquad \text{...(ii)}$$

and for rotational equilibrium of car, net torque about centre of gravity should be zero.

or $$N_1(a) = N_2(a) + f(h) \qquad \text{...(iii)}$$

From Eq. (iii), we can see that

$$N_2 = N_1 - \left(\frac{h}{a}\right) f = N_1 - \left(\frac{mv^2}{r}\right)\left(\frac{h}{a}\right) \qquad \text{...(iv)}$$

or $$N_2 < N_1$$

From Eq. (iv), we see that N_2 decreases as v is increased.

In critical case, $$N_2 = 0$$

and $$N_1 = mg \qquad \text{[From Eq. (i)]}$$

$\therefore$ $$N_1(a) = f(h) \qquad \text{[From Eq. (iii)]}$$

or $$(mg)(a) = \left(\frac{mv^2}{r}\right)(h) \quad \text{or} \quad v = \frac{gra}{h}$$

Now, if $v > \sqrt{\frac{gra}{h}}$, $N_2 < 0$, and the car topples outwards.

Therefore, for a safe turn without toppling $v \le \sqrt{\frac{gra}{h}}$.

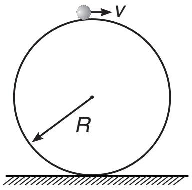
7. From the above discussion, we can conclude that while taking a turn on a level road there are two critical speeds, one is the maximum speed for sliding $(= \sqrt{\mu rg})$ and another is maximum speed for toppling $\left(= \sqrt{\frac{gra}{h}}\right)$. One should keep ones car's speed less than both for neither to slide nor to overturn.

8. **Motion of a ball over a smooth solid sphere**

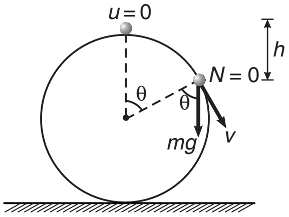
Suppose a small ball of mass m is given a velocity v over the top of a smooth sphere of radius R. The equation of motion for the ball at the topmost point will be

$$mg - N = \frac{mv^2}{R} \quad \text{or} \quad N = mg - \frac{mv^2}{R}$$

From this equation, we see that the value of N decreases as v increases. Minimum value of N can be zero. Hence,



(a)



(b)

$$0 = mg - \frac{mv_{\max}^2}{R} \quad \text{or} \quad v_{\max} = \sqrt{Rg}$$

So, ball will lose contact with the sphere right from the beginning if velocity of the ball at topmost point $v > \sqrt{Rg}$. If $v < \sqrt{Rg}$ it will lose contact after moving certain distance over the sphere. Now, let us find the angle θ where the ball loses contact with the sphere if velocity at topmost point is just zero. Fig. (b)

$$h = R(1 - \cos\theta) \qquad \text{...(i)}$$

$$v^2 = 2gh \qquad \text{...(ii)}$$

$$mg\cos\theta = \frac{mv^2}{R} \qquad (\text{as} \quad N = 0) \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

Thus, the ball can move on the sphere maximum upto $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

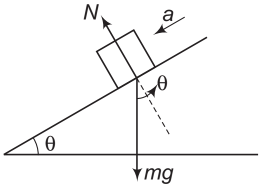
Exercise : Find angle θ where the ball will lose contact with the sphere, if velocity at topmost point is $u = \frac{v_{\max}}{2} = \frac{\sqrt{gR}}{2}$.

Ans. $\theta = \cos^{-1}\left(\frac{3}{4}\right) = 41.4°$

Hint: Only Eq. (ii) will change as,

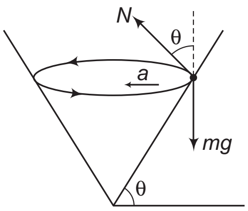
$$v^2 = u^2 + 2gh \qquad (u \neq 0)$$

9. In the following two figures, surface is smooth. So, only two forces N and mg are acting. But direction of acceleration are different.



$a = g\sin\theta$

(a)



$a = v^2/R$

(b)

Net force perpendicular to acceleration should be zero. So, in the first figure.

$$N = mg\cos\theta$$

and in the second figure, $N\cos\theta = mg$

Solved Examples

TYPED PROBLEMS

Type 1. *Based on vertical circular motion*

Concept

(i) Vertical circular motion is a non-uniform circular motion in which speed of the particle continuously keeps on changing. Therefore, a_t and a_r both are there. In moving upwards, speed decreases. So, a_t is in opposite direction of velocity. In moving downwards, speed increases. So, a_t is in the direction of velocity.

(ii) In circular motion normally, we resolve the forces in two directions, radial and tangential.

Here only two forces act on the particle, tension (T) and weight (mg). Tension is always in the radial direction (towards centre). So, resolve 'mg' along radial and tangential directions.

(iii) Weight (mg) is a constant force, while tension (T) is variable. It is maximum at the bottommost point and minimum at the topmost point.

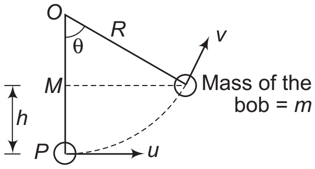
Example 1 *In the figure shown,* $u = \sqrt{6\,gR}\ (> \sqrt{5\,gR})$

Find h, v, a_r, a_t , T *and* F_{net} *when*

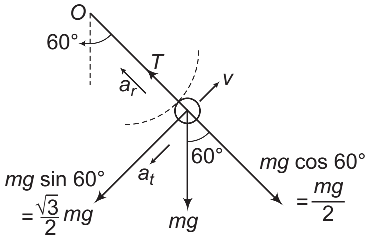
(a) $\theta = 60°$

(b) $\theta = 90°$

(c) $\theta = 180°$



Solution (a) **When $\theta = 60°$**



In the figure, we can see that,

$$h = PM = OP - OM = R - R\cos\theta$$

$$= R - R\cos 60° = R - \frac{R}{2} \qquad \textbf{Ans.}$$

$$v = \sqrt{u^2 - 2\,gh} = \sqrt{6\,gR - 2\,g\left(\frac{R}{2}\right)} = \sqrt{5\,gR} \qquad \textbf{Ans.}$$

$$a_r = \frac{v^2}{R} = \frac{(\sqrt{5gR})^2}{R} = 5g$$ **Ans.**

$$a_t = \frac{F_t}{m} = \frac{\sqrt{3}\,mg}{2m} = \frac{\sqrt{3}g}{2}$$

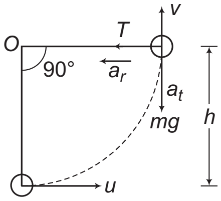
$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(5g)^2 + \left(\frac{\sqrt{3}g}{2}\right)^2} = \frac{\sqrt{103}}{2}\,g$$

$$T - \frac{mg}{2} = ma_r = m\,(5g)$$

$\therefore$ $$T = 5.5\,mg$$ **Ans.**

$$F_{\text{net}} = ma = \frac{\sqrt{103}}{2}\,mg$$ **Ans.**

(b) **When $\theta = 90°$**



$$h = R$$ **Ans.**

$$v = \sqrt{u^2 - 2gh} = \sqrt{6gR - 2gR}$$

$$= 2\sqrt{gR}$$ **Ans.**

$$a_r = \frac{v^2}{R} = \frac{(2\sqrt{gR})^2}{R} = 4g$$

$$a_t = \frac{F_t}{m} = \frac{mg}{m} = g$$

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(4g)^2 + (g)^2} = \sqrt{17}g$$

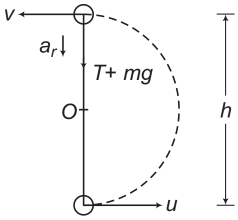
$$T = m\,a_r = m\,(4g)$$

$$= 4\,mg$$ **Ans.**

$$F_{\text{net}} = ma = m\,(\sqrt{17}\,g)$$

$$= \sqrt{17}\,mg$$ **Ans.**

(c) **When $\theta = 180°$**



$$h = 2R$$ **Ans.**

$$v = \sqrt{u^2 - 2gh}$$

$$= \sqrt{6gR - 2g \times 2R}$$

$$= \sqrt{2gR}$$ **Ans.**

$$a_r = \frac{v^2}{R} = \frac{(\sqrt{2gR})^2}{R}$$

$$= 2g$$ **Ans.**

$$a_t = \frac{F_t}{m} = 0$$ (as $F_t = 0$, both forces are radial)

$$a = a_r$$ (as $a_t = 0$)

$$= 2g$$ **Ans.**

$$T + mg = ma_r = m\,(2g)$$

$\therefore$ $$T = mg$$

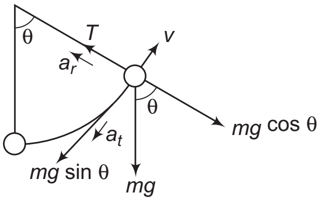
Note *This is the minimum tension during the motion.*

$$F_{net} = ma = m\,(2g) = 2\,mg$$ **Ans.**

Note Points

(i) $F_{net} (= ma)$ is also the vector sum of two forces. T and mg acting on the body.

(ii)



In general, $$a_t = \frac{F_t}{m} = \frac{mg\sin\theta}{m} = g\sin\theta$$

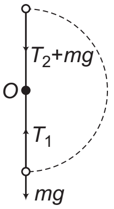
At $\theta = 60°$, 90° and 180°, this value is $\frac{\sqrt{3}}{2}g, g$ and zero.

Similarly, $T - mg\cos\theta = ma_r = \frac{mv^2}{R} \Rightarrow T = mg\cos\theta + \frac{mv^2}{R}$

(iii) At topmost and bottommost points, both forces act in radial direction.

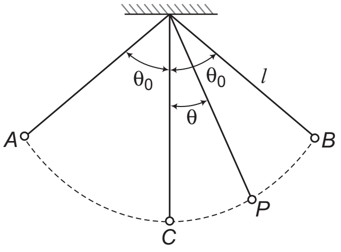
So, $$F_t = 0$$

$\Rightarrow$ $$a_t = \frac{F_t}{m} = 0$$



Type 2. *Based on motion of a pendulum*

Concept



Motion of a pendulum is the part of a vertical circular motion. It is the case of oscillation in vertical circular motion.Therefore velocity at bottommost point C should be less than or equal to $\sqrt{2gl}$.

At extreme positions A and B where, $\theta = \pm\theta_0$, $v = 0$, $T \neq 0$, $a_r = 0 \cdot T$

Therefore, $a_t = g\sin\theta_0$ and $T = mg\cos\theta_0$

At the bottommost point C, where $\theta = 0°$

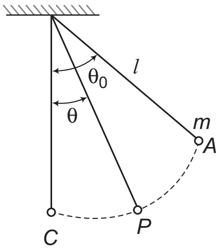
$$v = \text{maximum}$$
$$a_r = \text{maximum}$$
$$T = \text{maximum}$$

and $$a_t = 0$$

At some intermediate point P, where $\theta = \theta$, neither of the terms discussed above is zero,

$$h = l\cos\theta - l\cos\theta_0.$$
$$v = \sqrt{2gh}$$
$$a_r = \frac{v^2}{l},\ a_t = g\sin\theta,\ a = \sqrt{a_r^2 + a_t^2},$$

$F_{net} = ma$ and $T - mg\cos\theta = \frac{mv^2}{l} = ma_r$



Example 2 *A ball of mass 'm' is released from point A where, $\theta_0 = 53°$. Length of pendulum is 'l'. Find v, a_r, a_t, a, T and F_{net} at*

(a) point A

(b) point C

(c) pont P, where $\theta = 37°$

Solution (a) **At pont *A***

$$v = 0 \quad \Rightarrow \quad a_r = \frac{v^2}{R} = 0 \qquad (R = l)$$

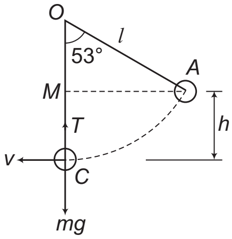
$$a_t = g \sin \theta_0 = g \sin 53° = \frac{4}{5} g$$

$$a = a_t = \frac{4}{5} g$$

$$T = mg \cos \theta_0 = mg \cos 53° = \frac{3}{5} mg$$

$$F_{net} = ma = \frac{4}{5} mg$$

(b) **At point *C***



$$h = OC - OM = l - l \cos 53°$$

$$= l - \frac{3}{5} l = \frac{2}{5} l$$

$$v = \sqrt{2\,gh} = \sqrt{2\,g\left(\frac{2}{5} l\right)} = \sqrt{\frac{4}{5} gl}$$

$$a_r = \frac{v^2}{R} = \frac{\left(\sqrt{\frac{4}{5} gl}\right)^2}{l} = \frac{4}{5} g$$

$$a_t = 0$$

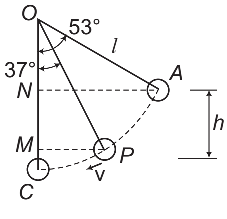
$$a = a_r = \frac{4}{5} g$$

$$T - mg = \frac{mv^2}{R} = \frac{4}{5} mg$$

$$\therefore \qquad T = \frac{9}{5} mg$$

$$F_{net} = ma = \frac{4}{5} mg$$

(c) At point *P*



$$h = OM - ON = l \cos 37° - l \cos 53°$$

$$= \frac{4}{5} l - \frac{3}{5} l = \frac{l}{5}$$

$$v = \sqrt{2\,gh} = \sqrt{2\,g\left(\frac{l}{5}\right)} = \sqrt{\frac{2}{5} gl}$$

$$a_r = \frac{v^2}{R} = \frac{\left(\sqrt{\frac{2}{5}gl}\right)^2}{l} = \frac{2}{5}g$$

$$a_t = g\sin\theta = g\sin 37° = \frac{3}{5}g$$

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{2}{5}g\right)^2 + \left(\frac{3}{5}g\right)^2} = \frac{\sqrt{13}}{5}g$$

$$T - mg\cos\theta = ma_r$$

or $$T - mg\cos 37° = m\left(\frac{2}{5}g\right)$$

$\therefore$ $$T = \frac{6}{5}mg$$

$$F_{\text{net}} = ma = \frac{\sqrt{13}}{5}mg$$

Miscellaneous Examples

Example 3 *A particle of mass m starts moving in a circular path of constant radius r, such that its centripetal acceleration a_c is varying with time t as $a_c = k^2rt^2$, where k is a constant. What is the power delivered to the particle by the forces acting on it ?*

[IIT JEE 1994]

Solution As $a_c = (v^2/r)$ so $(v^2/r) = k^2rt^2$

$\therefore$ Kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}mk^2r^2t^2$

Now, from work-energy theorem

$$W = \Delta K = \frac{1}{2}mk^2r^2t^2 - 0 \qquad [\text{as at } t = 0, K = 0]$$

So, $$P = \frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}mk^2r^2t^2\right) = mk^2r^2t$$ **Ans.**

Alternate solution : Given that $a_c = k^2rt^2$, so that

$$F_c = ma_c = mk^2rt^2$$

Now, as $a_c = (v^2/r)$, so $(v^2/r) = k^2rt^2$

or $v = krt$

So, that $a_t = (dv/dt) = kr$

i.e. $F_t = ma_t = mkr$

Now, as $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_t$

So, $$P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = (\mathbf{F}_c + \mathbf{F}_t)\cdot\mathbf{v}$$

In circular motion, $\mathbf{F}_c$ is perpendicular to $\mathbf{v}$ while $\mathbf{F}_t$ parallel to it, so

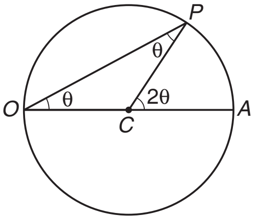
$$P = F_t v \qquad [\text{as } \mathbf{F}_c \cdot \mathbf{v} = 0]$$

$$\therefore \qquad P = mk^2r^2t$$ **Ans.**

Example 4 *If a point moves along a circle with constant speed, prove that its angular speed about any point on the circle is half of that about the centre.*

Solution Let, O be a point on a circle and P be the position of the particle at any time t, such that

$$\angle POA = \theta. \quad \text{Then,} \quad \angle PCA = 2\theta$$



Here, C is the centre of the circle.

Angular velocity of P about O is $\omega_O = \dfrac{d\theta}{dt}$

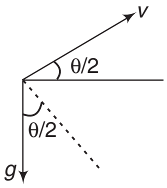
and angular velocity of P about C is,

$$\omega_C = \frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt}$$

or $$\omega_C = 2\omega_O$$ **Proved.**

Example 5 *A particle is projected with a speed u at an angle θ with the horizontal. What is the radius of curvature of the parabola traced out by the projectile at a point where the particle velocity makes an angle $\dfrac{\theta}{2}$ with the horizontal.*

Solution Let v be the velocity at the desired point. Horizontal component of velocity remains unchanged. Hence,



$$v \cos\frac{\theta}{2} = u\cos\theta$$

$$\therefore \qquad v = \frac{u\cos\theta}{\cos\dfrac{\theta}{2}} \qquad \dots \text{(i)}$$

Radial acceleration is the component of acceleration perpendicular to velocity or

$$a_n = g\cos\left(\frac{\theta}{2}\right)$$

$$\therefore \qquad \frac{v^2}{R} = g\cos\left(\frac{\theta}{2}\right) \qquad \dots \text{(ii)}$$

Substituting the value of v from Eq. (i) in Eq. (ii), we have radius of curvature

$$R = \frac{\left[\dfrac{u\cos\theta}{\cos\left(\dfrac{\theta}{2}\right)}\right]^2}{g\cos\left(\dfrac{\theta}{2}\right)} = \frac{u^2\cos^2\theta}{g\cos^3\left(\dfrac{\theta}{2}\right)}$$ **Ans.**

Example 6 *A point moves along a circle with a speed $v = kt$, where $k = 0.5\ m/s^2$. Find the total acceleration of the point at the moment when it has covered the n^{th} fraction of the circle after the beginning of motion, where $n = \frac{1}{10}$.*

Solution $v = \frac{ds}{dt} = kt$ or $\int_0^s ds = k\int_0^t t\, dt \Rightarrow \therefore\ s = \frac{1}{2}kt^2$

For completion of nth fraction of circle,

$$s = 2\pi rn = \frac{1}{2}kt^2 \quad \text{or} \quad t^2 = (4\pi nr)/k \qquad \text{...(i)}$$

Tangential acceleration $= a_t = \frac{dv}{dt} = k$ ...(ii)

Normal acceleration $= a_n = \frac{v^2}{r} = \frac{k^2t^2}{r}$ ...(iii)

Substituting the value of t^2 from Eq. (i), we have

or $a_n = 4\pi nk$

$$\therefore \quad a = \sqrt{(a_t^2 + a_n^2)} = [k^2 + 16\pi^2 n^2 k^2]^{1/2}$$

$$= k\,[1 + 16\pi^2 n^2]^{1/2}$$

$$= 0.50\,[1 + 16 \times (3.14)^2 \times (0.10)^2]^{1/2}$$

$$= 0.8\ \text{m/s}^2$$

Ans.

Example 7 *In a two dimensional motion of a body, prove that tangential acceleration is nothing but component of acceleration along velocity.*

Solution Let velocity of the particle be,

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

Acceleration

$$\mathbf{a} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}}$$

Component of **a** along **v** will be,

$$\frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{v_x\frac{dv_x}{dt} + v_y\cdot\frac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}} \qquad \text{... (i)}$$

Further, tangential acceleration of particle is rate of change of speed.

$$\text{or} \quad a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{v_x^2 + v_y^2}\right) \quad \text{or} \quad a_t = \frac{1}{2\sqrt{v_x^2 + v_y^2}}\left[2v_x\cdot\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt}\right]$$

$$\text{or} \quad a_t = \frac{v_x\cdot\frac{dv_x}{dt} + v_y\cdot\frac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}} \qquad \text{... (ii)}$$

From Eqs. (i) and (ii), we can see that

$$a_t = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}$$

or Tangential acceleration = component of acceleration along velocity. **Hence proved.**

Exercises

LEVEL 1

Assertion and Reason

Directions *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

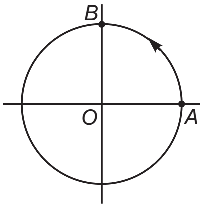
(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : A car moving on a horizontal rough road with velocity v can be stopped in a minimum distance d. If the same car, moving with same speed v takes a circular turn, then minimum safe radius can be $2d$.

Reason : $d = \frac{v^2}{2\mu g}$ and minimum safe radius $= \frac{v^2}{\mu g}$

2. Assertion : A particle is rotating in a circle with constant speed as shown. Between points A and B, ratio of average acceleration and average velocity is angular velocity of particle about point O.



Reason : Since speed is constant, angular velocity is also constant.

3. Assertion : A frame moving in a circle with constant speed can never be an inertial frame.

Reason : It has a constant acceleration.

4. Assertion : In circular motion, dot product of velocity vector (**v**) and acceleration vector (**a**) may be positive, negative or zero.

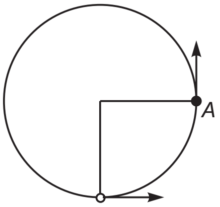
Reason : Dot product of angular velocity vector and linear velocity vector is always zero.

5. Assertion : Velocity and acceleration of a particle in circular motion at some instant are:

$\mathbf{v} = (2\hat{\mathbf{i}})\ \text{ms}^{-1}$ and $\mathbf{a} = (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ \text{ms}^{-2}$, then radius of circle is 2 m.

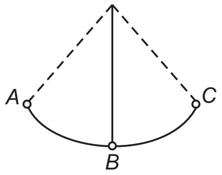
Reason : Speed of particle is decreasing at a rate of 1 ms^{-2}.

6. Assertion : In vertical circular motion, acceleration of bob at position A is greater than 'g'.



Reason : Net acceleration at A is resultant of tangential and radial components of acceleration.

7. Assertion : A pendulum is oscillating between points A, B and C. Acceleration of bob at points A or C is zero.



Reason : Velocity at these points is zero.

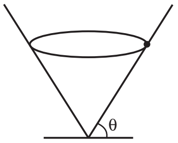
8. Assertion : Speed of a particle moving in a circle varies with time as, $v = (4t - 12)$. Such type of circular motion is not possible.

Reason : Speed cannot change linearly with time.

9. Assertion : Circular and projectile motions both are two dimensional motion. But in circular motion, we cannot apply $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ directly, whereas in projectile motion we can.

Reason : Projectile motion takes place under gravity, while in circular motion gravity has no role.

10. Assertion : A particle of mass m takes uniform horizontal circular motion inside a smooth funnel as shown. Normal reaction in this case is not $mg \cos\theta$.



Reason : Acceleration of particle is not along the surface of funnel.

11. Assertion : When water in a bucket is whirled fast overhead, the water does not fall out at the top of the circular path.

Reason : The centripetal force in this position on water is more than the weight of water.

Objective Questions

Single Correct Option

1. A particle is revolving in a circle with increasing its speed uniformly. Which of the following is constant ?

(a) Centripetal acceleration
(b) Tangential acceleration
(c) Angular acceleration
(d) None of these

2. A particle is moving in a circular path with a constant speed. If θ is the angular displacement, then starting from $\theta = 0$, the maximum and minimum change in the linear momentum will occur when value of θ is respectively

(a) 45° and 90°
(b) 90° and 180°
(c) 180° and 360°
(d) 90° and 270°

3. A simple pendulum of length l has maximum angular displacement θ. Then maximum kinetic energy of a bob of mass m is

(a) $\frac{1}{2} mgl$
(b) $\frac{1}{2} mgl \cos\theta$
(c) $mgl (1 - \cos\theta)$
(d) $\frac{1}{2} mgl \sin\theta$

4. A particle of mass m is fixed to one end of a light rigid rod of length l and rotated in a vertical circular path about its other end. The minimum speed of the particle at its highest point must be

(a) zero (b) $\sqrt{gl}$

(c) $\sqrt{1.5\, gl}$ (d) $\sqrt{2\, gl}$

5. A simple pendulum of length l and mass m is initially at its lowest position. It is given the minimum horizontal speed necessary to move in a circular path about the point of suspension. The tension in the string at the lowest position of the bob is

(a) $3\, mg$ (b) $4\, mg$

(c) $5\, mg$ (d) $6\, mg$

6. A point moves along a circle having a radius 20 cm with a constant tangential acceleration $5\ \text{cm/s}^2$. How much time is needed after motion begins for the normal acceleration of the point to be equal to tangential acceleration?

(a) 1 s (b) 2 s

(c) 3 s (d) 4 s

7. A ring of mass (2π) kg and of radius 0.25 m is making 300 rpm about an axis through its perpendicular to its plane. The tension in newton developed in ring is approximately

(a) 50 (b) 100

(c) 175 (d) 250

8. A car is moving on a circular level road of curvature 300 m. If the coefficient of friction is 0.3 and acceleration due to gravity is $10\ \text{m/s}^2$, the maximum speed of the car can be

(a) 90 km/h (b) 81 km/h

(c) 108 km/h (d) 162 km/h

9. A string of length 1 m is fixed at one end with a bob of mass 100 g and the string makes $\left(\frac{2}{\pi}\right)$ rev s^{-1} around a vertical axis through a fixed point. The angle of inclination of the string with vertical is

(a) $\tan^{-1}\left(\frac{5}{8}\right)$ (b) $\tan^{-1}\left(\frac{3}{5}\right)$

(c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) $\cos^{-1}\left(\frac{5}{8}\right)$

10. In the previous question, the tension in the string is

(a) $\frac{5}{8}$ N (b) $\frac{8}{5}$ N

(c) $\frac{50}{8}$ N (d) $\frac{80}{5}$ N

11. A small particle of mass 0.36 g rests on a horizontal turntable at a distance 25 cm from the axis of spindle. The turntable is accelerated at a rate of $\alpha = \frac{1}{3}$ rad s^{-2}. The frictional force that the table exerts on the particle 2 s after the startup is

(a) 40 μN (b) 30 μN

(c) 50 μN (d) 60 μN

12. A simple pendulum of length l and bob of mass m is displaced from its equilibrium position O to a position P so that height of P above O is h. It is then released. What is the tension in the string when the bob passes through the equilibrium position O? Neglect friction. v is the velocity of the bob at O.

(a) $m\left(g+\frac{v^2}{l}\right)$ (b) $\frac{2\,mgh}{l}$

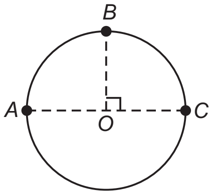
(c) $mg\left(1+\frac{h}{l}\right)$ (d) $mg\left(1+\frac{2\,h}{l}\right)$

13. Two particles revolve concentrically in a horizontal plane in the same direction. The time required to complete one revolution for particle A is 3 min, while for particle B is 1 min. The time required for A to complete one revolution relative to B is

(a) 2 min (b) 1 min

(c) 1.5 min (d) 1.25 min

14. Three particles A, B and C move in a circle in anticlockwise direction with speeds 1 ms^{-1}, 2.5 ms^{-1} and 2 ms^{-1} respectively. The initial positions of A, B and C are as shown in figure. The ratio of distance travelled by B and C by the instant A, B and C meet for the first time is



(a) 3 : 2 (b) 5 : 4 (c) 3 : 5 (d) data insufficient

Subjective Questions

1. A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 m/s and is increasing uniformly at 8 m/s^2. Determine the magnitude of its acceleration at this instant.

2. A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed?

3. A particle is projected with a speed u at an angle θ with the horizontal. Consider a small part of its path near the highest position and take it approximately to be a circular arc. What is the radius of this circle? This radius is called the radius of curvature of the curve at the point.

4. Find the maximum speed at which a truck can safely travel without toppling over, on a curve of radius 250 m. The height of the centre of gravity of the truck above the ground is 1.5 m and the distance between the wheels is 1.5 m, the truck being horizontal.

5. A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.

6. Show that the angle made by the string with the vertical in a conical pendulum is given by $\cos\theta = \frac{g}{L\omega^2}$, where L is the length of the string and ω is the angular speed.

7. A boy whirls a stone of small mass in a horizontal circle of radius 1.5 m and at height 2.9 m above level ground. The string breaks and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion ?

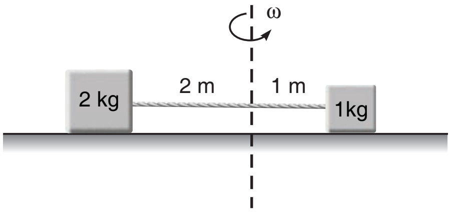
8. A block of mass m is kept on a horizontal ruler. The friction coefficient between the ruler and the block is μ. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end.

(a) What can the maximum constant angular speed be for which the block does not slip ?

(b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration α, at what angular speed will the block slip ?

9. A thin circular wire of radius R rotates about its vertical diameter with an angular frequency ω. Show that a small bead on the wire remains at its lowermost point for $\omega \le \sqrt{g/R}$. What is angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

10. Two blocks tied with a massless string of length 3 m are placed on a rotating table as shown. The axis of rotation is 1 m from 1 kg mass and 2 m from 2 kg mass. The angular speed $\omega = 4$ rad/s. Ground below 2 kg block is smooth and below 1 kg block is rough. ($g = 10\ \text{m/s}^2$)



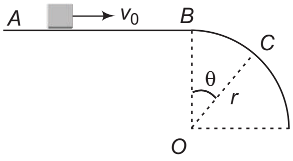
(a) Find tension in the string, force of friction on 1 kg block and its direction.

(b) If coefficient of friction between 1 kg block and ground is $\mu = 0.8$. Find maximum angular speed so that neither of the blocks slips.

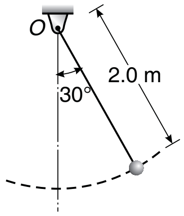
(c) If maximum tension in the string can be 100 N, then find maximum angular speed so that neither of the blocks slips.

Note *Assume that in part (b) tension can take any value and in parts (a) and (c) friction can take any value.*

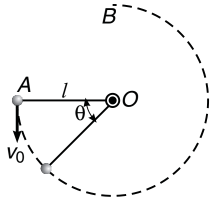
11. A small block slides with velocity $0.5\sqrt{gr}$ on the horizontal frictionless surface as shown in the figure. The block leaves the surface at point C. Calculate angle θ in the figure.



12. The bob of the pendulum shown in figure describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown. Find the velocity and the acceleration of the bob in that position.



13. The sphere at A is given a downward velocity v_0 of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length $l = 2$ m attached to a support at O. Determine the angle θ at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.

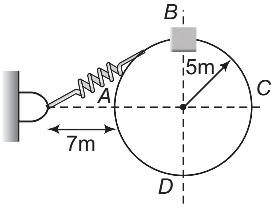


LEVEL 2

Objective Questions

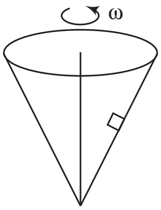
Single Correct Option

1. A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m. The spring lying in the plane of the circular track and having spring constant $200\,\text{Nm}^{-1}$ is undeformed when the collar is at A. If the collar starts from rest at B, the normal reaction exerted by the track on the collar when it passes through A is



(a) 360 N
(b) 720 N
(c) 1440 N
(d) 2880 N

2. A particle is at rest with respect to the wall of an inverted cone rotating with uniform angular velocity ω about its central axis. The surface between the particle and the wall is smooth. Regarding the displacement of particle along the surface up or down, the equilibrium of particle is

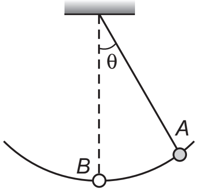


(a) stable
(b) unstable
(c) neutral
(d) None of these

3. A rough horizontal plate rotates with angular velocity ω about a fixed vertical axis. A particle of mass m lies on the plate at a distance $\frac{5a}{4}$ from this axis. The coefficient of friction between the plate and the particle is $\frac{1}{3}$. The largest value of ω^2 for which the particle will continue to be at rest on the revolving plate is

(a) $\frac{g}{3a}$ (b) $\frac{4g}{5a}$ (c) $\frac{4g}{9a}$ (d) $\frac{4g}{15a}$

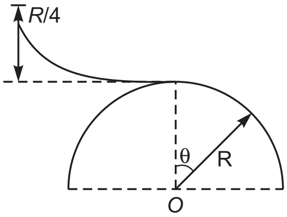
4. A ball attached to one end of a string swings in a vertical plane such that its acceleration at point A (extreme position) is equal to its acceleration at point B (mean position). The angle θ is



(a) $\cos^{-1}\left(\frac{2}{5}\right)$ (b) $\cos^{-1}\left(\frac{4}{5}\right)$

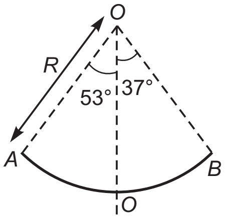
(c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) None of these

5. A skier plans to ski a smooth fixed hemisphere of radius R. He starts from rest from a curved smooth surface of height $\left(\frac{R}{4}\right)$. The angle θ at which he leaves the hemisphere is



(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\frac{5}{\sqrt{3}}$ (c) $\cos^{-1}\left(\frac{5}{6}\right)$ (d) $\cos^{-1}\left[\frac{5}{2\sqrt{3}}\right]$

6. A section of fixed smooth circular track of radius R in vertical plane is shown in the figure. A block is released from position A and leaves the track at B. The radius of curvature of its trajectory just after it leaves the track at B is ?



(a) R (b) $\frac{R}{4}$ (c) $\frac{R}{2}$ (d) $\frac{R}{3}$

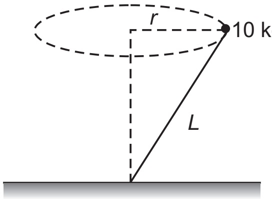
7. A particle is projected with velocity u horizontally from the top of a smooth sphere of radius a so that it slides down the outside of the sphere. If the particle leaves the sphere when it has fallen a height $\frac{a}{4}$, the value of u is

(a) $\sqrt{ag}$
(b) $\frac{\sqrt{ag}}{4}$
(c) $\frac{\sqrt{ag}}{2}$
(d) $\frac{\sqrt{ag}}{3}$

8. A particle of mass m describes a circle of radius r. The centripetal acceleration of the particle is $\frac{4}{r^2}$. What will be the momentum of the particle ?

(a) $2\frac{m}{r}$
(b) $2\frac{m}{\sqrt{r}}$
(c) $4\frac{m}{\sqrt{r}}$
(d) None of these

9. A 10 kg ball attached at the end of a rigid massless rod of length 1 m rotates at constant speed in a horizontal circle of radius 0.5 m and period of 1.58 s, as shown in the figure. The force exerted by the rod on the ball is ($g = 10$ ms^{-2})



(a) 158 N
(b) 128 N
(c) 110 N
(d) 98 N

10. A disc is rotating in a room. A boy standing near the rim of the disc of radius R finds the water droplet falling from the ceiling is always falling on his head. As one drop hits his head, other one starts from the ceiling. If height of the roof above his head is H, then angular velocity of the disc is

(a) $\pi\sqrt{\frac{2gR}{H^2}}$
(b) $\pi\sqrt{\frac{2gH}{R^2}}$
(c) $\pi\sqrt{\frac{2g}{H}}$
(d) None of these

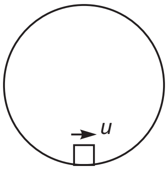
11. In a clock, what is the time period of meeting of the minute hand and the second hand?

(a) 59 s
(b) $\frac{60}{59}$ s
(c) $\frac{59}{60}$ s
(d) $\frac{3600}{59}$ s

12. A particle of mass m starts to slide down from the top of the fixed smooth sphere. What is the tangential acceleration when it breaks off the sphere ?

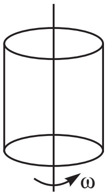
(a) $\frac{2g}{3}$
(b) $\frac{\sqrt{5}g}{3}$
(c) g
(d) $\frac{g}{3}$

13. A particle is given an initial speed u inside a smooth spherical shell of radius R so that it is just able to complete the circle. Acceleration of the particle, when its velocity is vertical, is



(a) $g\sqrt{10}$ (b) g
(c) $g\sqrt{2}$ (d) $g\sqrt{6}$

14. An insect of mass $m = 3$ kg is inside a vertical drum of radius 2 m that is rotating with an angular velocity of 5 rad s^{-1}. The insect doesn't fall off. Then, the minimum coefficient of friction required is



(a) 0.5
(b) 0.4
(c) 0.2
(d) None of the above

15. A simple pendulum is released from rest with the string in horizontal position. The vertical component of the velocity of the bob becomes maximum, when the string makes an angle θ with the vertical. The angle θ is equal to

(a) $\frac{\pi}{4}$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{3}$

16. A particle is moving in a circle of radius R in such a way that at any instant the normal and tangential component of its acceleration are equal. If its speed at $t = 0$ is v_0. The time taken to complete the first revolution is

(a) $\frac{R}{v_0}$ (b) $\frac{R}{v_0}e^{-2\pi}$

(c) $\frac{R}{v_0}(1 - e^{-2\pi})$ (d) $\frac{R}{v_0}(1 + e^{-2\pi})$

17. A particle is moving in a circular path in the vertical plane. It is attached at one end of a string of length l whose other end is fixed. The velocity at lowest point is u. The tension in the string is $\mathbf{T}$ and acceleration of the particle is $\mathbf{a}$ at any position. Then $\mathbf{T} \cdot \mathbf{a}$ is zero at highest point if

(a) $u > \sqrt{5gl}$
(b) $u = \sqrt{5gl}$
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

18. In the above question, $\mathbf{T} \cdot \mathbf{a}$ is positive at the lowest point for

(a) $u \leq \sqrt{2gl}$ (b) $u = \sqrt{2gl}$
(c) $u < \sqrt{2gl}$ (d) any value of u

More than One Correct Options

1. A ball tied to the end of the string swings in a vertical circle under the influence of gravity.

(a) When the string makes an angle 90° with the vertical, the tangential acceleration is zero and radial acceleration is somewhere between minimum and maximum

(b) When the string makes an angle 90° with the vertical, the tangential acceleration is maximum and radial acceleration is somewhere between maximum and minimum

(c) At no place in circular motion, tangential acceleration is equal to radial acceleration

(d) When radial acceleration has its maximum value, the tangential acceleration is zero

2. A small spherical ball is suspended through a string of length l. The whole arrangement is placed in a vehicle which is moving with velocity v. Now, suddenly the vehicle stops and ball starts moving along a circular path. If tension in the string at the highest point is twice the weight of the ball then (assume that the ball completes the vertical circle)

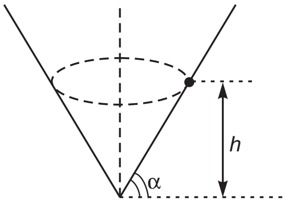
(a) $v = \sqrt{5\,gl}$

(b) $v = \sqrt{7\,gl}$

(c) velocity of the ball at highest point is $\sqrt{gl}$

(d) velocity of the ball at the highest point is $\sqrt{3\,gl}$

3. A particle is describing circular motion in a horizontal plane in contact with the smooth surface of a fixed right circular cone with its axis vertical and vertex down. The height of the plane of motion above the vertex is h and the semi-vertical angle of the cone is α. The period of revolution of the particle



(a) increases as h increases | (b) decreases as h decreases

(c) increases as α increases | (d) decreases as α increases

4. In circular motion of a particle,

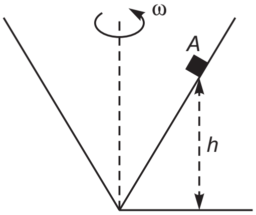
(a) particle cannot have uniform motion

(b) particle cannot have uniformly accelerated motion

(c) particle cannot have net force equal to zero

(d) particle cannot have any force in tangential direction

5. A smooth cone is rotated with an angular velocity ω as shown. A block A is placed at height h. Block has no motion relative to cone. Choose the correct options, when ω is increased.

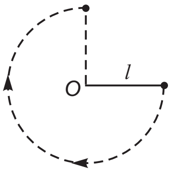


(a) net force acting on block will increase | (b) normal reaction acting on block will increase

(c) h will increase | (d) normal reaction will remain unchanged

Comprehension Based Questions

Passage 1 (Q.Nos. 1 to 2)



A ball with mass m is attached to the end of a rod of mass M and length l. The other end of the rod is pivoted so that the ball can move in a vertical circle. The rod is held in the horizontal position as shown in the figure and then given just enough a downward push so that the ball swings down and just reaches the vertical upward position having zero speed there. Now answer the following questions.

1. The change in potential energy of the system (ball + rod) is

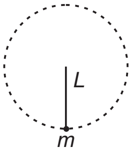
(a) mgl (b) $(M+m)\,gl$ (c) $\left(\frac{M}{2}+m\right)gl$ (d) $\frac{(M+m)}{2}\,gl$

2. The initial speed given to the ball is

(a) $\sqrt{\frac{Mgl+2mgl}{m}}$ (b) $\sqrt{2\,gl}$ (c) $\sqrt{\frac{2Mgl+mgl}{m}}$ (d) None of these

Note *Attempt the above question after studying chapter of rotational motion.*

Passage 2 (Q.Nos. 3 to 5)



A small particle of mass m attached with a light inextensible thread of length L is moving in a vertical circle. In the given case particle is moving in complete vertical circle and ratio of its maximum to minimum velocity is 2 : 1.

3. Minimum velocity of the particle is

(a) $4\sqrt{\frac{gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$

(c) $\sqrt{\frac{gL}{3}}$ (d) $3\sqrt{\frac{gL}{3}}$

4. Kinetic energy of the particle at the lower most position is

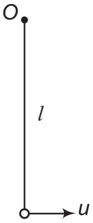
(a) $\frac{4\,mgL}{3}$ (b) $2\,mgL$ (c) $\frac{8\,mgL}{3}$ (d) $\frac{2\,mgL}{3}$

5. Velocity of the particle when it is moving vertically downward is

(a) $\sqrt{\frac{10\,gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$ (c) $\sqrt{\frac{8\,gL}{3}}$ (d) $\sqrt{\frac{13\,gL}{3}}$

Match the Columns

1. A bob of mass m is suspended from point O by a massless string of length l as shown. At the bottommost point it is given a velocity $u=\sqrt{12gl}$ for $l = 1$ m and $m = 1$ kg, match the following two columns when string becomes horizontal ($g = 10\text{ ms}^{-2}$)



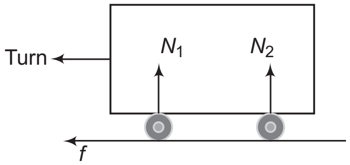
Column I	Column II (SI units)
(a) Speed of bob	(p) 10
(b) Acceleration of bob	(q) 20
(c) Tension in string	(r) 100
(d) Tangential acceleration of bob	(s) None

2. Speed of a particle moving in a circle of radius 2 m varies with time as $v = 2t$ (SI units). At $t = 1$ s match the following two columns :

Column I	Column II (SI units)
(a) $\mathbf{a} \cdot \mathbf{v}$	(p) $2\sqrt{2}$
(b) $\lvert \mathbf{a} \times \boldsymbol{\omega} \rvert$	(q) 2
(c) $\mathbf{v} \cdot \boldsymbol{\omega}$	(r) 4
(d) $\lvert \mathbf{v} \times \mathbf{a} \rvert$	(s) None

Here, symbols have their usual meanings.

3. A car is taking turn on a rough horizontal road without slipping as shown in figure. Let F is centripetal force, f the force of friction, N_1 and N_2 are two normal reactions. As the speed of car is increased, match the following two columns.



Column I	Column II
(a) N_1	(p) will increase
(b) N_2	(q) will decrease
(c) F/f	(r) will remain unchanged
(d) f	(s) cannot say anything

4. Position vector (with respect to centre) velocity vector and acceleration vector of a particle in circular motion are $\mathbf{r} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$ m, $\mathbf{v} = (4\hat{\mathbf{i}} - a\hat{\mathbf{j}})$ ms^{-1} and $\mathbf{a} = (-6\hat{\mathbf{i}} + b\hat{\mathbf{j}})$ ms^{-2}. Speed of particle is constant. Match the following two columns.

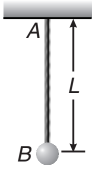
Column I	Column II (SI units)
(a) Value of a	(p) 8
(b) Value of b	(q) 3
(c) Radius of circle	(r) 5
(d) $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})$	(s) None

5. A particle is rotating in a circle of radius $R = \left(\frac{2}{\pi}\right)$ m, with constant speed 1 ms^{-1}. Match the following two columns for the time interval when it completes $\frac{1}{4}$th of the circle.

Column I	Column II (SI units)
(a) Average speed	(p) $\frac{\sqrt{2}}{\pi}$
(b) Average velocity	(q) $2\frac{\sqrt{2}}{\pi}$
(c) Average acceleration	(r) $\sqrt{2}$
(d) Displacement	(s) 1

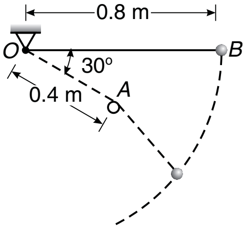
Subjective Questions

1. Bob B of the pendulum AB is given an initial velocity $\sqrt{3Lg}$ in horizontal direction. Find the maximum height of the bob from the starting point,



(a) if AB is a massless rod, (b) if AB is a massless string.

2. A small sphere B of mass m is released from rest in the position shown and swings freely in a vertical plane, first about O and then about the peg A after the cord comes in contact with the peg. Determine the tension in the cord



(a) just before the sphere comes in contact with the peg.
(b) just after it comes in contact with the peg.

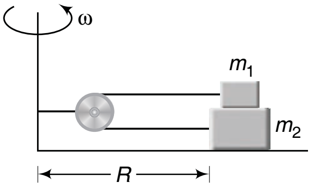
3. A particle of mass m is suspended by a string of length l from a fixed rigid support. A sufficient horizontal velocity $v_0 = \sqrt{3gl}$ is imparted to it suddenly. Calculate the angle made by the string with the vertical when the acceleration of the particle is inclined to the string by 45°.

4. A turn of radius 20 m is banked for the vehicles going at a speed of 36 km/h. If the coefficient of static friction between the road and the tyre is 0.4. What are the possible speeds of a vehicle so that it neither slips down nor skids up ? ($g = 9.8 \text{ m/s}^2$)

5. A particle is projected with a speed u at an angle θ with the horizontal. Find the radius of curvature of the parabola traced out by the projectile at a point, where the particle velocity makes an angle $\frac{\theta}{2}$ with the horizontal.

6. A particle is projected with velocity $20\sqrt{2}$ m/s at 45° with horizontal. After 1 s, find tangential and normal acceleration of the particle. Also, find radius of curvature of the trajectory at that point. (Take $g = 10 \text{ m/s}^2$)

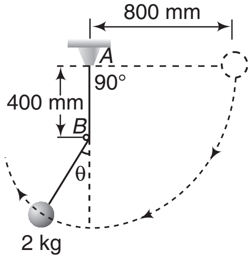
7. If the system shown in the figure is rotated in a horizontal circle with angular velocity ω. Find ($g = 10 \text{ m/s}^2$)



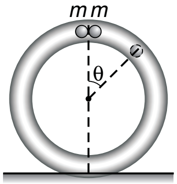
(a) the minimum value of ω to start relative motion between the two blocks.
(b) tension in the string connecting m_1 and m_2 when slipping just starts between the blocks.

The coefficient of friction between the two masses is 0.5 and there is no friction between m_2 and ground. The dimensions of the masses can be neglected. (Take $R = 0.5\text{ m}$, $m_1 = 2\text{ kg}$, $m_2 = 1\text{ kg}$)

8. The simple 2 kg pendulum is released from rest in the horizontal position. As it reaches the bottom position, the cord wraps around the smooth fixed pin at B and continues in the smaller arc in the vertical plane. Calculate the magnitude of the force R supported by the pin at B when the pendulum passes the position $\theta = 30°$. ($g = 9.8\ \text{m/s}^2$)

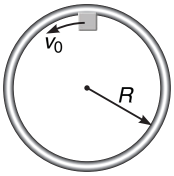


9. A circular tube of mass M is placed vertically on a horizontal surface as shown in the figure. Two small spheres, each of mass m, just fit in the tube, are released from the top. If θ gives the angle between radius vector of either ball with the vertical, obtain the value of the ratio M/m if the tube breaks its contact with ground when $\theta = 60°$. Neglect any friction.

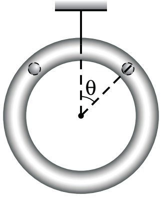


10. A table with smooth horizontal surface is turning at an angular speed ω about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the centre. Find the speed of the particle with respect to the table as its distance from the centre becomes L.

11. A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius R. At time $t = 0$, block is moving along the inside of the ring (i.e. in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ. Find the speed of the block at time t.



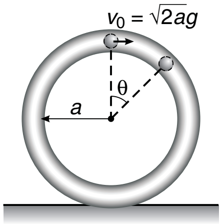
12. A ring of mass M hangs from a thread and two beads of mass m slides on it without friction. The beads are released simultaneously from the top of the ring and slides down in opposite sides. Show that the ring will start to rise, if $m > \frac{3M}{2}$.



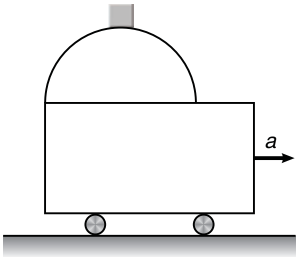
13. A smooth circular tube of radius R is fixed in a vertical plane. A particle is projected from its lowest point with a velocity just sufficient to carry it to the highest point. Show that the time taken by the particle to reach the end of the horizontal diameter is $\sqrt{\frac{R}{g}}\ln(1+\sqrt{2})$.

Hint : $\int \sec\theta \cdot d\theta = \ln(\sec\theta + \tan\theta)$

14. A heavy particle slides under gravity down the inside of a smooth vertical tube held in vertical plane. It starts from the highest point with velocity $\sqrt{2ag}$, where a is the radius of the circle. Find the angular position θ (as shown in figure) at which the vertical acceleration of the particle is maximum.



15. A vertical frictionless semicircular track of radius 1 m is fixed on the edge of a movable trolley (figure). Initially, the system is rest and a mass m is kept at the top of the track. The trolley starts moving to the right with a uniform horizontal acceleration $a = 2g/9$. The mass slides down the track, eventually losing contact with it and dropping to the floor 1.3 m below the trolley. This 1.3 m is from the point where mass loses contact. ($g = 10\ \text{m/s}^2$)



(a) Calculate the angle θ at which it loses contact with the trolley and

(b) the time taken by the mass to drop on the floor, after losing contact.

Answers

Introductory Exercise 10.1

1. Variable **2.** speed, acceleration, magnitude of acceleration

3. (a) 4.0 cm/s^2 (b) 2.0 cm/s^2 (c) $2\sqrt{5}$ cm/s^2 **4.** $\frac{2\sqrt{2}}{\pi}$ **5.** $\frac{1}{2}$ s **6.** 45°

7. (a) 21.65 ms^{-2} (b) 7.35 ms^{-1} (c) 12.5 ms^{-2}

Introductory Exercise 10.2

1. $\tan^{-1}\left(\frac{1}{4}\right)$ **2.** 0.25 **3.** (a) 17 ms^{-1} **4.** No **5.** He should apply the brakes

6. (a) 39.6 rpm (b) 150 N **7.** (a) 24 kN (b) 50 ms^{-1} (c) 32 kN

Introductory Exercise 10.3

1. (a) $\frac{5}{3}R$ (b) $\sqrt{\frac{2}{3}gR}$ **2.** (a) 60° (b) $\frac{mg}{2}$ **3.** (a) $\sqrt{3\,gR}$ (b) 2 mg (c) 8 mg **4.** 7 ms^{-1}

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (b) **3.** (c) **4.** (b) **5.** (b) **6.** (a) **7.** (d) **8.** (c) **9.** (c) **10.** (a)
11. (a)

Single Correct Option

1. (c) **2.** (c) **3.** (c) **4.** (a) **5.** (d) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (c) **12.** (d) **13.** (c) **14.** (b)

Subjective Questions

1. 9.5 ms^{-2} **2.** 4.7 rad/s **3.** $\frac{u^2\cos^2\theta}{g}$ **4.** 35 m/s

5. $\sqrt{\frac{g}{R\cos\alpha}}$ **7.** 113 ms^{-2} **8.** (a) $\sqrt{\frac{\mu g}{L}}$ (b) $\left[\left(\frac{\mu g}{L}\right)^2-\alpha^2\right]^{\frac{1}{4}}$ **9.** 60°

10. (a) T = 64 N, f = 48 N (outwards) (b) 1.63 rad/s (c) 5 rad/s

11. $\theta=\cos^{-1}\left(\frac{3}{4}\right)$ **12.** 5.66 ms^{-1}, 16.75 ms^{-2} **13.** $\sin^{-1}\left(\frac{1}{4}\right)$

LEVEL 2

Single Correct Option

1. (c) **2.** (b) **3.** (d) **4.** (c) **5.** (c) **6.** (c) **7.** (c) **8.** (b) **9.** (b) **10.** (c)
11. (d) **12.** (b) **13.** (a) **14.** (c) **15.** (b) **16.** (c) **17.** (b) **18.** (d)

More than One Correct Options

1. (b,d) **2.** (b,d) **3.** (a,d) **4.** (a, b, c) **5.** (d)

Comprehension Based Questions

1. (c) **2.** (d) **3.** (b) **4.** (c) **5.** (a)

Match the Columns

1. (a) → (p) (b) → (s) (c) → (r) (d) → (p)

2. (a) → (r) (b) → (p) (c) → (s) (d) → (r)

3. (a) → (q) (b) → (p) (c) → (r) (d) → (p)

4. (a) → (s) (b) → (s) (c) → (r) (d) → (s)

5. (a) → (s) (b) → (q) (c) → (r) (d) → (q)

Subjective Questions

1. (a) $\frac{3L}{2}$ (b) $\frac{40L}{27}$ **2.** (a) $\frac{3mg}{2}$ (d) $\frac{5mg}{2}$ **3.** $\theta = \frac{\pi}{2}$ **4.** $4.2\ \text{ms}^{-1} \le v \le 15\ \text{ms}^{-1}$

5. $\frac{u^2 \cos^2 \theta}{g \cos^3(\theta/2)}$ **6.** $a_t = -2\sqrt{5}\ \text{ms}^{-2}$, $a_n = 4\sqrt{5}\ \text{ms}^{-2}, R = 25\sqrt{5}\ \text{m}$ **7.** (a) $\omega_{\min} = 6.32$ rad/s (b) $T = 30$ N

8. 45 N **9.** $\frac{M}{m} = \frac{1}{2}$ **10.** $v = \omega\sqrt{L^2 - a^2}$ **11.** $\frac{v_0}{1 + \frac{\mu v_0 t}{R}}$ **14.** $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

15. (a) 37° (b) 0.38 s

Hints & Solutions

1. Basic Mathematics

Subjective Questions

2. (c) 2 sin 45° cos 15°

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\therefore \quad \frac{C+D}{2} = 45° \quad \text{or} \quad C+D = 90° \qquad \text{...(i)}$$

$$\frac{C-D}{2} = 15° \quad \text{or} \quad C-D = 30° \qquad \text{...(ii)}$$

Solving these two equations, we get

$$C = 60°, \; D = 30°$$

$$\therefore \quad 2\sin 45° \cos 15° = \sin 60° + \sin 30°$$

$$= \left(\frac{\sqrt{3}+1}{2}\right) \qquad \textbf{Ans.}$$

(d) Apply

$$\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$$

$$C = 60° \text{ and } D = 30°$$

$$\therefore\; 2\sin 15° \cos 45° = \sin 60° - \sin 30°$$

$$= \left(\frac{\sqrt{3}-1}{2}\right) \qquad \textbf{Ans.}$$

4. (c) and (d) parts : Refer (c) and (d) parts of example-1.2

6. (e) $y = (\sin 2x - x)$

$$\frac{dy}{dx} = 2\cos 2x - 1$$

and $$\frac{d^2y}{dx^2} = -4\sin 2x$$

Putting $\frac{dy}{dx} = 0$, we get

$$2\cos 2x - 1 = 0$$

$$\therefore \quad \cos 2x = \frac{1}{2}$$

or $$2x = \pm\, 60° \pm \frac{\pi}{3}$$

or $$\pi/2 \le x \le \pi/2$$

At $2x = +\,60°, \frac{d^2y}{dx^2}$ is –ve, so value of y is maximum. At $2x = -\,60°, \frac{d^2y}{dx^2}$ is positive. So value of y is minimum.

$$\therefore \quad y_{\max} = \sin(+60°) - \frac{\pi}{6}$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \text{ at } 2x = \frac{\pi}{3}$$

or $$x = \pi/6$$

and $$y_{\min} = \sin(-60°) + \frac{\pi}{6}$$

at $$x = -\frac{\pi}{6}$$

$$= \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) \qquad \textbf{Ans.}$$

2. Measurement and Errors

INTRODUCTORY EXERCISE 2.1

1. (a) $13.214 + 234.6 + 7.0350 + 6.38$

$$= 261.229$$
$$= 261.2$$

(rounding off to smallest one number of decimal place)

(b) $1247 + 134.5 + 450 + 78$

$$= 1909.5$$
$$= 1910$$

(rounding off to smallest one number of decimal place)

2. (a) $16.235 \times 0.217 \times 5$

$$= 17.614975$$
$$= 20$$

(rounding off to minimum one number of significant figure)

(b) 0.00435×4.6

$$= 0.02001$$
$$= 0.020$$

(rounding off to minimum two number of significant figures)

Exercises

Objective Questions

3. $A = l \times b = 3.124 \times 3.002$

$$= 9.378248 \text{ m}^2$$
$$= 9.378 \text{ m}^2$$

(rounding off to four significant digits) **Ans.**

4. $V = lbt = 12 \times 6 \times 2.45$

$$= 176.4 \text{ cm}^3 = 2 \times 10^2 \text{ cm}^3$$

(rounding off to one significant digit of breadth) **Ans.**

5. $V = \frac{4}{3}\pi R^3$

$\therefore$ (% error in V) = 3 (% error in R)

$$= 3\ (1\ \%)$$
$$= 3\ \%$$ **Ans.**

6. $\rho = \frac{m}{V} = \frac{m}{l^3} = ml^{-3}$

$\therefore$ Maximum % error in ρ = (% error in m) + 3 (% error in l)

7. $K = \frac{1}{2}mv^2$

$\therefore$ % error in K = (% error in m) + 2 (% error in v)

8. $P = \frac{F}{A} = \frac{F}{L^2} = FL^{-2}$

Permissible error (%) in P = (% error in F) +2 (% error in L)

9. $H = i^2Rt$

% error in H = 2 (% error in i) + (% error in R) + (% error in t)

10. $g = \frac{GM}{R^2}$

or $\quad g \propto R^{-2}$

$\therefore$ % change in g = (−2) (% change in R)

$$= (-2)(-2) = +\ 4\%$$

Rotational kinetic energy,

$$K = \frac{L^2}{2I} \qquad \ldots\text{(i)}$$

L = angular momentum = constant

$$I = \frac{2}{5}mR^2 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$K \propto R^{-2}$$

$\therefore$ % change in K = (−2) (% change in R)

$$= (-2)\ (-2) = +\ 4\ \%$$ **Ans.**

11. % error in $A = \frac{1}{2}$ (% error in p) + $\frac{1}{2}$ (% error in q) +2 (% error in r) +3 (% error in s)

12. $T = \frac{t}{n}$ and $\Delta T = \frac{\Delta t}{n}$

$$\therefore \quad \frac{\Delta T}{T} \times 100 = \frac{\Delta t}{t} \times 100$$
$$= \frac{0.1}{2 \times 100} \times 100 = 0.05\ \%$$

$$\frac{\Delta l}{l} \times 100 = \frac{0.1 \text{ cm}}{100 \text{ cm}} \times 100$$
$$= 0.1\ \%$$

Now, $\quad T = 2\pi\sqrt{\frac{l}{g}}$

or $$g = \frac{4\pi^2 l}{T^2} \propto \frac{l}{T^2}$$

$\therefore$ % error in g = (% error in l) +2 (% error in T)

$$= 0.1\,\% + 2\,(0.05\,\%) = 0.2\,\%$$ **Ans.**

13. Number 25 has infinite number of significant figures. Therefore we will round off to least number of significant figures or three significant figures in the measurement 1.76 kg.

14. $T = \frac{t}{n} \Rightarrow \Delta T = \frac{\Delta t}{n}$

$$\therefore \quad \frac{\Delta T}{T} \times 100 = \frac{\Delta t}{t} \times 100$$

$$= \frac{0.2}{25} \times 100$$

$$= 0.8\%$$ **Ans.**

Subjective Questions

6.
$$a = 6.75 \times 10^3 \text{cm}$$
$$b = 4.52 \times 10^2 \text{ cm}$$
$$= 0.452 \times 10^3 \text{ cm}$$
$$= 0.45 \times 10^3 \text{ cm}$$ (upto 2 places of decimal)

$$\therefore \; a + b = (6.75 \times 10^3 + 0.45 \times 10^3) \text{ cm}$$
$$= 7.20 \times 10^3 \text{ cm}$$

7. We have $\frac{25.2 \times 1374}{33.3} = 1039.7838$

Out of the three numbers given in expression 25.2 and 33.3 have 3 significant digits and 1374 has four. The answer should have three significant digits. Rounding 1039.7838… to three significant digits, it becomes 1040. Thus we write

$$\frac{25.2 \times 1374}{33.3} = 1040$$

8. $(4.0 \times 10^{-4}) - (0.025 \times 10^{-4})$
$$= 4.025 \times 10^{-4}$$
$$= 4.0 \times 10^{-4}$$

upto one decimal place **Ans.**

9. (a) 2.3 kg, (b) 0.02 g

10. $V = \pi r^2 l$
$$= (\pi)\,(0.046 \text{ cm})^2\,(21.7 \text{ cm})$$
$$= 0.1443112 \text{ cm}^3 = 0.14 \text{ cm}^3$$

(rounding off to two significant digits)

Ans.

11. $V = a^3 = (2.342 \text{ m})^3 = 12.84578569 \text{ m}^3$

$= 12.85 \text{ m}^3$ (rounding off to four significant digits)

$S = a^2 = (2.342)^2 \text{ m}^2 = 5.484964 \text{ m}^2$

$= 5.485 \text{ m}^2$ (rounding off to four significant digits)

Ans.

12. $\rho = \frac{m}{V} = \frac{9.23}{1.1}$

$= 8.3909090 \text{ kg/m}^3 = 8.4 \text{ kg/m}^3$

(rounding off to two significant digits) **Ans.**

13. $V = (4.234 \times 1.005 \times 2.01) \text{ m}^3$

$= 8.5528917 \text{ m}^3 = 8.55 \text{ m}^3$

(rounding off to three significant digits) **Ans.**

14. $S = 4\pi r^2 = 4\pi\,(2.1)^2 = 55.4 \text{ cm}^2$

$$\frac{\Delta S}{S} = 2 \times \frac{\Delta r}{r} \Rightarrow \therefore \quad \Delta S = 2 \times \frac{\Delta r}{r} \times S$$

$$= 2 \times \frac{0.5}{2.1} \times 55.4 = 26.4 \text{ cm}^2$$

$$\therefore \quad (S \pm \Delta S) = (55.4 \pm 26.4) \text{ cm}^2$$ **Ans.**

15. $(50°\text{C} \pm 0.5°\text{C}) - (20°\text{C} \pm 0.5°\text{C})$

$$= (30°\text{C} \pm 1°\text{C})$$ **Ans.**

16. The percentage error in V is 5% and in I it is 2%. The total error in R would therefore be 5% + 2% = 7%

17. $$\frac{\Delta\rho}{\rho} \times 100 = \left[2\left(\frac{\Delta r}{r}\right) + \left(\frac{\Delta R}{R}\right) + \left(\frac{\Delta l}{l}\right)\right] \times 100$$

$$= \left[\frac{2 \times 0.02}{0.2} + \frac{2}{60} + \frac{0.1}{150}\right] \times 100$$

$$= 23.4\ \%$$ **Ans.**

18. % error in ρ = 3 (% error in α) +2 (% error in β)

$$+ \frac{1}{2}(\text{ \% error in } \gamma) + (\text{\% error in } \eta)$$

19. $g = 4\pi^2 L/T^2$

Here, $T = \frac{t}{n}$ and $\Delta T = \frac{\Delta t}{n}$. Therefore, $\frac{\Delta T}{T} = \frac{\Delta t}{t}$.

The errors in both L and t are the least count errors. Therefore,

$$(\Delta g/g) = (\Delta L/L) + 2(\Delta T/T)$$
$$= (\Delta L/L) + 2(\Delta t/t)$$
$$= \frac{0.1}{20.0} + 2\left(\frac{1}{90}\right) = 0.027$$

Thus, the percentage error in g is

$$(\Delta g/g) \times 100 = 2.7\%$$

3. Experiments

INTRODUCTORY EXERCISE 3.1

1. $$\text{LC} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$$

$$= \frac{1\text{ mm}}{10} = 0.1\text{ mm} = 0.01\text{ cm}$$

$$\text{Positive zero error} = N + x\,(\text{LC})$$

$$= 0 + 5 \times 0.01$$

$$= 0.05\text{ cm}$$

$$\text{Diameter} = 3.2 + 4 \times 0.01 = 3.24\text{ cm}$$

$$\text{Actual diameter} = 3.24 - 0.05 = 3.19\text{ cm}$$

2. $$(N + m)\,\text{VSD} = (N)\,\text{MSD}$$

$$\Rightarrow \quad 1\,\text{VSD} = \left(\frac{N}{N + m}\right)\text{MSD}$$

$$\text{LC} = 1\,\text{MSD} - 1\,\text{VSD} = 1\,\text{MSD} - \left(\frac{N}{N + m}\right)\text{MSD}$$

$$= \frac{m}{N + m}\,\text{MSD} = \left(\frac{1}{1 + N/m}\right)\text{MSD}$$

Now, least count will be minimum for $m = 1$.

INTRODUCTORY EXERCISE 3.2

1. $$\text{LC} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

$$= \frac{1\text{mm}}{100} = 0.01\text{ mm}$$

$$\text{Linear scale reading} = 10\,(\text{pitch}) = 10\text{ mm}$$

$$\text{circular scale reading} = n\,(\text{LC}) = 65 \times 0.01$$

$$= 0.65\text{ mm}$$

$$\therefore \quad \text{Total reading} = (10 + 0.65)\text{ mm} = 10.65\text{ mm}$$

2. $$\text{LC} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

$$= \frac{1\text{mm}}{50} = 0.02\text{ mm}$$

$$\text{Positive zero error} = n_1\,(\text{LC})$$

or, $$e = 6 \times 0.02 = 0.12\text{ mm}$$

$$\text{Linear scale reading} = 3\,(\text{pitch}) = 3\text{ mm}$$

$$\text{Circular scale reading} = n_2\,(\text{LC}) = 31 \times 0.02$$

$$= 0.62\text{ mm}$$

$$\text{Measured diameter of wire} = (3 + 0.62)\text{ mm}$$

$$= 3.62\text{ mm}$$

$\therefore$ Actual diameter of wire

$$= 3.62\text{ mm} - 0.12\text{ mm} = 3.50\text{ mm}$$

INTRODUCTORY EXERCISE 3.3

1. A pendulum which has a time period of two seconds is called a second's pendulum.

2. Because $T = 2\pi\sqrt{L/g}$ is based on the assumption that $\sin\theta \cong \theta$ which is true only for small amplitude.

3. No, the time period does not depend on any of the given three properties of the bob.

4. $$L = \left(\frac{g}{4\pi^2}\right)T^2$$

5. The length of the pendulum used in clocks increases in summer and hence T increases whereas in winter, the length of the pendulum decreases, so T decreases. T increases means clock goes slow.

6. Invar is an alloy which has a very small coefficient of linear thermal expansion. Hence, the time period does not change appreciably with the change of temperature.

7. During the draining of the sand, the period first increases due to change in effective length, then decreases and finally attains a value that it had when the sphere was full of sand.

8. $$g_{\text{moon}} = \frac{g_{\text{earth}}}{6} = \frac{g}{6} \quad \Rightarrow \quad T = \text{constant}$$

$\therefore$ l should be made $\frac{l}{6}$ at moon

$$\left(\text{because } T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l/6}{g/6}}\right)$$

INTRODUCTORY EXERCISE 3.4

1. $$Y = \frac{4FL}{\pi d^2 l} = \frac{4 \times 1.0 \times 9.8 \times 2}{(\pi)\,(0.4 \times 10^{-3})^2\,(0.8 \times 10^{-3})}$$

$$= 1.94 \times 10^{11}\text{ N/m}^2$$

Further, $$Y = \frac{4MgL}{\pi d^2 l}$$

$$\Rightarrow \quad \frac{\Delta Y}{Y} = 2\left(\frac{\Delta d}{d}\right) + \frac{\Delta l}{l}$$

or $$\Delta Y = \left[2\left(\frac{\Delta d}{d}\right) + \left(\frac{\Delta l}{l}\right)\right] \times Y$$

$$= \left[2 \times \left(\frac{0.01}{0.4}\right) + \left(\frac{0.05}{0.8}\right)\right] \times 1.94 \times 10^{11}$$

$$= 0.22 \times 10^{11}\text{ N/m}^2$$

INTRODUCTORY EXERCISE 3.5

1. Let Δl be the end correction.

Given that fundamental tone for a length 0.1m and first overtone for the length is 0.35 m.

$$f = \frac{v}{4(0.1 + \Delta l)}$$

$$= \frac{3v}{4(0.35 + \Delta l)}$$

Solving this equation, we get

$$\Delta l = 0.025\,\text{m}$$

2. With end correction,

$$f = n\left[\frac{v}{4\,(l + e)}\right], \text{ (where, } n = 1, 3, \ldots)$$

$$= n\left[\frac{v}{4\,(l + 0.6\,r)}\right]$$

Because, $e = 0.6\,r$, where r is radius of pipe.

For first resonance, $n = 1$

$$\therefore \quad f = \frac{v}{4\,(l + 0.6\,r)}$$

or

$$l = \frac{v}{4f} - 0.6r$$

$$= \left[\left(\frac{336 \times 100}{4 \times 512}\right) - 0.6 \times 2\right]\text{cm}$$

$$= 15.2 \text{ cm}$$

INTRODUCTORY EXERCISE 3.6

1. $$\rho = \frac{\pi d^2 V}{4lI} = \frac{(3.14)(2.00 \times 10^{-3})^2(100.0)}{(4)(31.4)(10.0) \times 10^{-2}}$$

$1.00 \times 10^{-4}\ \Omega$ - m (to three significant figures)

2. $$\frac{\Delta\rho}{\rho} \times 100 = \left[2\frac{\Delta d}{d} + \frac{\Delta V}{V} + \frac{\Delta l}{l} + \frac{\Delta I}{I}\right] \times 100$$

$$= \left[2 \times \left(\frac{0.01}{2.00}\right) + \left(\frac{0.1}{100.0}\right) + \left(\frac{0.1}{31.4}\right) + \left(\frac{0.1}{10.0}\right)\right] \times 100$$

$$= 2.41\,\%$$

$$R = \frac{V}{I}$$

$$\Rightarrow \quad \frac{\Delta R}{R} \times 100 = \left[\left(\frac{\Delta V}{V}\right) + \left(\frac{\Delta I}{I}\right)\right] \times 100$$

$$= \left[\frac{0.1}{100} + \frac{0.1}{10}\right] \times 100 = 1.1\%$$

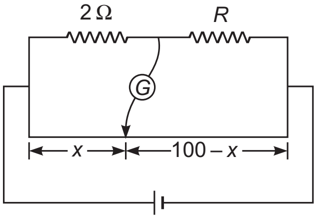
3. We will require a voltmeter, an ammeter, a test resistor and a variable battery to verify Ohm's law.

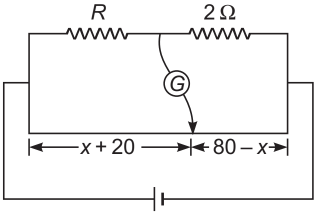
Voltmeter which is made by connecting a high resistance with a galvanometer is connected in parallel with the test resistor.

Further, an ammeter which is formed by connecting a low resistance in parallel with galvanometer is required to measure the current through test resistor.

INTRODUCTORY EXERCISE 3.7

1. $R > 2\,\Omega \quad \Rightarrow \quad \therefore \quad 100 - x > x$





Applying $\quad \dfrac{P}{Q} = \dfrac{R}{S}$

We have $$\frac{2}{R} = \frac{x}{100 - x} \quad \ldots\text{(i)}$$

$$\frac{R}{2} = \frac{x + 20}{80 - x} \quad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get, $R = 3\,\Omega$

$\therefore$ Correct option is (a).

2. Using the concept of balanced, Wheatstone bridge, we have,

$$\frac{P}{Q} = \frac{R}{S}$$

$$\therefore \quad \frac{X}{(52 + 1)} = \frac{10}{(48 + 2)}$$

$$\therefore \quad X = \frac{10 \times 53}{50} = 10.6\,\Omega$$

$\therefore$ Correct option is (b).

3. Slide wire bridge is most sensitive when the resistance of all the four arms of bridge is same.

Hence, B is the most accurate answer.

INTRODUCTORY EXERCISE 3.8

1. $\frac{P}{Q} = \frac{R}{X}$

$$\Rightarrow \qquad X = \left(\frac{Q}{P}\right) R = \left(\frac{1}{10}\right) R$$

R lies between 142 Ω and 143 Ω.

Therefore, the unknown resistance X lies between 14.2 Ω and 14.3 Ω.

2. Experiment can be done in similar manner but now K_2 should be pressed first then K_1.

3. BC, CD and BA are known resistances.

The unknown resistance is connected between A and D.

Exercises

Objective Questions

7. $\frac{R}{80} = \frac{20}{80}$

$$\therefore \qquad R = 20\ \Omega \qquad \textbf{Ans.}$$

11. Thermal capacity $= ms$

$$= (0.04\text{ kg})\ (4.2 \times 10^2\text{ J kg}^{-1}\ {}^{\circ}\text{C}^{-1})$$

$$= 16.8\text{ J}/{}^{\circ}\text{C} \qquad \textbf{Ans.}$$

12. Intercept $= \frac{1}{f}$

Therefore, $f = \frac{1}{\text{Intercept}} = \frac{1}{0.5}$

$$= 2\text{ m}$$

13. Deflection is zero for $R = 324\ \Omega$

Now, $X = \left(\frac{Q}{P}\right) R = \left(\frac{1}{100}\right)(324)$

$$= 3.24\ \Omega$$

14. $1\text{ MSD} = \frac{1}{10}(1\text{ cm}) = 1\text{ mm}$

$$10\text{ VSD} = 8\text{ MSD}$$

$$1\text{ VSD} = \frac{8}{10}\text{ MSD}$$

Least count = 1 MSD – 1 VSD

$$\therefore \qquad \text{LC} = 1\text{ mm} - \frac{8}{10}\text{ mm}$$

$$\therefore \qquad \text{LC} = \frac{2}{10}\text{ mm}$$

$$\therefore \qquad \text{LC} = \frac{2}{100}\text{ cm} = 0.02\text{ cm} \qquad \textbf{Ans.}$$

15. 50 VSD = 49 MSD

$$1\text{ VSD} = \frac{49}{50}\text{ MSD}$$

$$\therefore \qquad \text{VC} = 1\text{ MSD} - 1\text{ VSD}$$

$$\therefore \qquad \text{VC} = 1\text{ MSD} - \frac{49}{50}\text{ MSD}$$

$$\therefore \qquad \text{VC} = \frac{1}{50}\text{ MSD}$$

$$\therefore \qquad 1\text{ MSD} = 50\text{ (VC)}$$

$$= 50\ (0.001\text{ cm})$$

$$\therefore \qquad 1\text{ MSD} = 0.5\text{ mm} \qquad \textbf{Ans.}$$

16. $\text{LC} = \frac{1}{n}\text{ MSD}$

Here, n = number of vernier scale divisions

$$0.005\text{ cm} = \frac{1}{n}\left(\frac{1}{10}\text{ cm}\right)$$

$$n = \frac{0.1}{0.005} = \frac{1000}{50}$$

$$\therefore \qquad n = 20 \qquad \textbf{Ans.}$$

17. 100 VCD = 99 MSD

$$\therefore \qquad 1\text{ VSD} = \frac{99}{100}\text{ MSD}$$

$$\text{LC} = 1\text{ MSD} - 1\text{ VSD}$$

$$\therefore \qquad \text{LC} = 1\text{ MSD} - \frac{99}{100}\text{ MSD}$$

$$\therefore \qquad \text{LC} = 0.01\text{ MSD}$$

$$\therefore \qquad \text{LC} = 0.01(1\text{ mm}) = 0.01\text{ mm} \qquad \textbf{Ans.}$$

18. $1\text{ VSD} = \frac{0.8\text{ cm}}{10} = 0.08\text{ cm}$

$$1\text{ MSD} = 0.1\text{ cm}$$

$$\therefore \qquad \text{LC} = 1\text{ MSD} - 1\text{ VSD}$$

$$= 0.1\text{ cm} - 0.08\text{ cm}$$

$$= 0.02\text{ cm}$$

19. $A = l \times b$

$$= 10 \times 1.0 = 10\text{ cm}^2$$

$$\frac{\Delta A}{A} = \frac{\Delta l}{l} + \frac{\Delta b}{b}$$

$$\therefore \qquad \Delta A = \pm\left(\frac{\Delta l}{l} + \frac{\Delta b}{b}\right) \times A$$

$$= \pm \left(\frac{0.1}{10.0} - \frac{0.01}{1.00}\right) \times 10$$

$$= \pm\ 0.2\ \text{cm}^2$$

21. Distance moved in one rotation = 0.5 mm

$$\text{Least count, LC} = \frac{0.5\ \text{mm}}{50\ \text{divisions}} = 0.01\ \text{mm}$$

Screw gauge has negative zero error.

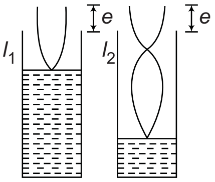
This error is (50 – 20) 0.01 mm or (30) (0.01) mm.

Thickness of plate

$$= (2 \times 0.5\ \text{mm}) + (30 + 20)\ (0.01\ \text{mm})$$

$$= 1.5\ \text{mm}$$ **Ans.**

22.



$$l_1 + e = \frac{\lambda}{4}$$

$$l_2 + e = \frac{3\lambda}{4}$$

Solving these two equations, we get

$$e = \frac{l_2 - 3l_1}{2}$$

23. $e = \dfrac{l_2 - 3l_1}{2}$

$$\therefore \quad e = 1\ \text{cm} = \frac{1}{100}\ \text{m}$$

$$\therefore \quad \frac{1}{100} = \frac{l_2 - 3\ (0.15)}{2}$$

$$\therefore \quad l_2 = 0.47\ \text{m}$$

$$\therefore \quad l = 47\ \text{cm}$$ **Ans.**

24. $\lambda = \dfrac{v}{f} = \dfrac{340}{340} = 1\ \text{m} = 100\ \text{cm}$

Length of air columns may be,

$\dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}$.... or 25 cm, 75 cm 125 cm....

Minimum height of water column

= 120 – maximum height of air column

= 120 – 75 = 45 cm

25. $f = \dfrac{v}{\lambda}$ ...(i)

Now, $l_1 = \dfrac{\lambda}{4}, l_2 = \dfrac{3\lambda}{4}$

$$\Rightarrow \quad l_2 - l_1 = \frac{\lambda}{2}$$

or $\lambda = 2\ (l_2 - l_1)$

Substituting in Eq. (i), we have

$$f = \frac{v}{2\ (l_2 - l_1)}$$

$$\Rightarrow \quad f \propto \frac{1}{l_2 - l_1}$$

$$\therefore \quad \frac{f_1}{f_2} = \frac{l'_2 - l'_1}{l_2 - l_1} = \frac{90 - 30}{30 - 10} = \frac{3}{1}$$

26. Heat lost by aluminium = 500 × s × (100 – 46.8) cal

$\therefore$ Heat lost = 26600 s

Heat gained by water and calorimeter

$$= 300 \times 1 \times (46.8 - 30) + 500 \times 0.093 \times (46.8 - 30)$$

$\therefore$ Heat gained = 5040 + 781.2 = 5821.2

Now, heat lost = Heat gained

$\therefore$ 26600 s = 5821.2

$\therefore$ s ≈ 0.22 cal $\text{g}^{-1}\ (^\circ\text{C})^{-1}$

27. Heat lost = Heat gained

$$\therefore \quad m_1 s_1 \Delta T_1 = m_2 s_2 \Delta T_2$$

$$\therefore \quad s_1 = \frac{m_2 s_2 \Delta T_2}{m_1 \Delta T_1}$$

$$= \frac{0.5 \times 4.2 \times 10^3 \times 3}{0.2 \times 77}\ \text{J kg}^{-1}\ {}^\circ\text{C}^{-1}$$

$$\therefore \quad s_1 = 0.41 \times 10^3\ \text{J kg}^{-1}\ {}^\circ\text{C}^{-1}$$ **Ans.**

28. Heat lost = Heat gained

$$0.20 \times 10^3 \times s(150 - 40) = 150 \times 1 \times (40 - 27) + 0.025 \times 10^3 \times (40 - 27)$$

$\{\because\ s_{\text{water}} = 1\ \text{cal g}^{-1}\ (^\circ\text{C})^{-1}\}$

$\therefore$ 22000 s = 1950 + 325

$\therefore$ 22000 s = 2275

$\therefore$ s = 0.10 cal $\text{g}^{-1}\ (^\circ\text{C})^{-1}$ **Ans.**

29. $\dfrac{R_1}{R_2} = \dfrac{50}{50} = 1$

$$\therefore \quad R_1 = R_2 = R \quad ...(i)$$

When 24 Ω is connected in parallel with R_2, then the balance point is 70 cm, so

$$\frac{R_P}{R} = \frac{\left(\dfrac{24R}{24 + R}\right)}{R} = \frac{30}{70}$$

$(\because\ R_P < R)$

$$\therefore \quad \frac{24}{24 + R} = \frac{3}{7}$$

$\therefore \quad 168 = 72 + 3R$

$\therefore \quad 96 = 3R$

$\therefore \quad R = 32\ \Omega$ **Ans.**

30. $\frac{R_1 + 10}{R_2} = \frac{50}{50} = 1$

$\therefore \quad R_1 + 10 = R_2$ ...(i)

Again, $\frac{R_1}{R_2} = \frac{40}{60} = \frac{2}{3}$

$\therefore \quad 3R_1 = 2R_2$

Substituting the value of R_2 from Eq. (i), we get

$3R_1 = 2(R_1 + 10)$

$\therefore \quad R_1 = 20\ \Omega$ **Ans.**

31. $\frac{R_1}{R_2} = \frac{20}{80} = \frac{1}{4}$

$\therefore \quad R_2 = 4R_1$

$\therefore \quad \frac{R_1 + 15}{R_2} = \frac{40}{60} = \frac{2}{3}$

$\therefore \quad \frac{R_1 + 15}{4R_1} = \frac{2}{3}$

$\therefore \quad R_1 = 9\ \Omega$ **Ans.**

32. $\frac{X}{Y} = \frac{20}{80} = \frac{1}{4}$

$\therefore \quad Y = 4X$

Since, $\frac{4X}{Y} = \frac{l}{100 - l}$

$\therefore \quad \frac{4X}{4X} = \frac{l}{100 - l}$

$\therefore \quad 2l = 100$

$\therefore \quad l = 50$ cm **Ans.**

33. For meter bridge to be balanced

$$\frac{P}{Q} = \frac{40}{60} = \frac{2}{3}$$

$\therefore \quad P = \frac{2}{3}Q$

When Q is shunted, i.e. a resistance of $10\ \Omega$ is connected in parallel across Q, the net resistance becomes $\frac{10Q}{10 + Q}$.

Now, the balance point shifts to 50 cm, i.e.

$$\frac{P}{\left(\frac{10Q}{10 + Q}\right)} = 1$$

$\therefore \quad \frac{2}{3} = \frac{10}{10 + Q}$

$\therefore \quad 20 + 2Q = 30$

$\therefore \quad Q = 5\ \Omega$

and $P = \frac{10}{3}\ \Omega$ **Ans.**

Subjective Questions

2. The bridge method is better because it is the null point method which is superior to all other methods.

3. Because the graph in this case is a straight line.

4. In the case of second resonance, energy gets distributed over a larger region and as such second resonance becomes feebler.

5. The bridge becomes insensitive for too high or too low values and the readings become undependable. When determining low resistance, the end resistance of the meter bridge wire and resistance of connecting wires contribute towards the major part of error.

6. No, the resistance of the connecting wires is itself of the order of the resistance to be measured. It would create uncertainty in the measurement of low resistance.

7. 20 VSD = 19 MSD

$\therefore \quad 1\text{ VSD} = \frac{19}{20}\text{ MSD}$

$$\text{LC} = 1\text{ MSD} - 1\text{ VSD}$$

$$= 1\text{ MSD} - \frac{19}{20}\text{ MSD} = \frac{1}{20}\text{ MSD}$$

$$= \frac{1\text{ cm}}{20} = 0.05\text{ cm}$$

8. The value of one main scale division $= \frac{1}{20}$ cm

Number of divisions on vernier scale = 20

$$\text{LC} = \frac{\text{Value of one main scale division}}{\text{Number of divisions on vernier scale}}$$

$$= \frac{\frac{1}{20}}{20} = \frac{1}{400} = 0.0025\text{ cm}$$ **Ans.**

9. Least count $= \frac{\text{Value of one main scale division}}{\text{Number of divisions on vernier scale}}$

$$= \frac{1\text{ mm}}{10} = \frac{0.1\text{ cm}}{10}$$

$= 0.01$ cm **Ans.**

Reading (diameter)

= MS reading + (coinciding VS reading × Least Count)

$= 4.3 \text{ cm} + (7 \times 0.01)$

$= 4.3 + 0.07 = 4.37 \text{ cm}$

Diameter = 4.37 cm

$\therefore$ Radius $= \dfrac{4.37}{2} = 2.185$ cm

$= 2.18$ cm

To the required number of significant figures.

10. Pitch of the screw $= \dfrac{1\text{mm}}{2}$

$= 0.5$ mm

Least count $= \dfrac{0.5}{50}$

$= 0.01$ mm **Ans.**

Reading = Linear scale reading + (coinciding circular scale × least count)

$= 3.0 \text{ mm} + (32 \times 0.01) = 3.0 + 0.32$

$= 3.32$ mm **Ans.**

11. $\text{LC} = \dfrac{0.5 \text{ mm}}{50} = 0.01$ mm

Thickness $= 5 \times 0.5 \text{ mm} + 34 \times 0.01$ mm

$= 2.84$ mm

12. $\text{LC} = \dfrac{1 \text{ mm}}{50} = 0.02$ mm

Negative zero error $= (50 - 44) \times 0.02 = 0.12$ mm

Thickness

$= (3 \times 1) \text{ mm} + (26 \times 0.02) \text{ mm} + 0.12$ mm

$= 3.64$ mm

13. $\text{LC} = \dfrac{1 \text{ mm}}{100} = 0.01$ mm

The instrument has a positive zero error,

$e = + n\,(\text{LC}) = + (6 \times 0.01) = + 0.06$ mm

Linear scale reading $= 2 \times (1 \text{ mm}) = 2$ mm

Circular scale reading

$= 62 \times (0.01 \text{ mm}) = 0.62$ mm

$\therefore$ Measured reading

$= 2 + 0.62 = 2.62$ mm

or True reading

$= 2.62 - 0.06$

$= 2.56$ mm

14. The instrument has a negative error,

$e = (-5 \times 0.01)$ cm

or $e = -\ 0.05$ cm

Measured reading

$= (2.4 + 6 \times 0.01) = 2.46$ cm

True reading = Measured reading $- e$

$= 2.46 - (-0.05)$

$= 2.51$ cm

Therefore, diameter of the sphere is 2.51 cm.

15. We have,

Least count of vernier callipers

$= \dfrac{1\text{mm}}{10} = 0.1 \text{ mm} = 0.01$ cm

Side of cube = (10) (1mm) + (1) (LC)

or $a = 10 \text{ mm} + 0.1$ mm

or $a = 10.1$ mm

or $a = 1.01$ cm

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{m}{a^3}$$

$$= \frac{2.736}{(1.01)^3}$$

$= 2.65553 \text{ g/cm}^3$

$= 2.66 \text{ g/cm}^3$ **Ans.**

4. Units and Dimensions

Exercises

Single Correct Option

1. $L = mvR = \dfrac{nh}{2\pi}$

$\therefore \quad [L] = [h] = [mvR]$

2. Velocity gradient is change in velocity per unit depth.

3. Coefficient of friction is unitless and dimensionless.

4. Dipole moment = (charge) × (distance)

Electric flux = (electric field) × (area)

Hence, the correct option is (d).

5. $[\eta] = \left[\dfrac{F}{av}\right] = \left[\dfrac{\text{MLT}^{-2}}{\text{LLT}^{-1}}\right]$

6. $a = \dfrac{F}{t}$

$b = \dfrac{F}{t^2}$

7. $R = \dfrac{l}{\sigma A} \Rightarrow \sigma = \dfrac{l}{RA}$

From $\quad H = I^2Rt \quad ,$

we have $\quad R = \dfrac{H}{I^2t}$

$$\therefore \quad [\sigma] = \left[\frac{lI^2t}{HA}\right] = \left[\frac{\text{LA}^2\text{T}}{\text{ML}^2\text{T}^{-2}\text{L}^2}\right] = [\text{M}^{-1}\text{L}^{-3}\text{T}^3\text{A}^2]$$

8. $\phi = Bs = \dfrac{F}{IL} \cdot s = \left[\dfrac{\text{MLT}^{-2}\text{L}^2}{\text{AL}}\right]$

$[g] = \text{LT}^{-2}]$

10. If unit of length and time is double, then value of g will be halved.

11. $B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$

But $\quad B = \dfrac{F}{il} \quad (F = ilB)$

$\therefore \quad \dfrac{F}{il} = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$

$\therefore \quad [\mu_0] = \left[\dfrac{F}{i^2}\right]$ **Ans.**

12. $\because \quad \omega k$ is dimensionless

$\therefore \quad [k] = \left[\dfrac{1}{\omega}\right] = [\text{T}]$

13. Let $E^a v^b F^c = km$

Then,

$$[\text{ML}^2\text{T}^{-2}]^a\,[\text{LT}^{-1}]^b\,[\text{MLT}^{-2}]^c = [\text{M}]$$

Equating the powers, we get

$a = 1, b = -2$ and $c = 0$ **Ans.**

14. $[\text{F}]^a\,[\text{L}]^b\,[\text{T}]^c = [\text{M}]$

$\therefore \quad [\text{MLT}^{-2}]^a\,[\text{L}]^b\,[\text{T}]^c = [\text{M}]$

Equating the powers we get,

$a = 1, b = -1$

and $\quad c = 2$ **Ans.**

15. $L = I\omega = \dfrac{nh}{2\pi}$

$\therefore \quad \dfrac{h}{I} = [\omega]$

16. $\because \quad \dfrac{2\pi x}{\lambda}$ is dimensionless.

$\therefore \quad [\lambda] = [x] = [\text{L}] = [\text{A}]$

17. In option (b), all three are related to each other.

18. $$[Y] = \left[\frac{X}{Z^2}\right] = \left[\frac{\text{Capacitance}}{(\text{Magnetic induction})^2}\right] = \left[\frac{\text{M}^{-1}\text{L}^{-2}\text{Q}^2\text{T}^2}{\text{M}^2\text{Q}^{-2}\text{T}^{-2}}\right] = [\text{M}^{-3}\text{L}^{-2}\text{T}^4\text{Q}^4]$$

19. $C = \dfrac{\Delta q}{\Delta V} = \dfrac{\varepsilon_0 A}{d}$

or $\quad \varepsilon_0 \dfrac{A}{L} = \dfrac{\Delta q}{\Delta V}$

or $\quad \varepsilon_0 = \dfrac{(\Delta q)\,L}{A.(\Delta V)}$

$X = \varepsilon_0 L \dfrac{\Delta V}{\Delta t}$

$$= \frac{(\Delta q)L}{A\,(\Delta V)}\,L\,\frac{\Delta V}{\Delta t}$$

but $[A] = [L^2]$

∴ $X = \dfrac{\Delta q}{\Delta t}$ = current

20. $\left[\dfrac{\alpha Z}{k\theta}\right] = [M^0L^0T^0]$

$$[\alpha] = \left[\frac{k\theta}{Z}\right]$$

Further $[p] = \left[\dfrac{\alpha}{\beta}\right]$

∴ $[\beta] = \left[\dfrac{\alpha}{p}\right] = \left[\dfrac{k\theta}{Zp}\right]$

Dimensions of $k\theta$ are that to energy. Hence,

$$[\beta] = \left[\frac{ML^2T^{-2}}{LML^{-1}T^{-2}}\right]$$
$$= [M^0L^2T^0]$$

More than One Correct Options

1. (a) Torque and work both have the dimensions $[ML^2T^{-2}]$.

(d) Light year and wavelength both have the dimensions of length i.e. [L].

2. Reynold's number and coefficient of friction are dimensionless quantities.

Curie is the number of atoms decaying per unit time and frequency is the number of oscillations per unit time. Latent heat and gravitational potential both have the same dimension corresponding to energy per unit mass.

3. (a) $L = \dfrac{\phi}{i}$ or henry $= \dfrac{\text{weber}}{\text{ampere}}$

(b) $e = -L\left(\dfrac{di}{dt}\right) \Rightarrow \therefore\ L = -\dfrac{e}{(di/dt)}$

or henry $= \dfrac{\text{volt-second}}{\text{ampere}}$

(c) $U = \dfrac{1}{2}Li^2 \Rightarrow \therefore\ L = \dfrac{2U}{i^2}$

or henry $= \dfrac{\text{joule}}{(\text{ampere})^2}$

(d) $U = \dfrac{1}{2}Li^2 = i^2\,Rt$

∴ $L = Rt$ or henry = ohm-second

4. $F = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{q_1\,q_2}{r^2}$

$$[\varepsilon_0] = \frac{[q_1][q_2]}{[F][r^2]} = \frac{[IT]^2}{[MLT^{-2}][L^2]}$$
$$= [M^{-1}\,L^{-3}\,T^4I^2]$$

Speed of light, $c = \dfrac{1}{\sqrt{\varepsilon_0\,\mu_0}}$

∴ $[\mu_0] = \dfrac{1}{[\varepsilon_0][c]^2} = \dfrac{1}{[M^{-1}\,L^{-3}\,T^4\,I^2][LT^{-1}]^2}$

$$= [MLT^{-2}I^{-2}]$$

5. CR and $\dfrac{L}{R}$ both are time constants. Their units is second.

∴ $\dfrac{1}{CR}$ and $\dfrac{R}{L}$ have the SI unit (second)$^{-1}$. Further, resonance frequency $\omega = \dfrac{1}{\sqrt{LC}}$

Match the Columns

1. (a) $U = \dfrac{1}{2}\,kT$

$\Rightarrow\ [ML^2T^{-2}] = [k][K]$

$\Rightarrow\ [K] = [ML^2T^{-2}K^{-1}]$

(b) $F = \eta A\,\dfrac{dv}{dx}$

$\Rightarrow\ [\eta] = \dfrac{[MLT^{-2}]}{[L^2LT^{-1}L^{-1}]}$

$$= [ML^{-1}\,T^{-1}]$$

(c) $E = h\nu$

$\Rightarrow\ [ML^2T^2] = [h][T^{-1}]$

$\Rightarrow\ [h] = [ML^2T^{-1}]$

(d) $\dfrac{dQ}{dt} = \dfrac{k\,A\Delta\theta}{l}$

$\Rightarrow\ [k] = \dfrac{[ML^2T^{-3}L]}{[L^2K]} = [MLT^{-3}K^{-1}]$

2. Angular momentum $L = I\omega$

∴ $[L] = [I\omega] = [ML^2][T^{-1}] = [ML^2T^{-1}]$

Latent heat, $L = \dfrac{Q}{m}$ (as $Q = mL$)

$\Rightarrow\ [L] = \left[\dfrac{Q}{m}\right] = \left[\dfrac{ML^2T^{-2}}{M}\right] = [L^2T^{-2}]$

Torque $\tau = F \times r_\perp$

$\therefore \quad [\tau] = [F \times r_\perp] = [MLT^{-2}][L]$

$= [ML^2T^{-2}]$

Capacitance $C = \dfrac{1}{2}\dfrac{q^2}{U}$ $\quad \left(\text{as } U = \dfrac{1}{2}\dfrac{q^2}{C}\right)$

$\therefore \quad [C] = \left[\dfrac{q^2}{U}\right] = \left[\dfrac{Q^2}{ML^2T^{-2}}\right] = [M^{-1}L^{-2}T^2Q^2]$

Inductance $L = \dfrac{2U}{i^2}$ $\quad \left(\text{as } U = \dfrac{1}{2}Li^2\right)$

$\therefore \quad [L] = \left[\dfrac{U}{i^2}\right] = \left[\dfrac{Ut^2}{Q^2}\right]$ $\quad \left(\text{as } i = \dfrac{Q}{t}\right)$

$= \left[\dfrac{ML^2T^{-2}T^2}{Q^2}\right] = [ML^2Q^{-2}]$

Resistivity $\rho = \dfrac{RA}{l}$ $\quad \left(\text{as } R = \rho\dfrac{l}{A}\right)$

$= \left[\dfrac{H}{i^2t}\right]\left[\dfrac{A}{l}\right]$ $\quad (\text{as } H = i^2Rt)$

$= \left[\dfrac{Ht}{Q^2}\right]\left[\dfrac{A}{l}\right]$ $\quad \left(\text{as } i = \dfrac{Q}{t}\right)$

$= \left[\dfrac{ML^2T^{-2}TL^2}{Q^2L}\right] = [ML^3T^{-1}Q^{-2}]$

The correct table is as under

Column I	Column II
Angular momentum	$[ML^2T^{-1}]$
Latent heat	$[L^2T^{-2}]$
Torque	$[ML^2T^{-2}]$
Capacitance	$[M^{-1}L^{-2}T^2Q^2]$
Inductance	$[ML^2Q^{-2}]$
Resistivity	$[ML^3T^{-1}Q^{-2}]$

3. $t \equiv \dfrac{L}{R} \Rightarrow \therefore \; L \equiv tR \equiv$ ohm-second

$$U \equiv \frac{q^2}{2C}$$

$\therefore \quad C \equiv \dfrac{q^2}{U} \equiv$ coulomb²/joule

$$q \equiv CV$$

$\therefore \quad C \equiv \dfrac{q}{V} \equiv$ coulomb/volt

$$L \equiv \frac{-e}{di/dt}$$

$\therefore \quad L \equiv \dfrac{e\,(dt)}{(di)} \equiv$ volt-second/ampere

$$F = ilB$$

$\therefore \quad B \equiv \dfrac{F}{il} \equiv$ newton/ampere-metre

Column I	Column II
Capacitance	coulomb/(volt)$^{-1}$ coulomb^2 joule^{-1}
Inductance	ohm-second, volt second/ampere^{-1}
Magnetic induction	newton (ampere-metre)$^{-1}$

4. (a) $F = \dfrac{GM_eM_s}{r^2}$

= Gravitational force between sun and earth

$\Rightarrow \quad GM_eM_s = Fr^2$

$\therefore \quad [GM_eM_s] = [Fr^2]$

$= [MLT^{-2}][L^2] = [ML^3T^{-2}]$

(b) $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}}$ = rms speed of gas molecules

$\therefore \quad \dfrac{3RT}{M} = v_{\text{rms}}^2$

or $\quad \left[\dfrac{3RT}{M}\right] = [v_{\text{rms}}^2] = [LT^{-1}]^2 = [L^2T^{-2}]$

(c) $F = Bqv$ = magnetic force on a charged particle

$\therefore \quad \dfrac{F}{Bq} = v$

or $\quad \left[\dfrac{F^2}{B^2q^2}\right] = [v]^2 = [L^2T^{-2}]$

(d) $v_o = \sqrt{\dfrac{GM_e}{R_e}}$ = orbital velocity of earth's satellite

$\therefore \quad \dfrac{GM_e}{R_e} = v_o^2$

or $\quad \left[\dfrac{GM_e}{R_e}\right] = [v_o^2] = [L^2T^{-2}]$

(p) $W = qV \Rightarrow$ (Coulomb) (Volt) = Joule

or [(Volt) (Coulomb) (Metre)] = [(Joule) (Metre)]

$= [ML^2T^{-2}][L] = [ML^3T^{-2}]$

(q) [(kilogram) (metre)^3 (second)$^{-2}$] $= [ML^3T^{-2}]$

(r) [(metre)^2 (second)$^{-2}$] $= [L^2T^{-2}]$

(s) $U = \frac{1}{2}CV^2$

$\Rightarrow$ (farad) (volt)^2 = Joule

or $[(\text{farad})\,(\text{volt})^2\,(\text{kg})^{-1}] = [(\text{Joule})(\text{kg})^{-1}]$

$= [\text{ML}^2\text{T}^{-2}][\text{M}^{-1}] = [\text{L}^2\text{T}^{-2}]$

Subjective Questions

1. $[a] = [y] = [\text{L}]$

$\Rightarrow \quad [\omega t] = [\text{M}^0\text{L}^0\text{T}^0]$

$\therefore \quad [\omega] = \frac{1}{[t]} = [\text{T}^{-1}]$

θ is angle, which is dimension less.

2. $1\,\text{N} = 10^5$ dyne

$1\text{ m}^2 = 10^4\text{ cm}^2$

$$\therefore \quad 2.0 \times 10^{11}\,\frac{\text{N}}{\text{m}^2} = \frac{2.0 \times 10^{11} \times 10^5}{10^4}\,\frac{\text{dyne}}{\text{cm}^2} = 2.0 \times 10^{12}\text{ dyne/cm}^2$$

3. 1 dyne $= 10^{-5}$ N

1 cm $= 10^{-2}$ m

$\therefore$ 72 dyne/cm

$$= \frac{72 \times 10^{-5}}{10^{-2}}\,\frac{\text{N}}{\text{m}} = 0.072\,\frac{\text{N}}{\text{m}}$$

4. $h = \frac{E}{\gamma} = \frac{\text{J}}{\text{s}^{-1}} = \text{J-s}$

$$[h] = [\text{ML}^2\text{T}^{-2}][\text{T}] = [\text{ML}^2\text{T}^{-1}]$$

5. $$S_t = \left\{ut + \frac{1}{2}at^2\right\} + \left\{u\,(t-1) + \frac{1}{2}\,a\,(t-1)^2\right\} = \left(u + at - \frac{1}{2}a\right)$$

Equation is dimensionally correct.

6. (a) Young's modulus $= \frac{F/A}{\Delta l / l}$

Hence, the MKS units are N/m^2.

(c) Power of a lens (in dioptre)

$$= \frac{1}{f\,(\text{in metre})}$$

7. $[T] = [p^a d^b E^c] = [\text{ML}^{-1}\text{T}^{-2}]^a\,[\text{ML}^{-3}]^b[\text{ML}^2\text{T}^{-2}]^c$

Equating the powers of both sides, we have

$$a + b + c = 0 \qquad \text{... (i)}$$
$$-a - 3b + 2c = 0 \qquad \text{... (ii)}$$
$$-2a - 2c = 1 \qquad \text{... (iii)}$$

Solving these three equations, we have

$$a = -\frac{5}{6},\ b = \frac{1}{2}$$

and $$c = \frac{1}{3}$$

8. $[\text{Y}] = [\text{ML}^{-1}\text{T}^{-2}]$

$$\left[\frac{\text{MgL}}{\pi r^2 l}\right] = \left[\frac{\text{MLT}^{-2}\text{L}}{\text{L}^2\text{L}}\right] = [\text{ML}^{-1}\text{T}^{-2}]$$

Dimensions of RHS and LHS are same. Therefore, the given equation is dimensionally correct.

9. $[\text{E}] = k\,[\text{m}]^x\,[\text{n}]^y\,[\text{a}]^z$

where, k is a dimensionless constant.

$\therefore \quad [\text{ML}^2\text{T}^{-2}] = k\,[\text{M}]^x\,[\text{T}^{-1}]^y\,[\text{L}]^z$

Solving we get,

$$x = 1,\ y = 2$$

and $$z = 2$$

$$\therefore \quad E = k\,m\,n^2a^2$$

10. Since dimension of

$Fv = [\text{Fv}] = [\text{MLT}^{-2}][\text{LT}^{-1}] = [\text{ML}^2\text{T}^{-3}]$

So, $\left[\frac{\beta}{x^2}\right]$ should also be $[\text{ML}^2\text{T}^{-3}]$

$$\frac{[\beta]}{[x^2]} = [\text{ML}^2\text{T}^{-3}]$$

$[\beta] = [\text{ML}^4\text{T}^{-3}]$ **Ans.**

and $\left[\text{Fv} + \frac{\beta}{x^2}\right]$ will also have dimension $[\text{ML}^2\text{T}^{-3}]$, so LHS should also have the same dimension $[\text{ML}^2\text{T}^{-3}]$.

So $$\frac{[\alpha]}{[\text{t}^2]} = [\text{ML}^2\text{T}^{-3}]$$

$[\alpha] = [\text{ML}^2\text{T}^{-1}]$ **Ans.**

11. $[\text{b}] = [\text{V}] = [\text{L}^3]$

$$\frac{[\text{a}]}{[\text{V}]^2} = [\text{P}] = [\text{ML}^{-1}\text{T}^{-2}]$$

$\Rightarrow \quad [\text{a}] = [\text{ML}^5\text{T}^{-2}]$

12. $\left[\dfrac{\mathrm{a}}{\mathrm{RTV}}\right] = [\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0]$

$$\Rightarrow \quad [\mathrm{a}] = [\mathrm{RTV}] = [\mathrm{ML}^2\mathrm{T}^{-2}][\mathrm{L}^3] = [\mathrm{ML}^5\mathrm{T}^{-2}]$$

$$[\mathrm{b}] = [\mathrm{V}] = [\mathrm{L}^3]$$

13. (a) Magnetic flux, $\phi = Bs = \left(\dfrac{F}{qv}\right)(s)$

[as $F = Bqv$]

$$\therefore \quad [\phi] = \left[\frac{\mathrm{Fs}}{\mathrm{qv}}\right] = \left[\frac{\mathrm{MLT}^{-2}\mathrm{L}^2}{\mathrm{QLT}^{-1}}\right] = [\mathrm{ML}^2\mathrm{T}^{-1}\mathrm{Q}^{-1}]$$

(b) $[\text{Rigidity Modulus}] = \left[\dfrac{\mathrm{F}}{\mathrm{A}}\right] = \left[\dfrac{\mathrm{MLT}^{-2}}{\mathrm{L}^2}\right] = [\mathrm{ML}^{-1}\mathrm{T}^{-2}]$

14. $\left[\dfrac{dx}{\sqrt{a^2 - x^2}}\right]$ is dimensionless.

$\left[\dfrac{1}{a}\sin^{-1}\left(\dfrac{a}{x}\right)\right]$ has the dimension of $[\mathrm{L}^{-1}]$.

15. $\left[\dfrac{dx}{\sqrt{2ax - x^2}}\right]$ is dimensionless.

Therefore, $\left[a^n \sin^{-1}\left(\dfrac{x}{a} - 1\right)\right]$ should also be dimensionless. Hence, $n = 0$

16. (a) $[\text{density}] = [\mathrm{F}]^x [\mathrm{L}]^y [\mathrm{T}]^z$

$\therefore \quad [\mathrm{ML}^{-3}] = [\mathrm{MLT}^{-2}]^x [\mathrm{L}]^y [\mathrm{T}]^z$

Equating the powers we get,

$$x = 1, \; y = -4 \text{ and } z = 2$$

$\therefore \quad [\text{density}] = [\mathrm{FL}^{-4}\mathrm{T}^2]$

In the similar manner, other parts can be solved.

5. Vectors

INTRODUCTORY EXERCISE 5.1

3. Apply $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

4. Apply $S = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

5. $R = S \Rightarrow \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$= \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

Solving we get, $\cos\theta = 0$ or $\theta = 90°$

INTRODUCTORY EXERCISE 5.2

1. A or $|\mathbf{A}| = \sqrt{(3)^2 + (-4)^2 + (5)^2} = 5\sqrt{2}$ units

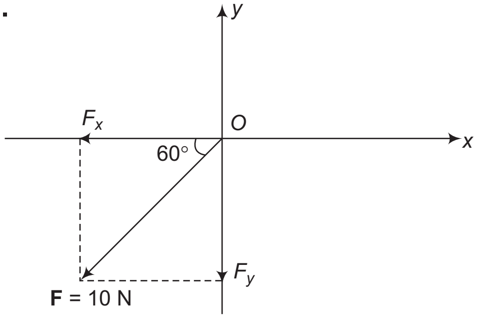
Directions of cosines are,

$$\cos\alpha = \frac{A_x}{A} = \frac{3}{5\sqrt{2}}$$

$$\cos\beta = \frac{A_y}{A} = \frac{-4}{5\sqrt{2}} \quad \text{and}$$

$$\cos\gamma = \frac{A_z}{A} = \frac{1}{\sqrt{2}}$$

2.



$F_x = 10\cos 60° = 5$ N (along negative x-direction)

$F_y = 10\sin 60° = 5\sqrt{3}$ N (along negative y-direction)

3. $\mathbf{A} - 2\mathbf{B} + 3\mathbf{C} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) - 2(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + 3\hat{\mathbf{k}}$

$$= (\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$\therefore \quad |\mathbf{A} - 2\mathbf{B} + 3\mathbf{C}| = \sqrt{(1)^2 + (3)^2}$

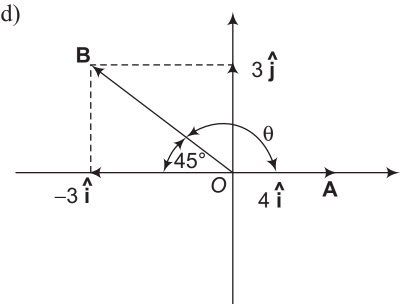
$$= \sqrt{10} \text{ units}$$

4. (a) Antiparallel vectors

(b) Perpendicular vectors

(c) **A** lies in x-y plane and **B** is along positive z- direction. So, they are mutually perpendicular vectors

(d)



$\theta = 135°$

INTRODUCTORY EXERCISE 5.3

3. $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$

4. $2\mathbf{A} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

$-3\mathbf{B} = -3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$$\Rightarrow \quad (2\mathbf{A}) \times (-3\mathbf{B}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -2 & 0 \\ 0 & -3 & -3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(6 - 0) + \hat{\mathbf{j}}(0 + 12) + \hat{\mathbf{k}}(-12 - 0) = 6\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$$

Exercises

Single Correct Option

6. $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$

$$= (3)(5)\cos 60° = 7.5$$

9. $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$

$$0 \le \sin\theta \le 1$$

$$\therefore \quad 0 \le |\mathbf{A} \times \mathbf{B}| \le AB$$

10. $W = \mathbf{F} \cdot \mathbf{s} = 9 + 16 = 25$ J **Ans.**

11. $\tau = \mathbf{r} \times \mathbf{F}$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(8 + 9) + \hat{\mathbf{j}}\,(6 - 12) + \hat{\mathbf{k}}\,(-9 - 4)$$

$= (17\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 13\hat{\mathbf{k}})$ N-m **Ans.**

12. $\sqrt{(0.5)^2 + (0.8)^2 + c^2} = 1$ **Ans.**

13. $R = \sqrt{A^2 + A^2 + 2\,AA\cos\theta}$

$R = A$ at $\theta = 120°$ **Ans.**

14. $(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 8\hat{\mathbf{k}}) \cdot (4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}) = 0$

$\therefore \quad 8 - 12 + 8\alpha = 0$

$\therefore \quad \alpha = \dfrac{1}{2}$ **Ans.**

15. $\mathbf{A} \cdot \mathbf{B} = 9 + 16 - 25 = 0$

$\therefore$ Angle between **A** and **B** is 90°. **Ans.**

16. $A + B = 7$

and $\quad A - B = 3$

$\therefore \quad A = 5$ and $B = 2$ **Ans.**

17. $\mathbf{P} \perp \mathbf{Q} = \mathbf{P} \cdot \mathbf{Q} = 0$

18. $\mathbf{s} = \mathbf{r}_B - \mathbf{r}_A$

$= (-2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - (3\hat{\mathbf{j}} - \hat{\mathbf{k}})$

$= -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ **Ans.**

20. $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$

$\Rightarrow \quad \mathbf{C} = -(\mathbf{A} + \mathbf{B})$

or $\quad |\mathbf{C}| = |-(\mathbf{A} + \mathbf{B})|$

or $\quad A \sim B \le C \le A + B$

21. Resultant is always inclined towards a vector of larger magnitude.

22. $A = |\mathbf{A}| = \sqrt{(3)^2 + (6)^2 + (2)^2}$

$= 7$

$\alpha = \cos^{-1}\left(\dfrac{A_x}{A}\right) = \cos^{-1}\left(\dfrac{3}{7}\right)$

= angle of **A** with positive x-axis.

Similarly, β and γ angles.

23. $\mathbf{R} = \mathbf{A} + \mathbf{B} = 12\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

$R = |\mathbf{R}| = \sqrt{(12)^2 + (5)^2}$

$= 13$

$\hat{\mathbf{R}} = \dfrac{\mathbf{R}}{R}$

24. Component of **A** along **B**

$= A\cos\theta$

$= \dfrac{\mathbf{A}\cdot\mathbf{B}}{B} = \dfrac{2+3}{\sqrt{1+1}} = \dfrac{5}{\sqrt{2}}$ **Ans.**

25. $C^2 = A^2 + B^2 + 2AB\cos\theta$

At $\theta = 90°$; $C^2 = A^2 + B^2$

26. $|\mathbf{A} \times \mathbf{B}| = \sqrt{3}\,\mathbf{A}\cdot\mathbf{B}$

$\Rightarrow \quad AB\sin\theta = \sqrt{3}\,AB\cos\theta$

or $\quad \tan\theta = \sqrt{3}$

$\Rightarrow \quad \theta = 60°$

$|\mathbf{A} + \mathbf{B}| = \sqrt{A^2 + B^2 + 2AB\cos 60°}$

$= \sqrt{A^2 + B^2 + AB}$

27. $(\mathbf{B} \times \mathbf{A})$ is $\perp$ to both **A** and **B**

$\therefore \quad (\mathbf{B} \times \mathbf{A}) \cdot \mathbf{A} = 0$

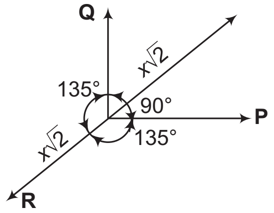
28. $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

Substituting the values of P, Q and R we get,

$\cos\theta = 0$

$\Rightarrow \quad \theta = 90°$

29.



$|\mathbf{P}| = |\mathbf{Q}| = x$

$|\mathbf{R}| = \sqrt{2}\,x$

30. $(\mathbf{P} + \mathbf{Q}) \cdot (\mathbf{P} - \mathbf{Q})$

$= P^2 + PQ\cos\theta - PQ\cos\theta - Q^2$

$= P^2 - Q^2$

Since dot product may be positive (if $P > Q$), negative (if $Q > P$) or zero (if $P = Q$). Therefore angle between $(\mathbf{P} + \mathbf{Q})$ and $(\mathbf{P} - \mathbf{Q})$ may be acute, obtuse or 90°.

31. Scalar triple product of three vectors

= volume of parallel applied $= 0$

$$\therefore \quad \begin{vmatrix} 2 & 3 & -2 \\ 5 & n & 1 \\ -1 & 2 & 3 \end{vmatrix} = 0$$

Solving this equation, we get

$n = 18$

32. $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$

$= \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b}$

$= 2\,(\mathbf{b} \times \mathbf{a})$

As $\mathbf{a} \times \mathbf{a}$ and $\mathbf{b} \times \mathbf{b}$ are two null vectors and $-\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$

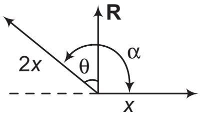
33. $R = \sqrt{(3P)^2 + (2P)^2 + 2\,(3P)\,(2P)\cos\theta}$ ...(i)

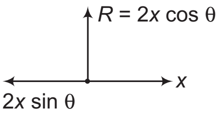
$2R = \sqrt{(6P)^2 + (2P)^2 + 2\,(6P)\,(2P)\cos\theta}$ ...(ii)

Solving these two equations we get,

$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \theta = 120°$$

34.





$2x\sin\theta = x$

$\therefore \quad \theta = 30°$

or $\quad \alpha = 120°$ **Ans.**

35. $\mathbf{A}\cdot\mathbf{B} = 0$

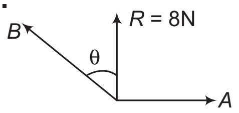
$\therefore \quad \mathbf{A} \perp \mathbf{B}$

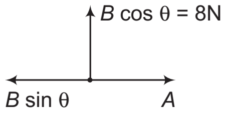
$\mathbf{A}\cdot\mathbf{C} = 0$

$\therefore \quad \mathbf{A} \perp \mathbf{C}$

A is perpendicular to both **B** and **C** and $\mathbf{B}\times\mathbf{C}$ is also perpendicular to both **B** and **C**. Therefore, **A** is parallel to $\mathbf{B}\times\mathbf{C}$.

36.





$A + B = 16$ ...(i)

$B\sin\theta = A$

and $\quad B\cos\theta = 8$

Squaring and adding these two equations, we get

$B^2 = A^2 + 64$ ...(ii)

Solving Eqs. (i) and (ii), we get

$A = 6\text{ N} \quad \text{and} \quad B = 10\text{ N}$ **Ans.**

37. $(\mathbf{A}+\mathbf{B}) \perp (\mathbf{A}-\mathbf{B})$

$\therefore \quad (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = 0$

$\therefore \quad A^2 + BA\cos\theta - AB\cos\theta - B^2 = 0$

or $\quad A = B$ **Ans.**

38. $A \sim B \le C \le A + B$

39. $A \sim B \le C \le A + B$

Match the Columns

1. (a) $AB\sin\theta = |AB\cos\theta|$

$\therefore \quad \tan\theta = \pm 1$

$\Rightarrow \quad v = 45°$

or $\quad 135°$ **Ans.**

(b) If **A** and **B** are parallel or antiparallel to each other then their cross product is a null vector and they may be said to be equal otherwise for any other angle,

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

(c) $\sqrt{A^2 + B^2 + 2AB\cos\theta}$

$= \sqrt{A^2 + B^2 - 2AB\cos\theta}$

$\therefore \quad \cos\theta = 0$

or $\quad \theta = 90°$ **Ans.**

(d) $A + B = C$ only at $\theta = 0°$

Subjective Questions

1. $\theta = \cos^{-1}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{ab}\right)$

2. If the angle between **A** and **B** is θ, the cross product will have a magnitude

$|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$

or $\quad 15 = 5 \times 6\sin\theta$

or $\quad \sin\theta = \dfrac{1}{2}$

Thus, $\quad \theta = 30°$

or $\quad 150°$

5. The angle between the force **F** and the displacement **r** is 180°. Thus, the work done is

$W = \mathbf{F}\cdot\mathbf{r}$

$= Fr\cos\theta$

$= (12\text{ N})\,(2.0\text{ m})\,(\cos 180°)$

$= -24\text{ N-m} = -24\text{ J}$

6. $\mathbf{A}\times\mathbf{B}$ is perpendicular to both **A** and **B**. So this is parallel or antiparallel to **C**. Now, cross product of two parallel or antiparallel vectors is zero. Hence,

$$\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0.$$

7. $\mathbf{A}\times\mathbf{B}$ is perpendicular to **A**. Now, dot product of two perpendicular vectors is zero. Hence,

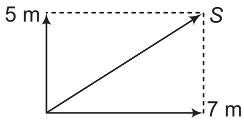
$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$$

8. $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$

$= [(5\cos 37°)\hat{\mathbf{i}} + (5\sin 37°)\hat{\mathbf{j}}] + 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

$= (4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) + 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

$= (7\hat{\mathbf{i}} + 5\hat{\mathbf{j}})\text{ m}$



$$S = \sqrt{(7)^2 + (5)^2} = \sqrt{74}\text{ m}$$

$$\tan\theta = \frac{5}{7}$$

or $$\theta = \tan^{-1}\left(\frac{5}{7}\right)$$

9. Suppose α is the angle between **A** and **B** and β the angle between **C** and **B**. But $\alpha \neq \beta$.

Given that,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{C}\cdot\mathbf{B}$$

$\Rightarrow$ $$AB\cos\alpha = CB\cos\beta$$

$\Rightarrow$ $$A\cos\alpha = C\cos\beta$$

or $$A \neq C \text{ as } \alpha \neq \beta$$

10. $\mathbf{A} + \mathbf{B} = \mathbf{R} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$,

$\mathbf{A} - \mathbf{B} = \mathbf{S} = \hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

Now, angle between **R** and **S** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{R}\cdot\mathbf{S}}{RS}\right)$$

11. Ratio of their coefficients should be same.

$\therefore$ $$\frac{2}{3} = \frac{3}{-a} = \frac{-4}{b}$$

$\therefore$ $$a = -4.5$$

and $$b = -6$$

12. $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 4 & -6 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(8-0) + \hat{\mathbf{j}}\,(-6-4) + \hat{\mathbf{k}}\,(0-4)$$

$$= 8\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

Area of parallelogram $= |\mathbf{A}\times\mathbf{B}|$

$$= \sqrt{(8)^2 + (10)^2 + (4)^2}$$

$$= 13.4 \text{ units}$$ **Ans.**

13. See the hint of Q-23 of objective type problems.

14. $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 2 & -3 & 7 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 2\,(0-2) - 3\,(0+1) + 7\,(1-0)$$

$$= -4 - 3 + 7 = 0$$

15. (a) $\mathbf{A} + \mathbf{B} = \mathbf{R}$

$$R_x = A_x + B_x$$

$\therefore$ $$B_x = R_x - A_x = 10 - 4 = 6 \text{ m}$$

Similarly,

$$B_y = R_y - A_y = 9 - 6 = 3 \text{ m}$$

(b) $$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(6)^2 + (3)^2}$$

$$= 3\sqrt{5} \text{ m}$$

$$\tan\theta = \frac{B_y}{B_x} = \frac{3}{6} = \frac{1}{2}$$

$\therefore$ $$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$

16. (a) $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{P} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

(b) $\mathbf{a} + \mathbf{b} - \mathbf{c} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} = \mathbf{Q}$

(c) Angle between **P** and **Q** is, $\theta = \cos^{-1}\left(\dfrac{\mathbf{P}\cdot\mathbf{Q}}{PQ}\right)$

17. $$\mathbf{R} = [(10\cos 30^\circ)\hat{\mathbf{i}} + (10\cos 60^\circ)\hat{\mathbf{j}}] + [(10\cos 60^\circ)\hat{\mathbf{i}} + (10\cos 30^\circ)\hat{\mathbf{j}}]$$

$$= 10(\cos 30^\circ + \cos 60^\circ)\hat{\mathbf{i}} + 10(\cos 60^\circ + \cos 30^\circ)\hat{\mathbf{j}}$$

$$= 10(2\cos 45^\circ\,\cos 15^\circ)\hat{\mathbf{i}} + 10(\cos 45^\circ\,\cos 15^\circ)\hat{\mathbf{j}}$$

$$= 20\cos 45^\circ\,\cos 15^\circ\,\hat{\mathbf{i}} + 20\cos 45^\circ\,\cos 15^\circ\,\hat{\mathbf{j}}$$

$$= R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

Here, $R_x = R_y = 10\sqrt{2}\cos 15^\circ$

$\therefore$ $$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2} = 20\cos 15^\circ$$

Since, $R_x = R_y$, therefore, $\theta = 45^\circ$

18. Their x components should be equal and opposite.

$\therefore$ $$6\cos\theta = -4$$

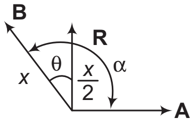
or $$\theta = \cos^{-1}\left(-\frac{2}{3}\right)$$

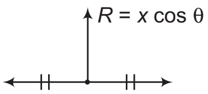
19. Component of **A** along another vector (say **B**) is given by, $A\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{B}$

20. Find $\mathbf{A}\times\mathbf{B}$ and then prove that $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{A} = 0$ and $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{B} = 0$.

It means $(\mathbf{A}\times\mathbf{B}) \perp$ to both **A** and **B**.

21.





$$R = x\cos\theta$$

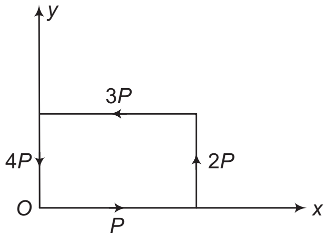
or $$\frac{x}{2} = x\cos\theta$$

$\therefore$ $$\cos\theta = \frac{1}{2}$$

or $$\theta = 60^\circ$$

$\therefore$ $$\alpha = 150^\circ$$ **Ans.**

22.



$$\mathbf{R} = (P\hat{\mathbf{i}} + 2P\hat{\mathbf{j}} - 3P\hat{\mathbf{i}} - 4P\hat{\mathbf{j}})$$
$$= (-2P\hat{\mathbf{i}} - 2P\hat{\mathbf{j}})$$
$$|\mathbf{R}| = \sqrt{(2P)^2 + (2P)^2}$$
$$= 2\sqrt{2}\,P \qquad \textbf{Ans.}$$

23. $R^2 = P^2 + Q^2 + 2PQ\cos\theta$ ....(i)

$S^2 = P^2 + Q^2 - 2PQ\cos\theta$ ...(ii)

Adding these two equations we get,

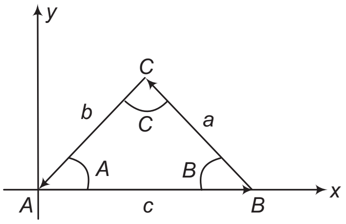
$$R^2 + S^2 = 2(P^2 + Q^2)$$

24. From polygon law of vector addition we can see that,

$$\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$$

$$\therefore\ c\hat{\mathbf{i}} + (a\sin B\hat{\mathbf{j}} - a\cos B\hat{\mathbf{i}}) + (-b\cos A\hat{\mathbf{i}} - b\sin A\hat{\mathbf{j}}) = 0$$

$$\therefore\ (c - a\cos B - b\cos A)\,\hat{\mathbf{i}} + (a\sin B - b\sin A)\,\hat{\mathbf{j}} = 0$$



Putting coefficient of $\hat{\mathbf{j}} = 0$, we find that

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Now taking B as origin and BC as the x-axis, we can prove other relation.

6. Kinematics

INTRODUCTORY EXERCISE 6.1

2. (a) $\mathbf{v}\cdot\mathbf{a} = 6-4-4 = -2\ \text{m}^2/\text{s}^3$

(b) Since dot product is negative. So angle between **v** and **a** is obtuse.

(c) As angle between **v** and **a** at this instant is obtuse, speed is decreasing.

INTRODUCTORY EXERCISE 6.2

1. **v** and **a** both are constant vectors. Further, these two vectors are antiparallel.

2. **a** is function of time and **v** and **a** are neither parallel nor antiparallel.

INTRODUCTORY EXERCISE 6.3

1. Distance may be greater than or equal to magnitude of displacement.

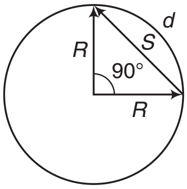
2. Constant velocity means constant speed in same direction. Further if any physical quantity has a constant value (here, the velocity) then its average value in any interval of time is equal to that constant value

3. Stone comes under gravity

$$\therefore \qquad F = mg$$

$$\text{or} \qquad a = \frac{F}{m} = g \qquad \textbf{Ans.}$$

4. In 15 s, it will rotate 90° or $\frac{1}{4}$th circle.



$$\therefore \quad \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{d}{t}$$

$$= \frac{(2\pi R/4)}{15} = \frac{\pi R}{30} = \frac{\pi(2.0)}{30}$$

$$= \left(\frac{\pi}{15}\right)\ \text{cm/s} \qquad \textbf{Ans.}$$

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{s}{t}$$

$$= \frac{\sqrt{2}\,R}{t} = \frac{(\sqrt{2})(2.0)}{15} = \frac{2\sqrt{2}}{15}\ \text{cm/s} \qquad \textbf{Ans.}$$

5. (a) On a curvilinear path (a path which is not straight line), even if speed is constant, velocity will change due to change in direction.

$$\therefore \qquad \mathbf{a} \neq 0$$

(b) (i) On a curved path velocity will definitely change (at least due to change in direction).

$$\therefore \qquad \mathbf{a} \neq 0$$

(ii) In projectile motion, path is curved, but acceleration is constant (= g).

(iii) Variable acceleration on curved path is definitely possible.

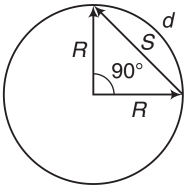
6. (a) $T = \dfrac{2\pi R}{v} = 25.13$ s **Ans.**

(b) (i) Since speed is constant. Therefore,

average speed = constant speed

= 1.0 cm/s **Ans.**

(ii) $\text{Average velocity} = \dfrac{s}{t} = \dfrac{\sqrt{2}R}{(T/4)} = \dfrac{4\sqrt{2}R}{T}$



$$= \frac{(4)(\sqrt{2})(4.0)}{25.13} = 0.9\ \text{cm/s} \qquad \textbf{Ans.}$$

(iii) Velocity vector will rotate 90°

$$\therefore \quad |\Delta\mathbf{v}| = \sqrt{v^2 + v^2 - 2vv\cos 90^\circ}$$

$$= \sqrt{2}\,v$$

$$\therefore \quad |\mathbf{a}_{\text{av}}| = \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{\sqrt{2}\,v}{(T/4)} = \frac{4\sqrt{2}\,v}{T}$$

$$= \frac{(4\sqrt{2})(1.0)}{25.13} = 0.23\ \text{cm/s}^2 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 6.4

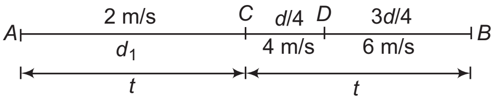
1. $\text{Average speed} = \dfrac{d}{t} = \dfrac{d_1 + d_2}{t_1 + t_2}$

$$= \frac{v_1t_1 + v_2t_2}{t_1 + t_2}$$

$$= \frac{(4\times 2) + (6\times 3)}{2 + 3}$$

$$= 5.2\ \text{m/s} \qquad \textbf{Ans.}$$

2.



$$2t = d_1 \qquad \text{...(i)}$$

$$\frac{(d/4)}{4} + \frac{(3d/4)}{6} = t$$

$$\Rightarrow \qquad \frac{3d}{16} = t \qquad \text{...(ii)}$$

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$$

$$= \frac{d_1 + d}{2t} = \frac{2t + \frac{16t}{3}}{2t}$$

$$= \frac{11}{3}\ \text{m/s}$$

INTRODUCTORY EXERCISE 6.5

1. S_t = (displacement upto t second) $-$ [displacement upto $(t-1)$ sec]

$$= \left(ut + \frac{1}{2}at^2\right) - \left[u\,(t-1) + \frac{1}{2}a\,(t-1)^2\right]$$

$$= u + at - \frac{1}{2}a$$

2. $$s_t = \left[ut + \frac{1}{2}at^2\right] - \left[u\,(t-1) + \frac{1}{2}a\,(t-1)^2\right]$$

$$= (u)(1) + \frac{1}{2}(a)(2t) - \frac{1}{2}(a)(1)^2$$

The first term is $(u)(1)$, which we are writing only u. Dimensions of u are $[\text{LT}^{-1}]$ and dimensions of 1 (which is actually 1 second) are $[\text{T}]$

$$\therefore \qquad [(u)(1)] = [\text{LT}^{-1}][\text{T}]$$

$$= [\text{L}] = s_t$$

Therefore dimension of (u) (1) are same as the dimensions of s_t. Same logic can be applied with other terms too.

5. In 4 s, it reaches upto the highest point and then changes its direction of motion.

$$s = ut + \frac{1}{2}at^2 = (40)(6) + \frac{1}{2}(-10)(6)^2$$

$$= 60\ \text{m} \qquad \textbf{Ans.}$$

$$d = |s|_{0-4} + |s|_{4-6} = \left|\frac{u^2}{2g}\right| + \left|\frac{1}{2}g\,(t - t_0)^2\right|$$

$$= \frac{(40)^2}{2 \times 10} + \frac{1}{2} \times 10 \times (6-4)^2 = 100\,\text{m} \qquad \textbf{Ans.}$$

6. $$u_{av} = \frac{s}{t} = \frac{ut + \frac{1}{2}at^2}{t} = u + \frac{1}{2}at \qquad \textbf{Ans.}$$

7. $v = u + at$

i.e. v-t function is linear. In linear function,

$$\text{average value} = \frac{(\text{final value} + \text{initial value})}{2}$$

$$\therefore \qquad v_{av} = \frac{v_f + v_i}{2} = \frac{v_1 + v_2}{2} \qquad \textbf{Ans.}$$

8. $$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 125}{10}} = 5\ \text{s}$$

$$v_{av} = \frac{s}{t} = \frac{125}{5} = 25\ \text{m/s} \qquad \text{(downwards)} \qquad \textbf{Ans.}$$

9. (a) $$S = ut + \frac{1}{2}at^2$$

$$= (2.5 \times 2) + \frac{1}{2}(0.5)(2)^2$$

$$= 6\ \text{m} = \text{distance also}$$

(b) $v = u + at$

$$7.5 = 2.5 + 0.5 \times t \quad \Rightarrow \quad t = 10\ \text{s}$$

(c) $$v^2 = u^2 + 2as$$

$$\therefore \qquad (7.5)^2 = (2.5)^2 + (2)(0.5)s$$

$$\Rightarrow \qquad s = 50\ \text{m} = \text{distance also}$$

10. (a) $h = \dfrac{u^2}{2g} = \dfrac{(50)^2}{2 \times 10} = 125\ \text{m}$

(b) Time of ascent $= \dfrac{u}{g} = \dfrac{50}{10} = 5\ \text{s}$

(c) $v^2 = u^2 + 2as = (50)^2 + 2(-10)\left(\dfrac{125}{2}\right)$

Solving, we get speed $v \approx 35$ m/s

INTRODUCTORY EXERCISE 6.6

1. (a) $a = \dfrac{dv}{dt} = 5 - 2t$

$$\text{At} \qquad t = 2\ \text{s}$$

$$a = 1\ \text{m/s}^2 \qquad \textbf{Ans.}$$

(b) $$x = \int_0^3 v\,dt = \int_0^3 (10 + 5t - t^2)\,dt$$

$$= \left[10\,t + 2.5\,t^2 - \frac{t^3}{3}\right]_0^3$$

$$= (10 \times 3) + (2.5)(3)^2 - \frac{(3)^3}{3}$$

$$= 43.5\ \text{m} \qquad \textbf{Ans.}$$

2. (a) Acceleration of particle,

$$a=\frac{dv}{dt}=(6+18t)\text{ cm/s}^2$$

At $t=3$ s,

$$a=(6+18\times 3)\text{cm/s}^2$$

$$=60\text{cm/s}^2$$

(b) Given, $v=(3+6t+9t^2)\text{cm/s}$

or $\frac{ds}{dt}=(3+6t+9t^2)$

or $ds=(3+6t+9t^2)\,dt$

∴ $\int_0^s ds=\int_5^8(3+6t+9t^2)\,dt$

∴ $s=[3t+3t^2+3t^3]_5^8$

or $s=1287$ cm

3. (a) Position, $x=(2t-3)^2$

Velocity, $v=\frac{dx}{dt}=4\,(2t-3)\text{m/s}$

and acceleration, $a=\frac{dv}{dt}=8\text{ m/s}^2$

At $t=2$ s,

$$x=(2\times 2-3)^2$$

$$=1.0\text{ m}$$

$$v=4\,(2\times 2-3)$$

$$=4\text{ m/s}$$

and $a=8\text{m/s}^2$

(b) At origin, $x=0$

or $(2t-3)=0$

∴ $v=4\times 0=0$

4. (a) At $t=0$, $x=2.0$ m **Ans.**

(b) $v=\frac{dx}{dt}=2t+6t^2$

At $t=0$, $v=0$ **Ans.**

(c) $a=\frac{dv}{dt}=2+12t$

At $t=2$ s

$a=26\text{ m/s}^2$ **Ans.**

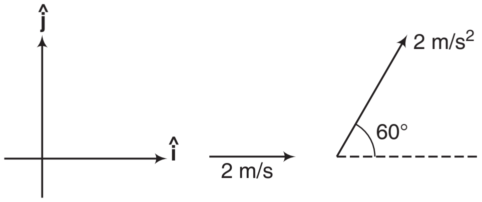
5. $v\propto t^{3/4}$

$$a=\frac{dv}{dt}\quad\Rightarrow\quad a\propto t^{-1/4}$$

$$s=\int vdt\quad\Rightarrow\quad s\propto t^{7/4}$$

INTRODUCTORY EXERCISE 6.7

1. $\mathbf{u}=2\hat{\mathbf{i}}$



$$\mathbf{a}=(2\cos 60°)\,\hat{\mathbf{i}}+(2\sin 60°)\,\hat{\mathbf{j}}$$

$$=(\hat{\mathbf{i}}+\sqrt{3}\,\hat{\mathbf{j}})$$

$t=2$ s

(i) $\mathbf{v}=\mathbf{u}+\mathbf{a}t=(2\hat{\mathbf{i}})+(\hat{\mathbf{i}}+\sqrt{3}\hat{\mathbf{j}})\,(2)$

$$=4\hat{\mathbf{i}}+2\sqrt{3}\,\hat{\mathbf{j}}$$

∴ $|\mathbf{v}|=\sqrt{(4)^2+(2\sqrt{3})^2}$

$=2\sqrt{7}$ m/s **Ans.**

(ii) $\mathbf{s}=\mathbf{u}\,t+\frac{1}{2}\,\mathbf{a}\,t^2=(2\hat{\mathbf{i}})\,(2)+\frac{1}{2}\,(\hat{\mathbf{i}}+\sqrt{3}\hat{\mathbf{j}})\,(2)^2$

$$=(6\hat{\mathbf{i}}+2\sqrt{3}\hat{\mathbf{j}})$$

∴ $|\mathbf{s}|=\sqrt{(6)^2+(2\sqrt{3})^2}$

$=4\sqrt{3}$ m **Ans.**

2. $\mathbf{v}=(2\hat{\mathbf{i}}+2t\hat{\mathbf{j}})$

(i) $\mathbf{a}=\frac{d\mathbf{v}}{dt}=(2\hat{\mathbf{j}})\text{ m/s}^2=\text{constant}$

∴ $\mathbf{v}=\mathbf{u}+\mathbf{a}\,t$ can be applied.

(ii) $\mathbf{s}=\int_0^1\mathbf{v}dt=\int_0^1(2\hat{\mathbf{i}}+2t\hat{\mathbf{j}})\,dt=[2t\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}\,]_0^1$

$=(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$ m **Ans.**

3. $\int_{2\hat{\mathbf{i}}}^{\mathbf{v}}d\mathbf{v}=\int\mathbf{a}dt=\int_0^t(2t\hat{\mathbf{i}}+3t^2\hat{\mathbf{j}})\,dt$

Solving we get $\mathbf{v}=2\hat{\mathbf{i}}+t^2\hat{\mathbf{i}}+t^3\hat{\mathbf{j}}$

At $t=1$ s, $\mathbf{v}=(3\hat{\mathbf{i}}+\hat{\mathbf{j}}\,)$ m/s

$$\int_0^t d\mathbf{r}=\int_0^t\mathbf{v}dt=\int_0^t[(2+t^2)\hat{\mathbf{i}}+t^3\hat{\mathbf{j}}]\,dt$$

⇒ $\mathbf{r}=\left(2t+\frac{t^3}{3}\right)\hat{\mathbf{i}}+\frac{t^4}{4}\hat{\mathbf{j}}$

At $t=1$ s, $\mathbf{r}=\frac{7}{3}\hat{\mathbf{i}}+\frac{1}{4}\,\hat{\mathbf{j}}$

Therefore, the co-ordinates are $\left(\frac{7}{3},\frac{1}{4}\right)$ m

INTRODUCTORY EXERCISE 6.8

1. (b) v = Slope of x - t graph

2. Distance travelled = displacement

$$= \text{area under } v\text{ - }t \text{ graph}$$

Acceleration = Slope of v - t graph

3. Acceleration = Slope of v - t graph

Distance travelled = displacement

$$= \text{area under } v\text{ - }t \text{ graph}$$

4. (a) Average velocity $= \frac{s}{t} = \frac{x_f - x_i}{t}$

$$= \frac{x_{10\text{ sec}} - x_{0\text{ sec}}}{10} = \frac{100}{10} = 10 \text{ m/s}$$

(b) Instantaneous velocity = Slope of x - t graph

5. From 0 to 20 s,

displacement s_1 = area under v - t graph

$= +\,50$ m (as v is negative)

From 20 to 40 s,

displacement s_2 = area under v - t graph

$= -\,50$ m (as v is negative)

Total distance travelled $= s_1 + |s_2| = 100$ m

Average velocity $= \frac{s}{t} = \frac{s_1 + s_2}{t}$

$$= \frac{50 - 50}{40} = 0$$

INTRODUCTORY EXERCISE 6.9

1. v_A = Slope of $A = \frac{2}{5} = 0.4$ m/s

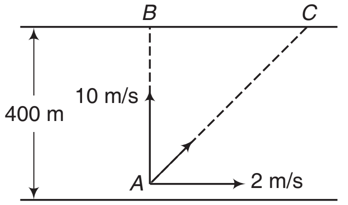
v_B = Slope of $B = \frac{12}{5} = 2.4$ m/s

$\therefore\ v_{AB} = v_A - v_B = -\,2$ m/s

2. $a_A = a_B = g$ (downwards)

$\therefore\quad a_{AB} = a_A - a_B = 0$ **Ans.**

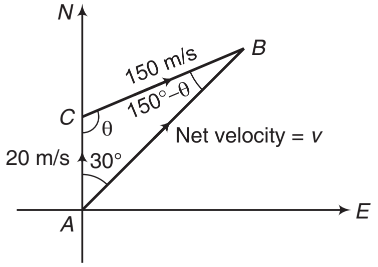
3. (a) $t = \frac{400}{10} = 40$ s **Ans.**



(b) $BC = (2 \text{ m/s})\,(t)$

$= 2 \times 40 = 80$ m **Ans.**

4. Applying sine law in ΔABC, we have

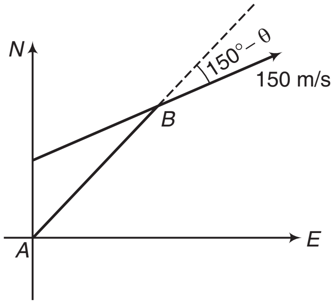


$$\frac{150}{\sin 30°} = \frac{v}{\sin\theta} = \frac{20}{\sin(150° - \theta)}$$

From first and third we have,

$$\sin(150° - \theta) = \frac{20 \sin 30°}{150} = \frac{1}{15}$$

$\therefore\quad 150° - \theta = \sin^{-1}\left(\frac{1}{15}\right)$ **Ans.**



Hence, we can see that direction of 150 m/s is $150° - \theta\left[\text{or } \sin^{-1}\left(\frac{1}{15}\right)\right]$, east of the line AB

(b) $150° - \theta = \sin^{-1}\left(\frac{1}{15}\right) = 3.8°$

$\therefore\quad \theta = 146.2°$

From first and second relation,

$$v = \frac{150 \sin\theta}{\sin 30°}$$

Substituting the values we get,

$$v = 167 \text{ m/s}$$

$\therefore\quad t = \frac{AB}{v}$

$$= \frac{500 \times 1000}{167 \times 60} \text{ min} = 49.9 \text{ min}$$

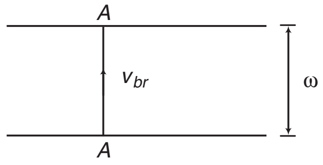
≈ 50 min **Ans.**

5. Let v_r = velocity of river

v_{br} = velocity of river in still water and

ω = width of river

Given, $t_{\min} = 10$ min or $\dfrac{\omega}{v_{br}} = 10$ … (i)



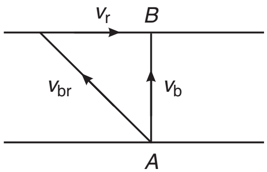
For minimum time

Drift in this case will be,

$$x = v_r t$$

$\therefore$ $120 = 10 v_r$ … (ii)

Shortest path is taken when v_b is along AB. In this case,



Shortest path

$$v_b = \sqrt{v_{br}^2 - v_r^2}$$

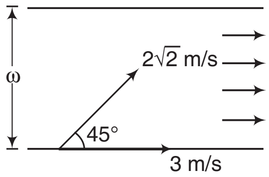
Now, $12.5 = \dfrac{\omega}{v_b} = \dfrac{\omega}{\sqrt{v_{br}^2 - v_r^2}}$ … (iii)

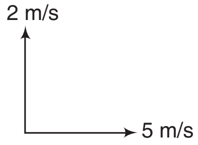
Solving these three equations, we get

$$v_{br} = 20 \text{ m/min},$$
$$v_r = 12 \text{ m/min}$$

and $\omega = 200$ m

6.





Net velocity of boatman

(a) $t = \dfrac{\omega}{2} = \dfrac{20}{2} = 10$ s

(b) Drift $= 5t = 50$ m

Exercises

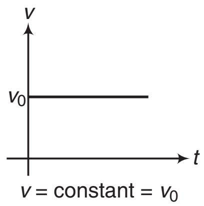
LEVEL 1

Assertion and Reason

1. $\mathbf{a} = -2(\hat{\mathbf{i}} - \hat{\mathbf{j}}) = -2\mathbf{v}$

i,e. **a** and **v** are constant vectors and antiparallel to each other. So motion is one dimensional. First retarded then accelerated in opposite direction.

2. In the given situation, v-t graph is a straight line. But s-t graph is also a straight line.



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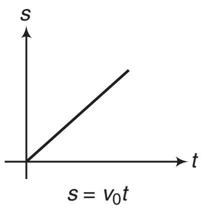


3. Average velocity $= \dfrac{s}{t}$

$$= \frac{\text{area of } v\text{-}t \text{ graph}}{t}$$

$$= \frac{\left(\dfrac{1}{2} v_0 t_0\right)}{t_0} = \frac{v_0}{2}$$

4. It is not necessary that if $v = 0$ then acceleration is also zero. If a particle is thrown upwards, then at highest point $v = 0$ but $a \neq 0$.

5. If acceleration is in opposite direction of velocity then speed will be decreasing even if magnitude of acceleration is increasing.

6. $a = 2t \neq$ constant

Therefore, velocity will not increase at a constant rate.

7. If a particle is projected upwards then $s = 0$, when it returns back to its initial position.

$\therefore$ Average velocity $= \dfrac{s}{t} = 0$

But its acceleration is constant $= g$

8. At point A, sign of slope of s-t graph is not changing. Therefore, sign of velocity is not changing.

9. $v_1 = \dfrac{ds_1}{dt} = (2 - 8t)$

$$v_2 = \frac{ds_2}{dt} = (-2 + 8t)$$

$$\therefore \quad v_{12} = v_1 - v_2 = (4 - 16t)$$

v_{12} does not keep on increasing.

10. $\mathbf{a} = \dfrac{d\,\mathbf{v}}{dt}$

So, direction of **a** and $d\,\mathbf{v}$ is same.

11. $t_1 = \sqrt{\dfrac{2\,(h/2)}{g}} = \sqrt{\dfrac{h}{g}}$

Velocity of second particle at height $\dfrac{h}{2}$ is,

$$v = \sqrt{u^2 - 2g\,(h/2)}$$

$$= \sqrt{gh - gh} = 0$$

or $\dfrac{h}{2}$ is its highest point.

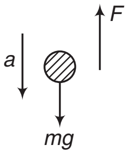
$$\therefore \quad t_2 = \frac{u}{g} = \frac{\sqrt{gh}}{g} = \sqrt{\frac{h}{g}}$$

$$\Rightarrow \quad t_1 = t_2$$

12.

$$a = \frac{mg - F}{m}$$

$$= g - \frac{F}{m} = \text{depends on } m$$



Since $m_1 \neq m_2$

$$\therefore \quad a_1 \neq a_2$$

$$\text{or} \quad t_1 \neq t_2$$

Single Correct Option

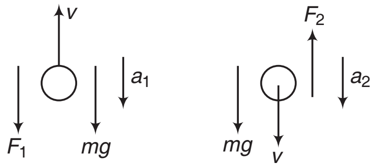
1. Packet comes under gravity. Therefore only force is mg.

$$a = \frac{F}{m} = \frac{mg}{m} = g \qquad \text{(downwards)}$$

2. Air resistance (let F) is always opposite to motion (or velocity)

Retardation in upward journey

$$a_1 = \frac{F_1 + mg}{m} = g + \frac{F_1}{m}$$



Upward journey

Downward journey

Acceleration in downward journey

$$a_2 = \frac{mg - F_2}{m} = g - \frac{F_2}{m}$$

Since, $\quad a_1 > a_2 \quad \Rightarrow \quad T_1 < T_2$

3. $v = \dfrac{2\pi R}{T} = \dfrac{(2\pi)\,(1)}{60} = \left(\dfrac{\pi}{30}\right)$ cm/s

In 15 s, velocity vector (of same magnitude) will rotate 90°.

$$\therefore \quad |\Delta\mathbf{v}| = |\mathbf{v}_f - \mathbf{v}_i| = \sqrt{v^2 + v^2 - 2vv\cos 90^\circ}$$

$$= \sqrt{2}\,v = \frac{\pi\sqrt{2}}{30} \text{ cm/s} \qquad \textbf{Ans.}$$

4. $h = \dfrac{u^2}{2g}$ or $u \propto \sqrt{h}$

5. $v_{\text{av}} = \dfrac{d}{t} = \dfrac{d_1 + d_2}{t_1 + t_2}$

$$21 = \frac{(18)\,(11) + (42)\,(v)}{60}$$

$$\Rightarrow \quad v = 25.3 \text{ m/s} \qquad \textbf{Ans.}$$

6. $v = \dfrac{dx}{dt} = 32 - 8t^2$

$$v = 0 \quad \text{at} \quad t = 2\text{ s}$$

$$a = \frac{dv}{dt} = -16\,t$$

at $\quad t = 2\text{s}, \quad a = -32 \text{ m/s}^2$ **Ans.**

7. $a = bt$

$$\therefore \quad \frac{dv}{dt} = bt$$

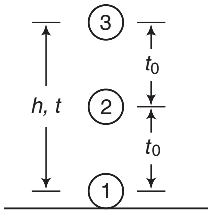
$$\text{or} \quad \int_{v_0}^{v} dv = \int_0^t (bt)\,dt$$

$$\therefore \quad v = v_0 + \frac{bt^2}{2}$$

$$s = \int_0^t v\,dt = \int_0^t \left(v_0 + \frac{bt^2}{2}\right) dt$$

$$= v_0 t + \frac{bt^3}{6}$$

8. $t = \sqrt{\dfrac{2h}{g}} = \sqrt{\dfrac{2 \times 5}{10}} = 1 \text{ s}$



Let t_0 is the interval between two drops. Then

$$2\,t_0 = t$$

$$\therefore \quad t_0 = 0.5 \text{ s}$$

2nd drop has taken t_0 time to fall. Therefore distance fallen,

$$d = \frac{1}{2} g t_0^2 = \left(\frac{1}{2}\right)(10)\,(0.5)^2$$

$$= 1.25 \text{ m}$$

$\therefore$ Height from ground $= h - d$

$$= 5 - 1.25$$

$$= 3.75 \text{ m}$$ **Ans.**

9. $\dfrac{1}{2} g t^2 = 20\,(t - 1) + \dfrac{1}{2} g (t - 1)^2$

Solving this equation we get,

$$\therefore \quad t = 1.5 \text{ s}$$

Now, $\quad d = 20\,(t - 1) + \dfrac{1}{2} g (t - 1)^2$

$$= 11.25 \text{ cm}$$ **Ans.**

10. $v_x = \dfrac{dx}{dt} = (8t - 2)$

$$\therefore \quad \int_{14}^{x} dx = \int_{2}^{t} (8t - 2)\, dt$$

$$x - 14 = (4t^2 - 2t) - (12)$$

$$\therefore \quad x = 4t^2 - 2t + 2 \qquad \text{...(i)}$$

$$v_y = \frac{dy}{dt} = 2$$

$$\therefore \quad \int_{4}^{y} dy = \int_{2}^{t} 2dt$$

$$\therefore \quad y - 4 = 2t - 4$$

$$\text{or} \quad y = 2t$$

$$\therefore \quad t = \frac{y}{2}$$

Substituting this value of t in Eq. (i) we have,

$$x = y^2 - y + 2$$ **Ans.**

11. $v_x = \dfrac{dx}{dt} = 5$

$$v_y = \frac{dy}{dt}$$

$$= (4t + 1)$$

At 45°, $\quad v_x = v_y$

$\Rightarrow \quad t = 1 \text{ s}$ **Ans.**

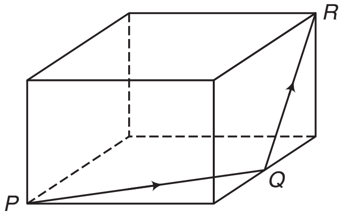
12. $h = \dfrac{1}{2} g T^2$

At $\dfrac{T}{3}$ second, distance fallen

$$d = \frac{1}{2} g \left(\frac{T}{3}\right) = \frac{h}{9}$$

$\therefore$ Height from ground $= h - d = \dfrac{8h}{9}$ **Ans.**

13. It should follow the path PQR.



$$PQR = \sqrt{5}\,a$$

$\therefore \quad t = \dfrac{\sqrt{5}\,a}{u}$ **Ans.**

14. **At 4 s**

$$u = at = 8 \text{ m/s}$$

$$s_1 = \frac{1}{2} a t^2 = \frac{1}{2} \times 2 \times 4^2 = 16 \text{ m}$$

From 4 s to 8 s

$$a = 0,\ v = \text{constant} = 8 \text{ m/s}$$

$$\therefore \quad s_2 = vt = (8)(4) = 32 \text{ m}$$

From 8 s to 12 s

$$s_3 = s_1 = 16 \text{ m}$$

$\therefore \quad s_{\text{Total}} = s_1 + s_2 + s_3 = 64 \text{ m}$ **Ans.**

15. Retardation is double. Therefore retardation time will be half.

Let t_0 = acceleration time

Then $\quad \dfrac{t_0}{2}$ = retardation time

$$t_0 + \frac{t_0}{2} = t$$

$$\therefore \quad t_0 = \frac{2t}{3}$$

Now, $s = s_1 + s_2$

$$= \left(\frac{1}{2}\right)(a)(t_0)^2 + \left(\frac{1}{2}\right)(2a)\left(\frac{t_0}{2}\right)^2$$

$$= \frac{3}{4}at_0^2$$

$$= \left(\frac{3}{4}\right)(a)\left(\frac{2t}{3}\right)^2$$

$$= \frac{at^2}{3}$$ **Ans.**

16. At the time of overtaking,

$$s_1 = s_2$$

$\therefore$ $$2ut + \frac{1}{2}at^2 = ut + \frac{1}{2}(2a)t^2$$

$\therefore$ $$t = \frac{2u}{a}$$

$\therefore$ $$s_1(\text{or } s_2) = (2u)\left(\frac{2u}{a}\right) + \frac{1}{2}(a)\left(\frac{2u}{a}\right)^2$$

$$= \frac{6u^2}{a}$$ **Ans.**

17. $t = \dfrac{80 \text{ m}}{30 \text{ m/s}} = \dfrac{8}{3}$ s.

Now in vertical direction

$$t = \frac{2u_r}{a_r}$$

or $$u_r = \frac{ta_r}{2} \qquad (a_r = g = 10 \text{ m/s}^2)$$

$$= \frac{(8/3)(10)}{2}$$

$$= \frac{40}{3} \text{ m/s}$$ **Ans.**

18. s = net area of v-t graph

At 2 s, net area = 0

$\therefore$ $s = 0$

and the particle crosses its initial position.

19. Total height $= 15 + \dfrac{u^2}{2g}$

$$= 15 + \frac{(10)^2}{2 \times 10}$$

or $h = 20$ m

Initial velocity $u = \sqrt{2gh}$

$$= \sqrt{2 \times 10 \times 20}$$

$$= 20 \text{ m/s}$$

Now applying

$v = u + at$, we have

$$v = (+20) + (-10)(3)$$

$$= -10 \text{ m/s}$$

$\therefore$ Velocity is 10 m/s, downwards.

20. $a = \dfrac{dv}{dt} = 0.2\, v^2$

$\therefore$ $$\int_{10}^{v} -5v^{-2} = \int_{0}^{2} dt$$

$\therefore$ $$\left[\frac{5}{v}\right]_{10}^{v} = 2$$

$\therefore$ $$\frac{5}{v} - \frac{5}{10} = 2$$

or $v = 2.0$ m/s **Ans.**

21. $d = d_1 + d_2$

$$= \frac{v_1^2}{2a_1} + \frac{v_2^2}{2a_2}$$

$$= \frac{(10)^2}{2 \times 2} + \frac{(20)^2}{(2 \times 1)}$$

$$= 225 \text{ m}$$ **Ans.**

22. $s_1 = s_2$

$\therefore$ $$(40)\,t - \frac{1}{2} \times 10 \times t^2 = (40)(t-2) - \frac{1}{2} \times 10 \times (t-2)^2$$

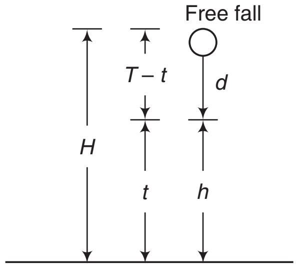
Solving this equation, we get

$$t = 5 \text{ s}$$

Then $$s_1 = (40)(5) - \frac{1}{2} \times 10 \times (5)^2$$

$$= 75 \text{ m}$$ **Ans.**

23. $d = \dfrac{1}{2}g\,(T - t)^2$



$\therefore$ $$h = H - d = H - \frac{1}{2}g\,(T - t)^2$$

24. $x = \dfrac{t^2}{2}$

$\therefore \quad v_x = \dfrac{dx}{dt} = t \quad y = \dfrac{x^2}{2} = \dfrac{t^4}{8}$

$\therefore \quad v_y = \dfrac{dy}{dt} = \dfrac{t^3}{2}$

At $t = 2$ s

$v_x = 2$ m/s

and $v_y = 4$ m/s

$\therefore \quad \mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ m/s

25. $\sqrt{x} = t + 3$

$\therefore \quad x = (t + 3)^2$

or $\quad v = \dfrac{dx}{dt}$

$= 2(t + 3)$

$\therefore$ v-t equation is linear **Ans.**

26. $\Delta v = v_f - v_i$ = area under a-t graph

$v_i = 0$

$\Rightarrow \quad v_f$ = area

$= 40 + 50$

$= 90$ m/s **Ans.**

27. $v^2 = 25 + 25$ s

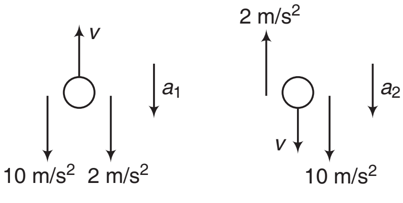
or $\quad v^2 = (5)^2 + 2\,(12.5)$ s

Now compare with $v^2 = u^2 + 2as$

28. Retardation during upward motion

$a_1 = 10 + 2$

$= 12$ m/s^2



Acceleration during downward motion,

$a_2 = 10 - 2 = 8$ m/s^2

$t = \sqrt{\dfrac{2s}{a}}$

or $\quad t \propto \dfrac{1}{\sqrt{a}}$

$\therefore \quad \dfrac{t_1}{t_2} = \sqrt{\dfrac{a_2}{a_1}} = \sqrt{\dfrac{8}{12}} = \sqrt{\dfrac{2}{3}}$ **Ans.**

29. $F = 3t^2 - 32$

$a = \dfrac{F}{m} = (0.3\,t^2 - 3.2)$

$\int_{10}^{v} dv = \int_{0}^{5} a\,dt = \int_{0}^{5} (0.3\,t^2 - 3.2)\,dt$

$v - 10 = -3.5$

$\therefore \quad v = 6.5$ m/s **Ans.**

30. $v^2 = u^2 - 2g\left(\dfrac{h}{2}\right)$

$\therefore \quad (10)^2 = u^2 - gh$

$\therefore \quad u^2 = (100 + gh)$

Now, $\quad h = \dfrac{u^2}{2g} = \dfrac{100 + gh}{2g}$

$h = 5 + \dfrac{h}{2}$ (as $2g = 20$ m/s^2)

$\therefore \quad h = 10$ m **Ans.**

Subjective Questions

1. (a) $\left|\dfrac{d\mathbf{v}}{dt}\right|$ is the magnitude of total acceleration.

While $\dfrac{d|\mathbf{v}|}{dt}$ represents the time rate of change of speed (called the tangential acceleration, a component of total acceleration) as $|\mathbf{v}| = v$.

(b) These two are equal in case of one dimensional motion.

2. (a) $x = 2t \quad \Rightarrow t = \dfrac{x}{2}$

$y = t^2 \quad$ or $\quad y = \left(\dfrac{x}{2}\right)^2$

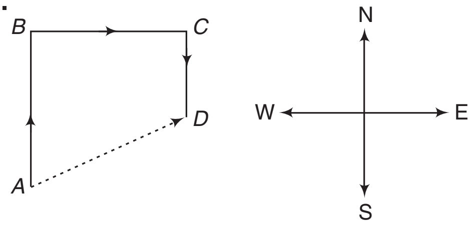
$\therefore \quad x^2 = 4y$ is the trajectory **Ans.**

(b) $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = (2t\hat{\mathbf{i}}) + (t^2\hat{\mathbf{j}})$

$\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = (2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}})$ units **Ans.**

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = (2\hat{\mathbf{j}})$ units **Ans.**

3.



(a) Distance $= AB + BC + CD$

$$= (500 + 400 + 200) = 1100 \text{ m}$$

(b) Displacement $= AD = \sqrt{(AB - CD)^2 + BC^2}$

$$= \sqrt{(500 - 200)^2 + (400)^2}$$

$$= 500 \text{ m}$$

(c) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$= \frac{1100}{20} = 55 \text{ m/min}$$

(d) Average velocity $= \dfrac{AD}{t}$

$$= \frac{500}{20} = 25 \text{ m/min (along } AD)$$

4. (a) The distance travelled by the rocket in 1 min (= 60 s) in which resultant acceleration is vertically upwards and 10 m/s^2 will be

$$h_1 = (1/2) \times 10 \times 60^2 = 18000 \text{ m} = 18 \text{ km} \quad \ldots\text{(i)}$$

and velocity acquired by it will be

$$v = 10 \times 60 = 600 \text{ m/s} \quad \ldots\text{(ii)}$$

Now, after 1 min the rocket moves vertically up with velocity of 600 m/s and acceleration due to gravity opposes its motion. So, it will go to a height h_2 till its velocity becomes zero such that

$$0 = (600)^2 - 2gh_2$$

or $\quad h_2 = 18000$ m [as $\quad g = 10$ m/s^2] ...(iii)

$$= 18 \text{ km}$$

So, from Eqs. (i) and (iii) the maximum height reached by the rocket from the ground

$h = h_1 + h_2 = 18 + 18 = 36$ km **Ans.**

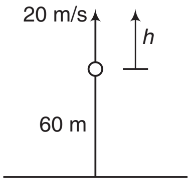
(b) As after burning of fuel the initial velocity from Eq. (ii) is 600 m/s and gravity opposes the motion of rocket, so the time taken by it to reach the maximum height (for which $v = 0$),

$$0 = 600 - gt$$

or $\quad t = 60$ s

i.e. after finishing fuel the rocket further goes up for 60 s, or 1 min. **Ans.**

5. $h = \dfrac{u^2}{2g} = \dfrac{(20)^2}{2 \times 10} = 20$ m



Let t be the time when particle collides with ground.

Then using the equation $s = ut + \frac{1}{2} at^2$

we have, $-60 = (20)\, t + \frac{1}{2} (-10)\, t^2$

Solving this equation, we get

$$t = 6 \text{ s}$$

(a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$= \frac{20 + 20 + 60}{6}$$

$= 16.67$ m/s **Ans.**

(b) Average velocity $= \dfrac{s}{t} = \dfrac{60}{6}$

$= 10$ m/s (downwards) **Ans.**

6. Average velocity $= \dfrac{\text{Total displacement}}{\text{Total time}}$

$$\therefore \quad 2.5\, v = \frac{vt_0 + 2vt_0 + 3vT}{t_0 + t_0 + T}$$

Solving, we get

$T = 4\, t_0$ **Ans.**

7. Retardation time is 8 s (double). Therefore, retardation should be half or 2 m/s^2.

s_1 = acceleration displacement

$$= \frac{1}{2} a_1 t_1^2 = \frac{1}{2} \times 4 \times (4)^2 = 32 \text{ m}$$

s_2 = retardation displacement

$$= \frac{1}{2} a_2 t_2^2$$

$$= \frac{1}{2} \times 2 \times (8)^2 = 64 \text{ m}$$

(a) $a_{av} = \dfrac{v_f - v_i}{t} = \dfrac{0 - 0}{12} = 0$ **Ans.**

(b) and *(c)* $d = s = s_1 + s_2 = 96$ m

$\therefore$ Average speed = Average velocity

$$= \frac{d}{t}$$

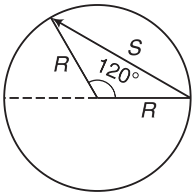
or $\quad \dfrac{s}{t} = \dfrac{96}{12} = 8$ m/s **Ans.**

8. $T = \dfrac{2\pi R}{v} = \dfrac{(2)\left(\dfrac{22}{7}\right)\left(\dfrac{21}{22}\right)}{1} = 6$ s

$$t = 2 \text{ s} = \frac{T}{3}$$

$\therefore$ Particle will rotate by 120°.

(a) $v_{av} = \frac{s}{t} = \frac{2R\sin\frac{\theta}{2}}{t}$ $(\theta = 120°)$



$$= \frac{\sqrt{3}\,R}{t} = \frac{21\sqrt{3}}{44}\ \text{m/s}$$ **Ans.**

(b) $$a_{av} = \frac{|\Delta \mathbf{v}|}{t} = \frac{\sqrt{v^2 + v^2 - 2vv\cos 120°}}{t}$$

$$= \frac{\sqrt{3}\,v}{t} = \frac{\sqrt{3}}{2}\ \text{m/s}^2$$ **Ans.**

9. At minimum distance, their velocities are same,

$\therefore$ $v_A = v_B$ or $u_A + a_A t = u_B + a_B t$

or $(3 + t) = (1 + 2t)$ or $t = 2$ s

Minimum distance, $d_{\min}$ = Initial distance – extra displacement of A upto this instant due to its greater speed

$= 10 - (s_A - s_B)$

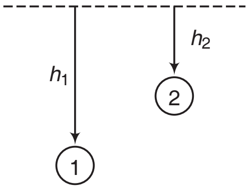
$= 10 + s_B - s_A$

$$= 10 + \left(u_B t + \frac{1}{2}a_B t^2\right) - \left(u_A t + \frac{1}{2}a_A t^2\right)$$

$$= 10 + \left[(1)(2) + \frac{1}{2}(2)(2)^2\right] - \left[(3)(2) + \frac{1}{2}(1)(2)^2\right]$$

$= 8$ m **Ans.**

10. Given $h_1 - h_2 = 10$ m



$$\therefore \quad \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 = 10$$

Solving equation, we get $t = 1.5$ s **Ans.**

11. At the time of collision,

$$s_1 = s_2$$

$$\therefore \quad v_0 t - \frac{1}{2}gt^2 = v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2$$

Solving this equation we get,

$$t = \frac{v_0}{g} + \frac{t_0}{2}$$ **Ans.**

12. (a) Velocity = slope of s-t graph

$\therefore$ Sign of velocity = sign of slope of s-t graph

(b) Let us discuss any one of them, let the portion cd, slope of s-t graph (= velocity) of this region is negative but increasing in magnitude. Therefore velocity is negative but increasing. Therefore sign of velocity and acceleration both are negative.

13. Displacement s = net area of v-t graph

$= 40 + 40 + 40 - 20 = 100$ m

Distance d = |Total area|

$= 40 + 40 + 40 + 20 = 140$ m

(a) Average velocity $= \frac{s}{t} = \frac{100}{14} = \frac{50}{7}$ m/s **Ans.**

(b) Average speed $= \frac{d}{t} = \frac{140}{14} = 10$ m/s **Ans.**

14. v_1 = speed of person

v_2 = speed of escalator

$$v_1 = \frac{l}{t_1} \quad \text{and} \quad v_2 = \frac{l}{t_2}$$

$$\therefore \quad t = \frac{l}{v_1 + v_2} = \frac{l}{\frac{l}{t_1} + \frac{l}{t_2}}$$

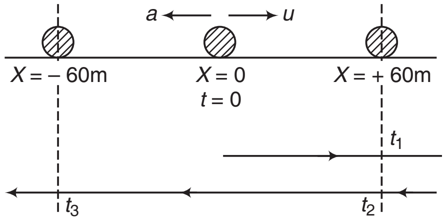
$$= \frac{t_1 t_2}{t_1 + t_2} = \frac{90 \times 60}{90 + 60} = 36\ \text{s}$$ **Ans.**

15. At time $t = t_1$, slope of s-t graph (= velocity) is positive and increasing. Therefore velocity is positive and acceleration is also positive.

At time $t = t_2$, slope is negative but decreasing (in magnitude). Therefore velocity is negative but decreasing in magnitude. Hence, acceleration is positive.

16. Comparing with $v = u + at$, we have,

$$u = 40\ \text{m/s} \quad \text{and} \quad a = -10\ \text{m/s}^2$$



Distance of 60 m from origin may be at $x = +60$ m and $x = -60$ m.

From the figure, we can see that at these two points particle is at three times, t_1, t_2 and t_3.

For $X = +60$ m or t_1 and t_2

$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow \qquad 60 = (+40)\,t + \frac{1}{2}(-10)\,t^2$$

Solving this equation, we get

$$t_1 = 2\,\text{s and } t_2 = 6\text{ s} \qquad \textbf{Ans.}$$

For $X = -60$ m or t_3

$$s = ut + \frac{1}{2}at^2$$

$$\therefore \qquad -60 = (+40)\,t + \frac{1}{2}(-10)\,t^2$$

Solving this equation, we get positive value of t as

$$t_3 = 2\,(2 + \sqrt{7})\text{ s} \qquad \textbf{Ans.}$$

17. Displacement = Area under velocity-time graph

Hence, $s_{OA} = \frac{1}{2} \times 2 \times 10 = 10$ m

$s_{AB} = 2 \times 10 = 20$ m

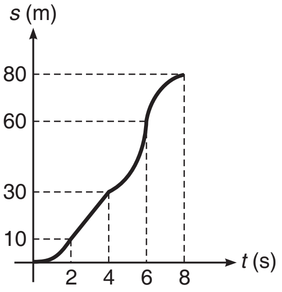
or $s_{OAB} = 10 + 20 = 30$ m

$s_{BC} = \frac{1}{2} \times 2(10 + 20) = 30$ m

or $s_{OABC} = 30 + 30 = 60$ m

and $s_{CD} = \frac{1}{2} \times 2 \times 20 = 20$ m

or $s_{OABCD} = 60 + 20 = 80$ m



Between 0 to 2 s and 4 to 6 s motion is accelerated, hence displacement-time graph is a parabola. Between 2 to 4 s motion is uniform, so displacement-time graph will be a straight line. Between 6 to 8 s motion is decelerated hence displacement-time graph is again a parabola but inverted in shape. At the end of 8 s velocity is zero, therefore, slope of displacement-time graph should be zero. The corresponding graph is shown in above figure.

18. (a) $a_{\text{av}} = \dfrac{v_f - v_i}{t} = \dfrac{v_{12} - v_6}{12 - 6}$

$$= \frac{(-10) - (20)}{6} = -5\text{ m/s}^2 \qquad \textbf{Ans.}$$

(b) $\Delta v = v_f - v_i$ = net area of a-t graph

$\therefore \qquad v_{14} - v_0 = 40 + 30 + 40 - 20 = 90$

But $\qquad v_0 = 0$

$\therefore \qquad v_{14} = 90$ m/s **Ans.**

19. (a) $\mathbf{v}_{av} = \dfrac{\mathbf{s}}{t} = \dfrac{\mathbf{r}_f - \mathbf{r}_i}{t}$

$$= \frac{(6\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}})}{4}$$

$$= (1.25\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}})\text{ m/s} \qquad \textbf{Ans.}$$

(b) $\mathbf{a}_{av} = \dfrac{\Delta\mathbf{v}}{t} = \dfrac{\mathbf{v}_f - \mathbf{v}_i}{t}$

$$= \frac{(2\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) - (4\hat{\mathbf{i}} + 6\hat{\mathbf{j}})}{4}$$

$$= (-0.5\hat{\mathbf{i}} + \hat{\mathbf{j}})\text{ m/s}^2 \qquad \textbf{Ans.}$$

(c) We cannot calculate the distance travelled from the given data.

20. In falling 5 m, first stone will take 1 s (from $h = \frac{1}{2}gt^2$)

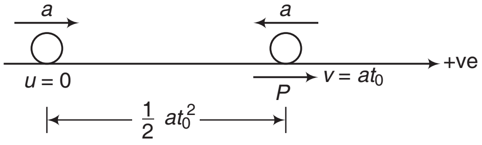
First stone, $\qquad t = \sqrt{\dfrac{2h}{g}} \qquad$ ...(i)

Second stone, $(t-1) = \sqrt{\dfrac{2\,(h - 2s)}{g}} \qquad$ ...(ii)

Solving these equations, we get

$$h = 45\text{ m} \qquad \textbf{Ans.}$$

21. If we start time calculations from point P and apply the equation,



$$s = ut + \frac{1}{2}at^2$$

$$\left(-\frac{1}{2}at_0^2\right) = (+at_0)\,t + \frac{1}{2}(-a)\,t^2$$

Solving we get, $t = (\sqrt{2} + 1)\,t_0$

$= 2.414\,t_0$

$\therefore$ Total time, $T = t + t_0$

$= (3.414)\,t_0$ **Ans.**

22. (a) $h = 15 + \dfrac{u^2}{2g}$

$$= 15 + \frac{(5)^2}{2 \times 10}$$

$$= 16.25 \text{ m} \qquad \textbf{Ans.}$$

(b) $t = \sqrt{\dfrac{2h}{g}} = \sqrt{\dfrac{2 \times 16.25}{10}} = 1.8 \text{ s}$ **Ans.**

23. (a) and (b) $60 = u\,(6) + \dfrac{1}{2} \times a \times (6)^2$ ...(i)

$$15 = u + (a)\,(6) \qquad \text{...(ii)}$$

Solving these two equations, we get

$$u = 5 \text{ m/s}$$

and $a = \dfrac{5}{3} \text{ m/s}^2$ **Ans.**

(c) $(5)^2 = (2)\left(\dfrac{5}{3}\right) s$ $(v^2 = 2\,as)$

$\therefore \quad s = 7.5 \text{ m}$ **Ans.**

24. $s = s_0 + ut + \dfrac{1}{2}at^2$

at $t = 0,\ s = s_0 = 2 \text{ m}$ ...(i)

at $t = 10 \text{ s},$

$$s = 0 = s_0 + ut + \frac{1}{2}at^2$$

or $0 = 2 + 10u + 50a$

or $10u + 50a = -2$ ...(ii)

$$v = u + at$$

$\therefore \quad 0 = u + 6a$ (at $t = 6$ s)

$\therefore \quad u + 6a = 0$ ...(iii)

Solving Eqs. (ii) and (iii) we get,

$$u = -1.2 \text{ m/s}$$

and $a = 0.2 \text{ m/s}^2$ **Ans.**

Now, $v = u + at$

$\therefore \quad v = (-1.2) + (0.2)\,(10)$

$$= 0.8 \text{ m/s} \qquad \textbf{Ans.}$$

25. **After 2 s**

$$\mathbf{v}_1 = \mathbf{u} + \mathbf{a}_1 t_1$$

$$= 0 + (2\hat{\mathbf{i}})\,(2) = (4\hat{\mathbf{i}}) \text{ m/s}$$

$$\mathbf{r}_1 = \mathbf{r}_i + \frac{1}{2}\mathbf{a}_1 t_1^2$$

$$= (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) + \frac{1}{2}(2\hat{\mathbf{i}})\,(2)^2$$

$$= (6\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \text{ m}$$

After next 2 s

(a) $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{a}_2 t_2$

$$= (4\hat{\mathbf{i}}) + (2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})\,(2)$$

$$= (8\hat{\mathbf{i}} - 8\hat{\mathbf{j}}) \text{ m/s}$$

(b) $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{v}_1 t_2 + \dfrac{1}{2}\mathbf{a}_2 t_2^2$

$$= (6\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) + (4\hat{\mathbf{i}})\,(2) + \frac{1}{2}(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})\,(2)^2$$

$$= (18\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) \text{ m}$$

$\therefore$ Co-ordinates are,

$$x = 18 \text{ m}$$

and $y = -4 \text{ m}$

26. $x = u_x t + \dfrac{1}{2}a_x t^2$

$\therefore \quad 29 = (0)\,(t) + \dfrac{1}{2} \times (4.0)\,t^2$

$\therefore \quad t^2 = \sqrt{14.5} \text{ s}^2 \quad \text{or} \quad t = 3.8 \text{ s}$

(a) $y = u_y t + \dfrac{1}{2}a_y t^2$

$$= (8)\,(3.8) + \frac{1}{2} \times 2 \times 14.5$$

$$= 44.9 \text{ m} \approx 45 \text{ m} \qquad \textbf{Ans.}$$

(b) $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$= (8\hat{\mathbf{j}}) + (4.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\,(3.8)$$

$$= (15.2\hat{\mathbf{i}} + 15.6\hat{\mathbf{j}})$$

$\therefore$ Speed $= |\mathbf{v}| = \sqrt{(15.2)^2 + (15.6)^2}$

$$= 22 \text{ m/s} \qquad \textbf{Ans.}$$

27. $\dfrac{dv}{ds} = -\dfrac{3\,\text{m/s}}{\text{m}}$

$$a = v \cdot \frac{dv}{ds}$$

$$= (10)\,(-3)$$

$$= -30 \text{ m/s}^2 \qquad \textbf{Ans.}$$

28. $v = 3t^2 - 6t \quad \Rightarrow \quad v = 0 \quad \text{at} \quad t = 2 \text{ s}$

For $t < 2$ s, velocity is negative. At $t = 2$ s, velocity is zero and for $t > 2$ s velocity is positive.

$\therefore \quad s_1 = \displaystyle\int_0^{3.5} v\,dt = \int_0^{3.5} (3t^2 - 6t)\,dt$

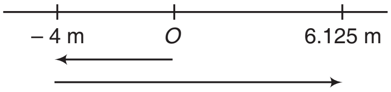
$$= 6.125 \text{ m}$$

= displacement upto 3.5 s

$$s_2 = \int_0^{2} v\,dt = \int_0^{2} (3t^2 - 6t)\,dt$$

$= -4$ m

= displacement upto 2 s



$\therefore \quad d$ = distance travelled in 3.5 s

$= 4 + 4 + 6.125$

$= 14.125$ m **Ans.**

$$\text{Average speed} = \frac{d}{t} = \frac{14.125}{3.5}$$

$= 4.03$ m/s **Ans.**

$$\text{Average velocity} = \frac{s_1}{t} = \frac{6.125}{3.5}$$

$= 1.75$ m/s **Ans.**

29. $a = \frac{4}{v}$ or $\frac{dv}{dt} = \frac{4}{v}$

$$\therefore \quad \int_6^v v\,dv = \int_2^t 4\,dt$$

$$\therefore \quad \frac{v^2}{2} - 18 = 4t - 8$$

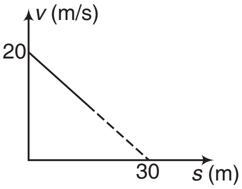
$$\therefore \quad v = \sqrt{8t + 20}$$

$$a = \frac{dv}{dt} = \frac{4}{\sqrt{8t + 20}}$$

At $t = 3$ s

$a = 0.603$ m/s^2 **Ans.**

30. v-s equation is



$$v = -\frac{2}{3}s + 20$$

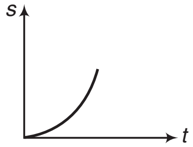
$\therefore \frac{dv}{ds} = -\frac{2}{3}$ per second

At $s = 15$ m, $v = 10$ m/s

$a = v \cdot \frac{dv}{ds} = -\frac{20}{3}$ m/s^2 **Ans.**

31. a = slope of v-t graph and s = area under v-t graph.

Further, for $t \le 2$ s, velocity (i.e. slope of s-t graph) is positive but increasing. Therefore, s-t graph is as under.



Same logic can be applied with other portions too.

32. Same as Q.No. 31.

33. Kinetic energy will first increase and then decrease.

$v = gt$ (in downward journey)

$$\therefore \quad \text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}mg^2t^2$$

Hence, KE *versus* t graph is a parabola.

34. Speed first increases. Then just after collision, it becomes half and now it decreases.

Pattern of velocity is same (with sign). In the answer, downward direction is taken as positive.

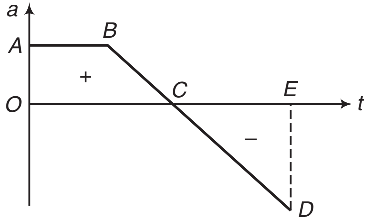
Further, $v = gt$

Hence, v-t graph is straight line.

35. Area of a-t graph gives change in velocity

$\therefore \quad \Delta v = v_f - v_i$ = area of a-t graph

Since, $v_f = v_i$



$\therefore$ Net area of a-t graph should be zero

$\therefore \quad$ Area $OABC$ = Area CDE

Substituting the values we get,

$t_E = (2 + \sqrt{3})$ s **Ans.**

36. (a) a = slope of v-t graph

(b) $s = r_f - r_i$ = net area of v-t graph

$\therefore \quad r_f = r_i$ + net area

$= 10 + 10 + 20 + 10 - 10 - 10$

$= 30$ m

(c) (i) **For** $0 \le t \le 2$ s

u = initial velocity = 0

s_0 = initial displacement = 10 m

a = slope of v-t graph = + 5 m/s^2

$$\therefore \quad s = s_0 + ut + \frac{1}{2}at^2 = 10 + 2.5\,t^2$$

(ii) **For** $4 \text{ s} \le t \le 8$ s

u = initial velocity = 10 m/s

= velocity at 4 s

s_0 = (10 m) + area of v-t graph upto 4 s

$= 10 + 10 + 20 = 40$ m

a = slope of v-t graph = –5 m/s^2

$$\therefore \quad s = s_0 + u(t-4) + \frac{1}{2}a(t-4)^2$$

$= 40 + 10(t - 4) - 2.5(t - 4)^2$ **Ans.**

37. (a) $a_1 = a_2 = -g$

$\therefore \quad a_{12} = a_1 - a_2 = 0$ **Ans.**

(b) $u_{21} = u_2 - u_1 = 20 - (-s)$

$= +25$ m/s **Ans.**

(c) $u_{12} = -u_{21} = -25$ m/s

Since, $a_{12} = 0$

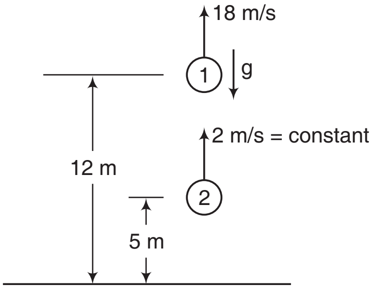
$\therefore \quad u_{12} =$ constant

$= -25$ m/s **Ans.**

(d) $a_{12} = 0$. Therefore, relative motion between them is uniform with constant velocity 25 m/s.

$\therefore \quad t = \dfrac{d}{v} = \dfrac{20}{25} = 0.8$ s **Ans.**

38. (a) When the two meet,



$s_2 = s_1 + 7$

or $\quad (2t) = (18t) - (4.9t^2) + 7$

Solving we get,

$t = 3.65$ s **Ans.**

$s_2 = 2 \times 3.65 = 7.3$ m

$\therefore$ Height $= 5 + 7.3$

$= 12.30$ m **Ans.**

(b) $v_{\text{ball}} = u - gt = 18 - 9.8 \times 3.65 = -17.77$ m/s

$\therefore$ Velocity of ball with respect to elevator

= velocity of ball − velocity of elevator

$= (-17.77) - (2)$

$= -19.77$ m/s

≈ -19.8 m/s **Ans.**

Negative sign indicates the downward direction.

39. (a) For truck

$$s_1 = \frac{1}{2} a_1 t^2$$

$$60 = \frac{1}{2} \times 2.2 \times t^2$$

$\therefore \quad t = 7.39$ s **Ans.**

(b) For automobile

$$s_2 = \frac{1}{2} a_2 t^2 = \frac{1}{2} \times 3.5 \times (7.39)^2$$

$= 95.5$ m

$\therefore$ Initial distance between them,

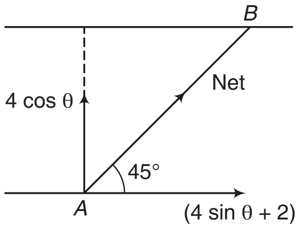
$= s_2 - s_1$

$= 35.5$ m **Ans.**

(c) $v_1 = a_1 t = (2.2)(7.39) = 16.2$ m/s

and $\quad v_2 = a_2 t = (3.5)(7.39) = 25.9$ m/s

40. Net velocity is along AB or at 45° if,

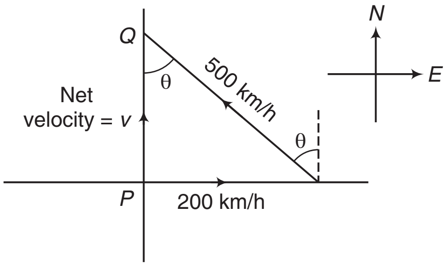


$$4\cos\theta = 4\sin\theta + 2$$

Solving this equation we get,

$\theta \approx 24.3°$ **Ans.**

41. (a) $\sin\theta = \dfrac{200}{500} = 0.4$



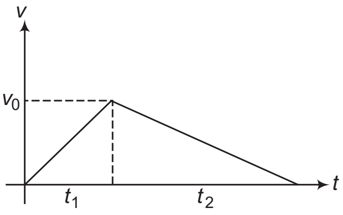
$\therefore \quad \theta = \sin^{-1}(0.4)$, west of north

(b) $\quad v = \sqrt{(500)^2 - (200)^2}$

$= 100\sqrt{21}$ km/h

$\therefore \quad t = \dfrac{PQ}{v} = \dfrac{1000}{100\sqrt{21}} = \dfrac{10}{\sqrt{21}}$ h **Ans.**

42.



$$\frac{v_0}{t_1}=x \quad\Rightarrow\quad t_1=\frac{v_0}{x}$$

$$\frac{v_0}{t_2}=y \quad\Rightarrow\quad t_2=\frac{v_0}{y}$$

$$t_1+t_2=v_0\left(\frac{1}{x}+\frac{1}{y}\right)=4 \qquad \text{...(i)}$$

Further,

area = displacement

$\therefore$ $\frac{1}{2}v_0\times t=s$

But numerically, $t=s=4$ units

$\therefore$ $v_0=2$ units

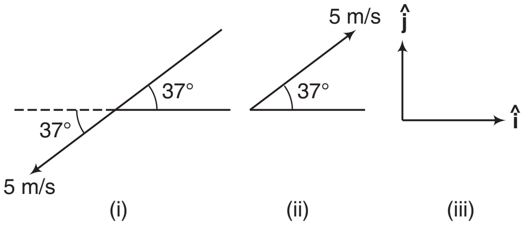
Substituting in Eq. (i) we get,

$$\frac{1}{x}+\frac{1}{y}=2.$$

LEVEL 2

Single Correct Option

1. Let velocity of rain is



(i) (ii) (iii)

$$\mathbf{v}_R=a\hat{\mathbf{i}}-b\hat{\mathbf{j}}$$

In first case, $\mathbf{v}_M=(-4\hat{\mathbf{i}}-3\hat{\mathbf{j}})$

$\therefore$ $\mathbf{v}_{RM}=\mathbf{v}_R-\mathbf{v}_M=(a+4)\,\hat{\mathbf{i}}+(-b+3)\,\hat{\mathbf{j}}$

It appears vertical

$\therefore$ $a+4=0$

or $a=-4$

In second case, $\mathbf{v}_M=(4\hat{\mathbf{i}}+3\hat{\mathbf{j}})$

$\therefore$ $\mathbf{v}_{RM}=\mathbf{v}_R-\mathbf{v}_M=(a-4)\,\hat{\mathbf{i}}+(-b-3\hat{\mathbf{j}})$

$=-8\hat{\mathbf{i}}+(-b-3\hat{\mathbf{j}})$

It appears at $\theta=\tan^{-1}\left(\frac{7}{8}\right)$

$\therefore$ $-b-3=-7$

$\therefore$ $b=4$

and speed of rain $=\sqrt{a^2+b^2}$

$=\sqrt{32}$ m/s **Ans.**

2. $\frac{dv}{dt}=a=-4v+8$

$\therefore$ $\frac{da}{dt}=-4\,\frac{dv}{dt}$

$=-4\,(-4v+8)$

$=16v-32$

$\therefore$ $\left(\frac{da}{dt}\right)_i=16\,v_i-32$

$=(16)\,(0)-32$

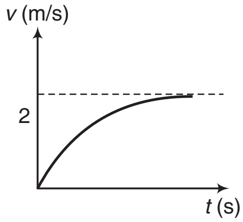
$=-32\text{ m/s}^2$

Further,

$$\int_0^v\frac{dv}{8-4v}=\int_0^t dt$$

Solving this equation we get,

$$v=2\,(1-e^{-4t})$$



Hence, v-t graph is exponentially increasing graph, terminating at 2 m/s.

3. For collision,

$\mathbf{r}_A=\mathbf{r}_B$ (at same instant)

$\therefore$ $(\mathbf{r}_i+\mathbf{v}_t)_A=(\mathbf{r}_i+\mathbf{v}_t)_B$

$\Rightarrow(5\hat{\mathbf{i}}+10\hat{\mathbf{j}}+5\hat{\mathbf{k}})\,t=30\hat{\mathbf{i}}+(10\hat{\mathbf{i}}+5\hat{\mathbf{j}}+5\hat{\mathbf{k}})\,t$

Equating the coefficients of x

$$5t=30+10t$$

$\Rightarrow$ $t=-$ ve

So, they will never collide.

4. $\frac{dv_y}{dt}=2t$

$\therefore$ $v_y=t^2$ or $\frac{dy}{dt}=t^2$

or $y=\frac{t^3}{3}$ ...(i)

and $x=v_0t \quad\Rightarrow\quad t=\frac{x}{v}$

Substituting in Eq. (i) we have,

$$y=\frac{x^3}{3v_0^3}$$ **Ans.**

5. $\frac{dt}{dx} = (2\alpha x + \beta)$

$$\therefore \qquad \frac{dx}{dt} = v = \left(\frac{1}{2\alpha x + \beta}\right)$$

$$a = \frac{dv}{dt} = -2\alpha\left(\frac{1}{2\alpha x + \beta}\right)^2 \cdot \frac{dx}{dt}$$

$$= -2\alpha\,(v^2)\,(v) = -2\alpha v^3 \qquad \textbf{Ans.}$$

6. $f = v \cdot \frac{dv}{dx} = a - bx$

$$\text{or} \qquad \int_0^v v\,dv = \int_0^x (a - bx)\,dx$$

$$\therefore \qquad v = \sqrt{2ax - bx^2} \qquad \text{...(i)}$$

At other station, $v = 0$

$$\Rightarrow \qquad x = \frac{2a}{b} \qquad \textbf{Ans.}$$

Further acceleration will change its direction when,

$$f = 0 \quad \text{or} \quad a - bx = 0 \quad \text{or} \quad x = \frac{a}{b}$$

At this x, velocity is maximum.

Using Eq. (i),

$$v_{\max} = \sqrt{2a\left(\frac{a}{b}\right) - b\left(\frac{a}{b}\right)^2} = \frac{a}{\sqrt{b}} \qquad \textbf{Ans.}$$

7. $a = \frac{F}{m} = \frac{-kx^2}{m}$

$$\therefore \qquad v \cdot \frac{dv}{dx} = -\frac{kx^2}{m}$$

$$\text{or} \qquad \int_0^v v\,dv = \int_a^0 -\frac{kx^2}{m}\,dx$$

$$\frac{v^2}{2} = \frac{ka^3}{3m}$$

$$\therefore \qquad v = \frac{2ka^3}{3m} \qquad \textbf{Ans.}$$

8. $a = v \cdot \frac{dv}{ds} = (4)\,(-\tan 60°)$

$$= -4\sqrt{3}\ \text{m/s}^2$$

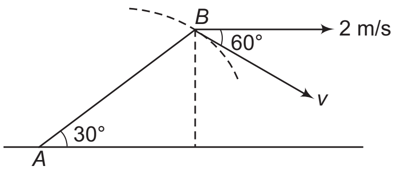
9. $x_1 = \frac{1}{2}gt^2 = 0.5\,gt^2$

$$x_1 + x_2 = \frac{1}{2}g\,(2t)^2 = 2\,gt^2$$

$$x_2 = 2gt^2 - x_1 = 1.5\,gt^2$$

$$\therefore \qquad x_2 - x_1 = gt^2 \quad \text{or} \quad t = \sqrt{\frac{x_2 - x_1}{g}}$$

10.



$$v \cos 60° = 2\ \text{m/s}$$

$$\therefore \qquad v = 4\ \text{m/s} \qquad \textbf{Ans.}$$

11. $90\ \text{km/h} = 90 \times \frac{5}{18} = 25\ \text{m/s}$

$180\ \text{km/h} = 108 \times \frac{5}{18} = 30\ \text{m/s}$

At maximum separation their velocities are same.

$\therefore$ Velocity of motorcycle = 25 m/s

or $\qquad at = 25$

or $\qquad t = 5$ s

But thief has travelled up to 7s.

s_1 = displacement of thief

$= v_1 t_1 = 25 \times 7 = 175$ m

s_2 = displacement of motorcycle

$= \frac{1}{2} \times a_2\, t_2^2$

$= \frac{1}{2} \times 5 \times (5)^2$

$= 62.5$ m

$\therefore$ Maximum separation

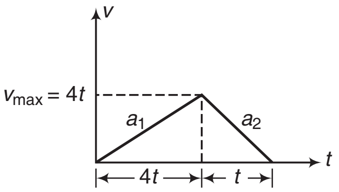
$= s_1 - s_2 = 112.5$ m **Ans.**

12. Relative velocity of A with respect to B should be along AB or absolute velocity components perpendicular to AB should be same.

$$\therefore \qquad \frac{2u}{3}\sin\theta = u \sin 30°$$

$$\therefore \qquad \theta = \sin^{-1}\left(\frac{3}{4}\right) \qquad \textbf{Ans.}$$

13. Deceleration is four times. Therefore, deceleration time should be $\frac{1}{4}$ th.



$$v_{\max} = (a_1)\,(4t) = (1)\,(4t) = 4t$$

Area of v-t graph = displacement

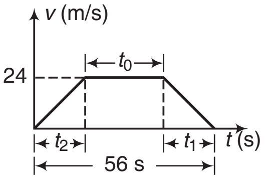
$$\therefore \qquad 200 = \frac{1}{2}(5t)(4t)$$

or $\qquad t = \sqrt{20}$ s

Total journey time $= 5t = 22.4$ s **Ans.**

14. Area of v-t graph = displacement

$$\therefore \quad 1032 = \frac{1}{2}(56 + t_0)(24) \quad \text{or} \quad t_0 = 30 \text{ s}$$

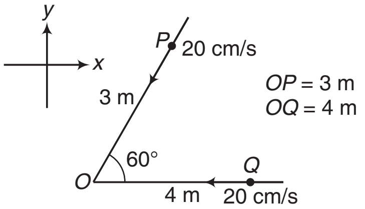


Deceleration time $t_1 = \dfrac{24}{4} = 6$ s

$\therefore$ Acceleration time $t_2 = 56 - t_0 - t_1 = 20$ s

$\therefore$ Acceleration $= \dfrac{24}{20} = 1.2 \text{ m/s}^2$ **Ans.**

15. $\mathbf{v}_Q = -20\hat{\mathbf{i}}$



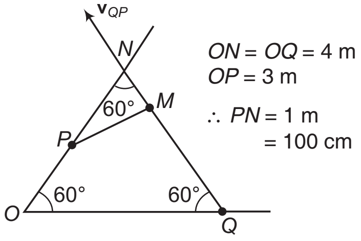
$$\mathbf{v}_P = -20\cos 60\hat{\mathbf{i}} - 20\sin 60°\hat{\mathbf{j}}$$
$$= -10\hat{\mathbf{i}} - 10\sqrt{3}\,\hat{\mathbf{j}}$$

Assuming P to be at rest,

$$\mathbf{v}_{QP} = \mathbf{v}_Q - \mathbf{v}_P = -10\hat{\mathbf{i}} + 10\sqrt{3}\,\hat{\mathbf{j}}$$

Now, $\quad \therefore \tan\theta = \dfrac{10\sqrt{3}}{10} = \sqrt{3}$ or $\theta = 60°$

where, θ is the angle of $\mathbf{v}_{QP}$ from x-axis towards positive y-axis.



Shortest distance $= PM = PN \sin 60°$

$$= (100)\frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ cm}$$

More than One Correct Options

1. (a) $a = -\alpha\sqrt{v}$

$$\therefore \qquad \frac{dv}{dt} = -\alpha\sqrt{v}$$

or $\qquad \displaystyle\int_0^t dt = -\frac{1}{\alpha}\int_{v_0}^0 v^{-1/2}\cdot dv$

$$\therefore \qquad t = \frac{2\sqrt{v_0}}{\alpha}$$ **Ans.**

(d) $a = -\alpha\sqrt{v}$

$$\therefore \qquad v\cdot\frac{dv}{ds} = -\alpha\sqrt{v}$$

$$\therefore \qquad \int_0^s ds = -\frac{1}{\alpha}\int_{v_0}^0 v^{1/2}\,dv$$

$$\therefore \qquad s = \frac{2v_0^{3/2}}{3\alpha}$$ **Ans.**

2. $a = -0.5t = \dfrac{dv}{dt}$

$$\therefore \qquad \int_{16}^v dv = \int_0^t -0.5t\,dt$$

$$\therefore \qquad v = 16 - 0.25t^2$$

$$s = \int v dt = 16t - \frac{0.25t^3}{3}$$

$v = 0$ when $16 - 0.25t^2 = 0$

or $\qquad t = 8$ s **Ans.**

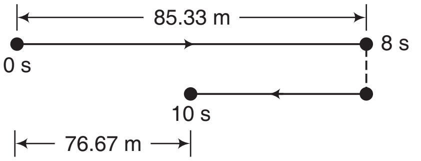
So direction of velocity changes at 8 s. Up to 8 s distance = displacement

$\therefore$ **At 4 s**

$$d = 16 \times 4 - \frac{0.25 \times (4)^3}{3} = 58.67 \text{ m}$$ **Ans.**

$$S_{8\text{s}} = 16 \times 8 - \frac{(0.25)(8)^3}{3}$$
$$= 85.33 \text{ m}$$

$$S_{10\text{s}} = 16 \times 10 - \frac{(0.25)(10)^3}{3}$$
$$= 76.67 \text{ cm}$$



Distance travelled in 10 s,

$$d = (85.33) + (85.33 - 76.67)$$
$$= 94 \text{ m}$$ **Ans.**

At 10 s

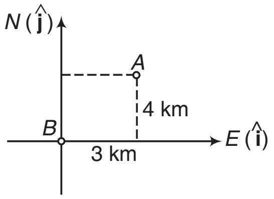
$$v = 16 - 0.25\,(10)^2 = -9\text{ m/s}$$

∴ Speed $= 9$ m/s **Ans.**

3. If $\mathbf{a}$ = constant

Then $|\mathbf{a}|$ is also constant or $\left|\dfrac{d\mathbf{v}}{dt}\right|$ = constant

4. $\mathbf{r}_B = 0$ at $t = 0$



$$\mathbf{r}_A = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$$

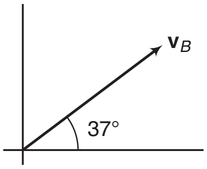
at $t = 0$

$$\mathbf{v}_A = (-20\hat{\mathbf{j}})$$

$$\mathbf{v}_B = (40\cos 37°)\hat{\mathbf{i}} + (40\sin 37°)\hat{\mathbf{j}}$$

$$= (32\hat{\mathbf{i}} + 24\hat{\mathbf{j}})$$

$$\mathbf{v}_{AB} = (-32\hat{\mathbf{i}} - 44\hat{\mathbf{j}})\text{ km/h}$$ **Ans.**



At time $t = 0$,

$$\mathbf{r}_{AB} = \mathbf{r}_A - \mathbf{r}_B = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$$

∴ At time $t = t$,

$$\mathbf{r}_{AB} = (\mathbf{r}_{AB}\text{ at } t = 0) + \mathbf{v}_{AB}\,t$$

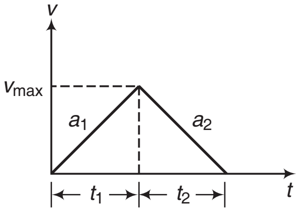
$$= (3 - 32t)\hat{\mathbf{i}} + (4 - 44t)\hat{\mathbf{j}}$$ **Ans.**

5. $a_1 t_1 = a_2 t_2$

$$v_{\max} = a_1 t_1 = a_2 t_2$$

$$s_1 = \text{Area of } v\text{-}t \text{ graph}$$

$$= \frac{1}{2}(t_1 + t_2)(a_1 t_1)$$

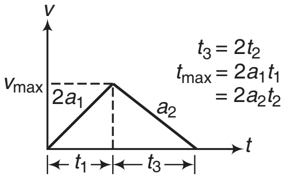


$$v_1 = \frac{s_1}{t_1 + t_2} = \frac{1}{2} a_1 t_1$$

$$s_2 = \frac{1}{2}(t_1 + t_3)\,v_{\max}$$

$$= \frac{1}{2}(t_1 + 2t_2)(2a_1 t_1)$$

$$v_2 = \frac{s_2}{(t_1 + t_3)} = a_1 t_1$$

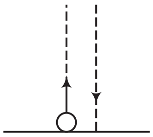


From the four relations we can see that

$$v_2 = 2v_1$$

and $2s_1 < s_2 < 4s_1$ **Ans.**

6. In the complete journey,



$s = 0$ and a = constant $= g$ (downwards)

7. $a = \dfrac{F}{m} = \dfrac{\alpha}{m}\,t$ ...(i)

or $a \propto t$

i,e. a-t graph is a straight line passing through origin.

If $u = 0$, then integration of Eq. (i) gives,

$$v = \frac{\alpha t^2}{2m} \quad \text{or} \quad v \propto t^2$$

Hence in this situation (when $u = 0$) v-t graph is a parabola passing through origin.

8. a-s equation corresponding to given graph is,

$$a = 6 - \frac{s}{5}$$

∴ $$v \cdot \frac{dv}{ds} = \left(6 - \frac{s}{5}\right)$$

or $$\int_0^v v\,dv = \int_0^s \left(6 - \frac{s}{5}\right) ds$$

or $$v = \sqrt{125 - \frac{s^2}{5}}$$

At $s = 10$ m, $v = 10$ m/s **Ans.**

Maximum values of v is obtained when

$$\frac{dv}{ds} = 0 \quad \text{which gives} \quad s = 30 \text{ m}$$

$$\therefore \qquad v_{\max} = \sqrt{12 \times 30 - \frac{(30)^2}{5}}$$

$$= \sqrt{180} \text{ m/s}$$ **Ans.**

9. $\mathbf{v}_{av} = \frac{\mathbf{s}}{t}$ and $v_{\text{av}} = \frac{d}{t}$

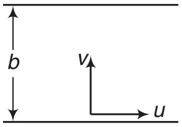
Now, $d \geq |\mathbf{s}|$

$\therefore$ $v_{\text{av}} \geq |\mathbf{v}_{\text{av}}|$

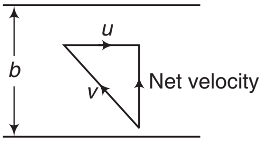
10. $v = at$ and $x = \frac{1}{2} at^2$ $(u = 0)$ i.e. v-t graph is a straight line passing through origin and x-t graph a parabola passing through origin.

11. For minimum time

$$\therefore \qquad t_{\min} = \frac{b}{v}$$



For reaching a point exactly opposite



Net velocity $= \sqrt{v^2 - u^2}$ (but $v > u$)

$$\therefore \qquad t = \frac{b}{\text{net velocity}}$$

12. For $t < T$, $v = -$ve

For $t > T$, $v = +$ve

At $t = T$, $v = 0$

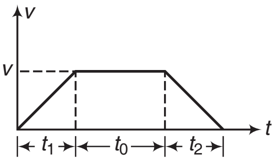
$\therefore$ Particle changes direction of velocity at $t = T$

S = Net area of v-t graph = 0

a = Slope of v-t graph = constant

13. $v = \alpha t_1 \Rightarrow t_1 = \frac{v}{\alpha}$

$v = \beta t_2 \Rightarrow t_2 = \frac{v}{\beta}$



$$\therefore \qquad t_0 = t - t_1 - t_2 = \left(t - \frac{v}{\alpha} - \frac{v}{\beta}\right)$$

Now, $$l = \frac{1}{2}\alpha t_1^2 + vt_0 + \frac{1}{2}\beta t_2^2$$

$$= \frac{1}{2}(\alpha)\left(\frac{v}{\alpha}\right)^2 + v\left(t - \frac{v}{\alpha} - \frac{v}{\beta}\right) + \frac{1}{2}(\beta)\left(\frac{v}{\beta}\right)^2$$

$$= vt - \frac{v^2}{2\alpha} - \frac{v^2}{2\beta}$$

$$\therefore \qquad t = \frac{l}{v} + \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

For t to be minimum its first derivation with respect to velocity be zero or,

$$0 = -\frac{l}{v^2} + \frac{\alpha + \beta}{2\alpha\beta}$$

$$\therefore \qquad v = \sqrt{\frac{2l\alpha\beta}{\alpha + \beta}}$$

14. $x = t^2 \Rightarrow v_x = \frac{dx}{dt} = 2t$

$\Rightarrow$ $a_x = \frac{dv_x}{dt} = 2$

$\Rightarrow$ $y = t^3 - 2t$

$\Rightarrow$ $v_y = \frac{dy}{dt} = 3t^2 - 2$

$\Rightarrow$ $a_y = \frac{dv_y}{dt} = 6t$

At $t = 0$, $v_x = 0$, $v_y = -2$, $a_x = 2$ and $a_y = 0$

$\therefore$ $\mathbf{v} = -2\hat{\mathbf{j}}$ and $\mathbf{a} = 2\hat{\mathbf{i}}$

or $\mathbf{v} \perp \mathbf{a}$

At $t = \sqrt{\frac{2}{3}}$, $v_y = 0$, $v_x \neq 0$.

Hence, the particle is moving parallel to x-axis.

15. $(14)^2 = (2)^2 + 2\,as$

$\therefore$ $2\,as = 192$ units

At mid point, $v^2 = (2)^2 + 2a\left(\frac{s}{2}\right)$

$$= 4 + \frac{192}{2} = 100$$

$\therefore$ $v = 10$ m/s **Ans.**

$XA : AY = 1 : 3$

$$\therefore \qquad XA = \frac{1}{4}s \quad \text{and} \quad AY = \frac{3}{4}s$$

$$v_1^2 = (2)^2 + 2a\left(\frac{s}{4}\right)$$

$$= 4 + \frac{192}{4} = 52$$

$\therefore \quad v_1 = \sqrt{52} \neq 5 \text{ m/s}$ **Ans.**

$10 = 2 + at_1 \qquad (v = u + at)$

$\therefore \quad t_1 = \frac{8}{a} \quad 14 = 10 + at_2$

$\therefore \quad t_2 = \frac{4}{a} \quad \text{or} \quad t_1 = 2\,t_2$ **Ans.**

$S_1 = (2t) + \frac{1}{2}\,a\,(t^2)$ = distance travelled in first half

$$S_2 = 2\,(2t) + \frac{1}{2}\,a\,(2t)^2$$

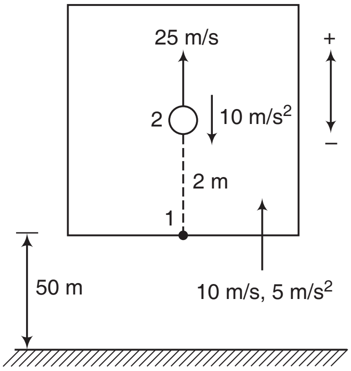
$S_3 = S_2 - S_1$ = distance travelled in second half

We can see that,

$$S_1 \neq \frac{S_3}{2}$$

Comprehension Based Questions

1. Velocity of ball with respect to elevator is 15 m/s (up) and elevator has a velocity of 10 m/s (up). Therefore, absolute velocity of ball is 25 m/s (upwards). Ball strikes the floor of elevator if,



$$S_1 = S_2 + 2$$

$\therefore \quad 10t + 2.5t^2 = 25t - 5t^2 + 2$

Solving this equation we get,

$t = 2.13$ s **Ans.**

2. If the ball does not collides, then it will reach its maximum height in time,

$$t_0 = \frac{u}{g} = \frac{25}{10} = 2.5 \text{ s}$$

Since, $t < t_0$, therefore as per the question ball is at its maximum height at 2.13 s.

$h_{\max} = 50 + 2 + 25 \times 2.13 - 5 \times (2.13)^2$

$= 82.56$ m **Ans.**

3. $S = 25 \times 2.13 - 5 \times (2.13)^2$

$= 30.56$ m **Ans.**

4. At maximum separation, their velocities are same

$\therefore \quad 25 - 10t = 10 + 5t$

or $\quad t = 1$ s

Maximum separation $= 2 + S_2 - S_1$

$= 2 + [25 \times 1 - 5 \times (1)^2] - [10 \times 1 + 2.5\,(1)^2]$

$= 9.5$ m **Ans.**

5. $u_A + a_A T = u_B - a_B T$

Putting $\quad T = 4$ s

we get $\quad 4\,(a_A + a_B) = u_B - u_A \qquad$ ....(i)

Now, $\quad S_A = S_B$

$\therefore \quad u_A t + \frac{1}{2}\,a_A t^2 = u_B\, t - \frac{1}{2}\,a_B\, t^2$

$\therefore \quad t = 2\,\frac{(u_B - u_A)}{(a_A + a_B)} = 2 \times 4 = 8$ s **Ans.**

6. $S_A = S_B$

$$5t + \frac{1}{2}\,a_A t^2 = 15t - \frac{1}{2}\,a_B t^2$$

or $\quad 10 + a_A\, t = 30 - a_B\, t$

$\therefore \quad (5 + a_A\, t) - (15 - a_B\, t) = 10$

or $\quad v_A - v_B = 10$ m/s **Ans.**

7. $8 = 6 + a_A T = 6 + 4a_A$

$\therefore \quad a_A = 0.5$ m/s

At 10 s,

$v_A = u_A + a_A\, t$

$= (6) + (0.5)\,(10)$

$= 11$ m/s **Ans.**

Match the Columns

1. In (a) and (b), if velocity is in the direction of acceleration (or of the same sign as that of acceleration) then speed increases. And if velocity is in opposite direction, then speed decreases.

   In (c) slope of *s*-*t* graph (velocity) is increasing. Therefore speed is increasing. In (d) slope of *s*-*t* graph is decreasing. Therefore, speed is decreasing.

2. If $\mathbf{v} \cdot \mathbf{a} = 0$, speed is constant because angle between **v** and **a** in this case is 90°.

   If $\mathbf{v} \cdot \mathbf{a}$ = positive then speed is increasing because angle between **v** and **a** in this case is acute.

   If $\mathbf{v} \cdot \mathbf{a}$ = negative then speed is increasing because angle between **v** and **a** in this case is obtuse.

3. In portion AB, we can see that velocity is positive and increasing. Similarly for other parts we can draw the conclusions.

4. (a) Average velocity $= \dfrac{S}{t} = \dfrac{\text{Area of } v\text{-}t \text{ graph}}{\text{Time}}$

$$= \frac{20}{4} = 5 \text{ m/s}$$

(b) Average acceleration $= \dfrac{v_f - v_i}{\text{time}}$

$$= \frac{v_{4\,\text{s}} - v_{1\,\text{s}}}{4-1} = \frac{0-5}{3} = -\frac{5}{3} \text{ m/s}^2$$

(c) Average speed $= \dfrac{d}{t} = \dfrac{|\text{Total area}|}{\text{Time}}$

$$= \frac{20+10}{6} = 5 \text{ m/s}$$

(d) Rate of change of speed at 4 s $= |\mathbf{a}|$

$$= |\text{slope of } v\text{-}t \text{ graph}| = 5 \text{ m/s}^2$$

5. (a) $x = 0$ at $t = 2\text{s}$

(b) $v = \dfrac{dx}{dt} = 10\,t$ and $a = \dfrac{dv}{dt} = 10$

$$v = a \text{ at 1s}$$

(c) $v = 10\,t$

$\therefore$ Velocity is positive all the time.

(d) $v = 0$ at $t = 0$ s

6. $v_x = \dfrac{dx}{dt} = 2t - 2$

$$a_x = \frac{dv_x}{dt} = 2$$

$$v_y = \frac{dy}{dt} = 2t - 4$$

$$a_y = \frac{dv_y}{dt} = 2$$

(a) It crosses y-axis when, $x = 0$

$\Rightarrow$ $t = 1$ s

At this instant v_y is -2 m/s.

(b) It crosses x-axis when, $y = 0$

$\Rightarrow$ $t = 2$ s

At this instant v_x is $+2$ m/s

(c) At $t = 0$, $v_x = -2$ m/s and $v_y = -4$ m/s

$\therefore$ Speed $= \sqrt{v_x^2 + v_y^2} = 2\sqrt{5}$ m/s

(d) At $t = 0$, $a_x = 2 \text{ m/s}^2$

and $a_y = 2 \text{ m/s}^2$

$$\therefore \quad a = \sqrt{a_x^2 + a_y^2}$$

$$= 2\sqrt{2} \text{ m/s}$$

Subjective Questions

1. $a_{\text{av}} = \dfrac{\Delta v}{\Delta t} = \dfrac{v_f - v_i}{\Delta t}$

$$= \frac{\sqrt{2g\,h_f} + \sqrt{2g\,h_i}}{\Delta t}$$

$$= \frac{\sqrt{2 \times 9.8 \times 2} + \sqrt{2 \times 9.8 \times 4}}{12 \times 10^{-3}}$$

$$= 1.26 \times 10^3 \text{ m/s}^2$$

Ans.

Note *v_f is upwards (+ve) and v_i is downwards (– ve).*

2. $v\,dv = a ds$

$$\therefore \quad \int_0^v v dv = \int_0^{12\text{ m}} a\,ds$$

$$\therefore \quad \frac{v^2}{2} = \text{area under } a\text{-}s \text{ graph from } s = 0 \text{ to } s = 12 \text{ m.}$$

$$= 2 + 12 + 6 + 4$$

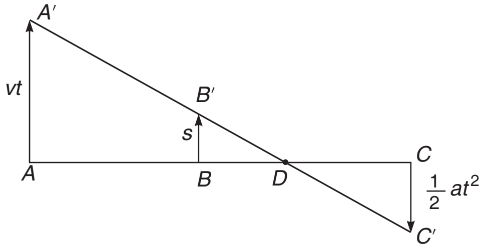
$$= 24 \text{ m}^2/\text{s}^2$$

or $v = \sqrt{48}$ m/s $= 4\sqrt{3}$ m/s **Ans.**

3. Let $AB = BC = d$

$BD = x$

and $BB' = s =$ displacement of point B.



From similar triangles we can write,

$$\frac{vt}{d+x} = \frac{s}{x} = \frac{\frac{1}{2}at^2}{d-x}$$

From first two equations we have,

$$1 + \frac{d}{x} = \frac{vt}{s}$$

or $\dfrac{d}{x} = \dfrac{vt}{s} - 1$ …(i)

From last two equations we have,

$$\frac{d}{x} - 1 = \frac{\frac{1}{2}at^2}{s}$$

or $\dfrac{d}{x} = \dfrac{\frac{1}{2}at^2}{s} + 1$ …(ii)

Equating Eqs. (i) and (ii) we have,

$$\frac{vt}{s} - 1 = \frac{\frac{1}{2}at^2}{s} + 1$$

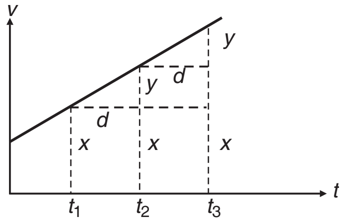
or $$\frac{vt - \frac{1}{2}at^2}{s} = 2$$

∴ $$s = \left(\frac{v}{2}\right)t - \frac{1}{2}\left(\frac{a}{2}\right)t^2$$

Comparing with $s = ut + \frac{1}{2}at^2$ we have,

Initial velocity of B is $+\frac{v}{2}$ and acceleration $-\frac{a}{2}$.

4. Let us draw v-t graph of the given situation, area of which will give the displacement and slope the acceleration.



$$s_2 - s_1 = x\,d + \frac{1}{2}yd \qquad \dots\text{(i)}$$

$$s_3 - s_2 = xd + yd + \frac{1}{2}yd \qquad \dots\text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we have

$$s_3 + s_1 - 2s_2 = yd$$

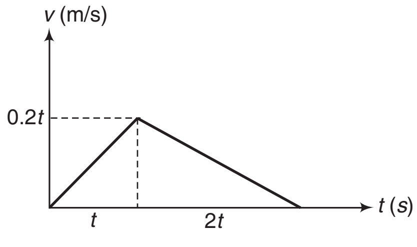
or $$s_3 + s_1 - 2\sqrt{s_1 s_3} = yd \qquad (s_2 = \sqrt{s_1 s_3})$$

Dividing by d^2 both sides we have,

$$\frac{(\sqrt{s_1} - \sqrt{s_3})^2}{d^2} = \frac{y}{d}$$

$= \text{slope of } v\text{-}t \text{ graph} = a.$ **Hence proved.**

5. Area of v-t graph = displacement



∴ $$\frac{1}{2}(3t)(0.2\,t) = 14$$ **Ans.**

Solving this equation, we get

$$3t = 20.5 \text{ s}$$

Note *Maximum speed 0.2 t is less than 2.5 m/s.*

6. Let t_0 be the breaking time and a the magnitude of deceleration.

80.5 km/h = 22.36 m/s, 48.3 km/h = 13.42 m/s.

In the first case,

$$56.7 = (22.36 \times t_0) + \frac{(22.36)^2}{2a} \qquad \dots\text{(i)}$$

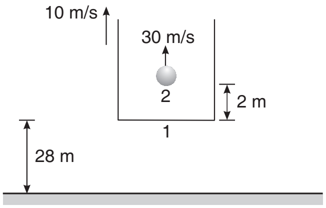
and $$24.4 = (13.42\,t_0) + \frac{(13.42)^2}{2a} \qquad \dots\text{(ii)}$$

Solving these two equations, we get

$$t_0 = 0.74 \text{ s}$$

and $$a = 6.2 \text{ m/s}^2$$ **Ans.**

7. Absolute velocity of ball = 30 m/s



(a) Maximum height of ball from ground

$$= 28 + 2 + \frac{(30)^2}{2 \times 9.8} = 76 \text{ m}$$

(b) Ball will return to the elevator floor when,

$s_1 = s_2 + 2$

or $$10t = (30t - 4.9\,t^2) + 2$$

Solving, we get $t = 4.2\,\text{s}$ **Ans.**

8. (a) Average velocity

$$= \frac{\text{Displacement}}{\text{Time}} = \frac{\int_0^5 v\,dt}{5}$$

$$= \frac{\int_0^5 (3t - t^2)\,dt}{5}$$

$$= -\,0.833 \text{ m/s}$$

(b) Velocity of particle = 0 at $t = 3$ s

i.e. at 3 s, particle changes its direction of motion.

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$= \frac{\text{(Distance from 0 to 3 s)} + \text{(Distance from 3 s to 5 s)}}{\text{Time}}$$

$$d_{0\text{-}3} = \int_0^3 (3t - t^2)\,dt = 4.5 \text{ m}$$

$$d_{3\text{-}5} = \int_3^5 (t^2 - 3t)\,dt = 8.67 \text{ m}$$

$$\therefore \quad \text{Average speed } = \frac{4.5 + 8.67}{5}$$

$$= 2.63 \text{ m/s}$$ **Ans.**

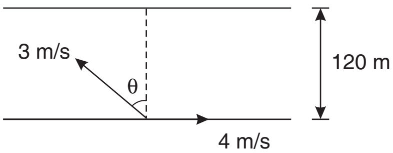
9. (a) $a = 2t - 2$ (from the graph)

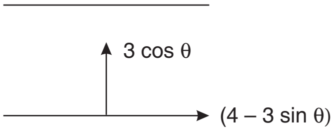
Now, $\int_0^v dv = \int_0^t a\,dt = \int_0^t (2t - 2)\,dt$

$\therefore \quad v = t^2 - 2t$

(b) $s = \int_2^4 v\,dt = \int_2^4 (t^2 - 2t)\,dt = 6.67$ m

10. (a)





Time to cross the river

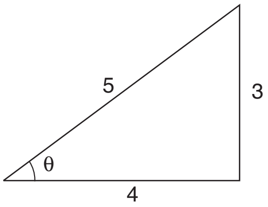
$$t_1 = \frac{120}{3\cos\theta} = \frac{40}{\cos\theta} = 40\sec\theta$$

Drift along the river $x = (4 - 3\sin\theta)\left(\dfrac{40}{\cos\theta}\right)$

$$= (160\sec\theta - 120\tan\theta)$$

To reach directly opposite, this drift will be covered by walking speed.

Time taken in this,



$$t_2 = \frac{160\sec\theta - 120\tan\theta}{1}$$

$$= 160\sec\theta - 120\tan\theta$$

$\therefore$ Total time taken

$$t = t_1 + t_2 = (200\sec\theta - 120\tan\theta)$$

For t to be minimum, $\dfrac{dt}{d\theta} = 0$

or $200\sec\theta\tan\theta - 120\sec^2\theta = 0$

or $\theta = \sin^{-1}(3/5)$

(b) $t_{\min} = 200\sec\theta - 120\tan\theta$ (where, $\sin\theta = \frac{3}{5}$)

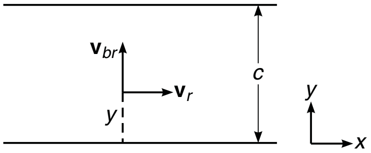
$$= 200 \times \frac{5}{4} - 120 \times \frac{3}{4}$$

$$= 250 - 90 = 160 \text{ s} = 2 \text{ min } 40 \text{ s}$$ **Ans.**

11. Given that $|\mathbf{v}_{br}| = v_y = \dfrac{dy}{dt} = u$ …(i)

$$|\mathbf{v}_r| = v_x = \frac{dx}{dt} = \left(\frac{2v_0}{c}\right) y \quad \text{…(ii)}$$

From Eqs. (i) and (ii) we have, $\dfrac{dy}{dx} = \dfrac{uc}{2v_0 y}$



or $\int_0^y y\,dy = \dfrac{uc}{2v_0}\int_0^x dx$ or $y^2 = \dfrac{ucx}{v_0}$ **Ans.**

At $y = \dfrac{c}{2}, x = \dfrac{cv_0}{4u}$

or $x_{\text{net}} = 2x = \dfrac{cv_0}{2u}$ **Ans.**

12. $a = v\dfrac{dv}{ds} = v$ (slope of v-s graph)

At $s = 50$ m

$v = 20$ m/s and $\dfrac{dv}{ds} = \dfrac{40}{100} = 0.4$ per sec

$\therefore \quad a = 20 \times 0.4 = 8 \text{ m/s}^2$

At $s = 150$ m

$v = (40 + 5) = 45$ m/s

and $\dfrac{dv}{ds} = \dfrac{10}{100} = 0.1$per sec

$\therefore \quad a = 45 \times 0.1 = 4.5 \text{ m/s}^2$

***a-s* graph**

From $s = 0$ to $s = 100$ m $v = 0.4\,s$

and $\dfrac{dv}{ds} = 0.4$

$\therefore \quad a = v.\dfrac{dv}{ds} = 0.16\,s$

i.e a - s graph is a straight line passing through origin of slope 0.16 per (sec)^2.

At $s = 100$ m, $a = 0.16 \times 100 = 16$ m/s^2

From $s = 100$ m to $s = 200$ m

$$v = 0.1s + 30$$

$$\frac{dv}{ds} = 0.1$$

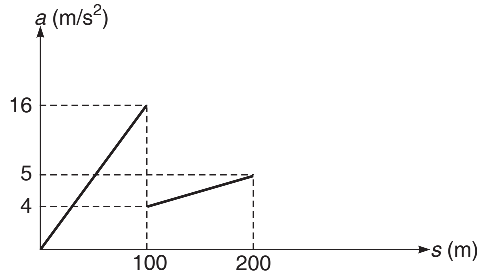
$$\therefore \quad a = v\frac{dv}{ds} = (0.1s + 30)(0.1) = (0.01s + 3)$$

i.e. a-s graph is straight line of slope 0.01 (sec)$^{-2}$ and intercept 3 m/s^2.

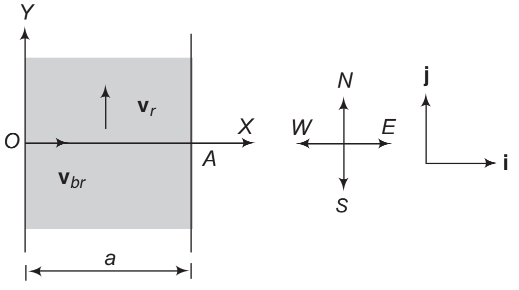
At $s = 100$ m, $\quad a = 4$ m/s^2

and at $s = 200$ m, $a = 5$ m/s^2

Corresponding a-s graph is as shown in figure.



13. (a) Let $\mathbf{v}_{br}$ be the velocity of boatman relative to river, $\mathbf{v}_r$ the velocity of river and $\mathbf{v}_b$ is the absolute velocity of boatman. Then,



$$\mathbf{v}_b = \mathbf{v}_{br} + \mathbf{v}_r$$

Given, $\quad |\mathbf{v}_{br}| = v$ and $\quad |\mathbf{v}_r| = u$

Now $\quad u = v_y = \frac{dy}{dt} = x(a - x)\frac{v}{a^2} \qquad \ldots\text{(i)}$

and $\quad v = v_x = \frac{dx}{dt} = v \qquad \ldots\text{(ii)}$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{dy}{dx} = \frac{x(a-x)}{a^2} \quad \text{or} \quad dy = \frac{x(a-x)}{a^2}\,dx$$

or $\quad \int_0^y dy = \int_0^x \frac{x(a-x)}{a^2}\,dx$

or $\quad y = \frac{x^2}{2a} - \frac{x^3}{3a^2} \qquad \ldots\text{(iii)}$

This is the desired equation of trajectory.

(b) Time taken to cross the river is

$$t = \frac{a}{v_x} = \frac{a}{v}$$

(c) When the boatman reaches the opposite side, $x = a$ or $v_y = 0$ [from Eq. (i)]

Hence, resultant velocity of boatman is v along positive x-axis or due east.

(d) From Eq. (iii)

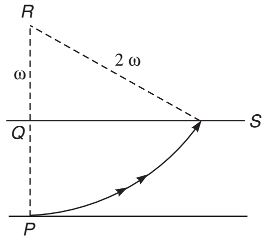
$$y = \frac{a^2}{2a} - \frac{a^3}{3a^2} = \frac{a}{6}$$

At $x = a$ (at opposite bank)

Hence, displacement of boatman will be

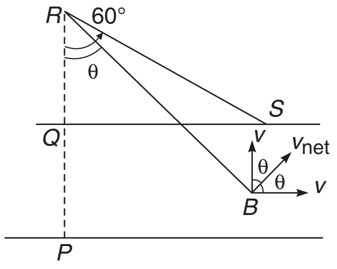
$$\mathbf{s} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \quad \text{or} \quad \mathbf{s} = a\hat{\mathbf{i}} + \frac{a}{6}\hat{\mathbf{j}}$$

14. (a) Since, the resultant velocity is always perpendicular to the line joining boat and R, the boat is moving in a circle of radius 2ω and centre at R.



(b) Drifting $= QS = \sqrt{4\omega^2 - \omega^2} = \sqrt{3}\omega$.

(c) Suppose at any arbitrary time, the boat is at point B.



$$V_{\text{net}} = 2v\cos\theta$$

$$\frac{d\theta}{dt} = \frac{V_{\text{net}}}{2\omega} = \frac{v\cos\theta}{\omega}$$

or $\frac{\omega}{v}\sec\theta\, d\theta = dt$

∴ $\int_0^t dt = \frac{\omega}{v}\int_0^{60°}\sec\theta\, d\theta$

∴ $t = \frac{\omega}{v}\left[\ln(\sec\theta + \tan\theta)\right]_0^{60°}$

or $t = \frac{1.317\omega}{v}$ **Ans.**

15. For $0 < s \le 60$ m

$$v = \frac{12}{60}s + 3 = 3 + \frac{s}{5}$$

$$\frac{dv}{dt} = \left(\frac{1}{5}\right)\cdot\frac{ds}{dt}$$

$$= \frac{1}{5}(v) = \frac{1}{5}\left(3 + \frac{s}{5}\right) = \frac{3}{5} + \frac{s}{25} \quad \text{...(i)}$$

or $a = \frac{3}{5} + \frac{s}{25}$

i.e. *a-s* graph is a straight line.

At $s = 0$, $a = \frac{3}{5}\ \text{m/s}^2 = 0.6\ \text{m/s}^2$

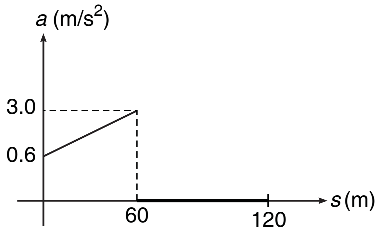
and at $s = 60$ m, $a = 3.0\ \text{m/s}^2$

For $s > 60$ m

$v =$ constant

∴ $a = 0$

Therefore, the corresponding *a-s* graph is shown in figure.



From Eq. (i), $\frac{dv}{dt} = \frac{v}{5}$

or $\int_3^v \frac{dv}{v} = \frac{1}{5}\int_0^t dt$

$$\ln\left(\frac{v}{3}\right) = \frac{t}{5}$$

∴ $v = 3e^{t/5}$ or $\int_0^{60} ds = 3\int_0^{t_1} e^{t/5}\, dt$

$60 = 15\left(e^{t_1/5} - 1\right)$ or $t_1 = 8.0$ s

Time taken to travel next 60 m with speed 15m/s will be $\frac{60}{15} = 4$ s

∴ Total time = 12.0 s **Ans.**

16. From the graph,

$$a = 22.5 - \frac{22.5}{150}\cdot s \quad \text{or}$$

$$\int_0^v v\cdot dv = \int_0^{60}\left(22.5 - \frac{22.5}{150}\times s\right) ds$$

∴ $\frac{v^2}{2} = 22.5\times 60 - \frac{22.5}{150}\times\frac{(60)^2}{2}$

∴ $v = 46.47$ m/s **Ans.**

17. (a) $u_x = 3\,\text{m/s}$

$a_x = -1.0\ \text{m/s}^2$

Maximum *x*-coordinate is attained after time

$$t = \left|\frac{u_x}{a_x}\right| = 3\ \text{s}$$

At this instant $v_x = 0$ and

$v_y = u_y + a_y t = 0 - 0.5\times 3 = -1.5$ m/s

∴ $\mathbf{v} = (-1.5\,\hat{\mathbf{j}})$ m/s **Ans.**

(b) $x = u_x t + \frac{1}{2}a_x t^2$

$= 3\times 3 + \frac{1}{2}(-1.0)(3)^2 = 4.5$ m

$y = u_y t + \frac{1}{2}a_y t^2 = 0 - \frac{1}{2}(0.5)(3)^2 = -2.25$ m

∴ $\mathbf{r} = (4.5\,\hat{\mathbf{i}} - 2.25\,\hat{\mathbf{j}})$ m **Ans.**

18. (b) $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$ and $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$`

$$\mathbf{v}\cdot\mathbf{a} = v_x a_x + v_y a_y$$

Further $v = \sqrt{v_x^2 + v_y^2}$

∴ $\frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{v_x a_x + v_y a_y}{\sqrt{v_x^2 + v_y^2}}$

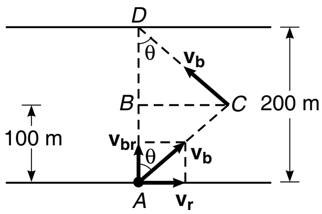
$$= \frac{dv}{dt} = a_t$$

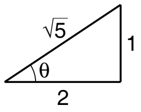
or component of **a** parallel to **v**

= tangential acceleration.

19. (a) $v_{br} = 4$ m/s, $v_r = 2$ m/s

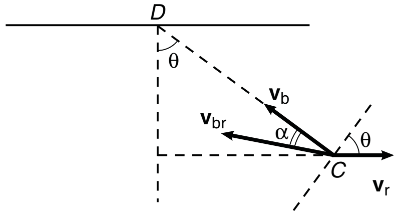
$$\tan\theta = \frac{BC}{AB} = \frac{|\mathbf{v}_r|}{|\mathbf{v}_{br}|} = \frac{2}{4} = \frac{1}{2}$$





In this case, $\mathbf{v}_b$ should be along *CD*.

$$\therefore \qquad v_r \cos\theta = v_{br} \sin\alpha$$



$$2\left(\frac{2}{\sqrt{5}}\right) = 4\sin\alpha$$

or $$\sin\alpha = \frac{1}{\sqrt{5}}$$

$$\therefore \qquad \alpha = \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

(b) $$t_1 = \frac{200}{|\mathbf{v}_{br}|} = \frac{200}{4} = 50 \text{ s}$$

$$DC = DB\sec\theta = (100)\frac{\sqrt{5}}{2} = 50\sqrt{5} \text{ m}$$

$$|\mathbf{v}_b| = |\mathbf{v}_{br}|\cos\alpha - |\mathbf{v}_r|\sin\theta$$

$$= 4\left(\frac{2}{\sqrt{5}}\right) - 2\left(\frac{1}{\sqrt{5}}\right) = \frac{6}{\sqrt{5}} \text{ m/s}$$

$$\therefore \qquad t_2 = \frac{t_1}{2} + \frac{DC}{|\mathbf{v}_b|} = 25 + \frac{50\sqrt{5}}{\frac{6}{\sqrt{5}}} = \frac{200}{3} \text{ s}$$

or $$\frac{t_2}{t_1} = \frac{4}{3}$$ **Ans.**

20. $\mathbf{v}_b$ = velocity of boatman = $\mathbf{v}_{br} + \mathbf{v}_r$

and $\mathbf{v}_c$ = velocity of child = $\mathbf{v}_r$

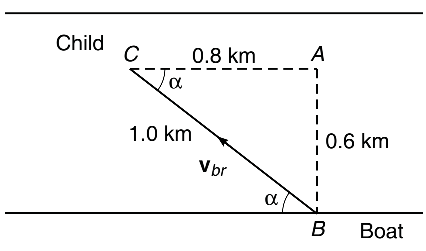
∴ $\mathbf{v}_{bc} = \mathbf{v}_b - \mathbf{v}_c = \mathbf{v}_{br}$

$\mathbf{v}_{bc}$ should be along BC.

i.e. $\mathbf{v}_{br}$ should be along BC,

where, $$\tan\alpha = \frac{0.6}{0.8} = \frac{3}{4}$$

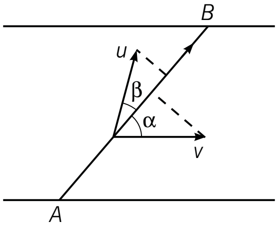
or $$\alpha = 37°$$ **Ans.**

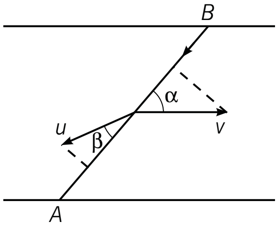


Further $$t = \frac{BC}{|\mathbf{v}_{br}|} = \frac{1}{20} \text{ h}$$

$$= 3 \text{ min}$$ **Ans.**

21. In order that the moving launch is always on the straight line AB, the components of velocity of the current and of the launch in the direction perpendicular to AB should be equal, i.e.





$$u\sin\beta = v\sin\alpha \qquad \text{...(i)}$$

$$S = AB = (u\cos\beta + v\cos\alpha)t_1 \qquad \text{...(ii)}$$

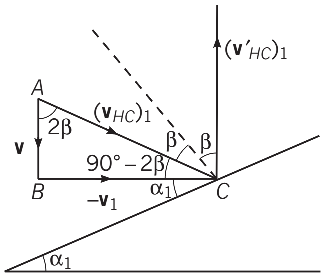
Further $$BA = (u\cos\beta - v\cos\alpha)t_2 \qquad \text{...(iii)}$$

$$t_1 + t_2 = t \qquad \text{...(iv)}$$

Solving these equations after proper substitution, we get

$$u = 8 \text{ m/s} \quad \text{and} \quad \beta = 12°$$ **Ans.**

22. Here, absolute velocity of hail stones $\mathbf{v}$ before colliding with wind screens is vertically downwards and velocity of hail stones with respect to cars after collision $\mathbf{v}'_{HC}$ is vertically upwards. Collision is elastic, hence, velocity of hail stones with respect to cars before collision $\mathbf{v}_{HC}$ and after collision $\mathbf{v}'_{HC}$ will make equal angles with the normal to the wind screen.



$(\mathbf{v}_{HC})_1$ = velocity of hail stones – velocity of car 1

$= \mathbf{v} - \mathbf{v}_1$

From the figure, we can see that

$$\beta + 90° - 2\beta + \alpha_1 = 90°$$

or $$\alpha_1 = \beta \quad \text{or} \quad 2\beta = 2\alpha_1$$

In ΔABC, $\tan 2\beta = \tan 2\alpha_1 = \dfrac{v_1}{v}$ …(i)

Similarly, we can show that

$$\tan 2\alpha_2 = \frac{v_2}{v} \quad \text{…(ii)}$$

From Eqs. (i) and (ii), we get

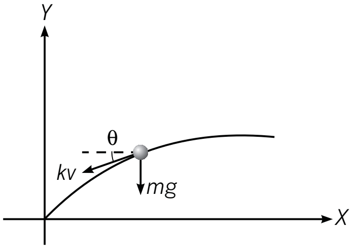
$$\frac{v_1}{v_2} = \frac{\tan 2\alpha_1}{\tan 2\alpha_2} = \frac{\tan 60°}{\tan 30°} = \frac{\sqrt{3}}{1/\sqrt{3}} = 3$$

$$\therefore \quad \frac{v_1}{v_2} = 3$$ **Ans.**

23. $a_x = \dfrac{dv_x}{dt} = -\dfrac{kv\cos\theta}{m} = -\dfrac{k}{m}v_x$

$$\therefore \quad \frac{dv_x}{v_x} = -\frac{k}{m}dt \text{ or } \int_{v_0\cos\theta_0}^{v_x}\frac{dv_x}{v_x} = -\frac{k}{m}\int_0^t dt$$

or $$v_x = v_0\cos\theta_0\, e^{-\frac{k}{m}t} \quad \text{…(i)}$$



Similarly,

$$a_y = \frac{dv_y}{dt} = -\frac{kv\sin\theta}{m} - g = -\left(\frac{k}{m}v_y + g\right)$$

or $$\int_{v_0\sin\theta_0}^{v_y}\frac{dv_y}{\frac{k}{m}v_y + g} = -\int_0^t dt$$

or $$\frac{m}{k}\left[\ln\left(\frac{k}{m}v_y + g\right)\right]_{v_0\sin\theta_0}^{v_y} = -t$$

or $$\frac{\left(\frac{k}{m}v_y + g\right)}{\left(\frac{k}{m}v_0\sin\theta_0 + g\right)} = e^{-\frac{k}{m}t}$$

or $$v_y = \frac{m}{k}\left[\left(\frac{k}{m}v_0\sin\theta_0 + g\right)e^{-\frac{k}{m}t} - g\right] \quad \text{…(ii)}$$

(b) Eq. (i) can be written as

$$\frac{dx}{dt} = v_0\cos\theta_0\, e^{-\frac{k}{m}t}$$

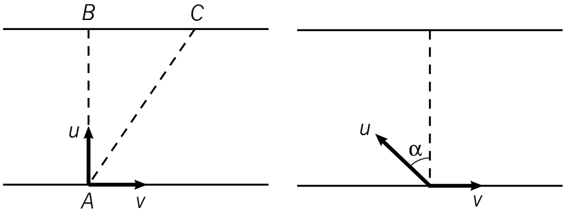
or $$\int_0^x dx = v_0\cos\theta_0\int_0^t e^{-\frac{k}{m}t}\,dt$$

or $$x = \frac{mv_0\cos\theta_0}{k}\left[1 - e^{-\frac{k}{m}t}\right]$$

$$x_m = \frac{mv_0\cos\theta_0}{k}$$

at $t = \infty$. **Ans.**

24. In the first case, $BC = vt_1$ and $w = ut_1$



In the second case,

$u\sin\alpha = v$ and $w = (u\cos\alpha)\,t_2$

Solving these four equations with proper substitution, we get

$w = 200$ m,

$u = 20$ m/min,

$v = 12$ m/min

and $\alpha = 36°50'$ **Ans.**

7. Projectile Motion

INTRODUCTORY EXERCISE 7.1

1. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$

$$\Rightarrow \quad (\mathbf{u}_1 + \mathbf{a}_1 t)\cdot(\mathbf{u}_2 + \mathbf{a}_2 t) = 0$$

$$\Rightarrow \quad (10\hat{\mathbf{i}} - 10t\hat{\mathbf{j}})\cdot(-20\hat{\mathbf{i}} - 10t\hat{\mathbf{j}}) = 0$$

$$\Rightarrow \quad -200 + 100t^2 = 0$$

$$\Rightarrow \quad t = \sqrt{2} \text{ sec}$$

2. It is two dimensional motion.

3. The uniform acceleration is **g**.

4. $\mathbf{u} = (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}})$ m/s, $\mathbf{a} = (-10\hat{\mathbf{j}})$ m/s^2, $t = 2$ sec

Now, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

5. $u_x = u_y = 20$ m/s, $a_y = -10$ m/s^2

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow \quad -25 = 20t - \frac{1}{2} \times 10 \times t^2$$

Solving this equation, we get the positive value of,

$$t = 5 \text{ sec}$$

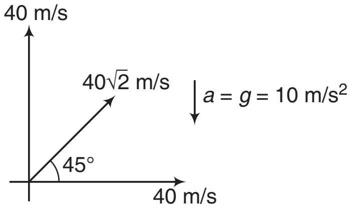
Now apply, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and $s_x = u_x t$

INTRODUCTORY EXERCISE 7.2

1. $\mathbf{u} = 40\hat{\mathbf{i}} + 40\hat{\mathbf{j}}$

$$\mathbf{a} = -10\hat{\mathbf{j}}$$

$$t = 2 \text{ s}$$



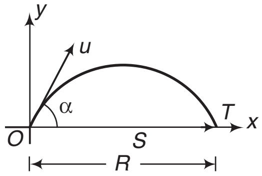
(a) Apply $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ as $\mathbf{a}$ = constant

(b) Apply $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

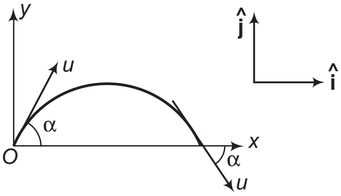
3. Average velocity

$$= \frac{s}{t} = \frac{R}{T} = \frac{(u_x T)}{T} = u_x$$

$$= u \cos \alpha$$



4. $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



$$= (u \cos \alpha \hat{\mathbf{i}} - u \sin \alpha \hat{\mathbf{j}}) - (u \cos \alpha \hat{\mathbf{i}} + u \sin \alpha \hat{\mathbf{j}})$$

$$= (-2u \sin \alpha)\hat{\mathbf{j}}$$

Therefore, change in velocity is $2u \sin \alpha$ in downward direction.

5. (a) $R = \dfrac{u^2 \sin 2\theta}{g}, \quad H = \dfrac{u^2 \sin^2\theta}{2g}$

and $\quad T = \dfrac{2u \sin\theta}{g}$

Here, $\quad u = 20\sqrt{2}$ m/s and $\theta = 45°$

(b) $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

where, $\quad \mathbf{u} = (20\hat{\mathbf{i}} + 20\hat{\mathbf{j}})$ m/s

and $\quad \mathbf{a} = (-10\hat{\mathbf{j}})$ m/s^2

(c) Horizontal component remains unchanged ($= 20\hat{\mathbf{i}}$) and vertical component is reversed in direction ($= -20\hat{\mathbf{j}}$)

$$\mathbf{v} = (20\hat{\mathbf{i}} - 20\hat{\mathbf{j}}) \text{ m/s}$$ **Ans.**

6. (a) Since, acceleration is constant. Therefore,

$$\mathbf{a}_{av} = \mathbf{a} = \mathbf{g} = (-10\hat{\mathbf{j}}) = \frac{\Delta\mathbf{v}}{\Delta t}$$

$$\therefore \quad \Delta\mathbf{v} = (-10\hat{\mathbf{j}})(\Delta t) = (-10\hat{\mathbf{j}})(3)$$

$$= (-10\hat{\mathbf{j}})(3) = (-30\hat{\mathbf{j}}) \text{ m/s}$$

∴ Change in velocity is 30 m/s, vertically downwards. **Ans.**

(b) $\mathbf{v}_{av} = \dfrac{\mathbf{s}}{t} = \dfrac{\mathbf{u}t + \frac{1}{2}\mathbf{a}t^2}{t} = \mathbf{u} + \dfrac{1}{2}\mathbf{a}t$

$$= (20\hat{\mathbf{i}} + 20\hat{\mathbf{j}}) + \frac{1}{2}(-10\hat{\mathbf{j}})(3)$$

$$= (20\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \text{ m/s}$$

$$\therefore \quad |\mathbf{v}_{av}| = \sqrt{(20)^2 + (5)^2}$$

$$= 20.62 \text{ m/s} \qquad \textbf{Ans.}$$

7. $T = \dfrac{2u \sin \theta}{g} = \dfrac{2 \times 20 \times \frac{1}{\sqrt{2}}}{10} = 2\sqrt{2}$ s

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20)^2 \sin 90°}{10}$$

$$= 40 \text{ m}$$

Now the remaining horizontal distance is $(50 - 40)$ m $= 10$ m. Let v is the speed of player, then

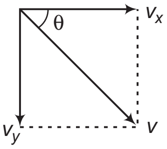
$$vT = 10 \quad \text{or} \quad v = \frac{10}{T} = \frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ m/s} \qquad \textbf{Ans.}$$

8. (a) Initial velocity is horizontal. So, in vertical direction it is a case of free fall.

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 100}{10}} = \sqrt{20} \text{ sec}$$

(b) $x = u_x t = 20\sqrt{20}$ m

(c) $v_x = u_x = 20$ m/s



$$v_y = u_y + a_y t$$

$$= 0 - 10\sqrt{20}$$

$$= -10\sqrt{20} \text{ m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = 49 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}(\sqrt{5})$$

9. $R = \dfrac{u^2 \sin 60°}{g} = 3$ km

$$\Rightarrow \qquad \frac{u^2}{g} = R_{\max} = \frac{3}{\sin 60°}$$

$$= 2\sqrt{3} \text{ km}$$

Since, $\quad R_{\max} < 5$ km

So, it can't hit the target at 5 km.

10. Comparing with,

$$y = x \tan \theta - \frac{gx^2}{2u^2}(1 + \tan^2 \theta)$$

$$\tan \theta = b, g = a$$

and $\qquad \dfrac{g}{2u^2}(1 + \tan^2 \theta) = c$

$$\therefore \qquad \frac{a(1 + b^2)}{2u^2} = c$$

$$\therefore \qquad u = \sqrt{\frac{a(1 + b^2)}{2c}}$$

INTRODUCTORY EXERCISE 7.3

1. $T = \dfrac{2u \sin(\alpha - \beta)}{g \cos \beta}$

$$= \frac{(2)(20\sqrt{2}) \sin(45° - 30°)}{(10)(\cos 30°)} = 1.69 \text{ s}$$

$$R = \frac{u^2}{g \cos^2 \beta}[\sin(2\alpha - \beta) - \sin \beta]$$

$$= \frac{(20\sqrt{2})^2}{(10)\cos^2 30°}[\sin(2 \times 45° - 30°) - \sin 30°]$$

$$= 39 \text{ m}$$

2. $T = \dfrac{2u \sin(\alpha + \beta)}{g \cos \beta}$

$$= \frac{2 \times 20\sqrt{2} \sin(45° + 30°)}{(10)\cos 30°} \approx 6.31 \text{ s}$$

$$R = \frac{u^2}{g \cos^2 \beta}[\sin(2\alpha + \beta) + \sin \beta]$$

$$= \frac{(20\sqrt{2})^2}{(10)\cos^2 30°}[\sin(2 \times 45° + 30°) + \sin 30°]$$

$$= 145.71 \text{ m}$$

3. Using the above equations with $\alpha = 0°$

$$T = \frac{2(20)\sin 30°}{(10)\cos 30°} = 2.31 \text{ s}$$

$$R = \frac{(20)^2}{(10)\cos^2 30°}[\sin 30° + \sin 30°]$$

$$= 53.33 \text{ m}$$

4. (a) Horizontal component of velocities of passenger and stone are same. Therefore relative velocity in horizontal direction is zero. Hence the relative motion is only in vertical direction.

(b) With respect to the man, stone has both velocity components, horizontal and vertical. Therefore path of the stone is a projectile.

5. (a) $\qquad a_1 = a_2 = g \qquad$ (downwards)

$\therefore \qquad$ Relative acceleration $= 0$

(b) $\mathbf{u}_{12} = \mathbf{u}_1 - \mathbf{u}_2$

$= (20\hat{\mathbf{j}}) - (20\hat{\mathbf{i}} + 20\hat{\mathbf{j}})$

$= (-20\hat{\mathbf{i}})$ m/s

or 20 m/s in horizontal direction

(c) $\mathbf{u}_{12} = (-20\hat{\mathbf{i}})$ m/s is constant.

Therefore relative motion is uniform.

$\therefore \quad d = |\mathbf{u}_{12}|\, t$

$= 20 \times 2 = 40$ m

6. The range is maximum at,

$$\alpha = \frac{\pi}{4} + \frac{\beta}{2} \quad \text{(given in the theory)}$$

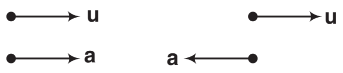
$$= 45° + \frac{30°}{2} = 60° \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

1. In the cases shown below path is straight line, even if **a** is constant

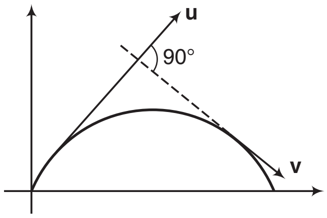


3. $T = \dfrac{2u\sin\theta}{g} \Rightarrow u\sin\theta = \dfrac{gT}{2}$

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{(gT/2)^2}{2g}$$

or $\quad H \propto T^2$

4.



5. In projectile motion along an inclined plane, normally we take x and y-axis along the plane and perpendicular to it. In that case, a_x and a_y both are non-zero.

7. $T = \dfrac{2u\sin\theta}{g} = 4$

$\therefore \quad (u\sin\theta) = 20$ m/s

$\therefore \quad H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{(20)^2}{20} = 20$ m $\qquad$ **Ans.**

8. $H = \dfrac{u^2\sin^2\theta}{2g}$

or $\quad u\sin\theta \propto \sqrt{H}$

$$\frac{(u\sin\theta)_1}{(u\sin\theta)_2} = \sqrt{\frac{4H}{H}} = 2$$

$$T = \frac{2u\sin\theta}{g} \Rightarrow T \propto u\sin\theta$$

$$\frac{T_1}{T_2} = \frac{(u\sin\theta)_1}{(u\sin\theta)_2} = \frac{2}{1}$$

$$R = u_x T$$

$\therefore \quad \dfrac{R_1}{R_2} = \dfrac{(u_x)_1}{(u_x)_2} \cdot \dfrac{T_1}{T_2} = \left(\dfrac{1}{2}\right)\left(\dfrac{2}{1}\right) = 1$

$\therefore \quad R_1 = R_2$

9. $\left|\dfrac{d\mathbf{v}}{dt}\right| = 10$ m/s^2

But $\quad \dfrac{d|\mathbf{v}|}{dt} \neq 10$ m/s^2

10. $v_x = u_x$

$$v_y = \pm\sqrt{u_y^2 - 2gh}$$

and $\quad v = \sqrt{v_x^2 + v_y^2}$

and $\quad u^2 = u_x^2 + u_y^2$

Single Correct Option

1. $\mathbf{a} = \mathbf{g}$ = constant for small heights.

2. $H_\theta = \dfrac{u^2\sin^2\theta}{2g}$

$$H_{90-\theta} = \frac{u^2\sin^2(90-\theta)}{2g} = \frac{u^2\cos^2\theta}{2g}$$

$\therefore \quad \dfrac{H_\theta}{H_{90-\theta}} = \dfrac{\sin^2\theta}{\cos^2\theta}$

3. $R_\theta = \dfrac{R_{45°}}{2}$

$\therefore \quad \dfrac{u^2\sin 2\theta}{g} = \dfrac{1}{2}\left(\dfrac{u^2}{g}\right)$ or $\sin 2\theta = \dfrac{1}{2}$

$\therefore \quad 2\theta = 30°$ or $\theta = 15°$

4. At 45°, range is maximum. At highest point, it has only horizontal component of velocity or 20 cos 45° ≈ 14 m/s.

5. $\mathbf{F} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{F} \perp \mathbf{v}$

Hence path is parabola.

6. $v_x = u_x$

$$\therefore \quad v \cos 30° = u \cos 60°$$

$$\text{or} \quad v = \frac{u}{\sqrt{3}}$$

Velocity has become $\frac{1}{\sqrt{3}}$ times. Therefore, kinetic energy will become 1/3 times.

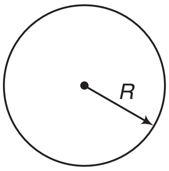
7. $T_1 = \dfrac{2u \sin\theta}{g}, \quad T_2 = \dfrac{2u \cos\theta}{g}$

$$R = \frac{2\,(u \sin\theta)\,(u \cos\theta)}{g}$$

$$= \frac{2\left(\frac{gT_1}{2}\right)\left(\frac{gT_2}{2}\right)}{g}$$

$$= \frac{1}{2}\, gT_1T_2 \qquad \textbf{Ans.}$$

8. $R_{\max} = \dfrac{v_0^2}{g}$ at $\theta = 45°$



$$\therefore \quad A_{\max} = \pi R_{\max}^2$$

9. Maximum range is obtained at 45°

$$\frac{u^2}{g} = 1.6 \quad \text{or} \quad u = 4 \text{ m/s}$$

$$T = \frac{2u \sin 45°}{g} = \frac{2 \times 4 \times (1/\sqrt{2})}{10}$$

$$= 0.4\sqrt{2} \text{ s}$$

Number of jumps in given time,

$$n = \frac{t}{T} = \frac{10\sqrt{2}}{0.4\sqrt{2}} = 25$$

∴ Total distance travelled = 1.6 × 25 = 40 m **Ans.**

10. $H_1 = \dfrac{u^2 \sin^2\theta}{2g}$

$$\therefore \quad 102 = \frac{(u^2) \sin^2 60°}{20}$$

$$\therefore \quad u = 52.2 \text{ m/s}$$

Other stone should be projected at 90° − θ or 30° from horizontal.

$$\therefore \quad H_2 = \frac{u^2 \sin^2 30°}{2g}$$

$$= \frac{(52.2)^2\,(1/4)}{20}$$

$$= 34 \text{ m} \qquad \textbf{Ans.}$$

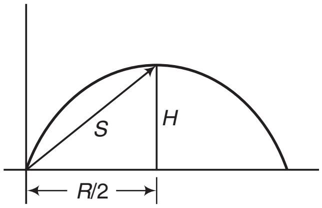
11. Using $s = ut + \frac{1}{2}at^2$ in vertical direction

$$\therefore \quad -70 = (50 \sin 30°)\, t + \frac{1}{2}(-10)\, t^2$$

On solving this equation, we get

$$t = 7 \text{ s} \qquad \textbf{Ans.}$$

12. $S = \sqrt{H^2 + R^2/4}$



$$\text{Average velocity} = \frac{S}{t} = \frac{S}{T/2} = \frac{2S}{T}$$

13. Velocity of train in the direction of train is also 30 m/s. So there is no relative motion in this direction. In perpendicular direction,

$$d = R = \frac{u^2 \sin 2\theta}{g}$$

$$= \frac{(30)^2 \sin 90°}{10} = 90 \text{ m} \qquad \textbf{Ans.}$$

14. For maximum value of y

$$\frac{dy}{dt} = 10 - 2t = 0$$

$$\Rightarrow \quad t = 5 \text{ s}$$

$$y_{\max} = (10)(5) - (5)^2$$

$$= 25 \text{ m} \qquad \textbf{Ans.}$$

15. $R = \dfrac{u^2}{g \cos^2 \beta}[\sin(2\alpha + \beta) + \sin\beta]$

$u = 50$ m/s, $g = 10$ m/s^2, $\alpha = 0°$,

$$R = \frac{(50)^2}{10 \cos^2 30°}[\sin(2 \times 0 + 30°) + \sin 30°]$$

$$= \frac{(2500)}{10 \times (3/4)}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= \frac{1000}{3} \text{ m} \qquad \textbf{Ans.}$$

16. $T = \dfrac{2u\sin\theta}{g} = \dfrac{(2)(8)\sin 53°}{10} = 1.28\ \text{s}$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(80)^2 \sin(106°)}{10} = 615.2\ \text{m}$$

Distance travelled by tank,

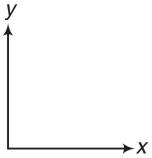
$$d = (5)\,T = (5)(1.28) = 6.4\ \text{m}$$

$\therefore$ Total distance $= (615.2 + 6.4)$ m

$= 621.6$ m **Ans.**

Subjective Questions

1. **At 45°**



$v_y = \pm v_x = \pm u_x = \pm 60\cos 60°$

$= \pm 30$ m/s

Now, $v_y = u_y + a_y t$

$\therefore$ $t = \dfrac{v_y - v_y}{a_y}$

$= \dfrac{(\pm 30) - 60\sin 60°}{-10}$

$\therefore$ $t_1 = 2.19$ s and $t_2 = 8.20$ s **Ans.**

2. Vertical component of initial velocity

$$u = 20\sqrt{2}\sin 45° = 20\ \text{m/s}$$

Now, apply $s = ut + \dfrac{1}{2}at^2$ (to find t) in vertical direction, with

$s = 15\,\text{m}$, $u = 20$ m/s and $a = -10\ \text{m/s}^2$ **Ans.**

3. $v_x = u_x = 20\cos 60° = 10$ m/s

Given, $v = \dfrac{u}{2}$

$\therefore$ $4v^2 = u^2$

or $4(v_x^2 + v_y^2) = u^2$

$\therefore$ $4[(10)^2 + v_y^2] = (20)^2$

or $v_y = 0$

Hence, it is the highest point

$\therefore$ $t = \dfrac{T}{2} = \dfrac{u\sin\theta}{g}$

$= \dfrac{(20)\sin 60°}{10}$

$= \sqrt{3}$ s **Ans.**

4. For collision to take place, relative velocity of A with respect to B should be along AB or their vertical components should be same.

Vertical components of A

$$v_1 = 10\sin 30° = 5\ \text{m/s}$$

and vertical component of B

$$v_2 = 5\sqrt{2}\cos 45° = 5\ \text{m/s}$$

Since, $v_1 = v_2$, so they may collide

Now the second condition is,

$R_1 + R_2 \ge d$ $(d = 15\ \text{m})$

$\therefore$ $R_1 = \dfrac{(10)^2\sin 60°}{10} = 8.66$ m

$R_2 = \dfrac{(5\sqrt{2})^2\sin 90°}{10} = 5$ m

Since, $R_1 + R_2 < d$, so they will not collide.

5. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ when $\mathbf{v}_1 \perp \mathbf{v}_2$

$\therefore$ $(\mathbf{u}_1 + \mathbf{a}_1 t)\cdot(\mathbf{u}_2 + \mathbf{a}_2 t) = 0$

$\therefore$ $(3\hat{\mathbf{i}} - 10t\hat{\mathbf{j}})\cdot(4\hat{\mathbf{i}} - 10t\hat{\mathbf{j}}) = 0$

or $t = \sqrt{0.12}$ s

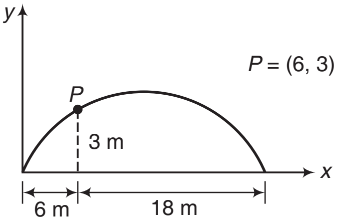
Now in vertical direction they have no relative motion and in horizontal direction their velocities are opposite.

$\therefore$ $d = 3t + 4t = 7t$

$= (7)(\sqrt{0.12})$ m

≈ 2.5 m **Ans.**

6. $y = x\left(1 - \dfrac{x}{R}\right)\tan\alpha$



$\therefore$ $\tan\alpha = \dfrac{yR}{x(R - x)}$

$= \dfrac{(3)(24)}{(6)(18)} = \dfrac{2}{3}$

$\therefore$ $\alpha = \tan^{-1}\left(\dfrac{2}{3}\right)$ **Ans.**

7. y-coordinate of particle is zero when,

$$4t - 5t^2 = 0$$

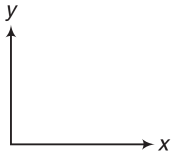
$\therefore$ $t = 0$ and 0.8 s

$x = 3t$

at $t = 0, x = 0$

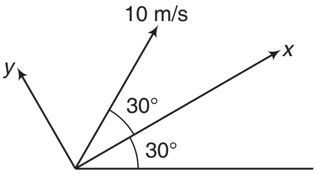
and at $t = 0.8$ s, $x = 2.4$ m **Ans.**

8.
$$T = \frac{2u \sin(\alpha - \beta)}{g \cos\beta} \qquad (\alpha = 60°, \beta = 30°)$$
$$= \frac{2 \times 10 \times \sin 30°}{(10) \cos 30°}$$
$$= \frac{2}{\sqrt{3}} \text{ s}$$

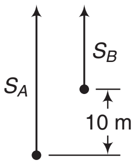


$$v_x = u_x = 10 \cos 60° = 5 \text{ m/s}$$
$$v_y = u_y + a_y t$$
$$= (10 \sin 60°) + (-10)(2/\sqrt{3})$$
$$= 5\sqrt{3} - \frac{20}{\sqrt{3}} = -\frac{5}{\sqrt{3}} \text{ m/s}$$
$$\therefore \quad = v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{25 + \frac{25}{3}}$$
$$= \frac{10}{\sqrt{3}} \text{ m/s}$$

9.
$$v_y = u_y + a_y t$$
$$= (10 \sin 30°) + (-g \cos 30°)\, T$$
$$= 5 - 10 \times \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}}$$
$$= -5 \text{m/s}$$



10. In vertical direction,



$$S_A = S_B + 10$$
$$\therefore \quad (10)\, t - \frac{1}{2} g t^2 = -\frac{1}{2} g t^2 + 10$$
$$\therefore \quad t = 1 \text{ s}$$

In horizontal direction,
$$d = |S_A| + |S_B|$$
$$= (10)\, t + 10t = 20t$$
$$= 20 \text{ m} \qquad (\text{as } t = 1 \text{ s})$$

11. Horizontal component of velocity always remains constant
$$\therefore \quad 40 \cos 60° = v \cos 30°$$
$$\text{or} \quad v = \frac{40}{\sqrt{3}} \text{ m/s}$$ **Ans.**

12. (a) If they collide in air then relative velocity of A with respect to B should be along AB or their vertical components should be same.
$$\therefore \quad 20 \sin\theta = 10$$
$$\text{or} \quad \theta = 30°$$ **Ans.**

(b)
$$x = |S_A| + |S_B|$$
$$= (20 \cos 30°)(t) + 0$$
$$= 20 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$$
$$= 5\sqrt{3} \text{ m}$$ **Ans.**

13.
$$v_x = u_x = 7.6 \text{ m/s}$$
$$v_y = \sqrt{u_y^2 - 2gh}$$
$$\therefore \quad u_y = \sqrt{v_y^2 + 2gh}$$
$$= \sqrt{(6.1)^2 + 2 \times 10 \times 9.1}$$
$$= 14.8 \text{ m/s}$$

(a) $$H = \frac{u_y^2}{2g} = \frac{(14.8)^2}{2 \times 10} \approx 11 \text{ m}$$

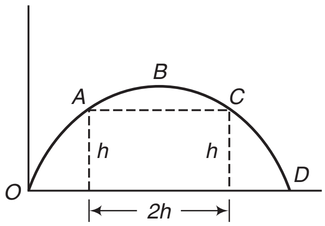
(b) $$R = \frac{2u_x u_y}{g} = \frac{2 \times 7.6 \times 14.8}{10} \approx 23 \text{ m}$$ **Ans.**

(c) $$u = \sqrt{u_x^2 + u_y^2}$$
$$= \sqrt{(7.6)^2 + (14.8)^2}$$
$$= 16.6 \text{ m/s}$$ **Ans.**

(d) $$\theta = \tan^{-1}\left(\frac{u_y}{u_x}\right)$$
$$= \tan^{-1}\left(\frac{14.8}{7.6}\right) \approx \tan^{-1}(2),$$

below horizontal. **Ans.**

14.
$$u_x^2 + u_y^2 = (2\sqrt{gh})^2 = 4gh \qquad \text{...(i)}$$



$$v_y = \sqrt{u_y^2 - 2gh} \qquad \text{...(ii)}$$

$$v_x = u_x \qquad \text{...(iii)}$$

Now for the projectile ABC, v_x and v_y are the initial components of velocity.

$$\therefore \quad 2h = \text{range} = \frac{2v_xv_y}{g} = \frac{2u_xv_y}{g}$$

$$\text{or} \quad u_x = \frac{gh}{v_y} \qquad \text{...(iv)}$$

Using Eqs. (ii) and (iv) for rewriting Eq. (i) we have,

$$\left(\frac{gh}{v_y}\right)^2 + (v_y^2 + 2gh) = 4gh$$

$$\therefore \quad v_y^4 - (2gh)\,v_y^2 + g^2h^2 = 0$$

$$\therefore \quad v_y^2 = \frac{2gh \pm \sqrt{4g^2h^2 - 4g^2h^2}}{2}$$

$$= gh$$

$$\therefore \quad v_y = \sqrt{gh}$$

Now, t_{AC} = time of projectile ABC

$$= \frac{2v_y}{g} = 2\sqrt{\frac{h}{g}} \qquad \textbf{Ans.}$$

15. Let v is the velocity at time t

Using $v_y = u_y + a_yt$

$$\therefore \quad v\sin\beta = u\sin\alpha - gt \qquad \text{...(i)}$$

$$v_x = u_x$$

$$\therefore \quad v\cos\beta = u\cos\alpha \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left(\frac{u\cos\alpha}{\cos\beta}\right)\sin\beta = u\sin\alpha - gt$$

On solving we get,

$$u = \frac{gt\cos\beta}{\sin(\alpha - \beta)}$$

16. $R - a = \dfrac{u^2\sin 2\alpha}{g}$

Multiplying with b we have,

$$bR - ab = \frac{bu^2\sin 2\alpha}{g} \qquad \text{...(i)}$$

$$R + b = \frac{u^2\sin 2\beta}{g}$$

Multiplying with a we have,

$$aR + ab = \frac{au^2\sin 2\beta}{g} \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii) and by putting

$$R = \frac{u^2\sin 2\theta}{g},$$

we get the result.

17. (a) $t = \dfrac{S_x}{u_x} = \dfrac{10}{5}$

$$= 2\text{ s}$$

(b) Vertical components of velocities are zero.

$$\therefore \quad h = \frac{1}{2}gt^2$$

$$= \frac{1}{2} \times 9.8 \times (2)^2$$

$$= 19.6\text{ m}$$

(c) $S_x = u_xt = 7.5 \times 2 = 15$ m

18. $u_x = 20\text{ km/h} = 20 \times \dfrac{5}{18} = 5.6\text{ m/s}$

$$u_y = 12\text{ km/h}$$

$$= 12 \times \frac{5}{18} = 3.3\text{ m/s}$$

Using $S = ut + \dfrac{1}{2}at^2$ in vertical direction, we have

$$-50 = (3.3)\,t + \frac{1}{2}(-10)\,t^2$$

Solving this equation we get,

$$t \approx 3.55\text{ s}$$

At the time of striking with ground,

$$v_x = u_x = 5.6\text{ m/s}$$

$$v_y = u_y + a_yt$$

$$= (3.3) + (-10)(3.55)$$

$$= 32.2\text{ m/s}$$

$$\therefore \quad \text{Speed} = \sqrt{(32.2)^2 + (5.6)^2}$$

$$\approx 32.7\text{ m/s}$$

19. $x = (u\sin\theta)\,T$

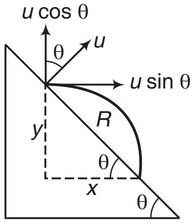
$$y = \left(\frac{1}{2}gT^2\right) - (u\cos\theta)\,T$$

$$\frac{y}{x} = \tan\theta = \frac{\frac{1}{2}gT^2 - (u\cos\theta)\,T}{(u\sin\theta)\,T}$$

$$\therefore \quad (2u\sin\theta)(\tan\theta) = gT - (2u\cos\theta)$$

$$T = \frac{2u\,[\sin\theta\tan\theta + \cos\theta]}{g}$$

$$= \frac{2u\sec\theta}{g}$$



Now, $R = (x)\sec\theta$

$$= (u\sin\theta\, T)\sec\theta$$

$$= (u\sin\theta)\left(\frac{2u\sec\theta}{g}\right)(\sec\theta)$$

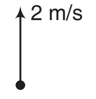
$$= \frac{2u^2\sin\theta\sec^2\theta}{g}$$

20. (a) Acceleration of stone is g or $10\ \text{m/s}^2$ in downward direction. Acceleration of elevator is $1\ \text{m/s}^2$ upwards. Therefore, relative acceleration of stone (with respect to elevator) is $11\ \text{m/s}^2$ downwards. Initial velocity is already given relative .

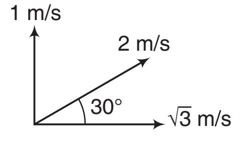
(b) $T = \dfrac{2u_r\sin\theta_r}{a_r} = \dfrac{(2)(2)\sin 30^\circ}{11}$

$= 0.18$ s . **Ans.**

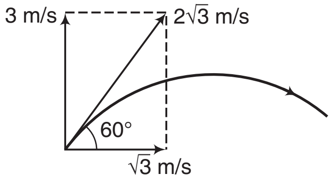
(c) $\mathbf{v}_S = \mathbf{v}_{SE} + \mathbf{v}_E$



Elevator



Stone relative to elevator



Absolute velocity of stone

(d) In that case relative acceleration between stone and elevator will be zero. So, with respect to elevator path is a straight line (uniform) with constant velocity of 20 m/s at 30°. But path, with respect to man on ground will remain unchanged.

21. (i) $x_A = x_B$

$\therefore\quad 10 + (u_1\cos\theta_1)\,t = 30 - (u_2\cos\theta_2)\,t$

$\therefore\quad (u_1\cos\theta_1 + u_2\cos\theta_2)\,t = 20$

(ii) $y_A = y_B$

$$\therefore\quad 10 + (u_1\sin\theta_1)\,t - \frac{1}{2}gt^2 = 20 + (u^2\sin\theta_2)\,t - \frac{1}{2}gt^2$$

or $\quad (u_1\sin\theta_1 - u_2\sin\theta_2)\,t = 10$

LEVEL 2

Single Correct Option

1. Relative acceleration between two is zero. Therefore, relative motion is uniform.

$$\mathbf{u}_{12} = \mathbf{u}_1 + \mathbf{u}_2$$

$$= (20\hat{\mathbf{j}}) - (20\cos 30^\circ\,\hat{\mathbf{i}} + 20\sin 30^\circ\,\hat{\mathbf{j}})$$

$$= (10\hat{\mathbf{j}} - 10\sqrt{3}\,\hat{\mathbf{i}})$$

$$|\mathbf{u}_{12}| = \sqrt{(10)^2 + (10\sqrt{3})^2} = 20\ \text{m/s}$$

$\therefore\quad d = |\mathbf{u}_{12}|\,t = 24$ m **Ans.**

2. $T = \sqrt{\dfrac{2h}{g}}$

$$d = (u)\,T = u\sqrt{\frac{2h}{g}}$$

$\therefore\quad d^2 = \dfrac{2hu^2}{g}$ **Ans.**

3. $R = \dfrac{u^2}{g\cos^2\beta}[\sin(2\alpha + \beta) + \sin\beta]$

$u = 10\ \text{m/s}$, $g = 10\ \text{m/s}^2$, $\beta = 30^\circ$ and $\alpha = 60^\circ$

4. $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$

$$\therefore\quad \frac{dy}{dx} = (\tan\theta) - \left(\frac{g}{u^2\cos^2\theta}\right)x$$

$\therefore\quad \dfrac{dy}{dx}$ *versus* x graph is a straight line with negative slope and positive intercept.

5. $\dfrac{dy}{dt} = (2\beta x)\cdot\dfrac{dx}{dt}$ and $\dfrac{d^2y}{dx^2} = 2\beta\left[\dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^2\right]$

$$\frac{d^2y}{dt^2} = \alpha = a_y$$

$$\frac{d^2x}{dt^2} = a_x = 0 \text{ and } \frac{dx}{dt} = v_x$$

$$\therefore\quad \alpha = 2\beta\cdot v_x^2$$

or $\quad v_x = \sqrt{\dfrac{\alpha}{2\beta}}$ **Ans.**

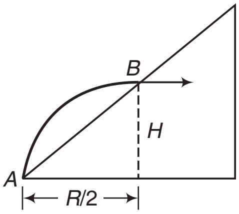
6. $$\frac{R}{2}=\frac{u^2\sin 2\,(60°)}{2g}$$

$$=\frac{\sqrt{3}\,u^2}{4g}$$

$$H=\frac{u^2\sin^2 60°}{2g}=\frac{3u^2}{8g}$$

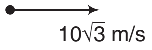
$$\therefore\qquad AB=\sqrt{\left(\frac{R}{2}\right)^2+H^2}$$

$$=\frac{\sqrt{21}\,u^2}{8g}$$

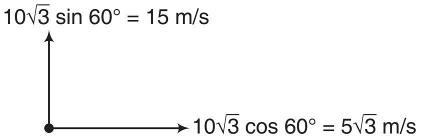


Ans.

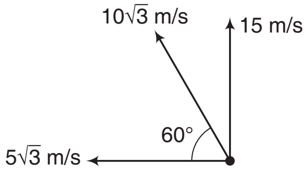
7.



Wedge



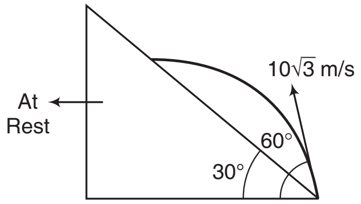
Particle



Particle with respect to wedge

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$=\frac{2\,(10\sqrt{3})\cdot\sin(60°-30°)}{10\cos 30°}=2\text{ s}$$



8. $$a_y=0=\frac{d^2y}{dt^2}$$

$$\frac{dx}{dt}=(2y+2)\cdot\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2}=(2y+2)\cdot\frac{d^2y}{dt^2}+2\left(\frac{dy}{dt}\right)^2$$

$$\therefore\qquad a_x=a=(2y+2)\,(0)+2\,(5)^2$$

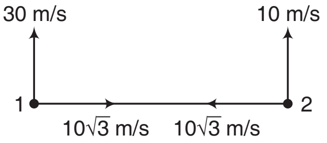
$$=50\text{ m/s}^2$$ **Ans.**

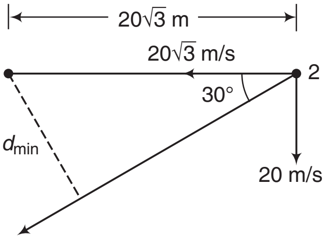
9. $$\tan\beta=\frac{H}{R/2}=\frac{2H}{R}$$

$$=\frac{(2u^2\sin^2\alpha)/2g}{(2u^2\sin\alpha\,\cos\alpha)/g}=\frac{\tan\alpha}{2}$$

$$\therefore\qquad \beta=\tan^{-1}\left(\frac{\tan\alpha}{2}\right)$$ **Ans.**

10. $a_1=a_2=g$ (downwards)





$$\therefore\qquad a_{12}=0$$

∴ Relative motion between them is uniform.

Relative velocity v_{21}

$$d_{\min}=20\sqrt{3}\,\sin 30°$$

$$=10\sqrt{3}\text{ m}$$ **Ans.**

More than One Correct Options

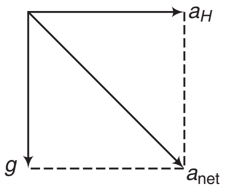
1. $\alpha+\beta=90°$ or $\beta=90°-\alpha$

$$h_1=\frac{u^2\sin^2\alpha}{2g}\quad\text{and}\quad h_2=\frac{u^2\cos^2\alpha}{2g}$$

$$t_1=\frac{2u\sin\alpha}{g}\quad\text{and}\quad t_2=\frac{2u\cos\alpha}{g}$$

$$R_1=R_2=\frac{2u^2\sin\alpha\,\cos\alpha}{g}=R$$

2. Since $u = 0$, motion of particle is a straight line in the direction of a_{net}.



$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 49}{9.8}}$$

$$= 3.16 \text{ s}$$ **Ans.**

3. Horizontal component of velocity remains unchanged

$\therefore \quad v \cos\theta = v' \cos(90 - \theta)$

or $\quad v' = v \cot\theta$

In vertical (y) direction,

$$v_y = u_y + a_y t$$

$$\therefore \quad t = \frac{v_y - u_y}{a_y}$$

$$= \frac{-v^1 \sin(90 - \theta) - v\sin\theta}{-g}$$

$$= \frac{(v \cot\theta) \cdot \cos\theta + v \sin\theta}{g}$$

$$= \frac{v \operatorname{cosec}\theta}{g}$$ **Ans.**

4. $u_x = v_x = 10 \text{ m/s}$

$$u_y = \sqrt{v_y^2 + 2gh}$$

$$= \sqrt{(10)^2 + (2)(10)(15)}$$

$$= 20 \text{ m/s}$$ **Ans.**

Angle of projection,

$$\theta = \tan^{-1}\left(\frac{u_y}{u_x}\right) = \tan^{-1}(2)$$

$$T = \frac{2u_y}{g} = \frac{(2)(20)}{10} = 4 \text{ s}$$

$$R = u_x T = (10)(4) = 40 \text{ m}$$

$$H = \frac{u_y^2}{2g} = \frac{(20)^2}{2 \times 10} = 20 \text{m}$$

5. $\dfrac{d\mathbf{v}}{dt} = \mathbf{a} = \text{constant} = \mathbf{g}$

$$\frac{d^2\mathbf{v}}{dt^2} = \frac{d\mathbf{a}}{dt} = 0 = \text{constant}$$

6. Horizontal component of velocity remains unchanged

$$X_{OA} = 20 \text{ m} = \frac{X_{AB}}{2}$$

$$\therefore \quad t_{OA} = \frac{t_{AB}}{2} = 1 \text{ s}$$

For *AB* projectile

$$T = 2 \text{ s} = \frac{2u_y}{g}$$

$$\therefore \quad u_y = 10 \text{ m/s}$$

$$H = \frac{u_y^2}{2g} = \frac{(10)^2}{2 \times 10} = 5 \text{ m}$$

$\therefore$ Maximum height of total projectile,

$$= 15 + 5 = 20 \text{ m}$$

$$t_{OB} = t_{OA} + t_{AB} = 1 + 2 = 3 \text{ s}$$

For complete projectile

$$T = 2(t_{OA}) + t_{AB}$$

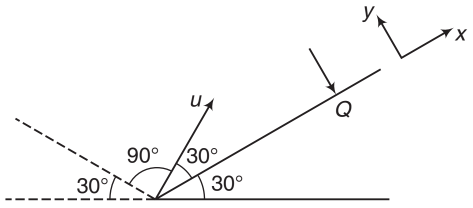
$$= 4 \text{ s} = \frac{2u_y}{g}$$

$$\therefore \quad u_y = 20 \text{ m/s}$$

$$u_x = \frac{AB}{t_{AB}} = \frac{40}{2} = 20 \text{ m/s}$$

Comprehension Based Questions

1. At Q, component parallel to OB becomes zero



$$v_x = u_x + a_x t$$

$$0 = (u \cos 30°) + (-g \sin 30°)\, t$$

$$\therefore \quad t = \frac{u}{g} \cot 30° = \frac{\sqrt{3}u}{g}$$ **Ans.**

2. $PQ = \text{range} = 2(PM) = 2a \cos 30°$

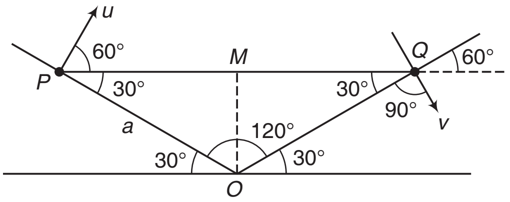
$$= (2)(4.9)\left(\frac{\sqrt{3}}{2}\right)$$

$$= 4.9\sqrt{3} \text{ m}$$

$$= \frac{u^2 \sin 2(60°)}{9.8}$$

$$\therefore \quad 4.9\sqrt{3} = \frac{u^2(\sqrt{3}/2)}{9.8}$$

$$\therefore \quad u = 9.8 \text{ m/s}$$ **Ans.**



Note *Velocity at P is making an angle of 60° with horizontal and velocity at Q is making an angle of 60° with horizontal. That is the reason PQ = range. Because under this condition, points P and Q lie on same horizontal line.*

Match the Columns

1. (a) $$\Delta x = (u_x)_2 \, (t-1) = (10)\,(2-1) = 10 \text{ m}$$

(b) $$y_1 = \frac{1}{2} g t^2 = \frac{1}{2} \times 10 \times (2)^2 = 20 \text{m}$$

$$y_2 = \frac{1}{2} g \, (t-1)^2 = 5 \text{ m}$$

$$\therefore \quad \Delta y = y_1 - y_2 = 15 \text{ m}$$

(c) $v_{x_1} = 0$, $v_{x_2} = 10$ m/s

$$\therefore \quad v_{x_2} - v_{x_1} = 10 \text{ m/s}$$

(d) $$v_{y_1} = gt = 10 \times 2 = 20 \text{ m/s}$$

$$v_{y_2} = g\,(t-1) = 10 \times 1 = 10 \text{ m/s}$$

$$\therefore \quad v_{y_1} - v_{y_2} = 10 \text{ m/s}$$

2. $$H = \frac{u_y^2}{2g} = 20 \text{ m}$$

$$\therefore \quad u_y = 20 \text{ m/s}$$

$$T = \frac{2u_y}{g} = 4 \text{ s}$$

$$R = u_x T = 40 \text{m}$$

$$\therefore \quad u_x = \frac{40}{T} = 10 \text{ m/s}$$

3. $$10 = \frac{u^2 \sin 2\,(15°)}{g}$$

$$\Rightarrow \quad \frac{u^2}{g} = 20 \text{ m}$$

(a) $$R_{\max} = \frac{u^2}{g} = 20 \text{ m}$$

(b) $$H_{\max} = \frac{u^2}{2g} \text{ when thrown vertically}$$

$$= 10 \text{ m}$$

(c) $R_{75°} = R_{15°} = 10$ m

(d) $$H_{30°} = \frac{u^2 \sin^2 30°}{2g}$$

$$= \frac{u^2}{8g} = \frac{20}{8} = 2.5 \text{ m}$$

4. $$T = \frac{2u_y}{g} \quad \Rightarrow \quad T \propto u_y$$

$$H = \frac{u_y^2}{2g}$$

$$\Rightarrow \quad H \propto u_y^2$$

$$R = u_x T$$

$$\Rightarrow \quad R \propto T$$

$$\Rightarrow \quad R \propto u_x$$

$$\Rightarrow \quad \tan\theta = \frac{u_y}{u_x}$$

$$\Rightarrow \quad \tan\theta \propto u_y$$

By doubling u_y, tan θ will become two times, not θ.

5. (a) $$\mathbf{a}_{av} = \mathbf{a} = (-g\hat{\mathbf{j}}) = \frac{\Delta \mathbf{v}}{\Delta t}$$

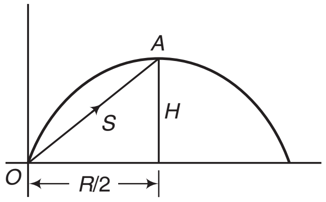
$$\therefore \quad \Delta \mathbf{v} = (-g\hat{\mathbf{j}})\,\Delta t$$

$$= (-g\hat{\mathbf{j}}) \left(\frac{T}{2}\right)$$

$$= (-g\hat{\mathbf{j}}) \left(\frac{u \sin\theta}{g}\right)$$

$$= (-u \sin\theta)\,\hat{\mathbf{j}}$$

(b) $$v_{av} = \frac{s}{t}$$



$$= \frac{\sqrt{(R/2)^2 + H^2}}{T/2}$$

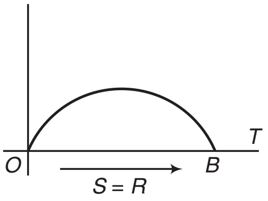
(c) $$\Delta \mathbf{v} = (-g\hat{\mathbf{j}})\,(\Delta t)$$

$$= (-g\hat{\mathbf{j}})\, T$$

$$= (-g\hat{\mathbf{j}}) \left(\frac{2u \sin\theta}{g}\right)$$

$$= (-2u \sin\theta)\,\hat{\mathbf{j}}$$

(d) $v_{av} = \dfrac{S}{t} = \dfrac{R}{T}$



$$= \frac{u_x T}{T}$$

$$= u_x = u \cos\theta$$

6. (a) $x_1 = (u_{x1})\, t = (30)\,(2) = 60$ m

$$x_2 = (130) + (u_{x2})\,(t-1)$$
$$= 130 + (-20)\,(1)$$
$$= 110 \text{ m}$$

$\therefore \qquad \Delta x = 50$ m

(b) $y_1 = u_{y1}t - \dfrac{1}{2}gt^2$

$$= (30)\,(2) - \frac{1}{2}(10)\,(2)^2$$
$$= 40 \text{ m}$$

$$y_2 = 75 + u_{y2}\,(t-1) - \frac{1}{2}g\,(t-1)^2$$
$$= 75 + 20 \times 1 - \frac{1}{2} \times 10 \times (1)^2$$
$$= 90 \text{ m}$$

$\therefore \quad \Delta y = 50$ m

(c) $v_{x1} = 30$ m/s

$$v_{x2} = -20 \text{ m/s}$$

$\therefore \qquad v_{x1} - v_{x2} = 50$ m/s

(d) $v_{y1} = u_{y1} + a_y t$

$$= (30) + (-10)\,(2)$$
$$= 10 \text{ m/s}$$
$$v_{y2} = u_{y2} + a_y\,(t-1)$$
$$= 20 + (-10)\,(1)$$
$$= 10 \text{ m/s}$$

$\therefore \quad v_{y1} - v_{y2} = 0$

7. $H = \dfrac{u_y^2}{2g}$ or $H \propto u_y^2$

Since H is same. Therefore, u_y is same for all three.

$\therefore \qquad T = \dfrac{2u_y}{g}$ or $t \propto u_y$

Since u_y is same. Therefore T is same for all.

$$R = u_x T$$

or $\qquad R \propto u_x \qquad$ (as $T \rightarrow$ same)

R for C is maximum. Therefore u_x is greatest for C.

$$u = \sqrt{u_y^2 + u_x^2} \qquad (u_y \rightarrow \text{same})$$

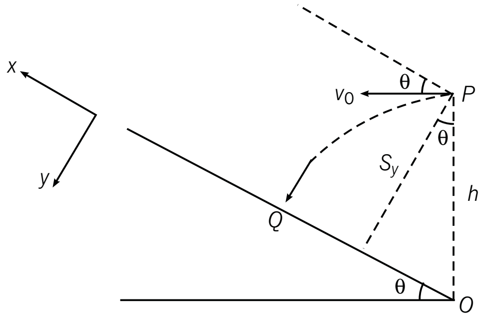
R is least for A. Therefore u_x and hence u is least for A.

Subjective Questions

1. $u_x = v_0 \cos\theta, u_y = v_0 \sin\theta, a_x = -g\sin\theta,$
$a_y = g\cos\theta$

At Q, $v_x = 0$

$\therefore \qquad u_x + a_x t = 0$



or $\qquad t = \dfrac{v_0 \cos\theta}{g \sin\theta} \qquad$ ...(i)

$$s_y = h\cos\theta$$

$\therefore \qquad u_y t + \dfrac{1}{2}a_y t^2 = h\cos\theta$

$\therefore \quad (v_0 \sin\theta)\left(\dfrac{v_0\cos\theta}{g\sin\theta}\right)$

$$+\frac{1}{2}(g\cos\theta)\left(\frac{v_0\cos\theta}{g\sin\theta}\right)^2 = h\cos\theta$$

Solving this equation we get,

$$v_0 = \sqrt{\frac{2gh}{2 + \cot^2\theta}} \qquad \textbf{Ans.}$$

2. Let v_x and v_y be the components of v_0 along x and y directions.

$$(v_x)(2) = 2$$

$\therefore \qquad v_x = 1$ m/s

$$v_y(2) = 10$$

or $\qquad v_y = 5$ m/s

$$v_0 = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{26} \text{ m/s}$$

$$\tan\theta = v_y/v_x = 5/1$$

$\therefore \qquad \theta = \tan^{-1}(5) \qquad$ **Ans.**

Note *We have seen relative motion between two particles. Relative acceleration between them is zero.*

3. $\mathbf{v}_1 = (u \cos \alpha)\,\hat{\mathbf{i}} + (u \sin \alpha - gt)\,\hat{\mathbf{j}}$

$\mathbf{v}_2 = (v \cos \beta)\,\hat{\mathbf{i}} + (v \sin \beta - gt)\,\hat{\mathbf{j}}$

These two velocity vectors will be parallel when, the ratio of coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are equal.

$$\therefore \qquad \frac{u \cos \alpha}{v \cos \beta} = \frac{u \sin \alpha - gt}{v \sin \beta - gt}$$

Solving we get,

$$t = \frac{uv \sin (\alpha - \beta)}{g(v \cos \beta - u \cos \alpha)} \qquad \textbf{Ans.}$$

4. At height 2 m, projectile will be at two times, which are obtained from the equation,

$$2 = (10 \sin 45°)t + \frac{1}{2}(-10)t^2$$

or $$2 = 5\sqrt{2}t - 5t^2$$

or $$5t^2 - 5\sqrt{2}t + 2 = 0$$

or $$t_1 = \frac{5\sqrt{2} - \sqrt{50-40}}{10} = \frac{5\sqrt{2} - \sqrt{10}}{10}$$

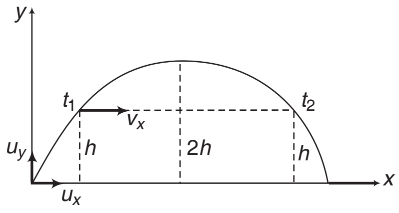
and $$t_2 = \frac{5\sqrt{2} + \sqrt{10}}{10}$$

Now $$d = (10 \cos 45°)(t_2 - t_1) = \frac{10}{\sqrt{2}}\left(\frac{2\sqrt{10}}{10}\right) = 4.47 \text{ m}$$

Distance of point of projection from first hurdle

$$= (10 \cos 45°)t_1 = \frac{10}{\sqrt{2}}\left(\frac{5\sqrt{2} - \sqrt{10}}{10}\right) = 5 - \sqrt{5} = 2.75 \text{ m} \qquad \textbf{Ans.}$$

5. $2h = \dfrac{u_y^2}{2g}$



or $$u_y = 2\sqrt{gh}$$

Now $$(t_2 - t_1)u_x = t_2 v_x \text{ or } \frac{v_x}{u_x} = \frac{t_2 - t_1}{t_2} \qquad \ldots\text{(i)}$$

Further $$h = u_y t - \frac{1}{2}gt^2$$

or $$gt^2 - 2u_y t + h = 0$$

or $$gt^2 - 4\sqrt{gh}\,t + 2h = 0$$

$$t_1 = \frac{4\sqrt{gh} - \sqrt{16gh - 8gh}}{2g} = (2 - \sqrt{2})\sqrt{\frac{h}{g}}$$

and $$t_2 = (2 + \sqrt{2})\sqrt{\frac{h}{g}}$$

Substituting in Eq. (i) we have,

$$\frac{v_x}{u_x} = \frac{2}{\sqrt{2} + 1} \qquad \textbf{Ans.}$$

6. (a) Time of descent $t = \sqrt{\dfrac{2H}{g}} = \sqrt{\dfrac{2 \times 400}{10}}$

$$= 8.94 \text{ s}$$

Now $$v_x = ay = \sqrt{5}\,y$$

or $$\frac{dx}{dt} = \sqrt{5}\left(\frac{1}{2}gt^2\right) = 5\sqrt{5}t^2$$

$$\therefore \qquad \int_0^x dx = 5\sqrt{5}\int_0^t t^2\, dt$$

or horizontal drift

$$x = \frac{5\sqrt{5}}{3}(8.94)^3 = 2663 \text{ m} \approx 2.67 \text{ km.}$$

(b) When particle strikes the ground

$$v_x = \sqrt{5}\,y = (\sqrt{5})(400) = 400\sqrt{5} \text{ m/s}$$

$$v_y = gt = 89.4 \text{ m/s}$$

$$\therefore \quad \text{Speed} = \sqrt{v_x^2 + v_y^2} = 899 \text{ m/s} \approx 0.9 \text{ km/s} \qquad \textbf{Ans.}$$

7. At $t = 0$, $\mathbf{v}_T = (10\hat{\mathbf{j}})$ m/s

$\mathbf{v}_{ST} = 10 \cos 37°\,\hat{\mathbf{k}} - 10 \sin 37°\,\hat{\mathbf{i}} = (8\hat{\mathbf{k}} - 6\hat{\mathbf{i}})$ m/s

$\therefore \quad \mathbf{v}_S = \mathbf{v}_{ST} + \mathbf{v}_T = (-6\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$ m/s

(a) At highest point vertical component $(\hat{\mathbf{k}})$ of $\mathbf{v}_S$ will become zero. Hence, velocity of particle at highest point will become $(-6\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s .

(b) Time of flight, $T = \dfrac{2v_Z}{g} = \dfrac{2 \times 8}{10} = 1.6$ s

$$x = x_i + v_x T = \frac{16}{\pi} - 6 \times 1.6 = -4.5 \text{ m}$$

$$y = (10)(1.6) = 16 \text{ m and } z = 0$$

Therefore, coordinates of particle where it finally lands on the ground are (– 4.5 m, 16 m , 0).

At highest point, $t = \frac{T}{2} = 0.8$ s

$$\therefore \quad x = \frac{16}{\pi} - (6)(0.8) = 0.3 \text{ m}$$

$$y = (10)(0.8) = 8.0 \text{ m}$$

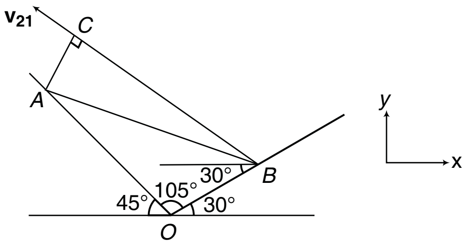
and $$z = \frac{v_Z^2}{2g} = \frac{(8)^2}{20} = 3.2 \text{ m}$$

Therefore, coordinates at highest point are, (0.3 m, 8.0 m, 3.2 m) **Ans.**

8. $|v_{21x}| = (v_1 + v_2)\cos 60° = 12$ m/s

$|v_{21y}| = (v_2 - v_1)\sin 60° = 4\sqrt{3}$ m/s

$\therefore v_{21} = \sqrt{(12)^2 + (4\sqrt{3})^2} = \sqrt{192}$ m/s

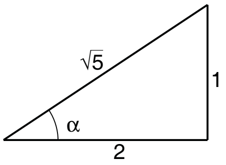


$$BC = (v_{21})\,t = 240 \text{ m}$$

$$AC = 70 \text{ m} \qquad \text{(Given)}$$

Hence, $AB = \sqrt{(240)^2 + (70)^2} = 250$ m **Ans.**

9. (a) Let, (x, y) be the coordinates of point C



$$x = OD = OA + AD$$

$$\therefore \quad x = \frac{10}{3} + y\cot 37° = \frac{10 + 4y}{3} \qquad \text{...(i)}$$

As point C lies on the trajectory of a parabola, we have

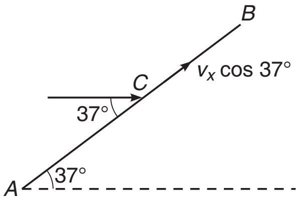
$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha) \qquad \text{...(ii)}$$

Given that, $\tan\alpha = 0.5 = \frac{1}{2}$

Solving Eqs. (i) and (ii), we get $x = 5$ m and $y = 1.25$ m.

Hence, the coordinates of point C are (5 m, 1.25 m). **Ans.**

(b) Let v_y be the vertical component of velocity of the particle just before collision at C.



Using $v_y = u_y + a_y t$, we have

$$v_y = u\sin\alpha - g\,(x/u\cos\alpha) \quad (\because \; t = x/u\cos\alpha)$$

$$= \frac{5\sqrt{5}}{\sqrt{5}} - \frac{10 \times 5}{(5\sqrt{5} \times 2/\sqrt{5})} = 0$$

Thus, at C, the particle has only horizontal component of velocity

$$v_x = u\cos\alpha = 5\sqrt{5} \times (2/\sqrt{5}) = 10 \text{ m/s}$$

Given, that the particle does not rebound after collision. So, the normal component of velocity (normal to the plane AB) becomes zero. Now, the particle slides up the plane due to tangential component $v_x \cos 37° = (10)\left(\frac{4}{5}\right) = 8$ m/s

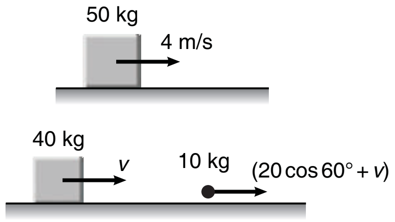
Let h be the further height raised by the particle. Then

$$mgh = \frac{1}{2}m(8)^2 \quad \text{or} \quad h = 3.2 \text{ m}$$

Height of the particle from the ground $= y + h$

$$\therefore \quad H = 1.25 + 3.2 = 4.45 \text{ m}$$ **Ans.**

10. For shell $u_z = 20\sin 60° = 17.32$ m/s



$$\therefore \quad z = u_z t - \frac{1}{2}gt^2 = (17.32 \times 2) - \left(\frac{1}{2} \times 9.8 \times 4\right)$$

or $$z = 15 \text{ m} \quad \Rightarrow \quad u_y = 0$$

$$\therefore \quad y = 0$$

For u_x conservation of linear momentum gives,

$$50 \times 4 = (40)(v) + 10(20\cos 60° + v)$$

or $$v = 2 \text{ m/s}$$

$$\therefore \quad u_x = (20\cos 60°) + 2 = 12 \text{ m/s}$$

$$\therefore \quad x = u_x t = (12)(2) = 24 \text{ m}$$

$$\therefore \quad \mathbf{r} = (24\hat{\mathbf{i}} + 15\hat{\mathbf{k}}) \text{ m}$$ **Ans.**

8. Laws of Motion

INTRODUCTORY EXERCISE 8.1

1. w_1 = weight of cylinder

w_2 = weight of plank

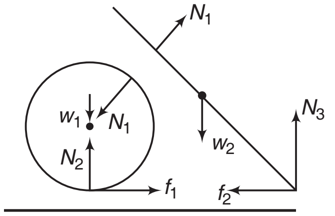
N_1 = normal reaction between cylinder and plank

N_2 = normal reaction on cylinder from ground

N_3 = normal reaction on plank from ground

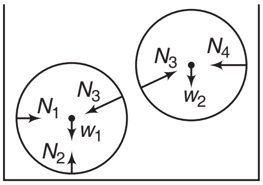
f_1 = force of friction on cylinder from ground

f_2 = force of friction on plank from ground

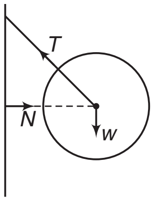


2. N = normal reactions

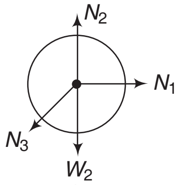
w = weights



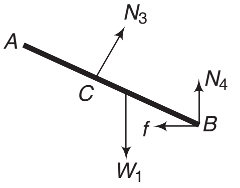
3.



4.



FBD of sphere
(i)



FBD of rod
(ii)

In the figure

N_1 = normal reaction between sphere and wall,

N_2 = normal reaction between sphere and ground

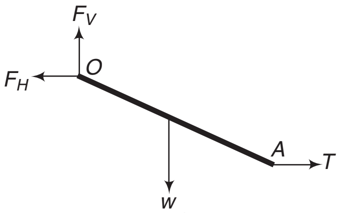
N_3 = normal reaction between sphere and rod and

N_4 = normal reaction between rod and ground

f = force of friction between rod and ground

No friction will act between sphere and ground because horizontal component of normal reaction from rod (on sphere) will be balanced by the horizontal normal reaction from the wall.

5.



FBD of rod

In the figure

T = tension in the string, W = weight of the rod,

F_V = vertical force exerted by hinge on the rod

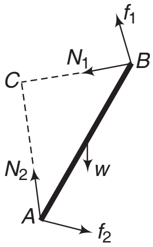
F_H = horizontal force exerted by hinge on the rod

6. **In the figure**

N_1 = Normal reaction at B,

f_1 = force of friction at B, N_2= normal reaction at A, f_2 = force of friction at A

w = weight of the rod.



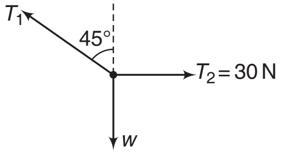
FBD of the rod

7.

Force	F_x	F_y
$\mathbf{F}_1$	$4\cos 30° = 2\sqrt{3}$ N	$4\sin 30° = 2$ N
$\mathbf{F}_2$	$-4\cos 60° = -2$ N	$4\sin 60° = 2\sqrt{3}$ N
$\mathbf{F}_3$	0	-6 N
$\mathbf{F}_4$	4 N	0

8. $T_1 \cos 45° = w$

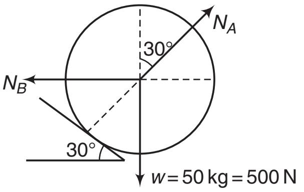
and $T_1 \sin 45° = 30$ N



$\therefore$ $w = 30$ N **Ans.**

9. $N_A \cos 30° = 500\text{ N}$ ...(i)

$N_A \sin 30° = N_B$ ...(ii)

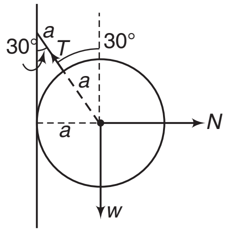


On solving these two equations, we get

$$N_A = \frac{1000}{\sqrt{3}}\text{ N}$$

and $N_B = \dfrac{500}{\sqrt{3}}\text{N}$ **Ans.**

10. Net force in vertical direction = 0

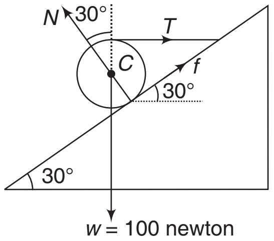


$\therefore\ T \cos 30° = w$ or $T = \dfrac{2w}{\sqrt{3}}$ **Ans.**

11. $w = 100\text{ N}$

Σ(forces in horizontal direction) = 0

$\therefore\ T + f \cos 30° = N \sin 30°$ ...(i)



Σ(Forces in vertical direction) = 0

$\therefore\ N \cos 30° + f \sin 30° = 100$ ...(ii)

Σ (moment of all forces about C) = 0

$\therefore\ (T)(R) = f(R)$

or $T = f$ ...(iii)

On solving these three equations, we get

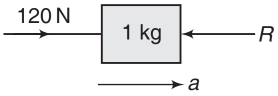
$$T = f = 26.8\text{N}$$

and $N = 100$ newton **Ans.**

INTRODUCTORY EXERCISE 8.2

1. (a)

$$a = \frac{\text{Net pushing force}}{\text{Total mass}} = \frac{120 - 50}{1 + 4 + 2} = 10\text{ m/s}^2$$



(b)

$$120 - R = 1 \times a = 10$$

$\therefore\ R = 110\text{ N}$

(c) $F_{\text{net}} = ma = (2)(10) = 20\text{ N}$

2. $T_4 = 4g$ and $T_1 = (1)\,g$ as $a = 0$

$\therefore\ \dfrac{T_4}{T_1} = 4$ **Ans.**

3. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{F + 10 - 20}{1 + 2}$ $(F = 20\text{ N})$

$= \dfrac{10}{3}\text{ m/s}^2$ **Ans.**

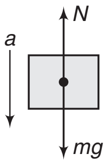
4. $m_A g = (2)(g) + (2)g \sin 30°$ as $a = 0$

$\therefore\ m_A = 3\text{ kg}$ **Ans.**

5. Since surface is smooth, acceleration of both is $g \sin\theta = 10 \sin 30° = 5\text{ m/s}^2$, down the plane.

The component of mg down the plane ($= mg \sin\theta$) provides this acceleration. So, normal reaction will be zero.

6. $N = \dfrac{mg}{4}$ is given



$$mg - \frac{mg}{4} = ma$$

$\therefore\ a = \dfrac{3g}{4}$

7. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$

$= \dfrac{(3 + 4 - 2 - 1)\,g}{3 + 4 + 2 + 1} = 4\text{ m/s}^2$ **Ans.**

4 kg $40 - T_1 = 4a \ \therefore\ T_1 = 24\text{ N}$ **Ans.**

3 kg $T_1 + 30 - T_2 = 3a \ \therefore\ T_2 = 42\text{ N}$ **Ans.**

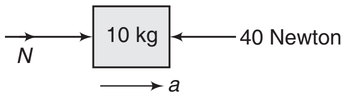
1 kg $T_3 - 10 = (1)(a) \ \therefore\ T_3 = 14\text{ N}$ **Ans.**

8. (a) $a = \dfrac{\text{Net pushing force}}{\text{Total mass}}$

$$= \frac{100-40}{6+4+10} = 3\ \text{m/s}^2$$

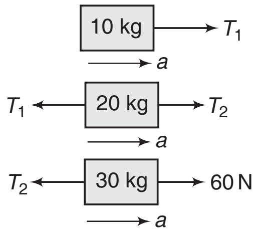
(b) Net force on any block = ma

(c) $N - 40 = 10\,a = 30$



$\therefore$ $N = 70$ newton **Ans.**

9. (a) $T_1 = 10a$, $T_2 - T_1 = 20a$, $60 - T_2 = 30a$



$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{60}{10+20+30}$$

$$= 1\ \text{m/s}^2$$

On solving, we get $T_1 = 10$ N, $T_2 = 30$ N **Ans.**

(b) $T_1 = 0$, $T_2 = 20a$, $60 - T_2 = 30a$

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{60}{20+30}$$

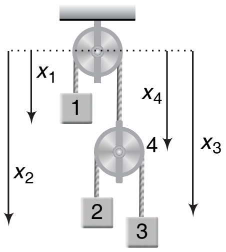
$$= 1.2\ \text{m/s}^2$$

On solving, we get

$T_2 = 24$ N **Ans.**

INTRODUCTORY EXERCISE 8.3

1. Points 1, 2, 3 and 4 are movable. Let their displacements from the fixed dotted line be x_1, x_2, x_3 and x_4 (ignoring the length over the pulley)



We have,

$x_1 + x_4 = l_1$ (length of first string) …(i)

and $(x_2 - x_4) + (x_3 - x_4) = l_2$ (length of second string)

or $x_2 + x_3 - 2x_4 = l_2$ …(ii)

On double differentiating with respect to time, we get

$a_1 + a_4 = 0$ …(iii)

and $a_2 + a_3 - 2a_4 = 0$ …(iv)

But $a_4 = -a_1$ [From Eq. (iii)]

We have, $a_2 + a_3 + 2a_1 = 0$

This is the required constraint relation between a_1, a_2 and a_3.

2. In above solution, we have found that

$$a_2 + a_3 + 2a_1 = 0$$

Similarly, we can find

$$v_2 + v_3 + 2v_1 = 0$$

Taking, upward direction as positive we are given

$$v_1 = v_2 = 1\ \text{m/s}$$

$\therefore$ $v_3 = -3$ m/s

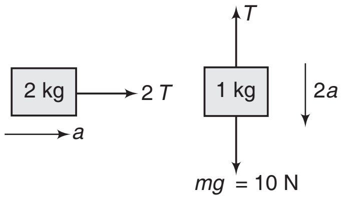
i.e. velocity of block 3 is 3 m/s (downwards).

3. **2 kg** $2T = 2\,(a)$ ...(i)

1 kg $10 - T = 1\,(2a)$ ...(ii)

On solving these two equations, we get

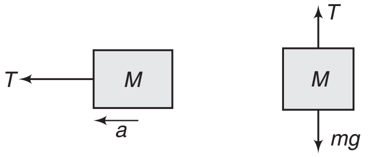
$$T = \frac{10}{3}\ \text{N}$$



and $2a = 2T = \dfrac{20}{3}\ \text{m/s}^2$

= acceleration of 1 kg block

4.



$T = Ma$ ...(i)

$Mg - T = Ma$ ...(ii)

On solving these two equations, we get

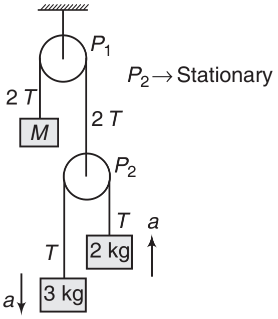
$$a = \frac{g}{2}$$

and $T = \dfrac{Mg}{2}$ **Ans.**

5. In the figure shown,

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{30 - 20}{3 + 2}$$

$$= 2 \text{ m/s}^2$$



2 kg $T - 20 = 2 \times a = 4$

$\therefore$ $T = 24$ N

Now, $Mg = 2T$

$\therefore$ $M = \frac{2T}{g} = \frac{48}{10}$

$= 4.8$ kg **Ans.**

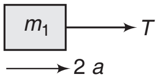
6. $T = m_1\,(2a) = (0.3)\,(2a) = 0.6a$ ...(i)

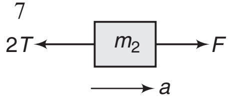
$F - 2T = m_2 a$

$\therefore$ $2T = 0.4 - 0.2a$ ...(ii)

On solving these two equations, we get

$$a = \frac{2}{7} \text{ m/s}^2$$ **Ans.**





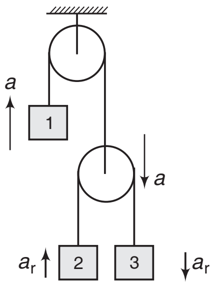
and $2T = \frac{12}{35}$ N **Ans.**

7. $a_3 = a + a_r = 6$ ...(i)

$a_2 = a_r - a = 4$ ...(ii)

On solving these two equations, we get

$$a = 1 \text{ m/s}^2$$

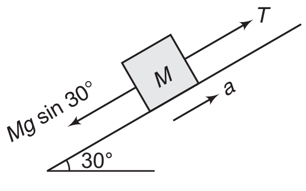


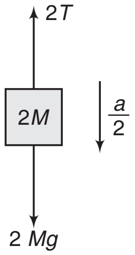
8. $T - Mg \sin 30° = Ma$ ...(i)

$2Mg - 2T = 2M\left(\frac{a}{2}\right)$ ...(ii)

Solving these equations, we get

$$a = \frac{g}{3}$$ **Ans.**





INTRODUCTORY EXERCISE 8.4

1. (a) $\mathbf{F}_A = -m_A \mathbf{a}_B = (4\hat{\mathbf{j}})$ N

(b) $\mathbf{F}_B = -m_B \mathbf{a}_A = (-4\hat{\mathbf{i}})$ N

2. Constant velocity means acceleration of frame is zero.

INTRODUCTORY EXERCISE 8.5

1. **Figure (a)** $N = mg = 40$ N

$\mu_s N = 24$ N

Since, $F < \mu_s N$

Block will remain stationary and

$f = F = 20$ N

Figure (b) $N = mg = 20$ N

$\mu_S N = 12$ N

and $\mu_K N = 8$ N

Since, $F > \mu_S N$, block will slide and kinetic friction (= 8 N) will act.

$$a = \frac{F - f}{m} = \frac{20 - 8}{2} = 6 \text{ m/s}^2$$

2. If $\theta \le$ angle of repose, the block is stationary, $a = 0$, $F_{\text{net}} = 0$ and $f = mg \sin \theta$.

If $\theta >$ angle of repose, the block will move,

$$f = \mu mg \cos \theta$$

$$a = \frac{mg \sin \theta - \mu mg \cos \theta}{m}$$

$$= g \sin \theta - \mu g \cos \theta \text{ and } F_{\text{net}} = ma$$

Further, $\mu = \tan \alpha$ (α = angle of repose = 45°)

$= \tan 45° = 1$

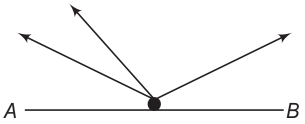
Exercises

LEVEL 1

Assertion and Reason

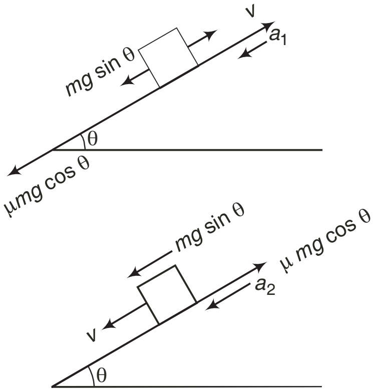
1. If net force is zero but net torque $\neq 0$, then it can rotate.

2. If three forces are above a straight line (say AB), then there should be some force below the line AB also to make their resultant $= 0$.



3. $a_1 = g \sin\theta + \mu g \cos\theta$

$$a_2 = g \sin\theta - \mu g \cos\theta$$



$$\therefore \quad \frac{a_1}{a_2} = \frac{1+\mu}{1-\mu} \quad \text{for} \quad \theta = 45^\circ$$

4. For $(-2\hat{\mathbf{i}})\ \text{m/s}^2$ component of acceleration, a horizontal force in $-$ve x- direction is required which can be provided only by the right vertical wall of the box.

5. By increasing F_1 limiting value of friction will increase. But it is not necessary that actual value of friction acting on block will increase.

7. **If $m_1 > m_2$**

$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{(m_1 - m_2)\, g}{m_1 + m_2}$$

FBD of m_1

$$m_1 g - T = m_1 a = \frac{m_1 (m_1 - m_2)\, g}{m_1 + m_2}$$

$$\therefore \quad T = \frac{2\, m_1 m_2}{(m_1 + m_2)}\, g$$

$$= m_1 g \left[\frac{2m_2}{m_1 + m_2}\right] = m_2 g \left[\frac{2m_1}{m_1 + m_2}\right]$$

$$= m_1 g \left[\frac{m_2 + m_2}{m_1 + m_2}\right] = m_2 g \left[\frac{m_1 + m_1}{m_1 + m_2}\right]$$

Since $m_1 > m_2, T < m_1 g$ but $T > m_2 g$

Similarly,

we can prove that

$$T < m_2 g$$

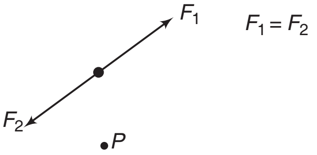
and $\quad T > m_1 g$

if $\quad m_2 > m_1$.

Therefore under all conditions T lies between $m_1 g$ and $m_2 g$.

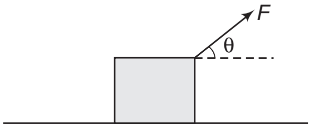
8. A frame moving with constant velocity $(a = 0)$ is inertial.

9. If vector sum of concurrent forces is zero, then all forces can be assumed a pair of two equal and opposite forces acting at one point.



Now, if we take moment about any general point P, then moment of F_1 is clockwise and moment of F_2 (of same magnitude) is anti-clockwise. Therefore net moment is zero.

10. $N = mg - F \sin\theta$



$$F \cos\theta = \mu N = \mu (mg - F \sin\theta)$$

$$\therefore \quad F = \frac{\mu mg}{\cos\theta + \mu \sin\theta} = f(\theta) \qquad \text{...(i)}$$

For F to be minimum, denominator should be maximum.

$$\text{or} \quad \frac{d}{d\theta}(\cos\theta + \mu \sin\theta) = 0$$

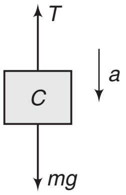
From here we get $\theta = \tan^{-1}(\mu) =$ angle of friction. With this value of θ we can also find that minimum value of force required from Eq. (i).

11. Friction (or you can say net force on man from ground) is in the direction of motion.

Single Correct Option

1. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$

$= \dfrac{(2+2-2)\,g}{2+2+2} = \dfrac{g}{3}$



FBD of *C*

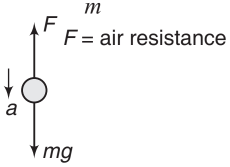
$$mg - T = ma = \frac{mg}{3}$$

$$\therefore \quad T = \frac{2}{3}\,mg$$

$$= \frac{2}{3}\,(20) = 13.3 \text{ N}$$

Ans.

2. $a = \dfrac{mg - F}{m} = g - \dfrac{F}{m}$



$$m_A > m_B$$

$\therefore \quad a_A > a_B$ and A will reach earlier

3. Only two forces are acting, mg and net contact force (resultant of friction and normal reaction) from the inclined plane. Since the body is at rest. Therefore these two forces should be equal and opposite.

$\therefore$ Net contact force $= mg$ (upwards)

4. $T_A = 10\,g = 100$ N

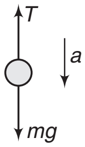
$$T_B \cos 30° = T_A$$

$$\therefore \quad T_B = \frac{200}{\sqrt{3}} \text{ N}$$

$$T_B \sin 30° = T_C$$

$$\therefore \quad T_C = \frac{100}{\sqrt{3}} \text{ N}$$

5. $a = \dfrac{mg - T}{m} = g - \dfrac{T}{m}$



$$a_{\min} = g - \frac{T_{\max}}{m}$$

$$= g - \frac{\frac{2}{3}mg}{m} = \frac{g}{3}$$

6. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{10 \times 10 - 5 \times 10}{10 + 5} = \dfrac{10}{3}$ m/s^2

5 kg $\quad T - 5 \times 10 = 5 \times a = \dfrac{50}{3}$

$$\therefore \quad T = \frac{200}{3} \text{ N}$$

This is also the reading of spring balance.

7. $t = \sqrt{\dfrac{2S}{a}} \propto \dfrac{1}{\sqrt{a}}$

$$\therefore \quad \frac{t_1}{t_2} = \sqrt{\frac{a_2}{a_1}}$$

$$\frac{2}{1} = \sqrt{\frac{g \sin\theta}{g \sin\theta - \mu_K\, g \cos\theta}} = \sqrt{\frac{1}{1-\mu_K}}$$

as, $\sin\theta = \cos\theta$ at 45°

On solving the above equation, we get

$$\mu_K = \frac{3}{4}$$

8. $F_1 = mg \sin\theta + \mu mg \cos\theta$

$F_2 = mg \sin\theta - \mu mg \cos\theta$

Given that $F_1 = 2\,F_2$

$\therefore (mg \sin\theta + \mu mg \cos\theta) = 2(mg \sin\theta - \mu mg \cos\theta)$

On solving, we get $\tan\theta = 3\mu$

9. During acceleration,

$$v = a_1 t_1 = \frac{F_1}{m}\, t_1$$

$$\therefore \quad F_1 = \frac{mv}{t_1} \qquad \text{...(i)}$$

During retardation,

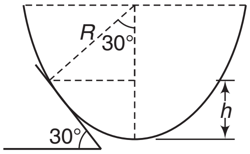
$$0 = v - a_2\, t_2 = v - \frac{F_2}{m}\, t_2$$

$$\therefore \quad F_2 = \frac{mv}{t_2} \qquad \text{...(ii)}$$

If $\quad t_1 = t_2$ then $F_1 = F_2$
$t_1 < t_2$ then $F_1 > F_2$ and
$t_1 > t_2$ then $F_1 < F_2$

10. Angle of repose $\theta = \tan^{-1}(\mu)$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$



So, particle may be placed maximum upto 30°, as shown in figure

$$h = R - R\cos 30° = \left(1 - \frac{\sqrt{3}}{2}\right) R$$

11. Net pulling force $F_1 = 15g - 5g$

$$F_1 = 10g = 100 \text{ N}$$

Net stopping force $F_2 = 0.2 \times 5 \times 10 = 10$ N

Since $F_1 > F_2$, therefore the system will move (anticlockwise) with an acceleration,

$$a = \frac{F_1 - F_2}{15 + 5 + 5} = \frac{90}{25}$$

$$= 3.6 \text{ m/s}^2$$

15 kg block

$$15 \times 10 - T_2 = 15 \times a = 15 \times 3.6$$

$$\therefore \quad T_2 = 96 \text{ N}$$

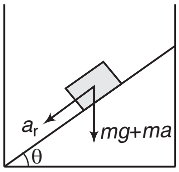
5 kg block

$$T_1 - 5 \times 10 = 5 \times a = 5 \times 3.6$$

$$\therefore \quad T_1 = 68 \text{ N}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{68}{96} = \frac{17}{24}$$

12. Acceleration of block with respect to lift



Pseudo force = (downward)

$$a_r = \frac{m(g + a)\sin\theta}{m} = (g + a)\sin\theta$$

$$S = \frac{1}{2} a_r t^2$$

$$\therefore \quad t = \sqrt{\frac{2S}{a_r}} = \sqrt{\frac{2L}{(g + a)\sin\theta}}$$

13. Since the block is resting (not moving)

$$\therefore \quad f = mg\sin\theta \neq \mu_S\, mg\cos\theta$$

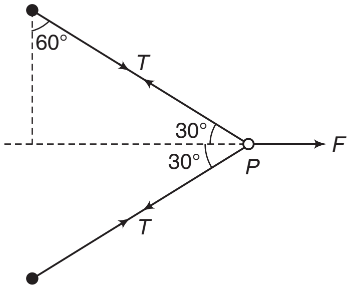
or

$$m = \frac{f}{g\sin\theta}$$

$$= \frac{10}{10 \times \sin 30°}$$

$$= 2 \text{ kg}$$ **Ans.**

14. At point P,



$$F = 2T\cos 30°$$

$$\therefore \quad T = \frac{F}{\sqrt{3}}$$

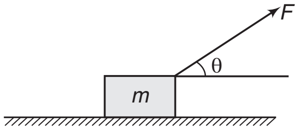
Acceleration of particle towards each other

$$= \frac{\text{component of } T \text{ towards each other particle}}{m}$$

$$= \frac{T\cos 60°}{m} = \frac{(F/\sqrt{3})(1/2)}{m} = \frac{F}{2\sqrt{3}\,m}$$

15. $\mu = \tan\phi$

$$N = mg - F\sin\theta$$



$$F\cos\theta = \mu N = (\tan\phi)(mg - F\sin\theta)$$

$$= \left(\frac{\sin\phi}{\cos\phi}\right)(mg - F\sin\theta)$$

On solving this equation, we get

$$F = \frac{mg\sin\phi}{\cos(\theta - \phi)}$$

16. $N = mg = 40$ N

$$\mu N = 0.2 \times 40 = 8 \text{ N}$$

At $\quad t = 2$ s, $F = 4$ N

Since $\quad F < \mu N$

$\therefore \quad f = F = 4$ N

17. $S = \frac{1}{2} at^2$

$$\therefore \quad t = \sqrt{\frac{2S}{a}} \quad \text{or} \quad t \propto \frac{1}{\sqrt{a}}$$

$$\therefore \quad \frac{t_1}{t_2} = \sqrt{\frac{a_2}{a_1}}$$

or

$$\frac{t}{2t} = \sqrt{\frac{g\sin\theta - \mu g\cos\theta}{g\sin\theta}}$$

On solving this equation, we get

$$\mu = \frac{3}{4}\tan\theta$$ **Ans.**

18. a_{mr} = acceleration of man relative to rope

$$= a_m - a_r$$

$$\therefore \quad a_m = a_{mr} + a_r$$

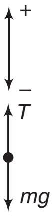
$$= (a) + (a)$$

$$= 2a$$

Now, $\quad T - mg = m(a_m)$

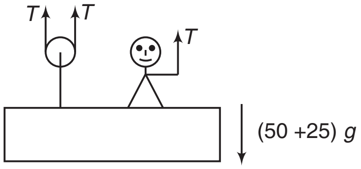
$$= m(2a)$$

$$\therefore \quad T = m(g + 2a)$$ **Ans.**



Note *Man slides down with a deceleration a relative to the rope. So,* $a_{mr} = +a$ *not* $-a$.

19. Steady rate means, net force = 0



$$\therefore \qquad 3T = 75\,g = 750$$

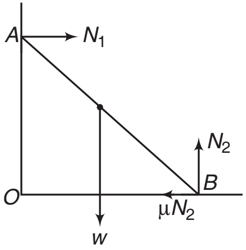
or $\qquad T = 250$ N **Ans.**

20. $N_1 = \mu N_2$ ...(i)

$$N_2 = w \qquad \text{...(ii)}$$

$$\Sigma M_B = 0$$

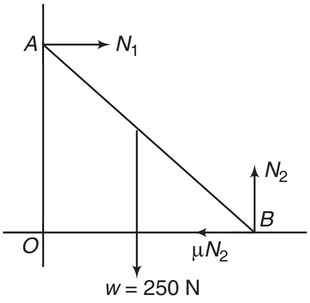
$$\therefore \qquad w\left(\frac{3}{2}\right) = N_1\,(4) \qquad \text{...(iii)}$$



On solving these equations, we get

$$\mu = \frac{3}{8}$$ **Ans.**

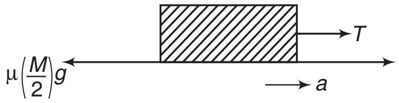
21. In the figure, N_2 is always equal to w or 250 N



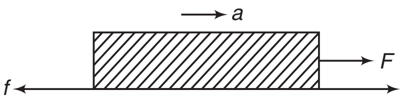
∴ Maximum value of friction available at B is μN_2 or 75 N.

22. $a = \dfrac{F - f}{M}$

$$= \frac{Mg - \mu mg}{M} = \frac{g}{2}$$



Now at the mid point



$$T - \mu\left(\frac{M}{2}\right)g = \frac{M}{2}$$

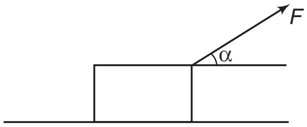
$$a = \left(\frac{M}{2}\right)\left(\frac{g}{2}\right)$$

Putting $\mu = \dfrac{1}{2}$, we get

$$T = \frac{Mg}{2}$$ **Ans.**

23. Angle of friction $\theta = \tan^{-1}(\mu)$

or $\qquad \theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30°$



Suppose the body is dragged by a force F acting at an angle α with horizontal. Then,

$$N = mg - F\sin\alpha$$

and $\qquad F\cos\alpha = \mu N = \mu\,(mg - F\sin\alpha)$

$$\therefore \qquad F = \frac{\mu mg}{\cos\alpha + \mu\sin\alpha} \qquad \text{...(i)}$$

F is minimum when denominator is maximum or

$$\frac{d}{d\alpha}(\cos\alpha + \mu\sin\alpha) = 0$$

or $\qquad -\sin\alpha + \mu\cos\alpha = 0$

or $\qquad \tan\alpha = \mu = \dfrac{1}{\sqrt{3}}$

∴ $\qquad \alpha = 30°$, the angle of friction θ

∴ At $\alpha = 30°$, force needed is minimum. Substituting the values in Eq. (i) we have,

$$F_{\min} = \frac{\left(\frac{1}{\sqrt{3}}\right)(25)\,(g)}{(\sqrt{3}/2) + (1/\sqrt{3})\,(1/2)}$$

$$= 12.5\ \text{g}$$

$$= 12.5\ \text{kg f}$$ **Ans.**

Note *From this example we can draw a conclusion that force needed to move the block is minimum when pulled at an angle equal to angle of friction.*

24. $N = mg = 40$ N

$$\mu N = 32 \text{ N}$$

Applied force $F = 30 \text{ N} < \mu\text{N}$

Therefore friction force,

$$f = F = 30 \text{ N}$$

∴ Net contact force from ground on block

$$= \sqrt{N^2 + f^2} = 50 \text{ N}$$ **Ans.**

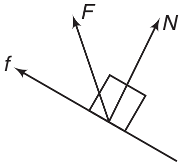
25. Maximum value of friction between B and ground,

$$= \mu N = \mu\,(m_A + m_B)\,g$$
$$= (0.5)\,(2 + 8)\,(10) = 50 \text{ N}$$

Since applied force $F = 25$ N is less than 50 N. Therefore system will not move and force of friction between A and B is zero.

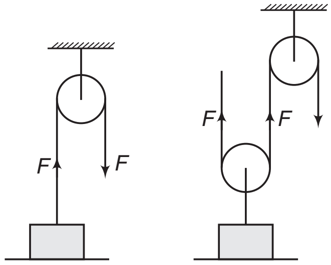
26. $mg \sin\theta = (10)\,(10)\,(3/5) = 60$ N

This 60 N > 30 N. Therefore friction force f will act in upward direction.



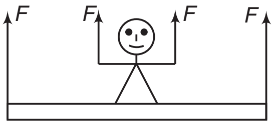
$F = \sqrt{N^2 + f^2}$ = net force by plane on the block.

27. If force applied by man is F. Then in first figure force transferred to the block is F, while in second figure force transferred to the block is $2F$.



(i) (ii)

28. Total upward force $= 4\left(\dfrac{mg}{2}\right) = 2\,mg$



Total downward force $= (m + m)\,g = 2\,mg$

∴ Net force = 0

$$F = \frac{mg}{2}$$

29. Maximum friction between two,

$$f_{\max} = \mu N = \mu mg$$

upper block moves only due to friction. Therefore its maximum acceleration may be,

$$a_{\max} = \frac{f_{\max}}{m} = \mu g$$

Relative motion between them will start when common acceleration becomes μg.

$$\therefore \quad \mu g = \frac{\text{Net force}}{\text{Total mass}} = \frac{at}{2m}$$

$$\therefore \quad t = \frac{2\mu\, mg}{a}$$ **Ans.**

30. $a_1 = g \sin 30° = g/2$

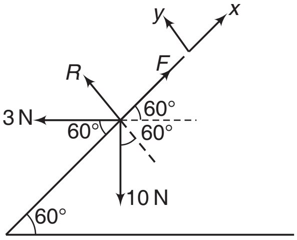
$$a_2 = g \sin 60°$$
$$= \sqrt{3}\, g/2$$
$$a_r = |\mathbf{a}_1 - \mathbf{a}_2|$$

Angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ is 30°

$$\therefore \quad a_r = \sqrt{a_1^2 + a_2^2 - 2\,a_1 a_2 \cos 30°} = \frac{g}{2}$$

Subjective Questions

1.



$$\Sigma F_x = 0$$

$$\therefore \quad F - 3\cos 60° - 10 \sin 60° = 0$$

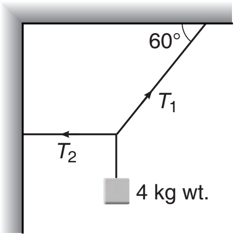
$$\therefore \quad F = 10.16 \text{ N}$$ **Ans.**

$$\Sigma F_y = 0$$

$$\therefore \quad R + 3 \sin 60° - 10 \cos 60° = 0$$

$$\therefore \quad R = 2.4 \text{ N}$$ **Ans.**

2. Resolving the tension T_1 along horizontal and vertical directions. As the body is in equilibrium,



$$T_1 \sin 60° = 4 \times 9.8 \text{ N} \qquad \text{...(i)}$$

$$T_1 \cos 60° = T_2 \qquad \text{...(ii)}$$

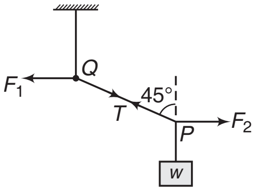
$$T_1 = \frac{4 \times 9.8}{\sin 60°} = \frac{4 \times 9.8 \times 2}{\sqrt{3}}$$

$$= 45.26 \text{ N} \qquad \textbf{Ans.}$$

$$T_2 = T_1 \cos 60° = 45.26 \times 0.5$$

$$= 22.63 \text{ N} \qquad \textbf{Ans.}$$

3. (a) **At *P***



At *P*, $F_2 = T \cos 45° = \dfrac{T}{\sqrt{2}}$ ...(i)

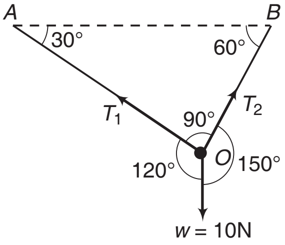
$w = T \cos 45° = \dfrac{T}{\sqrt{2}}$ ...(ii)

At *Q*, $F_1 = \dfrac{T}{\sqrt{2}}$ ...(iii)

From these three equations, we can see that

$$F_1 = F_2 = w = \frac{T}{\sqrt{2}} = \frac{60}{\sqrt{2}} = 30\sqrt{2} \text{ N} \qquad \textbf{Ans.}$$

4. Various forces acting on the ball are as shown in figure. The three concurrent forces are in equilibrium. Using Lami's theorem,



$$\frac{T_1}{\sin 150°} = \frac{T_2}{\sin 120°} = \frac{10}{\sin 90°}$$

or $$\frac{T_1}{\sin 30°} = \frac{T_2}{\sin 60°} = \frac{10}{1}$$

$\therefore$ $T_1 = 10 \sin 30° = 10 \times 0.5$

$= 5 \text{ N}$ **Ans.**

and $T_2 = 10 \sin 60° = 10 \times \dfrac{\sqrt{3}}{2}$

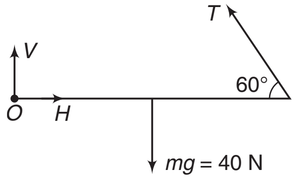
$= 5\sqrt{3} \text{ N}$ **Ans.**

5. $H = T \cos 60° = \dfrac{T}{2}$ ...(i)

$V + T \sin 60° = 40$ ...(ii)

$$\Sigma M_0 = 0$$

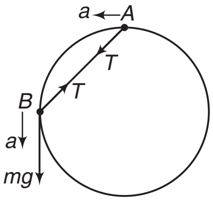
$\therefore$ $40\left(\dfrac{l}{2}\right) = T\ (l \sin 60°)$ or $T = \dfrac{40}{\sqrt{3}}$ N



$H = \dfrac{20}{\sqrt{3}}$ N and $V = 20$ N

$\therefore$ Net hinge force $= \sqrt{H^2 + V^2} = \dfrac{40}{\sqrt{3}}$ N **Ans.**

6.



For A $T \cos 45° = ma$

or $T = \sqrt{2}\, ma$ ...(i)

For B $mg - T \cos 45° = ma$

$\therefore$ $mg - ma = ma$ or $a = \dfrac{g}{2}$

Substituting in Eq. (i), we get $T = \dfrac{mg}{\sqrt{2}}$

7. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{8}{1 + 2 + 1}$

$= 2 \text{ m/s}^2$

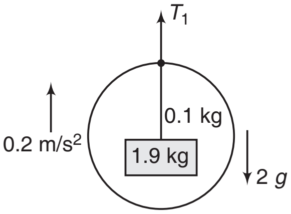
$\boldsymbol{T_B}$ $8 - T_B = m_1 a = (1)\,(2)$

$\therefore$ $T_B = 6$ N

$\boldsymbol{T_A}$ $T_A = m_2 a = (2)\,(2)$

$\therefore$ $= 4$ N **Ans.**

8. (a) $T_1 - 2g = 2a$

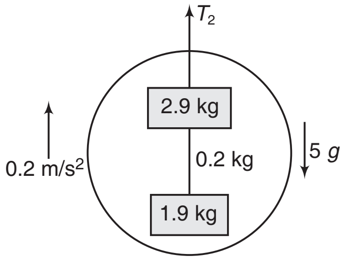


$\therefore$ $T_1 = 2\,(g + a)$

$= 2\,(9.8 + 0.2)$

$= 20$ N

(b) $T_2 - 5g = 5a$



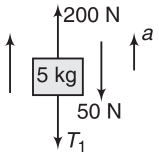
$$\therefore \quad T_2 = 5\,(g + a)$$
$$= 5\,(9.8 + 0.2)$$
$$= 50 \text{ N}$$

9. (a) $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$

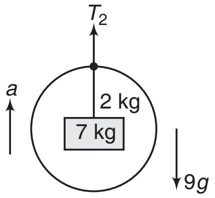
$$= \frac{200 - 16 \times 9.8}{16}$$
$$= 2.7 \text{ m/s}^2$$

(b) $200 - 50 - T_1 = 5a$

$$\therefore \quad T_1 = 200 - 5 \times 2.7 - 50$$
$$= 136.5 \text{ N}$$



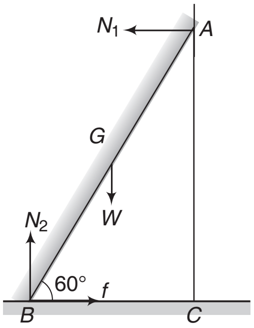
(c) $T_2 - 9g = 9a$



$$\therefore \quad T_2 = 9\,(g + a)$$
$$= 9\,(9.8 + 2.7)$$
$$= 112.5 \text{ N}$$

10. In figure, AB is a ladder of weight w which acts at its centre of gravity G.

$$\angle ABC = 60°$$
$$\therefore \quad \angle BAC = 30°$$



Let N_1 be the reaction of the wall, and N_2 the reaction of the ground. Force of friction f between the ladder and the ground acts along BC.

For horizontal equilibrium,

$$f = N_1 \qquad \text{...(i)}$$

For vertical equilibrium,

$$N_2 = w \qquad \text{...(ii)}$$

Taking moments about B, we get for equilibrium,

$$N_1(4 \cos 30°) - w(2 \cos 60°) = 0 \qquad \text{...(iii)}$$

Here, $w = 250 \text{N}$

Solving these three equations, we get

$$f = 72.17 \text{ N}$$

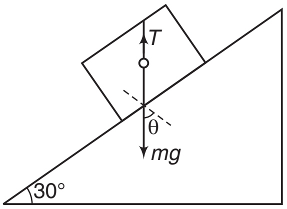
and $N_2 = 250 \text{ N}$

$$\therefore \quad \mu = \frac{f}{N_2} = \frac{72.17}{250}$$
$$= 0.288 \qquad \textbf{Ans.}$$

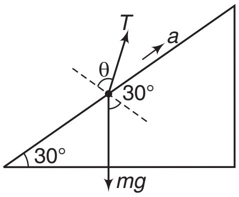
11. Constant velocity means net acceleration $= 0$. Therefore, net force should be zero. Only two forces T and mg are acting on the bob. So they should be equal and opposite.

Asked angle $\theta = 30°$

$$T = mg = (1)\,(10) = 10 \text{ N}$$



12. $T \sin\theta - mg \sin 30° = ma = \dfrac{mg}{2}$



$$\therefore \quad T \sin\theta = mg \qquad \text{...(i)}$$
$$T \cos\theta = mg \cos 30°$$

or $T \cos\theta = \dfrac{\sqrt{3}\, mg}{2}$ ...(ii)

Solving Eqs. (i) and (ii), we get

$$\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) \qquad \textbf{Ans.}$$

and $T = \dfrac{\sqrt{7}}{2}\, mg$

$$= 5\sqrt{7} \text{ N} \qquad \textbf{Ans.}$$

13. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$

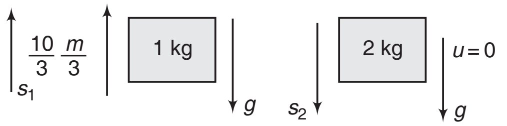
$$= \frac{2g - (1)\,g}{2+1}$$

$$= \frac{g}{3} = \frac{10}{3}\ \text{m/s}^2$$

After 1 s,

$$v = at = \frac{10}{3}\ \text{m/s}$$

At this moment string slacks (T` = 0)



String is again tight when,

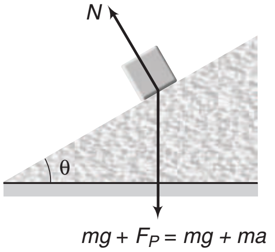
$$s_1 = s_2$$

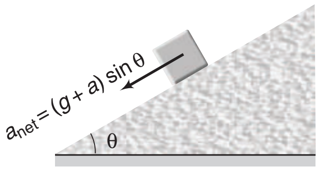
$$\therefore \quad \frac{10}{3}t - \frac{1}{2}gt^2 = \frac{1}{2}gt^2 \qquad (g = 10\ \text{m/s}^2)$$

On solving we get,

$$t = \frac{1}{3}\ \text{s}$$ **Ans.**

14. Since, acceleration of block w.r.t. wedge (an accelerating or non-inertial frame of reference) is to be find out.





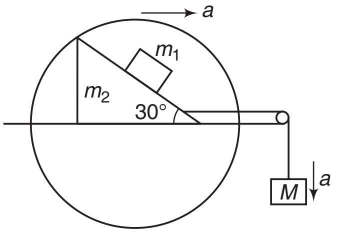
FBD of 'block' w.r.t. 'wedge' is shown in figure.

The acceleration would had been g sin θ (down the plane) if the lift were stationary or when only weight (i.e. mg) acts downwards.

Here, downward force is $m(g + a)$.

∴ Acceleration of the block (of course w.r.t. wedge) will be $(g + a)$ sin θ down the plane.

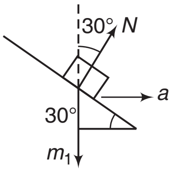
15. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$



$$a = \frac{Mg}{m_1 + m_2 + M} \qquad \text{...(i)}$$

m_1 $\quad N \cos 30° = m_1 g \qquad$ ...(ii)

$N \sin 30° = m_1 a \qquad$ ...(iii)



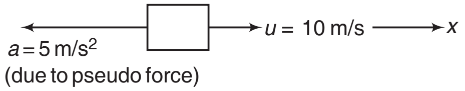
From Eqs. (ii) and (iii), we get

$$a = g \tan 30° = \frac{g}{\sqrt{3}}$$

Substituting this value in Eq. (i) we get,

$$M = 6.83\ \text{kg}$$ **Ans.**

16. (a) **With respect to box** (Non-inertial)



$$x = x_0 + u_x t + \frac{1}{2}a_x t^2$$

$$= x_0 + 10t - 2.5t^2$$ **Ans.**

$$v_x = u_x + a_x t = 10 - 5t$$ **Ans.**

(b) $v_x = 0$ at $2\text{s} = t_0$ (say)

∴ To return to the original position,

time taken $= 2t_0 = 4$ s **Ans.**

17. (a) **In car's frame** (non-inertial)

$a_x = -5\ \text{m/s}^2$ (due to pseudo force)

$u_x = 0$

$a_z = 0$

$u_z = 10\ \text{m/s}$

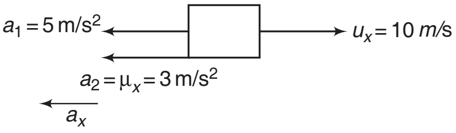
Now apply, $v = u + at$ and $s = s_0 + ut + \frac{1}{2}at^2$

in x and z-directions.

(b) **In ground frame** (inertial)

$$a_x = 0,\ u_x = 0,\ a_z = 0 \text{ and } u_z = 10\ \text{m/s}$$

18. **Relative to car** (non-inertial)



a_1 is due to pseudo force

$$a_x = -(5 + 3) = -8\ \text{m/s}^2$$

Block will stop when

$$v_x = 0 = u_x + a_x t = 10 - 8t$$

or at $t = 1.25$ s

So, for $t \le 1.25$ s

$$x = x_0 + u_x t + \frac{1}{2}a_x t^2$$

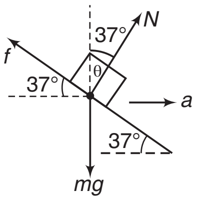
$$= x_0 + 10t - 4t^2$$

$$v_x = u_x + a_x t = 10 - 8t$$

After this $\mu_s g > 5\ \text{m/s}^2$ as $\mu_s > 0.5$

Therefore, now the block remains stationary with respect to car.

19.



$N \cos 37° + f \sin 37° = mg$ ...(i)

$N \sin 37° - f \cos 37° = ma$ ...(ii)

On solving these two equations, we get

$$f = 3.6\ m$$

$$= \frac{9}{25}mg$$ **Ans.**

20. $N = mg \cos 60°$

$$= (6)(10)\left(\frac{1}{2}\right) = 30\ \text{N}$$

$$\mu_S N = 18\ \text{N}$$

$$\mu_K N = 12\ \text{N}$$

Driving force $F = mg \sin 60°$

= force responsible to move the body downwards

$$= (6)(10)\left(\frac{\sqrt{3}}{2}\right)$$

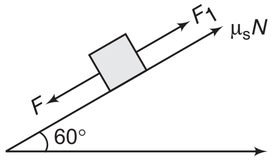
$$= 52\ \text{N}$$

(a) Force needed to keep the block stationary is

$F_1 = F - \mu_S N$ (upwards)

$= 52 - 18$

$= 34$ N (upwards) **Ans.**



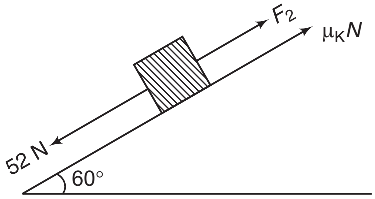
(b) If the block moves downwards with constant velocity ($a = 0$, $F_{net} = 0$), then kinetic friction will act in upward direction.

∴ Force needed,

$F_2 = F - \mu_K N$ (upwards)

$= 52 - 12$

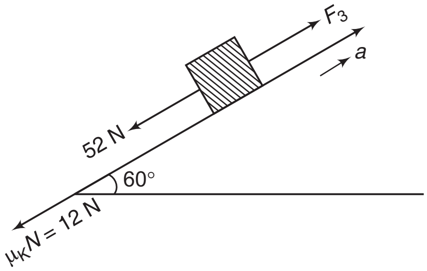
$= 40$ N (upwards) **Ans.**



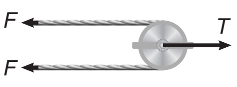
(c) Kinetic friction will act in downward direction

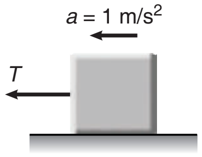
$$F_3 - 52 - 12 = ma = (6)(4)$$

∴ $F_3 = 88$ N (upwards) **Ans.**



21. As shown in Fig. when force F is applied at the end of the string, the tension in the lower part of the





string is also F. If T is the tension in string connecting the pulley and the block, then,

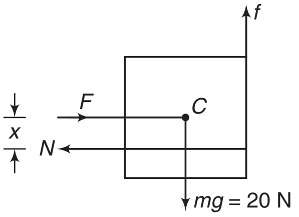
$$T = 2F$$

But $\quad T = ma = (200)(1) = 200 \text{ N}$

$\therefore \quad 2F = 200 \text{ N}$

or $\quad F = 100 \text{ N}$

22. $N = F = 40 \text{ N}$



Net moment about $C = 0$

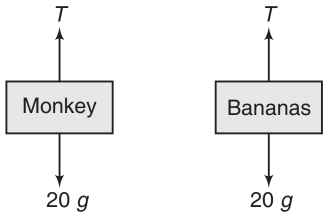
$\therefore$ Anti-clockwise moment of

f = clockwise moment of N

$$\therefore \quad (20)\left(\frac{20}{2}\right) = (40) \cdot x$$

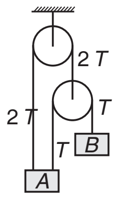
or $\quad x = 5 \text{ cm}$ **Ans.**

23. Force diagram on both sides is always similar. Therefore motion of both sides is always similar. For example, if monkey accelerates upwards, then $T > 20\,g$. But same T is on RHS also.



Therefore, bananas also accelerate upwards.

24. $T_A = 3T, T_B = T$



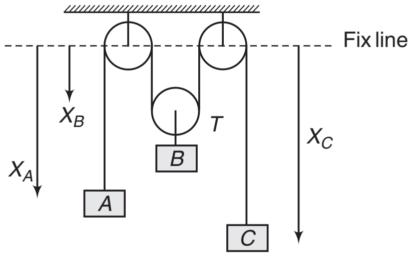
In such type of problems

$$a \propto \frac{1}{T}$$

$$\therefore \quad |a_B| = 3\,|a_A|$$

or $\quad a_B = -3a_A$, as a_A and a_B are in opposite directions.

25. $X_A + 2X_B + X_C$ = constant on double differentiating with respect to time, we get,



$$a_A + 2a_B + a_C = 0 \qquad \textbf{Ans.}$$

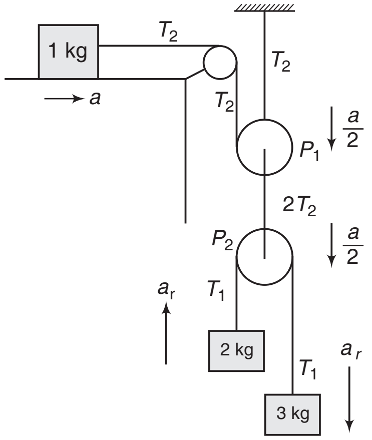
26. $P_2 \quad 2T_1 = 2T_2$

$\therefore \qquad T_1 = T_2 \qquad$ ...(i)

1 kg $\qquad T_2 = (1)(a) \qquad$ ...(ii)

2 kg $\qquad T_1 - 20 = 2\left(a_r - \dfrac{a}{2}\right) \qquad$ ...(iii)

3 kg $\qquad 30 - T_1 = 3\left(a_r + \dfrac{a}{2}\right) \qquad$ ...(iv)



On solving these equations, we get

$$T_1 = T_2 = \frac{120}{11} \text{ N}$$

$$a_1 = a = \frac{120}{11} \text{ m/s}^2$$

$$a_2 = a_r - \frac{a}{2} = -\frac{50}{11} \text{ m/s}^2$$

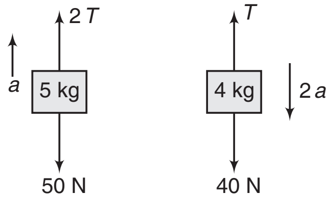
or $\qquad a_2 = \dfrac{50}{11} \text{ m/s}^2$ (downwards)

$$a_3 = a_r + \frac{a}{2} = \frac{70}{11} \text{ m/s}^2 \quad \text{(downwards)} \quad \textbf{Ans.}$$

27. $2T - 50 = 5a$ ...(i)

$40 - T = 4\,(2a)$ ...(ii)

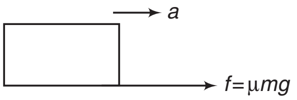
On solving these equations, we get



$$a = \frac{10}{7}\ \text{m/s}^2 \quad \text{or} \quad \frac{g}{7}$$ **Ans.**

and $$2a = \frac{20}{7}\ \text{m/s}^2 \quad \text{or} \quad \frac{2g}{7}$$ **Ans.**

28. $a = \dfrac{f}{m} = \mu g = 3\ \text{m/s}^2$



(a) Relative motion will stop when velocity of block also becomes 6 m/s by the above acceleration.

$$v = at$$

$$\therefore \quad t = \frac{v}{a} = \frac{6}{3} = 2\ \text{s}$$ **Ans.**

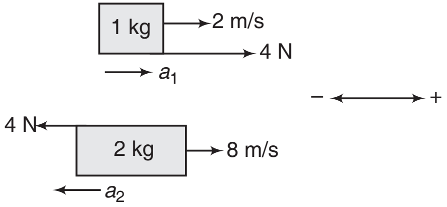
(b) $S = \dfrac{1}{2}at^2 = \dfrac{1}{2}(3)(2)^2 = 6\ \text{m}$ **Ans.**

29. 2 kg block has relative motion towards right. Therefore, maximum friction will acts on it towards left.

$$f = \mu N = (0.4)(1)(10) = 4\ \text{N}$$

$$a_1 = \frac{4}{1} = 4\ \text{m/s}^2$$

$$a_2 = -\frac{4}{2} = -2\ \text{m/s}^2$$



(a) Relative motion between them will stop when,

$$v_1 = v_2$$

$$\therefore \quad u_1 + a_1 t = u_2 + a_2 t \quad \text{...(i)}$$

$$\therefore \quad 2 + 4t = 8 - 2t \quad \text{...(ii)}$$

or $$t = 1\ \text{s}$$ **Ans.**

(b) Substituting value of 't' in Eq. (ii), either on RHS or on LHS, common velocity = 6 m/s **Ans.**

(c) $$s_1 = u_1 t + \frac{1}{2}a_1 t^2$$

$$= (2)(1) + \frac{1}{2}(4)(1)^2 = 4\ \text{m}$$ **Ans.**

$$s_2 = u_2 t + \frac{1}{2}a_2 t^2$$

$$= (8)(1) + \frac{1}{2}(-2)(1)^2 = 7\ \text{m}$$ **Ans.**

30. $N = 20\ \text{N}$ = applied force

$$\mu_S N = 16\ \text{N}$$

and $$\mu_K N = 12\ \text{N}$$

Driving force $F = mg$ (downwards)

Since, $F > \mu_S N$, block will slide downwards and kinetic friction of 12 N will act in upward direction

$$\therefore \quad a = \frac{F - \mu_K N}{m} \quad \text{(downwards)}$$

$$= \frac{20 - 12}{2} = 4\ \text{m/s}^2 \quad \text{(downwards)}$$ **Ans.**

31. $$f_{\max} = \mu N = \mu mg$$

$$= 0.8 \times 2 \times 10 = 16\ \text{N}$$

Block will not move till driving force $F = 2t$ becomes 16 N. This force becomes 16 N in 8 s

For $t \le 8$ s

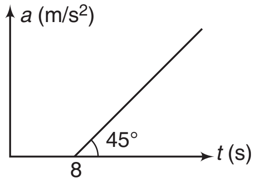
$$a = 0$$

For $t \ge 8$ s

$$a = \frac{F - f_{\max}}{m} = \frac{2t - 16}{2} \Rightarrow a = t - 8$$

i.e. a - t graph is a straight line of slope +1 and intercept −8.

Corresponding a - t graph is shown below



32. $N = mg = 60\ \text{N}$

$$\mu_S N = 36\ \text{N}$$

$$\mu_K N = 24\ \text{N}$$

Driving force $F = 4t$

Block will move when,

$$F = \mu_S N \quad \text{or} \quad 4t = 36 \Rightarrow t = 9\ \text{s}$$

For $t \le 9$ s

$$a = 0$$

For $t \geq 9$ s

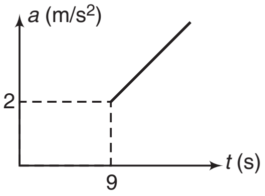
At 9 s block will start moving. Therefore kinetic friction will act

$$\therefore \quad a = \frac{F - \mu_K N}{m}$$
$$= \frac{4t - 24}{6}$$
$$= \left(\frac{2}{3}t - 4\right)$$

$\therefore$ a - t graph is a straight line

At $t = 9$ s , $a = 2$ m/s^2

The corresponding a - t graph is as shown in figure.



LEVEL 2

Single Correct Option

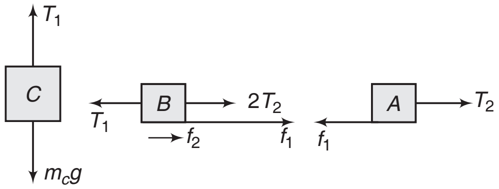
1. Maximum value of friction between A and B

$$(f_1)_{\max} = \mu N_1 = \mu m_A g$$
$$= 0.3 \times 50 \times 10$$
$$= 150 \text{ N}$$

Maximum value of friction between B and ground

$$(f_2)_{\max} = \mu N_2 = \mu\,(m_A + m_B)\,g$$
$$= (0.3)\,(120)\,(10) = 360 \text{ N}$$

Force diagram is as shown below



$$T_2 = (f_1)_{\max} = 150 \text{ N}$$
$$T_1 = 2T_2 + (f_1)_{\max} + (f_2)_{\max}$$
$$= 300 + 150 + 360 = 810 \text{ N}$$
$$T_1 = m_c g$$
$$\therefore \quad m_c = \frac{T_1}{g} = \frac{810}{10} = 81 \text{ kg} \qquad \textbf{Ans.}$$

2. $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = (8\hat{\mathbf{i}} - 4t\hat{\mathbf{j}})$

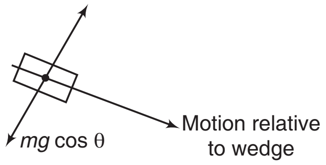
At 1 s $\quad \mathbf{F}_{\text{net}} = m\mathbf{a} = (1)\,(8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) = (8\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$
$$= \mathbf{W} + \mathbf{F}$$

where, **F** = force on cube

$$\therefore \quad \mathbf{F} = (8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) - \mathbf{w}$$
$$= (8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) - (-10\hat{\mathbf{j}})$$
$$= (8\hat{\mathbf{i}} + 6\hat{\mathbf{j}})$$

or
$$|\mathbf{F}| = \sqrt{(8)^2 + (6)^2}$$
$$= 10 \text{ N} \qquad \textbf{Ans.}$$

3. During motion of block, a component of its acceleration comes in the direction of $mg\cos\theta$. Therefore,



$$mg\cos\theta > N$$

4. Maximum friction available to m_2 is

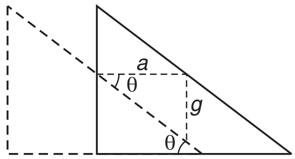
$$(f_{\max}) = \mu m_2 g$$

Therefore maximum acceleration which can be provided to m_2 by friction, (without the help of normal reaction from m_1) is

$$a_{\max} = \frac{f_{\max}}{m_2} = \mu g$$

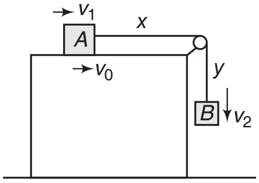
If $a > \mu g$, normal reaction from m_1 (on m_2) is non zero.

5. $\dfrac{a}{g} = \cot\theta$



$$\therefore \quad a = g\cot\theta \qquad \textbf{Ans.}$$

6. $x + y =$ constant



$$\therefore \quad \frac{dx}{dt} + \frac{dy}{dt} = 0$$

or
$$\left(-\frac{dx}{dt}\right) = \left(\frac{dy}{dt}\right)$$
$$\therefore \quad v_1 - v_0 = v_2$$

or
$$v_1 - v_2 = v_0 \qquad \textbf{Ans.}$$

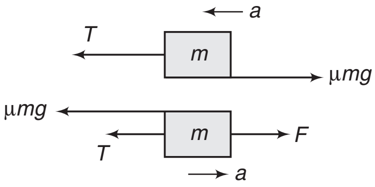
7. $$a_1 = \frac{m_2 g}{m_1 + m_2} = \frac{30}{7}\ \text{m/s}^2$$

$$a_2 = \frac{(m_1 - m_2)\,g}{m_1 + m_2} = \frac{10}{7}\ \text{m/s}^2$$

$$a_3 = \frac{m_2 g - m_1 g \sin 30°}{m_1 + m_2} = \frac{10}{7}\ \text{m/s}^2$$

$\therefore$ $a_1 > a_2 = a_3$ **Ans.**

8. $T - \mu mg = ma$



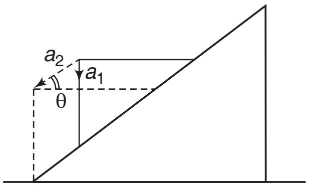
$\therefore$ $T = \mu mg + ma$

$F - T - \mu mg = ma$

$\therefore$ $F - \mu mg - ma - \mu mg = ma$

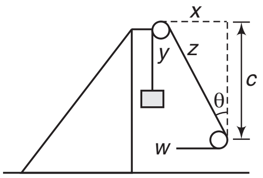
or $$a = \frac{F}{2m} - \mu g$$ **Ans.**

9. $\dfrac{a_1}{a_2} = \sin\theta$



$\therefore$ $a_1 = a_2 \sin\theta$

10. $z = \sqrt{x^2 + c^2}$



Now, $w + y + z = l$

or $w + y + \sqrt{x^2 + c^2} = l$

$$\therefore \quad \frac{dw}{dt} + \frac{dy}{dt} + \frac{x}{\sqrt{x^2 + c^2}} \cdot \frac{dx}{dt} = 0$$

or $$\left(-\frac{dw}{dt}\right) + \frac{x}{z}\left(-\frac{dx}{dt}\right) = \frac{dy}{dt} \quad \text{...(i)}$$

$$-\frac{dw}{dt} = -\frac{dx}{dt} = v_2$$

$$\frac{dy}{dt} = v_1$$

and $$\frac{x}{z} = \sin\theta$$

Substituting these values in Eq. (i) we have

$v_2\,(1 + \sin\theta) = v_1$ **Ans.**

11. **On the cylinder**

If N = normal reaction between cylinder and inclined plane

$N \sin\theta$ = horizontal component of N

$= ma$ ....(i)

$N \cos\theta$ = vertical component of N

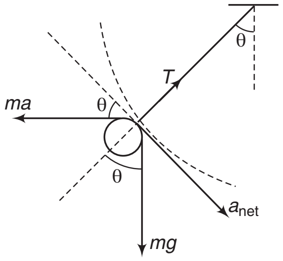
$= mg$ ...(ii)

Dividing Eq. (i) by (ii) we get,

$$\tan\theta = \frac{a}{g}$$

$\therefore$ $a = g \tan\theta$ **Ans.**

12. With respect to trolley means, assume trolley at rest and apply a pseudo force (= ma, towards left) on the bob.



$$a_{\text{net}} = \frac{mg \sin\theta - ma \cos\theta}{m}$$

$= g \sin\theta - a \cos\theta$ **Ans.**

13. $$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{(M - m)\,g}{(M + m)} \quad \text{...(i)}$$

Since, $M >> m$

$\therefore$ $M - m \approx M + m = M$

Substituting in Eq. (i), we have

$a \approx g$

FBD of *m* $T - mg = ma = mg$

$\therefore$ $T = 2\,mg$ **Ans.**

14. Let α = angle of repose

For $\theta \le \alpha$ Block is stationary and force of friction,

$$f = mg \sin\theta$$

or $\quad f \propto \sin\theta$

i.e. it is sine graph

For $\theta \ge \alpha$ Block slides downwards

$\therefore \quad f = \mu mg \cos\theta$

or $\quad f \propto \cos\theta$

i.e. now it is cosine graph

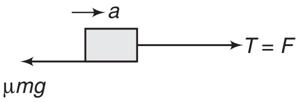
The correct alternative is therefore (b).

15. If they do not slip, then net force on system = F

$\therefore \quad$ Acceleration of system $a = \dfrac{F}{3m}$

$$T = F$$

FBD of *m* $\quad F - \mu mg = ma = \dfrac{F}{3}$



$\therefore \quad \mu mg = \dfrac{2F}{3} \quad$ or $\quad \mu = \dfrac{2F}{3mg}$ **Ans.**

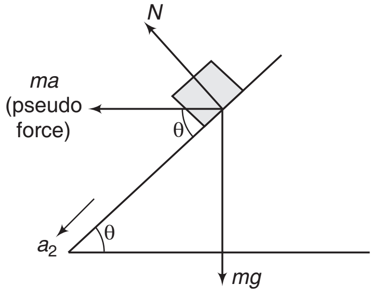
16. **FBD of *m* w.r.t. chamber**

Relative acceleration along the inclined plane

$$a_r = \frac{ma\cos\theta + mg\sin\theta}{m}$$

$$= (a\cos\theta + g\sin\theta)$$

$$t = \sqrt{\frac{2s}{a_r}} = \sqrt{\frac{2L}{a\cos\theta + g\sin\theta}}$$ **Ans.**



17. **FBD of *m* w.r.t. wedge**

Relative acceleration along the inclined plane

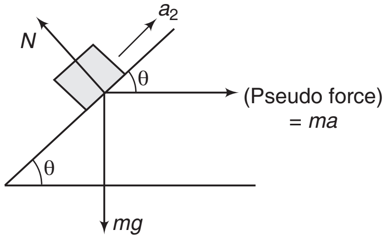
$$a_r = \frac{ma\cos\theta - mg\sin\theta}{m}$$

$$= a\cos\theta - g\sin\theta$$

$$= (10\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) - (10)\left(\frac{1}{2}\right)$$

$$= 10 \text{ m/s}^2$$

$$t = \sqrt{\frac{2s}{a_r}} = \sqrt{\frac{2 \times 1}{10}} = \frac{1}{\sqrt{5}} \text{ s}$$ **Ans.**

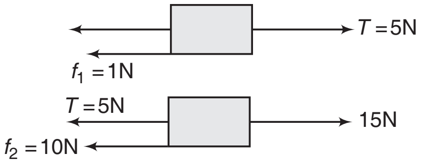


18. $(f_1)_{\max} = \mu_1 m_1 g = 6$ N

$(f_2)_{\max} = \mu_2 m_2 g = 10$ N

At $\quad t = 2$ s, $F' = 4$ N

Net pulling force, $F_1 = F` - F' = 11$ N



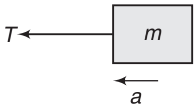
Total maximum resisting force,

$$F_2 = (f_1)_{\max} + (f_2)_{\max} = 16 \text{ N}$$

Since $F_1 < F_2$, system will not move and free body diagrams of the two block are as shown in figure.

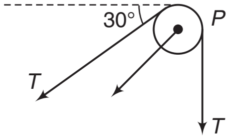
19. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$

$$= \frac{2mg\sin 30^\circ}{2m + m} = \frac{g}{3}$$



FBD of *m*

$$T = ma = mg/3$$



Resultant of tensions

$$\mathbf{R} = -(T\cos 30^\circ)\,\hat{\mathbf{i}} - (T\sin 30^\circ + T)\,\hat{\mathbf{j}}$$

Putting $T = mg/3$

$$\mathbf{R} = -\frac{\sqrt{3}}{6}\, mg\,\hat{\mathbf{i}} - \frac{mg}{2}\,\hat{\mathbf{j}}$$

Since, pulley P is in equilibrium. Therefore,

$$\mathbf{F} + \mathbf{R} = 0$$

where, $\mathbf{F}$ = force applied by clamp on pulley

$$\therefore \qquad \mathbf{F} = -\,\mathbf{R} = \frac{mg}{6}\,(\sqrt{3}\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}) \qquad \textbf{Ans.}$$

20. $(f_{2\text{ kg}})_{\max} = \mu_2 m_2 g = 0.6 \times 2 \times 10 = 12$ N

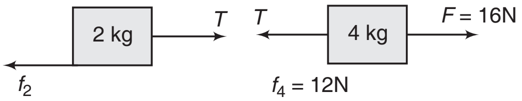
$(f_{4\text{ kg}})_{\max} = \mu_4 m_4 g = 0.3 \times 4 \times 10 = 12$ N

Net pulling force $F = 16$ N and

Net resistive force $F' = (f_{2\text{ kg}})_{\max} + (f_{4\text{ kg}})_{\max}$

$= 24$ N

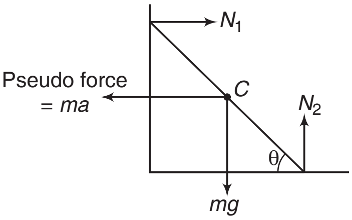
Since, $F < F'$, system will not move and free body diagrams of two blocks are as shown below.



4 kg $\qquad T + 12 = 16 \Rightarrow T = 4$ N

2 kg $\qquad f_2 = T = 4$ N $\qquad$ **Ans.**

21. **FBD of rod w.r. t. trolley**



$$N_1 = ma \qquad \text{...(i)}$$

$$N_2 = mg \qquad \text{...(ii)}$$

Net torque about point $C = 0$

$$\therefore \qquad N_1\left(\frac{l}{2}\sin\theta\right) = N_2\left(\frac{l}{2}\cos\theta\right)$$

or $\qquad (ma)(\sin\theta) = (mg)(\cos\theta)$

$\Rightarrow \qquad a = g\cot\theta \qquad$ **Ans.**

22. $v = 2t^2$

$$\therefore \qquad a = \frac{dv}{dt} = 4t$$

At $\qquad t = 1$ s,

$$a = 4 \text{ m/s}^2 \qquad \text{...(i)}$$

Limiting value of static friction,

$$f_L = \mu_S\, mg$$

$\therefore$ Maximum value of acceleration of coin which can be provided by the friction,

$$a_{\max} = \frac{f_L}{m} = \mu_S\, g \qquad \text{...(ii)}$$

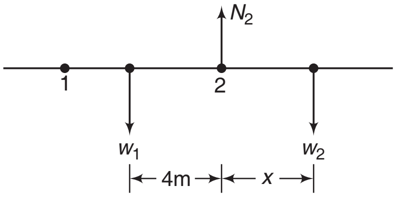
Equating Eqs. (i) and (ii), we get

$$4 = \mu_S\,(10)$$

$$\therefore \qquad \mu_S = 0.4 \qquad \textbf{Ans.}$$

23. Just at the time of tipping, normal reaction at 1 will become zero.

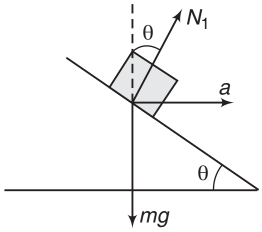
Σ (moments of all forces about point 2) = 0



$\therefore \qquad w_1\,(4) = w_2\,(x)$

$$\text{or} \qquad x = \frac{4w_1}{w_2} = \frac{4\,(10\,g)}{80\,g} = \frac{1}{2}\,m \qquad \textbf{Ans.}$$

24. In vertical direction, net force = 0



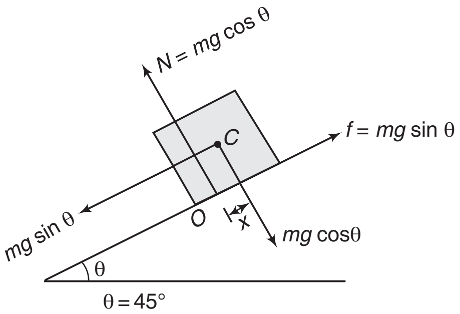
$$\therefore \qquad N_1\cos\theta = mg \quad \text{or} \quad N_1 = \frac{mg}{\cos\theta} \qquad \text{...(i)}$$

Under normal condition, normal reaction is,

$$N_2 = mg\cos\theta \qquad \text{...(ii)}$$

$$\therefore \qquad \frac{N_2}{N_1} = \cos^2\theta \qquad \textbf{Ans.}$$

25. Σ (moments of all forces about point C) = 0



$$\therefore \qquad f\left(\frac{a}{2}\right) = N\,(x)$$

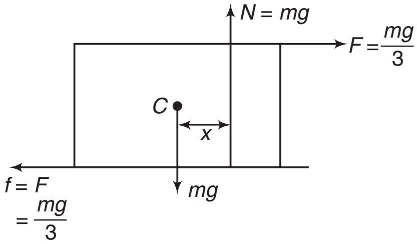
or $$(mg\sin\theta)\left(\frac{a}{2}\right) = (mg\cos\theta)\,x$$

At 45°, $\sin\theta = \cos\theta$

$$\therefore \qquad x = \frac{a}{2}$$

Hence, the normal reaction passes through O.

26.



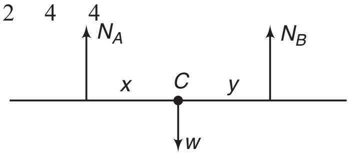
Σ (moments of all forces about point C) = 0

$$\therefore \qquad Nx = F\cdot\frac{a}{2} + f\cdot\frac{a}{2}$$

or $$(mg)\,x = \frac{mg}{3}\left(\frac{a}{2}\right) + \frac{mg}{3}\left(\frac{a}{2}\right)$$

or $$x = \frac{a}{3}$$ **Ans.**

27. $x = \frac{l}{2} - \frac{l}{4} = \frac{l}{4}$



$$y = \frac{l}{2} - \frac{l}{6} = \frac{l}{3}$$

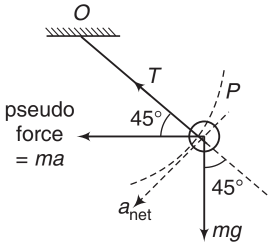
Σ (moments of all forces about point C) = 0

$$\therefore \qquad N_A\,x = N_B\,y$$

or $$\frac{N_A}{N_B} = \frac{y}{x} = \frac{4}{3}$$ **Ans.**

28. Let acceleration of box at this instant is 'a' (towards right).

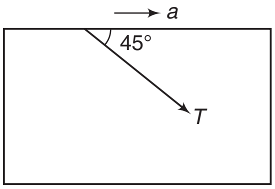
FBD of ball w.r.t. box



Net force in OP direction is zero

$$\therefore \qquad T + \frac{ma}{\sqrt{2}} = \frac{mg}{\sqrt{2}} \qquad \text{...(i)}$$

FBD of box w.r. t. ground



$$T\cos 45° = ma$$

or $$T = \sqrt{2}\,ma$$

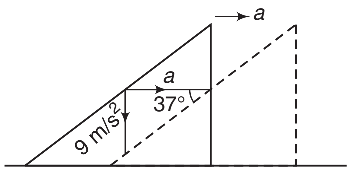
Substituting in Eq. (i), we get

$$a = \frac{g}{3}$$ **Ans.**

29. $\frac{9}{a} = \tan 37° = \frac{3}{4}$

$$\therefore \qquad a = 12\ \text{m/s}^2$$

Let N = normal reaction between rod and wedge. Then $N\sin 37°$ will provide the necessary ma force to the wedge



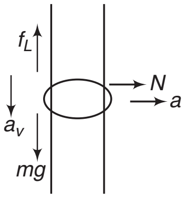
$$\therefore \qquad N\sin 37° = ma = (10)\,(12) = 120$$

$$\therefore \qquad N = \frac{120}{\sin 37°} = \frac{120}{0.6}$$

$$= 200\ \text{N}$$ **Ans.**

30. $N = ma$

$f_L = \mu N = 0.5\,ma$



Vertical acceleration,

$$a_v = \frac{mg - f_L}{m} = g - 0.5a$$

$$= 10 - 0.5\times 4$$

$$= 8\ \text{m/s}^2$$

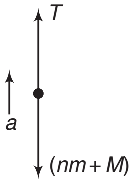
Applying $s = \frac{1}{2}at^2$

or $t = \sqrt{\frac{2s}{a}}$ in vertical direction, we have

$$t = \sqrt{\frac{2\times 1}{8}} = 0.5\ \text{s}$$ **Ans.**

31. $T - (nm + M)g = (nm + M)a$

$\therefore \quad n(mg + ma) = T - Mg - Ma$



or $$n = \frac{T - M(g + a)}{m(g + a)}$$

$$n_{\max} = \frac{T_{\max} - M(g + a)}{m(g + a)}$$

$$= \frac{2 \times 10^4 - 500(10 + 2)}{80(10 + 2)}$$

$$= 14.58$$

But answer will be 14. **Ans.**

32. It implies that the given surface is the path of the given projectile

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$= x\tan 60° - \frac{(10)x^2}{(2)(20)^2\cos^2 60°}$$

$$y = \sqrt{3}\,x - 0.05\,x^2 \qquad \text{...(i)}$$

Slope, $$\frac{dy}{dx} = \sqrt{3} - 0.1\,x \qquad \text{...(ii)}$$

At $y = 5\text{m}$

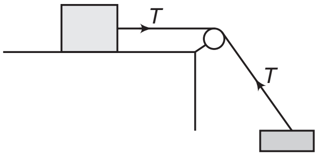
$$5 = \sqrt{3}\,x - 0.05\,x^2$$

or $$0.05\,x^2 - \sqrt{3}\,x + 5 = 0$$

$$x = \frac{\sqrt{3} \pm \sqrt{3 - 1}}{0.1} = \frac{\sqrt{3} \pm \sqrt{2}}{0.1}$$

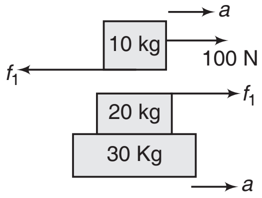
From Eq. (ii) slope at these two points are, $-\sqrt{2}$ and $\sqrt{2}$.

33. Horizontal displacement of both is same ($= l$). Horizontal force on A is complete T. But horizontal force on B is not complete T. It is component of T. So, horizontal acceleration of B will be less.



$$\therefore \qquad t_B > t_A$$

34. Maximum value of friction between 10 kg and 20 kg is



$$(f_1)_{\max} = 0.5 \times 10 \times 10 = 50\text{ N}$$

Maximum value of friction between 20 kg and 30 kg is

$$(f_2)_{\max} = (0.25)(10 + 20)(10) = 75\text{ N}$$

Now let us first assume that 20 kg and 30 kg move as a single block with 10 kg block. So, let us first calculate the requirement of f_1 for this

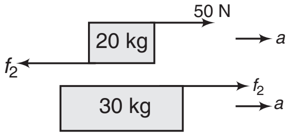
$$100 - f_1 = 10\,a$$

$$f_1 = 50\,a$$

On solving these two equations, we get

$$f_1 = 83.33\text{ N}$$

Since, it is greater than $(f_1)_{\max}$, so there is slip between 10 kg and other two blocks and 50 N will act here.



Now let us check whether there is slip between 20 kg and 30 kg or not. For this we will have to calculate requirement of f_2 for no slip condition.

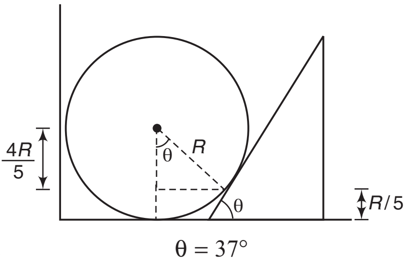
$$50 - f_2 = 20\,a \quad \text{and} \quad f_2 = 30\,a$$

On solving these two equations, we get

$$f_2 = 30\text{ N} \quad \text{and} \quad a = 1\text{ m/s}^2$$

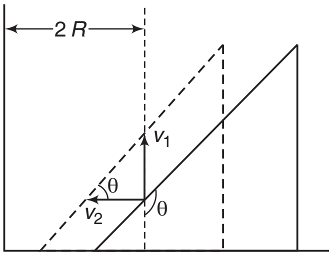
Since, f_2 is less than $(f_2)_{\max}$, so there is no slip between 20 kg and 30 kg and both move together with same acceleration of 1 m/s^2.

35. $$\cos\theta = \frac{4R/5}{R} = 0.8$$



$$\therefore \qquad \theta = 37°$$

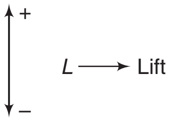
$$\frac{v_1}{v_2} = \tan\theta = \tan 37° = \frac{3}{4}$$



$$\therefore \qquad v_1 = \frac{3}{4}\, v_2 = \frac{3}{4} \times 20$$

$$= 15 \text{ m/s}$$ **Ans.**

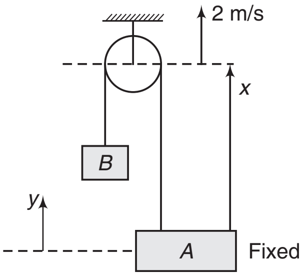
36. $v_{AL} = v_A - v_L$



$$\therefore \qquad v_A = v_{AL} + v_L$$

$$= (-2) + 2 = 0$$

Let z = length of string at some instant. Then,



$$-\frac{dz}{dt} = 2 \text{ m/s} \qquad \text{(Given)}$$

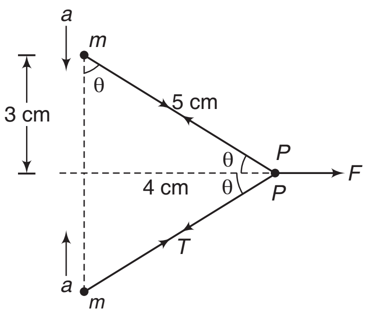
Now, $\quad y = x - (z - x) = 2x - z$

$$\therefore \qquad \frac{dy}{dt} = 2\,\frac{dx}{dt} + \left(-\frac{dz}{dt}\right)$$

$$= 2\,(2 \text{ m/s}) + 2 \text{ m/s}$$

$$= 6 \text{ m/s} = v_B$$ **Ans.**

37. Relative acceleration



$$a_r = 2a$$

$$\therefore \qquad a = \frac{a_r}{2} = 2.5 \text{ m/s}^2$$

At point *P*

$$2T \cos\theta = F$$

$$\therefore \qquad T = \frac{F}{2\cos\theta}$$

At mass *m*

Component of T along the other mass m is $T \sin\theta$

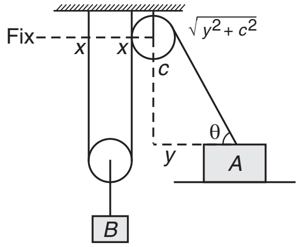
$$\therefore \qquad a = \frac{T\sin\theta}{m} = \left(\frac{F}{2\cos\theta}\right)\left(\frac{\sin\theta}{m}\right)$$

$$\therefore \qquad F = \frac{2\,ma}{\tan\theta} = \frac{2 \times 0.3 \times 2.5}{(3/4)}$$

$$= 2 \text{ N}$$ **Ans.**

38. $x + x + \sqrt{y^2 + c^2} = l$ = length of string

Differentiating w.r.t. time, we get



$$2\,\frac{dx}{dt} = \frac{y}{\sqrt{y^2 + c^2}}\left(-\frac{dy}{dt}\right)$$

or $$2\, v_B = \frac{1}{\cos\theta}\, v_A$$

$$\therefore \qquad v_A = 2\, v_B \cos\theta$$

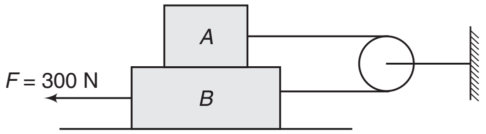
$$= (2)\,(10)\,(0.8)$$

$$= 16 \text{ m/s}$$ **Ans.**

39. Maximum force of friction between c and ground is

$$(f_c)_{\max} = (0.5)\,(60)\,(10) = 300 \text{ N}$$

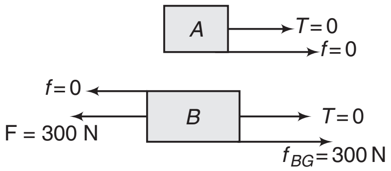
Since it is pulling the blocks by the maximum force (without moving). Therefore the applied force is $F = 300$ N



$$(f_{AB})_{\max} = 0.4 \times 60 \times 120 \times 10 = 240 \text{ N}$$

$$(f_{BG})_{\max} = 0.3 \times 120 \times 10 = 360 \text{ N}$$

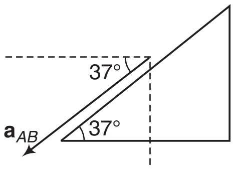
Since $(f_{BG})_{\max}$ is greater than 300 N, blocks will not move. Free body diagrams of block are as shown below.



40. Let $\mathbf{a}_B = a\hat{\mathbf{i}}$

Then, $\mathbf{a}_{AB} = \mathbf{a}_A - \mathbf{a}_B$

$= (15 - a)\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}}$



Since, $\mathbf{a}_{AB}$ is along the plane as shown in figure.

$$\therefore \qquad \tan 37^\circ = \frac{3}{4} = \frac{15}{15 - a}$$

Solving this equation, we get $a = -5$

or $\qquad \mathbf{a}_B = (-5\hat{\mathbf{i}})$

41. Acceleration,

$$a = \frac{2F - F}{m + m} = \frac{F}{2m} \qquad \text{(towards left)}$$

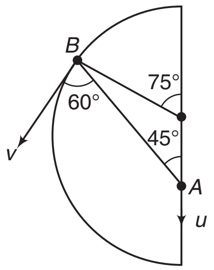
Horizontal forces on B gives the equation,

$$2F - N \sin 30^\circ = m \cdot a$$

$$\text{or} \qquad 2F - \frac{N}{2} = m\left(\frac{F}{2m}\right)$$

$$\therefore \qquad N = 3F \qquad \textbf{Ans.}$$

42. Distance AB = constant



$\therefore$ Component of v along BA = component of u along BA

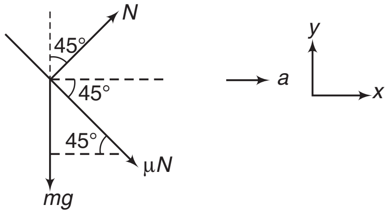
or $\qquad v \cos 60^\circ = u \cdot \cos 45^\circ$

or $\qquad v = \sqrt{2}\,u$ **Ans.**

43. Let a = maximum acceleration of A.

Under no slip condition acceleration of B is also a

FBD of *A* w.r. t. ground



$$\Sigma F_y = 0$$

$$\therefore \qquad \frac{N}{\sqrt{2}} = mg + \frac{\mu N}{\sqrt{2}} \qquad \text{...(i)}$$

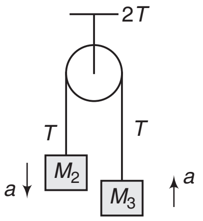
$$\Sigma F_x = ma$$

$$\therefore \qquad \frac{N}{\sqrt{2}} + \frac{\mu N}{\sqrt{2}} = ma \qquad \text{...(ii)}$$

Solving these two equations, we get

$$a = g\left(\frac{1 + \mu}{1 - \mu}\right) \qquad \textbf{Ans.}$$

44. **For M_2 and M_3**



$$a = \frac{M_2 g - M_3 g}{M_2 + M_3} = \frac{3M_3 g - M_3 g}{3M_3 + M_3}$$

$$= \frac{g}{2}$$

Now FBD of M_2 gives the equation,

$$M_2 g - T = M_2 \cdot a = \frac{M_2 g}{2}$$

$$\therefore \qquad T = \frac{M_2 g}{2}$$

$$\text{or} \qquad 2T = M_2 g$$

Now taking moments of forces about support point

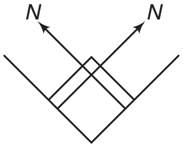
$$M_1 g\,(l_1) = (2T)\,l_2 = (M_2 g)\,(3l_1)$$

$$\therefore \qquad \frac{M_1}{M_2} = 3 \qquad \textbf{Ans.}$$

45. Resultant of N and N $(= \sqrt{2}\,N)$ is equal to $mg \cos \theta$

$$\therefore \qquad \sqrt{2}\,N = mg \cos \theta$$

$\therefore \quad N = \dfrac{mg\cos\theta}{\sqrt{2}}$



Now kinetic friction will act from two sides

$$\therefore \quad a = \frac{mg\sin\theta - 2\mu_k N}{m}$$

Substituting the value of N, we get

$$a = g(\sin\theta - \sqrt{2}\,\mu_k\cos\theta) \qquad \textbf{Ans.}$$

46. $f_1 \rightarrow$ force of friction between 2 kg and 3 kg

$$(f_1)_{\max} = 0.5 \times 3 \times 10 = 15\text{ N}$$

$f_2 \rightarrow$ force of friction between 2 kg and 1 kg

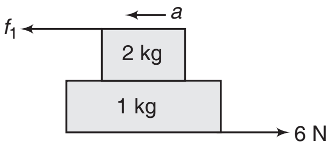
$$(f_2)_{\max} = 0.3 \times 5 \times 10 = 15\text{ N}$$

$f_3 \rightarrow$ force of friction between 1 kg and ground

$$(f_3)_{\max} = 0.1 \times 6 \times 10 = 6\text{ N}$$

When $F > 6$ N system will start moving with a common acceleration

$$a = \frac{F-6}{3+2+1} = \left(\frac{F}{6} - 1\right)\text{ m/s}^2$$



$$f_1 - 6 = (3)\,a = \frac{F}{2} - 3$$

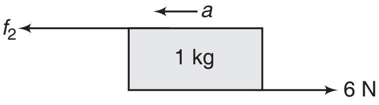
$$\therefore \quad f_1 = \left(6 + \frac{F}{2} - 3\right)$$

$$= \left(\frac{F}{2} + 3\right)$$

Since F is slightly greater than 6 N

$\therefore \quad f_1 < 15$ N or $< (f_1)_{\max}$

$\therefore$ No slipping will occur here

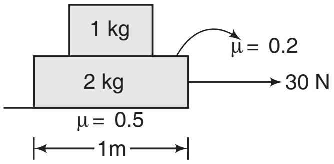


$$f_2 - 6 = (1)(a) = \frac{F}{6} - 1$$

$$\therefore \quad f_2 = \frac{F}{6} + 5$$

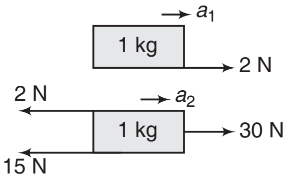
Again $f_2 < (f_2)_{\max}$. So no slip will take place here also.

47. $(f_1)_{\max}$ = between 1 kg and 2 kg



$$= 0.2 \times 1 \times 10 = 2\text{ N}$$

$(f_2)_{\max}$ = between 2 kg and ground



$$= 0.5 \times 3 \times 10 = 15\text{ N}$$

$$a_1 = \frac{2}{1} = 2\text{ m/s}^2$$

$$a_2 = \frac{30 - 15 - 2}{2}$$

$$= 6.5\text{ m/s}^2$$

$$a_r = a_2 - a_1$$

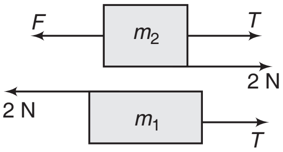
$$= 4.5\text{ m/s}^2$$

$$t = \sqrt{\frac{2S_r}{a_r}}$$

$$= \sqrt{\frac{2\times 1}{4.5}} = \frac{2}{3}\text{ s} \qquad \textbf{Ans.}$$

More than One Correct Options

1. Maximum value of friction between two blocks



$$f_{\max} = 0.2 \times 1 \times 10 = 2\text{N}$$

In critical case,

$$T = 2\text{ N}$$

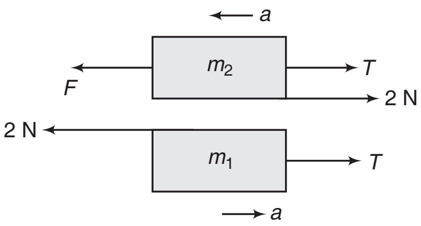
$$F = T + 2 = 4\text{ N}$$

$\therefore$ System is in equilibrium if $f \le 4$ N **Ans.**

For $F > 4$ N

$$F - (T + 2) = m_2 a = (1)(a) \qquad \text{...(i)}$$

$$T - 2 = m_1 a = (1)(a) \qquad \text{...(ii)}$$

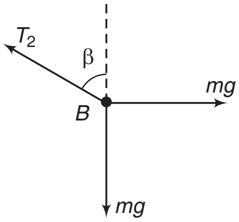


On solving these two equations, we get

$$T = \frac{F}{2}$$

When, $F = 6\text{ N},\ T = 3\text{ N}$ **Ans.**

2. Resultant of mg and mg is $\sqrt{2}\,mg$.



Therefore T_2 should be equal and opposite of this.

or $$T_2 = \sqrt{2}\,mg \quad \text{...(i)}$$

Further, $$T_2 \cos\beta = mg \quad \text{...(ii)}$$

and $$T_2 \sin\beta = mg \quad \text{...(iii)}$$

or $$\sin\beta = \cos\beta \Rightarrow \beta = 45°$$

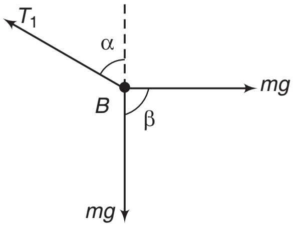
$$T_1 \cos\alpha = mg + T_2 \cos\beta$$

$$= mg + \sqrt{2}\,mg\left(\frac{1}{\sqrt{2}}\right)$$

or $$T_1 \cos\alpha = 2\,mg \quad \text{...(iv)}$$

$$T_1 \sin\alpha = T_2 \sin\beta = \sqrt{2}\,mg\left(\frac{1}{\sqrt{2}}\right)$$

$\therefore$ $$T_1 \sin\alpha = mg \quad \text{...(v)}$$



From Eqs. (iv) and (v), we get

$$\tan\alpha = \frac{1}{2} \quad \text{and} \quad T_1 = \sqrt{5}\,mg$$

$$\tan\beta = \tan 45° = 1 \quad \text{and} \quad T_2 = \sqrt{2}\,mg$$

$$\therefore\ \tan\beta = 2\tan\alpha \quad \text{and} \quad \sqrt{2}\,T_1 = \sqrt{5}\,T_2$$

3. a = slope of v - t graph

$$= -1\text{ m/s}^2$$

$\therefore$ $$\text{Retardation} = 1\text{ m/s}^2 = \frac{\mu mg}{m} = \mu g$$

or $$\mu = \frac{1}{g} = \frac{1}{10} = 0.1$$

If μ is half, then retardation a is also half. So using

$$v = u - at$$

or $$0 = u - at$$

or $$t = \frac{u}{a} \quad \text{or} \quad t \propto \frac{1}{a}$$

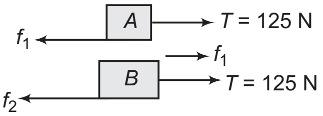
we can see that t will be two times.

4. Maximum force of friction between A and B

$$(f_1)_{\max} = 0.3 \times 60 \times 10 = 180\text{ N}$$

Maximum force of friction between B and ground

$$(f_2)_{\max} = 0.3 \times (60 + 40)\,g = 300\text{ N}$$

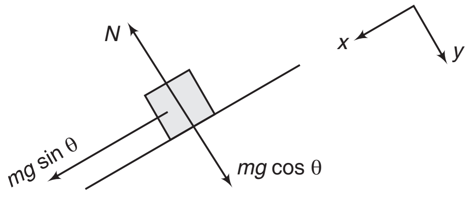


Both are stationary

$$f_1 = T = 125\text{ N}$$

$$f_2 = T + f_1 = 250\text{ N}$$

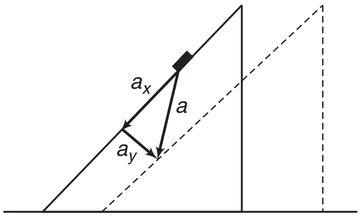
5. $$a_x = \frac{mg\sin\theta}{m} = g\sin\theta$$



It is also moving in y-direction

$\therefore$ $$mg\cos\theta > N$$

$$a_y = \frac{mg\cos\theta - N}{m}$$



Now, $$a = \sqrt{a_x^2 + a_y^2} > g\sin\theta$$

6. Maximum value of friction between A and B is

$$(f_1)_{\max} = 0.25 \times 3 \times 10 = 7.5 \text{ N}$$

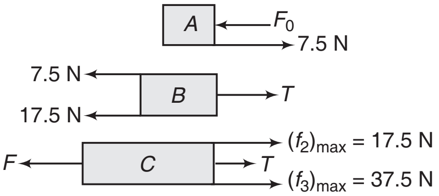
Maximum value of friction between B and C

$$(f_2)_{\max} = 0.25 \times 7 \times 10 = 17.5 \text{ N}$$

and maximum value of friction between C and ground,

$$(f_3)_{\max} = 0.25 \times 15 \times 10 = 37.5 \text{ N}$$

$$F_0 = \text{force on } A \text{ from rod}$$



If C is moving with constant velocity, then B will also move with constant velocity

For B, $T = 17.5 + 7.5 = 25$ N

For C, $F = 17.5 + 25 + 37.5 = 80$ N

For F = 200 N

Acceleration of B towards right

= acceleration of C towards left

= a (say)

Then $T - 7.5 - 17.5 = 4a$ ...(i)

$200 - 17.5 - 37.5 - T = 8a$ ...(ii)

On solving these two equations, we get

$$a = 10 \text{ m/s}^2$$

7. Since, $\mu_1 > \mu_2$

$\therefore$ $(f_1)_{\max} > (f_2)_{\max}$

Further if both move,

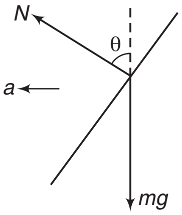
$$a = \frac{T - \mu mg}{m}$$

μ of block is less. Therefore, its acceleration is more.

8. $N \cos\theta = mg = 10$ ...(i)

$N \sin\theta = ma = 5$ ...(ii)

On solving these two equations, we get



$$N = 5\sqrt{5} \text{ N} \quad \text{and} \quad \tan\theta = \frac{1}{2}$$

9. $f_1 \rightarrow$ force of friction between 2 kg and 4 kg

$f_2 \rightarrow$ force of friction between 4 kg and ground

$$(f_{S_1})_{\max} = 0.4 \times 2 \times 10 = 8 \text{ N}$$
$$F_{K_1} = 0.2 \times 2 \times 10 = 4 \text{ N}$$
$$(f_{S_2})_{\max} = 0.6 \times 6 \times 10 = 36 \text{ N}$$
$$F_{K_2} = 0.4 \times 6 \times 10 = 24 \text{ N}$$

At $t = 1$ s, $F = 2$N < 36 N, therefore system remains stationary and force of friction between 2 kg and 4 kg is zero.

At $t = 4$ s, $F = 8$ N < 36 N. Therefore system is again stationary and force of friction on 4 kg from ground is 8 N.

At $t = 15$ s, $F = 30$ N < 36 N and system is stationary.

10. Net pulling force = 0

$$\Rightarrow \quad a = 0$$
$$T_1 = 1 \times g = 10 \text{ N}$$
$$T_3 = 2 \times g = 20 \text{N}$$
$$T_2 = 20 + T_1 = 30 \text{ N}$$

11. $f_{\max} = 0.3 \times 2 \times 10 = 6$ N

At t = 2 s, $F = 2 \text{ N} < f_{\max}$

$\therefore$ $f = F = 2$ N

At t = 8 s, $F = 8 \text{ N} > f_{\max}$

$\therefore$ $f = 6$ N

At t = 10 s, $F = 10 \text{ N} > f_{\max}$

$\therefore$ $f = 6$ N

$$a = \frac{F - f}{m} = \frac{10 - 6}{2} = 2 \text{ m/s}^2$$

$$F = f_{\max} = 6 \text{ N at 6 s}$$

For 6 s ≤ t ≤ 10 s

$$a = \frac{F - f}{m} = \frac{t - 6}{2} = 0.5t - 3$$

$$\int_0^v dv = \int a\,dt = \int_6^{10} (0.5t - 3)\,dt$$

$$v = 4 \text{ m/s}$$

After 10 s

$$a = \frac{F - f}{m} = \frac{10 - 6}{2} = 2 \text{ m/s}^2$$
$$= \text{constant}$$

$$\therefore \quad v' = v + at$$
$$= 4 + 2\,(12 - 10)$$
$$= 8 \text{ m/s}$$

12. Maximum force of friction between 2 kg and 4 kg

$$= 0.4 \times 2 \times 10 = 8 \text{ N}$$

2 kg moves due to friction. Therefore its maximum acceleration may be

$$a_{\text{max}} = \frac{8}{2} = 4 \text{ m/s}^2$$

Slip will start when their combined acceleration becomes 4 m/s^2

$$\therefore \qquad a = \frac{F}{m} \text{ or } 4 = \frac{2t}{6} \text{ or } t = 12 \text{ s}$$

At $t = 3$s

$$a_2 = a_4 = \frac{F}{m} = \frac{2t}{6} = \frac{2 \times 3}{6}$$

$$= 1 \text{ m/s}^2$$

Both a_2 and a_4 are towards right. Therefore pseudo forces F_1 (on 2 kg from 4 kg) and F_2 (on 4 kg from 2 kg) are towards left

$$F_1 = (2)\,(1) = 2\text{N}$$

$$F_2 = (4)\,(1) = 4 \text{ N}$$

From here we can see that F_1 and F_2 do not make a pair of equal and opposite forces.

13. See the hint of of Q.No-10 of Assertion and Reason type questions of Level-1.

Comprehension Based Questions

1. Let $\mu_K = \mu$, then $\mu_S = 2\mu$

According to first condition,

$$F + mg \sin\theta = \mu_S \, mg \cos\theta = 2\,\mu \, mg \cos\theta \qquad \text{...(i)}$$

According to second condition,

$$mg \sin\theta = F + \mu_K \, mg \cos\theta$$

$$= F + \mu \, mg \cos\theta \qquad \text{...(ii)}$$

Putting $\theta = 30°$, we get

$$F + mg/2 = 2\mu mg\left(\frac{\sqrt{3}}{2}\right)$$

$$\text{or} \qquad \sqrt{3}\,\mu \, mg = F + 0.5\, mg \qquad \text{...(iii)}$$

$$\frac{mg}{2} = F + \mu \, mg\left(\frac{\sqrt{3}}{2}\right)$$

$$\text{or} \qquad 0.5\sqrt{3}\,\mu \, mg = 0.5\, mg - F \qquad \text{...(iv)}$$

Dividing Eq. (iii) and (iv), we get

$$F = \frac{mg}{6}$$ **Ans.**

2. Substituting value of F in Eq. (iii), we have

$$\mu = \frac{2}{3\sqrt{3}} = \mu_K$$

$$\therefore \qquad \mu_S = 2\mu = \frac{4}{3\sqrt{3}}$$

3. $a = \dfrac{F + mg \sin\theta - \mu_K \, mg \cos\theta}{m}$

$$= \frac{(mg/6) + (mg/2) - \left(\dfrac{2}{3\sqrt{3}}\right) mg\left(\dfrac{\sqrt{3}}{2}\right)}{m}$$

$$= \frac{g}{3}$$ **Ans.**

4. $F' = mg \sin\theta + \mu_S \, mg \cos\theta$

$$= \frac{mg}{2} + \frac{4}{3\sqrt{3}} mg\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{7mg}{6}$$ **Ans.**

5. $F'' = mg \sin\theta + \mu_K \, mg \cos\theta$

$$= (mg/2) + \left(\frac{2}{3\sqrt{3}}\right) mg\left(\frac{\sqrt{3}}{2}\right)$$

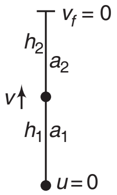
$$= \frac{5mg}{6}$$ **Ans.**

6. Acceleration $a_1 = \dfrac{1350 \times 9.8 - 1200 \times 9.8}{1200}$

$$= 1.225 \text{ m/s}^2$$

Retardation, $a_2 = \dfrac{1200\,g - 1000\,g}{1200}$

$$= 1.63 \text{ m/s}^2$$



$$h_1 + h_2 = 25 \qquad \text{...(i)}$$

$$v = \sqrt{2a_1 h_1} \quad \text{or} \quad \sqrt{2a_2 h_2}$$

$$\text{or} \qquad 2a_1 h_1 = 2a_2 h_2$$

$$\therefore \qquad \frac{h_1}{h_2} = \frac{a_2}{a_1} \qquad \text{...(ii)}$$

$$= \frac{1.63}{1.225} = 1.33$$

Solving these equations, we get

$$h_1 = 14.3 \text{ m}$$ **Ans.**

7. $v = \sqrt{2a_1 h_1} = \sqrt{2 \times 1.225 \times 14.3}$

$$= 5.92 \text{ m/s}$$ **Ans.**

8. $\tan\theta = \dfrac{8}{15} \quad \therefore \quad \theta = \tan^{-1}\left(\dfrac{8}{15}\right) = 28°$

$$(f_A)_{\text{max}} = 0.2 \times 170 \times 10 \times \cos 28°$$

$$= 300.2 \text{ N} \approx 300 \text{ N}$$

$$(f_B)_{\text{max}} = 0.4 \times 170 \times 10 \times \cos 28°$$

$$= 600.4 \text{ N} \approx 600 \text{ N}$$

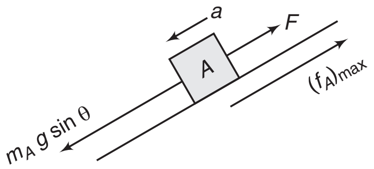
Now,

$$(m_A + m_B)\, g \sin\theta = (340)\,(10) \sin 28° = 1596 \text{ N}$$

Since this is greater than $(f_A)_{\text{max}} + (f_B)_{\text{max}}$, therefore blocks slides downward and maximum force of friction will act on both surfaces

$$\therefore \quad f_{\text{total}} = (f_A)_{\text{max}} + (f_B)_{\text{max}}$$
$$= 900 \text{ N}$$ **Ans.**

9. $$a = \frac{(m_A + m_B)\, g \sin\theta - f_{\text{total}}}{m_A + m_B}$$
$$= \frac{1596 - 900}{340} = 2.1 \text{ m/s}^2$$



F = force on connecting bar

$$m_A\, g \sin\theta - F - (f_A)_{\text{max}} = m_A a$$
$$\therefore \quad F = m_A g \sin\theta - (f_A)_{\text{max}} - m_A a$$
$$= 170 \times 10 \times \sin 28^\circ - 300 - 170 \times 2.1$$
$$= 141 \text{ N}$$ **Ans.**

Match the Columns

1. $F = 2t$

$$\mu_s\, mg = 20\,\mu_s$$
$$\mu_k\, mg = 20\,\mu_k$$

(a) Motion starts at 4 s

$$\therefore \quad F = \mu_s\, mg$$
$$\Rightarrow \quad (2)(4) = 20\,\mu_s$$
$$\therefore \quad \mu_s = 0.4$$

(b) At 4 s, when motion starts,

$$a = \frac{F - \mu_k\, mg}{m}$$
$$\therefore \quad 1 = \frac{8 - 20\,\mu_k}{2}$$

Solving we get, $\mu_k = 0.3$

(c) At $t = 0.1$ s, when motion has not started,

$$f = F = 2 \times 0.1 = 0.2 \text{ N}$$

(d) **At 8 s**

$$a = \frac{F - \mu_k\, mg}{m}$$
$$= \frac{2 \times 8 - 0.3 \times 20}{2}$$
$$= 5 \text{ m/s}^2$$
$$\therefore \quad \frac{a}{10} = 0.5 \text{ m/s}^2$$ **Ans.**

2. (a) At $\theta = 0^\circ$, driving force $F = 0$

$$\therefore \quad \text{friction} = 0$$

(b) At $\theta = 90^\circ$, N = 0

$\therefore$ Maximum friction $= \mu N = 0$ or friction $= 0$

(c) Angle of repose,

$$\theta_r = \tan^{-1}(\mu) = 45^\circ$$

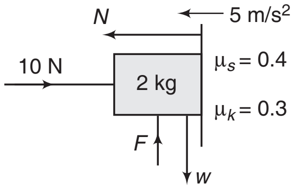
Since $\theta < \theta_r$, block is at rest and

$$f = mg \sin\theta = 2 \times 10 \sin 30^\circ = 10 \text{ N}$$

(d) $\theta > \theta_r$. Therefore block will be moving

$$f = \mu\, mg \cos\theta$$
$$= (1)(2)(10) \cos 60^\circ = 10 \text{ N}$$

4. (a) $N - 10 = ma = 5 \times 2$



$$\therefore \quad N = 20 \text{ N}$$
$$\mu_s N = 8 \text{ N}$$
$$\mu_k N = 6 \text{ N}$$
$$W = mg = 20 \text{ N}$$

(b) When $F = 15$ N

$$w - F = 5 \text{ N} \qquad \text{(downwards)}$$

This is less than $\mu_s N$

$$\therefore \quad f = 5 \text{ N} \qquad \text{(upwards)}$$

(c) $F = w - \mu_s N = 20 - 8 = 12$ N

(d) $F = w + \mu_S N = 20 + 8 = 28$ N

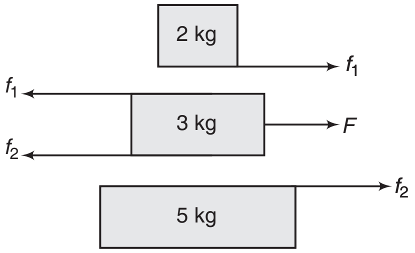
5. (a) Net pulling force F = net resisting frictional force at $C = 10$ N

(b) $f_c = 0$

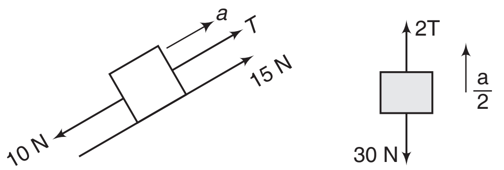
(c) $N_c = (m_B + m_C)\, g = 20$ N

(d) $T = F = 10$ N (everywhere)

7. Ground is smooth. So all blocks will move towards right 2 kg and 5 kg blocks due to friction.



8. $\mu mg \cos\theta = 15$ N



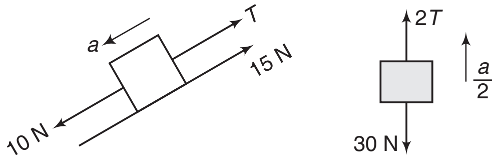
$$T - 10 - 15 = 2a \qquad \text{...(i)}$$

$$30 - 2T = 3\left(\frac{a}{2}\right) \qquad \text{...(ii)}$$

Solving these two equations we get,

$$a = -3.63 \text{ m/s}^2$$

So, if we take the other figure,



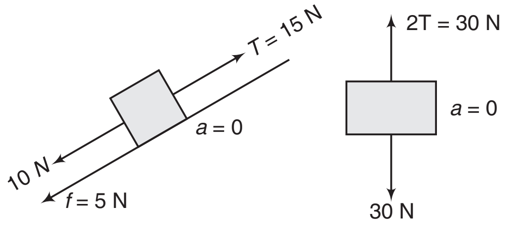
This figure is not feasible. Because for 'a' to be down the plane,

$$10 \text{ N} > T + 15$$

which is not possible

$$\therefore \qquad a = 0$$

and free body diagrams are as shown below.



Subjective Questions

1. It is just like a projectile motion with g to be replaced by $g \sin 45°$.

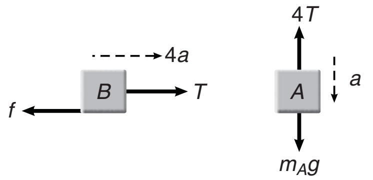
After 2 s,

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{\left(10 \sin 45° - \frac{g}{\sqrt{2}} \times 2\right) + (10 \cos 45°)^2}$$

$$= 10 \text{ m/s} \qquad \textbf{Ans.}$$

2. Suppose T be the tension in the string attached to block B. Then tension in the string connected to block A would be $4T$.



Similarly, if a be the acceleration of block A (downwards), then acceleration of block B towards right will be $4a$.

Equations of motion are

For block A, $\quad m_A g - 4T = m_A a$

or $\quad 50 - 4T = 5a \qquad \text{…(i)}$

For block B, $\quad T - f = 10(4a)$

or $\quad T - (0.1)(10)(10) = 40a$

or $\quad T - 10 = 40a \qquad \text{…(ii)}$

Solving Eqs. (i) and (ii), we get

$$a = \frac{2}{33} \text{ m/s}^2 \qquad \textbf{Ans.}$$

3. (a) When the truck accelerates eastward force of friction on mass is eastwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = (30 \times 1.8) = 54 \text{ N}$$

Since it is less than $\mu_s mg$

$$\therefore \qquad f = 54 \text{ N} \qquad \text{(eastwards)}$$

(b) When the truck accelerates westwards, force of friction is westwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = 30 \times 3.8 = 114 \text{ N}$$

Since it is greater than $\mu_s mg$. Hence

$$f = f_k = \mu_k mg = 60 \text{ N (westwards)} \qquad \textbf{Ans.}$$

4. Block B will fall vertically downwards and A along the plane.

Writing the equations of motion.

For block B,

$$m_B g - N = m_B a_B$$

or $\quad 60 - N = 6a_B \qquad \text{…(i)}$

$$(N + m_A g) \sin 30° = m_A a_A$$

or $\quad (N + 150) = 30\, a_A \qquad \text{…(ii)}$

Further $\quad a_B = a_A \sin 30°$

or $\quad a_A = 2a_B \qquad \text{…(iii)}$

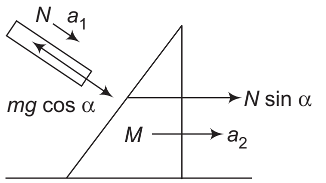
Solving these three equations, we get

(a) $a_A = 6.36 \text{ m/s}^2$ **Ans.**

(b) $a_{BA} = a_A \cos 30° = 5.5 \text{ m/s}^2$ **Ans.**

5. Let acceleration of m be a_1 (absolute) and that of M be a_2 (absolute).

Writing equations of motion.



For *m* $mg\cos\alpha - N = ma_1$ ...(i)

For *M*, $N\sin\alpha = Ma_2$ ...(ii)

Note *In the FBD only those forces which are along a_1 and a_2 have been shown.*

Constraint equation can be written as,

$$a_1 = a_2 \sin\alpha \qquad \text{...(iii)}$$

Solving above three equations, we get acceleration of rod,

$$a_1 = \frac{mg\cos\alpha\sin\alpha}{\left(m\sin\alpha + \dfrac{M}{\sin\alpha}\right)} \qquad \textbf{Ans.}$$

and acceleration of wedge

$$a_2 = \frac{mg\cos\alpha}{m\sin\alpha + \dfrac{M}{\sin\alpha}} \qquad \textbf{Ans.}$$

6. (a) N_2 and mg pass through G. N_1 has clockwise moment about G, so the ladder has a tendency to slip by rotating clockwise and the force of friction (f) at B is then up the plane.

(b) $\Sigma M_A = 0$

$$\therefore \quad fl = mg\left(\frac{l}{2}\sin 45°\right) \qquad \text{...(i)}$$

$$\Sigma F_V = 0$$

$$\therefore \quad mg = N_2\cos 45° + f\sin 45° \qquad \text{...(ii)}$$

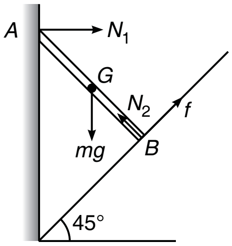
From Eqs. (i) and (ii),

$$N_2 = \frac{3}{2\sqrt{2}}mg$$

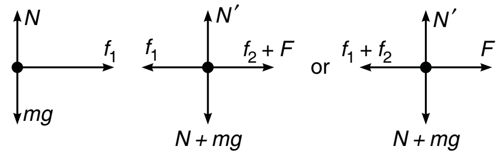
and $f = \dfrac{mg}{2\sqrt{2}}$

or $\mu_{\min} = \dfrac{f}{N_2}$

$$= \frac{1}{3} \qquad \textbf{Ans.}$$



7. Here f_1 = force of friction between man and plank and f_2 = force of friction between plank and surface.



FBD of man FBD of plank

For the plank not to move

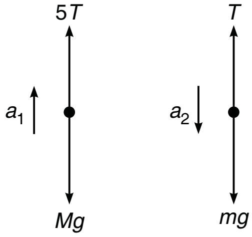
$$F - (f_2)_{\max} \le f_1 \le F + (f_2)_{\max}$$

or $F - \mu(M+m)g \le ma \le F + \mu(M+m)g$

or a should lie between $\dfrac{F}{m} - \dfrac{\mu(M+m)g}{m}$

and $\dfrac{F}{m} + \dfrac{\mu(M+m)g}{m}$ **Ans.**

8. Writing equations of motion



FBD of *M* FBD of *m*

For *M* $5T - Mg = Ma_1$ ...(i)

For *m*, $mg - T = ma_2$ ...(ii)

From constraint equation,

$$a_2 = 5a_1 \qquad \text{...(iii)}$$

Solving these equations, we get acceleration of M,

$$a_1 = \left(\frac{5m - M}{25m + M}\right)g$$

and of m, $a_2 = 5\left(\dfrac{5m - M}{25m + M}\right)g$

9. $2a_1s_1 = 2a_2s_2$ or $\dfrac{a_1}{a_2} = \dfrac{s_2}{s_1} = \dfrac{n}{m}$

or $\dfrac{g\sin\alpha}{\mu g\cos\alpha - g\sin\alpha} = \dfrac{m}{n}$

Solving it, we get

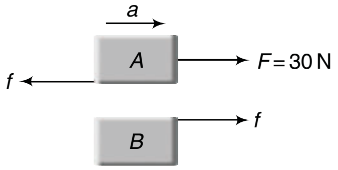
$$\mu = \left(\frac{m+n}{m}\right)\tan\alpha \qquad \textbf{Ans.}$$

10. Limiting friction between A and B

$$f_L = \mu N = 0.4 \times 100 = 40 \text{ N}$$

(a) Both the blocks will have a tendency to move together with same acceleration (say a).

So, the force diagram is as shown.



Equations of motion are,

$$30 - f = 10 \times a \qquad \ldots\text{(i)}$$

$$f = 25 \times a \qquad \ldots\text{(ii)}$$

Solving these two equations, we get

$$a = 0.857 \text{ m/s}^2$$

and $f = 21.42$ N

As this force is less than f_L, both the blocks will move together with same acceleration,

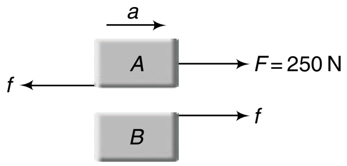
$$a_A = a_B = 0.857 \text{ m/s}^2$$

(b)
$$250 - f = 10a \qquad \ldots\text{(iii)}$$

$$f = 25a \qquad \ldots\text{(iv)}$$

Solving Eqs. (iii) and (iv), we get

$$f = 178.6 \text{ N}$$

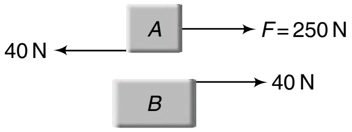


As $f > f_L$, slipping will take place between two blocks and

$$f = f_L = 40 \text{ N}$$

$$a_A = \frac{250 - 40}{10}$$

$$= 21.0 \text{ m/s}^2$$



$$a_B = \frac{40}{25} = 1.6 \text{ m/s}^2$$ **Ans.**

11. Normal reaction between A and B would be $N = mg \cos\theta$. Its horizontal component is $N \sin\theta$. Therefore, tension in cord CD is equal to this horizontal component.

Hence, $T = N \sin\theta = (mg \cos\theta)(\sin\theta)$

$$= \frac{mg}{2} \sin 2\theta$$ **Ans.**

12. Assuming that mass of truck >> mass of crate.

Retardation of truck

$$a_1 = (0.9)\, g = 9 \text{ m/s}^2$$

Retardation of crate

$$a_2 = (0.7)\, g = 7 \text{ m/s}^2$$

or relative acceleration of crate

$$a_r = 2 \text{ m/s}^2 .$$

Truck will stop after time

$$t_1 = \frac{15}{9} = 1.67 \text{ s}$$

and crate will strike the wall at

$$t_2 = \sqrt{\frac{2s}{a_r}} = \sqrt{\frac{2 \times 3.2}{2}} = 1.78 \text{ s}$$

As $t_2 > t_1$, crate will come to rest after travelling a distance

$$s = \frac{1}{2} a_r t_1^2 = \frac{1}{2} \times 2.0 \times \left(\frac{15}{9}\right)^2$$

$$= 2.77 \text{ m}$$ **Ans.**

13. $\mu_k mg = 0.2 \times 10 \times 10 = 20$ N

For t ≤ 0.2 s

Retardation $a_1 = \dfrac{F + \mu_k mg}{m}$

$$= \frac{20 + 20}{10} = 4 \text{ m/s}^2$$

At the end of 0.2 s,

$$v = u - a_1 t$$

$$v = 1.2 - 4 \times 0.2 = 0.4 \text{ m/s}$$

For t > 0.2 s

Retardation $a_2 = \dfrac{10 + 20}{10} = 3 \text{ m/s}^2$

Block will come to rest after time

$$t_0 = \frac{v}{a_2} = \frac{0.4}{3} = 0.13 \text{ s}$$

∴ Total time = 0.2 + 0.13 = 0.33 s **Ans.**

14. Block will start moving at, $F = \mu mg$

or $25t = (0.5)(10)(9.8) = 49$ N

∴ $t = 1.96$ s

Velocity is maximum at the end of 4 second.

$$\therefore \quad \frac{dv}{dt} = \frac{25t - 49}{10} = 2.5t - 4.9$$

$$\therefore \quad \int_0^{v_{\max}} dv = \int_{1.96}^{4} (2.5\,t - 4.9)\, dt$$

$$\therefore \quad v_{\max} = 5.2 \text{ m/s}$$ **Ans.**

For 4 s < ***t*** < 7 s

Net retardation $a_1 = \frac{49 - 40}{10} = 0.9\ \text{m/s}^2$

$\therefore \quad v = v_{\max} - a_1 t_1 = 5.2 - 0.9 \times 3 = 2.5\ \text{m/s}$

For ***t*** **> 7** s

Retardation $a_2 = \frac{49}{10} = 4.9\ \text{m/s}^2$

$\therefore \quad t = \frac{v}{a_2} = \frac{2.5}{4.9} = 0.51\ \text{s}$

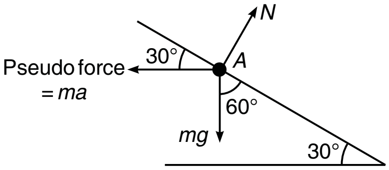
$\therefore$ Total time = (4 – 1.96) + (7 – 4) + (0.51)
= 5.55 s **Ans.**

15. Let *B* and *C* both move upwards (alongwith their pulleys) with speeds v_B and v_C then we can see that, *A* will move downward with speed, $2v_B + 2v_C$. So, with sign we can write,

$$\therefore \quad v_B = \frac{v_A}{2} - v_c$$

Substituting the values we have, $v_B = 0$ **Ans.**

16. FBD of *A* with respect to frame is shown in figure. *A* is in equilibrium under three concurrent forces shown in figure, so applying Lami's theorem



$$\frac{ma}{\sin (90 + 60)} = \frac{mg}{\sin (90 + 30)}$$

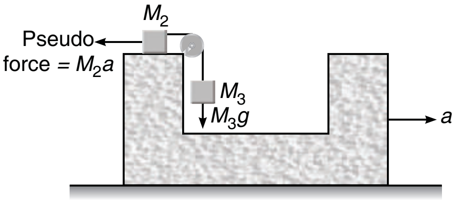
$$\therefore \quad a = \frac{g \cos 60°}{\cos 30°} = 5.66\ \text{m/s}^2$$

17. FBD of M_2 and M_3 in accelerated frame of reference is shown in figure.

Note *Only the necessary forces have been shown.*

Mass M_3 will neither rise nor fall if net pulling force is zero.

i.e. $M_2 a = M_3 g$



or $a = \frac{M_3}{M_2} g$

$\therefore \quad F = (M_1 + M_2 + M_3) a$

$= (M_1 + M_2 + M_3) \frac{M_3}{M_2} g$ **Ans.**

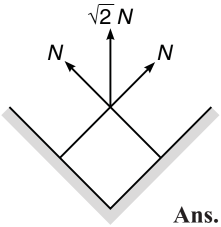
18. Retardation $a = \mu_k g = 0.15 \times 9.8 = 1.47\ \text{m/s}^2$

Distance travelled before sliding stops is,

$$s = \frac{v^2}{2a}$$

$$= \frac{(5)^2}{2 \times 1.47} \approx 8.5\ \text{m}$$ **Ans.**

19. $\sqrt{2}\, N = mg \cos \theta$



$\therefore \quad N = \frac{mg \cos \theta}{\sqrt{2}}$

$a = \frac{mg \sin \theta - 2\mu_k N}{m}$

$= g \sin \theta - \sqrt{2}\, \mu_k g \cos \theta$

$= g (\sin \theta - \sqrt{2} \mu_k \cos \theta)$ **Ans.**

20. $v \cdot \frac{dv}{dx} = \frac{\text{Net force}}{\text{mass}} = \frac{F - \mu_k \rho (L - x) g}{\rho L}$

$\therefore \quad \int_0^v v\, dv = \int_0^L \frac{F - \mu_k \rho (L - x) g}{\rho L} dx$

$\therefore \quad \frac{v^2}{2} = \frac{F}{\rho} - \mu_k gL + \frac{\mu_k gL}{2}$

$\therefore \quad v = \sqrt{\frac{2F}{\rho} - \mu_k gL}$ **Ans.**

21. (a) $v = a_1 t_1 = 2.6\ \text{m/s}$

$s_1 = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} \times 2 \times (1.3)^2 = 1.69\ \text{m}$

$s_2 = (2.2 - 1.69) = 0.51\ \text{m}$

Now, $s_2 = \frac{v^2}{2a_2}$

$\therefore \quad a_2 = \frac{v^2}{2s_1} = \frac{(2.6)^2}{2 \times 0.51} = 6.63\ \text{m/s}^2$

and $t_2 = \frac{v}{a_2} = 0.4\ \text{s}$

(b) Acceleration of package will be $2\ \text{m/s}^2$ while retardation will be $\mu_k g$ or $2.5\ \text{m/s}^2$ not $6.63\ \text{m/s}^2$.

For the package,

$$v = a_1 t_1 = 2.6 \text{ m/s} \Rightarrow s_1 = \frac{1}{2} a_1 t_1^2 = 1.69 \text{ m}$$

$$s_2 = vt_2 - \frac{1}{2} d_2 t_2^2 = 2.6 \times 0.4 - \frac{1}{2} \times 2.5 \times (0.4)^2$$

$$= 0.84 \text{ m}$$

∴ Displacement of package w.r.t. belt

$= (0.84 - 0.51) \text{ m} = 0.33 \text{ m}$ **Ans.**

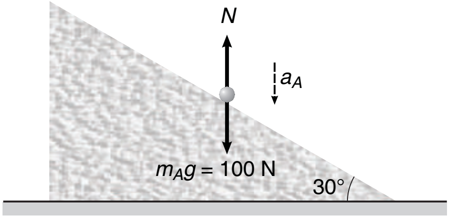
Alternate Solution For last 0.4 s

$$|a_r| = 6.63 - 2.5 = 4.13 \text{ m/s}^2$$

$$\therefore \quad s_r = \frac{1}{2}|a_r|t^2 = \frac{1}{2} \times 4.13 \times (0.4)^2$$

$$= 0.33 \text{ m}$$

22. Free body diagram of crate A w.r.t ground is shown in figure.



Equation of motion is

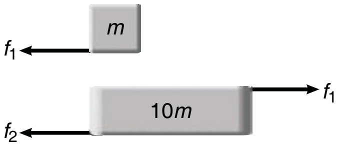
$$100 - N = 10a_A \qquad \ldots\text{(i)}$$

$$a_A = a \sin 30° = (2)\left(\frac{1}{2}\right) \quad \text{or} \quad a_A = 1 \text{m/s}^2$$

Substituting in Eq. (i), we get

$$N = 90 \text{ N}.$$

23. ***(a)*** Force of friction at different contacts are shown in figure.



Here, $f_1 = \mu_2 mg$

and $f_2 = \mu_1 (11\, mg)$

Given that $\mu_2 > 11\mu_1$

∴ $f_1 > f_2$

Retardation of upper block

$$a_1 = \frac{f_1}{m} = \mu_2 g$$

Acceleration of lower block

$$a_2 = \frac{f_1 - f_2}{m} = \frac{(\mu_2 - 11\mu_1)g}{10}$$

Relative retardation of upper block

$$a_r = a_1 + a_2 \quad \text{or} \quad a_r = \frac{11}{10}(\mu_2 - \mu_1)g$$

Now, $0 = v_{\min}^2 - 2a_r l$

$$\therefore \quad v_{\min} = \sqrt{2a_r l} = \sqrt{\frac{22(\mu_2 - \mu_1)gl}{10}}$$ **Ans.**

(b) $0 = v_{\min} - a_r t$

or $t = \dfrac{v_{\min}}{a_r} = \sqrt{\dfrac{20l}{11(\mu_2 - \mu_1)g}}$ **Ans.**

24. $v_r = \sqrt{v_1^2 + v_2^2}$

Retardation $a = \mu g$

∴ Time when slipping will stop is $t = \dfrac{v_r}{a}$

$$\text{or} \quad t = \frac{\sqrt{v_1^2 + v_2^2}}{\mu g} \qquad \ldots\text{(i)}$$

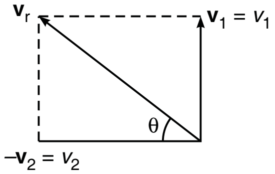
$$s_r = \frac{v_r^2}{2a} = \frac{v_1^2 + v_2^2}{2\mu g}$$

$$x_r = -s_r \cos\theta = -\left(\frac{v_1^2 + v_2^2}{2\mu g}\right)\left(\frac{v_2}{\sqrt{v_1^2 + v_2^2}}\right)$$

$$= \frac{-v_2\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

$$y_r = s_r \sin\theta = \left(\frac{v_1^2 + v_2^2}{2\mu g}\right)\left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}\right)$$

$$= \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$$



In time t, belt will move a distance $s = v_2 t$

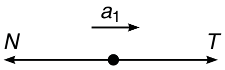
or $\dfrac{v_2\sqrt{v_1^2 + v_2^2}}{\mu g}$ in x-direction.

Hence, coordinate of particle,

$$x = x_r + s = \frac{v_2\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

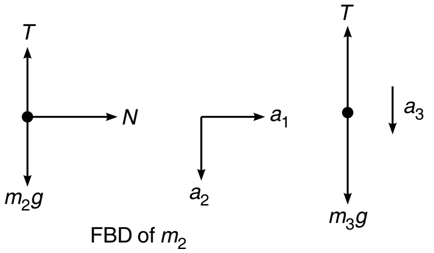
and $$y = y_r = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$$ **Ans.**

25. FBD of m_1 (showing only the horizontal forces)



Equation of motion for m_1 is

$$T - N = m_1 a_1 \quad \ldots\text{(i)}$$



FBD of m_2

Equations of motion for m_2 are

$$N = m_2 a_1 \quad \ldots\text{(ii)}$$

and $$m_2 g - T = m_2 a_2 \quad \ldots\text{(iii)}$$

Equation of motion for m_3 are

$$m_3 g - T = m_3 a_3 \quad \ldots\text{(iv)}$$

Further from constraint equation we can find the relation,

$$a_1 = a_2 + a_3 \quad \ldots\text{(v)}$$

We have five unknowns a_1, a_2, a_3, T and N solving, we get

$$a_1 = \frac{2m_1 m_3\, g}{(m_2 + m_3)(m_1 + m_2) + m_2 m_3}$$ **Ans.**

26. Writing equations of motion,

$$T - N = 3ma_1 \quad \ldots\text{(i)}$$

$$N = 2ma_1 \quad \ldots\text{(ii)}$$

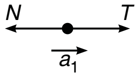
$$2mg - T = 2ma_2 \quad \ldots\text{(iii)}$$

$$T - \frac{mg}{2} = ma_3 \quad \ldots\text{(iv)}$$

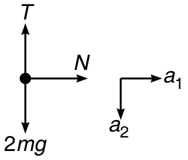
From constraint equation,

$$a_1 = a_2 - a_3 \quad \ldots\text{(v)}$$

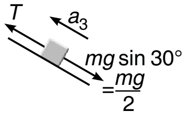
We have five unknowns. Solving the above five equations, we get



FBD of $3m$



FBD of $2m$



FBD of m

$$a_1 = \frac{3}{17} g, \quad a_2 = \frac{19}{34} g \text{ and } a_3 = \frac{13}{34} g$$

Acceleration of $m = a_3 = \frac{13}{34} g$,

Acceleration of $2m = \sqrt{a_1^2 + a_2^2} = \frac{\sqrt{397}}{34} g$

and acceleration of $3m = a_1 = \frac{3}{17} g$ **Ans.**

27. $a = \dfrac{m_A g}{m_A + M + m}$

For the equilibrium of B,

$$mg = \mu N = \mu(ma) = \frac{\mu m m_A g}{m_A + M + m}$$

$$\therefore \quad m_A = \frac{(M + m)m}{(\mu - 1)m}$$

$$m_A = \frac{(M + m)}{\mu - 1}$$ **Ans.**

Note $m_A > 0 \quad \therefore \quad \mu > 1$

9. Work, Energy and Power

INTRODUCTORY EXERCISE 9.1

1. $W = \mathbf{F} \cdot \mathbf{S} = \mathbf{F} \cdot (\mathbf{r}_f - \mathbf{r}_i)$

$$= (6\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot [(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}})]$$

$= -2\,\text{J}$

2. (a) $W_F = FS \cos 45°$

$$= (16)(2.2)\left(\frac{1}{\sqrt{2}}\right) = 24.9\ \text{J} \qquad \textbf{Ans.}$$

(b) $W_N = NS \cos 90° = 0$ **Ans.**

(c) $W_{mg} = (mg)(S) \cos 90° = 0$ **Ans.**

(d) Only three forces are acting. So, total work done is summation of all above work done.

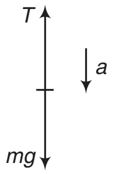
3. $W_T = (T)(x) \cos 0° = Tx$

$$W_W = (W)(x) \cos 90° = 0$$

$$W_N = (N)(x) \cos 90° = 0$$

$$W_F = (F)(x) \cos 180° = -Fx$$

4. $mg - T = ma = \dfrac{mg}{4}$



$\Rightarrow \qquad T = \dfrac{3mg}{4}$

$\therefore \qquad W_T = (T)(l)(\cos 180°)$

$$= -\frac{3}{4}\,mgl$$

5. $N = mg - F \sin 45° = 18 - \dfrac{F}{\sqrt{2}}$

Moving with constant speed means net force = 0

$$F \cos 45° = \mu N = \frac{1}{4}\left(18 - \frac{F}{\sqrt{2}}\right)$$

$\therefore \qquad \dfrac{4F}{\sqrt{2}} = 18 - \dfrac{F}{\sqrt{2}}$

$\therefore \qquad F = \dfrac{18\sqrt{2}}{5}\ \text{N}$

(a) $W_F = FS \cos 45°$

$$= \left(\frac{18\sqrt{2}}{5}\right)(2)\left(\frac{1}{\sqrt{2}}\right) = 7.2\ \text{J}$$

(b) $W_f = (\mu N)(S) \cos 180°$

$$= \left(\frac{1}{4}\right)\left(18 - \frac{F}{\sqrt{2}}\right)(2)(-1)$$

$$= -7.2\ \text{J}$$

(c) $W_{mg} = (mg)(S) \cos 90° = 0$

6. $W = \displaystyle\int_2^{-4} F dx = \int_2^{-4} (-2x)\, dx$

$$= [-x^2]_2^{-4} = -[16 - 4] = -12\ \text{J} \qquad \textbf{Ans.}$$

7. $W = \displaystyle\int_4^{2} F dx = \int_4^{2} \frac{4}{x^2}\, dx$

$$= -4\left[\frac{1}{x}\right]_4^2 = -4\left[\frac{1}{2} - \frac{1}{4}\right] = -1\ \text{J} \qquad \textbf{Ans.}$$

8. W = area under F-x graph

From $X = -4$ to $X = -2$

$F = -$ ve (from graph)

$S = +$ ve

$\therefore \qquad W_1 = -\dfrac{1}{2} \times 2 \times 10 = -10\ \text{J}$

From −2 to 4

$F = +$ ve

$S = +$ ve

$\therefore \qquad W_2 = +\dfrac{1}{2}(6 + 2)(10)$

$$= +40\ \text{J}$$

$\therefore \qquad W_T = W_1 + W_2 = 30\ \text{J}$ **Ans.**

9. (a) From $x = 10\,\text{m}$ to $x = 5\,\text{m}$

$S = -$ ve

$F = +$ ve (from graph)

$\therefore \qquad W_1 = -$ Area

$= -5 \times 3$

$= -15\ \text{J}$ **Ans.**

(b) From $x = 5\,\text{m}$ to $x = 10\ \text{m}$

$S = +$ ve

and $\qquad F = +$ ve

$\therefore \qquad W_2 = +\,\text{Area} = 5 \times 3$

$= 15\ \text{J}$ **Ans.**

(c) From $x = 10\,\text{m}$ to $x = 15\ \text{m}$

$S = +$ ve

$F = +$ ve

$\therefore \qquad W_3 = \text{Area}$

$= 3\ \text{J}$ **Ans.**

(d) From $x = 0$ to $x = 15\ \text{m}$

$S = +$ ve and $F = +$ ve

$$\therefore \quad W_4 = + \text{ Area}$$
$$= \frac{1}{2} \times 3 \times (12 + 6) = 27 \text{ J}$$ **Ans.**

10. (a) From $x = 0$ to $x = 3.0$ m

$$S = + \text{ ve}$$
$$F = + \text{ ve}$$
$$\therefore \quad W_1 = + \text{ Area}$$
$$= + 4 \text{ J}$$

(b) From $x = 3$ m to $x = 4$ m

$$F = 0 \quad \Rightarrow \quad W_2 = 0$$

(c) From $x = 4$ m to $x = 7$ m

$$S = + \text{ ve and } F = - \text{ ve}$$
$$\therefore \quad W_3 = + \text{ Area } = -1 \text{ J}$$

(d) From $x = 0$ to $x = 7$ m

$$W = W_1 + W_2 + W_3 = 3 \text{ J}$$

INTRODUCTORY EXERCISE 9.2

1. From work energy theorem,

$$W_{\text{net}} = W_{mg} + W_{\text{air}} = \frac{1}{2} m (v_f^2 - v_i^2)$$
$$\Rightarrow \quad 0 + W_{\text{air}} = \frac{1}{2} \times 0.1 [(6)^2 - (10)^2] = -3.2 \text{ J}$$

2. $W_{\text{All}} = \Delta \text{ KE} = K_f - K_i$

$$= \frac{1}{2} m (v_f^2 - v_i^2) = \frac{1}{2} \times 2 \times (0 - 20^2)$$
$$= -400 \text{ J}$$ **Ans.**

4. $v_{x=0} = 0 \quad : \quad v_{x=b} = \alpha \sqrt{b}$

$$\therefore \quad W_{\text{All}} = \Delta \text{ KE} = K_f - K_i$$
$$= \frac{1}{2} m (v_f^2 - v_i^2) = \frac{1}{2} \times m [(\alpha \sqrt{b})^2 - 0]$$
$$= \frac{1}{2} m \alpha^2 b$$ **Ans.**

5. $W_F = FS \cos 0° = 80 \times 4 \times 1 = 320$ J

$$W_{mg} = (mg)(S) \cos 180°$$
$$= (50)(4)(-1)$$
$$= -200 \text{ J}$$
$$K_f = W_{\text{All}} = 120 \text{ J}$$ **Ans.**

6. $K_f - K_i = W_F = \int F dx$

$$0 - \frac{1}{2} m v_0^2 = \int_0^x -Ax\,dx$$
$$\therefore \quad \frac{1}{2} m v_0^2 = \frac{Ax^2}{2} \quad \text{or} \quad x = v_0 \sqrt{\frac{m}{A}}$$ **Ans.**

7. (a) $T = F$, (b) $W_{\text{All forces}} = 40$ J

8. $v = \frac{ds}{dt} = (4t - 2)$

$$W_{\text{all}} = \Delta K = K_f - K_i = K_{2\text{ s}} - K_{0\text{ s}}$$
$$= \frac{1}{2} m (v_f^2 - v_i^2)$$
$$= \frac{1}{2} \times 2 [(4 \times 2 - 2)^2 - (4 \times 0 - 2)^2]$$
$$= 32 \text{ J}$$ **Ans.**

9. $W_{\text{all}} = \Delta K = K_f - K_i = K_f \qquad (\text{as } K_i = 0)$

$$\therefore \quad W_{mg} + W_{\text{chain}} = K_f$$
$$\text{or} \quad W_{\text{chain}} = K_f - W_{mg} = \frac{1}{2} m v_f^2 - mgh$$
$$= \frac{1}{2} \times 30 \times (0.4)^2 - 30 \times 10 \times 2$$
$$= -597.6 \text{ J}$$ **Ans.**

INTRODUCTORY EXERCISE 9.3

1. $\Delta U = -W$

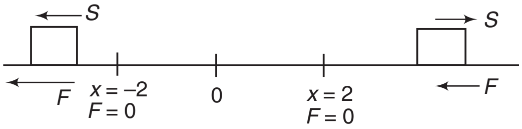
So, if work done by conservative force is positive then ΔU is negative or potential energy will decrease. But there is no straight forward rule regarding the kinetic energy.

2.
$$U_A = -60 \text{ J}$$
$$U_B = -20 \text{ J}$$
$$\therefore \quad U_B - U_A = 40 \text{ J}$$

INTRODUCTORY EXERCISE 9.4

1. $U = \frac{x^3}{3} - 4x + 6$

$$F = -\frac{dU}{dx} = -x^2 + 4$$
$$F = 0 \quad \text{at} \quad x = \pm 2 \text{ m}$$



For $x > 2$ m, $F = -$ ve i.e. displacement is in positive direction and force is negative. Therefore $x = 2$ is stable equilibrium position.

For $x < -2$ m, $F = -$ ve

i.e. force and displacement are in negative directions. Therefore, $x = -2$ m is unstable equilibrium position.

2. At A, $x = 0$ and $F = 0$

For $x > 0$, $F = +$ ve. i.e. force is in the direction of displacement. Hence A is unstable equilibrium position.

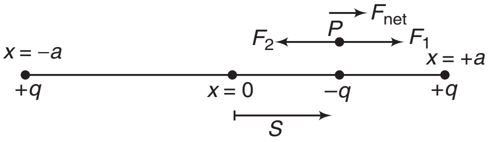
Same concept can be applied with E also.

At point C, $F = 0$.

For $x > x_C$, $F = -$ ve

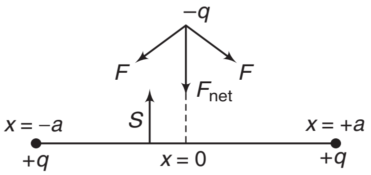
Displacement is positive and force is negative (in opposite direction of displacement). Therefore, C point is stable equilibrium point.

3. (a) At $x = 0$, $F = 0$. At P, attraction on $-q$ or $F_1 >$ attraction F_2



∴ Net force is in the direction of displacement.
So, equilibrium is unstable.

(b) F_{net} is in opposite direction of S. Therefore equilibrium is stable.



4. U = minimum $= -20$ J

at $x = 2$ m

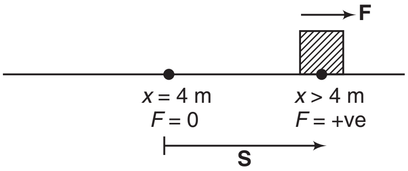
∴ $x = 2$ m is stable equilibrium position **Ans.**

5. $F = (x - 4)$

$F = 0$ at $x = 4$ m

When $x > 4$ m, $F = +$ ve

When displaced from $x = 4$ m (towards positive direction) force also acts in the same direction.



Therefore, equilibrium is unstable.

INTRODUCTORY EXERCISE 9.5

1. (a) $$P_{av} = \frac{W}{t} = \frac{\frac{1}{2}mv^2}{t} = \frac{\frac{1}{2}m(at)^2}{t}$$

$$= \frac{1}{2}ma^2 t = \frac{1}{2}\times 1\times(4)^2(2)$$

$= 16$ W **Ans.**

(b) $P_i = Fv = (ma)(at)$

$= ma^2t = (1)(4)^2(4)$

$= 64$ W **Ans.**

2. *(i)* $W = Pt = \frac{1}{2}mv^2$

(ii) $v = \sqrt{\frac{2Pt}{m}}$

(iii) Integrating the velocity, we will get displacement

∴ $$S = \sqrt{\frac{2P}{m}}\left(\frac{t^{3/2}}{3/2}\right) = \sqrt{\frac{8P}{9m}}\,t^{3/2}$$ **Ans.**

3. (a) $\text{KE} = W = \int P\,dt = \int 2t \cdot dt = t^2$

$$\frac{1}{2}mv^2 = t^2$$

∴ $$v = \sqrt{\frac{2}{m}}\,t$$ **Ans.**

(b) $P_{av} = \frac{W}{t} = \frac{t^2}{t} = t$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

1. F = constant

∴ $$a = \frac{F}{m} = \text{constant}$$

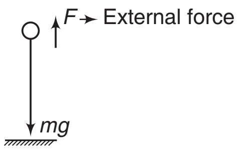
$$v = at = \frac{F}{m}t$$

$$P = F \cdot v = \frac{F^2}{m}t$$

∴ $P \neq$ constant

But $P \propto t$

3. In the figure, work done by conservative force (gravity force) is positive, potential energy is decreasing. But kinetic energy may increase, decrease or remain constant, depending on the value of F.



4. In non-uniform circular motion (when speed ≠ constant) work done by all the forces is not zero.

5. Slope of S-t graph is increasing.

Therefore, speed of the particle is increasing.

9. $W_{\text{All}} = \Delta K = K_f - K_i$

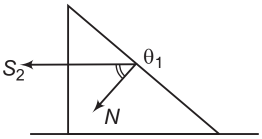
$$= \frac{1}{2} m (v_f^2 - v_i^2)$$

$v_f = v_{t_2}$ = Slope of S-t graph at t_2

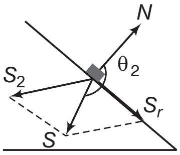
$v_i = v_{t_1}$ = Slope of S-t graph at t_1

These two slopes are not necessarily equal.

10.



θ_1 is acute

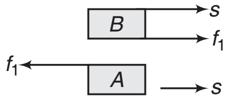


θ_2 is obtuse

11. $(W_{f_1})_{\text{on }A} = -\, f_1 S$

$(W_{f_1})_{\text{on }B} = +\, f_1 S$

$\therefore$ Net work done by $f_1 = 0$



12. Decrease in mechanical energy in first case

$$= E_i - E_f = \frac{1}{2} mv^2 - 0$$

$$= \frac{1}{2} mv^2$$

Decrease in mechanical energy in second case

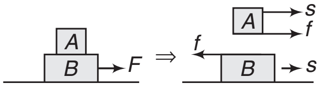
$$= E_i - E_f = \frac{1}{2} mv^2 - mgh$$

Further, μ does not depend on angle of inclination.

Single Correct Option

3. $(W_f)_A = +$ ve

$(W_f)_B = -$ ve



If there is no slip between A and B then f is static and total work done by static friction on system is zero..

4. $W = \mathbf{F} \cdot \mathbf{S} = \mathbf{F} \cdot (\mathbf{r}_f - \mathbf{r}_i)$

5. $W = \int_0^5 F dx = \int_0^5 (7 - 2x + 3x^2)\, dx$

$= 135$ J **Ans.**

6. $P_i = \mathbf{F} \cdot \mathbf{v}$

7. $P = \dfrac{W}{t} = \dfrac{mgh + \frac{1}{2} mv^2}{t}$

$$= \frac{(800 \times 10 \times 10) + \frac{1}{2} \times 800 \times (20)^2}{60}$$

$= 4000$ W **Ans.**

8. $E_f = 80\,\%$ of $E_i \Rightarrow mgh' = (0.8)\, mgh$

or $h' = 0.8\, h = 8$m **Ans.**

9. $K_i + U_i = K_f + U_f$

$\therefore \quad 0 + 6 = \frac{1}{2} \times 1 \times v^2 + 2$

$\therefore \quad v = 2\sqrt{2}$ m/s

10. Maximum range is obtained at 45°.

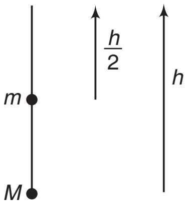
$$E = \frac{1}{2} mu^2 \quad \text{...(i)}$$

At highest point, $v = u \cos 45° = \dfrac{u}{\sqrt{2}}$

$$\therefore \quad E' = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{u}{\sqrt{2}}\right)^2$$

$$= \frac{\left(\frac{1}{2} mu^2\right)}{2} = \frac{E}{2}$$ **Ans.**

11. $W = Mgh + mg\dfrac{h}{2} = \left(M + \dfrac{m}{2}\right) gh$



12. (a) Velocity is decreasing. Therefore, acceleration (or net force) is opposite to the direction of motion.

(b) and (c): some other forces (other than friction) may also act which retard the motion.

13. Let retarding force is F

Then, $$Fx = \frac{1}{2} mv^2 \quad \text{...(i)}$$

and $$F(x') = \frac{1}{2} m (2v)^2 \quad \text{...(ii)}$$

Solving these two equations, we get

$$x' = 4x$$ **Ans.**

14. $K_i + U_i = K_f + U_f$

$$\therefore \quad \frac{1}{2} mv_0^2 + 0 = \frac{1}{2} mv^2 + mgh$$

$$\therefore \quad v = \sqrt{v_0^2 - 2gh}$$ **Ans.**

15. $a = \frac{F}{m}$

$$v = at = \left(\frac{F}{m}\right)(2)$$

$$P = F \cdot v = \frac{2F^2}{m}$$ **Ans.**

16. $a = \frac{F}{m}$

$$v = at = \frac{F}{m} t$$

$$P = F \cdot v = \left(\frac{F^2}{m}\right) t$$

or $$P \propto t$$

i.e. P-t graph is a straight line passing through origin.

17. $K = \frac{1}{2} mv^2 = \frac{1}{2} m (gt^2)$

i.e. $K \propto t^2$

i.e. during downward motion. K-t graph is a parabola passing through origin with K increasing with time. Then in upward journey K will decrease with time.

18. Upthrust = (Volume immersed) (density of liquid) g

$$= \left(\frac{5}{3000}\right)(1000)(10)$$

$$= \frac{50}{3} \text{ N}$$

weight = 50 N

$\therefore$ Applied force (upwards) = weight – upthrust

$$= 50 - \frac{50}{3} = \frac{100}{3} \text{ N}$$

$$W = FS = \frac{100}{3} \times 3$$

$$= 100 \text{ J}$$ **Ans.**

19. $S = 0$ (in vertical direction)

$$\Rightarrow \quad W = FS = 0$$

$$\therefore \quad P_{\text{av}} = \frac{W}{t} = 0$$

20. $k \propto \frac{1}{l}$

l of shorter part is less, therefore value of k is more

$$W = \frac{1}{2} kx^2$$

$\therefore$ $W_{\text{shorter part}}$ will be more.

21. Let θ_1 is the angle of $\mathbf{F}$ with positive x-axis.

$$\therefore \quad \tan\theta_1 = \frac{F_y}{F_x} = \frac{15}{20} = \frac{3}{4} = m_1 \quad \text{(say)}$$

Slope of given line, $m_2 = -\frac{\alpha}{3}$

$$W = 0 \quad \text{if} \quad \mathbf{F} \perp \mathbf{S}$$

or $$m_1 m_2 = -1$$

$$\therefore \quad \left(\frac{3}{4}\right)\left(-\frac{\alpha}{3}\right) = -1$$

$$\therefore \quad \alpha = 4$$ **Ans.**

22. Decrease in gravitational potential energy of block = increase in spring potential energy

$$\therefore \quad mg(x_m \sin\theta) = \frac{1}{2} K x_m^2$$

$$\therefore \quad x_m = \frac{2\, mg \sin\theta}{K}$$ **Ans.**

23. $\mathbf{v} = \frac{d\mathbf{S}}{dt} = (4\,t)\,\hat{\mathbf{i}}$

$$P = \mathbf{F} \cdot \mathbf{v} = 12\, t^2$$

$$\therefore \quad W = \int_0^2 P dt = \int_0^2 (12)\, t^2 dt$$

$$= 24 \text{ J}$$ **Ans.**

24. Decrease in potential energy = Work done against friction

$$\therefore \quad mg(h + d) = F \cdot d$$

Here F = average resistance

$$\Rightarrow \quad F = mg\left(1 + \frac{h}{d}\right)$$ **Ans.**

25. $F = kx \quad \Rightarrow \quad x = \dfrac{F}{k}$

Now, $U = \dfrac{1}{2}kx^2$

$$= \frac{1}{2}k\left(\frac{F}{k}\right)^2$$

or $U \propto \dfrac{1}{k}$

k_B is double. Therefore, U_B will be half.

26. $\dfrac{1}{2}mv^2 = \dfrac{1}{2}kX_m^2$

$$\therefore \quad X_m = \sqrt{\frac{m}{k}}\cdot v$$

$$= \sqrt{\frac{0.5}{50}} \times 1.5$$

$$= 0.15 \text{ m}$$ **Ans.**

27. F = resistance is same

$$\frac{1}{2}mv^2 = n\,(F \cdot d)$$

$$\Rightarrow \quad n \propto v^2$$

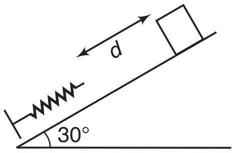
If v is doubled, n will become four times.

28. $E_i = E_f$

$$0 = \frac{1}{2} \times 10 \times (0.15)^2 - 0.1 \times 10 \times 0.15 + \frac{1}{2} \times 0.1 \times v^2$$

$$\therefore \quad v = 0.866 \text{ m/s}$$ **Ans.**

29. From $F = kx$



$$k = \frac{F}{x} = \frac{100}{1.0}$$

$$= 100 \text{ N/m}$$

Decrease in gravitational potential energy = increase in spring potential energy

$$\Rightarrow \quad 10 \times 10 \times (d + 2) \sin 30° = \frac{1}{2} \times 100 \times (2)^2$$

Solving, we get $d = 2$ m

$\therefore$ Total distance covered before coming momentarily to rest

$$= d + 2 = 4 \text{ m}$$ **Ans.**

30. Speed (and hence the kinetic energy) will increase as long as $mg\sin\theta > kx$.

31. (a) $W_N = NS \cos 90° = 0$

(b) $W_T = TS \cos 0° = (mg \sin\theta)\left(\dfrac{h}{\sin\theta}\right)$

$$= mgh$$

(c) $W_{mg} = (mg)\,(h) \cos 180° = -mgh$

(d) $W_{\text{Total}} = \Delta K = K_f - K_i = 0$

As block is moved slowly or $K_f = K_i$

32. Displacement of floor = 0

33. W = area under F-x graph

From $X = 1$ to $X = 3$, force and displacement both are positive. Therefore work done is positive.

$$W_1 = + \text{Area} = + 20 \text{ J}$$

From $X = 3$ to $X = 4$, force is negative but displacement is positive. Therefore work done is again positive.

$$\therefore \quad W_2 = - \text{Area} = - 5 \text{J}$$

From $X = 4$ to $X = 5$, force and displacement both are positive. Therefore work done is again positive.

$$\therefore \quad W_3 = + \text{Area} = + 5 \text{J}$$

$$W_{\text{Total}} = W_1 + W_2 + W_3 = 20 \text{ J}$$

34. $v = \dfrac{dx}{dt} = t^2$

$$W = \frac{1}{2}mv^2 = \frac{1}{2}mt^4$$

$$= \frac{1}{2} \times 2 \times (2)^4 = 16 \text{ J}$$ **Ans.**

35. At highest point,

$$\frac{1}{2}mu_x^2 = K$$

$$\therefore \quad u_x = \sqrt{\frac{2K}{m}}$$

$$R = 4\text{H}$$

$$\frac{2u_x u_y}{g} = \frac{4u_y^2}{2g}$$

$$\therefore \quad u_y = u_x = \sqrt{\frac{2K}{m}}$$

Now, $K_i = \dfrac{1}{2}mu^2 = \dfrac{1}{2}m\,(u_x^2 + u_y^2)$

$$= \frac{1}{2}m\left(\frac{2K}{m} + \frac{2K}{m}\right)$$

$$= 2K$$ **Ans.**

36. $\Delta K = W = \int P dt = \int_2^4 (3t^2 - 2t + 1)\, dt$

$= 46 \text{ J}$ **Ans.**

37. $K_f - K_i = W = \int F dx$

$\therefore \quad K_f = K_i + \int_{20}^{30} (-0.1x)\, dx$

$= \frac{1}{2} \times 10 \times (10)^2 - \left[0.1 \frac{x^2}{2}\right]_{20}^{30}$

$= 475 \text{ J}$ **Ans.**

38. KE = decrease in potential energy = mgh

or $\quad \text{KE} \propto m$

or $\quad \frac{(\text{KE})_1}{(\text{KE})_2} = \frac{12}{6} = \frac{2}{1}$

39. $W = U_f - U_i = \frac{1}{2} K (X_f^2 - X_i^2)$

$= \frac{1}{2} \times 5 \times 10^3 [(0.1)^2 - (0.05)^2]$

$= 18.75 \text{ J}$ **Ans.**

Subjective Questions

1. $K = \frac{p^2}{2m}$

$K' = \frac{p'^2}{2m} = \frac{(1.5p)^2}{2m}$

$= (2.25) \frac{p^2}{2m} = 2.25 K$

$\therefore \quad \% \text{ increase } = \frac{K' - K}{K} \times 100$

$= 125 \%$ **Ans.**

2. $p = \sqrt{2Km}$ or $P \propto K^{\frac{1}{2}}$

For small % changes,

% change in $p = \frac{1}{2}$ (% change in K)

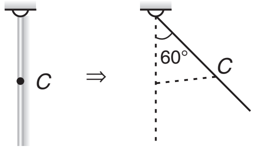
$= \frac{1}{2} (1 \%) = 0.5 \%$ **Ans.**

3. Total work done $= -\frac{1}{2} K (2x_0)^2 = -2Kx_0^2$

$\therefore$ Work done on one mass

$= \frac{-2Kx_0^2}{2} = -Kx_0^2$

4. For increase in gravitational potential energy of a rod we see the centre of the rod.



W = change in potential energy

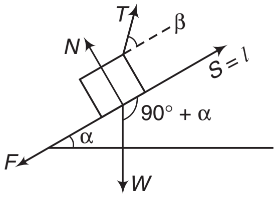
$= mg \frac{l}{2} (1 - \cos \theta)$

Substituting the values, we have

$W = (0.5)(9.8)\left(\frac{1.0}{2}\right)(1 - \cos 60°)$

$= 1.225 \text{ J}$ **Ans.**

5. $W_T = (T)(l) \cos \beta$



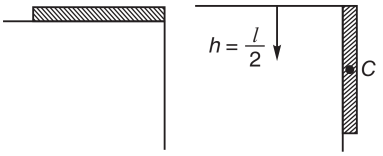
$W_N = (N)(l) \cos 90° = 0$

$W_W = (W)(l) \cos (90 + \alpha) = -W l \sin \alpha$

$W_f = (F)(l) \cos 180° = -Fl$

6. Decrease in potential energy of chain

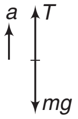
= increase in kinetic energy



$\therefore \quad mg \frac{l}{2} = \frac{1}{2} mv^2$

or $\quad v = \sqrt{gl}$

7. $T - mg = ma$



$\therefore \quad T = m(g + a) = 72(9.8 + 0.98)$

$= 776.16 \text{ N}$

(a) $\quad W_T = TS \cos 0°$

$= (776.16)(15)$

$= 11642 \text{ J}$ **Ans.**

(b) $W_{mg} = (mg)(S) \cos 180°$

$= (72 \times 9.8 \times 15)(-1)$

$= -10584 \text{ J}$ **Ans.**

(c) $K = W_{\text{Total}} = 11642 - 10584$

$= 1058 \text{ J}$ **Ans.**

(d) $K = \frac{1}{2} mv^2$

$$\therefore \quad v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2 \times 1058}{72}}$$

$= 5.42 \text{ m/s}$ **Ans.**

8. (a) $K = W = \int F dx = \int_0^2 (2.5 - x^2)\, dx$

$= 2.33 \text{ J}$ **Ans.**

(b) Maximum kinetic energy of the block is at a point where force changes its direction.

or $F = 0$

at $X = \sqrt{2.5}$

$= 1.58 \text{ m}$

$$\therefore \quad K_{\max} = \int_0^{1.58} (2.5 - X^2)\, dx$$

$= 2.635 \text{ J}$ **Ans.**

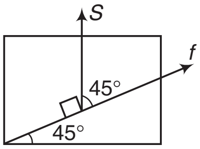
9. $W_f = f S \cos 45°$

Here, $f = mg \sin\theta$, because uniform velocity means net acceleration = 0 or net force = 0

$\therefore \quad W_f = (mg \sin\theta)(S)(\cos 45°)$

$= (1)(10)(\sin 45°)(S)(\cos 45°)$

If $S = vt = 2 \times 1 = 2 \text{ m}$



$$W_f = (1)(10)\left(\frac{1}{\sqrt{2}}\right)(2)\left(\frac{1}{\sqrt{2}}\right)$$

$= 10 \text{ J}$ **Ans.**

10. $v_{m_1} = v_{m_2} = v$ (say)

$$d_{m_1} = h_{m_2} = 4 \text{m}$$

Using the equation,

$E_i - E_f$ = Work done against friction

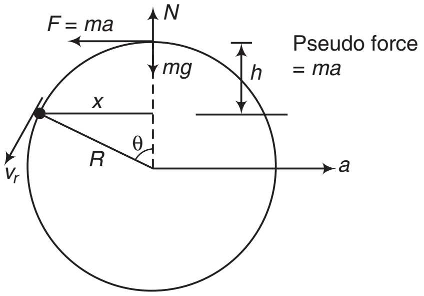
$$0 - \left[\frac{1}{2} \times (10 + 5)(v^2) - 5 \times 10 \times 4\right]$$

$= 0.2 \times 10 \times 10 \times 4$

Solving we get,

$$v = 4 \text{ m/s}$$

11. **FBD of particle w.r.t. sphere**



$$K_f = \frac{1}{2} mv_r^2 = W_{\text{All}}$$

or $\frac{1}{2} mv_r^2 = W_N + W_F + W_{mg}$

$= 0 + Fx + mgh$

$= (ma)(R\sin\theta) + mg\,[R\,(1 - \cos\theta)]$

$\therefore \quad v_r = \sqrt{2gR\,(1 + \sin\theta - \cos\theta)}$ **Ans.**

12. From constraint relations, we can see that

$$v_A = 2\, v_B$$

Therefore, $v_A = 2(0.3) = 0.6 \text{ m/s}$

as $v_B = 0.3 \text{ m/s}$ (given)

Applying $W_{nc} = \Delta U + \Delta K$

we get

$$-\mu\, m_A g S_A = -m_B g S_B + \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

Here, $S_A = 2S_B = 2 \text{ m}$ as $S_B = 1\text{m}$ (given)

$$\therefore \; -\mu(4.0)(10)(2) = -(1)(10)(1) + \frac{1}{2}(4)(0.6)^2 + \frac{1}{2}(1)(0.3)^2$$

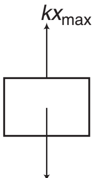
or $-80\mu = -10 + 0.72 + 0.045$

or $80\mu = 9.235$ or $\mu = 0.115$ **Ans.**

13. Let $x_{\max}$ = maximum extension of spring

Decrease in potential energy of A

= Increase in elastic potential energy of spring



$$\therefore \quad m_A g x_{\max} = \frac{1}{2} k x_{\max}^2$$

$$\therefore \quad x_{\max} = \frac{2m_A g}{k}$$

To just lift the block B,

$$k x_{\max} = m_B g$$

$$\therefore \quad 2m_A g = m_B g = mg$$

$\therefore \quad m_A = \frac{m}{2}$ **Ans.**

14. (a) $K_A + U_A = K_B + U_B$

$$\therefore \quad 0 + \frac{1}{2} \times 500 \times (0.5 - 0.1)^2$$

$$= \frac{1}{2} \times 10 \times v^2 + \frac{1}{2} \times 500 \times (0.3 - 0.1)^2$$

On solving, we get

$$v = 2.45 \text{ m/s} \qquad \textbf{Ans.}$$

(b) $CO = \sqrt{(BO)^2 + (BC)^2}$

$$= \sqrt{(30)^2 + (20)^2}$$

$$\approx 36 \text{ cm}$$

$$= 0.36 \text{ m}$$

Again applying the equation,

$$K_A + U_A = K_C + U_C$$

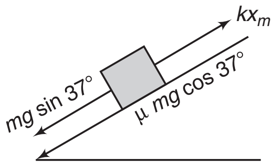
$$0 + \frac{1}{2} \times 500 \times (0.5 - 0.1)^2$$

$$= \frac{1}{2} \times 10 \times v^2 + \frac{1}{2} \times 500\,(0.36 - 0.1)^2$$

On solving, we get

$$v = 2.15 \text{ m/s} \qquad \textbf{Ans.}$$

15. Let X_m is maximum extension of spring. Then decrease in potential energy of M = increase in elastic potential energy of spring



$$\therefore \quad \frac{1}{2} kX_m^2 = MgX_m$$

or $\quad X_m = \dfrac{2\,Mg}{k}$

Now, $\quad kX_m = mg \sin 37° + \mu mg \cos 37°$

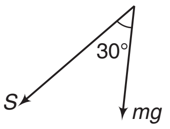
or $\quad 2\,Mg = (mg)\left(\dfrac{3}{5}\right) + \left(\dfrac{3}{4}\right) mg \left(\dfrac{4}{5}\right)$

$$\therefore \quad M = \frac{3}{5} m \qquad \textbf{Ans.}$$

16. (a) $\quad W_{mg} = (mg)\,(S) \cos 30°$

$$= (20)\,(2)\,(\sqrt{3}/2)$$

$$= 34.6 \text{ J} \qquad \textbf{Ans.}$$

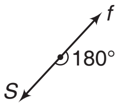


(b) $W_f = f\,S \cos 180°$

$$= (\mu mg \cos\theta)\,(S)\,(-1)$$

$$= -\left(\frac{1}{2}\right)(20)\,(\cos 60°)\,(2)$$

$$= -10 \text{ J} \qquad \textbf{Ans.}$$



17. $F = -\dfrac{dU}{dr} = \dfrac{A}{r^2}$ **Ans.**

18. (a) At $x = 6$ m,

$$U = (6 - 4)^2 - 16 = -12 \text{ J}$$

$$K = 8 \text{ J}$$

$$\therefore \quad E = U + K = -4 \text{ J} \qquad \textbf{Ans.}$$

(b) $U_{\text{min}} = -16$ J at $x = 4$ m

$$\therefore \quad K_{\text{max}} = E - U_{\text{min}}$$

$$= -4 + 16$$

$$= 12 \text{ J} \qquad \textbf{Ans.}$$

(c) $K = 0$

$$\therefore \quad U = E$$

or $\quad (x - 4)^2 - 16 = -4$

or $\quad x = (4 \pm 2\sqrt{3})$ m **Ans.**

(d) $F_x = -\dfrac{dU}{dx} = (8 - 2x)$ **Ans.**

(e) $F_x = 0$ at $x = 4$ m **Ans.**

19. Decrease in potential energy of 1 kg = increase in kinetic energy of both

$$\Rightarrow \quad 1 \times 10 \times 1 = \frac{1}{2} \times (4 + 1) \times v^2$$

$$\therefore \quad v = 2 \text{ m/s} \qquad \textbf{Ans.}$$

20. If A descends X then B will ascend $2x$. Further if speed of A at this instant is 2.5 m/s, then speed of B at this instant will be 5 m/s. Now,

Decrease in potential energy of A = increase in potential energy of B + increase in kinetic energy of both

$$\therefore \quad (300)\,x = (50)\,(2x) + \frac{1}{2}\left(\frac{300}{9.8}\right)(2.5)^2 + \frac{1}{2}\left(\frac{50}{9.8}\right)(5.0)^2$$

Solving we get,

$$x = 0.796 \text{ m} \qquad \textbf{Ans.}$$

21. If speed of sphere is v downwards then speed of wedge at this instant will be $v \cot\alpha$ in horizontal direction.

Now,

Decrease in potential energy of sphere = Increase in kinetic energy of both

$$\therefore \quad mgR = \frac{1}{2}mv^2 + \frac{1}{2}m\,(v\cot\alpha)^2$$

$$= \frac{1}{2}mv^2\,\text{cosec}^2\,\alpha$$

$$\therefore \quad v = \sqrt{2gR}\,\sin\alpha = \text{speed of sphere}$$

and speed of wedge $= v\cot\alpha$

$$= \sqrt{2gR}\,\cos\alpha$$ **Ans.**

22. If block drops 12 mm, then spring will further stretch by 24 mm. Now,

$$E_i = E_f$$

$$\therefore \quad \frac{1}{2}\times 1050\times(0.075)^2 = -45\times 10\times 0.012 + \frac{1}{2}\times 45\times v^2 + \frac{1}{2}\times 1050\times(0.099)^2$$

Solving we get, $v = 0.37$ m/s **Ans.**

23. If speed of block of 1.0 kg is 0.3 m/s then speed of 4.0 kg block at this instant would be 0.6 m/s. Applying,

$$E_i - E_f = \text{work done against friction}$$

$$\therefore \quad 0 - \left[\frac{1}{2}\times 1.0\times(0.3)^2 + \frac{1}{2}\times 4.0\times(0.6)^2 - 1\times 10\times 1\right] = \mu_k\times 4\times 10\times(2)$$

Solving this equation we get,

$$\mu_k = 0.12$$ **Ans.**

24. Retardation on horizontal surface,

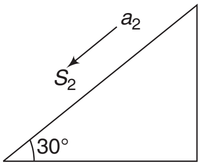
$$a_1 = \mu g = 0.15\times 10 = 1.5\ \text{m/s}^2$$

Velocity just entering before horizontal surface,

$$v = \sqrt{2a_1S_1}$$

$$= \sqrt{2\times 1.5\times 0.5}$$

$$= \sqrt{1.5}\ \text{m/s}$$



Acceleration on inclined plane,

$$a_2 = g\sin\theta - \mu g\cos\theta$$

$$= 10\times\frac{1}{2} - 0.15\times 10\times\frac{\sqrt{3}}{2}$$

$$= 3.7\ \text{m/s}$$

$$v = \sqrt{2a_2S_2} = \sqrt{1.5}$$

$$\therefore \quad S_2 = \frac{1.5}{2a_2} = \frac{1.5}{2\times 3.7} = 0.2\ \text{m}$$

$$h = S_2\sin 30° = 0.1\ \text{m}$$

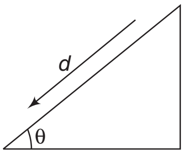
Now work done by friction

$= -$[initial mechanical energy]

$= -mgh = -(0.05)(10)(0.1)$

$= -0.05$ J **Ans.**

25. If A moves 1 m down the plane and its speed is v, then B will move 2m upwards and its speed will be $2v$.



Using the equation,

$$E_i - E_f = \text{Work done against friction}$$

$$\therefore \quad 0 - \left[\frac{1}{2}\times 30\times v^2 + \frac{1}{2}\times 5\times(2v)^2 + 5\times 10\times 2 - 30\times 10\times\frac{3}{5}\right]$$

$$= 0.2\times 30\times 10\times\frac{4}{5}\times 1$$

Solving this equation we get,

$$v = 1.12\ \text{m/s}$$ **Ans.**

Note $h = d\sin\theta = (1)\left(\frac{3}{5}\right)m = \frac{3}{5}m$

$$m_A = \frac{w_A}{g} = 30\ kg \text{ and } m_B = \frac{w_B}{g} = 5\ \text{kg}$$

26. (a) Thermal energy = Work done against friction

$= \mu_K\,mgd$

$= (0.25)(3.5)(9.8)(7.8)$ J

$= 66.88$ J **Ans.**

(b) Maximum kinetic energy

= work done against friction

$= 66.88$ J **Ans.**

(c) $\frac{1}{2}kx_m^2 =$ maximum kinetic energy

$$\frac{1}{2}\times 640\times x_m^2 = 66.88$$

$$x_m = 0.457\ \text{m} = 45.7\ \text{cm}$$ **Ans.**

LEVEL 2

Single Correct Option

1. $W_{\text{All}} = \frac{1}{2} mv^2$

$$\therefore \quad W_F + W_{mg} + W_N = \frac{1}{2} mv^2$$

$$\therefore \quad (5 \times 5) + \left(\frac{1}{2} \times 10 \times 5\right) + 0 = \frac{1}{2} \times \frac{1}{2} \times v^2$$

$$\therefore \quad v = 14.14 \text{ m/s} \qquad \textbf{Ans.}$$

2. $P = Fv = \text{constant}$

$$\therefore \quad F = \frac{P}{v}$$

$$\therefore \quad mv\left(\frac{dv}{ds}\right) = \frac{P}{v}$$

$$\therefore \quad \int_0^s ds = \frac{m}{P}\int_v^{2v} v^2 dv$$

Solving we get, $s = \frac{7mv^3}{3P}$ **Ans.**

3. $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{100}{1}$$

$$= 100 \text{ N/m}$$

$$E_i = E_f$$

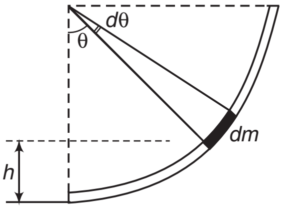
$$\therefore \quad \frac{1}{2} \times 10 \times v^2 = \frac{1}{2} \times 100 \times (2)^2 - (10)(10)(2 \sin 30°)$$

Solving we get,

$$v = \sqrt{20} \text{ m/s} \qquad \textbf{Ans.}$$

4. $dm = \left(\frac{m}{\pi/2}\right) d\theta = \left(\frac{2m}{\pi}\right) d\theta$

$$h = R(1 - \cos\theta)$$



$$dU_i = (dm)\,gh = \frac{2mgR}{\pi}(1 - \cos\theta)\,d\theta$$

$$\therefore \quad U_i = \int_0^{\pi/2} dU_i = \frac{2mgR}{\pi}\left(\frac{\pi}{2} - 1\right)$$

$$= mgR\left(1 - \frac{2}{\pi}\right)$$

Now, $\quad U_i + K_i = U_f + K_f$

$$\therefore \quad mgR\left(1 - \frac{2}{\pi}\right) = 0 + \frac{1}{2} mv^2$$

or $\quad v = \sqrt{2gR\left(1 - \frac{2}{\pi}\right)}$ **Ans.**

5. $T = 2mg$

As soon as string is cut T (on A) suddenly becomes zero. Therefore a force of $2mg$ acting on upward direction on A suddenly becomes zero.

So net force on it will become $2mg$ downwards.

$$\therefore \quad a_1 = \frac{2mg}{m} = 2g \qquad \text{(downwards)}$$

Spring force does not become instantly zero. So acceleration of B will not change abruptly.

or $\quad a_2 = 0$

6. Let X_m is maximum elongation of spring. Then,

increase in potential energy of spring = decrease in potential energy of C.

$$\therefore \quad \frac{1}{2} KX_m^2 = M_1 g X_m$$

or $\quad KX_m = \text{maximum spring force}$

$$= 2M_1 g = \mu_{\min} Mg$$

$$\therefore \quad \mu_{\min} = \frac{2M_1}{M} \qquad \textbf{Ans.}$$

7. $T_i = mg$ ...(i)

$$2kx = 2mg$$

$$\therefore \quad kx = mg$$

One kx force (acting in upward direction) is suddenly removed. So, net downward force on system will be kx or mg. Therefore, net downward acceleration of system,

$$a = \frac{mg}{2m} = \frac{g}{2}$$

Free body diagram of lower block gives the equation,

$$mg - T_f = ma = \frac{mg}{2}$$

$$\therefore \quad T_f = \frac{mg}{2} \qquad \text{..(ii)}$$

From these two equations, we get

$$\Delta T = \frac{mg}{2} \qquad \textbf{Ans.}$$

8. $F_{\text{net}} = mg \sin\theta - \mu mg \cos\theta$

$$= mg \sin\theta - 0.3\,xmg \cos\theta \qquad \text{...(i)}$$

At maximum speed $F_{\text{net}} = 0$. Because after this net force will become negative and speed will decrease.

From Eq. (i), $F_{\text{net}} = 0$ at

$$x = \frac{\tan\theta}{0.3} = \frac{3/4}{0.3} = 2.5 \text{ m} \qquad \textbf{Ans.}$$

9. At C, potential energy is minimum. So, it is stable equilibrium position.

Further,

$$F = -\frac{dU}{dr} = -(\text{Slope of } U\text{-}r \text{ graph})$$

Negative force means attraction and positive force means repulsion.

10. $P = \mathbf{F} \cdot \mathbf{v}$

$$= (-mg\,\hat{\mathbf{j}}) \cdot [u_x \hat{\mathbf{i}} + (u_y - gt)\,\hat{\mathbf{j}}]$$

$$= (-mgu_y) + mg^2 t$$

i.e. P *versus* t graph is a straight line with negative intercept and positive slope.

11. $E_i - E_f =$ Work done against friction

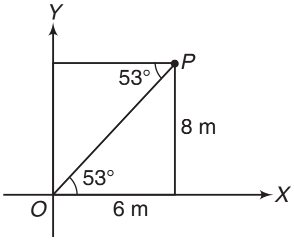
$$\therefore \frac{1}{2}kx^2 - \frac{1}{2}K\left(\frac{x}{2}\right)^2 = \mu mg\left(x + \frac{x}{2}\right)$$

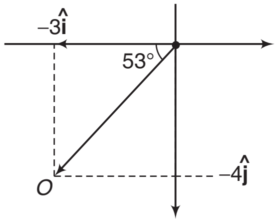
$$\therefore \qquad x = \frac{4\mu mg}{k} \qquad \textbf{Ans.}$$

12. $P = \mathbf{F} \cdot \mathbf{v} = Fv\cos\theta = TV\cos\theta$ **Ans.**

13. $\mathbf{F} = -\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}}\right]$

$$= (-3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$$





Since, particle was initially at rest. So, it will move in the direction of force.

We can see that initial velocity is in the direction of PO. So the particle will cross the X-axis at origin.

$$K_i + U_i = K_f + U_f$$

$$\therefore 0 + (3 \times 6 + 4 \times 8) = K_f + (3 \times 0 + 4 \times 0)$$

or $$K_f = 50 \text{ J} \qquad \textbf{Ans.}$$

14. $F = 0$ at $x = x_2$. When displaced from x_2 in negative direction, force is positive i.e. in the opposite direction of displacement. Similarly, when displaced in positive direction, force is negative.

15. $F_{\text{net}} = mg\sin\theta - \mu mg\cos\theta$

$$= mg\sin\theta - \mu_0 xg\cos\theta$$

$$a = \frac{F_{\text{net}}}{m} = g\sin\theta - \mu_0 xg\cos\theta$$

$$\therefore \qquad v \cdot \frac{dv}{dx} = g\sin\theta - \mu_0 xg\cos\theta$$

or $$\int_0^0 v\,dv = \int_0^{x_m} (g\sin\theta - \mu_0 xg\cos\theta)\,dx$$

Solving this equation we get,

$$x_m = \frac{2}{\mu_0}\tan\theta \qquad \textbf{Ans.}$$

16. Σ (Moments about C) = 0

$$\therefore \qquad (k_1 x)\,AC = (k_2 x)\,BC$$

$$\therefore \qquad \frac{AC}{BC} = \frac{k_2}{k_1} \qquad \text{...(i)}$$

$$AC + BC = l \qquad \text{...(ii)}$$

Solving these two equation we get,

$$AC = \left(\frac{k_2}{k_1 + k_2}\right) l \qquad \textbf{Ans.}$$

17. Work done by friction $= E_f - E_i$

$$= \frac{1}{2} \times 1 \times (2)^2 - 1 \times 10 \times 1$$

$$= -8 \text{ J} \qquad \textbf{Ans.}$$

18. $F = -\frac{dU}{dx} = \frac{12a}{x^{13}} - \frac{6b}{x^7}$

At equilibrium, $F = 0$ or $x = \left(\frac{2a}{b}\right)^{1/6}$

At this value of x, we can see that $\frac{d^2U}{dx^2}$ is positive.

So, potential energy is minimum or equilibrium is stable.

19. Σ (Moment about O) = 0

$$\therefore \quad (kx)\, l = mg\left(\frac{l}{2}\right) \quad \text{or } x = \frac{mg}{2k}$$

$$U = \frac{1}{2}kx^2 = \frac{(mg)^2}{8k}$$ **Ans.**

20. $E_i = E_f$

$$0 = m_2gh - m_1gh + \frac{1}{2}(m_1 + m_2)\, v^2$$

$$\therefore \quad v = \sqrt{2gh\left(\frac{m_1 - m_2}{m_1 + m_2}\right)}$$ **Ans.**

21. $F = -\dfrac{dU}{dx}$

or $\quad dU = -Fdx = (ax - bx^2)\, dx$

Assuming $U = 0$ at $x = 0$, and integrating the above equation we get,

$$U = \frac{ax^2}{2} - \frac{bx^3}{3}$$

$$U = 0 \text{ at } x = 0 \text{ and } x = \frac{3a}{2b}$$

For $x > \dfrac{3a}{2b}$, $b\dfrac{x^3}{3} > a\dfrac{x^2}{2}$ and U will become negative. So, option (c) is the most appropriate answer.

22. $W = FS$

F and S are same. Therefore,

$$\frac{W_A}{W_B} = \frac{1}{1}$$

From work energy theorem,

$$\frac{K_A}{K_B} = \frac{W_A}{W_B} = \frac{1}{1}$$

or $$\frac{1}{2}m_Av_A^2 = \frac{1}{2}m_B\, v_B^2$$

$$\therefore \quad \frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}} = \frac{2}{1}$$

23. Work done by conservative force

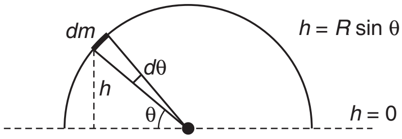
$$= -\Delta U$$
$$= U_i - U_f$$
$$= [k\,(1+1)] - [k\,(2+3)]$$
$$= -3k$$ **Ans.**

24. After falling on plank downward force on block is mg and upward force is kx. Kinetic energy will increase when $mg > kx$ and it will decrease when $kx > mg$. Therefore it is maximum when,

$$kx = mg \quad \text{or,} \quad x = \frac{mg}{k}$$

and this does not depend on h.

25. $dm = \left(\dfrac{m}{\pi}\right) d\theta$



$$\therefore \quad dU = (dm)\, gh = \left(\frac{m}{\pi}\, d\theta\right) gr\sin\theta$$

$$U_i = \int_0^{\pi} dU = \frac{2mgr}{\pi}$$

Now, $\quad K_i + U_i = K_f + U_f$

$$\therefore \quad 0 + \frac{2mgr}{\pi} = \frac{1}{2}mv^2 - mg\left(\frac{\pi r}{2}\right)$$

$$\therefore \quad v = \sqrt{2gr\left(\frac{2}{\pi} + \frac{\pi}{2}\right)}$$ **Ans.**

26. Maximum speed is at equilibrium where

$$F = kx \quad \Rightarrow \quad x = \frac{F}{k}$$

Now, $$F \cdot x = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

or $$F\left(\frac{F}{k}\right) = \frac{1}{2}mv^2 + \frac{1}{2}k\left(\frac{F}{k}\right)^2$$

Solving we get,

$$v = \frac{F}{\sqrt{mk}} = v_{\max}$$ **Ans.**

27. Increase in potential energy per unit time = decrease in kinetic energy of both

$$= -\frac{d}{dt}\left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= v_1\left(-m_1\frac{dv_1}{dt}\right) + v_2\left(-m_2\frac{dv_2}{dt}\right)$$

$$= v_1\,(-m_1\,a_1) + v_2\,(-m_2\,a_2)$$

or $$\frac{dU}{dt} = v_1\,(-F_1) + v_2\,(-F_2) \qquad \text{...(i)}$$

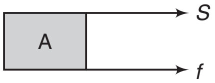
Here, $\quad -F_1 = -F_2 = kx = 200 \times 0.1 = 20\text{N}$

Substituting in Eq. (i) we have,

$$\frac{dU}{dt} = (4)\,(20) + (6)\,(20)$$

$$= 200 \text{ J/s}$$ **Ans.**

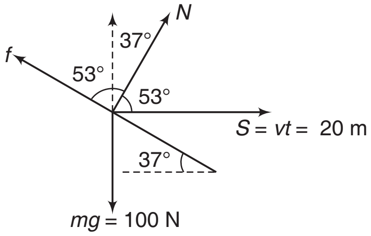
28. There is slip, so maximum friction will act on A in the direction of motion (or towards right)



$$f = \mu mg = 0.2 \times 45 \times 10 = 90\text{ N}$$
$$S = 40 - 10 - 10 = 20\text{ cm} = 0.2\text{ m}$$
$$\therefore \quad W = fS = 18\text{J} \qquad \textbf{Ans.}$$

29. Constant velocity means net force = 0

Using Lami's theorem in the figure,



We have,

$$\frac{N}{\sin(180° - 53°)} = \frac{100}{\sin 90°}$$
$$\therefore \quad N = 100 \sin 53° = 80\text{ N}$$

Now,
$$W_N = NS \cos 53°$$
$$= (80)(20)(0.6)$$
$$= 960\text{ J} \qquad \textbf{Ans.}$$

30. In both cases sudden changes in force by cutting the spring would be kx.

$$\therefore \quad a = \frac{\Delta F}{m} = \frac{kx}{m} \qquad \text{(in both case)}$$

In one case it is downwards and in other case it is upwards.

More than One Correct Options

1. $\mathbf{F} = -\left[\frac{\partial U}{\partial X}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}}\right] = (-7\hat{\mathbf{i}} - 24\hat{\mathbf{j}})\text{ N}$

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \left(-\frac{7}{5}\hat{\mathbf{i}} - \frac{24}{5}\hat{\mathbf{j}}\right)\text{ m/s}$$
$$|\mathbf{a}| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{24}{5}\right)^2} = 5\text{ m/s}^2 \qquad \textbf{Ans.}$$

Since, $\mathbf{a}$ = constant, we can apply,

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$= (8.6\hat{\mathbf{i}} + 23.2\hat{\mathbf{j}}) + \left(-\frac{7}{5}\hat{\mathbf{i}} - \frac{24}{5}\hat{\mathbf{j}}\right)(4)$$
$$= (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})\text{ m/s}$$
$$|\mathbf{v}| = \sqrt{(3)^2 + (4)^2} = 5\text{ m/s} \qquad \textbf{Ans.}$$

2. $F = -\frac{dU}{dx} = 5 - 200x$

At origin, $x = 0$

$$\therefore \quad F = 5\text{N}$$
$$a = \frac{F}{m} = \frac{5}{0.1} = 50\text{ m/s}^2$$

Mean position is at $F = 0$

or at, $x = \frac{5}{200} = 0.025$ m

$$a = \frac{F}{m} = \frac{5 - 200x}{0.1} = (50 - 2000x) \qquad \text{...(i)}$$

At 0.05 m from the origin,

$$x = +0.05\text{ m}$$

or
$$x = -0.05\text{ m}$$

Substituting in Eq. (i), we have,

$$|a| = 150\text{ m/s}^2$$

or
$$= 50\text{ m/s}^2$$

At 0.05 m from the mean position means,

$$x = 0.075$$

or
$$x = -0.025\text{ m}$$

Substituting in Eq. (i) we have,

$$|a| = 100\text{ m/s}^2 \qquad \textbf{Ans.}$$

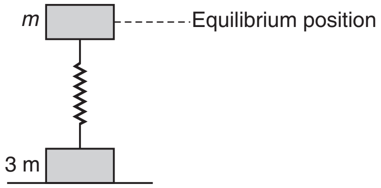
3. Spring force is always towards mean position. If displacement is also towards mean position, F and S will be of same sign and work done will be positive.

4. Work done by conservative force = $-\Delta U$

Work done by all the forces = ΔK

Work done by forces other than conservative forces = ΔE

5. At equilibrium



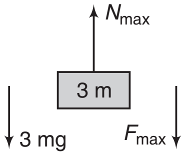
$$k\delta_0 = mg$$

or
$$\delta_0 = \frac{mg}{k}$$

where, δ_0 = compression

(b) $\delta_{\text{Total}} = \delta + \delta_0 = \dfrac{3mg}{k}$

$$F_{\max} = k\delta_{\max} = k\left(\frac{3mg}{k}\right) = 3mg \quad \text{(downward)}$$



∴ $N_{\max} = 3mg + F_{\max} = 6mg$ **Ans.**

(d) If $\delta > \dfrac{4mg}{k}$, then upper block will move a distance $x > \dfrac{4mg}{k} - \delta_0$ or $x > \dfrac{3mg}{k}$ from natural length.

Hence in this case, extension

$$x > \frac{3mg}{k}$$

or $F = kx > 3mg$ (upwards on lower block)

So lower block will bounce up.

6. (a) Decrease in potential energy of B = increase in spring potential energy

$$\therefore \quad 2mg\, x_m = \frac{1}{2} k x_m^2$$

$$\therefore \quad x_m = \frac{4mg}{k} \qquad \textbf{Ans.}$$

(b) $E_i = E_f$

$$0 = \frac{1}{2}(m + 2m)v^2 + \frac{1}{2} \times k \times \left(\frac{2mg}{k}\right)^2 - (2mg)\left(\frac{2mg}{k}\right)$$

$$\therefore \quad v = 2g\sqrt{\frac{m}{3k}}$$

(c) $a = \dfrac{kx_m - 2mg}{2m}$ (upwards)

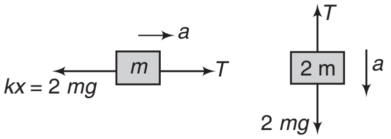
$$= \frac{k\left(\dfrac{4mg}{k}\right) - 2mg}{2m} = g$$

(d) $T - 2mg = ma$ ...(i)

$2mg - T = 2ma$ ...(ii)

Solving these two equations, we get

$a = 0$ and $T = 2mg$



8. Work done by conservative forces

$$= U_i - U_f = -20 + 10 = -10 \text{ J}$$

Work done by all the forces,

$$= K_f - K_i = 20 - 10 = 10 \text{ J}$$

9. (a) Work done by gravity in motion 1 is zero ($\theta = 90°$) and in motion 2 is negative ($\theta = 180°$).

(b) In both cases angle between N and S is acute.

(c) and (d) : Depending on the value of acceleration in motion 1, friction may act up the plane or down the plane. Therefore angle between friction and displacement may be obtuse or acute. So, work done by friction may be negative or positive.

Comprehension Based Questions

1. $U = E - K = 25 - K$

Since, $K \geq 0$

∴ $U \leq 25$ J

2. $U = E - K = -40 - K$

Since $K \geq 0$

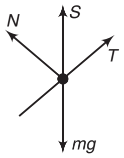
∴ $U \leq -40$ J

Match the Columns

1. $\mathbf{S} = \mathbf{r}_f - \mathbf{r}_i = (+2\hat{\mathbf{i}}) - (+4\hat{\mathbf{i}}) = -2\hat{\mathbf{i}}$

Now apply $W = \mathbf{F} \cdot \mathbf{S}$

2. Friction force = 0, as tension will serve that purpose

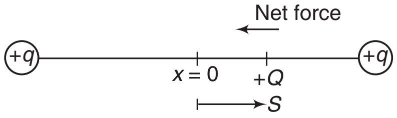


Angle between N and S or between T and S is acute

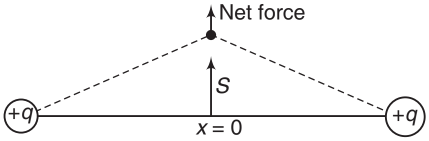
∴ W_T and W_N are positive. Angle between S and mg is 180°.

Therefore, W_{mg} is negative.

3. (a) Net force is towards the mean position $x = 0$ where, $F = 0$ when displaced from this position. Therefore, equilibrium is stable.

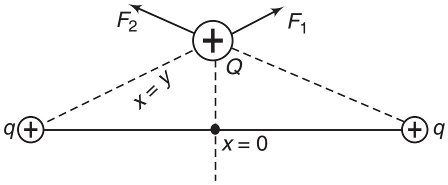


(b) Net force is away from the mean position. Therefore, equilibrium is unstable.



(c) Same logic can be applied as was applied in part (b).

(d) Net force is neither towards $x = 0$ nor away from $x = 0$. Therefore, equilibrium is none of the three.



4. (a) From A to B speed (or kinetic energy) will be increasing. Therefore net potential energy should decrease.

(b) From A to B, a part of decrease in gravitational potential energy goes in increasing the kinetic energy and rest goes in increasing the potential energy of spring.

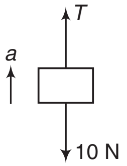
(c) and (d) : From B to C kinetic and gravitational potential energy are decreasing and spring potential energy is increasing.

∴ (decrease in kinetic energy) + (decrease in gravitational potential energy) = increase in spring potential energy.

5. Common acceleration of both blocks

$$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

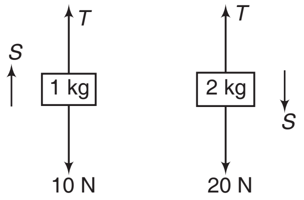
$$= \frac{20 - 10}{1 + 2} = \frac{10}{3}\ \text{m/s}^2$$



1 kg $\quad T - 10 = ma = 1 \times \frac{10}{3}$

∴ $\quad T = \frac{40}{3}$ N

$$S = \frac{1}{2}at^2 = \frac{1}{2} \times \frac{10}{3} \times (0.3)^2 = 0.15\ \text{m}$$



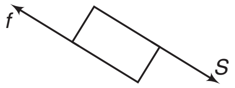
(a) W_{mg} on 2 kg block = (20) (S) cos 0°
$= (20)\,(0.15) = 3\,\text{J}$ **Ans.**

(b) W_{mg} on 1 kg block = (10) (S) cos 180°
$= (10)\,(0.15)\,(-1)$
$= -1.5\,\text{J}$ **Ans.**

(c) W_T on 2 kg block = (T) (S) cos 180°
$= \left(\frac{40}{3}\right)(0.15)\,(-1)$
$= -2$ J **Ans.**

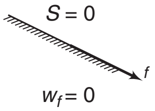
(d) W_T on 1 kg block = (T) (S) cos 0°
$= \left(\frac{40}{3}\right)(0.15)\,(1)$
$= +2$ J **Ans.**

6. (a)



$W_f = fS \cos 180° =$ negative

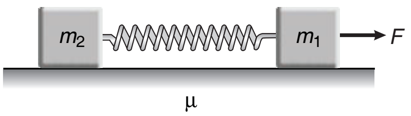
(b)



(c) and (d) : No solution is required.

Subjective Questions

1.



$(F - \mu m_1 g)x_m = \frac{1}{2}kx_m^2$

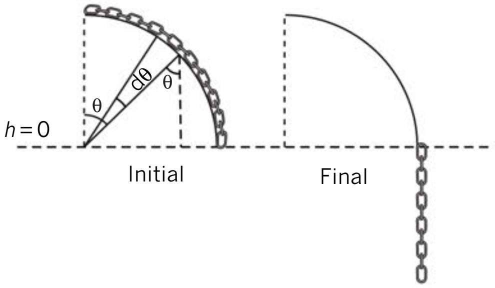
or $\quad kx_m = 2(F - \mu m_1 g)$

Second block will shift if $kx_m \geq \mu m_2 g$

∴ $\quad 2(F - \mu m_1 g) > \mu m_2 g$

or $\quad F > \left(m_1 + \frac{m_2}{2}\right)\mu g$ **Ans.**

2.



Initial PE,

$$U_i = \int_{\theta=0°}^{\theta=\pi/2} (r\,d\theta)(\rho)(g)(r\cos\theta)$$

$$= (\rho g\, r^2)\,[\sin\theta]_0^{\pi/2} = \rho g r^2$$

Final PE,

$$U_f = \left(\frac{\pi r}{2}\times\rho\right)(g)\left(-\frac{\pi r/2}{2}\right) = -\frac{\pi^2 r^2 \rho g}{8}$$

$$\Delta U = r^2\rho\, g\left(1+\frac{\pi^2}{8}\right)$$

$$\Delta U = \text{KE}$$

or $$r^2\rho g\left(1+\frac{\pi^2}{8}\right) = \frac{1}{2}\left(\frac{\pi r}{2}\right)(\rho)\, v^2$$

$\Rightarrow$ $$v = \sqrt{4rg\left(\frac{1}{\pi}+\frac{\pi}{8}\right)}$$

or $$v = \sqrt{rg\left(\frac{\pi}{2}+\frac{4}{\pi}\right)}$$ **Ans.**

3. For $t \le 0.2$ s

$$F = 800\text{ N} \quad \text{and} \quad v = \left(\frac{20}{0.3}\right)t$$

$\therefore$ $$P = Fv = (53.3\, t)\text{ kW}$$ **Ans.**

For $t > 0.2$ s

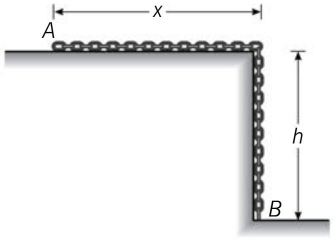
$$F = 800 - \left(\frac{800}{0.1}\right)(t-0.2) \text{ and } v = \frac{20}{0.3}t$$

$\therefore$ $$P = Fv = (160\,t - 533\,t^2)\text{ kW}$$

$$W = \int_0^{0.2}(53.3\,t)\,dt + \int_{0.2}^{0.3}(160\,t - 533\,t^2)\,dt$$

$$= 1.69\text{ kJ}$$ **Ans.**

4. At the instant shown in figure, net pulling force $= \frac{m}{l}gh$



Total mass being pulled $= \frac{m}{l}(x+h)$

$\therefore$ Acceleration $a = \dfrac{\text{Net pulling force}}{\text{Total mass being pulled}}$

$$= \frac{gh}{x+h}$$

$\therefore$ $$v\left(-\frac{dv}{dx}\right) = \frac{gh}{x+h}$$

or $$\int_0^v v\cdot dv = -gh\int_{(l-h)}^{0}\frac{dx}{x+h}$$

$\therefore$ $$\frac{v^2}{2} = gh\,[\ln(x+h)]_0^{l-h}$$

or $$\frac{v^2}{2} = gh\ln\left(\frac{l}{h}\right)$$

$\therefore$ $$v = \sqrt{2gh\ln(l/h)}$$ **Ans.**

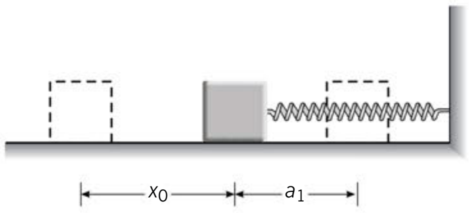
5. (a) From energy conservation principle,

Work done against friction = decrease in elastic PE

or $$f(x_0+a_1) = \frac{1}{2}k(x_0^2 - a_1^2)$$

or $$x_0 - a_1 = \frac{2f}{k} \qquad \text{...(i)}$$

From Eq. (i), we see that decrease of amplitude $(x_0 - a_1)$ is $\frac{2f}{k}$, which is constant and same for each cycle of oscillation.



(b) The block will come to rest when $ka = f$

or $$a = \frac{k}{f} \qquad \text{...(A)}$$

In the similar manner, we can write

$$a_1 - a_2 = \frac{2f}{k} \qquad \text{...(i)}$$

$$a_2 - a_3 = \frac{2f}{k} \qquad \text{...(ii)}$$

$$\dots\dots$$

$$a_{n-1} - a_n = \frac{2f}{k} \qquad \text{...(n)}$$

Adding Eqs. (i), (ii),… etc., we get

$$x_0 - a_n = n\left(\frac{2f}{k}\right)$$

or $$a_n = x_0 - n\left(\frac{2f}{k}\right) \qquad \text{...(B)}$$

Equating Eq. (A) and (B), we get

$$\frac{k}{f} = x_0 - n\left(\frac{2f}{k}\right)$$

or $$n = \frac{x_0 - \frac{k}{f}}{\frac{2f}{k}} = \frac{kx_0}{2f} - \frac{1}{2}$$

Number of cycles,

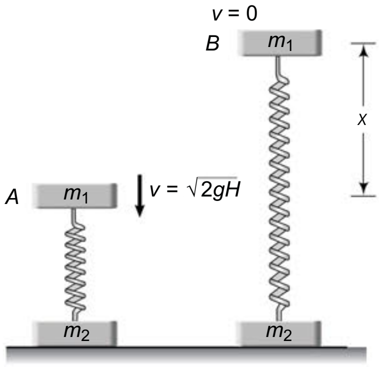
$$m = \frac{n}{2} = \frac{kx_0}{4f} - \frac{1}{4} = \frac{1}{4}\left(\frac{kx_0}{f} - 1\right) \qquad \textbf{Ans.}$$

6. Conservation of mechanical energy gives,

$$E_A = E_B$$

or $$\frac{1}{2}m_1v^2 = \frac{1}{2}kx^2 + m_1gx$$

or $$2m_1gH = kx^2 + 2m_1gx \qquad \text{...(i)}$$



The lower block will rebounce when

$$x > \frac{m_2g}{k} \qquad (kx = m_2g)$$

Substituting, $x = \frac{m_2g}{k}$ in Eq. (i), we get

$$2m_1gH = k\left(\frac{m_2g}{k}\right)^2 + 2m_1g\left(\frac{m_2g}{k}\right)$$

or $$H = \frac{m_2g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right)$$

Thus, $$H_{\min} = \frac{m_2g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right) \qquad \textbf{Ans.}$$

7. $E_i = E_f$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}kx^2$$

or $$k = \frac{3v^2m}{4x^2} \qquad \textbf{Ans.}$$

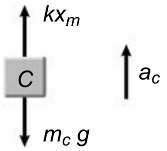
8. $\mu m_Ag = 0.8 \times 6 \times 10 = 48$ N

$(m_B + m_C)g = (1 + 2) \times 10 = 30$ N

Since $(m_B + m_C)g > \mu m_Ag, a_A = a_B = 0.$

From conservation of energy principle we can prove that maximum distance moved by C or maximum extension in the spring would be

$$x_m = \frac{2m_Cg}{k} = \frac{2 \times 1 \times 10}{1000}$$
$$= 0.02 \text{ m}$$



At maximum extension

$$a_C = \frac{kx_m - m_Cg}{m_C}$$

Substituting the values we have,

$$a_C = 10 \text{ m/s}^2. \qquad \textbf{Ans.}$$

9. Rate at which kinetic plus gravitational potential energy is dissipated at time t is actually the magnitude of power of frictional force at time t.

$$|P_f| = f.v = (\mu mg \cos\alpha)(at)$$
$$= (\mu mg \cos\alpha)[(g \sin\alpha - \mu g \cos\alpha)t]$$
$$= \mu mg^2 \cos\alpha(\sin\alpha - \mu \cos\alpha)t \qquad \textbf{Ans.}$$

10. From work-energy principle, $W = \Delta\text{KE}$

$$\therefore \quad Pt = \frac{1}{2} m (v^2 - u^2) \qquad (P = \text{power})$$

$$\text{or} \quad t = \frac{m}{2P} (v^2 - u^2) \qquad \ldots\text{(i)}$$

Further

$$F \cdot v = P$$

$$\therefore \quad m \cdot \frac{dv}{ds} \cdot v^2 = P$$

$$\text{or} \quad \int_u^v v^2\, dv = \frac{P}{m} \int_0^x ds$$

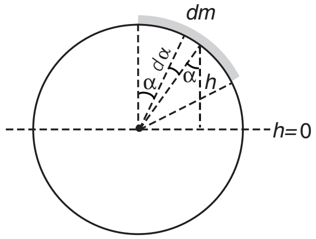
$$\therefore \quad (v^3 - u^3) = \frac{3P}{m} \cdot x$$

$$\text{or} \quad \frac{m}{P} = \frac{3x}{v^3 - u^3}$$

Substituting in Eq. (i)

$$t = \frac{3x\,(u + v)}{2\,(u^2 + v^2 + uv)} \qquad \textbf{Hence proved.}$$

11. (a) Mass per unit length $= \dfrac{m}{l}$



$$dm = \frac{m}{l} R\, d\alpha$$

$$h = R \cos\alpha$$

$$dU = (dm)\, gh = \frac{mgR^2}{l} \cos\alpha \cdot d\alpha$$

$$\therefore \quad U = \int_0^{l/R} dU = \frac{mgR^2}{l} \sin\left(\frac{l}{R}\right)$$

(b) $\text{KE} = U_i - U_f$

$$\text{Here,} \quad U_i = \frac{mgR^2}{l} \sin\left(\frac{l}{R}\right)$$

and

$$U_f = \int_\theta^{l/R + \theta} dU = \frac{mgR^2}{l} \left[\sin\left(\frac{l}{R} + \theta\right) - \sin\theta\right]$$

$$\therefore \quad \text{KE} = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

Ans.

(c) $$\frac{1}{2} mv^2 = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

$$\text{or} \quad v = \sqrt{\frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]}$$

$$v^2 = \frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

$$\therefore 2v \cdot \frac{dv}{dt} = \frac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right] \cdot \frac{d\theta}{dt}$$

$$\frac{dv}{dt} = \frac{\dfrac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \dfrac{l}{R}\right)\right]}{2v} \left(\frac{d\theta}{dt}\right) \qquad \ldots\text{(i)}$$

$$\text{Here} \quad \frac{\left(\dfrac{d\theta}{dt}\right)}{v} = \frac{\omega}{v} = \frac{1}{R}$$

Substituting in Eq. (i), we get

$$\frac{dv}{dt} = \frac{gR}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]$$

At $\quad t = 0, \ \theta = 0°$

$$\text{Hence,} \quad \frac{dv}{dt} = \frac{gR}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right] \qquad \textbf{Ans.}$$

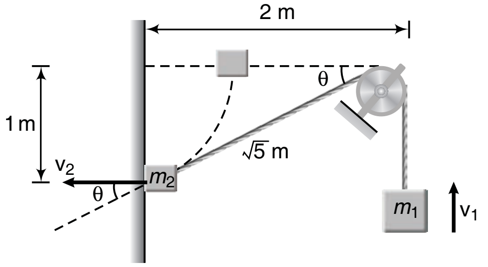
12. From conservation of energy,

$$m_2 g h_2 = m_1 g h_1 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$\text{Here,} \quad v_1 = v_2 \cos\theta$$

$$\therefore \ 2 \times 10 \times 1 = (0.5)\,(10)\,(\sqrt{5} - 1) + \frac{1}{2} \times 0.5 \times v_2^2 \times \left(\frac{2}{\sqrt{5}}\right)^2 + \frac{1}{2} \times 2 \times v_2^2$$

$$\therefore \quad 20 = 6.18 + 0.2\, v_2^2 + v_2^2$$



$$v_2 = 3.39 \text{ m/s} \qquad \textbf{Ans.}$$

$$\text{and} \quad v_1 = v_2 \cos\theta = \frac{2}{\sqrt{5}} \times 3.39$$

$$\text{or} \quad v_1 = 3.03 \text{ m/s} \qquad \textbf{Ans.}$$

13. Net retarding force $= kx + bMgx$

$$\therefore \text{ Net retardation} = \left(\frac{k + bMg}{M}\right) \cdot x$$

So, we can write

$$v \cdot \frac{dv}{dx} = -\left(\frac{k + bMg}{M}\right) \cdot x$$

or $$\int_{v_0}^{0} v \cdot dv = -\left(\frac{k + bMg}{M}\right) \int_0^x x\,dx$$

or $$x = \sqrt{\frac{M}{k + bMg}}\, v_0$$

Loss in mechanical energy

$$\Delta E = \frac{1}{2} M v_0^2 - \frac{1}{2} k x^2$$

or $$\Delta E = \frac{1}{2} M v_0^2 - \frac{k}{2}\left(\frac{M}{k + bMg}\right) v_0^2$$

or $$\Delta E = \frac{v_0^2}{2}\left[M - k\left(\frac{M}{k + bMg}\right)\right]$$

$$= \frac{v_0^2 b M^2 g}{2(k + bMg)}$$ **Ans.**

14. From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2} k x_i^2 + m g h_i = \frac{1}{2} m v_f^2$$

$$\therefore \quad v_f = \sqrt{2 g h_i + \frac{k}{m} x_i^2}$$

Substituting the values we have,

$$v_f = \sqrt{2 \times 9.8 \times 1.9 + \frac{2300}{0.12}(0.045)^2}$$

$$= 8.72 \text{ m/s}$$ **Ans.**

15. (a) From work energy theorem,

Work done by all forces = Change in kinetic energy

$$\therefore \quad Fx - \frac{1}{2} k x^2 = \frac{1}{2} m v^2$$

$$\therefore \quad v = \sqrt{\frac{2Fx - kx^2}{m}}$$

Substituting the values we have,

$$v = \sqrt{\frac{2 \times 20 \times 0.25 - 40 \times 0.25 \times 0.25}{0.5}}$$

$$= \sqrt{15} \text{ m/s} = 3.87 \text{ m/s}$$ **Ans.**

(b) From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2} m v_i^2 + \frac{1}{2} k x_i^2 = \frac{1}{2} k x_f^2$$

or $$x_f = \sqrt{\frac{m v_i^2}{k} + x_i^2}$$

$$= \sqrt{\frac{0.5 \times 15}{40} + (0.25)^2}$$

$$= 0.5 \text{ m (compression)}$$

$\therefore$ Distance of block from the wall

$$= (0.6 - 0.5) \text{ m}$$

$$= 0.1 \text{ m}$$ **Ans.**

10. Circular Motion

INTRODUCTORY EXERCISE 10.1

1. Direction of acceleration (acting towards centre) continuously keeps on changing. So, it is variable acceleration.

2. In uniform circular motion speed remains constant. In projectile motion (which is a curved path) acceleration remains constant.

3. (a) $$a_r = \frac{v^2}{R} = \frac{(2t)^2}{1.0} = 4t^2$$

At $t = 1$ s, $a_r = 4\ \text{cm/s}^2$ **Ans.**

(b) $$a_t = \frac{dv}{dt} = 2\ \text{cm/s}^2$$ **Ans.**

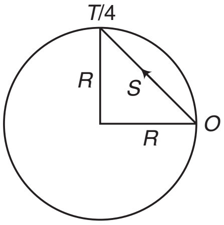
(c) $$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(4)^2 + (2)^2}$$

$$= 2\sqrt{5}\ \text{cm/s}^2$$ **Ans.**

4. $$v_{\text{av}} = \frac{s}{t} = \frac{\sqrt{2}\,R}{(T/4)}$$

$$= \frac{4\sqrt{2}\,R}{(2\pi R)/v}$$

$$\therefore \quad \frac{v_{\text{av}}}{v} = \frac{2\sqrt{2}}{\pi}$$ **Ans.**



5. $(R\omega^2) = R\alpha$

$$\therefore \quad \omega^2 = \alpha$$

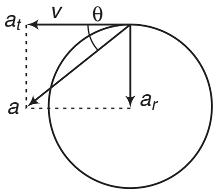
or $$(\alpha t)^2 = \alpha$$

$$\therefore \quad t = \frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{4}}$$

$$= 0.5\ \text{s}$$ **Ans.**

6. $$a_t = \frac{dv}{dt} = 8t$$

$$a_r = \frac{v^2}{R} = \frac{(4t^2)^2}{54} = \frac{8}{27}t^4$$



At $t = 3$ s,

$$a_t = 24\ \text{m/s}^2$$

$$a_r = 24\ \text{m/s}^2$$

$$\tan\theta = \frac{a_t}{a_r} = 1$$

$$\therefore \quad \theta = 45^\circ$$ **Ans.**

7. (a) $$a_r = a\cos 30^\circ = (25)\frac{\sqrt{3}}{2}$$

$$= 21.65\ \text{m/s}^2$$ **Ans.**

(b) $$a_r = \frac{v^2}{R} \quad \Rightarrow \quad v = \sqrt{a_r R}$$

or $$v = \sqrt{(21.65)\,(2.5)}$$

$$= 7.35\ \text{m/s}$$ **Ans.**

(c) $$a_t = a\sin 30^\circ = (25)\left(\frac{1}{2}\right) = 12.5\ \text{m/s}^2$$ **Ans.**

INTRODUCTORY EXERCISE 10.2

1. $$v = \frac{18 \times 5}{18} = 5\ \text{m/s}$$

$$\tan\theta = \frac{v^2}{Rg} = \frac{(5)^2}{10 \times 10} = \frac{1}{4}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{4}\right)$$ **Ans.**

2. $$v = \sqrt{\mu R g}$$

$$\therefore \quad \mu = \frac{v^2}{gR} = \frac{(5)^2}{10 \times 10}$$

$$= 0.25$$ **Ans.**

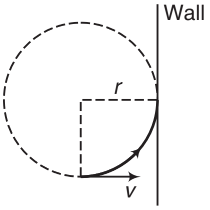
3. $$v = \sqrt{Rg\tan\theta} = \sqrt{50 \times 10 \times \tan 30^\circ}$$

$$\approx 17\ \text{m/s}$$ **Ans.**

4. $a \neq 0 \quad \Rightarrow \quad F_{\text{net}} = ma \neq 0$

Hence, particle is not in equilibrium.

5. If he applies the breaks, then to stop the car in a distance r



$$0 = v^2 - 2a_1 r$$

$$\therefore \quad a_1 = \frac{v^2}{2r} = \text{minimum retardation required.}$$

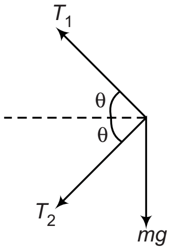
(by friction)

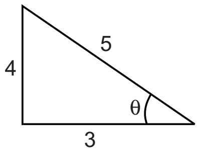
If he takes a turn of radius r, the centripetal acceleration required is

$$a_2 = \frac{v^2}{r} \quad \text{(provided again by friction)}$$

Since $a_1 < a_2$, it is better to apply brakes.

6.





$$T_1 \cos\theta + T_2 \cos\theta = mr\omega^2 \qquad \text{...(i)}$$

$$\omega = (2n\pi)$$

Here, n = number of revolutions per second.

Substituting the proper values in Eq. (i),

$$200 \times \left(\frac{3}{5}\right) + T_2 \times \left(\frac{3}{5}\right) = (4)(3)(2n\pi)^2$$

or $$600 + 3T_2 = 240\, n^2\pi^2 \qquad \text{...(ii)}$$

Further, $T_1 \sin\theta = T_2 \sin\theta + mg$

or $$200 \times \frac{4}{5} = T_2 \times \frac{4}{5} + 4 \times 10$$

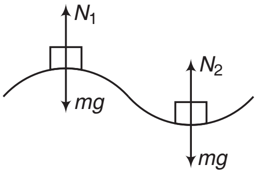
or $$800 = 4T_2 + 200 \qquad \text{...(iii)}$$

Solving Eqs. (ii) and (iii) we get,

$$T_2 = 150 \text{ N} \quad \text{and} \quad n = 0.66 \text{ rps}$$

$$= 39.6 \text{ rpm}$$

7. (a) $mg - N_1 = \dfrac{mv^2}{R}$



or $mg - \dfrac{mg}{2} = \dfrac{mv^2}{R}$

$$\therefore \quad \frac{mv^2}{R} = \frac{mg}{2} = \frac{16\text{kN}}{2} = 8\text{kN}$$

Now, $N_2 - mg = \dfrac{mv^2}{R}$ or $N_2 = mg + \dfrac{mv^2}{R}$

$$= mg + \frac{mg}{2} = \frac{3}{2}mg$$

$$= \frac{3}{2}(16 \text{ kN}) = 24 \text{ kN}$$ **Ans.**

(b) $mg - N = \dfrac{mv^2}{R}$ or $N = mg - \dfrac{mv^2}{R}$

$$0 = mg - \frac{mv_{\max}^2}{R}$$

$$\therefore \quad v_{\max} = \sqrt{gR} = \sqrt{10 \times 250} = 50 \text{ m/s}$$ **Ans.**

(c) $N_2 - mg = \dfrac{mv^2}{R}$

$$\therefore \quad N_2 = m\left(g + \frac{v^2}{R}\right)$$

$$= \frac{16 \times 10^3}{10}\left[10 + \frac{2500}{250}\right]$$

$$= 32 \times 10^3 \text{ N} = 32 \text{ kN}$$ **Ans.**

INTRODUCTORY EXERCISE 10.3

1. (a) String will slack at height h_1, discussed in article 10.5 where,

$$h_1 = \frac{u^2 + gR}{3g} = \frac{5}{3}R \qquad (\text{as } u^2 = 4gR)$$

$$v = \sqrt{u^2 - 2gh_1} = \sqrt{4gR - 2g \times \frac{5}{3}R}$$

$$= \sqrt{\frac{2}{3}gR}$$

2. (a) Velocity becomes zero at height h_2 discussed in article 10.5 where,

$$h_2 = \frac{u^2}{2g} = \frac{gR}{2g} = \frac{R}{2}$$

Now, $h_2 = R(1 - \cos\theta)$

$$\therefore \quad \frac{R}{2} = R(1 - \cos\theta) \Rightarrow \theta = 60°$$

(b) $T = mg\cos\theta = \dfrac{mg}{2}$ at $\theta = 60°$

3. (a) $v = \sqrt{u^2 - 2gh} = \sqrt{7gR - 2g(2R)} = \sqrt{3gR}$

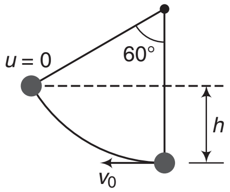
(b) $T + mg = \dfrac{mv^2}{R} = \dfrac{m}{R}(mgR)$

$$\therefore \quad T = 2mg$$

(c) $T - mg = \dfrac{mu^2}{R} = \dfrac{m}{R}(7gR)$

$$\therefore \quad T = 8mg$$

4. $h = l - l\cos 60° = \dfrac{l}{2} = 2.5$ m



$$v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 2.5} = 7 \text{ m/s}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

2. ***A* to *B*** $S = \sqrt{2}\,R$

and $|\Delta \mathbf{v}| = \sqrt{v^2 + v^2 - 2v \cdot v \cos 90°}$

$= \sqrt{2}\,v$

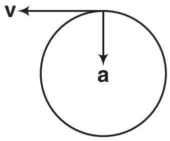
Average acceleration $= \dfrac{|\Delta \mathbf{v}|}{t} = \dfrac{\sqrt{2}\,v}{t}$

and average velocity $= \dfrac{S}{t} = \dfrac{\sqrt{2}\,R}{t}$

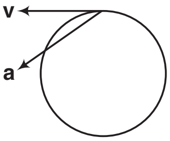
The desired ratio is $\dfrac{v}{R} = \omega$

3. Direction of acceleration keeps on changing. So, it is variable accelration. Further, it is accelerated so it is non-inertial.

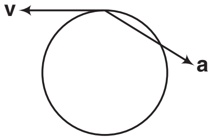
4.



v = constant



v is increasing



v is decreasing

In first figure, $\mathbf{v} \cdot \mathbf{a} = 0$

In second figure, $\mathbf{v} \cdot \mathbf{a}$ = positive and

In third figure, $\mathbf{v} \cdot \mathbf{a}$ = negative

Further, ω and $\mathbf{v}$ are always perpendicular, so

$$\omega \cdot \mathbf{v} = 0$$

5. Component of acceleration perpendicular to velocity is centripetal acceleration.

$$\therefore \quad a_c = \frac{v^2}{R} \quad \text{or} \quad R = \frac{v^2}{a_c} = \frac{(2)^2}{2} = 2 \text{ m}$$

6. **At *A***

Tangential acceleration is g. Radial acceleration is $\dfrac{v^2}{R}$.

$$\therefore \qquad a_{\text{net}} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}$$

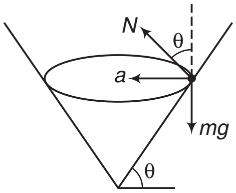
7. **At *A* and *C***, $v = 0$. Therefore, radial acceleration $\dfrac{v^2}{R}$ is zero. But tangential acceleration $g \sin\theta$ is non-zero.

8. For $t < 3$ sec, speed is negative which is not possible.,

9. In circular motion, $\mathbf{a} \neq$ constant, so we cannot apply $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ directly.

In vertical circular motion, gravity plays an important role.

10.



$$N \cos\theta = mg$$

$$\Rightarrow \qquad N = mg \sec\theta$$

$$\text{and} \qquad N \sin\theta = \frac{mv^2}{R}$$

11. Weight (plus normal reaction) provides the necessary centripetal force or weight is used in providing the centripetal force.

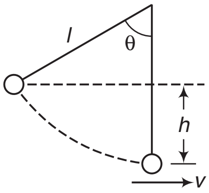
Single Correct Option

1. Magnitude of tangential acceleration is constant but its direction keeps on changing.

2. At $\theta = 180°$, $|\Delta \mathbf{P}| = 2\,mv$ = maximum

At $\theta = 360°$, $|\Delta \mathbf{P}| = 0$ = minimum

3. $h = l - l\cos\theta = l\,(1 - \cos\theta)$

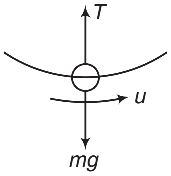


$$v^2 = 2gh = 2gl\,(1 - \cos\theta) = v^2_{\max}$$

$$\therefore \qquad K_{\max} = \frac{1}{2}mv^2 = mgl\,(1 - \cos\theta) \qquad \textbf{Ans.}$$

4. Rod does not slack (like string). So, minimum velocity at topmost point may be zero also.

5.



$$T - mg = \frac{mu^2}{R} = \frac{m(\sqrt{5gR})^2}{R}$$

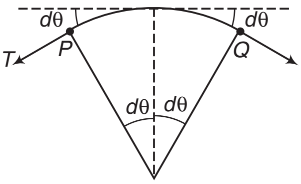
$\therefore \quad T = 6\,mg$ **Ans.**

6. $\frac{v^2}{R} = a_t = a$ (Here, $a_t = a$ say)

or $\quad \frac{(at)^2}{R} = a$

$\therefore \quad t = \sqrt{\frac{R}{a}} = \sqrt{\frac{20}{5}} = 2\text{ s}$ **Ans.**

7. cos $(d\theta)$ components of T are cancelled and sin $(d\theta)$ components towards centre provide the necessary centripetal force to small portion PQ.



$$\therefore \quad 2T\sin(d\theta) = (m_{PQ})(R)\,\omega^2$$

For small angle, $\sin d\theta \approx d\theta$

$$\therefore \quad 2T\,d\theta = \left(\frac{m}{2\pi}\right)(2\theta)(R)(2n\pi)^2$$

$$\therefore \quad T = 2\pi\,mn^2R$$

Substituting the values we get,

$$T = (2\pi)(2\pi)(300/60)^2(0.25)$$

$$\approx 250\text{ N}$$ **Ans.**

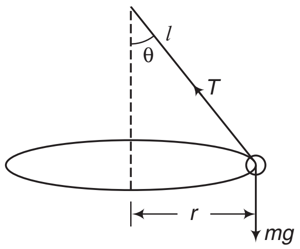
8. $\frac{mv^2}{R} = \mu mg$

$\therefore \quad v = \sqrt{\mu Rg} = \sqrt{0.3 \times 300 \times 10}$

$= 30\text{ m/s} = 108\text{ km/h}$ **Ans.**

9. $T\cos\theta = mg$ ...(i)

$T\sin\theta = mr\omega^2 = m(l\sin\theta)\,\omega^2$ ...(ii)



Solving these two equations we get,

$$\cos\theta = \frac{g}{l\omega^2} = \frac{g}{l\,(2\pi n)^2}$$

$$= \frac{10}{\left[2\pi \times \frac{2}{\pi}\right]^2} \quad (l = 1\text{ m})$$

$\therefore \quad \theta = \cos^{-1}(5/8)$ **Ans.**

10. $T = \frac{mg}{\cos\theta} = \frac{0.1 \times 10}{5/8} = \frac{8}{5}\text{ N}$ **Ans.**

11. $f = ma = m\sqrt{a_t^2 + a_r^2}$

$$= m\sqrt{(R\alpha)^2 + (R\omega^2)^2}$$

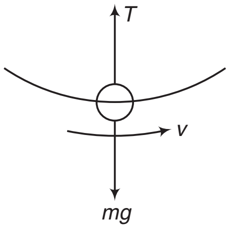
$$= m\sqrt{(R\alpha)^2 + [R(\alpha t)^2]^2}$$

$$= 0.36 \times 10^{-3}\sqrt{\left(0.25 \times \frac{1}{3}\right)^2 + \left[0.25\left(\frac{1}{3} \times 2\right)^2\right]^2}$$

$$= 50 \times 10^{-6}\text{ N}$$

$= 50\,\mu\text{N}$ **Ans.**

12. $v^2 = 2gh$



$$T - mg = \frac{mv^2}{l}$$

or $\quad T = mg + \frac{m(2gh)}{l}$

$= mg\left(1 + \frac{2h}{l}\right)$ **Ans.**

13. $(\omega_1 - \omega_2)\, t = 2\pi$

$$t = \frac{2\pi}{\omega_1 - \omega_2} = \frac{2\pi}{(2\pi/T_1) - (2\pi/T_2)}$$

$$\frac{T_1\, T_2}{T_2 - T_1} = \frac{3 \times 1}{3 - 1}$$

$= 1.5$ min **Ans.**

14. $\frac{d_B}{d_C} = \frac{v_B t}{v_C t} = \frac{2.5}{2} = \frac{5}{4}$ **Ans.**

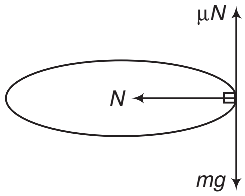
Subjective Questions

1. $a_t = 8\ \text{m/s}^2$

$$a_r = \frac{v^2}{R} = \frac{(16)^2}{50} = 5.12\ \text{m/s}^2$$

$$a = \sqrt{a_t^2 + a_r^2} = 9.5\ \text{m/s}^2$$ **Ans.**

2.



$$N = mR\omega^2 \qquad \text{...(i)}$$

$$\mu N = mg \qquad \text{...(ii)}$$

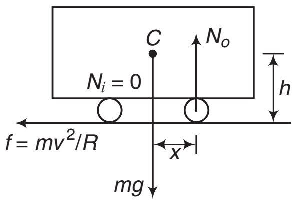
From these two equations we get,

$$\omega = \sqrt{\frac{g}{\mu R}}$$

$$= \sqrt{\frac{9.8}{0.15 \times 3}} = 4.7\ \text{rad/s}$$ **Ans.**

3. $a = \frac{v^2}{R} \Rightarrow R = \frac{v^2}{a} = \frac{(u\cos\theta)^2}{g}$ **Ans.**

4.



In critical case, normal reaction on inner wheel N_i will become zero. Normal reaction on outer wheel $N_o = mg$. Friction will provide the necessary centripetal force.

$$\therefore \quad f = \frac{mv^2}{R}$$

Taking moment about C

$$N_o\,(x) = f\,(h)$$

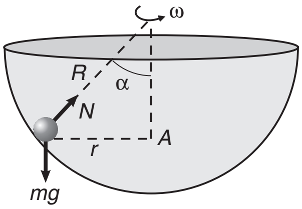
or $$(mg)\,x = \left(\frac{mv^2}{R}\right) h$$

$$\therefore \quad v = \sqrt{\frac{gRx}{h}}$$

$$= \sqrt{\frac{9.8 \times 250 \times 0.75}{1.5}}$$

$$= 35\ \text{m/s}$$

5. Let ω be the angular speed of rotation of the bowl.Two forces are acting on the ball.



1. normal reaction N
2. weight mg

The ball is rotating in a circle of radius $r (= R\sin\alpha)$ with centre at A at an angular speed ω. Thus,

$$N\sin\alpha = mr\omega^2$$

$$= mR\omega^2 \sin\alpha \qquad \text{...(i)}$$

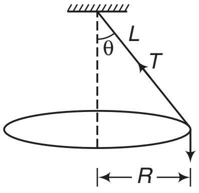
and $$N\cos\alpha = mg \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$$

$$\therefore \quad \omega = \sqrt{\frac{g}{R\cos\alpha}}$$

6. $R = L\sin\theta$



$$T\sin\theta = mR\omega^2 = m\,(L\sin\theta)\,\omega^2$$

$$\therefore \quad T = mL\omega^2$$

Now, $$T\cos\theta = mg$$

$$\therefore \quad \cos\theta = \frac{mg}{T} = \frac{mg}{mL\omega^2} = \frac{g}{L\omega^2}$$

Hence proved.

7. $t = \sqrt{\dfrac{2h}{g}} = \sqrt{\dfrac{2 \times 2.9}{9.8}} = 0.77$ s

Horizontal distance $x = vt$

$$\therefore \qquad v = \frac{x}{t} = \frac{10}{0.77}$$

$$\approx 13 \text{ m/s}$$

$$a = \frac{v^2}{R} = \frac{(13)^2}{1.5}$$

$$\approx 113 \text{ m/s}^2 \qquad \textbf{Ans.}$$

8. (a) The frictional force provides the necessary centripetal force.

$$\therefore \qquad mL\omega^2 = \mu mg$$

$$\text{or} \qquad \omega = \sqrt{\frac{\mu g}{L}}$$

(b) Net force of circular motion will be provided by the friction

$$\omega = \alpha t \qquad \text{...(i)}$$

$$F_{\text{net}} = m\sqrt{a_t^2 + a_n^2}$$

$$\therefore \qquad \mu mg = m\sqrt{(L\alpha)^2 + (L\omega^2)^2} \qquad \text{...(ii)}$$

Here, $a_t = L\alpha$ and $a_n = L\omega^2$

Substituting $\omega = \alpha t$ in Eq. (ii) we have,

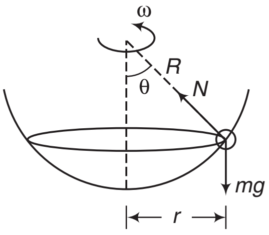
$$\mu g = \sqrt{L^2\alpha^2 + L^2\alpha^4 t^4}$$

$$\therefore \qquad t = \left(\frac{\mu^2 g^2 - L^2\alpha^2}{L^2\alpha^4}\right)^{\frac{1}{4}}$$

Substituting the value of t in Eq. (i) we have,

$$\omega = \left[\left(\frac{\mu g}{L}\right)^2 - \alpha^2\right]^{\frac{1}{4}}$$

9. $N \cos\theta = mg$



$$N \sin\theta = mr\omega^2$$

$$= m\,(R\sin\theta)\,\omega^2$$

Dividing these two equations,

$$\text{we get} \qquad \cos\theta = \frac{g}{R\,\omega^2} \qquad \text{...(i)}$$

$$\therefore \qquad \omega = \sqrt{\frac{g}{R\cos\theta}}$$

At lowermost point, $\theta = 0°$

$$\therefore \qquad \omega = \sqrt{\frac{g}{R}}$$

Substituting $\omega = \sqrt{2g/R}$ in Eqs. (i), we have,

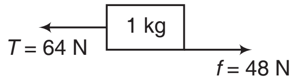
$$\cos\theta = \frac{1}{2}$$

$$\therefore \qquad \theta = 60° \qquad \textbf{Ans.}$$

10. (a)

$$T = m_2 r_2 \omega^2 = (2)\,(2)\,(4)^2 = 64 \text{ N}$$

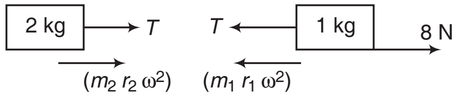
Centripetal force required to 1 kg block is



$$F_c = m_1 r_1 \omega^2 = (1)\,(1)\,(4)^2 = 16 \text{ N}$$

But available tension is 64 N. So, the extra force of 48 N is balanced by friction acting radially outward from the centre.

(b) $f_{\max} = \mu m_1 g = 0.8 \times 1 \times 10 = 8$ N



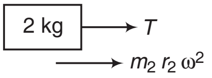
$$T = m_2 r_2\,\omega^2 = (2)\,(2)\,\omega^2 = 4\,\omega^2 \qquad \text{...(i)}$$

$$T - 8 = m_1\,r_1\,\omega^2 = (1)\,(1)\,(\omega^2) = \omega^2 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii) we get,

$$\omega = 1.63 \text{ rad/s} \qquad \textbf{Ans.}$$

(c) $T = m_2\,r_2\,\omega^2$



$$\therefore \quad 100 = (2)\,(2)\,\omega^2$$

$$\text{or} \quad \omega = 5 \text{ rad/s}$$

11. $v^2 = v_0^2 + 2gh = (0.5\sqrt{gr}\,)^2 + 2gr\,(1 - \cos\theta)$

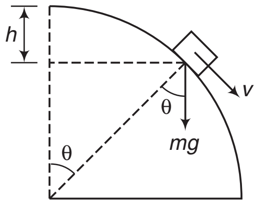
$$= (2.25\,gr - 2gr\cos\theta)$$

At the time of leaving contact, $N = 0$

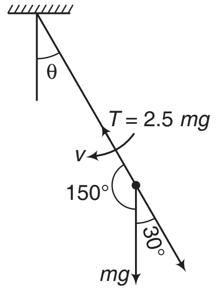
$$\therefore \quad mg\cos\theta = \frac{mv^2}{r} = 2.25\,mg - 2\,mg\cos\theta$$

$$\therefore \qquad \cos\theta = \frac{2.25}{3} = \frac{3}{4}$$

$$\therefore \qquad \theta = \cos^{-1}(3/4) \qquad \textbf{Ans.}$$



12. $2.5\,mg - mg\cos 30° = \dfrac{mv^2}{r}$



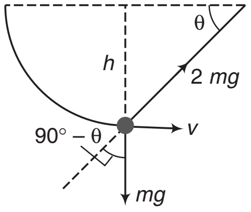
$$\therefore \qquad 1.63\,g = \frac{v^2}{r} = \frac{v^2}{2}$$

$$\therefore \qquad v = 5.66 \text{ m/s}$$

$$F_{\text{net}} = \sqrt{(2.5\,mg)^2 + (mg)^2 + (2)(2.5\,mg)(mg)\cos 150°}$$
$$= 1.7\,mg$$

$$a_{\text{net}} = \frac{F_{\text{net}}}{m} = 1.7\,g \approx 16.75 \text{ m/s}^2$$ **Ans.**

13. $v^2 = v_0^2 + 2\,gh = v_0^2 + 2\,gR\sin\theta$



$$= (5)^2 + 2 \times 10 \times 2\sin\theta$$
$$= (25 + 40\sin\theta)$$

Now, $\quad 2\,mg - mg\sin\theta = \dfrac{mv^2}{R}$

or $\quad 2g - g\sin\theta = \dfrac{v^2}{R}$

or $\quad 2 \times 10 - 10\sin\theta = \dfrac{25 + 40\sin\theta}{2}$

Solving this equation we get,

$$\theta = \sin^{-1}\left(\frac{1}{4}\right)$$ **Ans.**

LEVEL 2

Single Correct Option

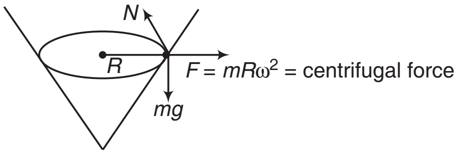
1. $E_A = E_B$

$$\therefore \qquad \frac{1}{2}mv_A^2 = \frac{1}{2} \times 200 \times (13 - 7)^2 \qquad (m = 2 \text{ kg})$$

$$\therefore \qquad v_A = 60 \text{ m/s}$$

At A, $\quad N = \dfrac{mv_A^2}{R} = \dfrac{(2)(60)^2}{S} = 1440$ N **Ans.**

2. Let us see the FBD with respect to rotating cone (non-inertial)



N, F and mg are balanced in the shown diagram. If displaced upwards, F will increase as R is increased. This will have a component up the plane. So, it will move upwards. Hence, the equilibrium is unstable.

3. $\mu mg = mR\omega^2$

$$\therefore \qquad \omega^2 = \frac{\mu g}{R} = \frac{(1/3)g}{5a/4} = \frac{4g}{15a}$$ **Ans.**

4. $a_A = g\sin\theta$

$$a_B = \frac{v^2}{R} = \frac{2gh}{R} = \frac{2gR(1 - \cos\theta)}{R}$$

Given, $\quad a_A = a_B$

$\therefore \quad \sin\theta = 2 - 2\cos\theta$

Squaring these two equations we have,

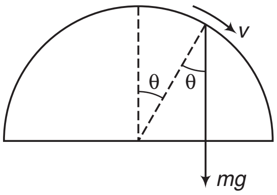
$$\sin^2\theta = 4 + 4\cos^2\theta - 8\cos\theta$$

or $\quad 1 - \cos^2\theta = 4 + 4\cos^2\theta - 8\cos\theta$

On solving this equation, we get

$\cos\theta = \dfrac{3}{5}$ or $\theta = \cos^{-1}(3/5)$ **Ans.**

5. At the time of leaving contact



$$N = 0$$

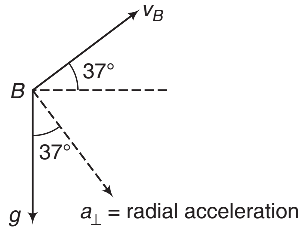
$$\therefore \qquad mg\cos\theta = \frac{mv^2}{R} = \frac{m(2gh)}{R}$$

$$\therefore \qquad \cos\theta = \frac{2h}{R} = \frac{2\left[\dfrac{R}{4} + R\,(1-\cos\theta)\right]}{R}$$

On solving this equation, we get

$$\cos\theta = 5/6 \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{5}{6}\right)$$ **Ans.**

6. $h_{AB} = R\cos 37° - R\cos 53°$



$= 0.8R - 0.6R$

$= 0.2\,R$

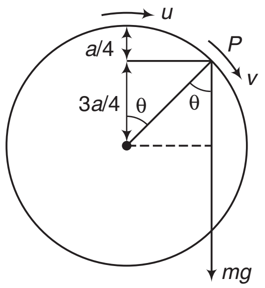
$$\therefore \qquad v_B = \sqrt{2gh_{AB}} = \sqrt{0.4\,gR}$$

$$a_\perp = g\cos 37° = (0.8g) = \frac{v_B^2}{r} = \frac{0.4\,gR}{r}$$

$\therefore \qquad r =$ radius of curvature at B

$$= \frac{R}{2}$$

7. At the time of leaving contact at P,



$$N = 0$$

$$\therefore \qquad mg\cos\theta = \frac{mv^2}{a} = \frac{m\,(u^2 + 2gh)}{a}$$

$$\therefore \qquad g\,(3a/4) = \frac{u^2 + 2g\,(a/4)}{a}$$

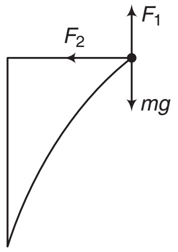
$$\therefore \qquad u = \frac{\sqrt{ag}}{2}$$ **Ans.**

8. $\dfrac{v^2}{r} = \dfrac{4}{r^2}$

$$\therefore \qquad v = \frac{2}{\sqrt{r}}$$

$$P = mv = \frac{2m}{\sqrt{r}}$$ **Ans.**

9. $v = \dfrac{2\pi r}{T} = \dfrac{(2\pi)\,(0.5)}{1.58}$



≈ 2 m/s

$F_1 = mg = 100$ N

$$F_2 = \frac{mv^2}{r} = \frac{10 \times 4}{0.5} = 80 \text{ N}$$

$\therefore$ Net force by rod on ball

$$= \sqrt{F_1^2 + F_2^2} = 128 \text{ N}$$ **Ans.**

10. $t = T$

$$\therefore \qquad \sqrt{\frac{2H}{g}} = \frac{2\pi}{\omega}$$

$$\therefore \qquad \omega = 2\pi\sqrt{\frac{g}{2H}} = \pi\sqrt{\frac{2g}{H}}$$ **Ans.**

11. $\omega_1 t - \omega_2 t = 2\pi$

$$\therefore \qquad t = \frac{2\pi}{\omega_1 - \omega_2}$$

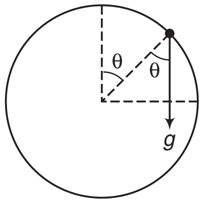
$$= \frac{2\pi}{(2\pi/T_1) - (2\pi/T_2)}$$

$$= \frac{T_1 T_2}{T_2 - T_1}$$

$$= \frac{(3600)\,(60)}{(3600) - (60)}$$

$$= \frac{3600}{59}\ \text{s}$$ **Ans.**

12. Particle breaks off the sphere at $\cos\theta = \dfrac{2}{3}$

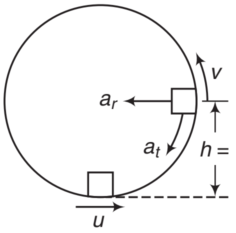


The tangential acceleration at this instant is

$$g\sin\theta = g\sqrt{1-\cos^2\theta}$$

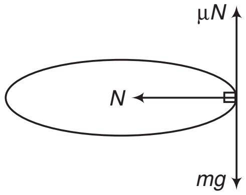
$$= g\sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}\,g$$ **Ans.**

13. $u = \sqrt{5gR}$



$$v^2 = u^2 - 2gh = (5gR) - 2gR$$
$$= 3gR$$
$$a_r = \frac{v^2}{R} = 3g$$
$$a_t = g$$
$$\therefore \quad a = \sqrt{a_r^2 + a_t^2}$$
$$= g\sqrt{10} \qquad \textbf{Ans.}$$

14. $N = mR\omega^2$



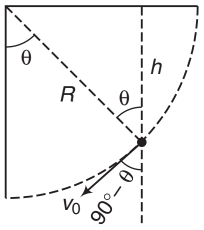
$$\mu N = mg$$

From these two equations. we get

$$\mu = \frac{g}{R\omega^2}$$
$$= \frac{10}{2 \times (5)^2} = 0.2 \qquad \textbf{Ans.}$$

15. $h = R \cos\theta$

$$v_0 = \sqrt{2gh} = \sqrt{2gR\cos\theta}$$



Vertical component of velocity is,

$$v = v_0 \sin\theta$$
$$= \sin\theta\sqrt{2gR\cos\theta}$$

For v to be maximum

$$\frac{dv}{d\theta} = 0$$

$$\therefore \quad [\cos\theta\sqrt{2gR\cos\theta}\,] + \frac{\sin\theta}{2\sqrt{2gR\cos\theta}}(-2gR\sin\theta) = 0$$

$$\therefore \quad 4gR\cos^2\theta - 2gR\sin^2\theta = 0$$

or $$2\cos^2\theta - (1 - \cos^2\theta) = 0$$

or $$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \textbf{Ans.}$$

16. $a_t = a_r$

$$\therefore \quad \frac{dv}{dt} = \frac{v^2}{R} \qquad (v = \text{speed})$$

$$\therefore \quad \int_{v_0}^{v} v^{-2}\,dv = \frac{1}{R}\int_0^t dt$$

$$\left[\frac{1}{v_0} - \frac{1}{v}\right] = \frac{t}{R}$$

or $$\frac{1}{v} = \frac{1}{v_0} - \frac{t}{R} = \frac{R - v_0 t}{R v_0}$$

or $$v = \frac{R v_0}{R - v_0 t}$$

$$\therefore \quad \frac{dx}{dt} = \frac{R v_0}{R - v_0 t} \quad (x = \text{distance travelled})$$

or $$\int_0^x dx = \int_0^t \frac{R v_0}{R - v_0 t}\,dt$$

$$x = R v_0\left(-\frac{1}{v_0}\right)\left[\ln(R - v_0 t)\right]_0^t$$
$$= -R\ln\left(1 - \frac{v_0 t}{R}\right)$$

$$\therefore \quad 1 - \frac{v_0 t}{R} = e^{-x/R}$$

or $$t = t = \frac{R}{v_0}(1 - e^{-x/R})$$

Putting $x = 2\pi R$, we get

$$t = \frac{R}{v_0}(1 - e^{-2\pi}) \qquad \textbf{Ans.}$$

17. If $u > \sqrt{5gl}$, then **T** and **a** are in same direction. Hence, **T** · **a** is positive.

If $u = \sqrt{5gl}$, then $T = 0$

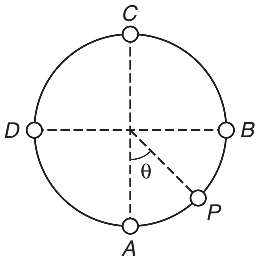
∴ **T** · **a** = 0

18. At lowest point **T** and **a** are always in same direction (towards centre).

∴ **T** · **a** is always positive.

More than One Correct Options

1. Radial acceleration is given by,



$$a_r = \frac{v^2}{R}$$

At A, speed is maximum.

Therefore, a_r is maximum.

At C, speed is minimum.

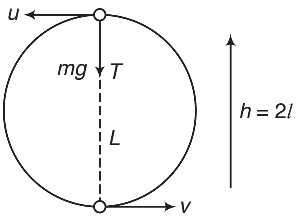
Therefore, a_r is minimum.

Tangential acceleration is $g \sin \theta$.

At point $B, \theta = 90°$.

Therefore, tangential acceleration is maximum ($= g$).

2. $T + mg = \frac{mu^2}{l}$



$$\therefore \qquad 2\,mg + mg = \frac{mu^2}{l}$$

$$\text{or} \qquad u = \sqrt{3gl}$$

$$v^2 = u^2 + 2\,gh = 3gl + 2g\,(2l)$$

$$= 7\,gl$$

$$\therefore \qquad v = \sqrt{7gl}$$

3. $R = h \cot \alpha$

$$N \cos \alpha = mg$$

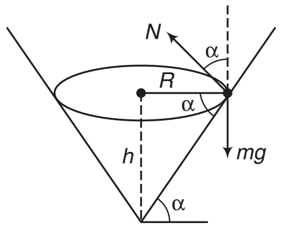
$$N \sin \alpha = \frac{mv^2}{R}$$

Solving these two equations, we get

$$v = \sqrt{Rg \tan \alpha}$$

$$= \sqrt{(h \cot \alpha)\,(g \tan \alpha)}$$

$$= \sqrt{gh}$$



Now,

$$T = \frac{2\pi R}{v} = \frac{2\pi\, h \cot \alpha}{\sqrt{gh}}$$

$$= 2\pi \sqrt{\frac{h}{g}} \cot \alpha$$

$$\therefore \qquad T \propto \sqrt{h}$$

$$\text{and} \qquad T \propto \cot \alpha$$

5. $N \cos \theta = mg$ ...(i)

$N \sin \theta = mR\omega^2$ ...(ii)

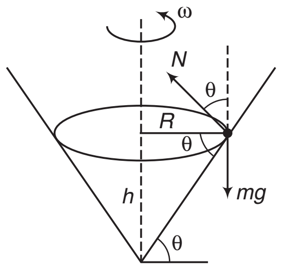
From Eq. (i),

$$N = \frac{mg}{\cos\theta} = \text{constant}$$

as $\qquad \theta = \text{constant}$

Net force is the resultant of N and mg and both forces are constant.

Hence, net force is constant.



$$\omega = \sqrt{\frac{g}{R} \tan\theta}$$

$$\text{But} \qquad R = \frac{h}{\tan \theta}$$

$$\therefore \qquad \omega = \sqrt{\frac{g}{h}} \tan \theta$$

$$\text{or} \qquad \omega \propto \frac{1}{\sqrt{h}} \qquad (\theta = \text{constant})$$

Comprehension Based Questions

1. $\Delta U = \Delta U_{\text{rod}} + \Delta U_{\text{ball}}$

$$= Mg\left(\frac{l}{2}\right) + mgl$$

$$= \left(\frac{M}{2} + m\right) gl \qquad \textbf{Ans.}$$

2. $\omega = \frac{v}{l}$

Now, decrease in rotational kinetic energy = increase in potential energy

$$\therefore \quad \frac{1}{2} I\omega^2 = \left(\frac{M}{2} + m\right) gl$$

$$\text{or} \quad \frac{1}{2}\left[\frac{Ml^2}{3} + ml^2\right]\left(\frac{v}{l}\right)^2 = \left(\frac{M}{2} + m\right) gl$$

$$\therefore \quad v = \sqrt{\frac{\left(\frac{M}{2} + m\right) gl}{\left(\frac{M}{6} + \frac{m}{2}\right)}}$$

3. Maximum velocity is at bottommost point and minimum velocity is at topmost point.

$$\frac{\sqrt{u_{\min}^2 + 2g\,(2L)}}{u_{\min}} = \frac{2}{1}$$

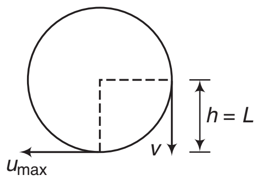
On solving, we get

$$u_{\min} = 2\sqrt{\frac{gL}{3}} \qquad \textbf{Ans.}$$

4. $u_{\max} = 2\,u_{\min} = 4\sqrt{\frac{gL}{3}}$

$$\therefore \quad K_{\max} = \frac{1}{2} m u_{\max}^2 = \frac{8\,mgL}{3} \qquad \textbf{Ans.}$$

5. $v = \sqrt{u_{\max}^2 - 2g\,(L)}$



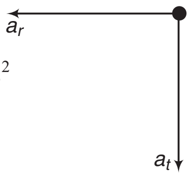
$$= \sqrt{\frac{16\,gL}{3} - 2gL} = \sqrt{\frac{10\,gL}{3}} \qquad \textbf{Ans.}$$

Match the Columns

1. (a) $v = \sqrt{u^2 - 2gh} = \sqrt{12gl - 2gl}$

$$= \sqrt{10gl} = \sqrt{10 \times 10 \times 1} = 10 \text{ m/s} \qquad \textbf{Ans.}$$

(b) $a_r = \frac{v^2}{R}$ or $\frac{v^2}{l}$



$$= \frac{(10)^2}{1} = 100 \text{ m/s}^2$$

$$a_t = g = 10 \text{ m/s}^2$$

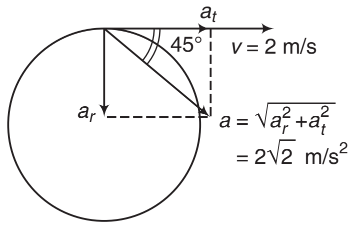
$$\therefore \quad a = \sqrt{a_r^2 + a_t^2}$$

$$= 100.49 \text{ m/s}^2$$

(c) $T = \frac{mv^2}{l} = \frac{(1)\,(10)^2}{1} = 100 \text{ N}$

(d) $a_t = g = 10 \text{ m/s}^2$

2. $a_r = \frac{v^2}{R} = \frac{(2t)^2}{2} = 2t^2$



$$a_t = \frac{dv}{dt} = 2 \text{ m/s}^2$$

$$v = 2t$$

$$\omega = \frac{v}{R} = \frac{2t}{2} = t$$

At 1 s,

$$a_r = 2 \text{ m/s}^2,$$

$$a_t = 2 \text{ m/s}^2,$$

$$v = 2 \text{ m/s}$$

and $\qquad \omega = 1 \text{ rad/s}$

(a) $\mathbf{a} \cdot \mathbf{v} = av \cos 45°$

$$= (2\sqrt{2})\,(2)\,(1\sqrt{2}) = 4 \text{ m}^2/\text{s}^3$$

(b) $|\mathbf{a} \times \omega| = a\omega \sin 90°$

$$= (2\sqrt{2})\,(1)\,(1) = 2\sqrt{2} \text{ m/s}^3$$

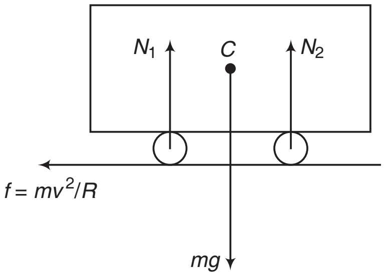
(c) $\mathbf{v} \cdot \omega = 0$ as $\theta = 90°$

(d) $|\mathbf{v} \times \mathbf{a}| = va \sin 45° = 4 \text{ m}^2/\text{s}^3$

3. $f = \frac{mv^2}{R} = F$

$$\therefore \quad \frac{F}{f} = 1$$

With increase in speed f will increase but $\frac{F}{f}$ will remain same.



Further, with increase in the value of v, friction f will increase. Therefore, anticlockwise moment of f about C will also increase. Hence, clockwise moment of N_2 should also increase. Thus, N_2 will increase. But $N_1 + N_2 = mg$. So, N_1 will decrease.

4. $r = |\mathbf{r}| = \sqrt{(3)^2 + (-4)^2} = 5 \text{ m}$

$$v = |\mathbf{v}| = \sqrt{16 + a^2}$$

$$a_r = |\mathbf{a}| = \sqrt{36 + b^2}$$

In uniform circular motion, **v** is always perpendicular to **a**.

$$\therefore \quad \mathbf{v} \cdot \mathbf{a} = 0$$

$$\text{or} \quad -24 - ab = 0$$

$$\therefore \quad ab = 24 \quad \text{...(i)}$$

$$a_r = \frac{v^2}{r}$$

$$\therefore \quad \sqrt{36 + b^2} = \frac{16 + a^2}{5} \quad \text{...(ii)}$$

Solving these two equations, we can find the values of a and b.

(d) **r**, **v** and **a** lie in same plane. But $\mathbf{v} \times \mathbf{a}$ is perpendicular to this plane.

$$\therefore \quad \mathbf{r} \perp \mathbf{v} \times \mathbf{a}$$

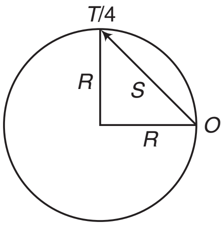
$$\text{or} \quad \mathbf{r} \cdot (\mathbf{v} \times \mathbf{a}) = 0$$

5. (a) Since, speed = constant

$\therefore$ Average speed = this constant value = 1 m/s.

(b) Average velocity $= \dfrac{S}{t}$

$$= \frac{\sqrt{2}R}{T/4} = \frac{4\sqrt{2}R}{(2\pi R/v)} = \frac{4\sqrt{2}\,v}{2\pi} = \frac{2\sqrt{2}}{\pi} \text{ m/s}$$



(c) $|\mathbf{a}_{\text{av}}| = \dfrac{|\Delta \mathbf{v}|}{t}$

$$= \frac{\sqrt{v_f^2 + v_i^2 - 2v_f v_i \cos 90°}}{(T/4)}$$

$$= \frac{4\sqrt{2}v}{T} \qquad (\text{as } v_i = v_f = v)$$

$$= \frac{(4\sqrt{2})\,v}{(2\pi R/v)} = \frac{2\sqrt{2}v^2}{\pi R}$$

$$= \sqrt{2} \text{ m/s}^2$$

(d) $S = \sqrt{2}R = \sqrt{2}\left(\dfrac{2}{\pi}\right)$

$$= \frac{2\sqrt{2}}{\pi} \text{ m}$$

Subjective Questions

1. (a) Applying conservation of energy

$$mgh = \frac{1}{2} m (\sqrt{3Lg})^2$$

$$\therefore \quad h = \frac{3L}{2} \qquad \textbf{Ans.}$$

(b) Since, $\sqrt{3Lg}$ lies between $\sqrt{2Lg}$ and $\sqrt{5Lg}$, the string will slack in upper half of the circle. Assuming that string slacks when it makes an angle θ with horizontal. We have

$$mg \sin\theta = \frac{mv^2}{L} \qquad \text{...(i)}$$

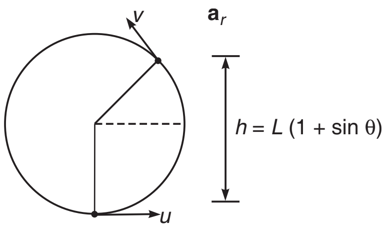
$$v^2 = (\sqrt{3gL})^2 - 2gL\,(1 + \sin\theta) \qquad \text{...(ii)}$$

Solving Eq. (i) and Eq. (ii), we get

$$\sin\theta = \frac{1}{3}$$

$$\text{and} \quad v^2 = \frac{gL}{3}$$

Maximum height of the bob from starting point,



$$H = L\,(1 + \sin\theta) + \frac{v^2 \sin^2 (90° - \theta)}{2g}$$

$$= \frac{4L}{3} + \left(\frac{gL}{6g}\right)\cos^2\theta = \frac{4L}{3} + \frac{4L}{27}$$

$$= \frac{40}{27}L \qquad \textbf{Ans.}$$

Note *Maximum height in part (b) is less than that in part (a), think why?*

2. $h = 0.8 \sin 30° = 0.4$ m

$$\therefore \quad v^2 = 2gh$$

(a) Just before,

$$T_1 - mg \sin 30° = \frac{mv^2}{R_1} \qquad (R_1 = 0.8 \text{ m})$$

$$T_1 = \frac{mg}{2} + \frac{m\,(2g)\,(0.4)}{0.8}$$

$$= \frac{3mg}{2} \qquad \textbf{Ans.}$$

(b) Just after,

$$T_2 - mg \sin 30° = \frac{mv^2}{R_2} \qquad (R_2 = 0.4 \text{ m})$$

$$T_2 = \frac{mg}{2} + \frac{m\,(2g)\,(0.4)}{0.4}$$

$$\text{or} \quad T_2 = \frac{5mg}{2} \qquad \textbf{Ans.}$$

3. $h = l(1 - \cos\theta)$

$v^2 = v_0^2 - 2gh = 3gl - 2gl(1 - \cos\theta) = gl(1 + 2\cos\theta)$

At 45° means radial and tangential components of acceleration are equal.

$$\therefore \quad \frac{v^2}{l} = g \sin\theta$$

$$\text{or} \quad 1 + 2\cos\theta = \sin\theta$$

Solving the equation we get, $\theta = 90°$ or $\dfrac{\pi}{2}$ **Ans.**

4. Banking angle, $\theta = \tan^{-1}\left(\dfrac{v^2}{Rg}\right)$

36 km/h = 10 m/s

$$\therefore \quad \theta = \tan^{-1}\left(\frac{100}{20 \times 9.8}\right) = 27°$$

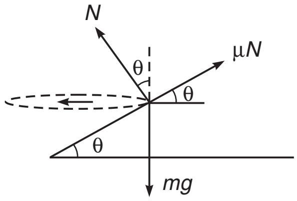
Angle of repose,

$$\theta_r = \tan^{-1}(\mu) = \tan^{-1}(0.4) = 21.8°$$

Since $\theta > \theta_r$, vehicle cannot remain in the given position with $v = 0$. At rest it will slide down. To find minimum speed, so that vehicle does not slip down, maximum friction will act up the plane.

To find maximum speed, so that the vehicle does not skid up, maximum friction will act down the plane.

Minimum Speed



Equation of motion are,

$$N\cos\theta + \mu N \sin\theta = mg \qquad \text{...(i)}$$

$$N\sin\theta - \mu N \cos\theta = \frac{m}{R}v_{\min}^2 \qquad \text{...(ii)}$$

Solving these two equations, we get

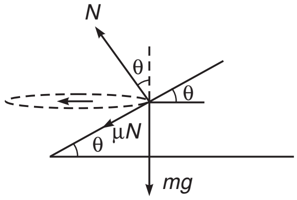
$$v_{\min} = 4.2 \text{ m/s} \qquad \textbf{Ans.}$$

Maximum Speed

Equations of motion are,

$$N\cos\theta - \mu N \sin\theta = mg \qquad \text{...(iii)}$$

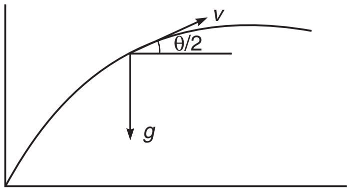
$$N\sin\theta + \mu N \cos\theta = \frac{m}{R}v_{\max}^2 \qquad \text{...(iv)}$$



Solving these two equations, we have

$$v_{\max} = 15 \text{ m/s} \qquad \textbf{Ans.}$$

5. Let v be the velocity at that instant. Then, horizontal component of velocity remains unchanged.



$$\therefore \quad v\cos\frac{\theta}{2} = u\cos\theta \quad \text{or} \quad v = \frac{u\cos\theta}{\cos\frac{\theta}{2}}$$

Tangential component of acceleration of this instant will be,

$$a_t = g\cos(\pi/2 + \theta/2) = -g\sin\theta/2$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{g^2 - g^2\sin^2\frac{\theta}{2}}$$

$$= g\cos\frac{\theta}{2}$$

Since, $a_n = \dfrac{v^2}{R}$

or $R = \dfrac{v^2}{a_n} = \dfrac{\left(\dfrac{u\cos\theta}{\cos\dfrac{\theta}{2}}\right)^2}{g\cos\dfrac{\theta}{2}} = \dfrac{u^2\cos^2\theta}{g\cos^3\left(\dfrac{\theta}{2}\right)}$ **Ans.**

6. **After 1 s** $\mathbf{v} = \mathbf{u} + \mathbf{a}t = 20\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}}$,

$v = \sqrt{500}$ m/s $= 10\sqrt{5}$ m/s

$\mathbf{a} = -10\,\hat{\mathbf{j}}$

$a_t = a\cos\theta$

$= \dfrac{\mathbf{a}\cdot\mathbf{v}}{v} = \dfrac{-100}{10\sqrt{5}}$

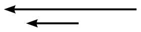
$= -2\sqrt{5}$ m/s^2 **Ans.**

$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(10)^2 - (2\sqrt{5})^3}$

$= \sqrt{80}$ m/s^2 $= 4\sqrt{5}$ m/s^2 **Ans.**

$R = \dfrac{v^2}{a_n} = \dfrac{(10\sqrt{5})^2}{4\sqrt{5}} = 25\sqrt{5}$ m. **Ans.**

7. (a) Force diagrams of m_1 and m_2 are as shown below



(Only horizontal forces have been shown)

Equations of motion are

$$T + \mu m_1 g = m_1 R\omega^2 \qquad \ldots\text{(i)}$$

$$T - \mu m_1 g = m_2 R\omega^2 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we have

$$\omega = \sqrt{\frac{2m_1\,\mu g}{(m_1 - m_2)R}}$$

Substituting the values, we have

$\omega_{\min} = 6.32$ rad/s **Ans.**

(b) $T = m_2 R\omega^2 + \mu m_1 g$

$= (1)\,(0.5)\,(6.32)^2 + (0.5)\,(2)\,(10)$

≈ 30 N **Ans.**

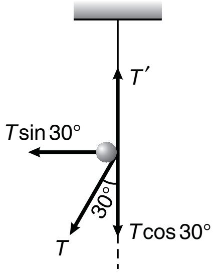
8. Speed of bob in the given position,

$v = \sqrt{2gh}$

Here, $h = (400 + 400\cos 30°)$ mm

$= 746$ mm $= 0.746$ m

$\therefore$ $v = \sqrt{2\times 9.8\times 0.746} = 3.82$ m/s



Now $T - mg\cos\theta = \dfrac{mv^2}{r}$

$T = 2\times 9.8\times\cos 30° + \dfrac{2\times(3.82)^2}{(0.4)}$

or $T = 90$ N

$\therefore$ $R = T\sin 30° = 45$ N **Ans.**

$T' = T\cos 30°$

9. Speed of each particle at angle θ is,

$v = \sqrt{2gh}$ (from energy conservation)

where, $h = R(1-\cos\theta)$

$\therefore$ $v = \sqrt{2gR(1-\cos\theta)}$

$$N + mg\cos\theta = \frac{mv^2}{R}$$

or $N + mg\cos\theta = 2mg(1-\cos\theta)$

or $N = 2mg - 3mg\cos\theta$ …(i)

The tube breaks its contact with ground when $2N\cos\theta > Mg$

Substituting, $2N\cos\theta = Mg$

or $4mg\cos\theta - 6mg\cos^2\theta = Mg$

Substituting, $\theta = 60°$

$$2mg - \frac{3mg}{2} = Mg$$

or $\dfrac{M}{m} = \dfrac{1}{2}$ **Ans.**

Note *Initially normal reaction on each ball will be radially outward and later it will be radially inward, so that normal reactions on tube is radially outward to break it off from the ground.*

10. At distance x from centre,

Centrifugal force $= mx\,\omega^2$

$\therefore$ Acceleration $a = x\,\omega^2$

or $v.\dfrac{dv}{dx} = x\,\omega^2$

or $$\int_0^v v\,dv = \omega^2 \int_a^L x\,dx$$

or $$\frac{v^2}{2} = \frac{\omega^2}{2}(L^2 - a^2)$$

or $$v = \omega\sqrt{L^2 - a^2}$$ **Ans.**

11. $$N = \frac{mv^2}{R}$$

$$f_{\max} = \mu N = \frac{\mu m v^2}{R}$$

∴ Retardation $$a = \frac{f_{\max}}{m} = \frac{\mu v^2}{R}$$

∴ $$\left(-\frac{dv}{dt}\right) = \frac{\mu v^2}{R}$$

or $$\int_{v_0}^{v} \frac{dv}{v^2} = -\frac{\mu}{R}\int_0^t dt$$

or $$v = \frac{v_0}{1 + \frac{\mu v_0 t}{R}}$$ **Ans.**

12. Let R be the radius of the ring

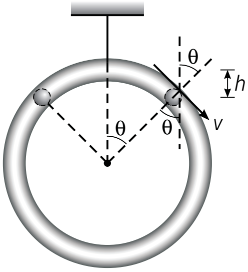
$$h = R(1 - \cos\theta)$$

$$v^2 = 2gh = 2gR(1 - \cos\theta)$$

$$\frac{mv^2}{R} = N + mg\cos\theta$$

or $$N = 2mg(1 - \cos\theta) - mg\cos\theta$$

$$N = 2mg - 3mg\cos\theta$$



In the critical condition, tension in the string is zero and net upward force on the ring

$$F = 2N\cos\theta = 2mg(2\cos\theta - 3\cos^2\theta) \qquad \ldots\text{(i)}$$

F is maximum when $\frac{dF}{d\theta} = 0$

or $$-2\sin\theta + 6\sin\theta\cos\theta = 0$$

or $$\cos\theta = \frac{1}{3}$$

Substituting in Eq. (i)

$$F_{\max} = 2mg\left(2 \times \frac{1}{3} - 3 \times \frac{1}{9}\right) = \frac{2}{3}mg$$

$$F_{\max} > Mg$$

or $$\frac{2}{3}mg > Mg$$

or $$m > \frac{3}{2}M$$ **Hence proved.**

13. Minimum velocity of particle at the lowest position to complete the circle should be $\sqrt{4gR}$ inside a tube.

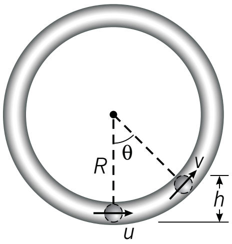
So, $$u = \sqrt{4gR}$$

$$h = R(1 - \cos\theta)$$

∴ $$v^2 = u^2 - 2gh$$

or $$v^2 = 4gR - 2gR(1 - \cos\theta) = 2gR(1 + \cos\theta)$$

or $$v^2 = 2gR\left(2\cos^2\frac{\theta}{2}\right)$$



or $$v = 2\sqrt{gR}\cos\frac{\theta}{2}$$

From $$ds = v \cdot dt$$

We get $$R\,d\theta = 2\sqrt{gR}\cos\frac{\theta}{2} \cdot dt$$

or $$\int_0^t dt = \frac{1}{2}\sqrt{\frac{R}{g}}\int_0^{\pi/2} \sec\left(\frac{\theta}{2}\right) d\theta$$

or $$t = \sqrt{\frac{R}{g}}\left[\ln\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

or $$t = \sqrt{\frac{R}{g}}\ln(1 + \sqrt{2})$$ **Hence proved.**

14. At position θ,

$$v^2 = v_0^2 + 2gh$$

where, $$h = a(1 - \cos\theta)$$

∴ $$v^2 = (\sqrt{2ag})^2 + 2ag(1 - \cos\theta)$$

or $$v^2 = 2ag(2 - \cos\theta) \qquad \ldots\text{(i)}$$

$$N + mg\cos\theta = \frac{mv^2}{a}$$

or $\quad N + mg\cos\theta = 2mg(2 - \cos\theta)$

or $\quad N = mg(4 - 3\cos\theta)$

Net vertical force,

$$F = N\cos\theta + mg$$

$$= mg(4\cos\theta - 3\cos^2\theta + 1)$$

This force (or acceleration) will be maximum when $\dfrac{dF}{d\theta} = 0$

or $\quad -4\sin\theta + 6\sin\theta\cos\theta = 0$

So, either

$$\sin\theta = 0,$$

$$\theta = 0°$$

or $\quad \cos\theta = \dfrac{2}{3},$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$\theta = 0°$ is unacceptable

Therefore, the desired position is at

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$ **Ans.**

15. (a) Let v_r be the velocity of mass relative to track at angular position θ.

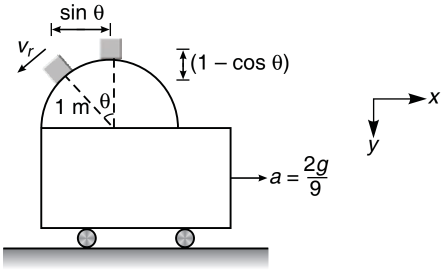
From work energy theorem, KE of particle relative to track

= Work done by force of gravity + work done by pseudo force

$\therefore \quad \dfrac{1}{2}mv_r^2 = mg(1 - \cos\theta) + m\left(\dfrac{2g}{9}\right)\sin\theta$

or $\quad v_r^2 = 2g(1 - \cos\theta) + \dfrac{4g}{9}\sin\theta \quad$ ...(i)

Particle leaves contact with the track where $N = 0$



or $\quad mg\cos\theta - m\left(\dfrac{2g}{9}\right)\sin\theta = mv_r^2$

or $\quad g\cos\theta - \dfrac{2g}{9}\sin\theta = 2g(1 - \cos\theta) + \dfrac{4g}{9}\sin\theta$

or $\quad 3\cos\theta - \dfrac{6}{9}\sin\theta = 2$

Solving this, we get $\quad \theta \approx 37°$ **Ans.**

(b) From Eq. (i), we have

$$v_r = \sqrt{2g(1 - \cos\theta) + \frac{4g}{9}\sin\theta}$$

or $\quad v_r = 2.58$ m/s at $\theta = 37°$

Vertical component of its velocity is

$$v_y = v_r\sin\theta$$

$$= 2.58 \times \frac{3}{5}$$

$$= 1.55 \text{ m/s}$$

Now, $\quad 1.3 = 1.55t + 5t^2 \quad \left(\because s = ut + \dfrac{1}{2}gt^2\right)$

or $\quad 5t^2 + 1.55t - 1.3 = 0$

or $\quad t = 0.38$ s **Ans.**

JEE Main and Advanced

Previous Years' Questions (2018-13)

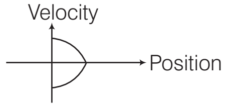
JEE Main

1. The density of a material in the shape of a cube is determined by measuring three sides of the cube and its mass. If the relative errors in measuring the mass and length are respectively 1.5% and 1% , the maximum error in determining the density is **(2018)**

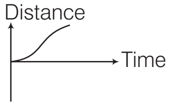
(a) 2.5% (b) 3.5% (c) 4.5% (d) 6%

2. All the graphs below are intended to represent the same motion. One of them does it incorrectly. Pick it up. **(2018)**

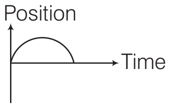
(a)



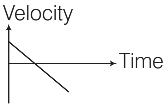
(b)



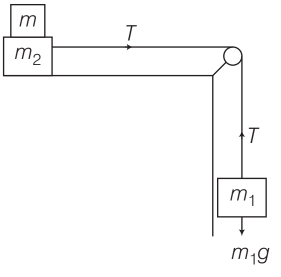
(c)



(d)



3. Two masses $m_1 = 5\,\text{kg}$ and $m_2 = 10\,\text{kg}$ connected by an inextensible string over a frictionless pulley, are moving as shown in the figure. The coefficient of friction of horizontal surface is 0.15. The minimum weight m that should be put on top of m_2 to stop the motion is **(2018)**



(a) 18.3 kg (b) 27.3 kg

(c) 43.3 kg (d) 10.3 kg

4. A particle is moving in a circular path of radius a under the action of an attractive potential $U = -\frac{k}{2r^2}$. Its total energy is **(2018)**

(a) $-\frac{k}{4a^2}$ (b) $\frac{k}{2a^2}$

(c) zero (d) $-\frac{3}{2}\frac{k}{a^2}$

5. In a collinear collision, a particle with an initial speed v_0 strikes a stationary particle of the same mass. If the final total kinetic energy is 50% greater than the original kinetic energy, the magnitude of the relative velocity between the two particles after collision, is **(2018)**

(a) $\frac{v_0}{4}$ (b) $\sqrt{2}\,v_0$ (c) $\frac{v_0}{2}$ (d) $\frac{v_0}{\sqrt{2}}$

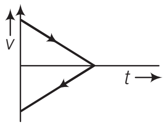
6. A particle is moving with a uniform speed in a circular orbit of radius R in a central force inversely proportional to the nth power of R. If the period of rotation of the particle is T, then : **(2018)**

(a) $T \propto R^{3/2}$ for any n (b) $T \propto R^{\frac{n}{2}+1}$

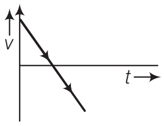
(c) $T \propto R^{(n+1)/2}$ (d) $T \propto R^{n/2}$

7. A body is thrown vertically upwards. Which one of the following graphs correctly represent the velocity *vs* time? **(2017)**

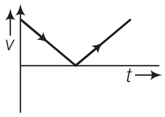
(a)



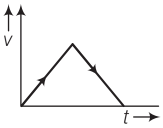
(b)



(c)



(d)



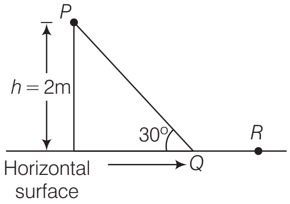
8. A body of mass $m = 10^{-2}$kg is moving in a medium and experiences a frictional force $F = -kv^2$. Its initial speed is $v_0 = 10\ \text{ms}^{-1}$. If, after 10 s, its energy is $1/8\ mv_0^2$, the value of k will be **(2017)**

(a) 10^{-3} kgs^{-1} (b) 10^{-4} kgm^{-1}
(c) 10^{-1} kgm^{-1}s^{-1} (d) 10^{-3} kgm^{-1}

9. A time dependent force $F = 6t$ acts on a particle of mass 1 kg. If the particle starts from rest, the work done by the force during the first 1 s will be **(2017)**

(a) 22 J (b) 9 J (c) 18 J (d) 4.5 J

10. A point particle of mass m, moves along the uniformly rough track PQR as shown in the figure. The coefficient of friction, between the particle and the rough track equals μ. The particle is released from rest, from the point P and it comes to rest at a point R. The energies lost by the ball, over the parts PQ and QR of the track, are equal to each other, and no energy is lost when particle changes direction from PQ to QR. The values of the coefficient of friction μ and the distance x (= QR), are respectively close to **(2016)**



(a) 0.2 and 6.5 m (b) 0.2 and 3.5 m
(c) 0.29 and 3.5 m (d) 0.29 and 6.5 m

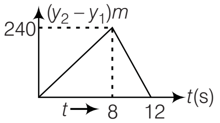
11. A person trying to loose weight by burning fat lifts a mass of 10 kg upto a height of 1 m 1000 times. Assume that the potential energy lost each time he lowers the mass is dissipated. How much fat will he use up considering the work done only when the weight is lifted up? Fat supplies 3.8×10^7 J of energy per kg which is converted to mechanical energy with a 20% efficiency rate. (Take, $g = 9.8\ \text{ms}^{-2}$) **(2016)**

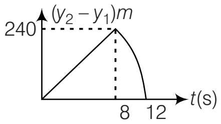
(a) 2.45×10^{-3} kg (b) 6.45×10^{-3} kg
(c) 9.89×10^{-3} kg (d) 12.89×10^{-3} kg

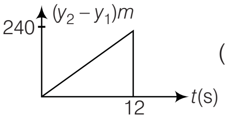
12. The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{L/g}$. Measured value of L is 20.0 cm known to 1 mm accuracy and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1s resolution. The accuracy in the determination of g is **(2015)**

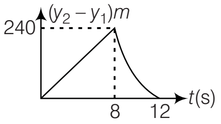
(a) 3% (b) 2% (c) 1% (d) 5%

13. Two stones are thrown up simultaneously from the edge of a cliff 240 m high with initial speed of 10 m/s and 40 m/s, respectively. Which of the following graph best represents the time variation of relative position of the second stone with respect to the first?
(Assume stones do not rebound after hitting the ground and neglect air resistance, take $g = 10\,\text{m/s}^2$) **(2015)**

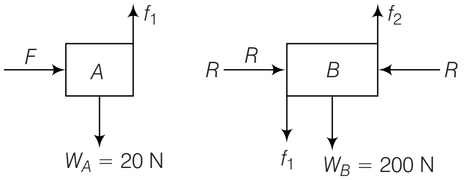
(a)


(b)


(c)


(d)


14. Given in the figure are two blocks A and B of weight 20 N and 100 N, respectively. These are being pressed against a wall by a force F as shown in figure. If the coefficient of friction between the blocks is 0.1 and between block B and the wall is 0.15, the frictional force applied by the wall in block B is **(2015)**



(a) 100 N (b) 120 N (c) 80 N (d) 150 N

15. The current voltage relation of diode is given by $I = (e^{1000V/T} - 1)$ mA, where the applied voltage V is in volt and the temperature T is in kelvin. If a student makes an error measuring ± 0.01 V while measuring the current of 5 mA at 300 K, what will be the error in the value of current in mA? **(2014)**

(a) 0.2 mA (b) 0.02 mA
(c) 0.5 mA (d) 0.05 mA

16. A student measured the length of a rod and wrote it as 3.50 cm. Which instrument did he use to measure it?

(a) A meter scale **(2014)**
(b) A vernier caliper where the 10 divisions in vernier scale matches with 9 divisions in main scale and main scale has 10 divisions in 1 cm
(c) A screw gauge having 100 divisions in the circular scale and pitch as 1 mm and pitch as 1 mm
(d) A screw gauge having 50 divisions in the circular scale

17. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle to hit the ground, is n times that taken by it to reach the highest point of its path. The relation between H, u and n is **(2014)**

(a) $2gH = n^2u^2$ (b) $gH = (n-2)^2 u^2$
(c) $2gH = nu^2(n-2)$ (d) $gH = (n-2)^2 u^2$

18. A block of mass m is placed on a surface with a vertical cross-section given by $y = x^3/6$. If the coefficient of friction is 0.5, the maximum height above the ground at which the block can be placed without slipping is **(2014)**

(a) 1/6 m (b) 2/3 m
(c) 1/3 m (d) 1/2 m

19. When a rubber band is stretched by a distance x, it exerts a restoring force of magnitude $F = ax + bx^2$, where a and b are constants. The work done in stretching the unstretched rubber band by L is **(2014)**

(a) $aL^2 + bL^3$ (b) $1/2(aL^2 + bL^3)$
(c) $\dfrac{aL^2}{2} + \dfrac{bL^3}{3}$ (d) $\dfrac{1}{2}\left(\dfrac{aL^2}{2} + \dfrac{bL^3}{3}\right)$

20. Let $[\varepsilon_0]$ denote the dimensional formula of the permittivity of vacuum. If M = mass, L = length, T = Time and A = electric current, then **(2013)**

(a) $[\varepsilon_0] = [M^{-1}L^{-3}T^2A]$
(b) $[\varepsilon_0] = [M^{-1}L^{-3}T^4A^2]$
(c) $[\varepsilon_0] = [M^{-2}L^2T^{-1}A^{-2}]$
(d) $[\varepsilon_0] = [M^{-1}L^2T^{-1}A^2]$

21. A projectile is given an initial velocity of $(\mathbf{i} + 2\mathbf{j})$ m/s, where, **i** is along the ground and **j** is along the vertical. If $g = 10\ \text{m/s}^2$, then the equation of its trajectory is **(2013)**

(a) $y = x - 5x^2$ (b) $y = 2x - 5x^2$
(c) $4y = 2x - 5x^2$ (d) $4y = 2x - 25x^2$

Answer *with* Explanations

1. **(c)** ∴ Density, $\rho = \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{M}{L^3}$ or $\rho = \dfrac{M}{L^3}$

⇒ Error in density $\dfrac{\Delta\rho}{\rho} = \dfrac{\Delta M}{M} + \dfrac{3\Delta L}{L}$

So, maximum % error in measurement of ρ is

$$\frac{\Delta\rho}{\rho} \times 100 = \frac{\Delta M}{M} \times 100 + \frac{3\Delta L}{L} \times 100$$

or % error in density = 1.5 + 3 × 1

% error = 4.5%

2. **(b)** If velocity versus time graph is a straight line with negative slope, then acceleration is constant and negative.

With a negative slope distance-time graph will be parabolic $\left(s = ut - \dfrac{1}{2}at^2\right)$.

So, option (b) will be incorrect.

3. **(b)** Motion stops when pull due to m_1 ≤ force of friction between m and m_2 and surface.

⇒ $m_1 g \le \mu(m_2 + m)g$

⇒ $5 \times 10 \le 0.15(10 + m) \times 10$

⇒ $m \ge 23.33$ kg

Here, nearest value is 27.3 kg

So, $m_{\min} = 27.3$ kg

4. **(c)** ∴ Force $= -\dfrac{dU}{dr}$

⇒ $F = -\dfrac{d}{dr}\left(\dfrac{-k}{2r^2}\right) = -\dfrac{k}{r^3}$

As particle is on circular path, this force must be centripetal force.

⇒ $|F| = \dfrac{mv^2}{r}$

So, $$\frac{k}{r^3} = \frac{mv^2}{r}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{k}{2r^2}$$

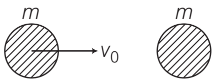
$\therefore$ Total energy of particle = KE + PE

$$= \frac{k}{2r^2} - \frac{k}{2r^2} = 0$$

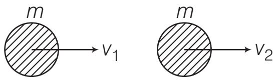
Total energy = 0

5. **(b)** Momentum is conserved in all type of collisions. Final kinetic energy is 50% more than initial kinetic energy

$$\Rightarrow \quad \frac{1}{2}mv_2^2 + \frac{1}{2}mv_1^2 = \frac{150}{100} \times \frac{1}{2}mv_0^2 \qquad \ldots\text{(i)}$$



Before collision



After collision

Conservation of momentum gives,

$$mv_0 = mv_1 + mv_2$$

$$v_0 = v_2 + v_1 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we have

$$v_1^2 + v_2^2 + 2v_1v_2 = v_0^2$$

$$\Rightarrow \quad 2v_1v_2 = \frac{-v_0^2}{2}$$

$$\therefore \quad (v_1 - v_2)^2 = (v_1 + v_2)^2 - 4v_1v_2$$

$$= 2v_0^2$$

or $$v_{\text{rel}} = \sqrt{2}v_0$$

6. **(c)** $\therefore$ Force = Mass × Acceleration $= m\omega^2R$

and given, $F \propto \frac{1}{R^n} \quad \Rightarrow F = \frac{k}{R^n}$

So, we have

$$\frac{k}{R^n} = m\left(\frac{2\pi}{T}\right)^2 R$$

$$\Rightarrow \quad T^2 = \frac{4\pi^2 m}{k} \cdot R^{n+1}$$

$$\Rightarrow \quad T \propto R^{\frac{n+1}{2}}$$

7. **(b)** Initially velocity keeps on decreasing at a constant rate, then it increases in negative direction with same rate.

8. **(b)** Given, force, $F = -kv^2$

$\therefore$ Acceleration, $a = \frac{-k}{m}v^2$

or $$\frac{dv}{dt} = \frac{-k}{m}v^2 \quad \Rightarrow \quad \frac{dv}{v^2} = -\frac{k}{m}\,.\,dt$$

Now, with limits, we have

$$\int_{10}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt$$

$$\Rightarrow \quad \left(-\frac{1}{v}\right)_{10}^{v} = -\frac{k}{m}t$$

$$\Rightarrow \quad \frac{1}{v} = 0.1 + \frac{kt}{m}$$

$$\Rightarrow \quad v = \frac{1}{0.1 + \frac{kt}{m}} = \frac{1}{0.1 + 1000k}$$

$$\Rightarrow \quad \frac{1}{2} \times m \times v^2 = \frac{1}{8} \times v_0^2$$

$$\Rightarrow \quad v = \frac{v_0}{2} = 5$$

$$\Rightarrow \quad \frac{1}{0.1 + 1000\,k} = 5$$

$$\Rightarrow \quad 1 = 0.5 + 5000k$$

$$\Rightarrow \quad k = \frac{0.5}{5000}$$

$$\Rightarrow \quad k = 10^{-4} \text{ kg/m}$$

9. **(d)** From Newton's second law, $\frac{\Delta p}{\Delta t} = F$

$$\Rightarrow \quad \Delta p = F\Delta t \;\Rightarrow\; p = \int dp = \int_0^1 F\,dt$$

$$\Rightarrow \quad p = \int_0^1 6\,t\,dt = 3\text{ kg}\left(\frac{\text{m}}{\text{s}}\right)$$

Also, $\Delta k = \frac{\Delta p^2}{2m} = \frac{3^2}{2 \times 1} = 4.5$

So, work done = Δk = 4.5 J

10. **(c)** As energy loss is same, thus

$$\mu\, mg\cos\theta \cdot (PQ) = \mu mg \cdot (QR)$$

$$\therefore \quad QR = (PQ)\cos\theta$$

$$\Rightarrow \quad QR = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3} \approx 3.5 \text{ m}$$

Further,decrease in potential energy = loss due to friction

$$\therefore \quad mgh = (\mu mg\cos\theta)d_1 + (\mu mg)d_2$$

$$m \times 10 \times 2 = \mu \times m \times 10 \times \frac{\sqrt{3}}{2} \times 4 + \mu \times m \times 10 \times 2\sqrt{3}$$

$$\Rightarrow \quad 4\sqrt{3}\,\mu = 2$$

$$\Rightarrow \quad \mu = \frac{1}{2\sqrt{3}} = 0.288 = 0.29$$

11. **(d)** Work done in lifting mass = (10 × 9.8 × 1) × 1000

If m is mass of fat burnt, then energy

$$= m \times 3.8 \times 10^7 \times \frac{20}{100}$$

Equating the two, we get

$$\therefore \quad m = \frac{49}{3.8} \approx 12.89 \times 10^{-3} \text{ kg}$$

12. (a) Time period is given by, $T = \dfrac{t}{n}$

Further, $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$\Rightarrow \quad g = \frac{(4\pi^2)L}{T^2} = \frac{(4\pi^2)(L)}{\left(\dfrac{t}{n}\right)^2} = (4\pi^2 n^2)\frac{L}{t^2}$$

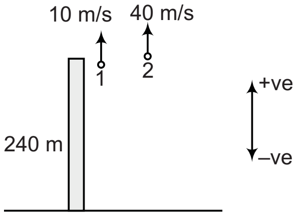
Percentage error in the value of 'g' will be

$$\frac{\Delta g}{g} \times 100 = \left(\frac{\Delta L}{L}\right) \times 100 + 2\left(\frac{\Delta t}{t}\right) \times 100$$

$$= \frac{0.1}{20} \times 100 + 2 \times \left(\frac{1}{90}\right) \times 100 = 2.72\%$$

$\therefore$ The nearest answer is 3%.

13. (b)



Let us first find, time of collision of two particles with ground in putting proper values in the equation

$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad -240 = 10t_1 - \frac{1}{2} \times 10 \times t_1^2$$

Solving, we get the position value of $t_1 = 8$ sec

Therefore, the first particle will strike the ground at 8 sec.

Similarly, $-240 = 40t_2 - \dfrac{1}{2} \times 10 \times t_2^2$

Solving this equation, we get positive value of $t_2 = 2$ s

Therefore, second particle strikes the ground at 12 sec.

If y is measured from ground. Then,

from 0 to 8 s $y_1 = 240 + s_1$

$$= 240 + u_1 t + \frac{1}{2}a_1 t^2$$

or $\quad y_1 = 240 + 10t - \dfrac{1}{2} \times 10 \times t^2$

Similarly, $\quad y_2 = 240 + 40t - \dfrac{1}{2} \times 10 \times t^2$

$$\Rightarrow \quad t_2 - y_1 = 30t$$

$\therefore (y_2 - y_1)$ versus t graph is a straight line passing through origin

At $t = 8$ s, $y_2 - y_1 = 240$ m

From 8 s to 12 s $y_1 = 0$

$$\Rightarrow \quad y_2 = 240 + 40t - \frac{1}{2} \times 10 \times t^2$$

$$= 240 + 40t - 5t$$

$$\therefore \quad (y_2 - y_1) = 240 + 40t - 5t^2$$

Therefore, $(y_2 - y_1)$ *versus* t graph is parabolic, substituting the values we can check that at $t = 8$ sec, $y_2 - y_1$ is 240 m and at $t = 12$ sec, $y_2 - y_1$ is zero.

14. (b) **Note** It is not given in the question, best assuming that both blocks are in equilibrium. The free body diagram of two blocks is as shown below,

Reaction force, R = applied force F

For vertical equilibrium of A;

f_1 = friction between two blocks $= W_A = 20$ N

For vertical equilibrium of B;

f_2 = friction between block B and wall

$$= W_B + f_1 = 100 + 20 = 120 \text{ N}$$

15. (a) Given, $\quad I = (e^{1000V/T} - 1)$ mA, $dV = \pm\, 0.01$ V

$$T = 300 \text{ K}$$

So, $\quad I = e^{1000V/T} - 1$

$$\Rightarrow \quad I + 1 = e^{1000V/T}$$

Taking log on both sides, we get

$$\log(I + 1) = \frac{1000\,V}{T}$$

On differentiating, $\quad \dfrac{dI}{I + 1} = \dfrac{1000}{T}dV$

$$dI = \frac{1000}{T} \times (I + 1)\, dV$$

$$\Rightarrow \quad dI = \frac{1000}{300} \times (5 + 1) \times 0.01$$

$$= 0.2 \text{ mA}$$

So, error in the value of current is 0.2 mA.

16. (c) If student measures 3.50 cm, it means that there is an uncertainly of order 0.01 cm.

For vernier scale with 1 MSD $= \dfrac{1}{10}$ cm

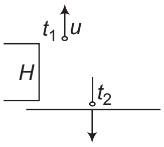
and $\quad$ 9 MSD = 10 VSD

LC of vernier caliper = 1MSD – 1VSD

$$= \frac{1}{10}\left(1 - \frac{9}{10}\right)$$

$$= \frac{1}{100} \text{ cm} = 0.01 \text{ cm}$$

17. **(c)** Time taken to reach the maximum height, $t_1 = \dfrac{u}{g}$



If t_2 is the time taken to hit the ground, then

i.e. $$-H = ut_2 - \frac{1}{2}gt_2^2$$

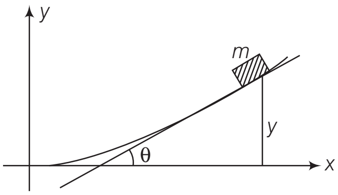
But $$t_2 = nt_1 \qquad \text{[Given]}$$

So, $$-H = u\frac{nu}{g} - \frac{1}{2}g\frac{n^2u^2}{g^2}$$

$$-H = \frac{nu^2}{g} - \frac{1}{2}\frac{n^2u^2}{g}$$

$$\Rightarrow \quad 2gH = nu^2(n-2)$$

18. **(a)** A block of mass m is placed on a surface with a vertical cross-section, then



$$\tan\theta = \frac{dy}{dx}\frac{d\left(\dfrac{x^3}{6}\right)}{dx} = \frac{x^2}{2}$$

At limiting equilibrium, we get

$$\mu = \tan\theta \Rightarrow 0.5 = \frac{x^2}{2}$$

$$\Rightarrow \quad x^2 = 1 \Rightarrow x = \pm 1$$

Now, putting the value of x in $y = \dfrac{x^3}{6}$, we get

When $x = 1$	When $x = -1$
$y = \dfrac{(1)^3}{6} = \dfrac{1}{6}$	$y = \dfrac{(-1)^3}{6} = \dfrac{-1}{6}$

So, the maximum height above the ground at which the block can be placed without slipping is 1/6 m.

19. **(c)** **Thinking Process** We know that change in potential energy of a system corresponding to a conservative internal force as

$$U_f - U_i = -W = -\int_i^f \mathbf{F}\cdot d\mathbf{r}$$

Given, $$F = ax + bx^2$$

We know that work done in stretching the rubber band by L is $|dW| = |Fdx|$

$$|W| = \int_0^L (ax + bx^2)\,dx = \left[\frac{ax^2}{2}\right]_0^L + \left[\frac{bx^3}{3}\right]_0^L$$

$$= \left[\frac{aL^2}{2} - \frac{a\times(0)^2}{2}\right] + \left[\frac{b\times L^3}{3} - \frac{b\times(0)^3}{3}\right]$$

$$= |W| = \frac{aL^2}{2} + \frac{bL^3}{3}$$

20. **(b)** From Coulomb's law, $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{R^2}$

$$\varepsilon_0 = \frac{q_1q_2}{4\pi FR^2}$$

Substituting the units, we have

$$\varepsilon_0 = \frac{\text{C}^2}{\text{N-m}^2} = \frac{[\text{AT}]^2}{[\text{MLT}^{-2}]\,[\text{L}^2]}$$

$$= [\text{M}^{-1}\text{L}^{-3}\text{T}^4\text{A}^2]$$

21. **(b)** Initial velocity $= (\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ m/s

Magnitude of initial velocity,

$$u = \sqrt{(1)^2 + (2)^2} = \sqrt{5}\ \text{m/s}$$

Equation of trajectory of projectile is

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta) \qquad \left[\tan\theta = \frac{y}{x} = \frac{2}{1} = 2\right]$$

$$\therefore \quad y = x \times 2 - \frac{10(x)^2}{2(\sqrt{5})^2}[1 + (2)^2]$$

$$= 2x - \frac{10(x^2)}{2\times 5}(1+4)$$

$$= 2x - 5x^2$$

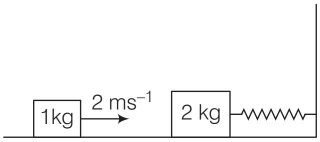
JEE Advanced

1. Two vectors $\mathbf{A}$ and $\mathbf{B}$ are defined as $\mathbf{A} = a\hat{\mathbf{i}}$ and $\mathbf{B} = a(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}})$, where a is a constant and $\omega = \pi/6 \text{ rad s}^{-1}$. If $|\mathbf{A} + \mathbf{B}| = \sqrt{3}|\mathbf{A} - \mathbf{B}|$ at time $t = \tau$ for the first time, the value of τ, in seconds, is ....

[Numerical Value, 2018]

2. A spring block system is resting on a frictionless floor as shown in the figure. The spring constant is 2.0 Nm^{-1} and the mass of the block is 2.0 kg. Ignore the mass of the spring. Initially, the spring is in an unstretched condition. Another block of mass 1.0 kg moving with a speed of 2.0 ms^{-1} collides elastically with the first block. The collision is such that the 2.0 kg block does not hit the wall. The distance, in metres, between the two blocks when the spring returns to its unstretched position for the first time after the collision is ....

[Numerical Value, 2018]



Paragraph X (Q. Nos. 3-4)

In electromagnetic theory, the electric and magnetic phenomena are related to each other. Therefore, the dimensions of electric and magnetic quantities must also be related to each other. In the questions below, $[\mathbf{E}]$ and $[\mathbf{B}]$ stand for dimensions of electric and magnetic fields respectively, while $[\varepsilon_0]$ and $[\mu_0]$ stand for dimensions of the permittivity and permeability of free space, respectively. $[\text{L}]$ and $[\text{T}]$ are dimensions of length and time, respectively. All the quantities are given in SI units. **(2018)**

3. The relation between $[E]$ and $[B]$ is

(a) $[E] = [B]\ [\text{L}]\ [\text{T}]$ (b) $[E] = [B][\text{L}]^{-1}[\text{T}]$

(c) $[E] = [B][\text{L}][\text{T}]^{-1}$ (d) $[E] = [B][\text{L}]^{-1}[\text{T}]^{-1}$

4. The relation between $[\varepsilon_0]$ and $[\mu_0]$ is

(a) $[\mu_0] = [\varepsilon_0][\text{L}]^2[\text{T}]^{-2}$ (b) $[\mu_0] = [\varepsilon_0][\text{L}]^{-2}[\text{T}]^2$

(c) $[\mu_0] = [\varepsilon_0]^{-1}[\text{L}]^2[\text{T}]^{-2}$ (d) $[\mu_0] = [\varepsilon_0]^{-1}[\text{L}]^{-2}[\text{T}]^2$

Paragraph A (Q. Nos. 5-6)

If the measurement errors in all the independent quantities are known, then it is possible to determine the error in any dependent quantity. This is done by the use of series expansion and truncating the expansion at the first power of the error. For example, consider the relation $z = x/y$. If the errors in x, y and z are Δx, Δy and Δz respectively, then

$$z \pm \Delta z = \frac{x \pm \Delta x}{y \pm \Delta y} = \frac{x}{y}\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

The series expansion for $\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$, to first power in $\Delta y/y$, is $1 \mp (\Delta y/y)$. The relative errors in independent variables are always added. So, the error in z will be

$$\Delta z = z\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right)$$

The above derivation makes the assumption that $\Delta x/x \ll 1$, $\Delta y/y \ll 1$. Therefore, the higher powers of these quantities are neglected. **(2018)**

5. Consider the ratio $r = \dfrac{(1-a)}{(1+a)}$ to be determined by measuring a dimensionless quantity a. If the error in the measurement of a is Δa $(\Delta a/a \ll 1)$, then what is the error Δr in determining r?

(a) $\dfrac{\Delta a}{(1+a)^2}$ (b) $\dfrac{-2\Delta a}{(1+a)^2}$ (c) $\dfrac{2\Delta a}{(1-a)^2}$ (d) $\dfrac{2a\Delta a}{(1-a^2)}$

6. In an experiment, the initial number of radioactive nuclei is 3000. It is found that 1000 ± 40 nuclei decayed in the first 1.0s. For $|x| \ll 1$, $\ln(1+x) = x$ up to first power in x. The error $\Delta\lambda$, in the determination of the decay constant λ in s^{-1}, is

(a) 0.04 (b) 0.03 (c) 0.02 (d) 0.01

7. A particle of mass m is initially at rest at the origin. It is subjected to a force and starts moving along the X-axis. Its kinetic energy K changes with time as $dK/dt = \gamma t$, where γ is a positive constant of appropriate dimensions. Which of the following statements is (are) true?

(More than One Correct Option, 2018)

(a) The force applied on the particle is constant
(b) The speed of the particle is proportional to time
(c) The distance of the particle from the origin increases linearly with time
(d) The force is conservative

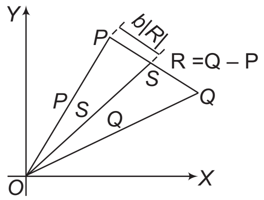
8. A solid horizontal surface is covered with a thin layer of oil. A rectangular block of mass $m = 0.4$ kg is at rest on this surface. An impulse of 1.0 N s is applied to the block at time $t = 0$, so that it starts moving along the X-axis with a velocity $v(t) = v_0 e^{-t/\tau}$, where v_0 is a constant and $\tau = 4$ s. The displacement of the block, in metres, at $t = \tau$ is .... (Take, $e^{-1} = 0.37$). **(Numerical Value, 2018)**

9. A ball is projected from the ground at an angle of 45° with the horizontal surface. It reaches a maximum height of 120 m and returns to the ground. Upon hitting the ground for the first time, it loses half of its kinetic energy. Immediately after the bounce, the velocity of the ball makes an angle of 30° with the horizontal surface. The maximum height it reaches after the bounce, in metres, is .... **(Numerical Value, 2018)**

10. A person measures the depth of a well by measuring the time interval between dropping a stone and receiving the sound of impact with the bottom of the well. The error in his measurement of time is $\delta T = 0.01$ s and he measures the depth of the well to be $L = 20$ m. Take the acceleration due to gravity $g = 10\ \text{ms}^{-2}$ and the velocity of sound is $300\ \text{ms}^{-1}$. Then the fractional error in the measurement, $\frac{\delta L}{L}$, is closest to **(Single Correct Option, 2017)**

(a) 1% (b) 5% (c) 3% (d) 0.2%

11. Three vectors **P**,**Q** and **R** are shown in the figure. Let S be any point on the vector **R**. The distance between the points P and S is b [R]. The general relation among vectors **P**,**Q** and **S** is **(Single Correct Option, 2017)**



(a) $\mathbf{S} = (1-b^2)\mathbf{P} + b\mathbf{Q}$ (b) $\mathbf{S} = (b-1)\mathbf{P} + b\mathbf{Q}$
(c) $\mathbf{S} = (1-b)\mathbf{P} + b\mathbf{Q}$ (d) $\mathbf{S} = (1-b)\mathbf{P} + b^2\mathbf{Q}$

12. A flat plane is moving normal to its plane through a gas under the action of a constant force F. The gas is kept at a very low pressure. The speed of the plate v is much less than the average speed u of the gas molecules. Which of the following option(s) is/are true? **(More than One Correct Option, 2017)**

(a) At a later time the external force F balances the resistive force
(b) The plate will continue to move with constant non-zero acceleration, at all time
(c) The resistive force experienced by the plate is proportional to v
(d) The pressure differnce between the leading and trailing faces of the plate is proportional to uv

13. A length-scale (l) depends on the permittivity (ε) of a dielectric material, Boltzmann's constant (k_B), the absolute temperature (T), the number per unit volume (n) of certain charged particles, and the charge (q) carried by each of the particles. Which of the following expression (s) for l is (are) dimensionally correct? **(More than One Correct Option, 2016)**

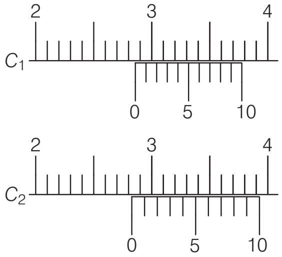
(a) $l = \sqrt{\left(\frac{nq^2}{\varepsilon k_B T}\right)}$ (b) $l = \sqrt{\left(\frac{\varepsilon k_B T}{nq^2}\right)}$

(c) $l = \sqrt{\left(\frac{q^2}{\varepsilon n^{2/3} k_B T}\right)}$ (d) $l = \sqrt{\left(\frac{q^2}{\varepsilon n^{1/3} k_B T}\right)}$

14. In an experiment to determine the acceleration due to gravity g, the formula used for the time period of a periodic motion is $T = 2\pi\sqrt{\frac{7(R-r)}{5g}}$. The values of R and r are measured to be (60 ± 1) mm and (10 ± 1) mm, respectively. In five successive measurements, the time period is found to be 0.52 s, 0.56 , 0.57 s, 0.54 s and 0.59 s. The least count of the watch used for the measurement of time period is 0.01 s. Which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**

(a) The error in the measurement of r is 10%
(b) The error in the measurement of T is 3.57%
(c) The error in the measurement of T is 2%
(d) The error in the measurement of g is 11%

15. There are two vernier callipers both of which have 1 cm divided into 10 equal divisions on the main scale. The vernier scale of one of the callipers (C_1) has 10 equal divisions that correspond to 9 main scale divisions. The vernier scale of the other callipers (C_2) has 10 equal divisions that correspond to 11 main scale divisions. The readings of the two callipers are shown in the figure. The measured values (in cm) by callipers C_1 and C_2 respectively, are **(Single Correct Option, 2016)**



(a) 2.87 and 2.87 (b) 2.87 and 2.83
(c) 2.85 and 2.82 (d) 2.87 and 2.86

16. A uniform wooden stick of mass 1.6 kg of length l rests in an inclined manner on a smooth, vertical wall of height $h\,(< l)$ such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/l and the frictional force f at the bottom of the stick are ($g = 10\,\text{ms}^{-2}$) **(Single Correct Option, 2016)**

(a) $\dfrac{h}{l} = \dfrac{\sqrt{3}}{16}$, $f = \dfrac{16\sqrt{3}}{3}$N (b) $\dfrac{h}{l} = \dfrac{3}{16}$, $f = \dfrac{16\sqrt{3}}{3}$N
(c) $\dfrac{h}{l} = \dfrac{3\sqrt{3}}{16}$, $f = \dfrac{8\sqrt{3}}{3}$N (d) $\dfrac{h}{l} = \dfrac{3\sqrt{3}}{16}$, $f = \dfrac{16\sqrt{3}}{3}$N

17. The energy of a system as a function of time t is given as $E(t) = A^2 \exp(-\alpha t)$, where $\alpha = 0.2\,\text{s}^{-1}$. The measurement of A has an error of 1.25%. If the error in the measurement of time is 1.50%, the percentage error in the value of $E(t)$ at $t = 5$ s is **(Single Integer Type, 2015)**

18. Planck's constant h, speed of light c and gravitational constant G are used to form a unit of length L and a unit of mass M. Then, the correct option(s) is/are **(More than One Correct Option, 2015)**

(a) $M \propto \sqrt{c}$ (b) $M \propto \sqrt{G}$
(c) $L \propto \sqrt{h}$ (d) $L \propto \sqrt{G}$

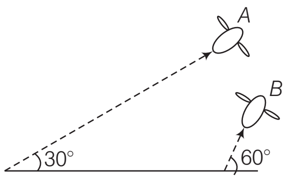
19. Consider a vernier callipers in which each 1 cm on the main scale is divided into 8 equal divisions and a screw gauge with 100 divisions on its circular scale. In the vernier callipers, 5 divisions of the vernier scale coincide with 4 divisions on the main scale and in the screw gauge, one complete rotation of the circular scale moves it by two divisions on the linear scale. Then, **(More than One Correct Option, 2015)**

(a) if the pitch of the screw gauge is twice the least count of the vernier callipers, the least count of the screw gauge is 0.01 mm
(b) if the pitch of the screw gauge is twice the least count of the Vernier callipers, the least count of the screw gauge is 0.05 mm
(c) if the least count of the linear scale of the screw gauge is twice the least count of the vernier callipers, the least count of the screw gauge is 0.01 mm
(d) if the least count of the linear scale of the screw gauge is twice the least count of the vernier callipers, the least count of the screw gauge is 0.005 mm.

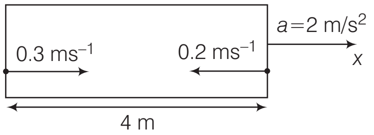
20. A particle of unit mass is moving along the x-axis under the influence of a force and its total energy is conserved. Four possible forms of the potential energy of the particle are given in Column I (a and U_0 are constants). Match the potential energies in Column I to the corresponding statements in Column II. **(Matching Type, 2015)**

Column I	Column II
A. $U_1(x) = \frac{U_0}{2}\left[1-\left(\frac{x}{a}\right)^2\right]^2$	P. The force acting on the particle is zero at $x = a$
B. $U_2(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2$	Q. The force acting on the particle is zero at $x = 0$
C. $U_3(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 \exp\left[-\left(\frac{x}{a}\right)^2\right]$	R. The force acting on the particle is zero at $x = -a$
D. $U_4(x) = \frac{U_0}{2}\left[\frac{x}{a} - \frac{1}{3}\left(\frac{x}{a}\right)^3\right]$	S. The particle experiences an attractive force towards $x = 0$ in the region $\lvert x\rvert < a$
	T. The particle with total energy $\frac{U_0}{4}$ can oscillate about the point $x = -a$.

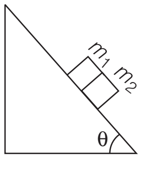
21. Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in figure. The speed of A is $100\sqrt{3}\ \text{ms}^{-1}$. At time $t = 0$ s, an observer in A finds B at a distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at $t = t_0$, A just escapes being hit by B, t_0 in seconds is **(Single Integer Type, 2014)**

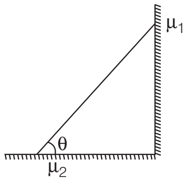


22. A rocket is moving in a gravity free space with a constant acceleration of $2\ \text{ms}^{-2}$ along $+x$ direction (see figure). The length of a chamber inside the rocket is 4 m. A ball is thrown from the left end of the chamber in $+x$ direction with a speed of $0.3\ \text{ms}^{-1}$ relative to the rocket. At the same time, another ball is thrown in $-x$ direction with a speed of $0.2\ \text{ms}^{-1}$ from its right end relative to the rocket. The time in seconds when the two balls hit each other is **(Single Integer Type, 2014)**



23. A block of mass $m_1 = 1$ kg another mass $m_2 = 2$ kg are placed together (see figure) on an inclined plane with angle of inclination θ. Various values of θ are given in Column I. The coefficient of friction between the block m_1 and the plane is always zero. The coefficient of static and dynamic friction between the block m_2 and the plane are equal to $\mu = 0.3$. In Column II expressions for the friction on the block m_2 are given. Match the correct expression of the friction in Column II with the angles given in Column I, and choose the correct option. The acceleration due to gravity is denoted by g.





[Useful information $\tan(5.5°) \approx 0.1$; $\tan(11.5°) \approx 0.2$; $\tan(16.5°) \approx 0.3$]

(Matching Type, 2014)

Column I	Column II
P. $\theta = 5°$	1. $m_2 g \sin\theta$
Q. $\theta = 10°$	2. $(m_1 + m_2) g \sin\theta$
R. $\theta = 15°$	3. $\mu m_2 g \cos\theta$
S. $\theta = 20°$	4. $\mu (m_1 + m_2) g \cos\theta$

Codes

	P	Q	R	S		P	Q	R	S
(a)	1,	1,	1,	3	(b)	2,	2,	2,	3
(c)	2,	2,	2,	4	(d)	2,	2,	3,	3

24. In the figure, a ladder of mass m is shown leaning against a wall. It is in static equilibrium making an angle θ with the horizontal floor. The coefficient of friction between the wall and the ladder is μ_1 and that between the floor and the ladder is μ_2. The normal reaction of the wall on the ladder is N_1 and that of the floor is N_2. If the ladder is about to slip, then

(More than One Correct Option, 2014)

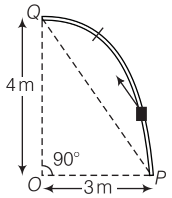
(a) $\mu_1 = 0, \mu_2 \neq 0$ and $N_2 \tan\theta = \frac{mg}{2}$

(b) $\mu_1 \neq 0, \mu_2 = 0$ and $N_1 \tan\theta = \frac{mg}{2}$

(c) $\mu_1 \neq 0, \mu_2 \neq 0$ and $N_2 = \frac{mg}{1+\mu_1\mu_2}$

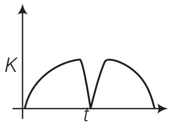
(d) $\mu_1 = 0, \mu_2 \neq 0$ and $N_1 \tan\theta = \frac{mg}{2}$

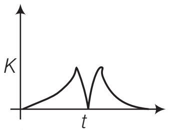
25. Consider an elliptically shaped rail PQ in the vertical plane with $OP = 3$ m and $OQ = 4$ m. A block of mass 1 kg is pulled along the rail from P to Q with a force of 18 N, which is always parallel to line PQ (see figure). Assuming no frictional losses, the kinetic energy of the block when it reaches Q is $(n \times 10)$ J. The value of n is (take acceleration due to gravity = 10 ms^{-2}) **(Single Integer Type, 2014)**

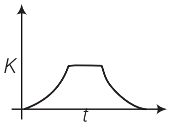


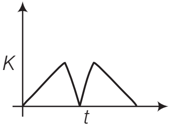
26. A tennis ball is dropped on a horizontal smooth surface. It bounces back to its original position after hitting the surface. The force on the ball during the collision is proportional to the length of compression of the ball. Which one of the following sketches describes the variation of its kinetic energy K with time t most appropriately? The figures are only illustrative and not to the scale.

(Single Correct Option, 2014)

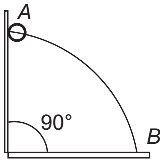
(a)


(b)


(c)


(d)


27. A wire, which passes through the hole in a small bead, is bent in the form of quarter of a circle. The wire is fixed vertically on ground as shown in the figure. The bead is released from near the top of the wire and it slides along the wire without friction. As the bead moves from A to B, the force it applies on the wire is



(Single Correct Option, 2014)

(a) always radially outwards
(b) always radially inwards
(c) radially outwards initially and radially inwards later
(d) radially inwards initially and radially outwards later

28. Using the expression $2d \sin\theta = \lambda$, one calculates the values of d by measuring the corresponding angles θ in the range 0 to 90°. The wavelength λ is exactly known and the error in θ is constant for all values of θ. As θ increases from 0°, then

(Single Correct Option, 2013)

(a) the absolute error in d remains constant
(b) the absolute error in d increases
(c) the fractional error in d remains constant
(d) the fractional error in d decreases

29. Match the Column I with Column II and select the correct answer using the codes given below the column. **(Matching Type, 2013)**

	Column I		Column II
P.	Boltzmann constant	1.	$[ML^2T^{-1}]$
Q.	Coefficient of viscosity	2.	$[ML^{-1}T^{-1}]$
R.	Planck constant	3.	$[MLT^{-3}K^{-1}]$
S.	Thermal conductivity	4.	$[ML^2T^{-2}K^{-1}]$

Codes

	P	Q	R	S		P	Q	R	S
(a)	3	1	2	4	(b)	3	2	1	4
(c)	4	2	1	3	(d)	4	1	2	3

30. The diameter of a cylinder is measured using a vernier callipers with no zero error. It is found that the zero of the vernier scale lies between 5.10 cm and 5.15 cm of the main scale. The vernier scale has 50 divisions equivalent to 2.45 cm. The 24th division of the vernier scale exactly coincides with one of the main scale divisions. The diameter of the cylinder is **(Single Correct Option, 2013)**

(a) 5.112 cm (b) 5.124 cm
(c) 5.136 cm (d) 5.148 cm

31. The work done on a particle of mass m by a force, $K\left[\dfrac{x}{(x^2+y^2)^{3/2}}\,\hat{\mathbf{i}}+\dfrac{y}{(x^2+y^2)^{3/2}}\,\hat{\mathbf{j}}\right]$ (K being a constant of appropriate dimensions), when the particle is taken from the point $(a, 0)$ to the point $(0, a)$ along a circular path of radius a about the origin in the x-y plane is **(Single Correct Option 2013)**

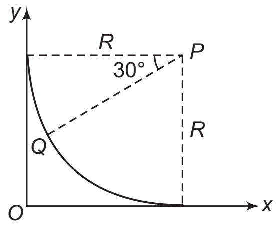
(a) $\dfrac{2K\pi}{a}$ (b) $\dfrac{K\pi}{a}$ (c) $\dfrac{K\pi}{2a}$ (d) 0

32. A particle of mass 0.2 kg is moving in one dimension under a force that delivers a constant power 0.5 W to the particle. If the initial speed (in ms^{-1}) of the particle is zero, the speed (in ms^{-1}) after 5 s is **(Single Integer Type, 2013)**

Passage (Q. Nos. 33 & 34)

A small block of mass 1 kg is released from rest at the top of a rough track. The track is a circular arc of radius 40 m. The block slides along the track without toppling and a frictional force acts on it in the direction opposite to the instantaneous velocity. The work done in overcoming the friction up to the point Q, as shown in the figure, is 150 J.

(Take the acceleration due to gravity, $g = 10\ \text{ms}^{-2}$) **(Passage Type, 2013)**



33. The speed of the block when it reaches the point Q, is

(a) $5\ \text{ms}^{-1}$ (b) $10\ \text{ms}^{-1}$
(c) $10\sqrt{3}\ \text{ms}^{-1}$ (d) $20\ \text{ms}^{-1}$

34. The magnitude of the normal reaction that acts on the block at the point Q is

(a) 7.5 N (b) 8.6 N (c) 11.5 N (d) 22.5 N

Answer *with* Explanations

1. (2.0) $\mathbf{A} = a\hat{\mathbf{i}}$ and $\mathbf{B} = a\cos\omega\hat{\mathbf{i}} + a\sin\omega t\hat{\mathbf{j}}$

$$\mathbf{A}+\mathbf{B} = (a + a\cos\omega t)\hat{\mathbf{i}} + a\sin\omega t\ \hat{\mathbf{j}}$$

$$\mathbf{A}-\mathbf{B} = (a - a\cos\omega t)\hat{\mathbf{i}} + a\sin\omega t\ \hat{\mathbf{j}}$$

$$|\mathbf{A}+\mathbf{B}| = \sqrt{3}\,|\mathbf{A}-\mathbf{B}|$$

$$\frac{\sqrt{(a + a\cos\omega t)^2 + (a\sin\omega t)^2}}{\sqrt{(a - a\cos\omega t)^2 + (a\sin\omega t)^2}} = \sqrt{3}$$

$$\Rightarrow \qquad 2\cos\frac{\omega t}{2} = \pm\sqrt{3}\times 2\sin\frac{\omega t}{2}$$

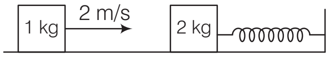
$$\tan\frac{\omega t}{2} = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \frac{\omega t}{2} = n\pi \pm \frac{\pi}{6}$$

$$\frac{\pi}{12}t = n\pi \pm \frac{\pi}{6}$$

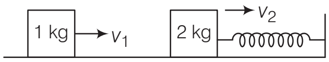
$$t = (12n \pm 2)\text{s}$$

$$= 2\text{s}, 10\text{ s}, 14\text{ s and so on.}$$

2. (2.09 m) **Just Before Collision,**



Just After Collision



Let velocities of 1 kg and 2 kg blocks just after collision be v_1 and v_2 respectively.

From momentum conservation principle,

$$1 \times 2 = 1v_1 + 2v_2 \qquad \ldots\text{(i)}$$

Collision is elastic. Hence $e = 1$ or relative velocity of separation = relation velocity of approach.

$$v_2 - v_1 = 2 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$v_2 = \frac{4}{3}\text{ m/s},\ v_1 = \frac{-2}{3}\text{ m/s}$$

2 kg block will perform SHM after collision,

$$t = \frac{T}{2} = \pi\sqrt{\frac{m}{k}} = 3.14\text{ s}$$

$$\text{Distance} = |v_1|t = \frac{2}{3} \times 3.14 = 2.093 = 2.09\text{ m}$$

3. **(c)** In terms of dimension, $F_e = F_m$

$$\Rightarrow \qquad qE = qvB \quad \text{or} \quad E = vB$$

$$[\text{E}] = [\text{B}][\text{LT}^{-1}]$$

4. **(d)** $c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$

$$c^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\mu_0 = \varepsilon_0^{-1} \cdot c^{-2}$$

$$[\mu_0] = [\varepsilon_0]^{-1}[\text{L}^{-2}\ \text{T}^2]$$

5. **(b)** $r = \dfrac{1-a}{1+a}$

$\ln\ r = \ln(1-a) - \ln(1+a)$

Differentiating, we get

$$\frac{dr}{r} = -\frac{da}{1-a} - \frac{da}{1+a}$$

or, we can write

$$\frac{\Delta r}{r} = -\left[\frac{\Delta a}{1-a} + \frac{\Delta a}{1+a}\right]$$

$$\frac{\Delta r}{r} = \frac{-2\,\Delta a}{1-a^2}$$

$$\text{or} \qquad \Delta r = -\left(\frac{2\Delta a}{1-a^2}\right)(r) = \frac{-2\,\Delta a}{(1+a)^2}$$

6. **(c)** $N = N_0 e^{-\lambda t}$

$$\ln N = \ln N_0 - \lambda t$$

Differentiating w.r.t λ , we get

$$\frac{1}{N} \cdot \frac{dN}{d\lambda} = 0 - t$$

$$\Rightarrow \qquad |d\lambda| = \frac{dN}{Nt} = \frac{40}{2000 \times 1} = 0.02$$

7. **(a, b)** $K = \dfrac{1}{2}mv^2 \Rightarrow \dfrac{dK}{dt} = mv\dfrac{dv}{dt}$

Given, $\dfrac{dK}{dt} = \gamma t \quad \Rightarrow \quad mv\dfrac{dv}{dt} = \gamma t$

$$\Rightarrow \qquad \int_0^v v\,dv = \int_0^t \frac{\gamma}{m} t\,dt \quad \Rightarrow \quad \frac{v^2}{2} = \frac{\gamma}{m}\frac{t^2}{2}$$

$$\Rightarrow \qquad v = \sqrt{\frac{\gamma}{m}}\,t \quad \Rightarrow \quad a = \frac{dv}{dt} = \sqrt{\frac{\gamma}{m}}$$

$$\therefore \qquad F = ma = \sqrt{\gamma m} = \text{constant}$$

$$\therefore \qquad V = \frac{ds}{dt} = \sqrt{\frac{\gamma}{m}}\,t \quad \Rightarrow \quad s = \sqrt{\frac{\gamma}{m}}\,\frac{t^2}{2}$$

NOTE *Force is constant. In the website of IIT, option (d) is given correct. In the opinion of author all constant forces are not necessarily conservative. For example : viscous force at terminal velocity is a constant force but it is not conservative.*

8. **(6.30)** Linear impulse, $J = mv_0$

$$\therefore \qquad v_0 = \frac{J}{m} = 2.5\text{ m/s}$$

$$\therefore \qquad v = v_0 e^{-t/\tau}$$

$$\frac{dx}{dt} = v_0 e^{-t/\tau}$$

$$\int_0^x dx = v_0 \int_0^\tau e^{-t/\tau} dt$$

$$x = v_0\left[\frac{e^{-t/\tau}}{-1/\tau}\right]_0^\tau$$

$$x = 2.5\,(-4)\,(e^{-1} - e^0)$$

$$= 2.5(-4)\,(0.37 - 1)$$

$$x = 6.30\text{ m}$$

9. **(30)** $\because H = \dfrac{u^2\sin^2 45°}{2g} = 120\text{ m}$

$$\Rightarrow \qquad \frac{u^2}{4g} = 120\text{ m}$$

If speed is v after the first collision, then speed should remain $\dfrac{1}{\sqrt{2}}$ times, as kinetic energy has reduced to half.

$$\Rightarrow \qquad v = \frac{u}{\sqrt{2}}$$

$$\therefore \qquad h_{\text{max}} = \frac{v^2\sin^2 30°}{2g}$$

$$= \frac{(u/\sqrt{2})^2\sin^2 30°}{2g}$$

$$= \left(\frac{u^2/4g}{4}\right) = \frac{120}{4} = 30\text{ m}$$

10. **(a)** ∵ $t = \sqrt{\frac{L}{5}}$

$$dt = \frac{1}{\sqrt{5}}\,\frac{1}{2}L^{-1/2}dL + \left(\frac{1}{300}dL\right)$$

$$dt = \frac{1}{2\sqrt{5}}\,\frac{1}{\sqrt{20}}\,dL + \frac{dL}{300} = 0.01$$

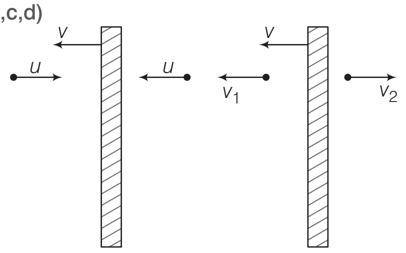
$$dL\left(\frac{1}{20} + \frac{1}{300}\right) = 0.01$$

$$dL\left[\frac{15}{300}\right] = 0.01,\; dL = \frac{3}{16}$$

$$\frac{dL}{L} \times 100 = \frac{3}{16} \times \frac{1}{20} \times 100 = \frac{15}{16} \simeq 1\%$$

11. **(c)** $\mathbf{S} = \mathbf{P} + b\mathbf{R} = \mathbf{P} + b(\mathbf{Q} - \mathbf{P}) = \mathbf{P}(1-b) + b\mathbf{Q}$

12. **(a,c,d)**



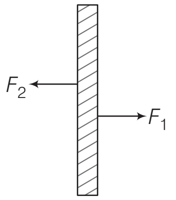
Just before the collision Just after the collision

$$v_1 = u + 2v \;\Rightarrow\; \Delta v_1 = (2u + 2v)$$

$$\Rightarrow \quad F_1 = \frac{dp_1}{dt} = pA(u+v)(2u+2v) = 2pA(u+v)^2$$

$$v_2 = (u + 2v) \;\Rightarrow\; \Delta v_2 = (2u - 2v)$$

$$F_2 = \frac{dp_2}{dt} = pA(u-v)(2u-2v) = 2pA(u-v)^2$$



[ΔF is the net force due to the air molecules on the plate]

$$\Delta F = 2pA(4uv) = 8pAuv$$

$$P = \frac{\Delta F}{A} = 8p(uv)$$

$F_{\text{net}} = (F - \Delta F) = ma$ [m is mass of the plate]

$$F - (8pAu)v = ma$$

13. **(b, d)** $[n] = [\mathrm{L}^{-3}]$, $[q] = [\mathrm{AT}]$

$$[\varepsilon] = [\mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{A}^2\mathrm{T}^4]$$

$$[T] = [\mathrm{L}]$$

$$[l] = [\mathrm{L}]$$

$$[k_B] = [\mathrm{M}^1\mathrm{L}^2\mathrm{T}^{-2}\mathrm{K}^{-1}]$$

(a) RHS

$$= \sqrt{\frac{[\mathrm{L}^{-3}\mathrm{A}^2\mathrm{T}^2]}{[\mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^4\mathrm{A}^2]\,[\mathrm{ML}^2\mathrm{T}^{-2}\mathrm{K}^{-1}]\,[\mathrm{K}]}}$$

$$= \sqrt{\frac{[\mathrm{L}^{-3}\mathrm{A}^2\mathrm{T}^2]}{[\mathrm{L}^{-1}\mathrm{T}^2\mathrm{A}^2]}} = \sqrt{[\mathrm{L}^{-2}]} = [\mathrm{L}^{-1}]$$

Wrong

(b) RHS

$$= \sqrt{\frac{[\mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^4\mathrm{A}^2]\,[\mathrm{ML}^2\mathrm{T}^{-2}\mathrm{K}^{-1}]\,[\mathrm{K}]}{[\mathrm{L}^{-3}]\,[\mathrm{A}^2\mathrm{T}^2]}}$$

$$= \sqrt{\frac{[\mathrm{L}^{-1}\mathrm{T}^2\mathrm{A}^2]}{[\mathrm{L}^{-3}\mathrm{T}^2\mathrm{A}^2]}} = [\mathrm{L}] \text{ Correct}$$

$$\text{(c) RHS} = \sqrt{\frac{[\mathrm{A}^2\mathrm{T}^2]}{[\mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^4\mathrm{A}^2]\,[\mathrm{L}^{-2}]\,[\mathrm{ML}^2\mathrm{T}^{-2}\mathrm{K}^{-1}]\,[\mathrm{K}]}}$$

$$= \sqrt{[\mathrm{L}^3]} \text{ Wrong}$$

$$\text{(d) RHS} = \sqrt{\frac{[\mathrm{A}^2\mathrm{T}^2]}{[\mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^4\mathrm{A}^2]\,[\mathrm{L}^{-1}]\,[\mathrm{M}^1\mathrm{L}^2\mathrm{T}^{-2}\mathrm{K}^{-1}]}}$$

$$= \sqrt{\frac{[\mathrm{A}^2\mathrm{T}^2]}{[\mathrm{L}^{-2}\mathrm{T}^2\mathrm{A}^2]}} = [\mathrm{L}] \text{ Correct}$$

14. **(a,b,d)** Mean time period

$$= \frac{0.52 + 0.56 + 0.57 + 0.54 + 0.59}{5}$$

= 0.556 ≅ 0.56 sec as per significant figures

Error in reading $= |T_{\text{mean}} - T_1| = 0.04$

$$|T_{\text{mean}} - T_2| = 0.00$$

$$|T_{\text{mean}} - T_3| = 0.01$$

$$|T_{\text{mean}} - T_4| = 0.02$$

$$|T_{\text{mean}} - T_5| = 0.03$$

Mean error = 0.1/5 = 0.02 % error in

$$T = \frac{\Delta T}{T} \times 100 = \frac{0.02}{0.56} \times 100 = 3.57\%$$

% error in $r = \frac{1}{10} \times 100 = 10\%$

% error in $R = \frac{1}{60} \times 100 = 1.67\%$

% error in

$$\frac{\Delta g}{g} \times 100 = \frac{\Delta(R) + \Delta(r)}{R - r} \times 100 + 2 \times \frac{\Delta T}{T}$$

$$= \frac{2}{50} \times 100 + 2 \times 3.57$$

$$= 4\% + 7\% = 11\%$$

15. **(b)** For vernier C_1

10 VSD = 9 MSD = 9 mm

1 VSD = 0.9 mm

⇒ LC = 1 MSD − 1 VSD

= 1 mm − 0.9 mm = 0.1 mm

Reading of C_1 = MSR + (VSR) (LC)

$= 28\text{ mm} + (7)(0.1)$

Reading of $C_1 = 28.7\text{ mm} = 2.87\text{ cm}$

For vernier C_2 : the vernier C_2 is abnormal

So, we have to find the reading form basics.

The point where both of the marks are matching :

distance measured from main scale = distance measured from vernier scale

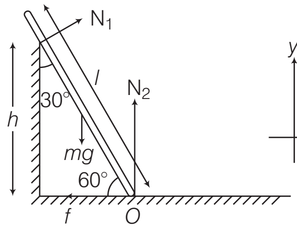
$28\text{ mm} + (1\text{ mm})(8) = (28\text{ mm} + x) + (1.1\text{ mm})(7)$

Solving we get, $x = 0.3\text{ mm}$

So, reading of $C_2 = 28\text{ mm} + 0.3\text{ mm}$

$= 2.83\text{ cm}$

16. (d)



$$\Sigma F_x = 0$$

$$N_1 \cos 30° - f = 0 \quad \ldots\text{(i)}$$

$$\sum F_y = 0$$

$$N_1 \sin 30° + N_2 - mg = 0 \quad \ldots\text{(ii)}$$

$$\sum \tau_0 = 0$$

$$mg\frac{l}{2}\cos 60° - N_1 \frac{h}{\cos 30°} = 0 \quad \ldots\text{(iii)}$$

Also, given $N_1 = N_2$ ...(iv)

Solving Eqs. (i), (ii), (iii) and (iv) we have

$$\frac{h}{l} = \frac{3\sqrt{3}}{16}$$

and $f = \dfrac{16\sqrt{3}}{3}$

17. (4) $E(t) = A^2 e^{-\alpha t}$ ...(i)

$$\alpha = 0.2\text{ s}^{-1}$$

$$\left(\frac{dA}{A}\right) \times 100 = 1.25\%$$

$$\left(\frac{dt}{t}\right) \times 100 = 1.50$$

$\Rightarrow$ $(dt \times 100) = 1.5t = 1.5 \times 5 = 7.5$

$$\therefore \left(\frac{dE}{E}\right) \times 100 = \pm 2\left(\frac{dA}{A}\right) \times 100 \pm \alpha\,(dt \times 100)$$

$$= \pm 2\,(1.25) \pm 0.2\,(7.5)$$

$$= \pm 2.5 \pm 1.5 = \pm 4\%$$

18. (c, d) $M \propto h^a c^b G^c$

$$M \propto (ML^2T^{-1})^a (LT^{-1})^b (M^{-1}L^3T^{-2})^c$$

$$\propto M^{a-c} L^{2a+b+3c} T^{-a-b-2c}$$

$$a - c = 1 \quad \ldots\text{(i)}$$

$$2a + b + 3c = 0 \quad \ldots\text{(ii)}$$

$$a + b + 2c = 0 \quad \ldots\text{(iii)}$$

On solving (i), (ii), (iii),

$$a = \frac{1}{2},\ b = +\frac{1}{2},\ c = -\frac{1}{2}$$

$\therefore M \propto \sqrt{c}$ only → (a) is correct.

In the same way we can find that, $L \propto h^{1/2} c^{-3/2} G^{1/2}$

$L \propto \sqrt{h}$, $L \propto \sqrt{G}$ → (c), (d) are also correct.

19. (b,c) **For vernier callipers**

$$1\text{ MSD} = \frac{1}{8}\text{ cm}$$

$$5\text{ VSD} = 4\text{ MSD}$$

$$\therefore\ 1\text{ VSD} = \frac{4}{5}\text{ MSD} = \frac{4}{5} \times \frac{1}{8} = \frac{1}{10}\text{ cm}$$

Least count of vernier callipers = 1 MSD − 1 VSD

$$= \frac{1}{8}\text{ cm} - \frac{1}{10}\text{ cm} = 0.025\text{ cm}$$

Fore screw gauge

Pitch of screw gauge $= 2 \times 0.025 = 0.05$ cm

Least count of screw gauge $= \dfrac{0.05}{100}$ cm

$= 0.005$ mm

Least count of linear scale of screw gauge = 0.05 mm

Pitch $= 0.05 \times 2 = 0.1$ cm

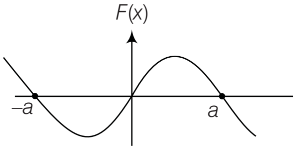
Least count of screw gauge $= \dfrac{0.1}{100}$ cm $= 0.01$ mm

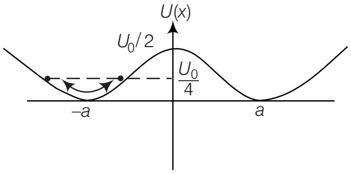
20. A → P,Q,R,T, B → Q,S, C → P,Q,R,S, D → P,R,T

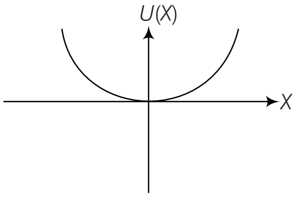
(A) $F_x = \dfrac{-dU}{dx} = -\dfrac{2U_0}{a^3}[x-a][x][x+a]$

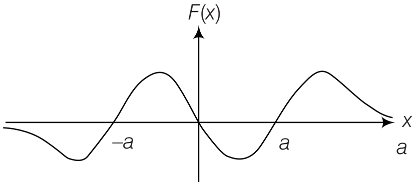
$F = 0$ at $x = 0$, $x = a$, $x = -a$

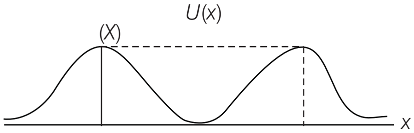
and $U = 0$ at $x = -a$ and $x = a$



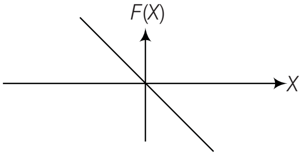




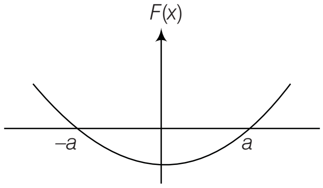


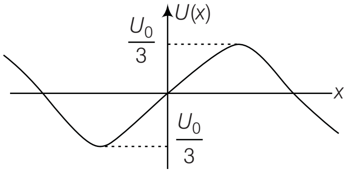


(B) $F_x = -\frac{dU}{dx} - U_0\left(\frac{x}{a}\right)$



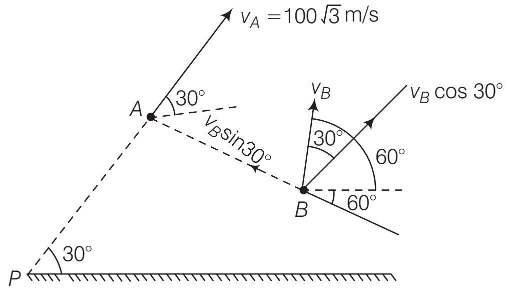
(C) $F_x = -\frac{dU}{dx} = U_0 \frac{e^{-x^2/x^2}}{a^3}[x]\,[x-a]\,[x+a]$





(D) $F_x = -\frac{dU}{dx} = -\frac{U_0}{2a^3}[(x-a)(x+a)]$

21. **(5)** Relative velocity of *B* with respect to *A* is perpendicular to line *PA*. Therefore, parallel to *PA*, velocity components of *A* and *B* should be same.



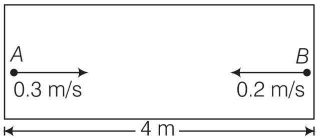
$$\Rightarrow \quad v_A = 100\sqrt{3} = v_B \cos 30°$$

$$\Rightarrow \quad v_B = 200 \text{ m/s}$$

As *A* and *B* just hit

$$t_0 = \frac{500}{v_B \sin 30°} = \frac{500}{200 \times 1/2} = 5 \text{ s}$$

22. **(2)**



Relative to rocket neither of ball has any all elevation. So, if balls meet in time *t* then,

$$0.2t + 0.3t = 4$$

$$\Rightarrow \quad 0.5t = 4$$

$$\Rightarrow \quad t = 8 \text{ s}$$

23. **(d)** Block will not slip if

$$(m_1 + m_2)\, g \sin\theta \le \mu m_2 g \cos\theta$$

$$\Rightarrow \quad 3\sin\theta \le \left(\frac{3}{10}\right)(2)\cos\theta$$

$$\tan\theta \le 1/5$$

$$\Rightarrow \quad \theta \le 11.5°$$

(P) $\theta = 5°$ friction is static

$$f = (m_1 + m_2)\, g \sin\theta$$

(Q) $\theta = 10°$ friction is static

$$f = (m_1 + m_2)\, g \sin\theta$$

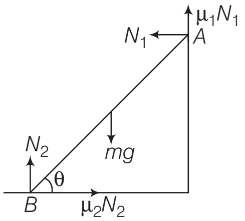
(R) $\theta = 15°$ friction is kinetic

$$f = \mu m_2 g \cos\theta$$

(S) $\theta = 20°$ friction is kinetic

$$\Rightarrow \quad f = \mu m_2 g \cos\theta$$

24. **(c, d)** μ_2 can never be zero for equilibrium.



When $\mu_1 = 0$, we have

$$N_1 = \mu_2 N_2 \qquad \ldots\text{(i)}$$

$$N_2 = mg \qquad \ldots\text{(ii)}$$

$$\tau_B = 0 \Rightarrow mg\frac{L}{2}\cos\theta = N_1 L\sin\theta$$

$$\Rightarrow \qquad N_1 = \frac{mg\cot\theta}{2}$$

$$\Rightarrow \qquad N_1\tan\theta = \frac{mg}{2}$$

When, $\mu_1 \neq 0$ we have

$$\mu_1 N_1 + N_2 = mg \qquad \ldots\text{(iii)}$$

$$\mu_2 N_2 = N_1 \qquad \ldots\text{(iv)}$$

$$\Rightarrow \qquad N_2 = \frac{mg}{1+\mu_1\mu_2}$$

25. **(5)** From work-energy theorem,

Work done by all forces = change in kinetic energy

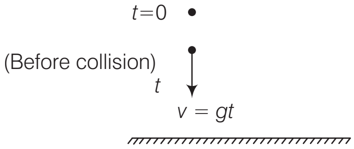
or $\qquad W_F + W_{mg} = K_f - K_i$

$$18 \times 5 + (1 \times 10)(-4) = K_f$$

$$90 - 40 = K_f \text{ or } K_f = 50$$

$$\text{J} = 5 \times 10 \text{ J}$$

26. **(b)**



$$K = \frac{1}{2}mg^2t^2$$

$K \propto t^2$ Therefore, K-t graph is parabola.

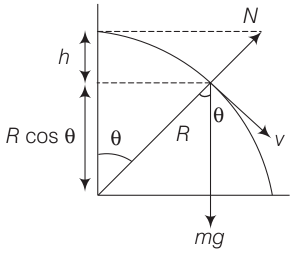
During collision retarding force is just like the spring force ($F \propto x$), therefore kinetic energy first decreases to elastic potential energy and then increases.

Most appropriate graph is therefore (b).

27. **(d)** $h = R - R\cos\theta$

Using conservation of energy,

$$mgR(1-\cos\theta) = \frac{1}{2}mv^2$$



Radial force equation is

$$mg\cos\theta - N = \frac{mv^2}{R}$$

Here, N = normal force on bead by wire

$$N = mg\cos\theta - \frac{mv^2}{R} = mg(3\cos\theta - 2)$$

$$N = 0 \text{ at } \cos\theta = \frac{2}{3}$$

$\Rightarrow$ Normal force act radially outward on bead, if $\cos\theta > \frac{2}{3}$ and normal force act radially inward on bead, if $\cos\theta < 2/3$

$\therefore$ Force on ring is opposite to normal force on bead.

28. **(d)** $2d\sin\theta = \lambda \Rightarrow d = \frac{\lambda}{2\sin\theta}$

Differentiate $\partial(d) = \frac{\lambda}{2}\partial(\text{cosec}\theta)$

$$\partial(d) = \frac{\lambda}{2}(-\text{cosec}\,\theta\cot\theta)\,\partial\theta$$

$$\partial(d) = \frac{-\lambda\cos\theta}{2\sin^2\theta}.\partial\theta$$

So, $\qquad \frac{\Delta d}{d} = -\frac{\cos\theta}{\sin\theta}, \Delta\theta$

As θ increases, $\frac{\cos\theta}{\sin\theta}$, decreases

$\Rightarrow \qquad \left|\frac{\Delta d}{d}\right|$ decreases

Alternate Solution

$$d = \frac{\lambda}{2\sin\theta}$$

$$\ln d = \ln\lambda - \ln 2 - \ln\sin\theta$$

$$\frac{\Delta(d)}{d} = 0 - 0 - \frac{1}{\sin\theta} \times \cos\theta\,(\Delta\theta)$$

Fractional error, $| + (d)| = \cot\theta\,\Delta\theta|$

Absolute error, $\Delta d = (d\cot\theta)\,\Delta\theta = \frac{\lambda}{2\sin\theta} \times \frac{\cos\theta}{\sin\theta}\cdot\Delta\theta$

$$\Delta d = \frac{\cos\theta}{\sin^2\theta}\cdot\lambda\cdot\Delta\theta$$

29. **(c)** (P) $U = \frac{1}{2} kT$

$\Rightarrow$ $[ML^2T^{-2}] = [k]\,K$

$\Rightarrow$ $[K] = [ML^2T^{-2}K^{-1}]$

(Q) $F = \eta A \frac{dv}{dx}$

$\Rightarrow$ $[\eta] = \frac{[MLT^{-2}]}{[L^2LT^{-1}L^{-1}]} = [ML^{-1}\ T^{-1}]$

(R) $E = h\nu$

$\Rightarrow$ $[ML^2T^2] = [h]\,[T^{-1}]$

$\Rightarrow$ $[h] = [ML^2T^{-1}]$

(S) $\frac{dQ}{dt} = \frac{k\,A\Delta\theta}{l}$

$\Rightarrow$ $[k] = \frac{[ML^2T^{-3}L]}{[L^2K]} = [MLT^{-3}K^{-1}]$

30. **(b)** 1 MSD = 5.15 cm − 5.10 cm = 0.05 cm

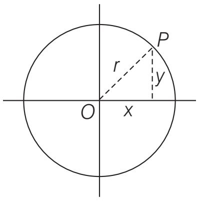
$1\text{ VSD} = \frac{2.45\text{ cm}}{50} = 0.049\text{ cm}$

$\therefore$ LC = 1 MSD − 1 VSD = 0.01 cm

Hence, diameter of cylinder = (Main scale reading) + (Vernier scale reading) (LC)

= 5.10 + (24) (0.001) = 5.124 cm

31. **(d)**



$$\mathbf{r} = \mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

$$\mathbf{F} = \frac{k}{(x^2 + y^2)^{3/2}}$$

$$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = \frac{k}{r^3}(\mathbf{r})$$

Since, **F** is along **r** or in radial direction. Therefore, work done is zero.

32. **(5)** $W = \frac{1}{2} mv^2 \Rightarrow Pt = \frac{1}{2} mv^2$

$\therefore$ $v = \sqrt{\frac{2Pt}{m}} = \sqrt{\frac{2 \times 0.5 \times 5}{0.2}} = 5\text{ m/s}$

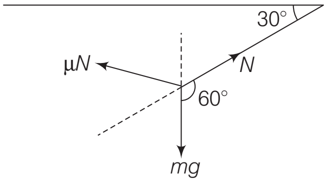
33. **(b)** Height fallen up to Q is $R\sin 30° = \frac{R}{2} = 20\text{ m}$

$E_i - E_f$ = work done against friction

$\therefore$ $0 - \left[mgh + \frac{1}{2}mv^2 \right] = 150$

$\therefore$ $(1)\,(10)\,(20) - \frac{1}{2} \times 1 \times v^2 = 150$ or $v = 10\text{ m/s}$

34. **(a)** $N = mg\cos 60° = \frac{mv^4}{R}$



$\therefore$ $N = \frac{mv^2}{R} + \frac{mg}{2}$

$= \frac{(1)(10)^2}{40} + \frac{(1)(10)}{2} = 7.5\text{ N}$

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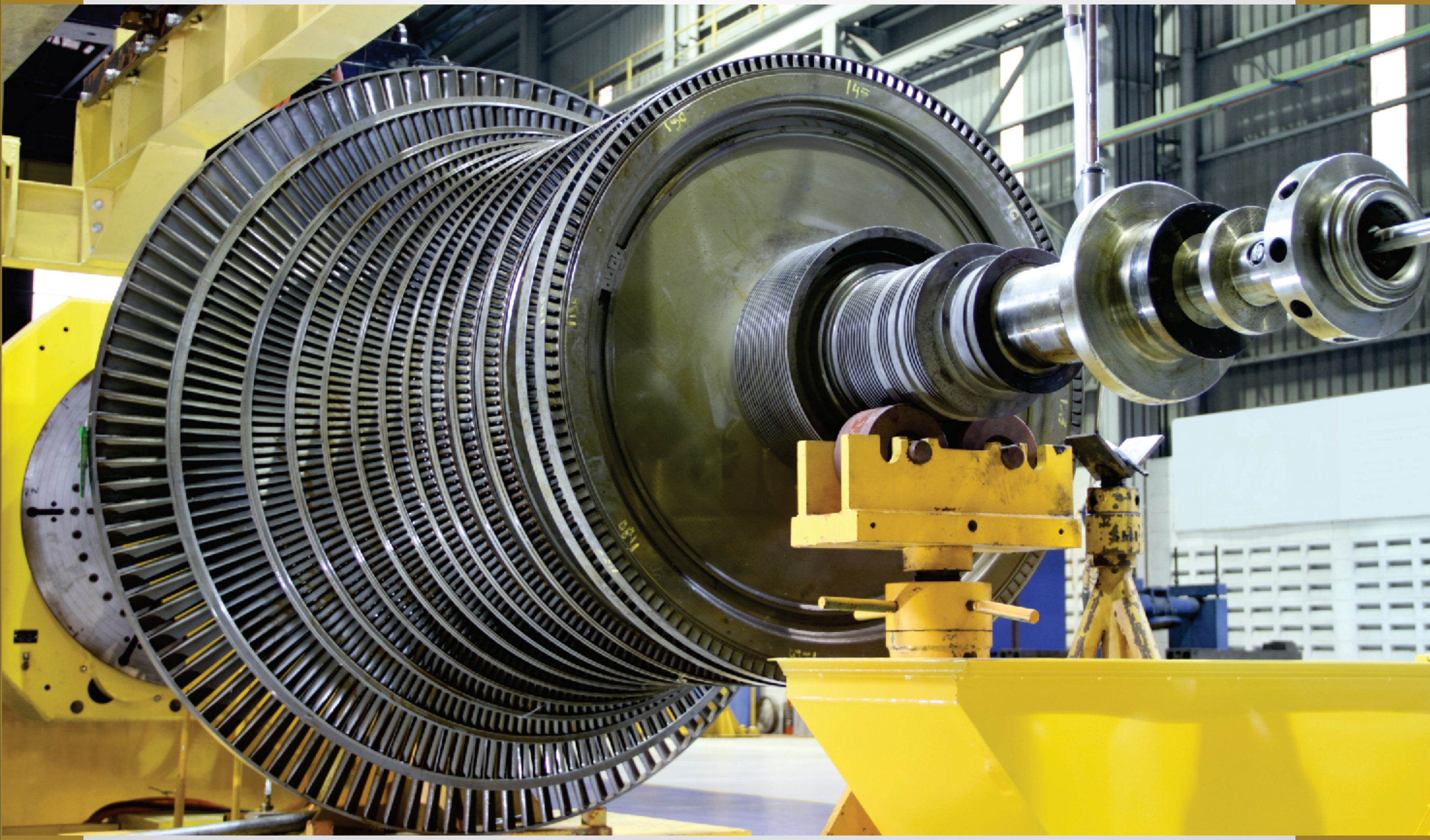

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced.
The NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions.

I am very thankful to (Dr.) Mrs. Sarita Pandey, Mr. Anoop Dhyani and Mr. Nisar Ahmad

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

Centre of Mass

Centre of mass of a two particle system, Centre of mass of a rigid body.

Rotational Motion

Basic concepts of rotational motion; Moment of a force, Torque, Angular momentum, conservation of angular momentum and its applications; Moment of inertia, Radius of gyration. Values of moments of inertia for simple geometrical objects, Parallel and perpendicular axes theorems and their applications. Rigid body rotation, Equations of rotational motion.

Gravitation

The universal law of gravitation. Acceleration due to gravity and its variation with altitude and depth. Kepler's laws of planetary motion. Gravitational potential energy; Gravitational potential, Escape velocity. Orbital, velocity of a satellite, Geo-stationary satellites.

Properties of Solids and Liquids

Elastic behaviour, Stress-strain relationship, Hooke's Law, Young's modulus, Bulk modulus, modulus of rigidity. Pressure due to a fluid column; Pascal's law and its applications. Viscosity, Stokes' law, Terminal velocity, Streamline and turbulent flow, Reynolds number. Bernoulli's principle and its applications. Surface energy and surface tension, Angle of contact, Application of surface tension – drops, bubbles and capillary rise.

Oscillations

Periodic motion – period, frequency, displacement as a function of time. Periodic functions. Simple harmonic motion (SHM) and its equation; Phase; Oscillations of a spring -restoring force and force constant; energy in SHM – Kinetic and potential energies; Simple pendulum – derivation of expression for its time period; Free, forced and damped oscillations, resonance.

JEE Advanced

General

Determination of g using simple pendulum. Young's modulus by Searle's method.

Centre of Mass and Collision

System of particles, Centre of mass and its motion, Impulse, Elastic and inelastic collisions.

Gravitation

Law of gravitation, Gravitational potential and field, Acceleration due to gravity, Motion of planets and satellites in circular orbits, Escape velocity.

Rotational Motion

Rigid body, moment of inertia, Parallel and perpendicular axes theorems, Moment of inertia of uniform bodies with simple geometrical shapes, Angular momentum, Torque, Conservation of angular momentum, Dynamics of rigid bodies with fixed axis of rotation, Rolling without slipping of rings, cylinders and spheres, Equilibrium of rigid bodies, Collision of point masses with rigid bodies.

Oscillations

Linear and angular simple harmonic motions.

Properties of Solids and Liquids

Hooke's law, Young's modulus. Pressure in a fluid, Pascal's law, Buoyancy, Surface energy and surface tension, capillary rise, Viscosity (Poiseuille's equation excluded), Stoke's law, Terminal velocity, Streamline flow, Equation of continuity, Bernoulli's theorem and its applications.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

11

Centre of Mass Linear Momentum and Collision

Chapter Contents

11.1 Centre of Mass
11.2 Motion of the Centre of Mass
11.3 Law of Conservation of Linear Momentum
11.4 Variable Mass
11.5 Linear Impulse
11.6 Collision

11.1 Centre of Mass

When we consider the motion of a system of particles, there is one point in it which behaves as though the entire mass of the system (i.e. the sum of the masses of all the individual particles) is concentrated there and its motion is the same as would ensue if the resultant of all the forces acting on all the particles were applied directly to it. This point is called the centre of mass (COM) of the system. The concept of COM is very useful in solving many problems, in particular, those concerned with collision of particles.

Position of Centre of Mass

First of all we find the position of COM of a system of particles. Just to make the subject easy we classify a system of particles in three groups:

1. System of two particles.
2. System of a large number of particles and
3. Continuous bodies.

Now, let us take them separately.

Position of COM of Two Particles

Centre of mass of two particles of mass m_1 and m_2 separated by a distance of d lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

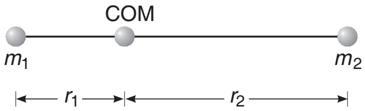


Fig. 11.1

i.e. $$r \propto \frac{1}{m}$$

or $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$m_1 r_1 = m_2 r_2$$

or $$\boxed{r_1 = \left(\frac{m_2}{m_2 + m_1}\right) d \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) d}$$

Here, r_1 = distance of COM from m_1

and r_2 = distance of COM from m_2

From the above discussion, we see that

$r_1 = r_2 = \frac{d}{2}$ if $m_1 = m_2$, i.e. COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$, i.e. COM is nearer to the particle having larger mass.

Example 11.1 *Two particles of masses 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.*

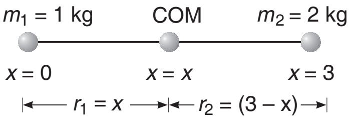


Fig. 11.2

Solution Since, both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at $x = x$, then

$r_1 =$ distance of COM from the particle of mass 1 kg $= x$

and $r_2 =$ distance of COM from the particle of mass 2 kg

$$= (3 - x)$$

Using
$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or
$$\frac{x}{3 - x} = \frac{2}{1} \quad \text{or} \quad x = 2\text{ m}$$

Thus, the COM of the two particles is located at $x = 2$ m.

Position of COM of a Large Number of Particles

If we have a system consisting of n particles, of mass $m_1, m_2, \ldots, m_n$ with $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ as their position vectors at a given instant of time. The position vector $\mathbf{r}_{\text{COM}}$ of the COM of the system at that instant is given by:

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum\limits_{i=1}^{n} m_i\mathbf{r}_i}{\sum\limits_{i=1}^{n} m_i} \quad \text{or} \quad \mathbf{r}_{\text{COM}} = \frac{\sum\limits_{i=1}^{n} m_i\mathbf{r}_i}{M}$$

Here, $M = m_1 + m_2 + \ldots + m_n$ and $\Sigma m_i\mathbf{r}_i$ is called the first moment of the mass.

Further,
$$\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

and
$$\mathbf{r}_{\text{COM}} = x_{\text{COM}}\hat{\mathbf{i}} + y_{\text{COM}}\hat{\mathbf{j}} + z_{\text{COM}}\hat{\mathbf{k}}$$

So, the cartesian co-ordinates of the COM will be

$$x_{\text{COM}} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum\limits_{i=1}^{n} m_ix_i}{\Sigma m_i} \quad \text{or} \quad x_{\text{COM}} = \frac{\sum\limits_{i=1}^{n} m_ix_i}{M}$$

Similarly,
$$y_{\text{COM}} = \frac{\sum\limits_{i=1}^{n} m_i y_i}{M}$$

and
$$z_{\text{COM}} = \frac{\sum\limits_{i=1}^{n} m_i z_i}{M}$$

Example 11.2 *The position vector of three particles of masses* $m_1 = 1\ kg$, $m_2 = 2\ kg$ *and* $m_3 = 3\ kg$ *are* $\mathbf{r}_1 = (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})\ m$, $\mathbf{r}_2 = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\ m$ *and* $\mathbf{r}_3 = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})\ m$ *respectively. Find the position vector of their centre of mass.*

Solution The position vector of COM of the three particles is given by

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}$$

Substituting the values, we get

$$\mathbf{r}_{\text{COM}} = \frac{(1)(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (2)(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + 3(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{1 + 2 + 3}$$

$$= \frac{9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{6}$$

$$\mathbf{r}_{\text{COM}} = \frac{1}{2}(3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\ \text{m}$$

Example 11.3 *Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find the position of centre of mass of the particles.*

Solution Assuming *D* as the origin, *DC* as *x*-axis and *DA* as *y*-axis, we have

$$m_1 = 1\,\text{kg}, (x_1, y_1) = (0, 1\,\text{m})$$

$$m_2 = 2\,\text{kg}, (x_2, y_2) = (1\,\text{m}, 1\,\text{m})$$

$$m_3 = 3\,\text{kg}, (x_3, y_3) = (1\,\text{m}, 0)$$

and

$$m_4 = 4\,\text{kg}, (x_4, y_4) = (0, 0)$$

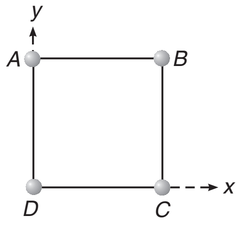


Fig. 11.3

Coordinates of their COM are

$$x_{\text{COM}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{5}{10} = \frac{1}{2}\ \text{m} = 0.5\ \text{m}$$

Similarly,

$$y_{\text{COM}} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{3}{10}\ \text{m} = 0.3\ \text{m}$$

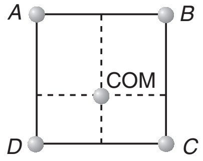


Fig. 11.4

$$\therefore \quad (x_{\text{COM}}, y_{\text{COM}}) = (0.5\ \text{m}, 0.3\ \text{m})$$

Thus, position of COM of the four particles is as shown in Fig. 11.4.

Position of COM of Continuous Bodies

If we consider the body to have continuous distribution of matter the summation in the formula of COM is replaced by integration. Suppose x, y and z are the co-ordinates of a small element of mass dm, we write the co-ordinates of COM as

$$x_{\text{COM}} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}$$

$$y_{\text{COM}} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

and

$$z_{\text{COM}} = \frac{\int z\,dm}{\int dm} = \frac{\int z\,dm}{M}$$

Here, dm is the mass of small element and (x, y, z) co-ordinates of COM of this element.

Note *In most of the cases element is a particle. In this case, COM of this particle lies over the particle itself.*

Let us take an example.

Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

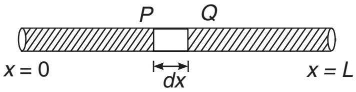


Fig. 11.5

Mass per unit length of the rod $= \dfrac{M}{L}$

Hence, the mass of the element PQ of length dx situated at $x = x$ is $dm = \dfrac{M}{L}\,dx$

The coordinates of the element PQ are $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{\text{COM}} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L (x)\left(\frac{M}{L}\,dx\right)}{M} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{\text{COM}} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $z_{\text{COM}} = 0$

i.e. the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$. Or it lies at the centre of the rod.

Proceeding in the similar manner, we can find the COM of certain rigid bodies. Centre of mass of some well known rigid bodies are given below :

1. Centre of mass of a uniform rectangular, square or circular plate lies at its centre.

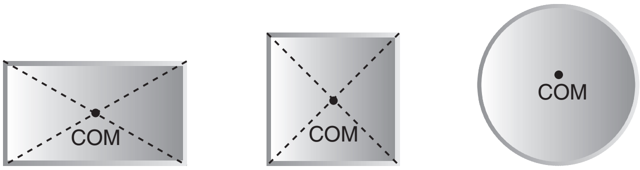


Fig. 11.6

2. Centre of mass of a uniform **semicircular ring** lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry where R is the radius of the ring.

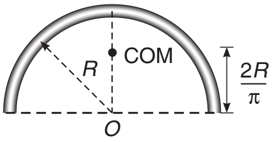


Fig. 11.7

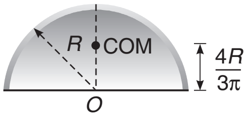


Fig. 11.8

3. Centre of mass of a uniform **semicircular disc** of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in Fig. 11.8.

4. Centre of mass of a **hemispherical shell** of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure 11.9.

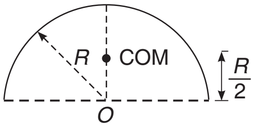


Fig. 11.9

5. Centre of mass of a **solid hemisphere** of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry.

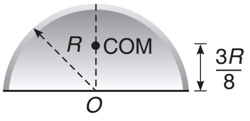


Fig. 11.10

Example 11.4 *A rod of length L is placed along the x-axis between* $x = 0$ *and* $x = L$*. The linear mass density (mass/length)* ρ *of the rod varies with the distance x from the origin as* $\rho = a + bx$*. Here, a and b are constants. Find the position of centre of mass of this rod.*

Solution Mass of element PQ of length dx situated at $x = x$ is

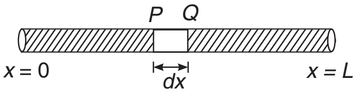


Fig. 11.11

$$dm = \rho\, dx = (a + bx)\, dx$$

The COM of the element has co-ordinates $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{\text{COM}} = \frac{\int_0^L x\, dm}{\int_0^L dm} = \frac{\int_0^L (x)(a + bx)\, dx}{\int_0^L (a + bx)\, dx}$$

$$= \frac{\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^L}{\left[ax + \frac{bx^2}{2}\right]_0^L} \quad \text{or} \quad x_{\text{COM}} = \frac{3aL + 2bL^2}{6a + 3bL}$$

The y-coordinate of COM of the rod is

$$y_{\text{COM}} = \frac{\int y\, dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $z_{\text{COM}} = 0$

Hence, the centre of mass of the rod lies at $\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$ **Ans.**

Extra Points to Remember

- For a laminar type (2-dimensional) body the formulae for finding the position of centre of mass are as under:

(i) $$\mathbf{r}_{\text{COM}} = \frac{A_1\mathbf{r}_1 + A_2\mathbf{r}_2 + \ldots + A_n\mathbf{r}_n}{A_1 + A_2 + .. + A_n}$$

(ii) $$x_{\text{COM}} = \frac{A_1x_1 + A_2x_2 + \ldots + A_nx_n}{A_1 + A_2 + \ldots + A_n}$$

$$y_{\text{COM}} = \frac{A_1y_1 + A_2y_2 + \ldots + A_ny_n}{A_1 + A_2 + \ldots + A_n}$$

and $$z_{\text{COM}} = \frac{A_1z_1 + A_2z_2 + \ldots + A_nz_n}{A_1 + A_2 + \ldots + A_n}$$

Here, A stands for the area.

- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(i) $$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 - m_2} \quad \text{or} \quad \mathbf{r}_{\text{COM}} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2}{A_1 - A_2}$$

(ii)
$$x_{COM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2} \quad \text{or} \quad x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$y_{COM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2} \quad \text{or} \quad y_{COM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

and
$$z_{COM} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2} \quad \text{or} \quad z_{COM} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$$

Here, m_1, A_1,$\mathbf{r}_1$ x_1, y_1 and z_1 are the values for the whole mass while m_2, A_2,$\mathbf{r}_2$, x_2, y_2 and z_2 are the values for the mass which has been removed.

Example 11.5 *Find the position of centre of mass of the uniform lamina shown in figure.*

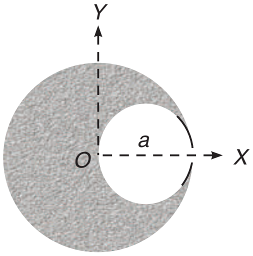


Fig. 11.12

Solution Here,

$$A_1 = \text{area of complete circle} = \pi a^2$$

$$A_2 = \text{area of small circle}$$

$$= \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

$(x_1, y_1) =$ coordinates of centre of mass of large circle $= (0, 0)$

$(x_2, y_2) =$ coordinates of centre of mass of small circle $= \left(\frac{a}{2}, 0\right)$

Using
$$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

we get
$$x_{COM} = \frac{(\pi a^2)(0) - \left(\frac{\pi a^2}{4}\right)\left(\frac{a}{2}\right)}{\pi a^2 - \frac{\pi a^2}{4}}$$

$$= \frac{-\left(\frac{1}{8}\right)}{\left(\frac{3}{4}\right)} a = -\frac{a}{6}$$

and $y_{COM} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$.

INTRODUCTORY EXERCISE 11.1

1. What is the difference between centre of mass and centre of gravity?
2. The centre of mass of a rigid body always lies inside the body. Is this statement true or false ?
3. The centre of mass always lies on the axis of symmetry if it exists. Is this statement true or false?
4. If all the particles of a system lie in y-z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or false?
5. What can be said about the centre of mass of a solid hemisphere of radius r without making any calculation. Will its distance from the centre be more than $r/2$ or less than $r/2$?
6. All the particles of a body are situated at a distance R from the origin. The distance of the centre of mass of the body from the origin is also R. Is this statement true or false?
7. Three particles of masses 1 kg, 2 kg and 3 kg are placed at the corners A, B and C respectively of an equilateral triangle ABC of edge 1 m. Find the distance of their centre of mass from A.
8. Find the distance of centre of mass of a uniform plate having semicircular inner and outer boundaries of radii a and b from the centre O.

   Hint : Distance of COM of semicircular plate from centre is $\dfrac{4r}{3\pi}$.

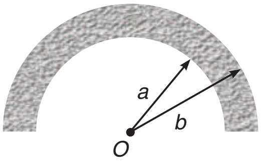


Fig. 11.13

9. Find the position of centre of mass of the section shown in figure 11.14.

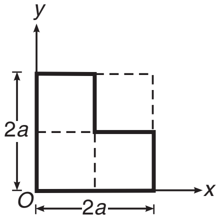


Fig. 11.14

Note *Solve the problem by using both the formulae:*

$$\text{(i) } x_{\text{COM}} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} \quad \text{and} \quad \text{(ii) } x_{\text{COM}} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

10. Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find square of distance of their centre of mass from A.
11. A square lamina of side a and a circular lamina of diameter a are placed touching each other as shown in Fig. 11.15. Find distance of their centre of mass from point O, the centre of square.

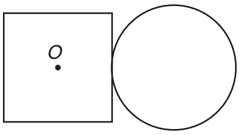


Fig. 11.15

12. The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 \dfrac{x^2}{l^2}$. Find the position of centre of mass of rod.
13. A straight rod of length L has one of its end at the origin and the other at $x = L$. If the mass per unit length of the rod is given by Ax where A is a constant, where is its centre of mass?

11.2 Motion of the Centre of Mass

Let us consider the motion of a system of n particles of individual masses $m_1, m_2, \ldots, m_n$ and total mass M. It is assumed that no mass enters or leaves the system during its motion, so that M remains constant. Then, as we have seen in the above article, we have the relation

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{M}$$

or $$M\mathbf{r}_{\text{COM}} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n$$

Differentiating this expression with respect to time t, we have

$$M\frac{d\mathbf{r}_{\text{COM}}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_n\frac{d\mathbf{r}_n}{dt}$$

Since, $$\frac{d\mathbf{r}}{dt} = \text{velocity}$$

Therefore, $$M\mathbf{v}_{\text{COM}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \qquad \ldots\text{(i)}$$

or velocity of the COM is

$$\mathbf{v}_{\text{COM}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n}{M}$$

or $$\boxed{\mathbf{v}_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i\mathbf{v}_i}{M}}$$

Further, $m\mathbf{v}$ = momentum of a particle $\mathbf{p}$. Therefore, Eq. (i) can be written as

$$\mathbf{p}_{\text{COM}} = \mathbf{p}_1 + \mathbf{p}_1 + \ldots + \mathbf{p}_n$$

or $$\boxed{\mathbf{p}_{\text{COM}} = \sum_{i=1}^{n} \mathbf{p}_i}$$

Differentiating Eq. (i) with respect to time t, we get

$$M\frac{d\mathbf{v}_{\text{COM}}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \ldots + m_n\frac{d\mathbf{v}_n}{dt}$$

or $$M\mathbf{a}_{\text{COM}} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n \qquad \ldots\text{(ii)}$$

or $$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n}{M}$$

or $$\boxed{\mathbf{a}_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i\mathbf{a}_i}{M}}$$

Further, in accordance with Newton's second law of motion $\mathbf{F} = m\mathbf{a}$. Hence, Eq. (ii) can be written as

$$\mathbf{F}_{\text{COM}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$$

or

$$\boxed{\mathbf{F}_{\text{COM}} = \sum_{i=1}^{n} \mathbf{F}_i}$$

Thus, as pointed out earlier also, the centre of mass of a system of particles moves as though it were a particle of mass equal to that of the whole system with all the external forces acting directly on it.

Extra Points to Remember

- Students are often confused over the problems of centre of mass. They cannot answer even the basic problems of COM. For example, let us take a simple problem: two particles one of mass 1 kg and the other of 2 kg are projected simultaneously with the same speed from the roof of a tower, the one of mass 1 kg vertically upwards and the other vertically downwards. What is the acceleration of centre of mass of these two particles? When I ask this question in my first class of centre of mass, three answers normally come from the students g, $\frac{g}{3}$ and zero. The correct answer is g. Because

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

Here, $\mathbf{a}_1 = \mathbf{a}_2 = g$ (downwards)

$\therefore$ $\mathbf{a}_{\text{COM}} = \dfrac{(1)(g) + (2)(g)}{1 + 2} = g$ (downwards)

The idea behind this is that apply the basic equations when asked anything about centre of mass. Just as a revision I am writing below all the basic equations of COM at one place.

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$x_{\text{COM}} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$$

$$y_{\text{COM}} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n}$$

$$z_{\text{COM}} = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}$$

$$\mathbf{v}_{\text{COM}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\mathbf{P}_{\text{COM}} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n}{m_1 + m_2 + \ldots + m_n}$$

and $\mathbf{F}_{\text{COM}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$

Example 11.6 *Two particles A and B of masses 1 kg and 2 kg respectively are projected in the directions shown in figure with speeds* $u_A = 200\ m/s$ *and* $u_B = 50\ m/s$. *Initially they were 90 m apart. They collide in mid air and stick with each other. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant.* ($g = 10\ m/s^2$)

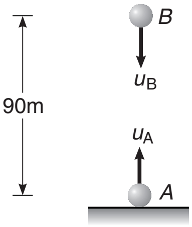


Fig. 11.16

Solution Using $m_A r_A = m_B r_B$

or $$(1)(r_A) = (2)(r_B)$$

or $$r_A = 2r_B \qquad \text{...(i)}$$

and $$r_A + r_B = 90\ \text{m} \qquad \text{...(ii)}$$

Solving these two equations, we get

$$r_A = 60\ \text{m} \quad \text{and} \quad r_B = 30\ \text{m}$$

i.e. COM is at height 60 m from the ground at time $t = 0$.

Further, $$\mathbf{a}_{\text{COM}} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B}$$

$$= g = 10\ \text{m/s}^2 \qquad \text{(downwards)}$$

as $$\mathbf{a}_A = \mathbf{a}_g = g \qquad \text{(downwards)}$$

$$\mathbf{u}_{\text{COM}} = \frac{m_A \mathbf{u}_A + m_B \mathbf{u}_B}{m_A + m_B}$$

$$= \frac{(1)(200) - (2)(50)}{1 + 2} = \frac{100}{3}\ \text{m/s} \qquad \text{(upwards)}$$

Let, h be the height attained by COM beyond 60 m. Using,

$$v_{\text{COM}}^2 = u_{\text{COM}}^2 + 2a_{\text{COM}}h$$

or $$0 = \left(\frac{100}{3}\right)^2 - (2)(10)h$$

or $$h = \frac{(100)^2}{180} = 55.55\ \text{m}$$

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 55.55 = 115.55\ \text{m}$$ **Ans.**

Example 11.7 *In the arrangement shown in figure,* $m_A = 2\ kg$ *and* $m_B = 1\ kg$. *String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.*

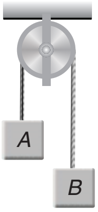


Fig. 11.17

Solution Net pulling force on the system is $(m_A - m_B)g$

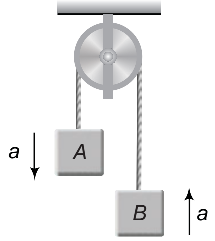


Fig. 11.18

or
$$(2-1)g = g$$

Total mass being pulled is $m_A + m_B$ or 3 kg

$$\therefore \qquad a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{g}{3}$$

$$\text{Now,} \qquad \mathbf{a}_{\text{COM}} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B} = \frac{(2)(a)-(1)(a)}{1+2} = \frac{a}{3} = \frac{g}{9} \qquad \text{(downwards)}$$

Alternate Method

Free body diagram of block A is shown in Fig. 11.19.

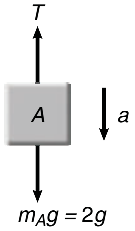


Fig. 11.19

$$2g - T = m_A(a)$$

or
$$T = 2g - m_A a$$
$$= 2g - (2)\left(\frac{g}{3}\right) = \frac{4g}{3}$$

Free body diagrams of A and B both are as shown in Fig. 11.20.

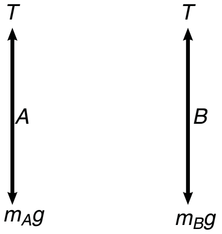


Fig. 11.20

$$\mathbf{a}_{\text{COM}} = \frac{\text{Net force on both the blocks}}{m_A + m_B}$$
$$= \frac{(m_A + m_B)g - 2T}{2+1}$$
$$= \frac{3g - \frac{8g}{3}}{3} = \frac{g}{9} \qquad \text{(downwards)}$$

Example 11.8 *Two blocks A and B of equal masses are released on two sides of a fixed wedge C as shown in figure. Find the acceleration of centre of mass of blocks A and B. Neglect friction.*

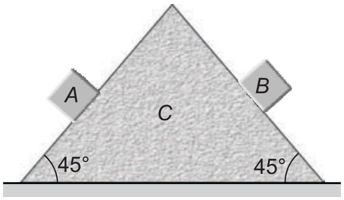


Fig. 11.21

Solution Acceleration of both the blocks will be $g \sin 45°$ or $\frac{g}{\sqrt{2}}$ at right angles to each other. Now,

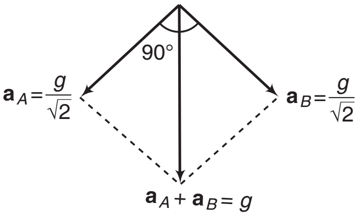

90°
$\mathbf{a}_A = \frac{g}{\sqrt{2}}$
$\mathbf{a}_B = \frac{g}{\sqrt{2}}$
$\mathbf{a}_A + \mathbf{a}_B = g$

Fig. 11.22

$$\mathbf{a}_{\text{COM}} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B}$$

Here, $m_A = m_B$

$\therefore$ $$\mathbf{a}_{\text{COM}} = \frac{1}{2}(\mathbf{a}_A + \mathbf{a}_B) = \frac{1}{2} g \qquad \text{(downwards)}$$

INTRODUCTORY EXERCISE 11.2

1. A block of mass 1 kg is at $x = 10$ m and moving towards negative x-axis with velocity 6 m/s. Another block of mass 2 kg is at $x = 12$ m and moving towards positive x-axis with velocity 4 m/s at the same instant. Find position of their centre of mass after 2 s.
2. Two particles of masses 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$s.
3. There are two masses m_1 and m_2 placed at a distance l apart. Let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C. Find the distance, from C where new centre of mass will be located.
4. At one instant, the centre of mass of a system of two particles is located on the x-axis at $x = 3.0$ m and has a velocity of $(6.0 \text{ m/s})\hat{\mathbf{j}}$. One of the particles is at the origin, the other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 12.0$ m.
   (a) What is the mass of the particle at the origin ?
   (b) Calculate the total momentum of this system.
   (c) What is the velocity of the particle at the origin ?
5. A stone is dropped at $t = 0$. A second stone, with twice the mass of the first, is dropped from the same point at $t = 100$ ms.
   (a) How far below the release point is the centre of mass of the two stones at $t = 300$ ms ? (Neither stone has yet reached the ground).
   (b) How fast is the centre of mass of the two-stone system moving at that time ?
6. Two blocks A and B of equal masses are attached to a string passing over a smooth pulley fixed to a wedge as shown in figure. Find the magnitude of acceleration of centre of mass of the two blocks when they are released from rest. Neglect friction.

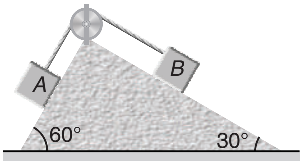


Fig. 11.23

11.3 Law of Conservation of Linear Momentum

The product of mass and the velocity of a particle is defined as its linear momentum ($\mathbf{p}$). So,

$$\boxed{\mathbf{p} = m\mathbf{v}}$$

The magnitude of linear momentum may be written as

$$p = mv \quad \text{or} \quad p^2 = m^2v^2 = 2m\left(\frac{1}{2}mv^2\right) = 2mK$$

Thus,
$$\boxed{p = \sqrt{2Km} \quad \text{or} \quad K = \frac{p^2}{2m}}$$

Here, K is the kinetic energy of the particle. In accordance with Newton's second law,

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

Thus,
$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

In case the external force applied to a particle (or a body) be zero, we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = 0 \quad \text{or} \quad \mathbf{p} = \text{constant}$$

showing that in the absence of an external force, the linear momentum of a particle (or the body) remains constant. This is called the law of conservation of linear momentum. The law may be extended to a system of particles or to the centre of mass of a system of particles. For example, for a system of particles it takes the form :

If net force (or the vector sum of all the forces) on a system of particles is zero, the vector sum of linear momentum of all the particles remain conserved, or

If
$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n = 0$$

Then,
$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots + \mathbf{p}_n = \text{constant}$$

The same is the case for the centre of mass of a system of particles, i.e. if

$$\mathbf{F}_{\text{COM}} = 0, \ \mathbf{p}_{\text{COM}} = \text{constant}.$$

Thus, the law of conservation of linear momentum can be applied to a single particle, to a system of particles or even to the centre of mass of the particles.

The law of conservation of linear momentum enables us to solve a number of problems which can not be solved by a straight application of the relation $\mathbf{F} = m\mathbf{a}$.

For example, suppose a particle of mass m initially at rest, suddenly explodes into two fragments of masses m_1 and m_2 which fly apart with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Obviously, the forces resulting in the explosion of the particle must be internal forces, since no external force has been applied. In the absence of the external forces, therefore, the momentum must remain conserved and we should have

$$m\mathbf{v}_i = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

Since, the particle was initially at rest, $\mathbf{v}_i = 0$ and therefore,

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = 0$$

$$\mathbf{v}_1 = -\frac{m_2}{m_1}\mathbf{v}_2 \quad \text{or} \quad \frac{|\mathbf{v}_1|}{|\mathbf{v}_2|} = \frac{m_2}{m_1}$$

Showing at once that the velocities of the two fragments must be inversely proportional to their masses and in opposite directions along the same line. This result could not possibly be arrived at from the relation $\mathbf{F} = m\mathbf{a}$, since we know nothing about the forces that were acting during the explosion. Nor, could we derive it from the law of conservation of energy.

Extra Points to Remember

- If law of conservation of linear momentum is applied to a single particle, then we can explain it like this.
  If net force or net external force (as there is no internal force on a single particle) acting on the particle is zero then its linear momentum remains constant. If mass of the particle is constant, then velocity of the particle is also constant. If the particle is at rest then it will remain at rest for ever. If it is moving then it will continue to be moving with constant velocity.
- If law of conservation of momentum is applied to a system of particles (may be a rigid body also) then the law is like this :
  It net force or net external force (as summation of net internal forces acting on a system of particles is already zero) on a system of particles (or on the centre of mass of the system of particles) is zero, then total momentum of the system of particles (or momentum of centre of mass of the system of particles) remains constant. If total mass of the system of particles is constant then velocity of centre of mass will also remain constant.
- If a projectile explodes in air in different parts, the path of the centre of mass remains unchanged. This is because during explosion no external force (except gravity) acts on the centre of mass. The situation is as shown in figure.

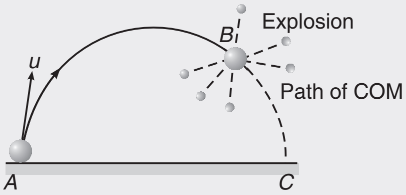


Fig. 11.24

Path of COM is parabola, even though the different parts travel in different directions after explosion. This situation continues till the first particle strikes the ground. Because, after that force behaviour of system of particles will change.

Example 11.9 *Linear momentum of particle is increased by*

(a) 100 % *(b)* 1 %

without changing its mass. Find percentage increase in its kinetic energy in both cases.

Solution (a) Relation between kinetic energy K and momentum p is given by:

$$K = \frac{p^2}{2m} \qquad \text{......(i)}$$

Now, momentum is increased by 100%.

So, new momentum, $p' = 2p$

$$\therefore \qquad K' = \frac{(p')^2}{2m} = \frac{(2p)^2}{2m} = 4\left[\frac{p^2}{2m}\right] = 4K \qquad \text{[From Eq. (i)]}$$

Now percentage change in kinetic energy,

$$= \left(\frac{\text{Final value} - \text{Initial value}}{\text{Initial value}}\right) \times 100$$

$$= \left(\frac{K' - K}{K}\right) \times 100 = \left(\frac{4K - K}{K}\right) \times 100$$

$$= +300\%$$ **Ans.**

Plus sign indicates that, with increase in linear momentum, kinetic energy will also increase.

(b) $K = \frac{p^2}{2m}$ or $K \propto p^2$ (as $m =$ constant)

Here, power of K is 1 and power of p is 2. For small changes, we can write it like this

(1) (% change in K) = (2) (% change in p)

or $$\text{\% change in } K = (2)(1\%) = +2\%$$ **Ans.**

Example 11.10 *Kinetic energy of a particle is increased by*

(a) 50 % *(b)* 1 %

Find percentage change in linear momentum.

Solution (a) $p = \sqrt{2Km}$ …(i)

Kinetic energy is increased by 50 %. So, the new value of kinetic energy is

$$K' = 1.5K$$

$$\therefore \qquad P' = \sqrt{2K'm}$$

$$= \sqrt{2(1.5K)m} = \sqrt{1.5}\,(\sqrt{2Km})$$

$$= 1.22\sqrt{2Km} = 1.22\,p \qquad \text{[From Eq. (i)]}$$

So, the percentage change in momentum is

$$= \left(\frac{p' - p}{p}\right) \times 100 = \left(\frac{1.22p - p}{p}\right) \times 100 = +22\%$$ **Ans.**

(b) $p = \sqrt{2Km}$ or $p \propto \sqrt{K}$ or $p \propto K^{\frac{1}{2}}$ (as $m =$ constant)

Here, power of p is 1 and the power of K is $\frac{1}{2}$.

For small percentage changes we can write as

$$(1)\ (\text{\% change in } P) = \left(\frac{1}{2}\right) (\text{\% change in } K)$$

or $$\text{\% change in } p = \frac{1}{2}(1\%) = +0.5\%$$ **Ans.**

Example 11.11 *Two blocks A and B of masses 1 kg and 2 kg are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Find the ratio of their,*

(a) speed

(b) magnitude of momentum and

(c) kinetic energy at any instant.

Solution (a) Net force on the system of two blocks is zero. Therefore, linear momentum of the system will remain constant. Initially, they are at rest. So, at any instant the net momentum will be zero.

$$\therefore \qquad \mathbf{p}_1 + \mathbf{p}_2 = 0$$

$$\Rightarrow \qquad \mathbf{p}_1 = -\mathbf{p}_2 \qquad \ldots\text{(i)}$$

$$\text{or} \qquad m_1 \mathbf{v}_1 = -m_2 \mathbf{v}_2$$

$$\text{or} \qquad (1)\mathbf{v}_1 = -2\mathbf{v}_2$$

$$\text{or} \qquad \frac{|\mathbf{v}_1|}{|\mathbf{v}_2|} = 2 \qquad \textbf{Ans.}$$

(b) From Eq. (i),

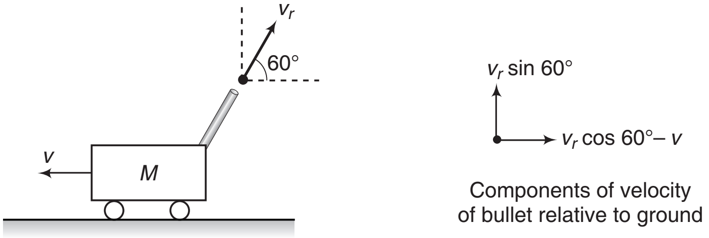
$$\frac{|\mathbf{p}_1|}{|\mathbf{p}_2|} = 1$$

(c) $K = \dfrac{p^2}{2m}$ or $K \propto \dfrac{p^2}{m}$

$$\therefore \qquad \frac{K_1}{K_2} = \left(\frac{p_1}{p_2}\right)^2 \left(\frac{m_2}{m_1}\right) = (1)^2 \left(\frac{2}{1}\right) = 2 \qquad \textbf{Ans.}$$

Example 11.12 *A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun.*

Solution Let the recoil speed of gun is v. Taking gun + bullet as the system. Net external force on the system in horizontal direction is zero. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum in horizontal direction, we get



Components of velocity of bullet relative to ground

Fig. 11.25

$$Mv - m(v_r \cos 60° - v) = 0$$

$$\therefore \qquad v = \frac{mv_r \cos 60°}{M + m} \quad \text{or} \quad v = \frac{mv_r}{2(M + m)} \qquad \textbf{Ans.}$$

Example 11.13 *A projectile of mass 3 m is projected from ground with velocity $20\sqrt{2}\ m/s$ at $45°$. At highest point it explodes into two pieces. One of mass 2 m and the other of mass m. Both the pieces fly off horizontally in opposite directions. Mass 2 m falls at a distance of 100 m from point of projection. Find the distance of second mass from point of projection where it strikes the ground.* $(g = 10\ m/s^2)$

Solution Range of the projectile in the absence of explosion

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20\sqrt{2})^2 \sin 90°}{10} = 80\ \text{m}$$

The path of centre of mass of projectile will not change, i.e. x_{COM} is still 80 m. Now, from the definition of centre of mass

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or

$$80 = \frac{(m)(x_1) + (2m)(100)}{m + 2m}$$

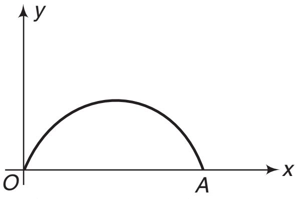


Fig. 11.26

Solving this equation, we get $x_1 = 40\ \text{m}$

Therefore, the mass m will fall at a distance $x_1 = 40\,\text{cm}$ from point of projection. **Ans.**

INTRODUCTORY EXERCISE 11.3

1. Three particles of masses 20 g, 30 g and 40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s. When due to their mutual interaction, the first particle comes to rest, the second acquires a velocity $(10\hat{\mathbf{i}} + 20\hat{\mathbf{k}})$ cm/s. What is then the velocity of the third particle?
2. A boy of mass 25 kg stands on a board of mass 10 kg which in turn is kept on a frictionless horizontal ice surface. The boy makes a jump with a velocity component 5 m/s in a horizontal direction with respect to the ice. With what velocity does the board recoil? With what rate are the boy and board separating from each other?
3. Find the ratio of the linear momenta of two particles of masses 1.0 kg and 4.0 kg if their kinetic energies are equal.
4. A uranium-238 nucleus, initially at rest, emits an alpha particle with a speed of 1.4×10^7 m/s. Calculate the recoil speed of the residual nucleus thorium-234. Assume that the mass of a nucleus is proportional to the mass number.
5. A man of mass 50 kg starts moving on the earth and acquires a speed of 1.8 m/s. With what speed does the earth recoil? Mass of earth $= 6 \times 10^{24}$ kg.
6. A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find velocity of trolley and total energy produced by man.
7. A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal masses. One fragment, whose initial speed is zero falls vertically. How far from the gun does the other fragment land, assuming a level terrain? Take $g = 10\ \text{m/s}^2$?

11.4 Variable Mass

In our discussion of the conservation of linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those systems whose mass is variable, i.e. those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping, thereby continuously decreasing its mass.

In such problems you have nothing to do but apply a thrust force ($\mathbf{F}_t$) to the main mass in addition to the all other forces acting on it. This thrust force is given by,

$$\mathbf{F}_t = \mathbf{v}_{\text{rel}}\left(\pm\frac{dm}{dt}\right)$$

Here, $\mathbf{v}_{\text{rel}}$ is the velocity of the mass gained or mass ejected relative to the main mass. In case of rocket this is sometimes called the exhaust velocity of the gases. $\frac{dm}{dt}$ is the rate at which mass is increasing or decreasing.

The expression for the thrust force can be derived from the conservation of linear momentum in the absence of any external forces on a system as follows :

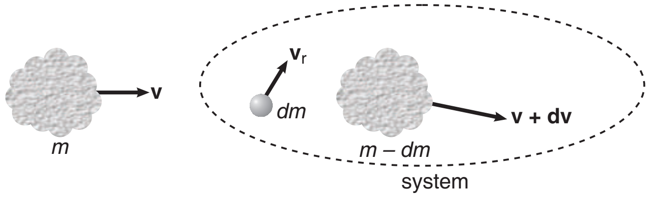


Fig. 11.27

Suppose at some moment mass of a body is m and its velocity is $\mathbf{v}$. After some time interval dt its mass becomes $(m - dm)$ and velocity becomes $\mathbf{v} + d\mathbf{v}$. The mass dm is ejected with relative velocity $\mathbf{v}_r$. Absolute velocity of mass 'dm' is therefore $(\mathbf{v}_r + \mathbf{v} + d\mathbf{v})$. If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\mathbf{p}_i = \mathbf{p}_f$$

or $$m\mathbf{v} = (m - dm)(\mathbf{v} + d\mathbf{v}) + dm(\mathbf{v}_r + \mathbf{v} + d\mathbf{v})$$

or $$m\mathbf{v} = m\mathbf{v} + md\mathbf{v} - dm\mathbf{v} - (dm)(d\mathbf{v}) + dm\mathbf{v} + \mathbf{v}_r dm + (dm)(d\mathbf{v})$$

∴ $$md\mathbf{v} = -\mathbf{v}_r dm \quad \text{or} \quad m\left(\frac{d\mathbf{v}}{dt}\right) = \mathbf{v}_r\left(-\frac{dm}{dt}\right)$$

Here, $$m\left(\frac{d\mathbf{v}}{dt}\right) = \text{thrust force } (\mathbf{F}_1)$$

and $$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

Problems Related to Variable Mass can be Solved in Following Three Steps

1. Make a list of all the forces acting on the main mass and apply these forces on it.
2. Apply an additional thrust force $\mathbf{F}_t$ on the mass, the magnitude of which is $\left|\mathbf{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\mathbf{v}_r$ in case the mass is increasing and in the direction of $-\mathbf{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

$$\mathbf{F}_{\text{net}} = m\frac{d\mathbf{v}}{dt} \qquad (m = \text{mass at that particular instant})$$

Rocket Propulsion

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially let us assume that the velocity of the rocket is u.

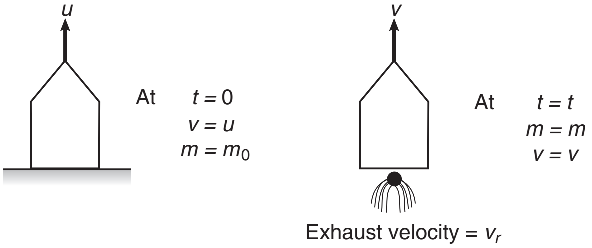


Fig. 11.28

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket

$$F_t = v_r\left(-\frac{dm}{dt}\right) \qquad \text{(upwards)}$$

2. Weight of the rocket

$$w = mg \qquad \text{(downwards)}$$

3. Net force on the rocket

$$F_{\text{net}} = F_t - W \qquad \text{(upwards)}$$

or

$$F_{\text{net}} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or

$$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

or $$\int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$$

or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$$

Thus, $$\boxed{v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)} \qquad \ldots\text{(i)}$$

Note *(i)* $F_t = v_r\left(-\frac{dm}{dt}\right)$ *is upwards, as* v_r *is downwards and* $\frac{dm}{dt}$ *is negative.*

(ii) If gravity is ignored and initial velocity of the rocket $u = 0$*, Eq. (i) reduces to* $v = v_r \ln\left(\frac{m_0}{m}\right)$.

Example 11.14 *(a) A rocket set for vertical firing weighs* 50 *kg and contains* 450 *kg of fuel. It can have a maximum exhaust velocity of* 2 *km/s. What should be its minimum rate of fuel consumption*

(i) to just lift it off the launching pad?

(ii) to give it an initial acceleration of 20 m/s^2*?*

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 *kg/s after whole of the fuel is consumed? (Take* $g = 9.8\ m/s^2$*)*

Solution (a) (i) To just lift it off the launching pad

weight = thrust force

or $$mg = v_r\left(\frac{-dm}{dt}\right)$$

or $$\left(\frac{-dm}{dt}\right) = \frac{mg}{v_r}$$

Substituting the values, we get

$$\left(\frac{-dm}{dt}\right) = \frac{(450+50)(9.8)}{2\times 10^3}$$

$$= 2.45 \text{ kg/s}$$ **Ans.**

(ii) Net acceleration $a = 20\ \text{m/s}^2$

∴ $$ma = F_t - mg$$

or $$a = \frac{F_t}{m} - g$$

or $$a = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

This gives $$\left(-\frac{dm}{dt}\right) = \frac{m(g+a)}{v_r}$$

Substituting the values, we get

$$\left(-\frac{dm}{dt}\right)=\frac{(450+50)(9.8+20)}{2\times 10^3}$$

$$= 7.45 \text{ kg/s}$$ **Ans.**

(b) The rate of fuel consumption is 10 kg/s.

So, the time for the consumption of entire fuel is

$$t=\frac{450}{10}=45\,\text{s}$$

Using Eq. (i), i.e. $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Here, $u = 0$, $v_r = 2\times 10^3$ m/s, $m_0 = 500$ kg and $m = 50$ kg

Substituting the values, we get

$$v = 0-(9.8)(45)+(2\times 10^3)\ln\left(\frac{500}{50}\right)$$

or $$v = -441 + 4605.17$$

or $$v = 4164.17 \text{ m/s}$$

or $$v = 4.164 \text{ km/s}$$ **Ans.**

INTRODUCTORY EXERCISE 11.4

1. A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/s. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also, calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is $(g = 9.8 \text{ m/s}^2)$

   (i) 2 kg/s (ii) 20 kg/s

2. A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning ceases after one minute, find the maximum velocity of the rocket. (Take g as constant at 10 ms^{-2})

3. A rocket is moving vertically upward against gravity. Its mass at time t is $m = m_0 - \mu t$ and it expels burnt fuel at a speed u vertically downward relative to the rocket. Derive the equation of motion of the rocket but do not solve it. Here, μ is constant.

4. A rocket of initial mass m_0 has a mass $m_0(1 - t/3)$ at time t. The rocket is launched from rest vertically upwards under gravity and expels burnt fuel at a speed u relative to the rocket vertically downward. Find the speed of rocket at $t = 1$.

11.5 Linear Impulse

Consider a constant force **F** which acts for a time *t* on a body of mass *m,* thus, changing its velocity from **u** to **v**. Because the force is constant, the body will travel with constant acceleration **a** where

$$\mathbf{F} = m\mathbf{a}$$

and
$$\mathbf{a}t = \mathbf{v} - \mathbf{u}$$

hence,
$$\frac{\mathbf{F}}{m}t = \mathbf{v} - \mathbf{u}$$

or
$$\mathbf{F}t = m\mathbf{v} - m\mathbf{u}$$

The product of constant force **F** and the time *t* for which it acts is called the **impulse** (**J**) of the force and this is equal to the change in linear momentum which it produces.

Thus,
$$\boxed{\text{Impulse } (\mathbf{J}) = \mathbf{F}t = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i}$$

Instantaneous Impulse There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g. a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (*i.e.,* impulse) can be known by measuring the initial and final momenta. Thus, we can write

$$\boxed{\mathbf{J} = \int \mathbf{F}dt = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i}$$

Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force-time (F-t) graph in the same time interval.

In one dimensional motion, we can simply write as,

$$\boxed{J = \int F\,dt = \Delta p = p_f - p_i}$$

In this equation all vector quantities (J, F and p) are taken with proper signs.

Further, if mass is constant, then we can write,

$$p_f - p_i \text{ as } m(v_f - v_i)$$

Now depending on the nature of force there are following three cases :

Case 1 If force is constant, then linear impulse can be obtained by multiplying this constant force with the given time interval. Now, this impulse is change in linear momentum.

Case 2 If force is a function of time, then linear impulse can be obtained by integrating this linear function of time in the given time interval. Result of this integration (or linear impulse) is equal to the change in linear momentum.

Case 3 If force *versus* time graph is given, then linear impulse can be obtained by the area under F-t graph. This area (or the linear impulse) is equal to the change in linear momentum.

Note *Normally, the word impulse is used when a large force acts for a short interval of time but the equations discussed above can be used even if the time interval is large.*

Example 11.15 *A ball of mass* 200 *g is projected with a velocity of* 30 *m/s at* 30° *from horizontal. Using the concept of impulse, find change in velocity in* 2 *s. Take* $g=10\,m/s^2$.

Solution In air, a constant force (= weight = $m\mathbf{g}$) will act on the ball. Therefore, the linear impulse can be obtained directly by multiplying this constant force with the given time interval. Further, this linear impulse is equal to the change in linear momentum. Thus,

$$\mathbf{J}=\mathbf{F}\times t=\Delta\mathbf{p}=m\Delta\mathbf{v}$$

$$\Rightarrow \qquad \Delta\mathbf{v}=\left(\frac{t}{m}\right)\mathbf{F}=\left(\frac{t}{m}\right)m\mathbf{g}$$

$$=\mathbf{g}\times t=(10\,\text{m/s}^2\text{ downwards})(2\text{ s})$$

$$=20\,\text{m/s} \qquad \text{(downwards)}$$ **Ans.**

Example 11.16 *A time varying force,* $F=2t$ *is acting on a particle of mass* 2 *kg moving along x-axis. velocity of the particle is* 4 *m/s along negative x-axis at time* $t=0$. *Find the velocity of the particle at the end of* 4 *s.*

Solution Given force is a function of time.

So, linear impulse can be obtained by integration. Further, this impulse is equal to change in linear momentum.

Thus, $$J=\int F\,dt=\Delta p=p_f-p_i=m(v_f-v_i)$$

or $$v_f=\frac{1}{m}\int F\,dt+v_i=\frac{1}{2}\int_0^4(2t)\,dt-4 \qquad (v_i=-4\text{ m/s})$$

$$=\frac{1}{2}[t^2]_0^4-4=\frac{1}{2}[16-0]-4$$

$$=+4\,\text{m/s}$$ **Ans.**

Therefore, the final velocity is 4 m/s along positive x-direction.

Example 11.17 *A particle of mass* 2 *kg is initially at rest. A force starts acting on it in one direction whose magnitude changes with time. The force time graph is shown in figure. Find the velocity of the particle at the end of* 10 *s.*

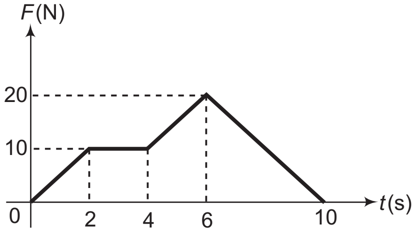


Fig. 11.29

Solution Using impulse = Change in linear momentum (or area under F-t graph)

We have, $$m(v_f-v_i)=\text{Area}$$

$$\text{or} \qquad 2(v_f - 0) = \frac{1}{2} \times 2 \times 10 + 2 \times 10 + \frac{1}{2} \times 2 \times (10+20) + \frac{1}{2} \times 4 \times 20$$

$$= 10 + 20 + 30 + 40 \quad \text{or} \quad 2v_f = 100$$

$$\therefore \qquad v_f = 50 \text{ m/s}$$ **Ans.**

Example 11.18 *A bullet of mass 10^{-3} kg strikes an obstacle and moves at 60° to its original direction. If its speed also changes from 20 m/s to 10 m/s. Find the magnitude of impulse acting on the bullet.*

Solution Mass of the bullet $m = 10^{-3}$ kg

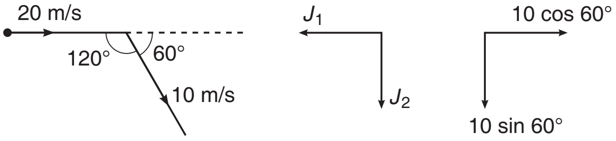


Fig. 11.30

Consider components parallel to J_1.

$$J_1 = 10^{-3} [-10 \cos 60° - (-20)] \quad \text{or} \quad J_1 = 15 \times 10^{-3} \text{ N-s}$$

Similarly, parallel to J_2, we have

$$J_2 = 10^{-3} [10 \sin 60° - 0] = 5\sqrt{3} \times 10^{-3} \text{ N-s}$$

The magnitude of resultant impulse is given by

$$J = \sqrt{J_1^2 + J_2^2} = 10^{-3} \sqrt{(15)^2 + (5\sqrt{3})^2} \quad \text{or} \quad J = \sqrt{3} \times 10^{-2} \text{N-s}$$ **Ans.**

INTRODUCTORY EXERCISE 11.5

1. A truck of mass 2×10^3 kg travelling at 4 m/s is brought to rest in 2 s when it strikes a wall. What force (assume constant) is exerted by the wall ?
2. A ball of mass m, travelling with velocity $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ receives an impulse $-3m\hat{\mathbf{i}}$. What is the velocity of the ball immediately afterwards ?
3. The net force *versus* time graph of a rocket is shown in figure. The mass of the rocket is 1200 kg. Calculate velocity of rocket, 16 seconds after starting from rest. Neglect gravity.

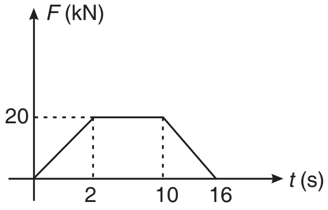


Fig. 11.31

4. A 5.0 g bullet moving at 100 m/s strikes a log. Assume that the bullet undergoes uniform deceleration and stops in 6.0 cm. Find (a) the time taken for the bullet to stop, (b) the impulse on the log and (c) the average force experienced by the log.

11.6 Collision

Contrary to the meaning of the term 'collision' in our everyday life, in physics it does not necessarily mean one particle 'striking' against other. Indeed two particles may not even touch each other and may still be said to collide. All that is implied is that as the particles approach each other,

(i) an impulse (a large force for a relatively short time) acts on each colliding particles.

(ii) the total momentum of the particles remain conserved.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

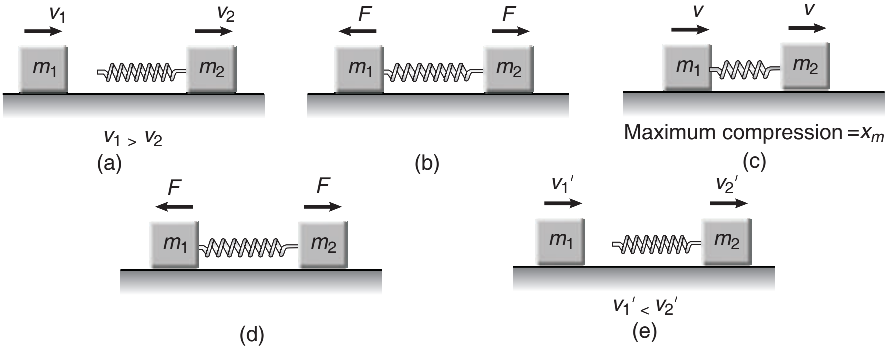


Fig. 11.32

Two blocks of masses m_1 and m_2 are moving with velocities v_1 and $v_2 (< v_1)$ along the same straight line in a smooth horizontal surface. A spring is attached to the block of mass m_2. Now, let us see what happens during the collision between two particles.

Figure (a) Block of mass m_1 is behind m_2. Since, $v_1 > v_2$, the blocks will collide after some time.

Figure (b) The spring is compressed. The spring force $F (= kx)$ acts on the two blocks in the directions shown in figure. This force decreases the velocity of m_1 and increases the velocity of m_2.

Figure (c) The spring will compress till velocity of both the blocks become equal. So, at maximum compression (say x_m) velocities of both the blocks are equal (say v).

Figure (d) Spring force is still in the directions shown in figure, i.e. velocity of block m_1 is further decreased and that of m_2 is increased. The spring now starts relaxing.

Figure (e) The two blocks are separated from one another. Velocity of block m_2 becomes more than the velocity of block m_1, i.e. $v_2' > v_1'$.

Equations Which can be Used in the Above Situation

Assuming spring to be perfectly elastic following two equations can be applied in the above situation.

(i) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, i.e.

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v = m_1 v_1' + m_2 v_2' \qquad \ldots\text{(i)}$$

(ii) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, i.e.

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}kx_m^2$$

$$=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2 \qquad \ldots\text{(ii)}$$

Note *In the above situation we have assumed, spring to be perfectly elastic, i.e. it regains its original shape and size after the two blocks are separated. In actual practice there is no such spring between the two blocks. During collision, both the blocks (or bodies) are slightly deformed. This situation is similar to the compression of the spring. Due to deformation, two equal and opposite forces act on both the blocks. These two forces redistribute their linear momentum in such a manner that both the blocks are separated from one another. The collision is said to be elastic if both the blocks regain their original shape and size completely after they are separated. On the other hand if the blocks do not return to their original form the collision is said to be inelastic. If the deformation is permanent and the blocks move together with same velocity after the collision, the collision is said to be perfectly inelastic.*

Types of Collision

Collision between two bodies may be classified in two ways:

1. Elastic collision and inelastic collision.
2. Head on collision or oblique collision.

As discussed earlier also collision between two bodies is said to be **elastic** if both the bodies come to their original shape and size after the collision, i.e. no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision. On the other hand, in an **inelastic** collision, the colliding bodies do not return to their original shape and size completely after collision and some part of the mechanical energy of the system goes to the deformation potential energy. Thus, only linear momentum remains conserved in case of an inelastic collision.

Further, a collision is said to be **head on (or direct)** if the directions of the velocity of colliding objects are along the line of action of the impulses, acting at the instant of collision. If just before collision, at least one of the colliding objects was moving in a direction different from the line of action of the impulses, the collision is called **oblique or indirect.**

Head on Elastic Collision

Let the two balls of masses m_1 and m_2 collide each other elastically with velocities v_1 and v_2 in the directions shown in Fig. 11.33(a). Their velocities become v_1' and v_2' after the collision along the same line. Applying conservation of linear momentum,

we get

(a) Before collision (b) After collision

Fig. 11.33

$$m_1v_1+m_2v_2=m_1v_1'+m_2v_2' \qquad \ldots\text{(iii)}$$

In an elastic collision kinetic energy before and after collision is also conserved. Hence,

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1{v_1}'^2+\frac{1}{2}m_2{v_2}'^2 \qquad \ldots\text{(iv)}$$

Solving Eqs. (iii) and (iv) for v_1' and v_2', we get

$$v_1'=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_1+\left(\frac{2m_2}{m_1+m_2}\right)v_2 \qquad \ldots\text{(v)}$$

and

$$v_2'=\left(\frac{m_2-m_1}{m_1+m_2}\right)v_2+\left(\frac{2m_1}{m_1+m_2}\right)v_1 \qquad \ldots\text{(vi)}$$

Special Cases

1. If $m_1=m_2$, then from Eqs. (v) and (vi), we can see that

$$v_1'=v_2 \text{ and } v_2'=v_1$$

i.e. when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., e.g.

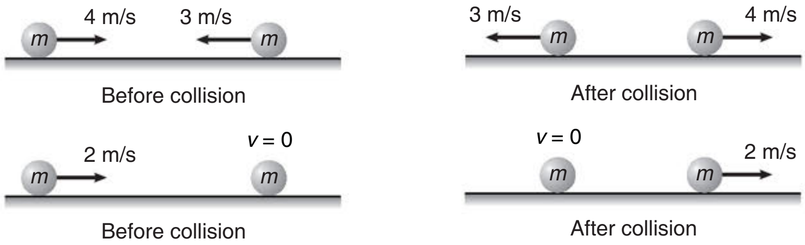


Fig. 11.34

2. If $m_1>>m_2$ and $v_1=0$. Then $\frac{m_2}{m_1}\approx 0$ with these two substitutions $\left(v_1=0 \text{ and } \frac{m_2}{m_1}=0\right)$

Fig. 11.35

we get the following two results

$$v_1'\approx 0 \quad \text{and} \quad v_2'\approx -v_2$$

i.e. the particle of mass m_1 remains at rest while the particle of mass m_2 bounces back with same speed v_2.

3. If $m_2>>m_1$ and $v_1=0$,

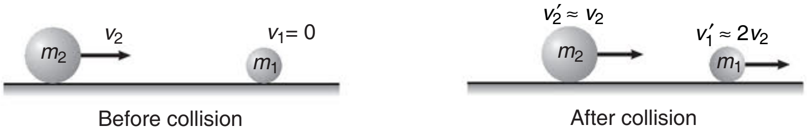


Fig. 11.36

with the substitution $\frac{m_1}{m_2} \approx 0$ and $v_1 = 0$, we get the results

$$v_1' \approx 2v_2 \quad \text{and} \quad v_2' \approx v_2$$

i.e. the mass m_1 moves with velocity $2v_2$ while the velocity of mass m_2 remains unchanged.

Note *It is important to note that Eqs. (v) and (vi) and their three special cases can be used only in case of a head on elastic collision between two particles. I have found that many students apply these two equations even if the collision is inelastic and do not apply these relations where clearly a head on elastic collision is given in the problem.*

Head on Inelastic Collision

As we have discussed earlier also, in an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

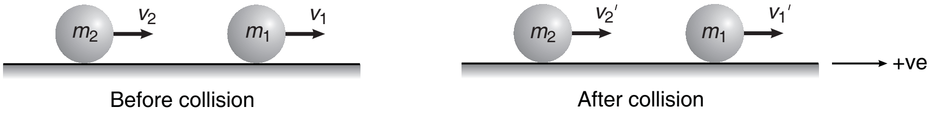


Fig. 11.37

Suppose the velocities of two particles of mass m_1 and m_2 before collision be v_1 and v_2 in the directions shown in figure. Let v_1' and v_2' be their velocities after collision. The law of conservation of linear momentum gives

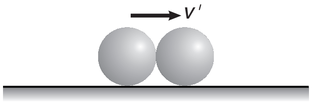


Fig. 11.38

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{...(vii)}$$

Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity, say v' as shown in Fig. 11.38. In this case, Eq. (vii) can be written as

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v'$$

or

$$\boxed{v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}} \qquad \text{...(viii)}$$

Newton's Law of Restitution

When two objects are in direct (head on) impact, the speed with which they separate after impact is usually less than or equal to their speed of approach before impact.

Experimental evidence suggests that the ratio of these relative speeds is constant for two given set of objects. This property formulated by Newton, is known as the law of restitution and can be written in the form

$$\boxed{\frac{\text{separation speed}}{\text{approach speed}} = e} \qquad \text{...(ix)}$$

The ratio e is called the coefficient of restitution and is constant for two particular objects.

In general $\boxed{0 \le e \le 1}$

$e=0$, for completely inelastic collision, as both the objects stick together. So, their separation speed is zero or $e=0$ from Eq. (ix).

$e=1$, for an elastic collision, as we can show from Eq. (v) and (vi), that

$$v_1' - v_2' = v_2 - v_1$$

or separation speed = approach speed

or $e=1$

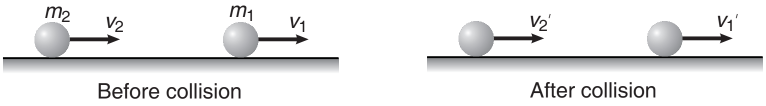


Fig. 11.39

Let us now find the velocities of two particles after collision if they collide directly and the coefficient of restitution between them is given as e.

Applying conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{...(x)}$$

Further, separation speed = e (approach speed)

or

$$v_1' - v_2' = e\,(v_2 - v_1) \qquad \text{...(xi)}$$

Solving Eqs. (x) and (xi), we get

$$\boxed{v_1' = \left(\frac{m_1 - em_2}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right) v_2} \qquad \text{...(xii)}$$

and

$$\boxed{v_2' = \left(\frac{m_2 - em_1}{m_1 + m_2}\right) v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right) v_1} \qquad \text{...(xiii)}$$

Special Cases

1. If collision is elastic, i.e. $e=1$, then

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

and

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1$$

which are same as Eqs. (v) and (vi).

2. If collision is perfectly inelastic, i.e. $e=0$, then

$$v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = v' \text{ (say)}$$

which is same as Eq. (viii).

3. If $m_1 = m_2$ and $v_1 = 0$, then

$$v_1' = \left(\frac{1+e}{2}\right) v_2 \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right) v_2 \qquad \ldots\text{(xiv)}$$

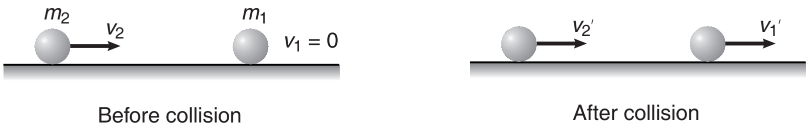


Fig. 11.40

Note *(i) If mass of one body is very-very greater than that of the other, then after collision velocity of heavy body does not change appreciably. (Whether the collision is elastic or inelastic).*

(ii) In the situation shown in figure if e is the coefficient of restitution between the ball and the ground, then after n^{th} collision with the floor the speed of ball will remain $e^n v_0$ and it will go upto a height $e^{2n} h$ or,

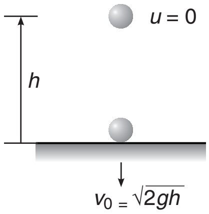


Fig. 11.41

$$v_n = e^n v_0 = e^n \sqrt{2gh}$$

and

$$h_n = e^{2n} h$$

EXERCISE *Derive the above two relations.*

Oblique Collision

During collision between two objects a pair of equal and opposite impulses act at the moment of impact. If just before impact at least one of the objects was moving in a direction different from the line of action of these impulses the collision is said to be oblique.

In the figure, two balls collide obliquely. During collision impulses act in the direction xx. Henceforth, we will call this direction as common normal direction and a direction perpendicular to it (i.e. yy) as common tangent.

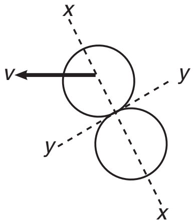


Fig. 11.42

Following four points are important regarding an oblique collision :

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual bodies do change along common normal direction. If mass of the colliding bodies remain constant during collision, then we can say that linear velocity of the individual bodies change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual bodies (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the bodies is zero during collision. Hence, net momentum of both the bodies remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, i.e. along common normal direction we can apply

Relative speed of separation = e (relative speed of approach)

Here, e is the coefficient of restitution between the particles.

Example 11.19 *Two blocks A and B of equal mass $m = 1.0$ kg are lying on a smooth horizontal surface as shown in figure. A spring of force constant $k = 200$ N/m is fixed at one end of block A. Block B collides with block A with velocity $v_0 = 2.0$ m/s. Find the maximum compression of the spring.*

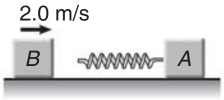


Fig. 11.43

Solution At maximum compression (x_m) velocity of both the blocks is same, say it is v. Applying conservation of linear momentum, we have

$$(m_A + m_B)v = m_B v_0$$

or $$(1.0 + 1.0)v = (1.0)\, v_0$$

or $$v = \frac{v_0}{2} = \frac{2.0}{2} = 1.0 \text{ m/s}$$

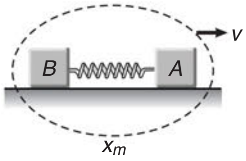


Fig. 11.44

Using conservation of mechanical energy, we have

$$\frac{1}{2} m_B v_0^2 = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2} k x_m^2$$

Substituting the values, we get

$$\frac{1}{2} \times (1) \times (2.0)^2 = \frac{1}{2} \times (1.0 + 1.0) \times (1.0)^2 + \frac{1}{2} \times (200) \times x_m^2$$

or $$2 = 1.0 + 100 x_m^2$$

or $$x_m = 0.1 \text{ m} = 10.0 \text{ cm}$$ **Ans.**

Example 11.20 *Two balls of masses m and $2\,m$ moving in opposite directions collide head on elastically with velocities v and $2v$. Find their velocities after collision.*

Solution Here, $v_1 = -v$, $v_2 = 2v$, $m_1 = m$ and $m_2 = 2m$.

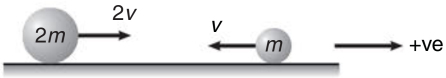


Fig. 11.45

Substituting these values in Eqs. (v) and (vi), we get

$$v_1' = \left(\frac{m - 2m}{m + 2m}\right)(-v) + \left(\frac{4m}{m + 2m}\right)(2v) \quad \text{or} \quad v_1' = \frac{v}{3} + \frac{8v}{3} = 3v$$

and $$v_2' = \left(\frac{2m - m}{m + 2m}\right)(2v) + \left(\frac{2m}{m + 2m}\right)(-v) \quad \text{or} \quad v_2' = \frac{2}{3}v - \frac{2}{3}v = 0$$

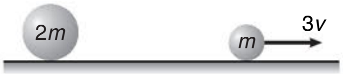


Fig. 11.46

i.e. the second ball (of mass $2m$) comes to a rest while the first (of mass m) moves with velocity $3v$ in the direction shown in Fig. 11.46.

Example 11.21 *Two pendulum bobs of masses m and 2m collide head on elastically at the lowest point in their motion. If both the balls are released from a height H above the lowest point, to what heights do they rise for the first time after collision?*

Solution Given, $m_1 = m, m_2 = 2m, v_1 = -\sqrt{2gH}$ and $v_2 = \sqrt{2gH}$

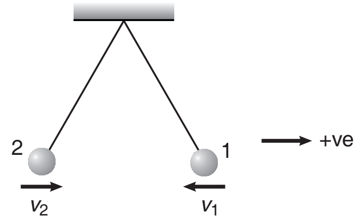


Fig. 11.47

Since, the collision is elastic. Using Eqs. (v) and (vi) discussed in the theory the velocities after collision are

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-\sqrt{2gH}) + \left(\frac{4m}{m+2m}\right)\sqrt{2gH}$$

$$= \frac{\sqrt{2gH}}{3} + \frac{4\sqrt{2gH}}{3} = \frac{5}{3}\sqrt{2gH}$$

and

$$v_2' = \left(\frac{2m-m}{m+2m}\right)(\sqrt{2gH}) + \left(\frac{2m}{m+2m}\right)(-\sqrt{2gH})$$

$$= \frac{\sqrt{2gH}}{3} - \frac{2\sqrt{2gH}}{3} = -\frac{\sqrt{2gH}}{3}$$

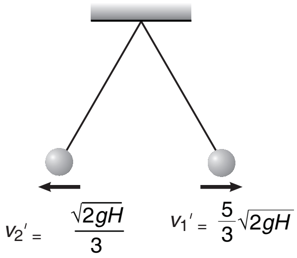


Fig. 11.48

i.e. the velocities of the balls after the collision are as shown in Fig. 11.48.

Therefore, the heights to which the balls rise after the collision are:

$$h_1 = \frac{(v_1')^2}{2g} \qquad \text{(using } v^2 = u^2 - 2gh\text{)}$$

or

$$h_1 = \frac{\left(\frac{5}{3}\sqrt{2gH}\right)^2}{2g} \quad \text{or} \quad h_1 = \frac{25}{9}H$$

and $$h_2 = \frac{(v_2')^2}{2g}$$

or $$h_2 = \frac{\left(\frac{\sqrt{2gH}}{3}\right)^2}{2g}$$

or $$h_2 = \frac{H}{9}$$

Note *Since, the collision is elastic, mechanical energy of both the balls will remain conserved, or*

$$E_i = E_f$$

$\Rightarrow$ $$(m + 2m)\, gH = mgh_1 + 2mgh_2$$

$\Rightarrow$ $$3mgH = (mg)\left(\frac{25}{9}H\right) + (2mg)\left(\frac{H}{9}\right)$$

$\Rightarrow$ $$3mgH = 3mgH$$

Example 11.22 *A ball of mass m moving at a speed v makes a head on inelastic collision with an identical ball at rest. The kinetic energy of the balls after the collision is $\frac{3}{4}$th of the original. Find the coefficient of restitution.*

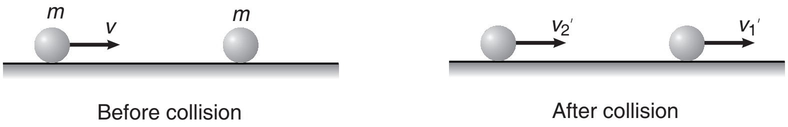


Fig. 11.49

Solution For the given conditions, we can use Eq. (xiv) or

$$v_1' = \left(\frac{1+e}{2}\right)v \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right)v$$

Given that $$K_f = \frac{3}{4}K_i$$

or $$\frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $$(1+e)^2 + (1-e)^2 = 3$$

or $$2 + 2e^2 = 3$$

or $$e^2 = \frac{1}{2}$$

or $$e = \frac{1}{\sqrt{2}}$$ **Ans.**

Example 11.23 *A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in figure. Assuming collision to be elastic, find the velocity of ball immediately after the collision.*

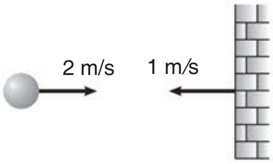
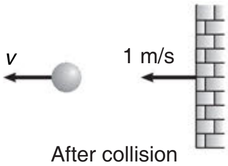


Fig. 11.50

Solution The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since, collision is elastic ($e = 1$).



After collision

Fig. 11.51

$$\text{separation speed} = \text{approach speed}$$

or
$$v - 1 = 2 + 1$$

or
$$v = 4 \text{ m/s}$$
Ans.

Example 11.24 *A ball of mass m hits a floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.*

Solution The component of velocity v_0 along common tangent direction $v_0 \sin \alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying

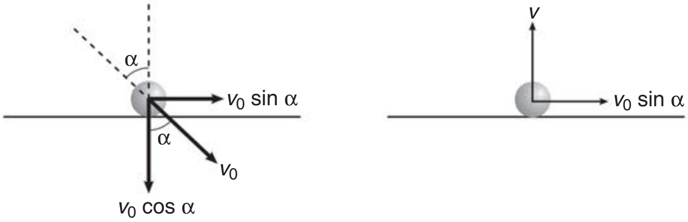


Fig. 11.52

Relative speed of separation = e (relative speed of approach)

along common normal direction, we get

$$v = ev_0 \cos \alpha$$

Thus, after collision components of velocity v' are $v_0 \sin \alpha$ and $ev_0 \cos \alpha$

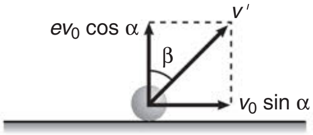


Fig. 11.53

∴
$$v' = \sqrt{(v_0 \sin \alpha)^2 + (ev_0 \cos \alpha)^2}$$

and
$$\tan \beta = \frac{v_0 \sin \alpha}{ev_0 \cos \alpha}$$

or
$$\tan \beta = \frac{\tan \alpha}{e}$$

Note *For elastic collision, $e = 1$*

∴ $v' = v_0$ *and* $\beta = \alpha$.

Example 11.25 *After perfectly inelastic collision between two identical balls moving with same speed in different directions, the speed of the combined mass becomes half the initial speed. Find the angle between the two before collision.*

Solution Let θ be the desired angle. Linear momentum of the system will remain conserved.

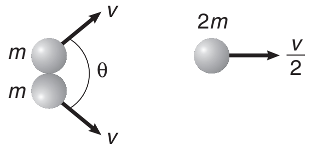


Fig. 11.54

Hence, $$p^2 = p_1^2 + p_2^2 + 2p_2 p_2 \cos\theta$$

or $$\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$$

or $$1 = 1 + 1 + 2\cos\theta \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$

∴ $$\theta = 120^\circ$$ **Ans.**

Example 11.26 *The coefficient of restitution between a snooker ball and the side cushion is* $\frac{1}{3}$*. If the ball hits the cushion and then rebounds at right angles to its original direction, show that the angles made with the side cushion by the direction of motion before and after impact are* 60° *and* 30° *respectively.*

Solution Let the original speed be u, in a direction making an angle θ with the side cushion. Using the law of restitution

Relative speed of separation along common normal direction

= e (relative speed of approach)

or $$v = \frac{1}{3}(u\sin\theta) \Rightarrow \frac{u}{v} = \frac{3}{\sin\theta}$$

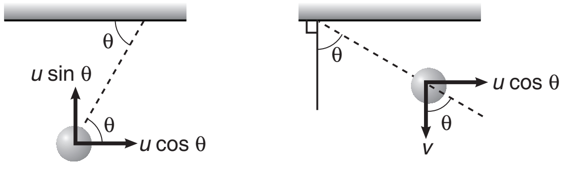


Fig. 11.55

After impact, $$\tan\theta = \frac{u\cos\theta}{v} = \frac{3\cos\theta}{\sin\theta}$$

⇒ $$\tan^2\theta = 3$$

⇒ $$\tan\theta = \sqrt{3}$$

⇒ $$\theta = 60^\circ$$

Therefore, the directions of motion before and after impact are at 60° and 30° to the cushion.

INTRODUCTORY EXERCISE 11.6

1. Two blocks of masses 3 kg and 6 kg respectively are placed on a smooth horizontal surface. They are connected by a light spring of force constant $k = 200$ N/m. Initially the spring is unstretched. The indicated velocities are imparted to the blocks. Find the maximum extension of the spring.

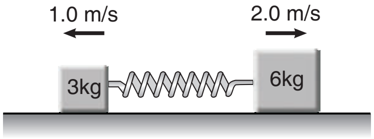


Fig. 11.56

2. A moving body of mass m makes a head on elastic collision with another body of mass $2m$ which is initially at rest. Find the fraction of kinetic energy lost by the colliding particle after collision.
3. What is the fractional decrease in kinetic energy of a body of mass m_1 when it makes a head on elastic collision with another body of mass m_2 kept at rest?
4. In one dimensional elastic collision of equal masses, the velocities are interchanged. Can velocities in a one dimensional collision be interchanged if the masses are not equal.
5. After an head on elastic collision between two balls of equal masses, one is observed to have a speed of 3 m/s along the positive x-axis and the other has a speed of 2 m/s along the negative x-axis. What were the original velocities of the balls ?
6. A ball of mass 1 kg moving with 4 ms^{-1} along $+x$-axis collides elastically with an another ball of mass 2 kg moving with 6 m/s is opposite direction. Find their velocities after collision.
7. Three balls A, B and C are placed on a smooth horizontal surface. Given that $m_A = m_C = 4m_B$. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls. All collisions are elastic.

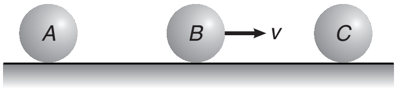


Fig. 11.57

8. Ball 1 collides directly with another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls?
9. A sphere A of mass m, travelling with speed v, collides directly with a stationary sphere B. If A is brought to rest and B is given a speed V, find (a) the mass of B (b) the coefficient of restitution between A and B?
10. A smooth sphere is moving on a horizontal surface with velocity vector $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ immediately before it hits a vertical wall. The wall is parallel to $\hat{\mathbf{j}}$ and the coefficient of restitution of the sphere and the wall is $e = \frac{1}{2}$. Find the velocity of the sphere after it hits the wall?
11. A ball falls vertically on an inclined plane of inclination α with speed v_0 and makes a perfectly elastic collision. What is angle of velocity vector with horizontal after collision.

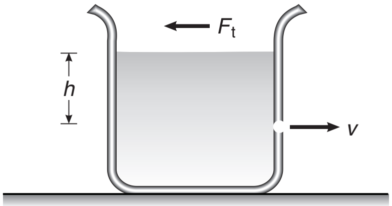
Final Touch Points

1. **Centre of mass frame of reference or *C*-frame of reference or zero momentum frame** A frame of reference carried by the centre of mass of an isolated system of particles (i.e. a system not subjected to any external forces) is called the centre of mass or *C*-frame of reference. In this frame of reference.

   (i) Position vector of centre of mass is zero.

   (ii) Velocity and hence momentum of centre of mass is also zero.

2. A liquid of density ρ is filled in a container as shown in figure. The liquid comes out from the container through a orifice of area *'a'* at a depth *'h'* below the free surface of the liquid with a velocity *v*. This exerts a thrust force in the container in the backward direction. This thrust force is given by



$$F_t = v_r\left(-\frac{dm}{dt}\right)$$

Here, $v_r = v$ (in forward direction)

and $\left(-\frac{dm}{dt}\right) = \rho a v$

as $\left(\frac{dV}{dt}\right)$ = Volume of liquid flowing per second

$= av$

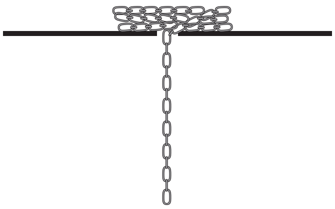
$\therefore$ $\left(-\frac{dm}{dt}\right) = \rho\left(\frac{dV}{dt}\right) = \rho a v$

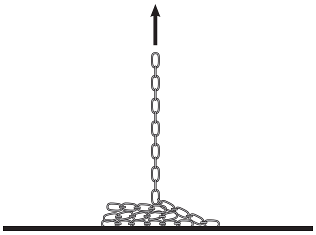
$\therefore$ $F_t = v\,(\rho a v)$

or $F_t = \rho a v^2$ (in backward direction)

Further, we will see in the chapter of fluid mechanics that $v = \sqrt{2gh}$.

3. Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in Fig. (a) or the end of the chain piled on the platform is lifted vertically as in Fig. (b). In both the cases, due to increase of mass in the portion of the chain which is moving with a velocity *v* at certain moment of time a thrust force acts on this part of the chain which is given by


(a)


(b)

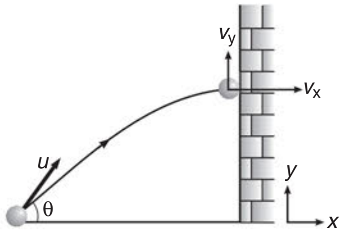
$$F_t = v_r \left(\frac{dm}{dt}\right). \quad \text{Here, } v_r = v \quad \text{and} \quad \frac{dm}{dt} = \lambda v$$

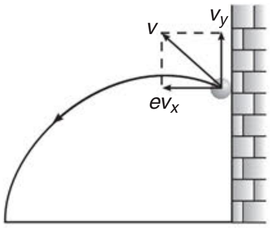
Here, v_r is upwards in case (a) and downwards in case (b). Thus,

$$F_t = \lambda v^2$$

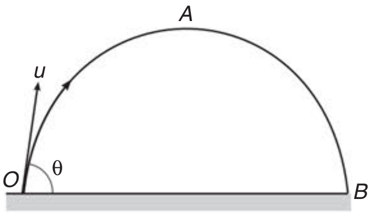
The direction of F_t is upwards in case (a) and downwards in case (b).

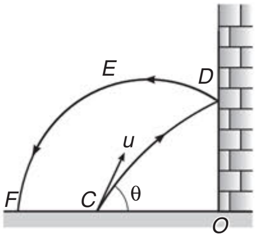
4. Suppose a ball is a projected with speed u at an angle θ with horizontal. It collides at some distance with a wall parallel to y-axis as shown in figure. Let v_x and v_y be the components of its velocity along x and y-directions at the time of impact with wall. Coefficient of restitution between the ball and the wall is e. Component of its velocity along y-direction (common tangent) v_y will remain unchanged while component of its velocity along x-direction (common normal) v_x will become ev_x is opposite direction.





Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and maximum height attained by the ball will remain same as it would had been in the absence of collision with the wall. Thus,





$$t_{OAB} = t_{CD} + t_{DEF} = T = \frac{2u \sin\theta}{g}$$

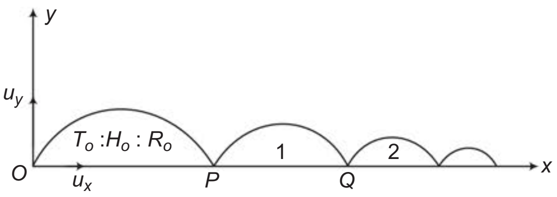
and $$h_A = h_E = \frac{u^2 \sin^2\theta}{2g}$$

Further, $CO + OF \leq$ Range or OB

It collision is elastic, then $CO + OF = \text{Range} = \dfrac{u^2 \sin 2\theta}{g}$

and if it is inelastic, $CO + OF <$ Range

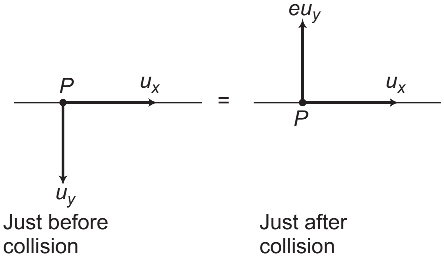
5. In the projectile motion as shown in figure,



$$T=\frac{2u_y}{g} \Rightarrow T \propto u_y$$

$$\Rightarrow \qquad H=\frac{u_y^2}{2g} \quad \Rightarrow \quad H \propto u_y^2$$

$$R=u_xT=u_x\left(\frac{2u_y}{g}\right) \Rightarrow R \propto u_xu_y$$



As shown in above figure, vertical component of velocity just after collision becomes eu_y or e times, while horizontal component remains unchanged. Hence, the next time, T will become e times (as $T \propto u_y$), H will become e^2 times (as $H \propto u_y^2$) and R will also becomes e times (as $R \propto u_xu_y$).

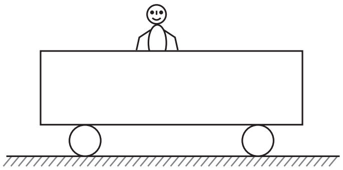
Thus, if T_0, H_0 and R_0 are the initial values then after first collision,

$$T_1=eT_0, \; H_1=e^2H_0 \text{ and } R_1=eR_0$$

Similarly after n-collisions,

$$T_n=e^nT_0, \; H_n=e^{2n}H_0 \text{ and } R_n=e^nR_0$$

6. Thrust force in variable mass system is nothing but a result of law of conservation of linear momentum. Let us take an example.



smooth

A boy is standing over a trolley kept over a smooth surface. If the boy throws a stone towards right, then mass of (trolley + boy) system is decreasing. Relative velocity of stone is towards right. So, a thrust force will act on (trolley + boy) system towards left (in a direction opposite to relative velocity, as mass is decreasing). Due to this thrust force (trolley + boy) system will move towards left.

And this is nothing but law of conservation of linear momentum. Initial momentum of system was zero. Therefore final momentum should also be zero. If momentum of stone is towards right, then momentum of (trolley + boy) system should be towards left (of same magnitude), to make total momentum equal to zero.

7. In perfectly inelastic collision, all bodies stick together and they have a common velocity given by:

$$\mathbf{v}_{\text{common}}=\frac{\text{Total momentum}}{\text{Total mass}}=\frac{\mathbf{p}_{\text{Total}}}{M_{\text{Total}}}$$

and this common velocity is also equal to the velocity of centre of mass of the system.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on law of conversation of linear momentum.*

Concept

(i) If net force on a system is zero, then linear momentum of the system (or centre of mass remains constant).

(ii) Normally the ground is given smooth. So, the net force on a system in horizontal direction will be zero and momentum of the system in horizontal direction will remain constant.

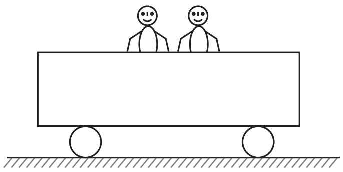
$\therefore$ Initial momentum = Final momentum

or $$p_i = p_f \qquad \text{(in horizontal direction)}$$

(iii) If all individual bodies/blocks of the system are initially at rest then, p_i will be zero. Therefore, total momentum of the system at any instant or p_f is also zero.

(iv) Since, we are using only one conservation law i.e. law of conservation of linear momentum, so we have only one equation and only one unknown.

Example 1 *A trolley of mass M is at rest over a smooth horizontal surface as shown in figure. Two boys each of mass 'm' are standing over the trolley. They jump from the trolley (towards right) with relative velocity v_r [relative to velocity of trolley just after jumping]*

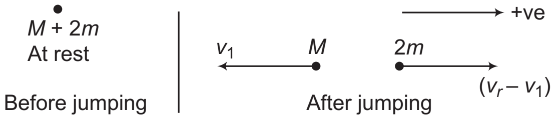


(a) together

(b) one after the other.

Find velocity of trolley in both cases.

Solution (a) Let velocity of trolley just after jumping is v_1 (towards left). Relative velocity of boys towards right is v_r. Therefore, there absolute velocity is $v_r - v_1$, towards right.



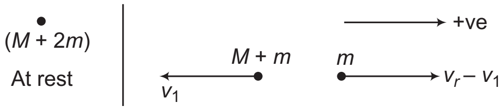
Net force on the system in horizontal direction is zero. Hence, linear momentum of the system in horizontal direction is zero.

or $$p_i = p_f$$

$\Rightarrow$ $$0 = 2m\,(v_r - v_1) - Mv_1$$

$\Rightarrow$ $$v_1 = \frac{2m\,v_r}{2m + M} \qquad \textbf{Ans.}$$

(b) Let v_1 be the velocity of trolley after the first boy jumps. Then,

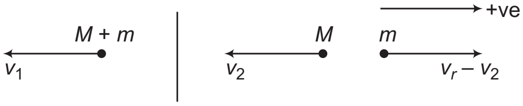


$$p_i = p_f$$

$$\Rightarrow \quad 0 = m\,(v_r - v_1) - (M + m)\,v_1$$

$$\Rightarrow \quad v_1 = \frac{mv_r}{M + 2m} \qquad \ldots\text{(i)}$$

Now, the second boy jumps from the moving trolley and let v_2 be the velocity of trolley after the second boy also jumps. Then,



$$p_i = p_f$$

$$\Rightarrow \quad -(M + m)v_1 = m\,(v_r - v_2) - Mv_2$$

$$\therefore \quad v_2 = \frac{mv_r + (M + m)\,v_1}{(M + m)}$$

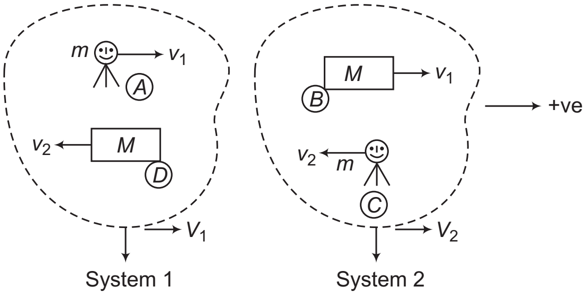
$$= \left(\frac{m}{M + m}\right) v_r + v_1$$

Substituting the value of v_1 from Eq. (i), we have

$$v_2 = mv_r\left[\frac{1}{M + m} + \frac{1}{M + 2m}\right]$$ **Ans.**

Example 2 *Two toy trains each of mass 'M' are moving in opposite directions with velocities v_1 and v_2 over two smooth rails. Two stuntmen of mass 'm' each are also moving with the trains (at rest w.r.t. trains). When trains are opposite to each other the stuntmen interchange their positions, then find the final velocities of the trains.*

Solution When the stuntmen are in air (after jumping) they also have horizontal velocities v_1 and v_2 in opposite directions. Initially (A) and (B) were together. Similarly (C) and (D) were together.



Now, (A) will fall over (D). So this is one system and suppose velocity of this system is V_1 (towards right or in positive direction) after jumping. Similarly, (C) will fall over (B) after jumping.

Let V_2 is the velocity of this system towards right after jumping.

Applying conservation of linear momentum.

System 1

$$p_i = p_f$$

$\Rightarrow$ $$m\,v_1 - M\,v_2 = (M+m)\,V_1$$

$\Rightarrow$ $$V_1 = \frac{m\,v_1 - M\,v_2}{M+m}$$ **Ans.**

System 2

$$p_i = p_f$$

$\Rightarrow$ $$M\,v_1 - m\,v_2 = (M+m)\,V_2$$

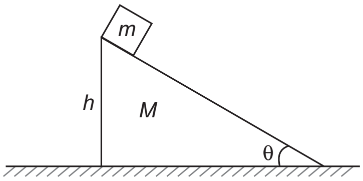
$\Rightarrow$ $$V_2 = \frac{Mv_1 - mv_2}{M+m}$$ **Ans.**

Type 2. *Based on conservation on linear momentum and mechanical energy.*

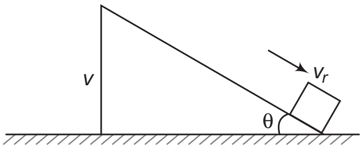
Concept

In these type of problems, all surfaces are given smooth. So, linear momentum in horizontal direction and mechanical energy of the system remains conserved. Since, we are applying two conservation laws. Therefore, number of equations are two. Hence, number of unknowns are also two.

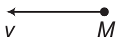
Example 3 *All surfaces shown in figure are smooth. Find velocity of wedge (of mass M) when the block (of mass m) reaches the bottom of the wedge.*

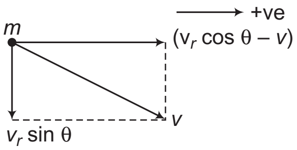


Solution When the block reaches to the bottom of the wedge, their velocities are as shown in figure. Here v is the absolute velocity (with respect to ground), but v_r is the relative velocity (relative to wedge).



Their absolute velocity components are as shown below:





$$v = \sqrt{(v_r \sin\theta)^2 + (v_r \cos\theta - v)^2}$$

Linear momentum in horizontal direction is conserved.

$$\therefore \qquad p_i = p_f$$

$$\text{or} \qquad 0 = m\,(v_r \cos\theta - v) - Mv \qquad \text{...(i)}$$

Mechanical energy is also conserved.

$$\therefore \qquad E_i = E_f$$

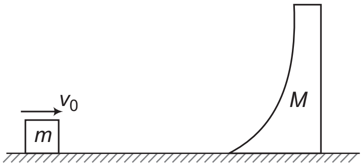
$$\text{or} \qquad mgh = \frac{1}{2} Mv^2 + \frac{1}{2} mv^2$$

$$\text{or} \qquad mgh = \frac{1}{2} Mv^2 + \frac{1}{2} m \;\; [(v_r \sin\theta)^2 + (v_r \cos\theta - v)^2] \qquad \text{...(ii)}$$

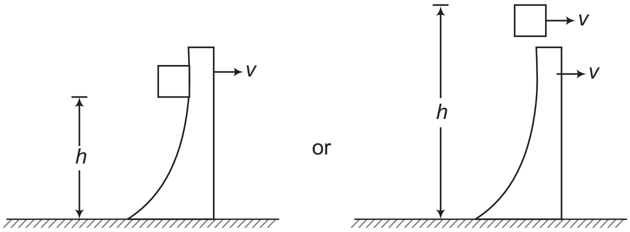
We have two equations and two unknowns v and v_r. Solving these equations, we can find the value of v.

Exercise: Solve these two equations to find the value of v.

Example 4 *All surfaces shown in figure are smooth. Wedge of mass 'M' is free to move. Block of mass 'm' is given a horizontal velocity v_0 as shown. Find the maximum height 'h' attained by 'm' (over the wedge or outside it).*



Solution At maximum height vertical component of velocity of 'm' will be zero. It will have only horizontal component of velocity and this is equal to the horizontal component of 'M' also (think why ?). So at the highest point figure is like this:



Applying law of conservation of linear momentum in horizontal direction,

$$p_i = p_f$$

$$\Rightarrow \qquad mv_0 = (M + m)\,v \qquad \text{...(i)}$$

Now, applying law of conservation of mechanical energy,

$$E_i = E_f$$

$$\therefore \qquad \frac{1}{2} mv_0^2 = \frac{1}{2} (M + m)\, v^2 + mgh \qquad \text{...(ii)}$$

We have two unknowns v and h.

So, we can find the value of 'h' by solving these two equations.

Exercise : Solve these two equations to find the value of 'h'.

Type 3. *Based on following four equations.*

$$\Sigma F_R = \Sigma F_L,\ \Sigma m_R a_R = \Sigma m_L a_L,\ \Sigma m_R v_R = \Sigma m_L v_L,\ \Sigma m_R x_R = \Sigma m_L x_L$$

In these equations, R stands for right hand side and L stands for left hand side. x is the absolute displacement with respect to ground.

Concept

If net force on a system in a particular direction is zero (normally in horizontal direction). And this can done by giving the ground smooth. Initially, the system is at rest. So, in this case individual bodies can move towards right or towards left, but centre of mass will remain stationary. Further, net force in horizontal direction is zero. Hence, total force towards right is equal to the total force towards left or,

$$\Sigma F_R = \Sigma F_L \qquad \text{...(i)}$$

or

$$\Sigma m_R a_R = \Sigma m_L a_L \qquad \text{...(ii)}$$

Now, integrating 'a' we will get 'v' and by further integrating 'v', we will get 'x'.

$\therefore$

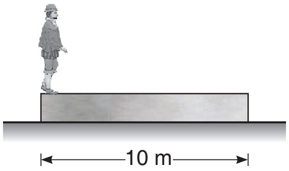
$$\Sigma m_R v_R = \Sigma m_L v_L \qquad \text{...(iii)}$$

and

$$\Sigma m_R x_R = \Sigma m_L x_L \qquad \text{...(iv)}$$

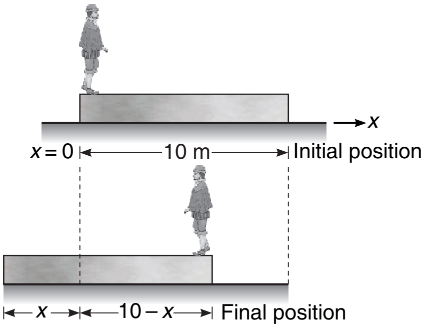
Note *If the system is initially not at rest but net force is zero, then Eqs. (i) and (ii) are still applicable but not the Eqs. (iii) and (iv). Think why ?*

Example 5 *A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.*



Solution Here, the system is man + plank. Net force on this system in horizontal direction is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in horizontal direction.

Let x be the displacement of the Plank. Assuming the origin, i.e. $x = 0$ at the position shown in figure.



As we said earlier also, the centre of mass will not move in horizontal direction (x-axis). Therefore, for centre of mass to remain stationary,

$$x_i = x_f$$

$$\frac{(60)(0) + 20\left(\frac{10}{2}\right)}{60 + 20} = \frac{(60)(10 - x) + 20\left(\frac{10}{2} - x\right)}{60 + 20}$$

or $$\frac{5}{4} = \frac{6\,(10 - x) + 2\left(\frac{10}{2} - x\right)}{8} = \frac{60 - 6x + 10 - 2x}{8}$$

or $$5 = 30 - 3x + 5 - x$$

or $$4x = 30$$

or $$x = \frac{30}{4}\text{ m} \quad \text{or} \quad x = 7.5\text{ m}$$

Note *The centre of mass of the plank lies at its centre.*

Alternate Method

x_L = displacement of plank towards left = x

m_L = mass of plank displaced towards left = 20 kg

x_R = displacement of man relative to ground towards right = $10 - x$

and m_R = mass of man displaced towards right = 60 kg

Applying $x_R m_R = x_L m_L$, we get

$$(10 - x)(60) = 20x$$

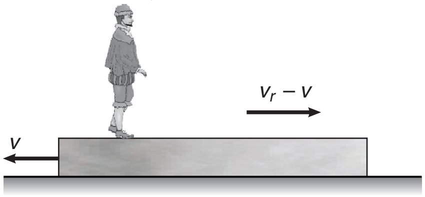
or $$x = 30 - 3x$$

or $$4x = 30$$

$\therefore$ $$x = \frac{30}{4} = 7.5\text{ m}$$

Example 6 *A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.*

Solution Absolute velocity of man = $v_r - v$ where v = recoil velocity of platform.



Taking the platform and the man as a system, net external force on the system in horizontal direction is zero. The linear momentum of the system remains constant. Initially both the man and the platform were at rest.

Hence, $$0 = m_1(v_r - v) - m_2 v$$

$\therefore$ $$v = \frac{m_1 v_r}{m_1 + m_2}$$

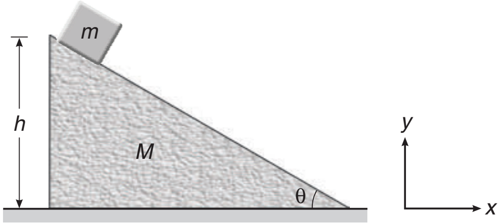
Alternate Method Using the equation,

$$\Sigma m_R v_R = \Sigma m_L v_L$$

we have, $$m_1(v_r - v) = m_2 v$$

$\therefore$ $$v = \frac{m_1 v_r}{m_1 + m_2}$$

Example 7 *A block of mass m is released from the top of a wedge of mass M as shown in figure. Find the displacement of wedge on the horizontal ground when the block reaches the bottom of the wedge. Neglect friction everywhere.*



Solution Here, the system is wedge + block. Net force on the system in horizontal direction (x-direction) is zero, therefore, the centre of mass of the system will not move in x-direction so we can apply,

$$x_R m_R = x_L m_L \qquad \text{...(i)}$$

Let x be the displacement of wedge. Then,

x_L = displacement of wedge towards left = x

m_L = mass of wedge moving towards left = M

x_R = displacement of block with respect to ground towards right = $h \cot \theta - x$

and m_R = mass of block moving towards right = m

Substituting in Eq. (i), we get

$$m(h \cot \theta - x) = xM$$

$$\therefore \qquad x = \frac{mh \cot \theta}{M + m} \qquad \textbf{Ans.}$$

Type 4. *Explosion of a bomb or a projectile.*

Concept

If a bomb or a projectile explodes in two or more than two parts, then it explodes due to internal forces. Therefore, net force or net external force is zero. Hence, linear momentum of the system can be conserved just before and just after explosion. By this momentum conservation equation, we can find the velocity of some unknown part. If explosion takes place in air, then during the explosion, the external force due to gravity (= weight) can be neglected, as the time of explosion is very short. So, impulse of this force is negligible and impulse is change in linear momentum.Hence, change in linear momentum is also negligible.

Example 8 *A bomb of mass '5m' at rest explodes into three parts of masses 2m, 2m and m. After explosion, the equal parts move at right angles with speed v each. Find speed of the third part and total energy released during explosion.*

Solution Let the two equal parts move along positive x and positive y directions and suppose the velocity of third part is $\mathbf{V}$. From law of conservation of linear momentum,

we have, $$\mathbf{p}_i = \mathbf{p}_f$$

$$\Rightarrow \qquad 0 = 2m(v\hat{\mathbf{i}}) + 2m(v\hat{\mathbf{j}}) + m\mathbf{V}$$

Solving this equation we have, $\mathbf{V} = -2v\hat{\mathbf{i}} - 2v\hat{\mathbf{j}}$

$\therefore$ Speed of this particle

$$= |\mathbf{V}| \text{ or } V$$
$$= \sqrt{(-2v)^2 + (-2v)^2}$$
$$= 2\sqrt{2}v$$

Ans.

Energy released during explosion,

$$= \text{kinetic energy of all three parts}$$
$$= \frac{1}{2}(2m)v^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}(m)(2\sqrt{2}v)^2$$
$$= 6mv^2$$

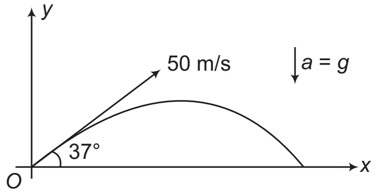
Ans.

Example 9 *A projectile of mass 3 kg is projected with velocity 50 m/s at 37° from horizontal. After 2 s, explosion takes place and the projectile breaks into two parts of masses 1 kg and 2 kg. The first part comes to rest just after explosion. Find,*

(a) the velocity of second part just after explosion.

(b) maximum height attained by this part. Take $g = 10\ m/s^2$

Solution



(a)
$$\mathbf{u} = (50\cos 37°)\,\hat{\mathbf{i}} + (50\sin 37°)\,\hat{\mathbf{j}}$$
$$= (40\,\hat{\mathbf{i}} + 30\,\hat{\mathbf{j}})\text{ m/s}$$
$$\mathbf{a} = (-10\hat{\mathbf{j}})\text{ m/s}^2 = \text{constant}$$

After $t = 2$ s
$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}}) + (-10\hat{\mathbf{j}})(2) = (40\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\text{ m/s}$$
$$\mathbf{S} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}})(2) + \frac{1}{2}(-10\hat{\mathbf{j}})(2)^2 = (80\hat{\mathbf{i}} + 40\hat{\mathbf{j}})\text{m}$$
$$s_y = 40\text{ m}$$

Hence, the explosion takes place at a height of 40 m and the velocity just before explosion is $(40\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s.

Velocity of first part just after explosion is zero and let velocity of second part just after explosion is $\mathbf{V}$, then from conservation of linear momentum,

$$\mathbf{p}_i = \mathbf{p}_f$$
$$\Rightarrow \quad 3(40\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) = (1)(0) + 2\mathbf{V}$$
$$\Rightarrow \quad \mathbf{V} = (60\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}})\text{ m/s}$$

Ans.

(b) Vertical component of velocity of second part just after collision is 15 m/s and explosion has taken place at a height of 40 m. Therefore total height of this part from ground.

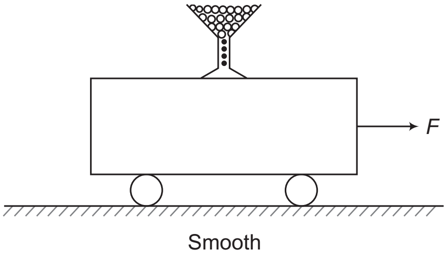
$$= 40 + \frac{v_y^2}{2g} = 40 + \frac{(15)^2}{2 \times 10} = 51.25\,\text{m}$$ **Ans.**

Type 5. *Based on variable mass system.*

Concept

In case of a variable mass system, a thrust force of magnitude $\left| v_r \times \frac{dm}{dt} \right|$ has to be applied on the system, whose mass is changing. Direction of this force is in the direction of $\mathbf{v}_r$ if mass is increasing and in the opposite direction of $\mathbf{v}_r$, if mass is decreasing, Further, $\mathbf{v}_r$ is zero if a mass is just dropped from a moving body. Because the dropped body has the same velocity as of the moving body at the time of dropping. So, no thrust force will act in this case.

Example 10 *A constant force F is applied on a trolley of initial mass m_0 kept over a smooth surface. Sand is poured gently over the trolley at a constant rate of (μ) kg/s. After time t, find*



(a) mass of the trolley (with sand)

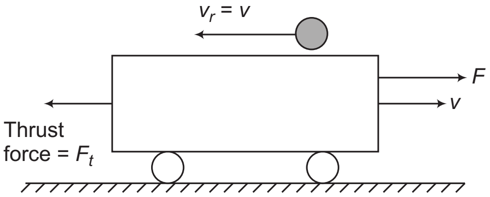
(b) net force on the trolley

(c) velocity of trolley

Solution (a) Mass of the trolley after time t is

$$m = \text{mass of trolley} + (\text{mass of sand poured per second})\ (\text{time})$$
$$= m_0 + \mu t$$

(b) Let v is the velocity of trolley at time t. Sand is poured gently. So, velocity of sand is zero or, relative velocity of sand is v in the opposite direction of velocity of trolley. Mass of trolley is increasing. So, thrust force on the trolley is in the direction of relative velocity or in the opposite direction of motion of trolley.



Net force on trolley $= F - F_t$

$$= F - v_r \frac{dm}{dt} = F - (v)(\mu)$$

or $$F_{\text{net}} = F - \mu v$$ **Ans.**

Note *In the above expression of F_{net}, velocity v is a function of time which has been asked in the next part.*

(c) $F_{\text{net}} = F - v\mu$

$\therefore$ $$ma = F - \mu v$$

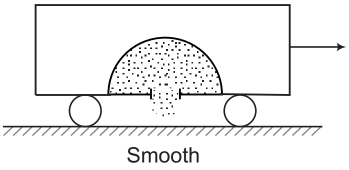
$\Rightarrow$ $$(m_0 + \mu t)\left(\frac{dv}{dt}\right) = F - \mu v$$

$\Rightarrow$ $$\int_0^v \frac{dv}{F - \mu v} = \int_0^t \frac{dt}{m_0 + \mu t}$$

Solving this equation we get

$$v = \frac{Ft}{m_0 + \mu t}$$ **Ans.**

Example 11 *A trolley of initial mass m_0 is kept over a smooth surface as shown in figure. A constant force F is applied on it. Sand kept inside the trolley drains out from its floor at a constant rate of (μ) kg/s. After time t find:*



(a) total mass of trolley and sand.

(b) net force on the trolley.

(c) velocity of trolley.

Solution (a) Total mass of trolley and sand

$$m = \text{initial mass} - \text{mass of sand drained out}$$

or $$m = m_0 - \mu t$$

(b) Let v is the velocity of trolley at time t. Then, velocity of sand drained out is also v or relative velocity is zero. Hence, no thrust force will act in this case. Therefore,

$$F_{\text{net}} = F$$

(c) $F_{\text{net}} = F$

or $$ma = F \quad \text{or} \quad m\frac{dv}{dt} = F$$

or $$(m_0 - \mu t)\frac{dv}{dt} = F$$

$\therefore$ $$\int_0^v dv = \int_0^t \frac{F dt}{m_0 - \mu t}$$

Solving this equation we get,

$$v = \frac{1}{\mu}\log_e\left(\frac{m_0}{m_0 - \mu t}\right)$$ **Ans.**

Type 6. *Problems based on linear impulse (J) and coefficient of restitution (e).*

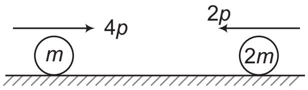
Concept

(i) During collision, equal and opposite impulses act on the two colliding bodies along common normal directions. This impulse changes the linear momentum of an individual body, but total momentum of the system remains constant, as the total linear impulse is zero. Further, this linear impulse is equal to change in linear momentum.

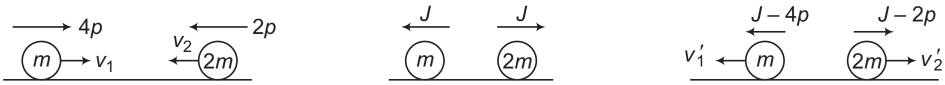
(ii) Coefficient of restitution between two colliding bodies is defined along common normal direction and this is given by:

$$e=\frac{\text{relative speed of separation}}{\text{relative speed of approach}}$$

Example 12 *Two balls of masses m and 2m and momenta 4p and 2p (in the directions shown) collide as shown in figure. During collision, the value of linear impulse between them is J. In terms of J and p find coefficient of restitution 'e'. Under what condition collision is elastic. Also find the condition of perfectly inelastic collision.*



Solution Directions of linear impulses on the two colliding bodies at the time of collision are shown in Fig. (b).



Before collision
(a)

At the time to collision
(b)

After collision
(c)

Linear impulse is equal to the change in linear momentum. Hence, momenta of the balls after collision are shown in Fig. (c).

Let v_1 and v_2 are their velocities before collision [in the directions shown in figure (a)] and v_1' and v_2' are the velocities after collision as shown in figure (c).

$$v=\frac{\text{linear momentum}}{\text{mass}}$$

$$e=\frac{\text{relative speed of separation}}{\text{relative speed of approach}}$$

or
$$e=\frac{v_1'+v_2'}{v_1+v_2}=\frac{\dfrac{J-4p}{m}+\dfrac{J-2p}{2m}}{\dfrac{4p}{m}+\dfrac{2p}{2m}}$$

or
$$e=\frac{3J-10p}{10p}=\frac{3J}{10p}-1 \qquad \textbf{Ans.}$$

For elastic collision,

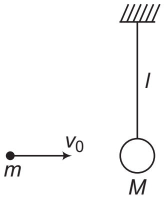
$$e=1 \quad\Rightarrow\quad \frac{3J}{10p}=2 \quad\text{or}\quad J=\frac{20}{3}p \qquad \textbf{Ans.}$$

For perfectly inelastic collision,

$$e=0 \quad\Rightarrow\quad \frac{3J}{10p}=1 \quad\text{or}\quad J=\frac{10}{3}p \qquad \textbf{Ans.}$$

Type 7. *Based on conservation of linear momentum and vertical circular motion.*

Concept



A ball of mass M is suspended from a massless string of length l as shown in figure. A bullet of mass m moving with velocity v_0 collides with the ball and sticks with the ball.

Now, velocity of the combined mass $(M+m)$ just after collision at the bottommost point (say it's u) can by obtained from law of conservation of linear momentum. After finding the value of u we can use the theory of vertical circular motion. For example, if $u \geq \sqrt{5gl}$, then the combined mass will complete the vertical circular motion.

If $\sqrt{2gl} < u < \sqrt{5gl}$, string will slack in upper half of the circle and if $0 < u \leq \sqrt{2gl}$, the combined mass will oscillate in lower half of the circle.

Example 13 *In the situation discussed above, find*

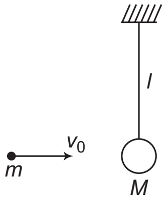
(a) velocity of combined mass just after collision at the bottommost point (or u).

(b) loss of mechanical energy during collision.

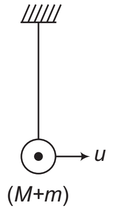
(c) minimum value of v_0 so that the combined mass completes the vertical circular motion.

Solution (a) Applying conservation of linear momentum, just before and just after collision.

$$p_i = p_f \quad \Rightarrow \quad mv_0 = (M+m)\,u \quad \Rightarrow \quad u = \frac{mv_0}{M+m}$$ **Ans.**



Just before collision



Just after collision

(b) Loss of mechanical energy during collision,

$$= E_i - E_f \qquad (E = \text{mechanical energy})$$

$$= \frac{1}{2} mv_0 - \frac{1}{2}(M+m)\,u^2$$

$$= \frac{1}{2} mv_0^2 - \frac{1}{2}(M+m)\left[\frac{mv_0}{M+m}\right]^2$$

$$= \frac{1}{2}\frac{Mm\,v_0^2}{(M+m)}$$ **Ans.**

(c) For completing the vertical circular motion, velocity at bottommost point, $u \geq \sqrt{5\,gl}$

$$\therefore \qquad \frac{mv_0}{M+m} \geq \sqrt{5\,gl} \quad \Rightarrow \quad \therefore \quad v_0 \geq \frac{M+m}{m}\sqrt{5\,gl}$$

or

$$(v_0)_{\min} = \frac{M+m}{m}\sqrt{5\,gl}$$ **Ans.**

Example 14 *A pendulum bob of mass* 10^{-2} *kg is raised to a height* 5×10^{-2} *m and then released. At the bottom of its swing, it picks up a mass* 10^{-3} *kg. To what height will the combined mass rise ?*

Solution Velocity of pendulum bob at bottom most point

$$v_1=\sqrt{2gh}=\sqrt{2\times10\times5\times10^{-2}}=1\text{ m/s}$$

When the bob picks up a mass 10^{-3} kg at the bottom, then by conservation of linear momentum the velocity of combined mass is given by

$$m_1v_1+m_2v_2=(m_1+m_2)\,v$$

$$10^{-2}\times1+10^{-3}\times0=(10^{-2}+10^{-3})\,v$$

or

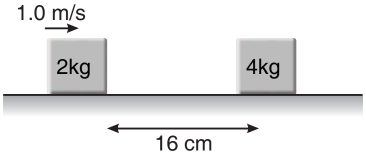
$$v=\frac{10^{-2}}{1.1\times10^{-2}}=\frac{10}{11}\text{ m/s}$$

Now,

$$h=\frac{v^2}{2g}=\frac{(10/11)^2}{2\times10}=4.1\times10^{-2}\text{ m}$$ **Ans.**

Miscellaneous Examples

Example 15 *The friction coefficient between the horizontal surface and each of the block shown in the figure is 0.2. The collision between the blocks is perfectly elastic. Find the separation between them when they come to rest.*
(Take $g=10\text{ m/s}^2$*).*



Solution Retardation, $a=\frac{\mu mg}{m}=\mu g=0.2\times10=2\text{ m/s}^2$

Velocity of first block before collision,

$$v_1^2=1^2-2\,(2)\times0.16=1-0.64$$

$$v_1=0.6\text{ m/s}$$

By conservation of momentum, $2\times0.6=2v_1'+4v_2'$

also $v_2'-v_1'=v_1$ for elastic collision

It gives $v_2'=0.4\text{ m/s},\quad v_1'=-0.2\text{ m/s}$

Now distance moved after collision

$$s_1=\frac{(0.4)^2}{2\times2}$$

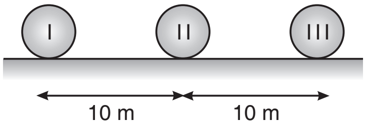
and

$$s_2=\frac{(0.2)^2}{2\times2}$$

∴

$$s=s_1+s_2=0.05\text{m}=5\text{ cm}$$ **Ans.**

Example 16 *Three identical balls, ball I, ball II and ball III are placed on a smooth floor on a straight line at the separation of* 10 *m between balls as shown in figure. Initially balls are stationary. Ball I is given velocity of* 10 *m/s towards ball II, collision between balls I and II is inelastic with coefficient of restitution* 0.5 *but collision between balls II and III is perfectly elastic. What is the time interval between two consecutive collisions between ball I and II*?



Solution Let velocity of I ball and II ball after collision be v_1 and v_2

$$v_2 - v_1 = 0.5 \times 10 \qquad \dots \text{(i)}$$

$$mv_2 + mv_1 = m \times 10 \qquad \dots \text{(ii)}$$

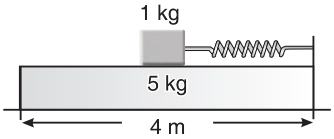
$$\Rightarrow \qquad v_2 + v_1 = 10$$

Solving Eqs. (i) and (ii), we get $v_1 = 2.5$ m/s, $v_2 = 7.5$ m/s

Ball II after moving 10 m collides with ball III elastically and stops. But ball I moves towards ball II. Time taken between two consecutive collisions

$$\frac{10}{7.5} - \frac{10 - 10 \times \frac{2.5}{7.5}}{2.5} = 4\,\text{s} \qquad \textbf{Ans.}$$

Example 17 *A plank of mass* 5 *kg is placed on a frictionless horizontal plane. Further a block of mass* 1 *kg is placed over the plank. A massless spring of natural length* 2 *m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. They system is now released from the rest. What is the velocity of the plank when block leaves the plank* ? *(The stiffness constant of spring is* 100 *N/m)*



Solution Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$ [M = mass of the plank, m = mass of the block]

$$\Rightarrow \qquad 100 = u^2 + 5v^2 \qquad \dots \text{(i)}$$

By conservation of momentum

$$mu + Mv = 0 \quad \Rightarrow \quad u = -5v \qquad \dots \text{(ii)}$$

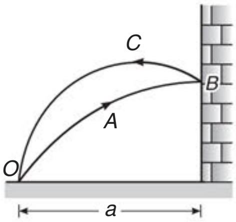
Solving Eqs. (i) and (ii) $\quad 30v^2 = 100 \quad \Rightarrow \quad v = \sqrt{\frac{10}{3}}$ m/s

From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls, velocity of plank = $\sqrt{\frac{10}{3}}$ m/s. **Ans.**

Example 18 *A ball is projected from the ground with speed u at an angle α with horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.*

Solution As we have discussed in the theory, the horizontal component of the velocity of ball during the path OAB is $u\cos\alpha$ while in its return journey BCO it is $eu\cos\alpha$. The time of flight T also remains unchanged. Hence,



$$T = t_{OAB} + t_{BCO}$$

or
$$\frac{2u\sin\alpha}{g} = \frac{a}{u\cos\alpha} + \frac{a}{eu\cos\alpha}$$

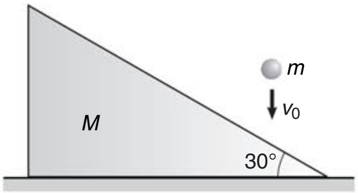
or
$$\frac{a}{eu\cos\alpha} = \frac{2u\sin\alpha}{g} - \frac{a}{u\cos\alpha}$$

or
$$\frac{a}{eu\cos\alpha} = \frac{2u^2\sin\alpha\cos\alpha - ag}{gu\cos\alpha}$$

$$e = \frac{ag}{2u^2\sin\alpha\cos\alpha - ag}$$

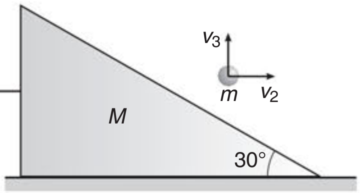
or
$$e = \frac{1}{\left(\frac{u^2\sin 2\alpha}{ag} - 1\right)}$$

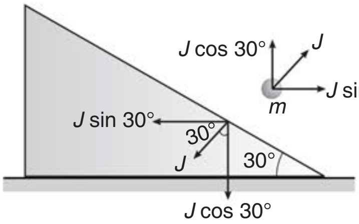
Example 19 *A ball of mass $m = 1$ kg falling vertically with a velocity $v_0 = 2$ m/s strikes a wedge of mass $M = 2$ kg kept on a smooth, horizontal surface as shown in figure. If impulse between ball and wedge during collision is J. Then make two equations which relate J with velocity components of wedge and ball. Also find impulse on wedge from ground during impact.*



Solution Given $M = 2$ kg and $m = 1$ kg

Let, J be the impulse between ball and wedge during collision and v_1, v_2 and v_3 be the components of velocity of the wedge and the ball in horizontal and vertical directions respectively.





Applying impulse = change in momentum

we get
$$J\sin 30° = Mv_1 = mv_2$$

or
$$\frac{J}{2} = 2v_1 = v_2 \qquad \text{...(i)}$$

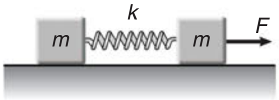
$$J \cos 30° = m\,(v_3 + v_0)$$

or $$\frac{\sqrt{3}}{2} J = (v_3 + 2) \qquad \text{...(ii)}$$

So these are two equations relating J and velocity components of wedge and ball.

Further, net vertical impulse on wedge should be zero. Therefore, impulse on wedge from ground is $J \sin 30°$ or $\frac{J}{2}$ in upward direction.

Example 20 *Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure. (a) Find the displacement of the centre of mass at time t. (b) If the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.*



Solution (a) The acceleration of the centre of mass is

$$a_{\text{COM}} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2} a_{\text{COM}}\, t^2 = \frac{Ft^2}{4m} \qquad \textbf{Ans.}$$

(b) Suppose the displacement of the first block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m} \quad \text{or} \quad \frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or $$x_1 + x_2 = \frac{Ft^2}{2m} \qquad \text{...(i)}$$

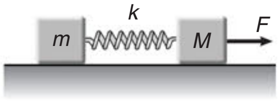
Further, the extension of the spring is $x_1 - x_2$. Therefore,

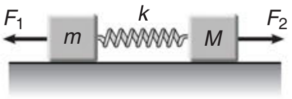
$$x_1 - x_2 = x_0$$

From Eqs. (i) and (ii),

$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right) \quad \text{and} \quad x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right).$$

Example 21 *A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.*





Solution The centre of mass of the system (two blocks + spring) moves with an acceleration $a = \frac{F}{m + M}$. Let us solve the problem in a frame of reference fixed to the centre of mass of the system. As this frame is accelerated with respect to the ground, we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is

$$F_1 = ma = \frac{mF}{m + M} \qquad \text{(towards left)}$$

and the net external force on M is

$$F_2 = F - Ma = F - \frac{MF}{m + M} = \frac{mF}{m + M} \qquad \text{(towards right)}$$

As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The extension of the spring will be maximum at this instant. Suppose, the left block is displaced through a distance x_1 and the right block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are

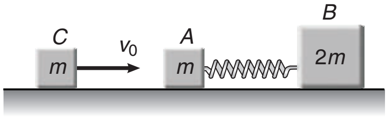
$$W = F_1x_1 + F_2x_2 = \frac{mF}{m + M}(x_1 + x_2)$$

This should be equal to the increase in the potential energy of the spring, as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m + M}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2 \quad \text{or} \quad x_1 + x_2 = \frac{2mF}{k(m + M)} \qquad \textbf{Ans.}$$

This is the maximum extension of the spring.

Example 22 *Two blocks A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third block C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0. Determine (i) the common velocity of A and B at time t_0, and (ii) the spring constant.*



Solution Initially, the blocks A and B are at rest and C is moving with velocity v_0 to the right. As masses of C and A are same and the collision is elastic the body C transfers its whole momentum mv_0 to body A and as a result the body C stops and A starts moving with velocity v_0 to the right. At this instant the spring is uncompressed and the body B is still at rest.

The momentum of the system at this instant $= mv_0$

Now, the spring is compressed and the body B comes in motion. After time t_0, the compression of the spring is x_0 and common velocity of A and B is v (say).

As external force on the system is zero, the law of conservation of linear momentum gives

$$mv_0 = mv + (2m)v \quad \text{or} \quad v = \frac{v_0}{3} \qquad \textbf{Ans.}$$

The law of conservation of energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2$$

or
$$\frac{1}{2}mv_0^2 = \frac{3}{2}mv^2 + \frac{1}{2}kx_0^2$$

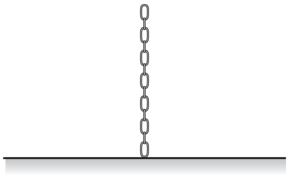
$$\frac{1}{2}mv_0^2 = \frac{3}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

∴
$$\frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 - \frac{1}{6}mv_0^2$$

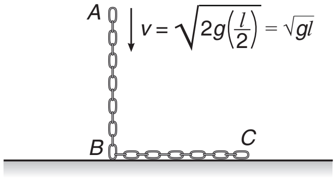
or
$$\frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$$

$$k = \frac{2}{3}\frac{mv_0^2}{x_0^2}$$ **Ans.**

Example 23 *A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.*



Solution Force exerted by the chain on the table consists of two parts:



1. Weight of the portion BC of the chain lying on the table,
$$W = \frac{mg}{2} \quad \text{(downwards)}$$
2. Thrust force $F_t = \lambda v^2$

Here, λ = mass per unit length of chain $= \frac{m}{l}$

$$v^2 = (\sqrt{gl})^2 = gl$$

∴
$$F_t = \left(\frac{m}{l}\right)(gl) = mg \qquad \text{(downwards)}$$

∴ Net force exerted by the chain on the table is

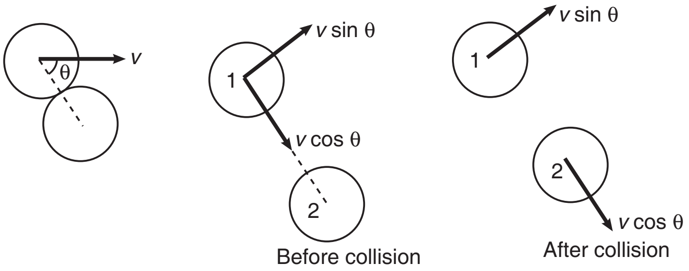
$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2}mg \qquad \text{(downwards)}$$

So, from Newton's third law the force exerted by the table on the chain will be $\frac{3}{2}mg$ (vertically upwards).

Note *Here, the thrust force (F_t) applied by the chain on the table will be vertically downwards, as $F_t = v_r\left(\frac{dm}{dt}\right)$ and in this expression v_r is downwards plus $\frac{dm}{dt}$ is positive. So, F_t will be downwards.*

Example 24 *A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.*

Solution In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v\cos\theta$ becomes zero after collision, while that of 2 becomes $v\cos\theta$. While the components along common tangent direction of both the balls remain unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given below.



Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v\sin\theta$	$v\sin\theta$	$v\cos\theta$	0
2	0	0	0	$v\cos\theta$

From the above table and figure, we see that both the balls move at right angles after collision with velocities $v\sin\theta$ and $v\cos\theta$.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

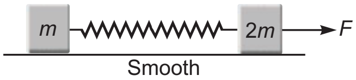
(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : Centre of mass of a rigid body always lies inside the body.
Reason : Centre of mass and centre of gravity coincide if gravity is uniform.

2. Assertion : A constant force F is applied on two blocks and one spring system as shown in figure. Velocity of centre of mass increases linearly with time.



Reason : Acceleration of centre of mass is constant.

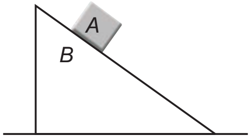
3. Assertion : To conserve linear momentum of a system, no force should act on the system.
Reason : If net force on a system is zero, its linear momentum should remain constant.

4. Assertion : A rocket moves forward by pushing the surrounding air backwards.
Reason : It derives the necessary thrust to move forward according to Newton's third law of motion.

5. Assertion : Internal forces cannot change linear momentum.
Reason : Internal forces can change the kinetic energy of a system.

6. Assertion : In case of bullet fired from gun, the ratio of kinetic energy of gun and bullet is equal to ratio of mass of bullet and gun.
Reason : Kinetic energy $\propto \frac{1}{\text{mass}}$; if momentum is constant.

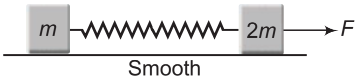
7. Assertion : All surfaces shown in figure are smooth. System is released from rest. Momentum of system in horizontal direction is constant but overall momentum is not constant.



Reason : A net vertically upward force is acting on the system.

8. Assertion : During head on collision between two bodies let Δp_1 is change in momentum of first body and Δp_2 the change in momentum of the other body, then $\Delta p_1 = \Delta p_2$.
Reason : Total momentum of the system should remain constant.

9. Assertion : In the system shown in figure spring is first stretched then left to oscillate. At some instant kinetic energy of mass m is K. At the same instant kinetic energy of mass $2m$ should be $\frac{K}{2}$.



Reason : Their linear momenta are equal and opposite and $K = \frac{p^2}{2m}$ or $K \propto \frac{1}{m}$.

10. Assertion : Energy can not be given to a system without giving it momentum.

Reason : If kinetic energy is given to a body it means it has acquired momentum.

11. Assertion : The centre mass of an electron and proton, when released moves faster towards proton.

Reason : Proton is heavier than electron.

12. Assertion : The relative velocity of the two particles in head-on elastic collision is unchanged both in magnitude and direction.

Reason : The relative velocity is unchanged in magnitude but gets reversed in direction.

13. Assertion : An object of mass m_1 and another of mass $m_2 (m_2 > m_1)$ are released from certain distance. The objects move towards each other under the gravitational force between them. In this motion, centre of mass of their system will continuously move towards the heavier mass m_2.

Reason : In a system of a heavier and a lighter mass, centre of mass lies closer to the heavier mass.

14. Assertion : A given force applied in turn to a number of different masses may cause the same rate of change in momentum in each but not the same acceleration to all.

Reason : $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ and $\mathbf{a} = \frac{\mathbf{F}}{m}$

15. Assertion : In an elastic collision between two bodies, the relative speed of the bodies after collision is equal to the relative speed before the collision.

Reason : In an elastic collision, the linear momentum of the system is conserved.

Objective Questions

Single Correct Option

1. A ball is dropped from a height of 10 m. Ball is embedded in sand through 1 m and stops.
(a) only momentum remains conserved
(b) only kinetic energy remains conserved
(c) both momentum and kinetic energy are conserved
(d) neither kinetic energy nor momentum is conserved

2. If no external force acts on a system
(a) velocity of centre of mass remains constant
(b) position of centre of mass remains constant
(c) acceleration of centre of mass remains non-zero and constant
(d) All of the above

3. When two blocks connected by a spring move towards each other under mutual interaction
(a) their velocities are equal
(b) their accelerations are equal
(c) the force acting on them are equal and opposite
(d) All of the above

4. If two balls collide in air while moving vertically, then momentum of the system is conserved because
(a) gravity does not affect the momentum of the system
(b) force of gravity is very less compared to the impulsive force
(c) impulsive force is very less than the gravity
(d) gravity is not acting during collision

5. When a cannon shell explodes in mid air, then identify the **incorrect** statement
(a) the momentum of the system is conserved at the time of explosion
(b) the kinetic energy of the system always increases
(c) the trajectory of centre of mass remains unchanged
(d) None of the above

6. In an inelastic collision
(a) momentum of the system is always conserved
(b) velocity of separation is less than the velocity of approach
(c) the coefficient of restitution can be zero
(d) All of the above

7. The momentum of a system is defined
(a) as the product of mass of the system and the velocity of centre of mass
(b) as the vector sum of the momentum of individual particles
(c) for bodies undergoing translational, rotational and oscillatory motion
(d) All of the above

8. The momentum of a system with respect to centre of mass
(a) is zero only if the system is moving uniformly
(b) is zero only if no external force acts on the system
(c) is always zero
(d) can be zero in certain conditions

9. Three identical particles are located at the vertices of an equilateral triangle. Each particle moves along a meridian with equal speed towards the centroid and collides inelastically.
(a) all the three particles will bounce back along the meridians with lesser speed.
(b) all the three particles will become stationary
(c) all the particles will continue to move in their original directions but with lesser speed
(d) nothing can be said

10. The average resisting force that must act on a 5 kg mass to reduce its speed from 65 to 15 ms^{-1} in 2s is
(a) 12.5 N (b) 125 N (c) 1250 N (d) None of these

11. In a carbon monoxide molecule, the carbon and the oxygen atoms are separated by a distance 1.2×10^{-10} m. The distance of the centre of mass from the carbon atom is
(a) 0.48×10^{-10} m (b) 0.51×10^{-10} m
(c) 0.74×10^{-10} m (d) 0.68×10^{-10} m

12. A bomb of mass 9 kg explodes into two pieces of masses 3 kg and 6 kg. The velocity of mass 3 kg is 16 ms^{-1}. The kinetic energy of mass 6 kg is

(a) 96 J (b) 384 J (c) 192 J (d) 768 J

13. A heavy ball moving with speed v collides with a tiny ball. The collision is elastic, then immediately after the impact, the second ball will move with a speed approximately equal to

(a) v (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{3}$

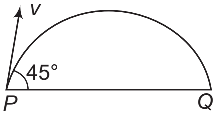
14. A loaded 20, 000 kg coal wagon is moving on a level track at 6 ms^{-1}. Suddenly 5000 kg of coal is dropped out of the wagon. The final speed of the wagon is

(a) 6 ms^{-1} (b) 8 ms^{-1} (c) 4.8 ms^{-1} (d) 4.5 ms^{-1}

15. A machine gun shoots a 40 g bullet at a speed of 1200 ms^{-1}. The man operating the gun can bear a maximum force of 144 N. The maximum number of bullets shot per second is

(a) 3 (b) 5 (c) 6 (d) 9

16. A projectile of mass m is fired with a velocity v from point P at an angle 45°. Neglecting air resistance, the magnitude of the change in momentum leaving the point P and arriving at Q is



(a) $\sqrt{2}\, mv$ (b) $2mv$ (c) $\frac{mv}{2}$ (d) $\frac{mv}{\sqrt{2}}$

17. A ball after freely falling from a height of 4.9 m strikes a horizontal plane. If the coefficient of restitution is 3/4, the ball will strike second time with the plane after

(a) $\frac{1}{2}$ s (b) 1 s (c) $\frac{3}{2}$ s (d) $\frac{3}{4}$ s

18. The centre of mass of a non uniform rod of length L, whose mass per unit length varies as $\rho = \frac{k \cdot x^2}{L}$ where k is a constant and x is the distance of any point from one end is (from the same end)

(a) $\left(\frac{3}{4}\right)L$ (b) $\left(\frac{1}{4}\right)L$ (c) $\left(\frac{1}{6}\right)L$ (d) $\left(\frac{2}{3}\right)L$

19. A boat of length 10 m and mass 450 kg is floating without motion in still water. A man of mass 50 kg standing at one end of it walks to the other end of it and stops. The magnitude of the displacement of the boat in metres relative to ground is

(a) zero (b) 1 m (c) 2 m (d) 5 m

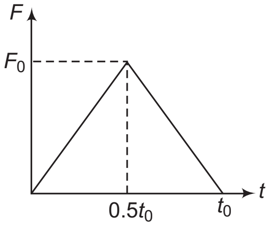
20. A man of mass M stands at one end of a stationary plank of length L, lying on a smooth surface. The man walks to the other end of the plank. If the mass of the plank is $M/3$, the distance that the man moves relative to the ground is

(a) $\frac{3L}{4}$ (b) $\frac{L}{4}$ (c) $\frac{4L}{5}$ (d) $\frac{L}{3}$

21. A ball of mass m moving at a speed v collides with another ball of mass $3m$ at rest. The lighter block comes to rest after collision. The coefficient of restitution is

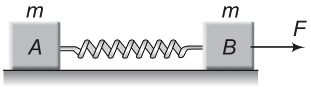
(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) None of these

22. A particle of mass m moving with velocity u makes an elastic one-dimensional collision with a stationary particle of mass m. They come in contact for a very small time t_0. Their force of interaction increases from zero to F_0 linearly in time $0.5t_0$, and decreases linearly to zero in further time $0.5\,t_0$ as shown in figure. The magnitude of F_0 is



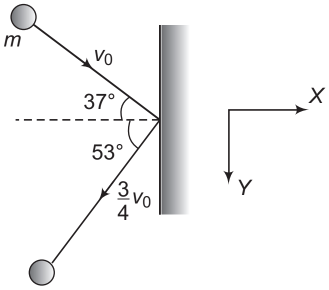
(a) $\frac{mu}{t_0}$
(b) $\frac{2mu}{t_0}$
(c) $\frac{mu}{2t_0}$
(d) None of these

23. Two identical blocks A and B of mass m joined together with a massless spring as shown in figure are placed on a smooth surface. If the block A moves with an acceleration a_0, then the acceleration of the block B is



(a) a_0
(b) $-a_0$
(c) $\frac{F}{m} - a_0$
(d) $\frac{F}{m}$

24. A ball of mass m moving with velocity v_0 collides a wall as shown in figure. After impact it rebounds with a velocity $\frac{3}{4}\,v_0$. The impulse acting on ball during impact is

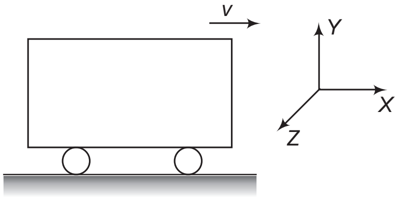


(a) $-\frac{m}{2}\,v_0\,\hat{\mathbf{j}}$
(b) $-\frac{3}{4}mv_0\,\hat{\mathbf{i}}$
(c) $\frac{-5}{4}\,mv_0\,\hat{\mathbf{i}}$
(d) None of the above

25. A steel ball is dropped on a hard surface from a height of 1 m and rebounds to a height of 64 cm. The maximum height attained by the ball after n^{th} bounce is (in m)

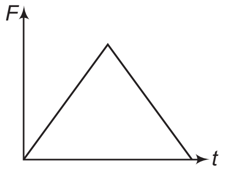
(a) $(0.64)^{2n}$
(b) $(0.8)^{2n}$
(c) $(0.5)^{2n}$
(d) $(0.8)^{n}$

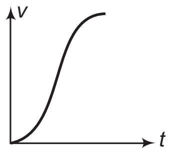
26. A car of mass 500 kg (including the mass of a block) is moving on a smooth road with velocity 1.0 ms^{-1} along positive x-axis. Now a block of mass 25 kg is thrown outside with absolute velocity of 20 ms^{-1} along positive z-axis. The new velocity of the car is (ms^{-1})

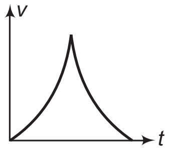


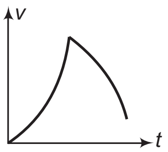
(a) $10\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{k}}$
(b) $10\,\hat{\mathbf{i}} - 20\,\hat{\mathbf{k}}$
(c) $\frac{20}{19}\hat{\mathbf{i}} - \frac{20}{19}\hat{\mathbf{k}}$
(d) $10\,\hat{\mathbf{i}} - \frac{20}{19}\,\hat{\mathbf{k}}$

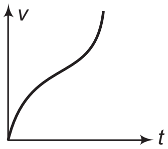
27. The net force acting on a particle moving along a straight line varies with time as shown in the diagram. Force is parallel to velocity. Which of the following graph is best representative of its speed with time ? (Initial velocity of the particle is zero)



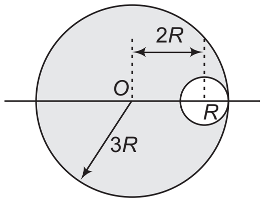
(a)


(b)


(c)


(d)


28. In the figure shown, find out centre of mass of a system of a uniform circular plate of radius $3R$ from O in which a hole of radius R is cut whose centre is at $2R$ distance from the centre of large circular plate



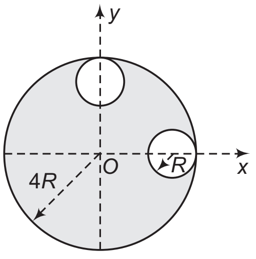
(a) $\frac{R}{2}$

(b) $\frac{R}{5}$

(c) $\frac{R}{4}$

(d) None of these

29. From the circular disc of radius $4R$ two small discs of radius R are cut off. The centre of mass of the new structure will be at



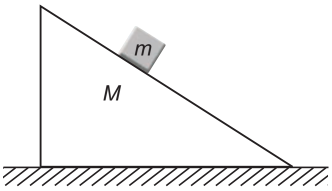
(a) $\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(b) $-\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(c) $-\hat{\mathbf{i}}\frac{R}{5}-\hat{\mathbf{j}}\frac{R}{5}$

(d) None of the above

30. A block of mass m rests on a stationary wedge of mass M. The wedge can slide freely on a smooth horizontal surface as shown in figure. If the block starts from rest



(a) the position of the centre of mass of the system will change

(b) the position of the centre of mass of the system will change along the vertical but not along the horizontal

(c) the total energy of the system will remain constant

(d) All of the above

31. A bullet of mass m hits a target of mass M hanging by a string and gets embedded in it. If the block rises to a height h as a result of this collision, the velocity of the bullet before collision is

(a) $v = \sqrt{2gh}$ (b) $v = \sqrt{2gh}\left[1 + \frac{m}{M}\right]$ (c) $v = \sqrt{2gh}\left[1 + \frac{M}{m}\right]$ (d) $v = \sqrt{2gh}\left[1 - \frac{m}{M}\right]$

32. A loaded spring gun of mass M fires a bullet of mass m with a velocity v at an angle of elevation θ. The gun is initially at rest on a horizontal smooth surface. After firing, the centre of mass of the gun and bullet system

(a) moves with velocity $\frac{v}{M}m$

(b) moves with velocity $\frac{vm}{M\cos\theta}$ in the horizontal direction

(c) does not move in horizontal direction

(d) moves with velocity $\frac{v\,(M-m)}{M+m}$ in the horizontal direction

33. Two bodies with masses $m_1$and m_2 ($m_1 > m_2$) are joined by a string passing over fixed pulley. Assume masses of the pulley and thread negligible. Then the acceleration of the centre of mass of the system ($m_1 + m_2$) is

(a) $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$ (b) $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$

(c) $\frac{m_1 g}{m_1 + m_2}$ (d) $\frac{m_2 g}{m_1 + m_2}$

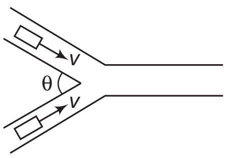
34. A rocket of mass m_0 has attained a speed equal to its exhaust speed and at that time the mass of the rocket is m. Then the ratio $\frac{m_0}{m}$ is (neglect gravity)

(a) 2.718 (b) 7.8 (c) 3.14 (d) 4

35. A jet of water hits a flat stationary plate perpendicular to its motion. The jet ejects 500g of water per second with a speed of 1 m/ s. Assuming that after striking, the water flows parallel to the plate, then the force exerted on the plate is

(a) 5 N (b) 1.0 N
(c) 0.5 N (d) 10 N

36. Two identical vehicles are moving with same velocity v towards an intersection as shown in figure. If the collision is completely inelastic, then



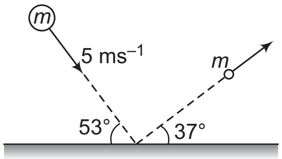
(a) the velocity of separation is zero

(b) the velocity of approach is $2v\sin\frac{\theta}{2}$

(c) the common velocity after collision is $v\cos\frac{\theta}{2}$

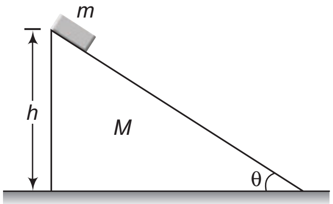
(d) All of the above

37. A ball of mass $m = 1$ kg strikes a smooth horizontal floor as shown in figure. The impulse exerted on the floor is



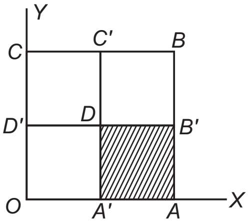
(a) 6.25 Ns (b) 1.76 Ns (c) 7.8 Ns (d) 2.2 Ns

38. A small block of mass m is placed at rest on the top of a smooth wedge of mass M, which in turn is placed at rest on a smooth horizontal surface as shown in figure**.** It h be the height of wedge and θ is the inclination, then the distance moved by the wedge as the block reaches the foot of the wedge is



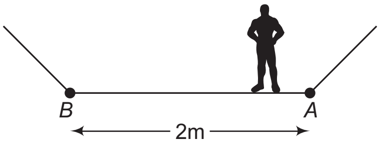
(a) $\dfrac{Mh\cot\theta}{M+m}$ (b) $\dfrac{mh\cot\theta}{M+m}$ (c) $\dfrac{Mh\,\text{cosec}\theta}{M+m}$ (d) $\dfrac{mh\,\text{cosec}\theta}{M+m}$

39. A square of side 4 cm and uniform thickness is divided into four squares. The square portion $A'\,AB'\,D$ is removed and the removed portion is placed over the portion $DB'\,BC'$. The new position of centre of mass is



(a) (2 cm, 2 cm) (b) (2 cm, 3 cm) (c) (2 cm, 2.5 cm) (d) (3 cm, 3 cm)

40. A boy having a mass of 40 kg stands at one end A of a boat of length 2 m at rest. The boy walks to the other end B of the boat and stops. What is the distance moved by the boat? Friction exists between the feet of the boy and the surface of the boat. But the friction between the boat and the water surface may be neglected. Mass of the boat is 15 kg.



(a) 0.49 m (b) 2.46 m (c) 1.46 m (d) 3.2 m

41. Three identical particles with velocities $v_0\,\hat{\mathbf{i}}, -3v_0\,\hat{\mathbf{j}}$ and $5v_0\,\hat{\mathbf{k}}$ collide successively with each other in such a way that they form a single particle. The velocity vector of resultant particle is

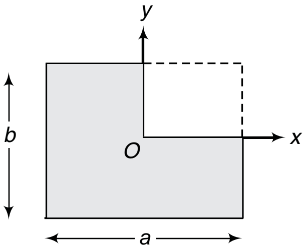
(a) $\dfrac{v_0}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (c) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (d) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

42. A mortar fires a shell of mass M which explodes into two pieces of mass $\dfrac{M}{5}$ and $\dfrac{4M}{5}$ at the top of the trajectory. The smaller mass falls very close to the mortar. In the same time the bigger piece lands a distance D from the mortar. The shell would have fallen at a distance R from the mortar if there was no explosion. The value of D is (neglect air resistance)

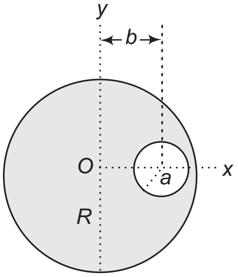
(a) $\dfrac{3R}{2}$ (b) $\dfrac{4R}{3}$ (c) $\dfrac{5R}{4}$ (d) None of these

Subjective Questions

1. Consider a rectangular plate of dimensions $a \times b$. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be zero.



2. The uniform solid sphere shown in the figure has a spherical hole in it. Find the position of its centre of mass.



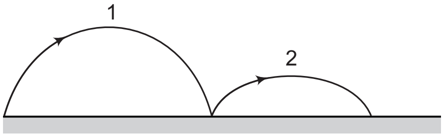
3. A gun fires a bullet. The barrel of the gun is inclined at an angle of 45° with horizontal. When the bullet leaves the barrel it will be travelling at an angle greater than 45° with the horizontal. Is this statement true or false?

4. Two blocks A and B of masses m_A and m_B are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Show that the kinetic energies of the blocks are, at any instant inversely proportional to their masses.

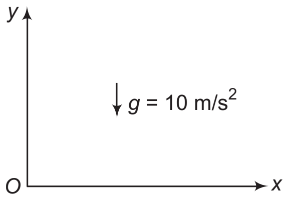
5. Show that in a head on elastic collision between two particles, the transference of energy is maximum when their mass ratio is unity.

6. A particle moving with kinetic energy K makes a head on elastic collision with an identical particle at rest. Find the maximum elastic potential energy of the system during collision.

7. A ball is projected from the ground at some angle with horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e.



8. *x-y* is the vertical plane as shown in figure. A particle of mass 1 kg is at (10 m, 20 m) at time $t = 0$. It is released from rest. Another particle of mass 2 kg is at (20 m, 40 m) at the same instant. It is projected with velocity $(10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s. After 1 s. Find



(a) acceleration,
(b) velocity and
(c) position of their centre of mass.

9. A system consists of two particles. At $t = 0$, one particle is at the origin; the other, which has a mass of 0.60 kg, is on the y-axis at $y = 80$ m. At $t = 0$, the centre of mass of the system is on the y-axis at $y = 24$ m and has a velocity given by $(6.0\ \text{m/s}^3)\,t^2\hat{\mathbf{j}}$.

(a) Find the total mass of the system.
(b) Find the acceleration of the centre of mass at any time t.
(c) Find the net external force acting on the system at $t = 3.0$ s.

10. A particle of mass 2 kg moving with a velocity $5\hat{\mathbf{i}}$ m/s collides head-on with another particle of mass 3 kg moving with a velocity $-2\hat{\mathbf{i}}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction. Find

(a) velocity of the centre of mass after the collision,
(b) velocity of the second particle after the collision,
(c) coefficient of restitution.

11. A rocket of mass 40 kg has 160 kg fuel. The exhaust velocity of the fuel is 2.0 km/s. The rate of consumption of fuel is 4 kg/s. Calculate the ultimate vertical speed gained by the rocket. $(g = 10\ \text{m/s}^2)$

12. A boy of mass 60 kg is standing over a platform of mass 40 kg placed over a smooth horizontal surface. He throws a stone of mass 1 kg with velocity $v = 10$ m/s at an angle of 45° with respect to the ground. Find the displacement of the platform (with boy) on the horizontal surface when the stone lands on the ground. Take $g = 10\ \text{m/s}^2$.

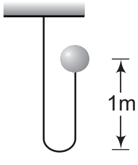
13. A man of mass m climbs to a rope ladder suspended below a balloon of mass M. The balloon is stationary with respect to the ground.

(a) If the man begins to climb the ladder at speed v (with respect to the ladder), in what direction and with what speed (with respect to the ground) will the balloon move ?
(b) What is the state of the motion after the man stops climbing ?

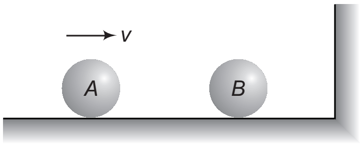
14. Find the mass of the rocket as a function of time, if it moves with a constant acceleration a, in absence of external forces. The gas escapes with a constant velocity u relative to the rocket and its mass initially was m_0.

15. A particle of mass $2m$ is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces. As a result of explosion one part comes to rest. Find the maximum height attained by the other part. Take $g = 10\ \text{m/s}^2$.

16. A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in figure. Find the impulse imparted by the string to the ball immediately after the string becomes taut. (Take $g = 10\ \text{m/s}^2$)



17. Two balls shown in figure are identical. Ball A is moving towards right with a speed v and the second ball is at rest. Assume all collisions to be elastic. Show that the speeds of the balls remain unchanged after all the collisions have taken place.



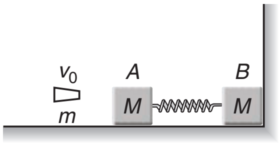
18. A particle of mass 0.1 kg moving at an initial speed v collides with another particle of same mass kept initially at rest. If the total energy becomes 0.2 J after the collision, what would be the minimum and maximum values of v ?

19. A particle of mass m moving with a speed v hits elastically another stationary particle of mass $2\,m$ on a smooth horizontal circular tube of radius r. Find the time when the next collision will take place?

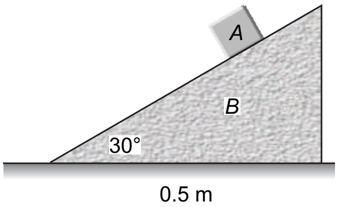
20. In a one-dimensional collision between two identical particles A and B, B is stationary and A has momentum p before impact. During impact, B gives an impulse J to A. Find the coefficient of restitution between A and B ?

21. Two billiard balls of same size and mass are in contact on a billiard table. A third ball of same mass and size strikes them symmetrically and remains at rest after the impact. Find the coefficient of restitution between the balls?

22. Two identical blocks each of mass $M = 9\,kg$ are placed on a rough horizontal surface of frictional coefficient $\mu = 0.1$. The two blocks are joined by a light spring and block B is in contact with a vertical fixed wall as shown in figure. A bullet of mass $m = 1$ kg and $v_0 = 10$ m/s hits block A and gets embedded in it. Find the maximum compression of spring. (Spring constant = 240 N/m, $g = 10\text{ m/s}^2$)



23. Block A has a mass of 5 kg and is placed on top of a smooth triangular block, B having a mass of 30 kg. If the system is released from rest, determine the distance, B moves when A reaches the bottom. Neglect the size of block A.



24. A trolley was moving horizontally on a smooth ground with velocity v with respect to the earth. Suddenly a man starts running from rear end of the trolley with a velocity (3/2) v with respect to the trolley. After reaching the other end, the man turns back and continues running with a velocity (3/2) v with respect to trolley in opposite direction. If the length of the trolley is L, find the displacement of the man with respect to earth when he reaches the starting point on the trolley. Mass of the trolley is equal to the mass of the man.

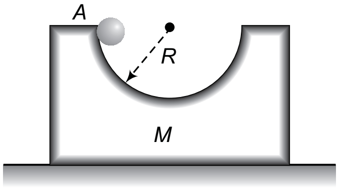
25. A 4.00 g bullet travelling horizontally with a velocity of magnitude 500 m/s is fired into a wooden block with a mass of 1.00 kg, initially at rest on a level surface. The bullet passes through the block and emerges with speed 100 m/s. The block slides a distance of 0.30 m along the surface from its initial position.

(a) What is the coefficient of kinetic friction between block and surface ?
(b) What is the decrease in kinetic energy of the bullet ?
(c) What is the kinetic energy of the block at the instant after the bullet has passed through it ? Neglect friction during collision of bullet with the block.

26. A bullet of mass 0.25 kg is fired with velocity 302 m/s into a block of wood of mass $m_1 = 37.5$ kg. It gets embedded into it. The block m_1 is resting on a long block m_2 and the horizontal surface on which it is placed is smooth. The coefficient of friction between m_1 and m_2 is 0.5. Find the displacement of m_1 on m_2 and the common velocity of m_1 and m_2. Mass $m_2 = 1.25$ kg.

27. A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length l. At the initial moment the wagon and the pendulum are at rest and the string is deflected through an angle α from the vertical. Find the velocity of the wagon when the pendulum passes through its mean position.

28. A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface shown in figure. A uniform cylinder of radius r and mass m is released from rest at the point A. The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reaches the bottom of the track ? How fast is the block moving when the cylinder reaches the bottom of the track ?

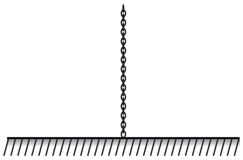


29. A ball of mass 50 g moving with a speed 2 m/s strikes a plane surface at an angle of incidence 45°. The ball is reflected by the plane at equal angle of reflection with the same speed. Calculate

(a) The magnitude of the change in momentum of the ball.

(b) The change in the magnitude of the momentum of the wall.

30. A uniform rope of mass m per unit length, hangs vertically from a support so that the lower end just touches the table top shown in figure. If it is released, show that at the time a length y of the rope has fallen, the force on the table is equivalent to the weight of a length $3y$ of the rope.



31. Sand drops from a stationary hopper at the rate of 5 kg/s on to a conveyor belt moving with a constant speed of 2 m/s. What is the force required to keep the belt moving and what is the power delivered by the motor, moving the belt ?

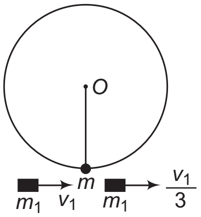
32. A 3.0 kg block slides on a frictionless horizontal surface, first moving to the left at 50 m/s. It collides with a spring as it moves left, compresses the spring and is brought to rest momentarily. The body continues to be accelerated to the right by the force of the compressed spring. Finally, the body moves to the right at 40 m/s. The block remains in contact with the spring for 0.020 s. What were the magnitude and direction of the impulse of the spring on the block? What was the spring's average force on the block?

33. Block A has a mass 3 kg and is sliding on a rough horizontal surface with a velocity $v_A = 2$ m/s when it makes a direct collision with block B, which has a mass of 2 kg and is originally at rest. The collision is perfectly elastic. Determine the velocity of each block just after collision and the distance between the blocks when they stop sliding. The coefficient of kinetic friction between the blocks and the plane is $\mu_k = 0.3$. (Take $g = 10\,\text{m/s}^2$)

LEVEL 2

Objective Questions

Single Correct Option



1. A pendulum consists of a wooden bob of mass m and length l. A bullet of mass m_1 is fired towards the pendulum with a speed v_1 and it emerges from the bob with speed $\frac{v_1}{3}$. The bob just completes motion along a vertical circle. Then v_1 is

(a) $\frac{m}{m_1}\sqrt{5gl}$ (b) $\frac{3m}{2m_1}\sqrt{5gl}$

(c) $\frac{2}{3}\left(\frac{m}{m_1}\right)\sqrt{5gl}$ (d) $\left(\frac{m_1}{m}\right)\sqrt{gl}$

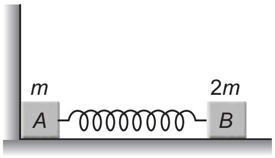
2. A bob of mass m attached with a string of length l tied to a point on ceiling is released from a position when its string is horizontal. At the bottom most point of its motion, an identical mass m gently stuck to it. Find the maximum angle from the vertical to which it rotates.

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$

(c) $\cos^{-1}\left(\frac{1}{4}\right)$ (d) 60°

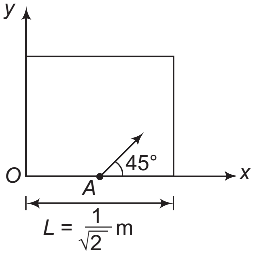
3. A train of mass M is moving on a circular track of radius R with constant speed v. The length of the train is half of the perimeter of the track. The linear momentum of the train will be

(a) zero (b) $\frac{2Mv}{\pi}$ (c) MvR (d) Mv



4. Two blocks A and B of mass m and $2\,m$ are connected together by a light spring of stiffness k. The system is lying on a smooth horizontal surface with the block A in contact with a fixed vertical wall as shown in the figure. The block B is pressed towards the wall by a distance x_0 and then released. There is no friction anywhere. If spring takes time Δt to acquire its natural length then average force on the block A by the wall is

(a) zero (b) $\frac{\sqrt{2mk}}{\Delta t}x_0$ (c) $\frac{\sqrt{mk}}{\Delta t}x_0$ (d) $\frac{\sqrt{3mk}}{\Delta t}x_0$

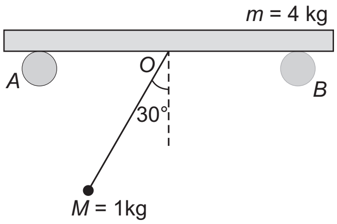


5. A striker is shot from a square carom board from a point A exactly at midpoint of one of the walls with a speed of 2 ms^{-1} at an angle of 45° with the x-axis as shown in the figure. The collisions of the striker with the walls of the fixed carom are perfectly elastic. The coefficient of kinetic friction between the striker and board is 0.2. The coordinate of the striker when it stops (taking point O to be the origin) is (in SI units)

(a) $\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}$ (b) $0, \frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}, 0$ (d) $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$

6. A ball of mass 1 kg is suspended by an inextensible string 1 m long attached to a point O of a smooth horizontal bar resting on fixed smooth supports A and B. The ball is released from rest from the position when the string makes an angle 30° with the vertical. The mass of the bar is 4 kg. The displacement of bar when ball reaches the other extreme position (in m) is



(a) 0.4
(b) 0.2
(c) 0.25
(d) 0.5

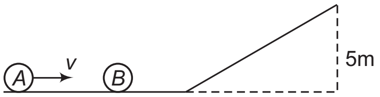
7. A ball falls vertically onto a floor with momentum p and then bounces repeatedly. If coefficient of restitution is e, then the total momentum imparted by the ball to the floor is

(a) $p(1+e)$
(b) $\frac{p}{1-e}$
(c) $p\left(\frac{1-e}{1+e}\right)$
(d) $p\left(\frac{1+e}{1-e}\right)$

8. A bullet of mass m penetrates a thickness h of a fixed plate of mass M. If the plate was free to move, then the thickness penetrated will be

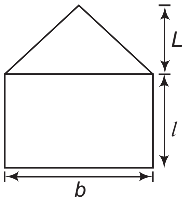
(a) $\frac{Mh}{M+m}$
(b) $\frac{2Mh}{M+m}$
(c) $\frac{mh}{2(M+m)}$
(d) $\frac{Mh}{2(M+m)}$

9. Two identical balls of equal masses A and B, are lying on a smooth surface as shown in the figure. Ball A hits the ball B (which is at rest) with a velocity $v = 16\text{ ms}^{-1}$. What should be the minimum value of coefficient of restitution e between A and B so that B just reaches the highest point of inclined plane? ($g = 10\text{ ms}^{-2}$)



(a) $\frac{2}{3}$
(b) $\frac{1}{4}$
(c) $\frac{1}{2}$
(d) $\frac{1}{3}$

10. The figure shows a metallic plate of uniform thickness and density. The value of l in terms of L so that the centre of mass of the system lies at the interface of the triangular and rectangular portion is

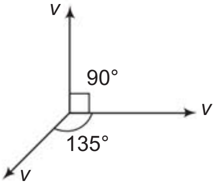


(a) $l = \frac{L}{3}$
(b) $l = \frac{L}{2}$
(c) $l = \frac{L}{\sqrt{3}}$
(d) $l = \sqrt{\frac{2}{3}}\,L$

11. Particle A makes a head on elastic collision with another stationary particle B. They fly apart in opposite directions with equal speeds. The mass ratio will be

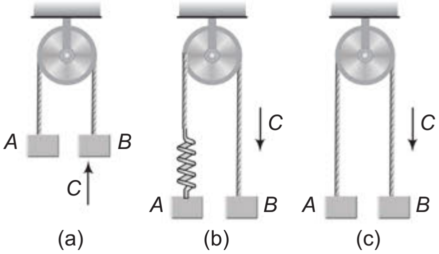
(a) $\frac{1}{3}$
(b) $\frac{1}{2}$
(c) $\frac{1}{4}$
(d) $\frac{2}{3}$

12. A particle of mass $4m$ which is at rest explodes into four equal fragments. All four fragments scattered in the same horizontal plane. Three fragments are found to move with velocity v as shown in the figure. The total energy released in the process is



(a) $mv^2(3-\sqrt{2})$
(b) $\frac{1}{2}mv^2(3-\sqrt{2})$
(c) $2mv^2$
(d) $\frac{1}{2}mv^2(1+\sqrt{2})$

13. In figures (a), (b) and (c) shown, the objects A, B and C are of same mass. String, spring and pulley are massless. C strikes B with velocity u in each case and sticks it. The ratio of velocity of B in case (a) to (b) to (c) is



(a) 1 : 1 : 1
(b) 3 : 3 : 2
(c) 3 : 2 : 2
(d) 1 : 2 : 3

14. A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\theta = 30°$ with horizontal. Then, the speed of the ladder's centre of mass must be

(a) $\frac{\sqrt{3}}{2}v$
(b) $\frac{v}{2}$
(c) v
(d) $2v$

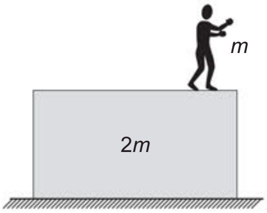
15. A body of mass 2 g, moving along the positive x-axis in gravity free space with velocity 20 cms^{-1} explodes at $x = 1$ m, $t = 0$ into two pieces of masses 2/3 g and 4/3 g. After 5s, the lighter piece is at the point (3m, 2m, – 4 m). Then the position of the heavier piece at this moment, in metres is

(a) (1.5, –1, –2)
(b) (1.5, –2, –2)
(c) (1.5, –1, –1)
(d) None of these

16. A body of mass m is dropped from a height of h. Simultaneously another body of mass $2m$ is thrown up vertically with such a velocity v that they collide at height $\frac{h}{2}$. If the collision is perfectly inelastic, the velocity of combined mass at the time of collision with the ground will be

(a) $\sqrt{\frac{5gh}{4}}$
(b) $\sqrt{gh}$
(c) $\sqrt{\frac{gh}{4}}$
(d) None of these

17. A man is standing on a cart of mass double the mass of man. Initially cart is at rest. Now, man jumps horizontally with velocity u relative to cart. Then work done by man during the process of jumping will be



(a) $\frac{mu^2}{2}$
(b) $\frac{3mu^2}{4}$
(c) mu^2
(d) None of these

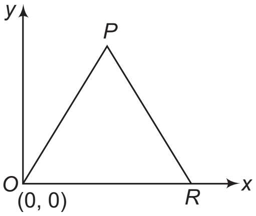
18. Two balls of equal mass are projected upwards simultaneously, one from the ground with initial velocity 50 ms^{-1} and the other from a 40m tower with initial velocity of 30 ms^{-1}. The maximum height attained by their COM will be

(a) 80 m
(b) 60 m
(c) 100 m
(d) 120 m

19. A particle of mass m and momentum $\mathbf{p}$ moves on a smooth horizontal table and collides directly and elastically with a similar particle (of mass m) having momentum $-2\,\mathbf{p}$. The loss (–) or gain (+) in the kinetic energy of the first particle in the collision is

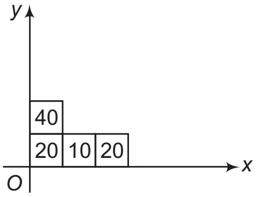
(a) $+\dfrac{p^2}{2m}$
(b) $-\dfrac{p^2}{4m}$
(c) $+\dfrac{3p^2}{2m}$
(d) zero

20. An equilateral triangular plate of mass $4m$ of side a is kept as shown. Consider two cases : (i) a point mass $4m$ is placed at the vertex P of the plate (ii) a point mass m is placed at the vertex R of the plate. In both cases the x-coordinate of centre of mass remains the same. Then x coordinate of centre of mass of the plate is



(a) $\dfrac{a}{3}$
(b) $\dfrac{a}{6}$
(c) $\dfrac{6a}{7}$
(d) $\dfrac{2a}{3}$

21. Four cubes of side a each of mass 40 g, 20 g, 10 g and 20 g are arranged in XY plane as shown in the figure. The coordinates of COM of the combination with respect to point O is

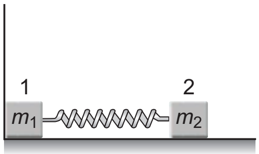


(a) $\dfrac{19a}{18}, \dfrac{17a}{18}$
(b) $\dfrac{17a}{18}, \dfrac{11a}{18}$
(c) $\dfrac{17a}{18}, \dfrac{13a}{18}$
(d) $\dfrac{13a}{18,}\ \dfrac{17a}{18}$

22. A particle of mass m_0, travelling at speed v_0, strikes a stationary particle of mass $2m_0$. As a result the particle of mass m_0 is deflected through 45° and has a final speed of $\dfrac{v_0}{\sqrt{2}}$. Then the speed of the particle of mass $2m_0$ after this collision is

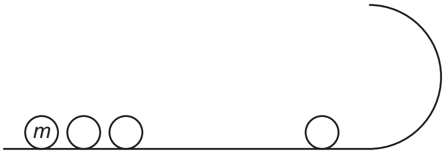
(a) $\dfrac{v_0}{2}$
(b) $\dfrac{v_0}{2\sqrt{2}}$
(c) $\sqrt{2}v_0$
(d) $\dfrac{v_0}{\sqrt{2}}$

23. Two bars of masses m_1 and m_2, connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Bar 2 is shifted by a small distance x_0 to the left and released. The velocity of the centre of mass of the system when bar 1 breaks off the wall is



(a) $x_0\sqrt{\dfrac{km_2}{m_1+m_2}}$
(b) $\dfrac{x_0}{m_1+m_2}\sqrt{km_2}$
(c) $x_0 k\dfrac{m_1+m_2}{m_2}$
(d) $x_0\dfrac{\sqrt{km_1}}{(m_1+m_2)}$

24. n elastic balls are placed at rest on a smooth horizontal plane which is circular at the ends with radius r as shown in the figure. The masses of the balls are $m, \frac{m}{2}, \frac{m}{2^2} \ldots\ldots\ldots\ldots \frac{m}{2^{n-1}}$ respectively. What is the minimum velocity which should be imparted to the first ball of mass m such that n^{th} ball completes the vertical circle



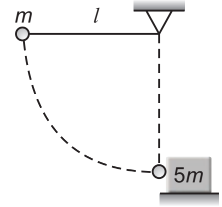
(a) $\left(\frac{3}{4}\right)^{n-1} \sqrt{5gr}$
(b) $\left(\frac{4}{3}\right)^{n-1} \sqrt{5gr}$
(c) $\left(\frac{3}{2}\right)^{n-1} \sqrt{5gr}$
(d) $\left(\frac{2}{3}\right)^{n-1} \sqrt{5gr}$

More than One Correct Options

1. A particle of mass m, moving with velocity v collides a stationary particle of mass $2m$. As a result of collision, the particle of mass m deviates by 45° and has final speed of $\frac{v}{2}$. For this situation mark out the correct statement (s).

(a) The angle of divergence between particles after collision is $\frac{\pi}{2}$
(b) The angle of divergence between particles after collision is less than $\frac{\pi}{2}$
(c) Collision is elastic
(d) Collision is inelastic

2. A pendulum bob of mass m connected to the end of an ideal string of length l is released from rest from horizontal position as shown in the figure. At the lowest point the bob makes an elastic collision with a stationary block of mass $5m$, which is kept on a frictionless surface. Mark out the correct statement(s) for the instant just after the impact.



(a) Tension in the string is $\frac{17}{9}mg$
(b) Tension in the string is 3 mg.
(c) The velocity of the block is $\frac{\sqrt{2gl}}{3}$
(d) The maximum height attained by the pendulum bob after impact is (measured from the lowest position) $\frac{4l}{9}$

3. A particle of mass m strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebound with velocity v making an angle ϕ with the floor. The coefficient of restitution between the particle and the floor is e. Then

(a) the impulse delivered by the floor to the body is $mu(1 + e)\sin\theta$
(b) $\tan\phi = e\tan\theta$
(c) $v = u\sqrt{1 - (1 - e^2)\sin^2\theta}$
(d) the ratio of the final kinetic energy to the initial kinetic energy is $\cos^2\theta + e^2\sin^2\theta$

4. A particle of mass m moving with a velocity $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ \text{ms}^{-1}$ collides with another body of mass M and finally moves with velocity $(-2\hat{\mathbf{i}} + \hat{\mathbf{j}})\ \text{ms}^{-1}$. Then during the collision

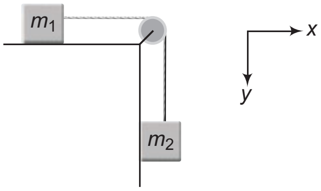
(a) impulse received by m is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(b) impulse received by m is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(c) impulse received by M is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(d) impulse received by M is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

5. All surfaces shown in figure are smooth. System is released from rest. x and y components of acceleration of COM are



(a) $(a_{\text{cm}})_x = \dfrac{m_1 m_2 g}{m_1 + m_2}$

(b) $(a_{\text{cm}})_x = \dfrac{m_1 m_2 g}{(m_1 + m_2)^2}$

(c) $(a_{\text{cm}})_y = \left(\dfrac{m_2}{m_1 + m_2}\right)^2 g$

(d) $(a_{\text{cm}})_y = \left(\dfrac{m_2}{m_1 + m_2}\right) g$

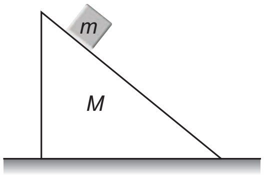
6. A block of mass m is placed at rest on a smooth wedge of mass M placed at rest on a smooth horizontal surface. As the system is released

(a) the COM of the system remains stationary

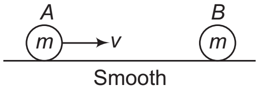
(b) the COM of the system has an acceleration g vertically downward

(c) momentum of the system is conserved along the horizontal direction

(d) acceleration of COM is vertically downward $(a < g)$



7. In the figure shown, coefficient of restitution between A and B is $e = \dfrac{1}{2}$, then



(a) velocity of B after collision is $\dfrac{v}{2}$

(b) impulse between two during collision is $\dfrac{3}{4}mv$

(c) loss of kinetic energy during the collision is $\dfrac{3}{16}mv^2$

(d) loss of kinetic energy during the collision is $\dfrac{1}{4}mv^2$

8. In case of rocket propulsion, choose the correct options.

(a) Momentum of system always remains constant

(b) Newton's third law is applied

(c) If exhaust velocity and rate of burning of mass is kept constant, then acceleration of rocket will go on increasing

(d) Newton's second law can be applied

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 2)

A block of mass 2 kg is attached with a spring of spring constant 4000 $\mathrm{Nm^{-1}}$ and the system is kept on smooth horizontal table. The other end of the spring is attached with a wall. Initially spring is stretched by 5 cm from its natural position and the block is at rest. Now suddenly an impulse of 4 $\mathrm{kg\text{-}ms^{-1}}$ is given to the block towards the wall.

1. Find the velocity of the block when spring acquires its natural length

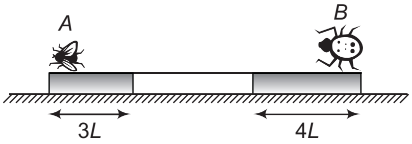
(a) 5 $\mathrm{ms^{-1}}$
(b) 3 $\mathrm{ms^{-1}}$
(c) 6 $\mathrm{ms^{-1}}$
(d) None of these

2. Approximate distance travelled by the block when it comes to rest for a second time (not including the initial one) will be (Take $\sqrt{45} = 6.70$)

(a) 30 cm
(b) 25 cm
(c) 40 cm
(d) 20 cm

Passage 2 (Q. Nos. 3 to 7)

A uniform bar of length $12L$ and mass $48\,m$ is supported horizontally on two fixed smooth tables as shown in figure. A small moth (an insect) of mass $8m$ is sitting on end A of the rod and a spider (an insect) of mass $16m$ is sitting on the other end B. Both the insects moving towards each other along the rod with moth moving at speed $2v$ and the spider at half this speed (absolute). They meet at a point P on the rod and the spider eats the moth. After this the spider moves with a velocity $\frac{v}{2}$ relative to the rod towards the end A. The spider takes negligible time in eating on the other insect. Also, let $v = \frac{L}{T}$ where T is a constant having value 4 s.



3. Displacement of the rod by the time the insect meet the moth is

(a) $\frac{L}{2}$
(b) L
(c) $\frac{3L}{4}$
(d) zero

4. The point P is at

(a) the centre of the rod
(b) the edge of the table supporting the end B
(c) the edge of the table supporting end A
(d) None of the above

5. The speed of the rod after the spider eats up the moth and moves towards A is

(a) $\frac{v}{2}$
(b) v
(c) $\frac{v}{6}$
(d) $2v$

6. After starting from end B of the rod the spider reaches the end A at a time

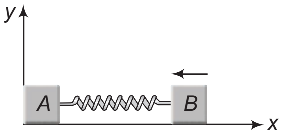
(a) 40 s
(b) 30 s
(c) 80 s
(d) 10 s

7. By what distance the centre of mass of the rod shifts during this time?

(a) $\frac{8L}{3}$
(b) $\frac{4L}{3}$
(c) L
(d) $\frac{L}{3}$

Match the Columns

1. Two identical blocks A and B are connected by a spring as shown in figure. Block A is not connected to the wall parallel to y-axis. B is compressed from the natural length of spring and then left. Neglect friction. Match the following two columns.



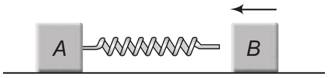
Column I	Column II
(a) Acceleration of centre of mass of two blocks	(p) remains constant
(b) Velocity of centre of mass of two blocks	(q) first increases then becomes constant
(c) x-coordinate of centre of mass of two blocks	(r) first decreases then becomes zero
(d) y-coordinate of centre of mass of two blocks	(s) continuously increases

2. One particle is projected from ground upwards with velocity 20 ms^{-1}. At the same time another identical particle is dropped from a height of 180 m but not along the same vertical line. Assume that collision of first particle with ground is perfectly inelastic. Match the following two columns for centre of mass of the two particles ($g = 10\ \text{ms}^{-2}$)

Column I	Column II
(a) Initial acceleration	(p) 5 SI units
(b) Initial velocity	(q) 10 SI units
(c) Acceleration at $t = 5$ s	(r) 20 SI units
(d) Velocity at $t = 5$ s	(s) 25 SI units

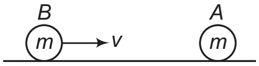
Note *Only magnitudes are given in column-II.*

3. Two identical blocks of mass 0.5 kg each are shown in figure. A massless elastic spring is connected with A. B is moving towards A with kinetic energy of 4 J. Match the following two columns. Neglect friction.



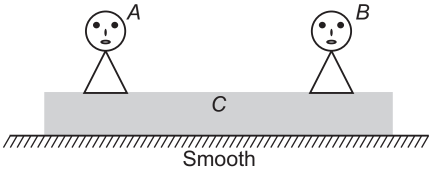
Column I	Column II
(a) Initial momentum of B	(p) zero
(b) Momentum of centre of mass of two blocks	(q) 1 kg-ms^{-1}
(c) Momentum of A at maximum compression	(r) 2 kg-ms^{-1}
(d) Momentum of B when spring is relaxed after compression	(s) 4 kg-ms^{-1}

4. Two identical balls A and B are kept on a smooth table as shown. B collides with A with speed v. For different conditions mentioned in Column I, match with speed of A after collision given in Column II.



Column I	Column II
(a) Elastic collision	(p) $\frac{3}{4}v$
(b) Perfectly inelastic collision	(q) $\frac{5}{8}v$
(c) Inelastic collision with $e=\frac{1}{2}$	(r) v
(d) Inelastic collision with $e=\frac{1}{4}$	(s) $\frac{v}{2}$

5. Two boys A and B of masses 30 kg and 60 kg are standing over a plank C of mass 30 kg as shown. Ground is smooth. Match the displacement of plank of Column II with the conditions given in Column I.

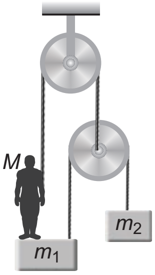


Column I	Column II
(a) A moves x towards right	(p) x towards right
(b) B moves x towards left	(q) $2x$ towards left
(c) A moves x towards right and B moves x towards left	(r) $\frac{x}{3}$ towards left
(d) A and B both move x towards right	(s) None

Note *All displacements mentioned in two columns are with respect to ground.*

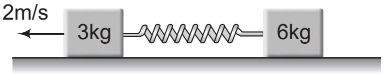
6. A man of mass M is standing on a platform of mass m_1 and holding a string passing over a system of ideal pulleys. Another mass m_2 is hanging as shown.

$$(m_2 = 20\text{kg},\ m_1 = 10\text{kg},\ g = 10\ \text{ms}^{-2})$$



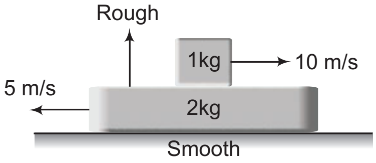
Column I	Column II
(a) Weight of man for equilibrium	(p) 100 N
(b) Force exerted by man on string to accelerate the COM of system upwards	(q) 150 N
(c) Force exerted by man on string to accelerate the COM of system downwards	(r) 500 N
(d) Normal reaction of platform on man in equilibrium	(s) 600 N

7. Two blocks of masses 3 kg and 6 kg are connected by an ideal spring and are placed on a frictionless horizontal surface. The 3 kg block is imparted a speed of 2 ms^{-1} towards left. (consider left as positive direction)



Column I	Column II
(a) When the velocity of 3 kg block is $\frac{2}{3}$ ms^{-1}	(p) velocity of centre of mass is $\frac{2}{3}$ ms^{-1}
(b) When the velocity of 6 kg block is $\frac{2}{3}$ ms^{-1}	(q) deformation of the spring is zero
(c) When the speed of 3 kg block is minimum	(r) deformation of the spring is maximum
(d) When the speed of 6 kg block is maximum	(s) both the blocks are at rest with respect to each other

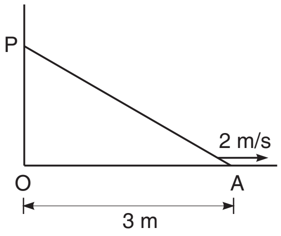
8. In a two block system shown in figure match the following

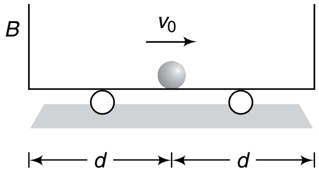


Column I	Column II
(a) Velocity of centre of mass	(p) Keep on changing all the time
(b) Momentum of centre of mass	(q) First decreases then become zero
(c) Momentum of 1 kg block	(r) Zero
(d) Kinetic energy of 2 kg block	(s) Constant

Subjective Questions

1. A ladder *AP* of length 5 m inclined to a vertical wall is slipping over a horizontal surface with velocity of 2 m/s, when *A* is at distance 3 m from ground. What is the velocity of COM at this moment?

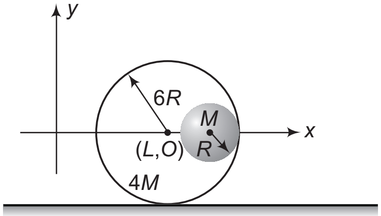


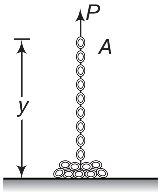


2. A ball of negligible size and mass m is given a velocity v_0 on the centre of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e. Determine the velocity of the ball and the cart just after the ball strikes A. Also, determine the total time needed for the ball to strike A, rebound, then strike B, and rebound and then return to the centre of the cart. Neglect friction.

3. Two point masses m_1 and m_2 are connected by a spring of natural length l_0. The spring is compressed such that the two point masses touch each other and then they are fastened by a string. Then the system is moved with a velocity v_0 along positive x-axis. When the system reached the origin, the string breaks ($t = 0$). The position of the point mass m_1 is given by $x_1 = v_0 t - A(1 - \cos \omega t)$ where A and ω are constants. Find the position of the second block as a function of time. Also, find the relation between A and l_0.

4. A small sphere of radius R is held against the inner surface of larger sphere of radius $6R$ (as shown in figure). The masses of large and small spheres are $4M$ and M respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the coordinates of the centre of the large sphere, when the smaller sphere reaches the other extreme position.



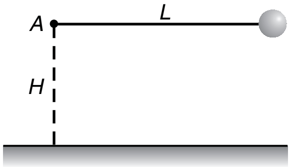


5. A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v_0, express in terms of the length y of chain which is off the floor at any given instant.

(a) The magnitude of the force P applied to end A.

(b) Energy lost during the lifting of the chain.

6. A is a fixed point at a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length $L(> H)$ has one end attached to A and other to a heavy particle. The particle is held at the level of A with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



7. A particle of mass 2 m is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s, explosion takes place and the particle is broken into two equal pieces.

As a result of explosion one part comes to rest. Find the maximum height attained by the other part. (Take $g = 10$ m/s^2)

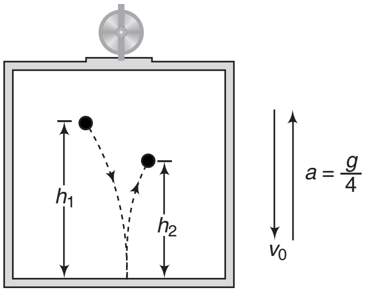
8. A sphere of mass m, impinges obliquely on a sphere, of mass M, which is at rest. Show that, if $m = eM$, the directions of motion of the spheres after impact are at right angles.

9. A gun of mass M (including the carriage) fires a shot of mass m. The gun along with the carriage is kept on a smooth horizontal surface. The muzzle speed of the bullet v_r is constant. Find

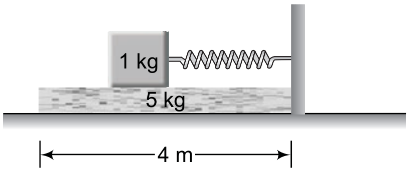
(a) The elevation of the gun with horizontal at which maximum range of bullet with respect to the ground is obtained.

(b) The maximum range of the bullet.

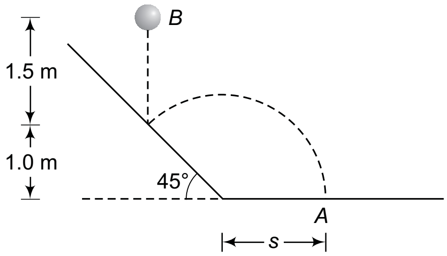
10. A ball is released from rest relative to the elevator at a distance h_1 above the floor. The speed of the elevator at the time of ball release is v_0. Determine the bounce height h_2 relative to elevator of the ball (a) if v_0 is constant and (b) if an upward elevator acceleration $a = \frac{g}{4}$ begins at the instant the ball is released. The coefficient of restitution for the impact is e.



11. A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. The system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is 100 N/m).

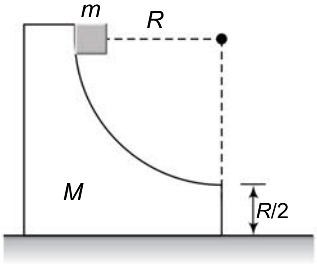


12. To test the manufactured properties of 10N steel balls, each ball is released from rest as shown and strikes a 45° inclined surface. If the coefficient of restitution is to be $e = 0.8$, determine the distance s, where the ball must strike the horizontal plane at A. At what speed does the ball strike at A?($g = 9.8 \text{ m/s}^2$)

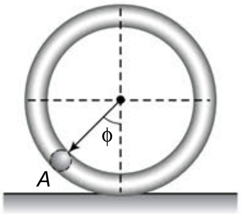


13. Two particles A and B of equal masses lie close together on a horizontal table and are connected by a light inextensible string of length l. A is projected vertically upwards with a velocity $\sqrt{10gl}$. Find the velocity with which it reaches the table again.

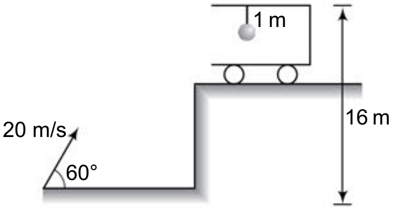
14. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown in figure. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path. Find the horizontal distance from the bottom of block where cube hits the table.



15. A thin hoop of mass M and radius r is placed on a horizontal plane. At the initial instant, the hoop is at rest. A small washer of mass m with zero initial velocity slides from the upper point of the hoop along a smooth groove in the inner surface of the hoop. Determine the velocity u of the centre of the hoop at the moment when the washer is at a certain point A of the hoop, whose radius vector forms an angle ϕ with the vertical (figure). The friction between the hoop and the plane should be neglected.



16. A shell of mass 1 kg is projected with velocity 20 m/s at an angle 60° with horizontal. It collides inelastically with a ball of mass 1 kg which is suspended through a thread of length 1 m. The other end of the thread is attached to the ceiling of a trolley of mass $\frac{4}{3}$ kg as shown in figure. Initially the trolley is stationary and it is free to move along horizontal rails without any friction. What is the maximum deflection of the thread with vertical? String does not slack. Take $g = 10\ \text{m/s}^2$.

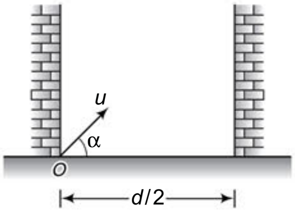


17. A small ball is projected at an angle α between two vertical walls such that in the absence of the wall its range would have been $5d$. Given that all the collisions are perfectly elastic, find

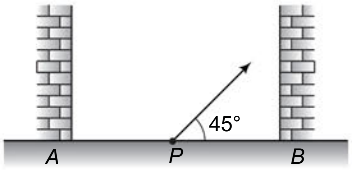
(a) maximum height attained by the ball.

(b) total number of collisions with the walls before the ball comes back to the ground, and

(c) point at which the ball finally falls. The walls are supposed to be very tall.

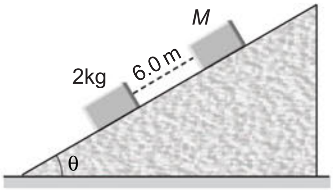


18. Two large rigid vertical walls A and B are parallel to each other and separated by 10 m. A particle of mass 10 g is projected with an initial velocity of 20 m/s at 45° to the horizontal from point P on the ground, such that $AP = 5$ m. The plane of motion of the particle is vertical and perpendicular to the walls. Assuming that all the collisions are perfectly elastic , find the maximum height attained by the particle and the total number of collisions suffered by the particle with the walls before it hits ground. Take $g = 10\ \text{m/s}^2$.

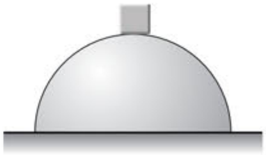


19. Two blocks of masses 2 kg and M are at rest on an inclined plane and are separated by a distance of 6.0 m as shown. The coefficient of friction between each block and the inclined plane is 0.25. The 2 kg block is given a velocity of 10.0 m/s up the inclined plane. It collides with M, comes back and has a velocity of 1.0 m/s when it reaches its initial position. The other block M after the collision moves 0.5 m up and comes to rest. Calculate the coefficient of restitution between the blocks and the mass of the block M.

[Take $\sin\theta \approx \tan\theta = 0.05$ and $g = 10\ \text{m/s}^2$]



20. A small block of mass m is placed on top of a smooth hemisphere also of mass m which is placed on a smooth horizontal surface. If the block begins to slide down due to a negligible small impulse, show that it will loose contact with the hemisphere when the radial line through vertical makes an angle θ given by the equation $\cos^3\theta - 6\cos\theta + 4 = 0$.



21. A ball is projected from a given point with velocity u at some angle with the horizontal and after hitting a vertical wall returns to the same point. Show that the distance of the point from the wall must be less than $\dfrac{eu^2}{(1+e)g}$, where e is the coefficient of restitution.

Answers

Introductory Exercise 11.1

1. $\mathbf{r}_{\text{COM}} = \dfrac{\sum\limits_{i=1}^{n} m_i \mathbf{r}_i}{\sum\limits_{i=1}^{n} m_i}$ while $\mathbf{r}_{\text{CG}} = \dfrac{\sum\limits_{i=1}^{n} w_i \mathbf{r}_i}{\sum\limits_{i=1}^{n} w_i}$ Here, w = weight (mg), $\mathbf{r}_{\text{COM}} = \mathbf{r}_{\text{CG}}$ when $\mathbf{g}$ = constant

2. False **3.** True **4.** True **5.** less than $\dfrac{r}{2}$

6. False **7.** $\dfrac{\sqrt{19}}{6}$ m **8.** $\dfrac{4}{3\pi}\left\{\dfrac{a^2 + ab + b^2}{a + b}\right\}$ **9.** $\left(\dfrac{5a}{6}, \dfrac{5a}{6}\right)$

10. 0.74 m^2 **11.** $\left(\dfrac{\pi}{\pi + 4}\right) a$ **12.** $x_{\text{COM}} = \dfrac{3l}{4}$ **13.** $\dfrac{2}{3} L$

Introductory Exercise 11.2

1. $x_{CM} = 12.67$ m **2.** zero **3.** $\dfrac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$ **4.** (a) 0.30 kg (b) (2.4 kg-ms^{-1})$\hat{\mathbf{j}}$ (c) (8.0 ms^{-1}) $\hat{\mathbf{j}}$

5. (a) 28 cm (b) 2.3 ms^{-1} **6.** $\left(\dfrac{\sqrt{3} - 1}{4\sqrt{2}}\right) g$

Introductory Exercise 11.3

1. $(2.5\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ cm/s **2.** 12.5 m/s in opposite direction, 17.5 m/s **3.** 1 : 2

4. 2.4×10^5 m/s **5.** 1.5×10^{-23} m/s **6.** 9 ms^{-1}, 1.08 kJ **7.** 60 m

Introductory Exercise 11.4

1. 1.225 kgs^{-1}, (i) 2.8 kms^{-1}, (ii) 3.6 kms^{-1} **2.** 1232.6 ms^{-1}

3. $(m_0 - \mu t)\dfrac{d^2x}{dt^2} = \mu u - (m_0 - \mu t)\, g$ **4.** $u \ln\left(\dfrac{3}{2}\right) - g$

Introductory Exercise 11.5

1. 4×10^3 N **2.** $-\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ **3.** 200 m/s **4.** (a) 1.2×10^{-3} s (b) 0.5 N-s (c) 417 N

Introductory Exercise 11.6

1. 30 cm **2.** $\dfrac{8}{9}$ **3.** $\dfrac{4m_1m_2}{(m_1 + m_2)^2}$ **4.** No

5. 2 ms^{-1} in negative x-axis, 3 m/s in positive x-axis.

6. $v_1 = \dfrac{28}{3}$ ms^{-1} (in negative x-direction) and $v_2 = \dfrac{2}{3}$ ms^{-1} (in positive x-direction)

7. Two **8.** $\dfrac{1}{3}$ **9.** (a) $\dfrac{mv}{V}$ (b) $\dfrac{V}{v}$ **10.** $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ **11.** 90° – 2α

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (d) **4.** (d) **5.** (b) **6.** (a) **7.** (c) **8.** (d) **9.** (a) **10.** (d)
11. (d) **12.** (d) **13.** (d) **14.** (a) **15.** (d)

Single Correct Option

1. (a) **2.** (a) **3.** (c) **4.** (b) **5.** (b) **6.** (d) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (d) **12.** (c) **13.** (b) **14.** (a) **15.** (a) **16.** (a) **17.** (c) **18.** (a) **19.** (b) **20.** (b)
21. (d) **22.** (b) **23.** (c) **24.** (c) **25.** (b) **26.** (c) **27.** (a) **28.** (c) **29.** (d) **30.** (d)
31. (c) **32.** (c) **33.** (b) **34.** (a) **35.** (c) **36.** (d) **37.** (a) **38.** (b) **39.** (c) **40.** (c)
41. (d) **42.** (c)

Subjective Questions

1. $\left(-\frac{a}{12}, -\frac{b}{12}\right)$ **2.** $x_{\text{COM}} = -\frac{a^3 b}{R^3 - a^3}$ **3.** True **6.** $\frac{K}{2}$

7. $\frac{1}{e}, \frac{1}{e}, \frac{1}{e^2}$

8. (a) $(-10\hat{\mathbf{j}})\,\text{ms}^{-2}$ (b) $\frac{10}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}})\,\text{ms}^{-1}$ (c) $\left(\frac{70}{3}\hat{\mathbf{i}} + 35\hat{\mathbf{j}}\right)\text{m}$

9. (a) 2.0 kg (b) $(12.0\ \text{ms}^{-2})t\,\hat{\mathbf{j}}$ (c) $(72.0\ \text{N})\hat{\mathbf{j}}$ **10.** (a) $0.8\,\hat{\mathbf{i}}$ m/s (b) $2.4\,\hat{\mathbf{i}}$ m/s (c) $\frac{4}{7}$

11. 2.82 kms^{-1} **12.** 10 cm **13.** (a) $\frac{mv}{M+m}$ (b) balloon will also stop moving

14. $m = m_0 e^{-(a/u)t}$ **15.** 35 m **16.** $2\sqrt{10}$ N·s

18. 2 ms^{-1}, $2\sqrt{2}$ ms^{-1} **19.** $\frac{2\pi r}{v}$

20. $\frac{2J}{P} - 1$ **21.** $\frac{2}{3}$

22. $\frac{1}{6}$ m

23. 71.4 mm **24.** $\frac{4L}{3}$

25. (a) 0.43 (b) 480 J (c) 1.28 J

26. 0.011mm, 1.94 m/s **27.** $2m\sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{gl}{M(M+m)}}$

28. $\frac{m(R-r)}{M+m}, m\sqrt{\frac{2g(R-r)}{M(M+m)}}$

29. (a) 0.14 kg-ms^{-1} (b) zero
31. 10 N, 20 W
32. 270 N-s (to the right), 13.5 kN (to the right)
33. 0.4 ms^{-1}, 2.4 ms^{-1}, 0.93 m

LEVEL 2

Single Correct Option

1. (b) **2.** (b) **3.** (b) **4.** (b) **5.** (a) **6.** (b) **7.** (d) **8.** (a) **9.** (b) **10.** (c)
11. (a) **12.** (a) **13.** (b) **14.** (c) **15.** (d) **16.** (d) **17.** (d) **18.** (c) **19.** (c) **20.** (b)
21. (a) **22.** (b) **23.** (b) **24.** (a)

More than One Correct Options

1. (b,d) **2.** (a,c,d) **3.** (all) **4.** (b,d) **5.** (b,c) **6.** (c,d) **7.** (b,c) **8.** (b,c,d)

Comprehension Based Questions

1. (b) **2.** (b) **3.** (d) **4.** (b) **5.** (c) **6.** (c) **7.** (a)

Match the Columns

1. (a) → (r) (b) → (q) (c) → (s) (d) → (p)
2. (a) → (q) (b) → (q) (c) → (p) (d) → (s)
3. (a) → (r) (b) → (r) (c) → (q) (d) → (p)
4. (a) → (r) (b) → (s) (c) → (p) (d) → (q)
5. (a) → (r) (b) → (p) (c) → (p) (d) → (s)
6. (a) → (r) (b) → (r,s) (c) → (p,q) (d) → (p)
7. (a) → (p,r,s) (b) → (p,r,s) (c) → (p) (d) → (p,q)
8. (a) → (r,s) (b) → (r,s) (c) → (q) (d) → (q)

Subjective Questions

1. $1.25\ \text{ms}^{-1}$

2. $v_{\text{ball}} = \left(\frac{eM - m}{M + m}\right) v_0$ (leftwards), $v_{\text{cart}} = \left(\frac{e + 1}{M + m}\right) m v_0$ (rightwards); $t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$

3. $x_2 = v_0 t + \frac{m_1}{m_2} A\,(1 - \cos\omega t)$, $l_0 = \left(\frac{m_1}{m_2} + 1\right) A$

4. $(L + 2R, 0)$

5. (a) $\frac{m}{l}(gy + v_0^2)$ (b) $\frac{myv_0^2}{2l}$

6. $\frac{H^5}{L^4}$

7. 35 m

9. (a) 45° (b) $\left(\frac{M}{M + m}\right)\frac{v_r^2}{g}$

10. (a) $e^2 h_1$ (b) $e^2 h_1$

11. $\sqrt{\frac{10}{3}}\ \text{ms}^{-1}$

12. 0.93 m, $6.6\ \text{ms}^{-1}$

13. $2\sqrt{gl}$

14. $R\sqrt{\frac{2(M + m)}{M}}$

15. $v = m\cos\phi\sqrt{\frac{2gr(1 + \cos\phi)}{(M + m)(M + m\sin^2\phi)}}$

16. 60°

17. (a) $\frac{u^2\sin^2\alpha}{2g}$ (b) nine (c) point O

18. 10 m, Four

19. 0.84, 15.12 kg

12

Rotational Mechanics

Chapter Contents

12.1 Introduction
12.2 Moment of Inertia
12.3 Angular Velocity
12.4 Torque
12.5 Rotation of a Rigid Body about a Fixed Axis
12.6 Angular Momentum
12.7 Conservation of Angular Momentum
12.8 Combined Translational and Rotational Motion of a Riggid Body
12.9 Uniform Pure Rolling
12.10 Instantaneous Axis of Rotation
11.11 Accelerated Pure Rolling
12.12 Angular Impulse
12.13 Toppling

12.1 Introduction

A particle means mass with negligible volume. A rigid body is made up of too many particles but distance between any two particles is always constant. In any type of motion of a rigid body this distance always remains constant. A particle has only translational motion. Even if a particle is rotating in a circle it has only translational motion and it has only translational kinetic energy $\frac{1}{2}mv^2$.

A rigid body may have either of the following three types of motions :

(i) Translational motion
(ii) Rotational motion
(iii) Translational plus rotational motion

In translational motion of the rigid body all particles of the rigid body have same linear displacement, same linear velocity and same linear acceleration. In rest two motions, different particles have different linear displacement, different linear velocity and different linear acceleration.

As far as translational motion is concerned we do not differentiate between a particle and a rigid body. This motion is already discussed in the chapter of kinematics. This is the reason, in the chapter of kinematics, sometimes we write: a particle is moving and sometimes we write: a block (or a body) is moving. Rest two motions are only defined for a rigid body. In the present chapter, we shall discuss these two motions of a rigid body.

12.2 Moment of Inertia

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating . The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion. If a body is at rest, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult it is to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis) and it depends on the mass as well as its distribution about that axis.

Moment of Inertia of a Single Particle

For a very simple case the moment of inertia of a single particle about an axis is given by,

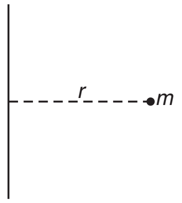


Fig. 12.1

$$\boxed{I = mr^2} \qquad \ldots\text{(i)}$$

Here, m is the mass of the particle and r its distance from the axis under consideration.

Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by,

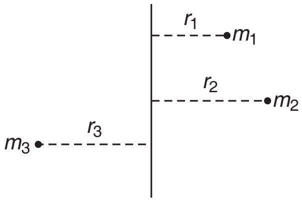


Fig. 12.2

$$\boxed{I = \sum_i m_i r_i^2} \qquad \ldots\text{(ii)}$$

where, r_i is the perpendicular distance from the axis to the i^{th} particle, which has a mass m_i.

For example, in Fig. 12.2:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Moment of Inertia of Rigid Bodies

For a continuous mass distribution such as found in a rigid body, we replace the summation of Eq. (ii) by an integral. If the system is divided into infinitesimal elements of mass dm and if r is the distance from a mass element to the axis of rotation, the moment of inertia is,

$$\boxed{I = \int r^2 \, dm}$$

where the integral is taken over the system.

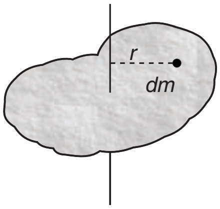
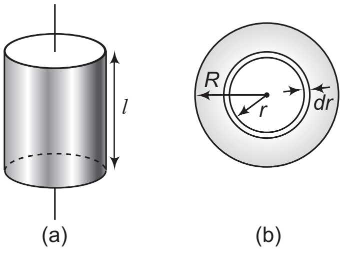


Fig. 12.3

Moment of Inertia of a Uniform Cylinder

Let us find the moment of inertia of a uniform cylinder about an axis through its centre of mass and perpendicular to its base. Mass of the cylinder is M and radius is R.

We first divide the cylinder into annular shells of width dr and length l as shown in figure. The moment of inertia of one of these shells is



(a) (b)

Fig. 12.4

$$dI = r^2 dm = r^2 \, (\rho \cdot dV)$$

Here, ρ = density of cylinder

and dV = volume of shell $= 2\pi r l \, dr$

$\therefore$ $dI = 2\pi \rho l r^3 \, dr$

The cylinder's moment of inertia is found by integrating this expression between 0 and R,

So, $$I = 2\pi\rho l \int_0^R r^3\,dr = \frac{\pi\rho l}{2} R^4 \qquad \ldots\text{(iii)}$$

The density ρ of the cylinder is the mass divided by the volume.

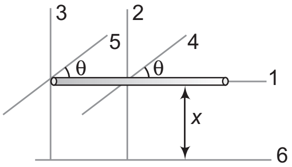
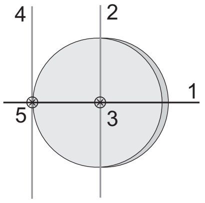
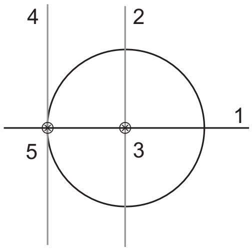
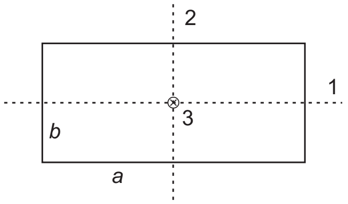
∴ $$\rho = \frac{M}{\pi R^2 l} \qquad \ldots\text{(iv)}$$

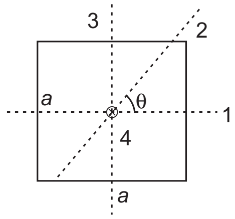
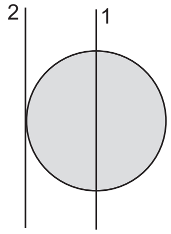
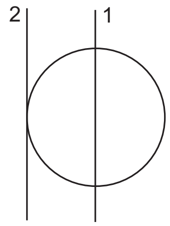
From Eqs. (iii) and (iv), we have

$$\boxed{I = \frac{1}{2} MR^2}$$

Proceeding in the similar manner we can find the moment of inertia of certain rigid bodies about some given axis. Moments of inertia of several rigid bodies with symmetry are listed in Table. 12.1.

Table 12.1

Thin rod		$I_1 = 0, I_2 = \frac{ml^2}{12}$ $I_3 = \frac{ml^2}{3}, I_4 = \frac{ml^2}{12}\sin^2\theta$ $I_5 = \frac{ml^2}{3}\sin^2\theta, I_6 = mx^2$
Circular disc		$I_1 = I_2 = \frac{mR^2}{4}$ $I_3 = I_1 + I_2 = \frac{mR^2}{2}$ $I_4 = I_2 + mR^2 = \frac{5}{4}mR^2$ $I_5 = I_3 + mR^2 = \frac{3}{2}mR^2$
Circular ring		$I_1 = I_2 = \frac{mR^2}{2}$ $I_3 = I_1 + I_2 = mR^2$ $I_4 = I_2 + mR^2 = \frac{3}{2}mR^2$ $I_5 = I_3 + mR^2 = 2mR^2$
Rectangular slab		$I_1 = \frac{mb^2}{12}$ $I_2 = \frac{ma^2}{12}$ $I_3 = I_1 + I_2$ $= \frac{m}{12}(a^2 + b^2)$

Square slab		$I_1 = I_2 = I_3 = \frac{ma^2}{12}$ $I_4 = I_1 + I_3 = \frac{ma^2}{6}$
Solid sphere		$I_1 = \frac{2}{5}mR^2$ $I_2 = I_1 + mR^2$ $= \frac{7}{5}mR^2$ m = mass of sphere
Hollow sphere		$I_1 = \frac{2}{3}mR^2$ $I_2 = I_1 + mR^2$ $= \frac{5}{3}mR^2$

Theorems on Moment of Inertia

There are two important theorems on moment of inertia, which, in some cases, enable the moment of inertia of a body to be determined about any general axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

Theorem of Parallel Axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.

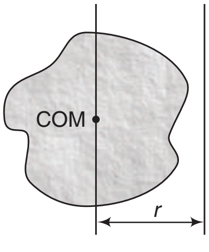


Fig. 12.5

Two such axes are shown in figure for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them then, these two are related by,

$$\boxed{I = I_{\text{COM}} + Mr^2}$$

We now present a proof of the above theorem.

Proof To prove this theorem, we consider two axes, both parallel to the z-axis, one through the center of mass and the other through a point P (Fig 12.6). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the centre of mass of the body; the coordinates of the centre of mass are then $x_{\text{cm}} = y_{\text{cm}} = z_{\text{cm}} = 0$. The axis through the centre of mass passes through this thin slice at point O and the parallel axis passes through point P, whose x and y coordinates are (a, b). The distance of this axis from the axis through the centre of mass is r, where $r^2 = a^2 + b^2$.

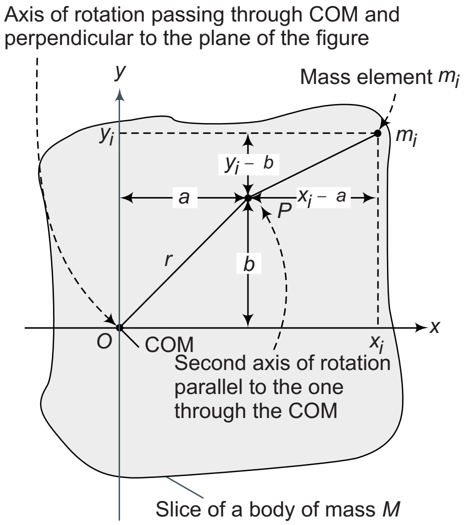

Axis of rotation passing through COM and perpendicular to the plane of the figure
y
Mass element m_i
m_i
y_i
$y_i - b$
a
$x_i - a$
P
r
b
x
O
COM
x_i
Second axis of rotation parallel to the one through the COM
Slice of a body of mass M

Fig. 12.6

We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then, the moment of inertia I_{COM} of the slice about the axis through the centre of mass (at O) is

$$I_{\text{COM}} = \sum_i m_i (x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i [(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slice, so we can extend the sums to include all particles in all slices. Then, I_P becomes the moment of inertia of the entire body for an axis through P. We then expand the squared terms, regroup and obtain

$$I_P = \sum_i m_i (x_i^2 + y_i^2) - 2a \sum_i m_i x_i - 2b \sum_i m_i y_i + (a^2 + y^2) \sum_i m_i$$

The first sum is I_{COM}. From the, definition of the centre of mass the second and third sums are proportional to x_{cm} and y_{cm}. These are zero because we have taken our origin to be the centre of mass. The final term is r^2 multiplied by the total mass or r^2. This completes our proof that

$$I_P = I_{\text{COM}} + Mr^2$$

Note *From the above theorem we can see that among several parallel axes, moment of inertia is least about an axis which passes through centre of mass. e. g.* I_2 *is least among* I_1, I_2 *and* I_3. *Similarly,* I_5 *is least among* I_4, I_5 *and* I_6.

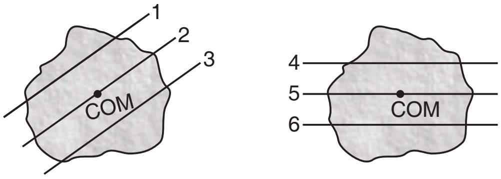


Fig. 12.7

Theorem of Perpendicular Axes

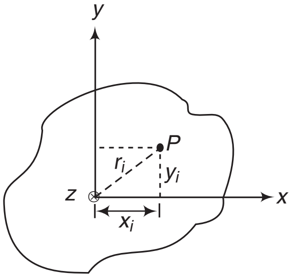


Fig. 12.8

This theorem is applicable only to the plane bodies (two dimensional). The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it. Let x and y axes be chosen in the plane of the body and z-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that

$$\boxed{I_z = I_x + I_y}$$

Proof Consider an arbitrary particle P of mass m_i, distant r_i from O and x_i and y_i are the perpendicular distances of point P from the x and y-axes respectively, we have

$$I_z = \sum_i m_i\, r_i^2,\, I_x = \sum_i m_i\, y_i^2 \text{ and } I_y = \sum_i m_i x_i^2$$

So that,

$$I_x + I_y = \sum_i m_i\, y_i^2 + \sum_i m_i x_i^2$$

$$= \sum_i m_i\,(y_i^2 + x_i^2) = \sum_i m_i r_i^2 = I_z$$

i.e. $$I_z = I_x + I_y$$ **Hence Proved**.

Radius of Gyration

Radius of gyration (K) of a body about an axis is the effective distance from this axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$\boxed{I = MK^2 \quad \text{or} \quad K = \sqrt{\frac{I}{M}}}$$

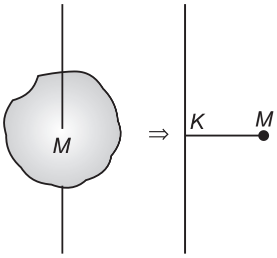


Fig. 12.9

e.g. radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

$$K = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

Extra Points to Remember

- Theorem of parallel axes is applicable for any type of rigid body whether it is a two dimensional or three dimensional, while the theorem of perpendicular axes is applicable for laminar type or two dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes (*x, y* and *z*) may be any point on the plane of body (it may even lie outside the body also). This point may or may not be the centre of mass of the body.
- If whole mass of the rigid body is kept at same distance *x* or *R* from the axis, then moment of intertia is mx^2 or mR^2, where *m* is the mass of whole body.

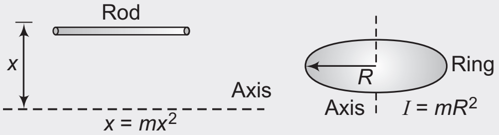


Fig. 12.10

- If a portion is symmetrically cut about an axis and mass of remaining portion is *M*. Then, moment of inertia of the remaining portion is same as the moment of inertia of the whole body of same mass *M*. e.g. in figure 12.11(a) moment of inertia of the section shown (a part of circular disc) about an axis perpendicular to its plane and passing through point *O* is $\frac{1}{2}MR^2$ as the moment of inertia of the complete disc is also $\frac{1}{2}MR^2$.

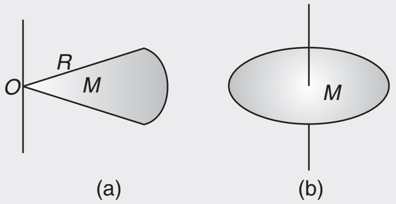


Fig. 12.11

Proof : Suppose the given section is $\frac{1}{n}$ th part of the disc, then mass of the disc will be *nM.*

$$I_{\text{disc}} = \frac{1}{2}(nM)R^2$$

$$\therefore \qquad I_{\text{section}} = \frac{1}{n} I_{\text{disc}} = \frac{1}{2} MR^2$$

- If whole mass of the rigid body is kept over the axis then, moment of inertia is zero. For example, moment of inertia of a thin rod about an axis passing through the rod is zero.

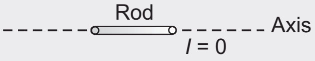


Fig. 12.12

Example 12.1 *Three particles of masses* 1g, 2g *and* 3g *are kept at points* (2cm, 0), (0,6 cm), (4cm, 3cm). *Find moment of inertia of all three particles (in* $gm\text{-}cm^2$*) about, (a) x-axis (b) y-axis (c) z-axis*

Solution (a) **About *x*-axis**

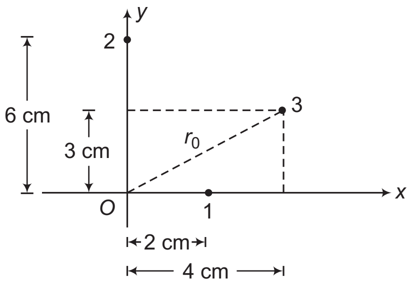


Fig. 12.13

$$I_x = I_1 + I_2 + I_3$$
$$= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here r = perpendicular distance of the particle from x-axis

$$\therefore \qquad I_1 = (1)(0)^2 + (2)(6)^2 + (3)(3)^2$$
$$= 99\text{g-cm}^2$$ **Ans.**

(b) **About *y*-axis**

$$I_y = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here, r = perpendicular distance of the particle from y-axis

$\therefore$ $$I_y = (1)(2)^2 + (2)(0)^2 + (3)(4)^2 = 52\,\text{g-cm}^2$$ **Ans.**

(c) **About z-axis** $$I_z = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here, r = perpendicular distance of the particle from z-axis.

$$r_0 = \sqrt{(3)^2 + (4)^2} = 5\,\text{cm}$$

$\therefore$ $$I_z = (1)(2)^2 + (2)(6)^2 + 3(5)^2 = 151\,\text{g-cm}^2$$ **Ans.**

Note *In the above example, from theorem of perpendicular axes, we can see that*

$$I_z = I_x + I_y$$

Example 12.2 *Three rods each of mass m and length l are joined together to form an equilateral triangle as shown in figure. Find the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of the triangle.*

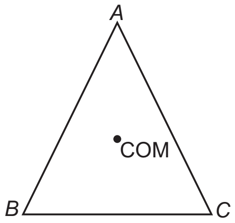


Fig. 12.14

Solution Moment of inertia of rod BC about an axis perpendicular to plane of triangle ABC and passing through the mid-point of rod BC (i.e. D) is

$$I_1 = \frac{ml^2}{12}$$

$$r = BD \tan 30°$$

or $$r = \left(\frac{l}{2}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{l}{2\sqrt{3}}$$

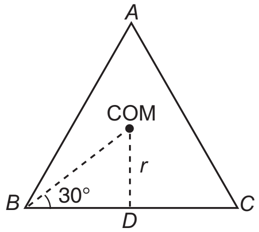


Fig. 12.15

From theorem of parallel axes, moment of inertia of this rod about the asked axis is

$$I_2 = I_1 + mr^2 = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}$$ **Ans.**

$\therefore$ Moment of inertia of all the three rods is

$$I = 3I_2 = 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}$$ **Ans.**

Example 12.3 *Find the moment of inertia of a solid sphere of mass M and radius R about an axis XX shown in figure. Also find radius of gyration about the given axis.*

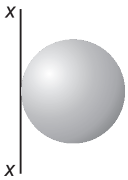


Fig. 12.16

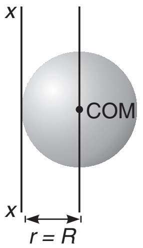


Fig. 12.17

Solution From theorem of parallel axis,

$$I_{XX} = I_{COM} + Mr^2 = \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2 \qquad \textbf{Ans.}$$

Radius of gyration, $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\frac{7}{5}MR^2}{M}} = \sqrt{\frac{7}{5}}R$

Note *If whole mass M is kept at a distance* $K\left(=\sqrt{\frac{7}{5}}R\right)$ *as a particle, then moment of inertia is again* $\frac{7}{5}MR^2$.

Example 12.4 *Consider a uniform rod of mass m and length 2l with two particles of mass m each at its ends. Let AB be a line perpendicular to the length of the rod and passing through its centre. Find the moment of inertia of the system about AB.*

Solution

$$I_{AB} = I_{\text{rod}} + I_{\text{both particles}}$$

$$= \frac{m(2l)^2}{12} + 2(ml^2)$$

$$= \frac{7}{3}ml^2 \qquad \textbf{Ans.}$$

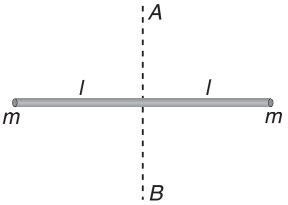


Fig. 12.18

Example 12.5 *Find the moment of inertia of the rod AB about an axis yy as shown in figure. Mass of the rod is m and length is l.*

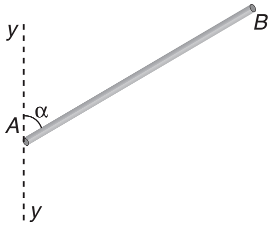


Fig. 12.19

Solution Mass per unit length of the rod $= \frac{m}{l}$

Mass of an element PQ of the rod is, $dm = \left(\frac{m}{l}\right)dx$

Perpendicular distance of this elemental mass about yy is, $r = x\sin\alpha$

$\therefore$ Moment of inertia of this small element of the rod (can be assumed as a point mass) about yy is,

$$dI = (dm)r^2 = \left(\frac{m}{l}dx\right)(x\sin\alpha)^2 = \left(\frac{m}{l}\sin^2\alpha\right)x^2\, dx$$

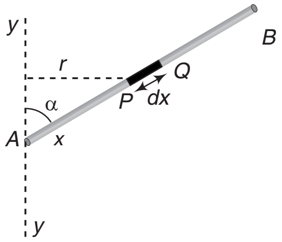


Fig. 12.20

∴ Moment of inertia of the complete rod,

$$I = \int_{x=0}^{x=l} dI = \frac{m}{l} \sin^2 \alpha \int_0^l x^2 \, dx = \frac{ml^2}{3} \sin^2 \alpha \qquad \textbf{Ans.}$$

Note (i) $I = 0$ if $\alpha = 0$ (ii) $I = \frac{ml^2}{3}$ if $\alpha = \frac{\pi}{2}$ or 90°

INTRODUCTORY EXERCISE 12.1

1. Find the radius of gyration of a rod of mass m and length $2l$ about an axis passing through one of its ends and perpendicular to its length.
2. A mass of 1 kg is placed at (1 m, 2 m, 0). Another mass of 2 kg is placed at (3 m, 4 m, 0). Find moment of inertia of both the masses about z-axis.
3. Four thin rods each of mass m and length l are joined to make a square. Find moment of inertia of all the four rods about any side of the square.
4. About what axis would a uniform cube have its minimum moment of inertia?
5. There are four solid balls with their centres at the four corners of a square of side a. The mass of each sphere is m and radius is r. Find the moment of inertia of the system about (i) one of the sides of the square (ii) one of the diagonals of the square.
6. A non-uniform rod AB has a mass M and length $2l$. The mass per unit length of the rod is mx at a point of the rod distant x from A. Find the moment of inertia of this rod about an axis perpendicular to the rod (a) through A (b) through the mid-point of AB.
7. The uniform disc shown in the figure has a moment of inertia of 0.6 kg-m^2 around the axis that passes through O and is perpendicular to the plane of the page. If a segment is cut out from the disc as shown, what is the moment of inertia of the remaining disc?

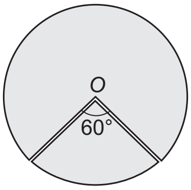


Fig. 12.21

8. If two circular disks of the same weight and thickness are made from metals having different densities. Which disk, if either will have the larger moment of inertia about its central axis.
9. Particles of masses 1g, 2g, 3g, ... 100g are kept at the marks 1cm, 2cm, 3cm, ..., 100cm respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.
10. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 the moment of inertia of the ring formed by the same rod about an axis passing through the centre of mass of the ring and perpendicular to the plane of the ring. Then find the ratio $\frac{I_1}{I_2}$.

12.3 Angular Velocity

Angular velocity is a vector quantity. It is represented by ω. Its SI unit is radian per second. It can be defined for following three situations :

(i) Angular velocity of a particle (in motion) about a fixed point.
(ii) Angular velocity of a rigid body in pure rotational motion.
(iii) Angular velocity of a rigid body in rotational and translational motion.

Angular Velocity of a Particle (in motion) about a Fixed Point

At a given instant a particle P has velocity **v**. It has position vector **r** with respect to a fixed point O as shown in figure.

After some time position vector has become **r′**. We can see two changes in its position vector.

First, its magnitude $|\mathbf{r}|$ has changed, second its direction has changed or we can say, its position vector has been rotated. If we resolve **v** along **r** and perpendicular to **r** then its two components $v_{\|}$ and $v_{\perp}$ have the following meanings.

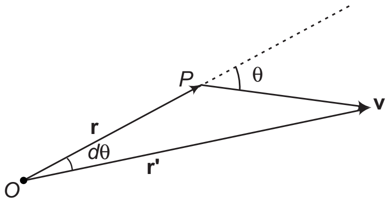


Fig. 12.22

$$v_{\|} = v\cos\theta = \frac{\mathbf{v}\cdot\mathbf{r}}{r}$$

$$= \text{component of } \mathbf{v} \text{ along } \mathbf{r} = \frac{d|\mathbf{r}|}{dt}$$

= rate by which magnitude of **r** changes

= rate by which distance of P from O changes.

$\theta =$ angle between **r** and **v**

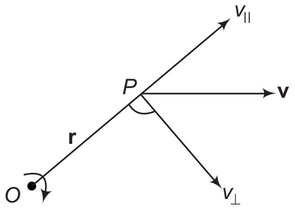


Fig. 12.23

$$\frac{v_{\perp}}{|\mathbf{r}|} = \frac{v_{\perp}}{r} = \frac{v_{\perp}}{OP} = \omega$$

= angular velocity of particle P about point O at this instant

= rate by which **r** rotates

If θ is acute, cos θ or $v_{\|}$ is positive i.e. distance of P from O is increasing. If θ is obtuse, cos θ or $v_{\|}$ is negative i.e. distance of P from O is decreasing. If θ is 90°, then cos θ or $v_{\|}$ is zero or distance of P from O is constant. For example, when particle rotates in a circle then with respect to centre, θ is always 90°. This is the reason, why distance of particle from centre always remains constant.With respect to any other point θ is sometimes acute and sometimes obtuse.Therefore, distance sometimes increases and sometimes decreases.

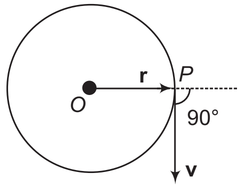


Fig. 12.24

$\mathbf{v}_{\perp} = \omega \times \mathbf{r}$ i.e. perpendicular component of velocity in vector form is the cross product of ω and **r**.

Direction of ω is given by right hand screw law. For example: in Fig. 12.24 rotation is clockwise. So, ω is perpendicular to paper inwards. If a particle moves in a straight line, then about any point lying on this line angle between **r** and **v** is 0° or 180°. Hence, $v_{\perp} = 0$ or $\omega = 0$.

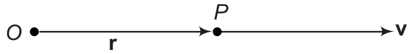


Fig. 12.25

Angular Velocity of a Rigid Body in Pure Rotational Motion

Consider a rigid body rotating about a fixed line AB. Consider a particle P. Draw a perpendicular PO to the axis of rotation. In time Δt, this particle moves to point Q.

Let $$\angle QOP = \Delta\theta$$

Then, we say that the particle has rotated through an angle $\Delta\theta$. In fact, all the particles of the rigid body have rotated the same angle $\Delta\theta$ or we can say that the whole body has rotated through an angle $\Delta\theta$.

The average angular velocity of the rigid body during the time interval Δt is

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

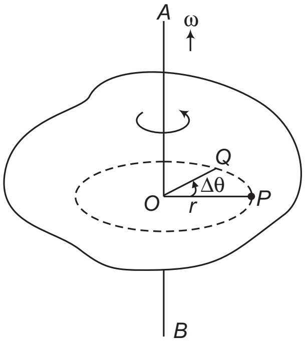


Fig. 12.26

The instantaneous angular velocity of the rigid body is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Direction of angular velocity is given by right hand rule. The direction of angular velocity is defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of rotation.

For example, direction of ω in the given figure is along the axis of rotation from B to A. The magnitude of angular velocity is called angular speed. However, we shall continue to use the word angular velocity.

Angular Velocity of a Rigid Body in Rotational and Translational Motion

Consider two particles A and B on a rigid body (in translational and rotational motion). In general, velocity of A is not equal to the velocity of B.

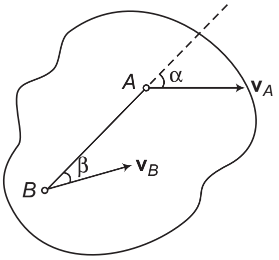


Fig. 12.27

or $$\mathbf{v}_A \neq \mathbf{v}_B$$

Find their components along AB and perpendicular to AB. Now,

(a) Along AB their components are always equal or,

$$v_A \cos\alpha = v_B \cos\beta$$

This is because, in a rigid body distance between two particles (here A and B) is always constant.

If the components are not equal then the distance AB will either increase (when $v_A \cos\alpha > v_B \cos\beta$) or decrease (when $v_B \cos\beta > v_A \cos\alpha$).

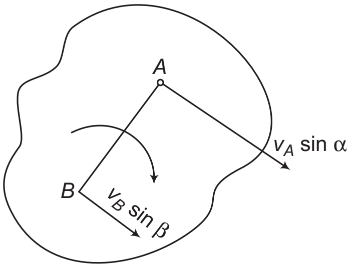


Fig. 12.28

(b) Find relative component perpendicular to AB and divide it by the distance AB to find angular velocity of the rigid body. In the given figure, the perpendicular components are $v_A \sin\alpha$ and $v_B \sin\beta$ (in the same direction). Suppose $v_A \sin\alpha > v_B \sin\beta$. Then, the relative component perpendicular to AB is

$$v_r = v_A \sin\alpha - v_B \sin\beta$$

$$\therefore \qquad \omega = \frac{v_r}{AB} = \frac{v_A \sin\alpha - v_B \sin\beta}{AB}$$

As, $v_A \sin\alpha > v_B \sin\beta$, so rotation of the body is clockwise and according to right hand rule, angular velocity vector is perpendicular to paper inwards. This direction is also shown like ⊗.

✓ Extra Points to Remember

- If a particle P is moving in a circle, its angular velocity about centre of the circle (ω_C) is two times the angular velocity about any point on the circumference of the circle (ω_O) or $\omega_C = 2\omega_O$

  This is because $\angle P'CP = 2\angle P'OP$ (by property of a circle)

  $$\omega_C = \frac{\angle P'CP}{t_{pp'}}, \quad \omega_O = \frac{\angle P'OP}{t_{pp'}}$$

  From these relations we can see that $\omega_C = 2\omega_O$.

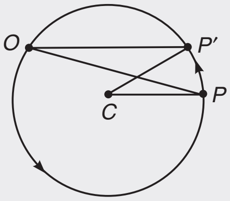
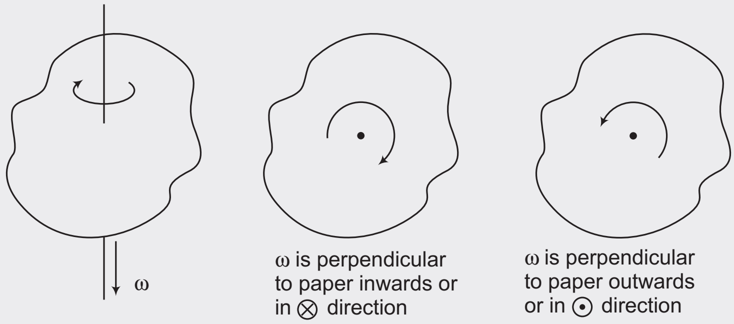


Fig. 12.29

- In pure translational motion of a rigid body its angular velocity will be zero.
- According to right hand rule direction of angular velocity in some cases (in pure rotational motion) have been shown below :



ω is perpendicular to paper inwards or in ⊗ direction

ω is perpendicular to paper outwards or in ⊙ direction

Fig. 12.30

Example 12.6 *Rod AB has a length L. Velocity of end A of the rod has velocity v_0 at the given instant.*

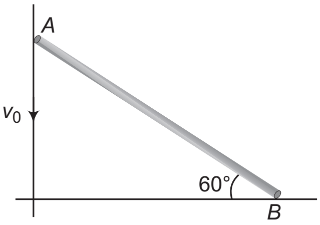


Fig. 12.31

(a) Which type of motion the rod has?
(b) Find velocity of end B at the given instant.
(c) Find the angular velocity of the rod.

Solution (a) The rod has rotational plus translational motion.

(b) Let velocity of end *B* is v_B in the direction shown in figure.

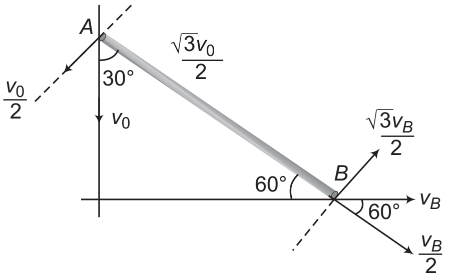


Fig. 12.32

Components of $\mathbf{v}_A$ and $\mathbf{v}_B$ along *AB* and perpendicular to *AB* are also shown in the same figure.

Rod is a rigid body. So, distance *AB* should remain constant or the components along *AB* should be same.

$$\therefore \qquad \frac{\sqrt{3}v_0}{2} = \frac{v_B}{2} \quad \text{or} \quad v_B = \sqrt{3}v_0 \qquad \textbf{Ans.}$$

(c) Perpendicular to *AB*, components are in opposite directions. So, the relative component will be

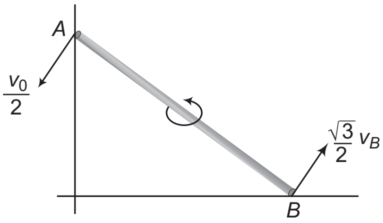


Fig. 12.33

$$v_r = \frac{v_0}{2} + \frac{\sqrt{3}v_B}{2}$$

Substituting $v_B = \sqrt{3}v_0$ we get, $\quad v_r = 2v_0$

Now angular velocity of the rod is $\omega = \dfrac{v_r}{AB}$

or $\omega = \dfrac{2v_o}{L}$ **Ans.**

Rotation of the rod is anticlockwise. Therefore, from right hand rule ω is perpendicular to paper outwards or in $\odot$ direction.

INTRODUCTORY EXERCISE 12.2

1. Find angular speed of second's clock.
2. Two points *P* and *Q*, diametrically opposite on a disc of radius *R* have linear velocities *v* and 2*v* as shown in figure. Find the angular speed of the disc.

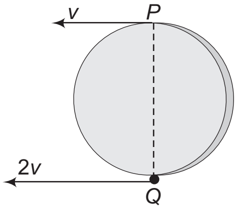


Fig. 12.34

3. A particle is located at (3 m, 4 m) and moving with $\mathbf{v} = (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ m/s. Find its angular velocity about origin at this instant.
4. In the figure shown, the instantaneous speed of end *A* of the rod is *v* to the left. Find angular velocity of the rod at given instant.

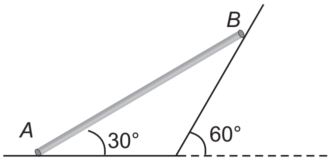


Fig. 12.35

12.4 Torque

Suppose a force **F** is acting on a particle *P* and let **r** be the position vector of this particle about some reference point *O*. The torque of this force **F**, about *O* is defined as,

$$\boxed{\tau = \mathbf{r} \times \mathbf{F}}$$

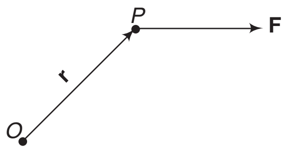


Fig. 12.36

This is a vector quantity having its direction perpendicular to both **r** and **F**, according to the rule of cross product.

Note *Here,* $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_O$

$\mathbf{r}_P$ = *position vector of point, where force is acting and*

$\mathbf{r}_O$ = *position vector of point about which torque is required.*

Torque of a Force about a Line

Consider a rigid body rotating about a fixed axis AB. Let **F** be a force acting on the body at point P. Take the origin O somewhere on the axis of rotation. The torque of **F** about O is

$$\tau = \mathbf{r} \times \mathbf{F}$$

Its component along AB is called the torque of **F** about AB. This is also equal to

$$\tau = F \times r_\perp$$

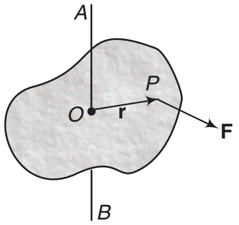


Fig. 12.37

Here, $r_\perp$ = perpendicular distance of point of application of force from the line AB

Extra Points to Remember

- When a rigid body is rotating about a fixed axis and a force is applied on it at some point then we are concerned with the component of torque of this force about the axis of rotation not with the net torque.
- The component of torque about axis of rotation is independent of the choice of the origin O, so long as it is chosen on the axis of rotation, i.e. we may choose point O anywhere on the line AB.
- Component of torque along axis of rotation AB is zero if
  (a) **F** is parallel to AB (b) **F** intersects AB at some point
- If **F** is perpendicular to AB, but does not intersect it, then component of torque about line AB = magnitude of force **F** × perpendicular distance of **F** from the line AB (called the lever arm or moment arm) of this torque.
- If there are more than one force $\mathbf{F}_1, \mathbf{F}_2, \ldots$ acting on a body, the total torque will be

$$\tau = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \ldots$$

But if the forces act on the same particle, one can add the forces and then take the torque of the resultant force, or

$$\tau = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \ldots)$$

Example 12.7 *Find the torque of a force* $\mathbf{F} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ *N about a point O. The position vector of point of application of force about O is* $\mathbf{r} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ *m.*

Solution Torque $\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$

$$= \hat{\mathbf{i}}\,(-9+2) + \hat{\mathbf{j}}(-1+6) + \hat{\mathbf{k}}(4-3)$$

or $\tau = (-7\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}})$ N-m **Ans.**

Example 12.8 *A small ball of mass 1.0 kg is attached to one end of a 1.0 m long massless string and the other end of the string is hung from a point O. When the resulting pendulum is making 30° from the vertical, what is the magnitude of net torque about the point of suspension?*
[Take $g = 10\ m/s^2$*]*

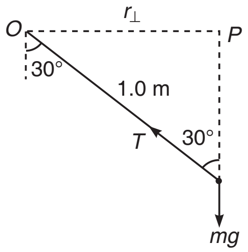


Fig. 12.38

Solution Two forces are acting on the ball

(*i*) tension (T) (*ii*) weight (mg)

Torque of tension about point O is zero, as it passes through O.

$$\tau_{mg} = F \times r_\perp$$

Here, $$r_\perp = OP = 1.0 \sin 30° = 0.5 \text{ m}$$

$\therefore$ $$\tau_{mg} = (mg)(0.5)$$

$$= (1)(10)(0.5) = 5 \text{ N-m}$$ **Ans.**

Example 12.9 *A force* $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *N is acting at point* $P(2 \text{ m}, -3 \text{ m}, 6 \text{ m})$. *Find torque of this force about a point* O *whose position vector is* $(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ *m.*

Solution $\tau = \mathbf{r} \times \mathbf{F}$ Here, $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_O = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = (2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ m

Now, $$\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = (-17\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \text{ N-m}$$ **Ans.**

INTRODUCTORY EXERCISE 12.3

1. A force $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ N is acting on a body at point (2 m, 4 m, –2 m). Find torque of this force about origin.
2. A particle of mass $m = 1$kg is projected with speed $u = 20\sqrt{2}$ m/s at angle $\theta = 45°$ with horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.
3. Point C is the centre of mass of the rigid body shown in figure. Find the total torque acting on the body about point C.

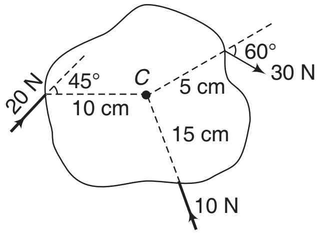


Fig. 12.39

4. Find the net torque on the wheel in figure about the point O if $a = 10$ cm and $b = 25$ cm.

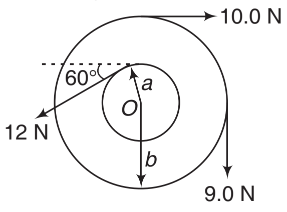


Fig. 12.40

12.5 Rotation of a Rigid Body about a Fixed Axis

When a body is rotating about a fixed axis, any point *P* located in the body travels along a circular path. Before, analyzing the circular motion of point *P,* we will first study the angular motion properties of a rigid body.

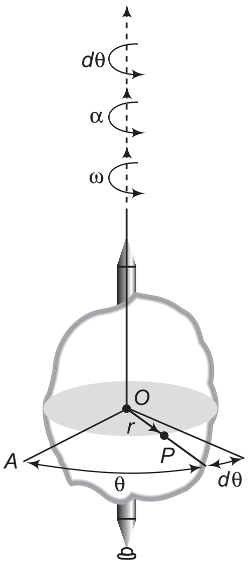


Fig. 12.41

Angular Motion

Since, a point is without dimension, it has no angular motion. Only lines or bodies undergo angular motion. Let us consider the angular motion of a radial line *r* located with the shaded plane.

Angular Position

The angular position of *r* is defined by the angle θ, measured between a fixed reference line *OA* and *r.*

Angular Displacement

The change in the angular position, often measured as a differential *d*θ is called the angular displacement. (Finite angular displacements are not vector quantities, although differential rotations *d*θ are vectors). This vector has a magnitude *d*θ and the direction of *d*θ is along the axis.

Specifically, the direction of *d*θ is determined by right hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb or *d*θ points upward.

Angular Velocity

The time rate of change in the angular position is called the angular velocity ω. Thus,

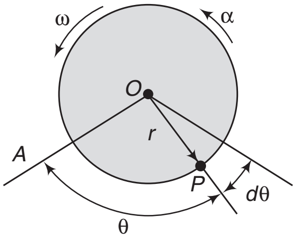


Fig. 12.42

$$\boxed{\omega = \frac{d\theta}{dt}} \qquad \ldots\text{(i)}$$

It is expressed here in scalar form, since its direction is always along the axis of rotation, i.e. in the same direction as *d*θ.

Angular Acceleration

The angular acceleration α measures the time rate of change of the angular velocity. Hence, the magnitude of this vector may be written as,

$$\alpha = \frac{d\omega}{dt} \qquad \ldots(ii)$$

It is also possible to express α as,

$$\alpha = \frac{d^2\theta}{dt^2}$$

The line of action of α is the same as that for ω, however its sense of direction depends on whether ω is increasing or decreasing with time. In particular, if ω is decreasing, α is called an angular deceleration and therefore, has a sense of direction which is opposite to ω.

Torque and Angular Acceleration for a Rigid Body

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}$$

Thus, for a rigid body we have the rotational analog of Newton's second law :

$$\Sigma\tau = I\alpha \qquad \ldots(iii)$$

Following two points are important regarding the above equation :

(i) The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration α is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

Rotation with Constant Angular Acceleration

If the angular acceleration of the body is constant then Eqs. (i) and (ii) when integrated yield a set of formulae which relate the body's angular velocity, angular position and time. These equations are similar to equations used for rectilinear motion. Table given ahead compares the linear and angular motion with constant acceleration.

Table 12.2

Straight line motion with constant linear acceleration	Fixed axis rotation with constant angular acceleration
$a =$ constant	$\alpha =$ constant
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = s_0 + ut + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2a(s - s_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Here, θ_0 and ω_0 are the initial values of the body's angular position and angular velocity respectively.

Note *If α is not constant then we will have to take help of either differentiation or integration (with limits). The equations involved are :*

Equations of Differentiation

$$\omega = \frac{d\theta}{dt} \text{ and } \alpha = \frac{d\omega}{dt} = \omega \frac{d\omega}{d\theta}$$

Equations of Integration

$$\int d\theta = \int \omega dt, \int d\omega = \int \alpha dt \text{ and } \int \omega d\omega = \int \alpha d\theta$$

Kinetic Energy of a Rigid Body Rotating about a Fixed Axis

Suppose a rigid body is rotating about a fixed axis with angular speed ω. Then, kinetic energy of the rigid body will be

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \, (\omega r_i)^2$$

$$= \frac{1}{2} \omega^2 \sum_i m_i r_i^2 = \frac{1}{2} I\omega^2 \quad (\text{as } \sum_i m_i r_i^2 = I)$$

Thus, $$\boxed{\text{KE} = \frac{1}{2} I\omega^2}$$

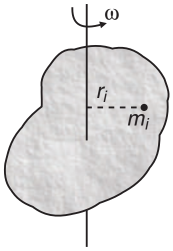


Fig. 12.43

Sometimes it is called the rotational kinetic energy.

Extra Points to Remember

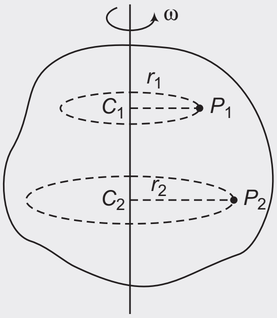


Fig. 12.44

- In pure rotational motion of a rigid body all particles rotate in circular paths except the particles lying on the axis.The particles lying on the axis are at rest.
- Planes of all circular paths are mutually parallel with their centres lying on the axis.
- Linear velocity of any particle is tangential to its own circle and its linear speed is,

  $$v = r\omega$$

  In this equation, ω is same for all particles (and that is also called ω of the rigid body).

  ∴ $$v \propto r$$

  In the figure shown, $r_2 > r_1 \Rightarrow \therefore v_2 > v_1$

  Further, r for the particles lying on the axis is zero. Therefore, their linear velocity is zero or they are at rest.
- Since every particle is rotating in a circle (except the particles lying on the axis). Hence acceleration of the particle has two components :

  (i) Radial component or centripetal acceleration given by

  $$a_r = r\omega^2 = \frac{v^2}{r}$$

  This component is always towards the centre and this can't be zero.

  (ii) Tangential component given by $a_t = \frac{dv}{dt} = r\alpha$ where, $\alpha = \frac{d\omega}{dt}$

  This component is tangential and it may be zero, positive or negative. If v or ω is constant then a_t is zero. If v or ω is increasing then this component is positive and in the direction of linear velocity. If v or ω is decreasing then this component is negative and in the opposite direction of linear velocity.

  (iii) Net acceleration of the particle is the resultant of these two perpendicular components.

Note *From the above discussions, we have seen that pure rotational motion of a rigid body is nothing but circular motion of its different particles. So, before solving the problems of this topic make yourself expert in circular motion.*

Example 12.10 *A solid sphere of mass 2 kg and radius 1 m is free to rotate about an axis passing through its centre. Find a constant tangential force F required to rotate the sphere with 10 rad/s in 2 s. Also find the number of rotations made by the sphere in that time interval.*

Solution Since, the force is constant, the torque produced by it and the angular acceleration α will be constant. Hence, we can apply

$$\omega = \omega_0 + \alpha t \quad \Rightarrow \quad 10 = 0 + (\alpha)(2) \quad \Rightarrow \quad \alpha = 5 \text{ rad/s}^2$$

Further, the force is tangential. Therefore, the perpendicular distance from the axis of rotation will be equal to the radius of the sphere.

$$\therefore \qquad \alpha = \frac{\tau}{I} = \frac{F \cdot R}{\frac{2}{5} mR^2} = \frac{5F}{2mR} \quad \text{or} \quad F = \frac{2mR\alpha}{5}$$

Substituting the value, we have $F = \dfrac{(2)(2)(1)(5)}{(5)} = 4 \text{ N}$ **Ans.**

Further, angle rotated $\theta = \dfrac{1}{2}\alpha t^2 = \dfrac{1}{2}(5)(2)^2 = 10 \text{ rad}$

$\therefore$ Number of rotations $n = \dfrac{\theta}{2\pi} = \dfrac{10}{2\pi} = \dfrac{5}{\pi}$ **Ans.**

Example 12.11 *The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where θ is in radians and t is in seconds. What are the angular velocities at*

(a) $t = 2.0$ s and *(b) $t = 4.0$ s ?*

(c) What is the average angular acceleration for the time interval that begins at $t = 2.0$ s and ends at $t = 4.0$ s?

(d) What are the instantaneous angular acceleration at the beginning and the end of this time interval?

Solution Angular velocity $\omega = \dfrac{d\theta}{dt} = \dfrac{d}{dt}(4t - 3t^2 + t^3)$ or $\omega = 4 - 6t + 3t^2$

(a) At $t = 2.0$ s, $\omega = 4 - 6 \times 2 + 3(2)^2$ or $\omega = 4$ rad/s **Ans.**

(b) At $t = 4.0$ s, $\omega = 4 - 6 \times 4 + 3(4)^2$ or $\omega = 28$ rad/s **Ans.**

(c) Average angular acceleration $\alpha_{av} = \dfrac{\omega_f - \omega_i}{t_f - t_i} = \dfrac{28 - 4}{4 - 2}$ or $\alpha_{av} = 12 \text{ rad/s}^2$ **Ans.**

(d) Instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(4 - 6t + 3t^2) \quad \text{or} \quad \alpha = -6 + 6t$$

At $t = 2.0$ s, $\alpha = -6 + 6 \times 2 = 6 \text{ rad/s}^2$ **Ans.**

At $t = 4.0$ s, $\alpha = -6 + 6 \times 4 = 18 \text{ rad/s}^2$ **Ans.**

Example 12.12 *A circular disc is rotating with an angular speed (in radian per sec)*

$$\omega = 2t^2$$

Given, $CP = 2\,m$

In terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$, *at* $t = 1$ s *find,*

(a) ω

(b) α

(c) linear velocity of the particle lying at P

(d) linear acceleration of the particle lying at P

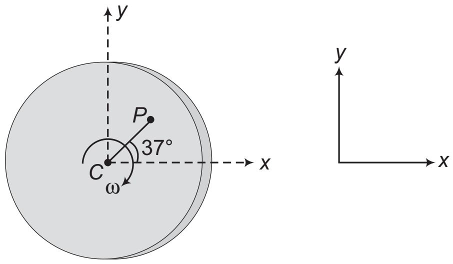


Fig. 12.45

Solution $\omega = 2t^2 \quad \Rightarrow \quad \alpha = \dfrac{d\omega}{dt} = 4t$

At $t = 1$s, $\omega = 2\,\text{rad/s}$ and $\alpha = 4\,\text{rad/s}^2$

For the particle at P, $r = CP = 2\,\text{m}$

(a) Rotation is clockwise. So, according to right hand rule, ω is perpendicular to paper inwards along negative z-direction.

$$\therefore \qquad \boldsymbol{\omega} = (-2\hat{\mathbf{k}})\,\text{rad/s}$$

(b) ω is increasing. So, α is also in the direction of ω.

$$\therefore \qquad \boldsymbol{\alpha} = (-4\hat{\mathbf{k}})\,\text{rad/s}^2$$

(c) $v = r\omega = (2)(2) = 4\,\text{m/s}$

This velocity is tangential to the doted circle of P as shown in figure.

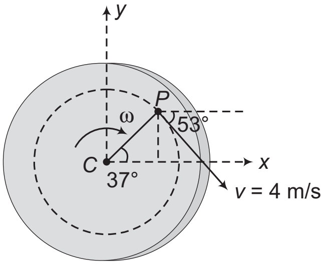


Fig. 12.46

$$\therefore \qquad \mathbf{v} = (4\cos 53°)\hat{\mathbf{i}} - (4\sin 53°)\hat{\mathbf{j}}$$

$$\text{or} \qquad \mathbf{v} = (2.4\,\hat{\mathbf{i}} - 3.2\,\hat{\mathbf{j}})\,\text{m/s}$$

(d) Acceleration of the particle has two components

(i) $a_r = r\omega^2$ (radial component)

$$= (2)(2)^2 = 8\,\text{m/s}^2$$

This components is towards centre C.

(ii) $a_t = r\alpha$ (tangential component)

$$= (2)(4) = 8\,\text{m/s}^2$$

This component is in the direction of linear velocity, as ω or v is increasing.

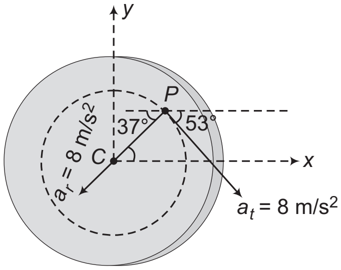


Fig. 12.47

$$\therefore \qquad \mathbf{a} = (8\cos 53° - 8\cos 37°)(\hat{\mathbf{i}}) + (8\sin 53° + 8\sin 37°)(-\hat{\mathbf{j}})$$

$$\text{or} \qquad \mathbf{a} = (-1.6\hat{\mathbf{i}} - 11.2\hat{\mathbf{j}})\,\text{m/s}^2 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 12.4

1. A wheel rotating with uniform angular acceleration covers 50 rev in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.
2. A body rotates about a fixed axis with an angular acceleration 1 rad/s^2. Through what angle does it rotates during the time in which its angular velocity increases from 5 rad/s to 15 rad/s?
3. A flywheel of moment of inertia 5.0 kg-m^2 is rotated at a speed of 10 rad/s. Because of the friction at the axis it comes to rest in 10 s. Find the average torque of the friction.
4. A wheel starting from rest is uniformly accelerated at 4 rad/s^2 for 10 s. It is allowed to rotate uniformly for the next 10 s and is finally brought to rest in the next 10 s. Find the total angle rotated by the wheel.
5. A wheel of mass 10 kg and radius 0.2 m is rotating at an angular speed of 100 rpm, when the motion is turned off. Neglecting the friction at the axis, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 rev. Assume wheel to be a disc.
6. A solid body rotates about a stationary axis according to the law $\theta = 6t - 2t^3$. Here, θ is in radian and t in seconds. Find
   (a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop,
   (b) the angular acceleration at the moment when the body stops.

Hint If $y = y(t)$, then mean/average value of y between t_1 and t_2 is $< y > = \dfrac{\int_{t_1}^{t_2} y(t)\,dt}{t_2 - t_1}$.

7. A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of 2 rad/s^2. At what time will the body have kinetic energy same as the initial value if the torque continues to act ?
8. A wheel whose moment of inertia is 0.03 kg m^2, is accelerated from rest to 20 rad/s in 5 s. When the external torque is removed, the wheel stops in 1 min. Find
   (a) the frictional torque, (b) the external torque.
9. A flywheel whose moment of inertia about its axis of rotation is 16 kg-m^2 is rotating freely in its own plane about a smooth axis through its centre. Its angular velocity is 9 rad s^{-1} when a torque is applied to bring it to rest in t_0 seconds. Find t_0 if
   (a) the torque is constant and of magnitude 4 N-m,
   (b) the magnitude of the torque after t seconds is given by kt.
10. A shaft is turning at 65 rad/s at time zero. Thereafter, angular acceleration is given by $\alpha = -10\ \text{rad/s}^2 - 5t\ \text{rad/s}^2$
    where t is the elapsed time.
    (a) Find its angular speed at $t = 3.0$ s.
    (b) How much angle does it turn in these 3s ?
11. The angular velocity of a gear is controlled according to $\omega = 12 - 3t^2$ where ω, in radian per second, is positive in the clockwise sense and t is the time in seconds. Find the net angular displacement $\Delta\theta$ from the time $t = 0$ to $t = 3$ s. Also, find the number of revolutions N through which the gear turns during the 3 s.

12.6 Angular Momentum

A mass moving in a straight line has linear momentum $(\mathbf{P})$. When a mass rotates about some point/axis, there is momentum associated with rotational motion called the angular momentum $(\mathbf{L})$. Just as net external force is required to change the linear momentum of an object a net external torque is required to change the angular momentum of an object. Keeping in view the problems asked in JEE, the angular momentum is classified in following three types.

Angular Momentum of a Particle about a Fixed Point

Suppose a particle A of mass m is moving with linear momentum $\mathbf{P} = m\mathbf{v}$. Its angular momentum $\mathbf{L}$ about point O is defined as

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times (m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v})}$$

Here, $\mathbf{r}$ is the radius vector of particle A about O at that instant of time. The magnitude of $\mathbf{L}$ is

$$\boxed{L = mvr \sin\theta = mvr_\perp}$$

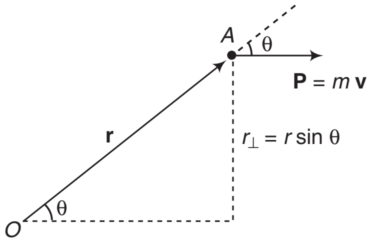


Fig. 12.48

Here, $r_\perp = r\sin\theta$ is the perpendicular distance of line of action of velocity $\mathbf{v}$ from point O. The direction of $\mathbf{L}$ is same as that of $\mathbf{r} \times \mathbf{v}$.

Direction of angular momentum can also be by right hand rule as shown below:

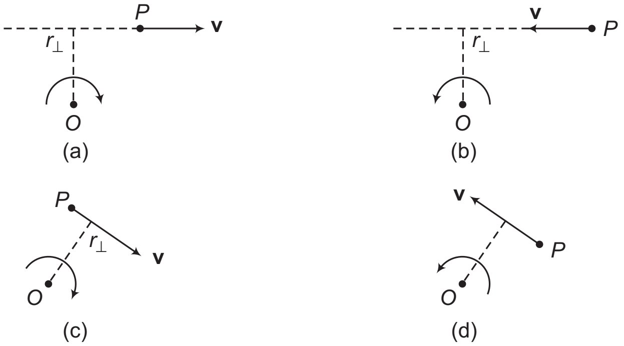


Fig. 12.49

In figures (a) and (c), rotation is clockwise. Hence, according to right hand rule angular momentum is perpendicular to paper inwards or in ⊗ direction.

In figures (b) and (d), rotation is anticlockwise. Hence, direction of angular momentum is perpendicular to paper outwards or in ʘ direction.

✓ Extra Points to Remember

- If line of action of velocity passes through the point O, then $r_\perp = 0$. Therefore, angular momentum is zero.

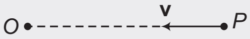
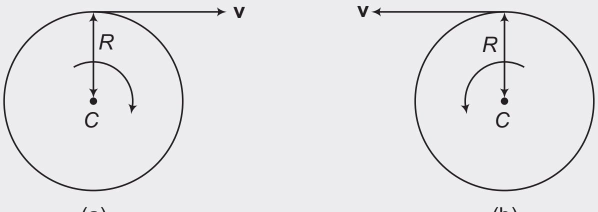


Fig. 12.50

- If a particle rotates in a circle then $r_\perp$ from centre is always equal to R (= radius of circle). Or, θ between $\mathbf{r}$ and $\mathbf{v}$ (or $\mathbf{P}$) is always 90°. Therefore, angular momentum about centre is $L = mvR$



(a) (b)

Fig. 12.51

In figure (a), direction of angular momentum is perpendicular to paper inwards ⊗ and in figure (b) outwards or ⊙.

- If a particle is moving with a constant velocity (speed and direction of velocity both are constant) then angular momentum about any point always remains constant. But this constant value will be different about different points. In the figure shown,

$L_{O_1} = 0$ as $r_\perp = 0$

$L_{O_2} = mvb$ (= constant), perpendicular to paper inwards

$L_{O_3} = mvc$ (= constant), perpendicular to paper outwards.

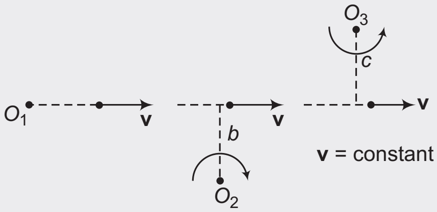


Fig. 12.52

Angular Momentum of a Rigid Body Rotating about a Fixed Axis

Suppose a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case $\mathbf{r}$ and $\mathbf{P}$ are perpendicular to each other and $\mathbf{r} \times \mathbf{P}$ is along the axis. Thus, component of $\mathbf{r} \times \mathbf{P}$ along the axis is mvr itself. The angular momentum of the whole rigid body about AB is the sum of components of all particles, i.e.

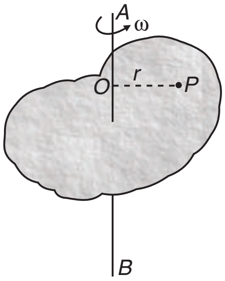


Fig. 12.53

$$L = \sum_i m_i r_i v_i$$

Here, $$v_i = r_i \omega$$

∴ $$L = \sum_i m_i r_i^2 \omega$$

or $$L = \omega \sum_i m_i r_i^2 \quad \text{or} \quad \boxed{L = I\omega} \qquad \left(\text{as } \sum_i m_i r_i^2 = I\right)$$

Here, I is the moment of inertia of the rigid body about AB.

Thus, $L = I\omega$ is the component of angular momentum of the whole rigid body about axis of rotation AB. Direction of this component is again given by right hand rule. For example, in the given figure $L = I\omega$ is upwards or along the axis of rotation from B to A.

Extra Points to Remember

- The vector relation $\mathbf{L} = I\omega$ is not correct in the above case because $\mathbf{L}$ and ω do not point in the same direction, but we could write $L_{AB} = I\omega$. If however the body is symmetric about the axis of rotation $\mathbf{L}$ and ω are parallel and we can write $(L = I\omega)$ in vector form as $\mathbf{L} = I\omega$.
- By symmetric we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation.
- Thus, remember that $\mathbf{L} = I\omega$ applies only to bodies that have symmetry about the (fixed) rotational axis. Here, $\mathbf{L}$ stands for total angular momentum. However the relation $L_{AB} = I\omega$ holds for any rigid body symmetrical or not that is rotating about a fixed axis.

Angular Momentum of a Rigid Body in Combined Rotation and Translation

Let O be the fixed point in an inertial frame of reference.

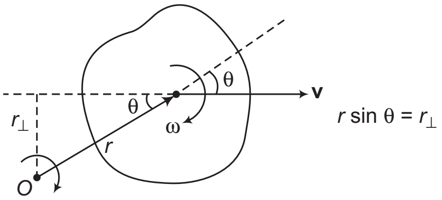


Fig. 12.54

Angular momentum of the rigid body about O is the vector sum of two terms (as discussed above).

(i) $mvr\sin\theta$ or $mvr_{\perp}$ (ii) $I\omega$

Here, $\mathbf{v}$ is the velocity of centre of mass. Moment of inertia I is about an axis passing through centre of mass. Directions of above two terms can be determined by right hand rule. If both the terms are in the same direction then these two terms are additive and if they are in opposite directions, then they are subtractive.

Example 12.13 *A particle of mass m is moving along the line $y = b, z = 0$ with constant speed v. State whether the angular momentum of particle about origin is increasing, decreasing or constant.*

Solution $|\mathbf{L}| = mvr\sin\theta = mvr_{\perp} = mvb$

$\therefore$ $|\mathbf{L}| =$ constant as m, v and b all are constants.

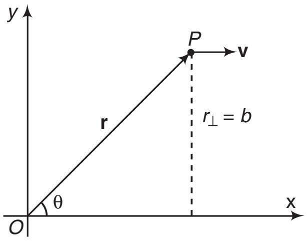


Fig. 12.55

Direction of $\mathbf{r} \times \mathbf{v}$ also remains the same. Therefore, angular momentum of particle about origin remains constant with due course of time.

Note *In this problem $|\mathrm{r}|$ is increasing, θ is decreasing but $r\sin\theta$, i.e. b remains constant. Hence, the angular momentum remains constant.*

Example 12.14 *A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of particle at any time t about O before the particle strikes the ground again.*

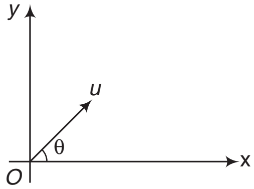


Fig. 12.56

Solution $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$

Here, $\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = (u\cos\theta)t\hat{\mathbf{i}} + (ut\sin\theta - \frac{1}{2}gt^2)\hat{\mathbf{j}}$

and $\mathbf{v}(t) = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = (u\cos\theta)\hat{\mathbf{i}} + (u\sin\theta - gt)\hat{\mathbf{j}}$

$$\therefore \quad \mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ (u\cos\theta)t & (u\sin\theta)t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

$$= m\left[(u^2\sin\theta\cos\theta)t - (u\cos\theta)gt^2 - (u^2\sin\theta\cos\theta)t + \frac{1}{2}(u\cos\theta)gt^2\right]\hat{\mathbf{k}}$$

$$= -\frac{1}{2}m(u\cos\theta)gt^2\ \hat{\mathbf{k}}$$ **Ans.**

Example 12.15 *A rod of mass 2kg and length 2m is rotating about its one end O with an angular velocity* $\omega = 4$ *rad/s. Find angular momentum of the rod about the axis rotation.*

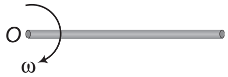


Fig. 12.57

Solution In pure rotational motion of a rigid body, component of total angular momentum about axis of rotation is given by

$$L = I\omega = \left(\frac{ml^2}{3}\right)\omega \qquad \left(I_0 = \frac{ml^2}{3}\right)$$

Substituting the values we have,

$$L = \frac{(2)(2)^2}{3}(4)$$

$$= \frac{32}{3}\ \text{kg-m}^2/\text{s}$$ **Ans.**

Direction of this component is perpendicular to paper inwards (from right hand rule), as the rotation is clockwise.

Example 12.16 *A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed* $\omega = \frac{v}{R}$ *in clockwise direction as shown in figure. Find the magnitude of the total angular momentum of the disc about bottommost point O of the disc.*

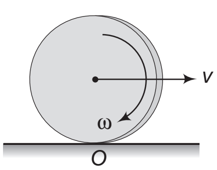


Fig. 12.58

Solution As we have discussed, angular momentum about O is the vector sum of two terms:

$$I\omega \text{ and } mvr_{\perp}$$

Here, $$I\omega = I_{cm}\omega = \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right) = \frac{1}{2}mvR \qquad \text{(perpendicular to paper inwards)}$$

and $$mvr_{\perp} = mv_c r_{\perp} = mRv \qquad \text{(perpendicular to paper inwards)}$$

Since, both the terms are in the same direction.

$$L_{\text{Total}} = \frac{1}{2}mvR + mvR$$

or $$L_{\text{Total}} = \frac{3}{2}mvR \qquad \text{(perpendicular to paper inwards)}$$

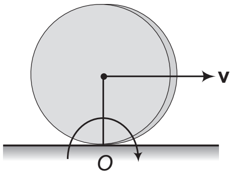


Fig. 12.59

INTRODUCTORY EXERCISE 12.5

1. A uniform rod of mass m is rotated about an axis passing through point O as shown. Find angular momentum of the rod about rotational axis.

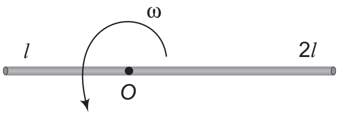


Fig. 12.60

2. A particle of mass 1 kg is moving along a straight line $y = x + 4$. Both x and y are in metres. Velocity of the particle is 2 m/s. Find magnitude of angular momentum of the particle about origin.
3. A particle of mass m is projected from the ground with an initial speed u at an angle α. Find the magnitude of its angular momentum at the highest point of its trajectory about the point of projection.
4. If the angular momentum of a body is zero about some point. Is it necessary that it will be zero about a different point?
5. A solid sphere of mass m and radius R is rolling without slipping as shown in figure. Find angular momentum of the sphere about z-axis.

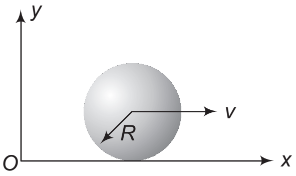


Fig. 12.61

6. In example number 12.16 suppose the disc starts rotating anticlockwise with the same angular velocity $\omega = \frac{v}{R}$, then what will be the angular momentum of the disc about bottommost point in this new situation?
7. Two particles each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is the same about any point taken as origin.

12.7 Conservation of Angular Momentum

As we have seen in Article 12.6, the angular momentum of a particle about some reference point O is defined as,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \ldots\text{(i)}$$

Here, **P** is the linear momentum of the particle and **r** its position vector with respect to the reference point O. Differentiating Eq. (i) with respect to time, we get

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} \qquad \ldots\text{(ii)}$$

Here, $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ (velocity of particle)

Hence, Eq. (ii) can be rewritten as,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} + \mathbf{v} \times \mathbf{p}$$

Now, $\mathbf{v} \times \mathbf{p} =$ a null vector, because **v** and **p** are parallel to each other and the cross product of two parallel vectors is a null vector. Thus,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \tau \quad \text{or} \quad \boxed{\tau = \frac{d\mathbf{L}}{dt}} \qquad \ldots\text{(iii)}$$

Which states that the time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it. This result is rotational analog of the equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, which states that the time rate of change of the linear momentum of a particle is equal to the force acting on it. Eq. (iii), like all vector equations, is equivalent to three scalar equations, namely

$$\tau_x = \left(\frac{dL}{dt}\right)_x, \quad \tau_y = \left(\frac{dL}{dt}\right)_y \quad \text{and} \quad \tau_z = \left(\frac{dL}{dt}\right)_z$$

The same equation can be generalised for a system of particles as, $\tau_{\text{ext}} = \frac{d\mathbf{L}}{dt}$. According to which the time rate of change of the total angular momentum of a system of particles about some reference point in an inertial frame of reference is equal to the sum of all external torques (of course the vector sum) acting on the system about the same reference point.

Now, suppose that $\tau_{\text{ext}} = 0$, then $\frac{d\mathbf{L}}{dt} = 0$, so that $\mathbf{L} =$ constant.

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega$$

It is possible for the moment of inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then L_z must remain constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as

$$I\omega = \text{constant} \quad \text{or} \quad \boxed{I_1\omega_1 = I_2\omega_2} \qquad \ldots\text{(iv)}$$

Note *According to above equation, if some mass moves away from the axis of rotation, its moment of inertia will increase. So, to conserve angular momentum or $I\omega$ its angular speed ω should decrease. Therefore, time period of rotation $\left(T = \frac{2\pi}{\omega}\right)$ should also increase.*

Example 12.17 *A wheel of moment of inertia I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary particle of mass m at its edge. Find the new angular speed of the wheel.*

Solution Net external torque on the system is zero. Therefore, angular momentum will remain conserved. Thus,

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad \omega_2 = \frac{I_1\omega_1}{I_2}$$

Here, $I_1 = I, \quad \omega_1 = \omega_0, \; I_2 = I + mR^2$

$\therefore$ $\omega_2 = \dfrac{I\omega_0}{I + mR^2}$ **Ans.**

INTRODUCTORY EXERCISE 12.6

1. A thin circular ring of mass M and radius R is rotating about its axis with an angular speed ω_0. Two particles each of mass m are now attached at diametrically opposite points. Find the new angular speed of the ring.
2. If the ice at the poles melts and flows towards the equator, how will it affect the duration of day-night?
3. When tall buildings are constructed on earth, the duration of day night slightly increases. Is this statement true or false?
4. If radius of earth is increased, without change in its mass, will the length of day increase, decrease or remain same?

12.8 Combined Translational and Rotational Motion of a Rigid Body

This is the most complex motion of a rigid body. But this can be simplified by splitting this motion into following two parts:

(i) pure translational motion with the velocity ($\mathbf{v}$) and acceleration ($\mathbf{a}$) of centre of mass.

(ii) pure rotational motion about an axis passing through centre of mass with angular velocity ω and angular acceleration α.

As we have discussed earlier also, different particles of the rigid body in this type of motion have different linear velocity and linear acceleration. So, now the question is how to find linear velocity and linear acceleration of a general particle P on the rigid body?

Method of Finding Linear Velocity of a General Particle *P*

Suppose linear velocity v (of centre of mass) and angular velocity ω (of the rigid body) are known to us and we wish to find linear velocity of a general particle P at a distance r from C.

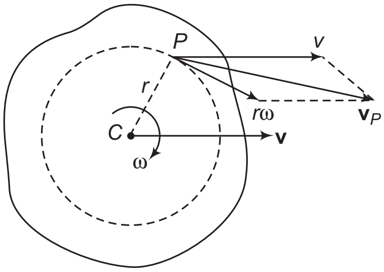


Fig. 12.62

As we know that

$$\mathbf{v}_{PC} = \mathbf{v}_P - \mathbf{v}_C$$

$$\therefore \qquad \mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{PC}$$

or, absolute velocity of P is the vector sum of $\mathbf{v}_C$ and $\mathbf{v}_{PC}$. Here, $\mathbf{v}_C = \mathbf{v}$ and motion of P with respect to C is a circle (dotted circle shown in figure). In circular motion, velocity of a particle is $r\omega$, tangential to its circle (in the direction of rotation). So, $\mathbf{v}_{PC}$ is $r\omega$ in the tangential direction as shown in Fig. 12.62. Thus net velocity of P is the vector sum of following two terms:

(i) v and (ii) $r\omega$

For different particles, values of v and ω are same. But values of r and therefore $r\omega$ and direction of $r\omega$ are different . This is the reason why different particles have different linear velocities.

Method of Finding Acceleration of a General Particle *P*

Suppose linear acceleration a (of centre of mass) angular velocity ω and angular acceleration α (of the rigid body) are known to us and we wish to find linear acceleration of a general particle P at a distance r from C.

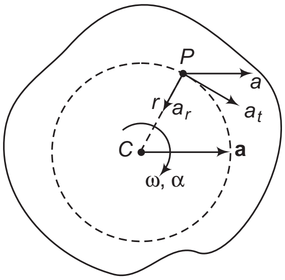


Fig. 12.63

As we know that

$$\mathbf{a}_{PC} = \mathbf{a}_P - \mathbf{a}_C$$

$$\therefore \qquad \mathbf{a}_P = \mathbf{a}_C + \mathbf{a}_{PC}$$

Thus, absolute acceleration of P is the vector sum of $\mathbf{a}_C$ and $\mathbf{a}_{PC}$.

Here, $\mathbf{a}_C = \mathbf{a}$ and motion of P with respect to C is a circle (dotted circle shown in figure). In circular motion, acceleration of a particle has two components:

(i) tangential component, a_t (ii) radial component, a_r

Here, $$a_t = r\alpha \qquad \left(\text{where, } \alpha = \frac{d\omega}{dt}\right)$$

This component is tangential in the direction of linear velocity if ω is increasing and in the opposite direction of linear velocity if ω is decreasing. Further, this component is zero if ω is constant.

$$a_r = r\omega^2$$

and this component is always towards centre C.

Hence, net acceleration of the particle P is the vector sum of the following three terms:

(i) a (ii) $a_t = r\alpha$ (iii) $a_r = r\omega^2$

For different particles, values of a, ω and α are same. But value of r is different. Therefore $r\alpha$, $r\omega^2$ and their directions are different. This is the reason why different particles have different linear accelerations.

Kinetic Energy of Rigid Body in Combined Translational and Rotational Motion

Here, two energies are associated with the rigid body. One is translational $\left(=\frac{1}{2} m\, v_{\text{COM}}^2\right)$ and another is rotational $\left(=\frac{1}{2} I_{\text{COM}}\, \omega^2\right)$. Thus, total kinetic energy of the rigid body is

$$\boxed{K = \frac{1}{2} m v_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}}\, \omega^2}$$

Example 12.18

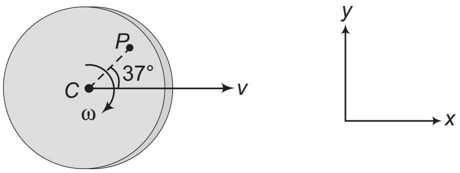


Fig. 12.64

In the figure shown $v = 2\ m/s$, $\omega = 5 rad/s$ *and* $CP = 1\ m$

In terms of $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *find linear velocity of particle P.*

Solution For particle P,

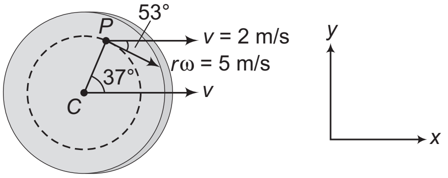


Fig. 12.65

$$r = CP = 1 \text{ m}$$

$$\Rightarrow \qquad r\omega = (1)(5) = 5 \text{ m/s}$$

Net velocity of P is the vector sum of v and $r\omega$ as shown in figure.

$$\therefore \qquad \mathbf{v}_P = 2\hat{\mathbf{i}} + (5\cos 53^\circ\, \hat{\mathbf{i}} - 5\sin 53^\circ\, \hat{\mathbf{j}})$$

$$= 2\hat{\mathbf{i}} + 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} = (5\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) \text{ m/s}$$

Ans.

Example 12.19 *A disc of radius R has linear velocity v and angular velocity* ω *as shown in the figure. Given* $v = r\omega$. *Find velocity of points A, B, C and D on the disc.*

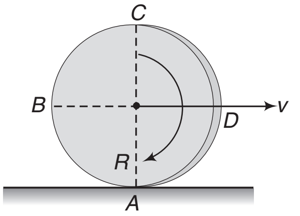


Fig. 12.66

Solution As stated in above article, velocity of any point of the rigid body in rotation plus translation is the vector sum of v (the velocity of centre of mass) and $r\omega$. Here, r is the distance of the point under consideration from the centre of mass of the body. Direction of this $r\omega$ is perpendicular to the line joining the point with centre of mass in the sense of rotation. Based on this, velocities of points *A, B, C* and *D* are as shown below

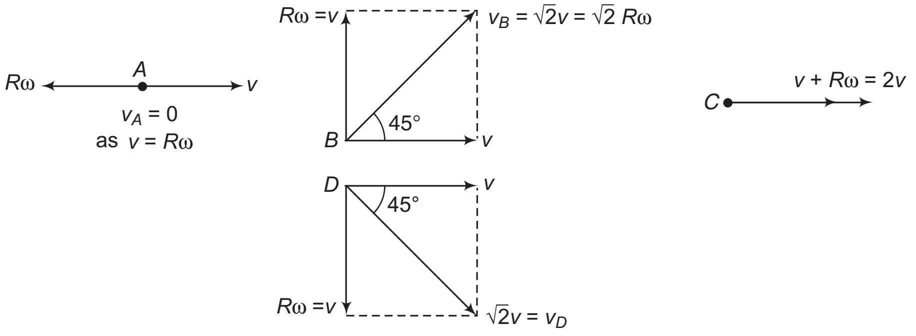


Fig. 12.67

Thus, v_A is zero, velocity of B and D is $\sqrt{2}v$ or $\sqrt{2}R\omega$ and velocity of C is $2v$ or $2R\omega$ in the directions shown in figure.

Example 12.20 *In the shown figure,*

$$a = 2\,m/s^2,\ \omega = (2t)\ rads^{-1}\ \text{and}\ CP = 1\ m.$$

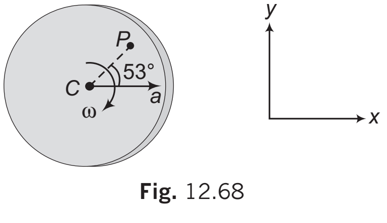


Fig. 12.68

In terms of $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$, *find linear acceleration of the particle at P at* $t = 1$ *s.*

Solution For particle at P

$$r = CP = 1\,\text{m}$$

$\Rightarrow$
$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(2t) = 2\,\text{rad/s}^2$$

At $t = 1\,\text{s}$,

$$\omega = 2\,\text{rad/s}$$

$$\alpha = 2\,\text{rad/s}^2$$

$$a_t = r\alpha = 2\,\text{m/s}^2$$

$$a_r = r\omega^2 = 4\,\text{m/s}^2$$

and
$$a = 2\,\text{m/s}^2$$

Net acceleration of P is the vector sum of three terms a, a_r and a_t as shown in figure below.

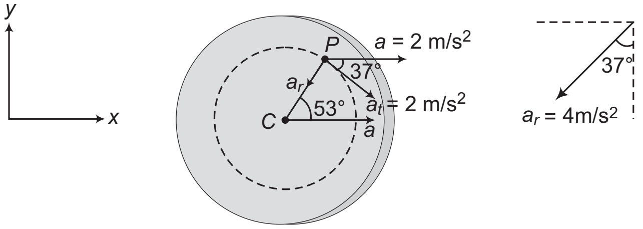


Fig. 12.69

$\therefore$
$$\mathbf{a}_P = 2\hat{\mathbf{i}} + (2\cos 37^\circ\,\hat{\mathbf{i}} - 2\sin 37^\circ\,\hat{\mathbf{j}}) + (-4\sin 37^\circ\,\hat{\mathbf{i}} - 4\cos 37^\circ\,\hat{\mathbf{j}})$$
$$= 2\hat{\mathbf{i}} + 1.6\hat{\mathbf{i}} - 1.2\hat{\mathbf{j}} - 2.4\hat{\mathbf{i}} - 3.2\hat{\mathbf{j}}$$
$$= (1.2\hat{\mathbf{i}} - 4.4\hat{\mathbf{j}})\,\text{m/s}^2$$

Ans.

INTRODUCTORY EXERCISE 12.7

1. In the figure shown, $\omega = \dfrac{v}{2R}$. In terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, find linear velocities of particles M, N, R and S.

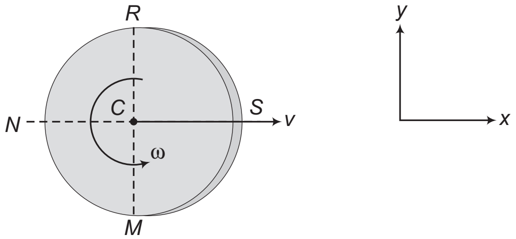


Fig. 12.70

2. In the same figure, if v and ω both are constant, then find linear accelerations of points M, N, R and S in terms of R, ω, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, where R is the radius of disc.

12.9 Uniform Pure Rolling

Pure rolling means no relative motion (or no slipping) at point of contact between two bodies.

For example, consider a disc of radius R moving with linear velocity v and angular velocity ω on a horizontal ground. The disc is said to be moving without slipping if velocities of points P and Q (shown in figure 12.71 (b) are equal, i.e.

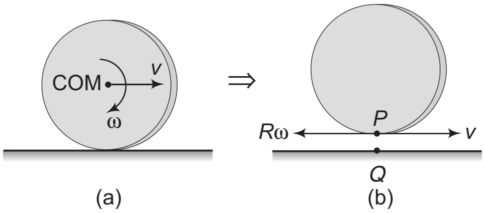


Fig. 12.71

$$v_P = v_Q$$

or $$v - R\omega = 0$$

or $$\boxed{v = R\omega}$$

If $v_P > v_Q$ or $v > R\omega$, the motion is said to be in **forward slipping** and if $v_P < v_Q$ or $v < R\omega$, the motion is said to be in **backward slipping** (or sometimes called forward English).

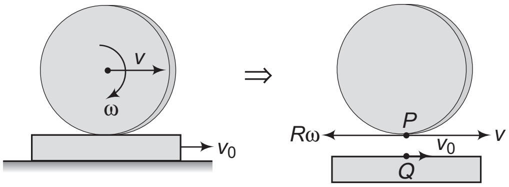


Fig. 12.72

In pure rolling ($v = R\omega$) over a stationary ground net velocity of bottommost point of the body is zero. In forward slip condition ($v > R\omega$) net velocity is in the direction of motion. In backward slip condition ($v < R\omega$) net velocity is in the opposite direction of motion.

Thus, $v = R\omega$ is the condition of pure rolling on a stationary ground. Sometimes it is simply said rolling. Suppose the base over which the disc in rolling, is also moving with some velocity (say v_0) then in that case condition of pure rolling is different.

For example, in the Fig. 12.72,

$$v_P = v_Q$$

or $$v - R\omega = v_0$$

Thus, in this case $v - R\omega \neq 0$, but $v - R\omega = v_0$. By uniform pure rolling we mean that v and ω are constant.

They are neither increasing nor decreasing.

Extra Points to Remember

In case of pure rolling on a stationary horizontal ground (when $v = R\omega$), following points are important to note:

- Distance moved by the centre of mass of the rigid body in one full rotation is $2\pi R$.

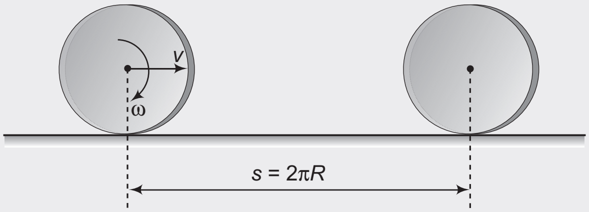


Fig. 12.73

This is because $$s = v \cdot T = (\omega R)\left(\frac{2\pi}{\omega}\right) = 2\pi R$$

In forward slipping $s > 2\pi R$ (as $v > \omega R$)

and in backward slipping $s < 2\pi R$ (as $v < \omega R$)

- The speed of a point on the circumference of the body at the instant shown in figure is $2v\sin\frac{\theta}{2}$ or $2R\omega\sin\frac{\theta}{2}$. i.e.

$$|\mathbf{v}_P| = v_p = 2v\sin\frac{\theta}{2} = 2R\omega\sin\frac{\theta}{2}$$

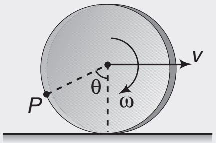


Fig. 12.74

This can be shown as :

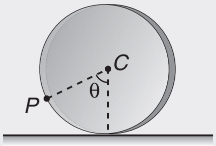


Fig. 12.75

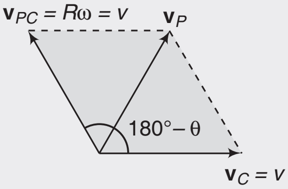


Fig. 12.76

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{PC}$$

$$\therefore \quad |\mathbf{v}_P| = \sqrt{v^2 + v^2 + 2v \cdot v\cos(180^\circ - \theta)} = 2v\sin\frac{\theta}{2}$$

- From the above expression we can see that :

$$v_A = 0 \quad \text{as} \quad \theta = 0^\circ$$
$$v_B = \sqrt{2}v \quad \text{as} \quad \theta = 90^\circ$$
and $$v_C = 2v \quad \text{as} \quad \theta = 180^\circ$$

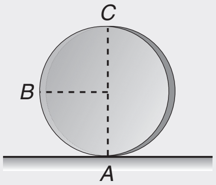


Fig. 12.77

- The path of a point on circumference is a cycloid and the distance moved by this point in one full rotation is $8R$.

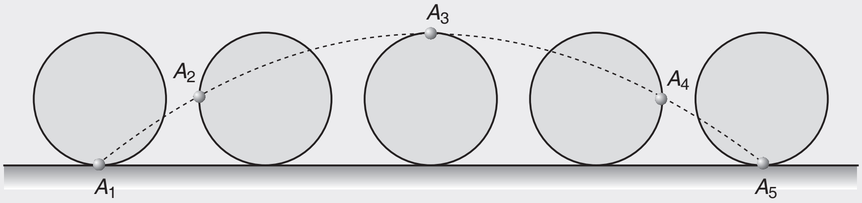


Fig. 12.78

In the figure, the dotted line is a cycloid and the distance $A_1 A_2 \dots A_5$ is $8R$. This can be proved as under. In Fig. 12.79

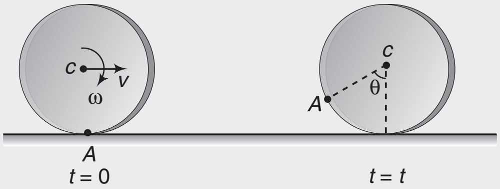


Fig. 12.79

$$\theta = \omega t$$

Speed of point A at this moment is, $v_A = 2R\omega \sin\frac{\theta}{2} = 2R\omega \sin\left(\frac{\omega t}{2}\right)$

Distance moved by it in time dt is, $ds = v_A\, dt = 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$

Therefore, total distance moved in one full rotation is,

$$s = \int_0^{T = 2\pi/\omega} ds \quad \text{or} \quad s = \int_0^{T = 2\pi/\omega} 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

On integration we get, $s = 8R$.

- $\frac{K_R}{K_T} = 1$ for a ring

  $= \frac{1}{2}$ for a disc

  $= \frac{2}{5}$ for a solid sphere

  $= \frac{2}{3}$ for a hollow sphere etc.

Here, K_R stands for rotational kinetic energy $\left(= \frac{1}{2} I\omega^2\right)$ and K_T for translational kinetic energy $\left(= \frac{1}{2} mv^2\right)$.

For example, for a disc

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mR^2\right)\left(\frac{v}{R}\right)^2$$

$$= \frac{1}{4} mv^2 \quad \text{and} \quad K_T = \frac{1}{2} mv^2$$

$$\therefore \quad \frac{K_R}{K_T} = \frac{1}{2}$$

Example 12.21 *A solid disc is rolling without slipping on a horizontal ground as shown in figure. Its total kinetic energy is 100 J. What is its translational and rotational kinetic energy.*

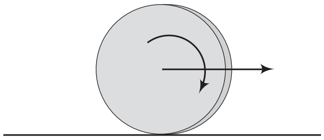


Fig. 12.80

Solution In case of pure rolling, ratio of rotational kinetic energy and translational kinetic energy is $\frac{1}{2}$.

or $\frac{K_R}{K_T} = \frac{1}{2} \Rightarrow \therefore K_R = \left(\frac{1}{1+2}\right)$(Total kinetic energy)

$$= \frac{1}{3}(100\,\text{J}) = \frac{100}{3}\,\text{J}$$ **Ans.**

Similarly, $$K_T = \left(\frac{2}{1+2}\right)(\text{Total kinetic energy})$$

$$= \frac{2}{3}(100\,\text{J}) = \frac{200}{3}\,\text{J}$$ **Ans.**

Example 12.22 *A disc of radius R starts at time $t = 0$ moving along the positive x-axis with linear speed v and angular speed ω. Find the x and y coordinates of the bottommost point at any time t.*

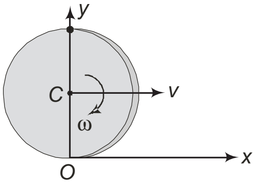


Fig. 12.81

Solution At time t the bottommost point will rotate an angle $\theta = \omega t$ with respect to the centre of the disc C. The centre C will travel a distance $s = vt$.

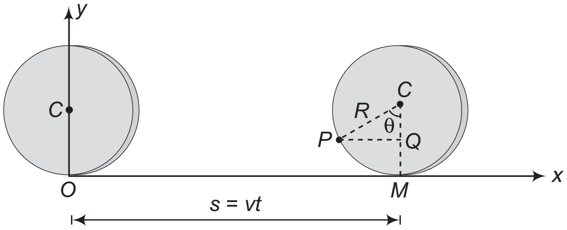


Fig. 12.82

In the figure, $PQ = R\sin\theta = R\sin\omega t$ and $CQ = R\cos\theta = R\cos\omega t$

Coordinates of point P at time t are,

$$x = OM - PQ = vt - R\sin\omega t$$

and $$y = CM - CQ = R - R\cos\omega t$$

$\therefore$ $$(x, y) \equiv (vt - R\sin\omega t, R - R\cos\omega t)$$

INTRODUCTORY EXERCISE 12.8

1. A solid sphere is rolling without slipping on a horizontal ground. Its rotational kinetic energy is 10 J. Find its translational and total kinetic energy.
2. Under forward slip condition, translational kinetic energy of a ring is greater than its rotational kinetic energy. Is this statement true or false?
3. In backward slip condition, translational kinetic energy of a disc may be equal to its rotational kinetic energy. Is this statement true of false?

12.10 Instantaneous Axis of Rotation

As we have seen that combined rotational and translational motion of a rigid body is the most complex motion. But this motion can be simplified and may be assumed to be in pure rotational motion (with same ω) about an axis called instantaneous axis of rotation.

Further as we know that, in pure rotational motion, points lying on the axis of rotation are at rest. Therefore, we can say that, instantaneous axis of rotation passes through those points which are at rest.

For example, in pure rolling over ground instantaneous axis of rotation (*IAOR*) passes through the bottommost point, as it is a point of zero velocity. Thus, the combined motion of rotation and translation can be assumed to be pure rotational motion about bottommost point with same angular speed ω.

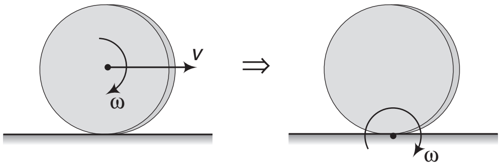


Fig. 12.83

Now, there are two uses of the concept of instantaneous axis of rotation.

(i) Velocity of any point P can obtained by a single term

$$v = r\omega$$

as in pure rotational motion this is the expression of velocity of any point P. Here r is the distance of P from instantaneous axis of rotation. With respect to O, point P is rotating in a circle with centre at O and radius as r. Velocity of P is tangential to this circle (or perpendicular to OP), in the direction of rotation.

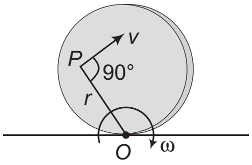


Fig. 12.84

(ii) We can find total kinetic energy of the body by a single term.

$$K = \frac{1}{2} I\omega^2$$

But here, I is the moment of inertia about instantaneous axis of rotation.

Example 12.23 *Using the concept of instantaneous axis of rotation. Find speed of particle P as shown in figure, under pure rolling condition.*

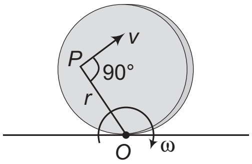


Fig. 12.85

Solution In pure rolling, combined rotation and translation motion may be assumed to be a pure rotational motion about an axis passing through bottommost point (with same ω) or instantaneous axis of rotation.

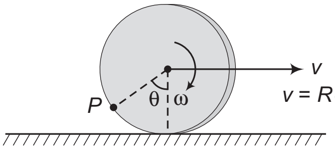


Fig. 12.86

$$|\mathbf{v}_P| = (OP)\omega \quad \text{Here, } OP = 2R\sin\frac{\theta}{2}$$

$$\therefore \qquad |\mathbf{v}_P| = \left(2R\sin\frac{\theta}{2}\right)\omega = 2R\omega\sin\left(\frac{\theta}{2}\right) = 2v\sin\frac{\theta}{2}$$

Example 12.24 *A disc is rolling (without slipping) on a horizontal surface. C is its centre and Q and P are two points equidistant from C. Let v_P, v_Q and v_C be the magnitude of velocities of points P, Q and C respectively, then* **(JEE 2004)**

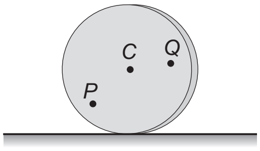


Fig. 12.87

(a) $v_Q > v_C > v_P$
(b) $v_Q < v_C < v_P$
(c) $v_Q = v_P, v_C = \frac{1}{2} v_P$
(d) $v_Q < v_C > v_P$

Solution In case of pure rolling bottom most point is the instantaneous centre of zero velocity.

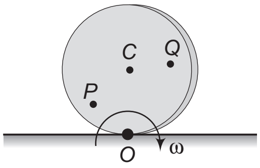


Fig. 12.88

Velocity of any point on the disc, $v = r\omega$, where r is the distance of point from O.

$$r_Q > r_C > r_P \quad \Rightarrow \quad \therefore \quad v_Q > v_C > v_P$$

Therefore, the correct option is (a).

INTRODUCTORY EXERCISE 12.9

1. A disc is rolling without slipping with linear velocity v as shown in figure. With the concept of instantaneous axis of rotation, find velocities of points A, B, C and D.

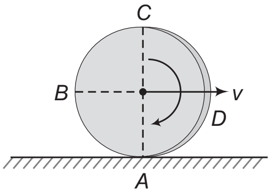


Fig. 12.89

2. A solid sphere is rolling without slipping as shown in figure. Prove that

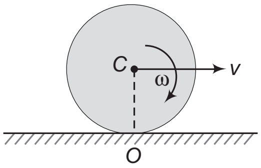


Fig. 12.90

$$\frac{1}{2}mv^2 + \frac{1}{2}I_C\,\omega^2 = \frac{1}{2}I_0\,\omega^2$$

12.11 Accelerated Pure Rolling

Till now we were discussing the uniform pure rolling in which v and ω were constants. Now, suppose an external force is applied to the rigid body, the motion will no longer remain uniform. The condition of pure rolling on a stationary ground is,

$$v = R\omega$$

Differentiating this equation with respect to time, we have

$$\frac{dv}{dt} = R \cdot \frac{d\omega}{dt}$$

or $$a = R\alpha$$

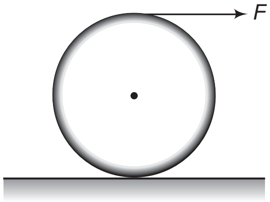


Fig. 12.91

Thus, in addition to $v = R\omega$ at every instant of time there is one additional condition, linear acceleration = R × angular acceleration or $a = R\alpha$ for pure rolling to take place. Here, friction plays an important role in maintaining the pure rolling. The friction may sometimes act in forward direction, sometimes in backward direction or under certain conditions it may be zero. Here, we should not forget the basic nature of friction, which is a self adjusting force (upto a certain maximum limit) and which has a tendency to stop the relative motion between two bodies in contact and here the relative motion stops when at every instant $v = R\omega$. To satisfy this equation all the time, $a = R\alpha$ equation should also be satisfied. Let us take an example illustrating the above theory.

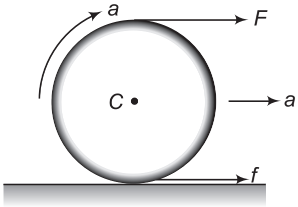


Fig. 12.92

Suppose a force F is applied at the topmost point of a rigid body of radius R, mass M and moment of inertia I about an axis passing through the centre of mass. Now, the applied force F can produce by itself:

(i) a linear acceleration a and

(ii) an angular acceleration α.

If $a = R\alpha$, then there is no need of friction and force of friction $f = 0$. If $a < R\alpha$, then to support the linear motion the force of friction f will act in forward direction. Similarly, if $a > R\alpha$, then to support the angular motion the force of friction will act in backward direction. So, in this case force of friction will be either backward, forward or even zero also. It all depends on M, I and R. For calculation purpose initially we can choose any direction of friction. Let we assume it in forward direction,

Let, a = linear acceleration, α = angular acceleration

then, $$a = \frac{F_{\text{net}}}{M} = \frac{F + f}{M} \qquad \text{...(i)}$$

$$\alpha = \frac{\tau_c}{I} = \frac{(F - f)R}{I} \qquad \text{...(ii)}$$

For pure rolling to take place, $$a = R\alpha \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get $$f = \frac{(MR^2 - I)}{(MR^2 + I)} \cdot F \qquad \text{...(iv)}$$

From Eq. (iv) following conclusions can be drawn

(i) If $I = MR^2$ (e.g. in case of a ring) $f = 0$ i.e. if a force F is applied on the top of a ring, the force of friction will be zero and the ring will roll without slipping.

(ii) If $I < MR^2$, (e.g. in case of a solid sphere or a hollow sphere), f is positive, i.e. force of friction will be forward.

(iii) If $I > MR^2$, f is negative, i.e. force of friction will be backwards. Although under no condition $I > MR^2$. (Think why?). So, force of friction is either in forward direction or zero. Here, it should be noted that the force of friction f obtained in Eq. (iv) should be less than the limiting friction (μMg) for pure rolling to take place. Further, we have seen that for $I < MR^2$ force of friction acts in forward direction. This is because α is more if I is small $\left(\alpha = \frac{\tau}{I}\right)$ i.e. to support the linear motion force of friction is in forward direction.

Note *It is often said that rolling friction is less than the sliding friction. This is because the force of friction calculated by equation number (iv) is normally less than the sliding friction ($\mu_k N$) and sometimes it is in forward direction, i.e. it supports the motion.*

Extra Points to Remember

- In accelerated pure rolling, $v = R\omega$ and $a = R\alpha$ are not the only conditions to be satisfied, sense of rotation is also important as per the direction of linear acceleration. Sense of rotation (or direction of α) should be as shown below in following two figures:

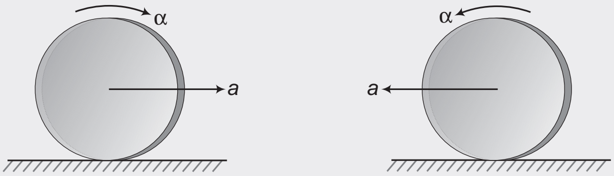


Fig. 12.93

- In following two figures accelerated pure rolling is not possible.

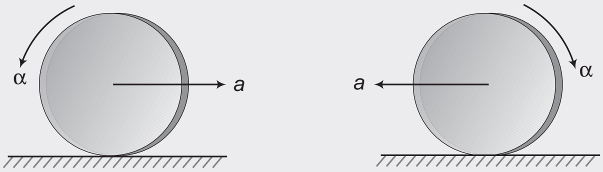


Fig. 12.94

- There are certain situations in which the direction of friction is fixed. For example in the following situations the force of friction is backwards. This is because linear acceleration 'a' due to the applied force (= $mg \sin\theta$ in first case) is in the direction shown in figure. So, direction of α should also be in the shown direction. We have only friction force which can provide 'α' in that direction. Hence, it should be in backward direction.

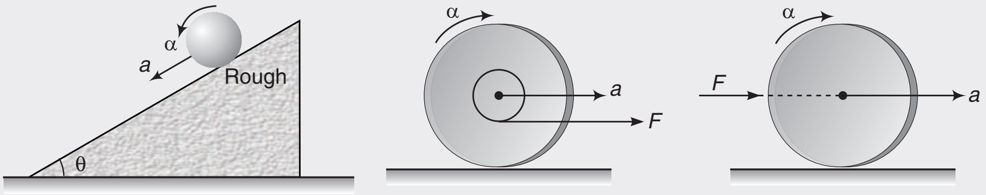


Fig. 12.95

Rolling on Rough Inclined Plane

A body of mass M, radius R and moment of inertia I is kept over a rough ground as shown in figure. In this case no external force is applied for accelerated pure rolling. But $mg\sin\theta$ is already acting at centre.

As we said earlier also, force of friction in this case will be backward. Equations of motion are

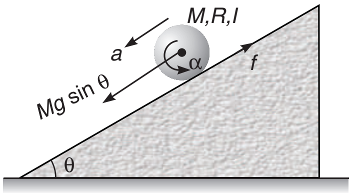


Fig. 12.96

$$a = \frac{Mg\sin\theta - f}{M} \qquad \text{...(i)}$$

$$\alpha = \frac{fR}{I} \qquad \text{...(ii)}$$

For pure rolling to take place,

$$a = R\alpha \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{Mg\sin\theta}{1 + \frac{MR^2}{I}} \qquad \text{...(iv)}$$

and

$$a = \frac{g\sin\theta}{1 + \frac{I}{MR^2}} \qquad \text{...(v)}$$

From Eq. (v), we can see that if a solid sphere and a hollow sphere of same mass and radius are released from a rough inclined plane and pure rolling is taking place, then the solid sphere reaches the bottom first because

$$I_{\text{solid}} < I_{\text{hollow}} \quad \text{or} \quad a_{\text{solid}} > a_{\text{hollow}} \Rightarrow \therefore \quad t_{\text{solid}} < t_{\text{hollow}}$$

Further, the force of friction calculated in Eq. (iv) for pure rolling to take place should be less than or equal to the maximum available friction $\mu Mg\cos\theta$.

or

$$\frac{Mg\sin\theta}{1 + \frac{MR^2}{I}} \le \mu\, Mg\cos\theta \quad \text{or} \quad \mu \ge \frac{\tan\theta}{1 + \frac{MR^2}{I}}$$

Thus, minimum value of friction required for pure rolling is

$$\mu_{\text{min}} = \frac{\tan\theta}{1 + MR^2/I}$$

If given value of $\mu > \mu_{\text{min}}$, then friction acting on the body is

$$f = \frac{Mg\sin\theta}{1 + MR^2/I}$$

and in this case linear acceleration of the body is

$$a = \frac{g\sin\theta}{1 + I/MR^2}$$

Example 12.25 *In the shown figure, accelerated pure rolling will takes place, if* $a = R\alpha$. *Find the case if*

(a) $a > R\alpha$ *(b)* $a < R\alpha$

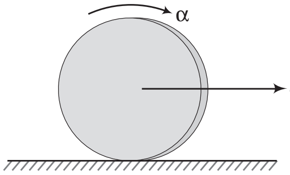


Fig. 12.97

Solution (a) If $a > R\alpha$, then at any instant $v > R\omega$. So, it is a case of forward slipping.

(b) If $a < R\alpha$, then at any instant $v < R\omega$. So, it is a case of backward slipping.

Example 12.26 *If accelerated pure rolling is taking place on a stationary ground, then work done by friction is always zero. Comment on this.*

Solution In pure rolling on stationary ground the bottommost point of the rigid body (where force of friction is acting) is at rest. Therefore, work done by friction is zero.

Example 12.27 *In the shown figure, M is mass of the body, R its radius and I the moment of inertia about an axis passing through centre. Find force of friction 'f' acting on the body (upwards), its linear acceleration 'a' (down the plane) and type of motion if:*

(a) $\mu = 0$ *(b)* $\mu < \mu_{\min}$ *(c)* $\mu > \mu_{\min}$

where, $\mu_{\min}$ *is the minimum value of coefficient of friction required for pure rolling.*

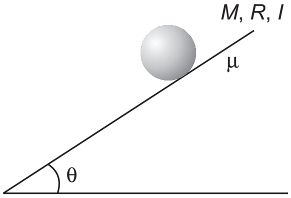


Fig. 12.98

Solution (a) If $\mu = 0$ then,

$$f = 0 \quad \text{and} \quad a = g\sin\theta = a_1 \text{ (say)}$$

and the motion is only translational.

(b) If $\mu < \mu_{\min}$, then maximum value of friction will act, as friction is insufficient to provide accelerated pure rolling or to stop the relative motion.

$$\therefore \qquad f = f_{\max} = \mu\, mg\cos\theta$$

$$\text{and} \qquad a = \frac{Mg\sin\theta - \mu\, Mg\cos\theta}{m}$$

$$= g\sin\theta - \mu g\cos\theta = a_2 \text{ (say)}$$

In this case, motion is rotation + translation with forward slip (as $a > R\alpha$).

(c) If $\mu > \mu_{\min}$, Then we have discussed in the above article that

$$f = \frac{Mg\sin\theta}{1 + \dfrac{MR^2}{I}} \quad \text{and} \quad a = \frac{g\sin\theta}{1 + \dfrac{I}{MR^2}} = a_3 \text{ (say)}$$

Motion in this case is rotation + translation with accelerated pure rolling.

Note *In the above example, we can see that* a_1 *and* a_2 *are independent of moment of inertia I, but* a_3 *depends on it.*

Example 12.28 *A tangential force F acts at the top of a thin spherical shell of mass m and radius R. Find the acceleration of the shell if it rolls without slipping.*

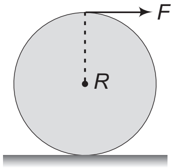


Fig. 12.99

Solution Let f be the force of friction between the shell and the horizontal surface.

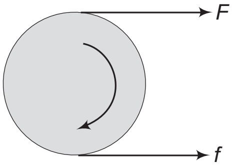


Fig. 12.100

For translational motion,

$$F + f = ma \qquad \dots \text{(i)}$$

For rotational motion,

$$FR - fR = I\alpha = I\frac{a}{R}$$

[$\because a = R\alpha$ for pure rolling]

$$\Rightarrow \qquad F - f = I\frac{a}{R^2} \qquad \dots \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2F = \left(m + \frac{I}{R^2}\right)a$$

$$= \left(m + \frac{2}{3}m\right)a = \frac{5}{3}ma$$

or

$$F = \frac{5}{6}ma \qquad \left[\because I_{\text{shell}} = \frac{2}{3}mR^2\right]$$

$$\Rightarrow \qquad a = \frac{6F}{5m}$$

Ans.

Example 12.29 *A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is* μ. *What is maximum value of F, for which there is no slipping?*

Solution $F - f = Ma \qquad \dots\text{(i)}$

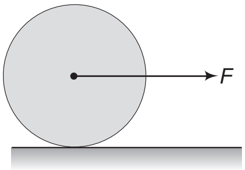


Fig. 12.101

$$\tau = I\alpha = I\left(\frac{a}{R}\right)$$

$$\Rightarrow \qquad f \cdot R = \frac{2}{5}MR^2\,\frac{a}{R}$$

$$\Rightarrow \qquad f = \frac{2}{5}Ma \qquad \dots\text{(ii)}$$

Solving Eqs. (i) and (ii) we get,

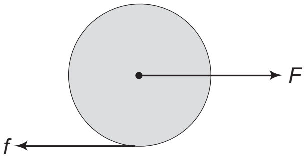


Fig. 12.102

$$f = \frac{2}{7}F$$

$$\frac{2}{7}F \le \mu mg$$

$$\Rightarrow \qquad F \le \frac{7}{2}\mu mg \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 12.10

1. Work done by friction in pure rolling is always zero. Is this statement true or false?
2. In the figure shown, a force F is applied at the top of a disc of mass 4 kg and radius 0.25 m. Find maximum value of F for no slipping.

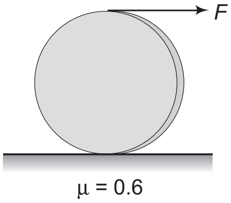


Fig. 12.103

3. In the figure shown a solid sphere of mass 4 kg and radius 0.25 m is placed on a rough surface. Find $(g = 10\ \text{ms}^2)$
   (a) minimum coefficient of friction for pure rolling to take place.
   (b) If $\mu > \mu_{\text{min}}$, find linear acceleration of sphere.
   (c) If $\mu = \frac{\mu_{\text{min}}}{2}$, find linear acceleration of cylinder.
   Here, μ_{min} is the value obtained in part (a).

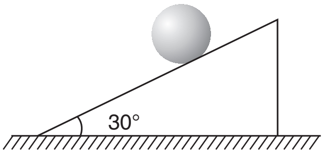


Fig. 12.104

4. A ball of mass M and radius R is released on a rough inclined plane of inclination θ. Friction is not sufficient to prevent slipping. The coefficient of friction between the ball and the plane is μ. Find:
   (a) the linear acceleration of the ball down the plane,
   (b) the angular acceleration of the ball about its centre of mass.
5. A spool is pulled by a force in vertical direction as shown in figure. What is the direction of friction in this case? The spool does not loose contact with the ground.

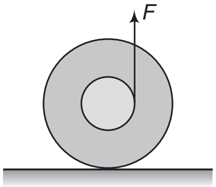


Fig. 12.105

12.12 Angular Impulse

In the previous chapter, we have learnt that linear impulse

$$= \mathbf{F} \cdot \Delta t = \mathbf{J} = \text{change in linear momentum } \Delta \mathbf{P}$$

or, $$\mathbf{J} = \Delta \mathbf{P} = \mathbf{P}_f - \mathbf{P}_i = m(\mathbf{v}_f - \mathbf{v}_i)$$

In one dimension, we can simply write as:

$$J = \Delta P = P_f - P_i = m(v_f - v_i)$$

If $v_i = 0$ and $v_f = v$, then

$$J = mv \quad \text{or} \quad \boxed{v = \frac{J}{m}}$$

In the similar manner, angular impulse

$$= \tau \cdot \Delta t = A \cdot I = \text{change in angular momentum } \Delta L$$

or $$A \cdot I = \tau \, \Delta t = \Delta L = L_f - L_i$$

But $$\tau = F \times r_\perp$$

∴ $$A \cdot I = F \times r_\perp \times \Delta t$$

$$= J \times r_\perp \qquad (\text{as } F \times \Delta t = J)$$

Thus, $A \cdot I = J \times r_\perp = L_f - L_i$

If $L_i = 0$, then $L_f = L = I\omega$

∴ $$A \cdot I = J \times r_\perp = I\omega \quad \text{or} \quad \boxed{\omega = \frac{J \times r_\perp}{I}}$$

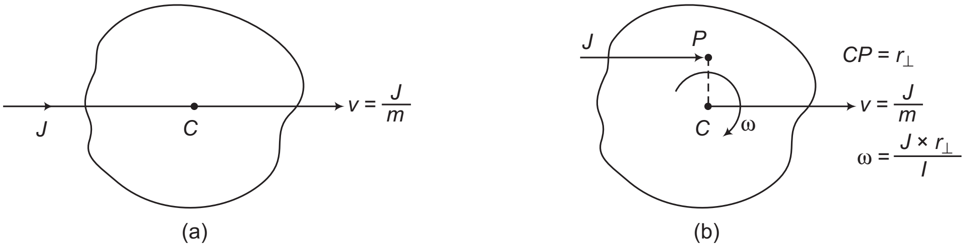


Fig. 12.106

In Fig. (a) A linear impulse J is applied at centre of mass C of the rigid body. Just after hitting, it will have only translational motion and its linear velocity will be given by

$$v = \frac{J}{m}$$

In Fig. (b) A linear impulse J is applied at point P, at a perpendicular distance $r_\perp = CP$. Just after hitting it will have both translational and rotational motion. Its linear velocity v and angular velocity ω will be given by

$$v = \frac{J}{m} \text{ and } \omega = \frac{J \times r_\perp}{I}$$

If $r_\perp$ is increased (keeping J to be constant) then v will remain same but ω will increase. So, the translational kinetic energy will have the same value but rotational kinetic energy will be more.

✓ Extra Points to Remember

- Angular impulse $A \cdot I = \tau \times \Delta t = \Delta L$

  Now, there are following three cases:

  (i) If torque is constant, then angular impulse can be obtained by directly multiplying this constant torque with the given time interval.

  (ii) If torque is a function of time then angular impulse can be obtained by integration.

  $$\therefore \qquad A \cdot I = \int_{t_i}^{t_f} \tau \, dt$$

  (iii) If torque *versus* time graph is given then angular impulse can be obtained by the area under that graph.

  In all three cases, angular impulse is equal to the change in angular momentum.

Example 12.30 *A solid sphere of mass M and radius R is hit by a cue at a height h above the centre C. For what value of h the sphere will roll without slipping ?*

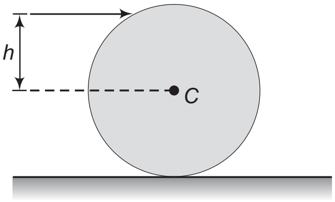


Fig. 12.107

Solution For rolling without slipping,

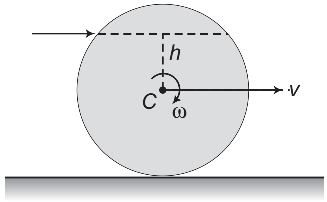


Fig. 12.108

$$v = R\omega$$

Here, v and ω are the values obtained just after hitting.

$$\therefore \qquad \frac{J}{M} = R\left(\frac{J \times r_{\perp}}{I}\right) \qquad \ldots\text{(i)}$$

Here, $r_{\perp} = h$ and $I = I_C = \frac{2}{5}MR^2$

Substituting these values in Eq. (i), we have

$$h = \frac{2}{5}R$$ **Ans.**

Example 12.31 *A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the result be affected in the case of a perfectly smooth inclined plane? The angle of inclination of the plane is* θ.

Solution Applying the equation (about bottommost point)

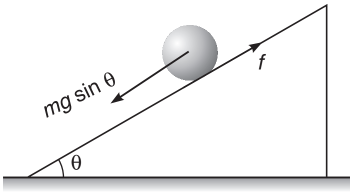


Fig. 12.109

Angular impulse = change in angular momentum about point of contact we have,

$$\tau = (mg\sin\theta)R = \text{constant}$$

$\therefore$ Angular impulse $= \tau \times t = (mg\sin\theta)Rt$

or $L = (mg\sin\theta)Rt$ **Ans.**

Note *There will be no change in the result even if body pure rolls or slides, as the torque of force of friction is zero about point of contact. So, it hardly matters whether the surface is rough or smooth.*

INTRODUCTORY EXERCISE 12.11

1. A cylinder is rolling down a rough inclined plane. Its angular momentum about the point of contact remains constant. Is this statement true or false?
2. A solid sphere and a hollow sphere both of same mass and same radius are hit by a cue at a height h above the centre C. In which case,

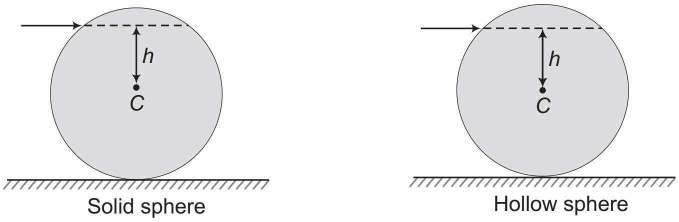


Fig. 12.110

(a) linear velocity will be more ?
(b) angular velocity will be more ?
(c) rotational kinetic energy will be more ?

Note *Linear impulse in both cases is same.*

12.13 Toppling

You might have seen in your practical life that if a force F is applied to a block A of smaller width and greater height it is more likely to topple down before sliding while if the same force F is applied to an another block B of broader base, chances of its sliding are more compared to its toppling. Have you ever thought why it happens so. To understand it in a better way let us take an example.

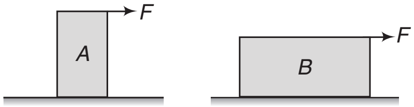


Fig. 12.111

Suppose a force F is applied at a height b above the base AE of the block. Further, suppose the friction f is sufficient to prevent sliding. In this case, if the normal reaction N also passes through C, then despite the fact that the block is in translational equilibrium ($F = f$ and $N = mg$), an unbalanced torque (due to the couple of forces F and f) is there.

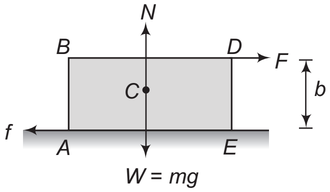


Fig. 12.112

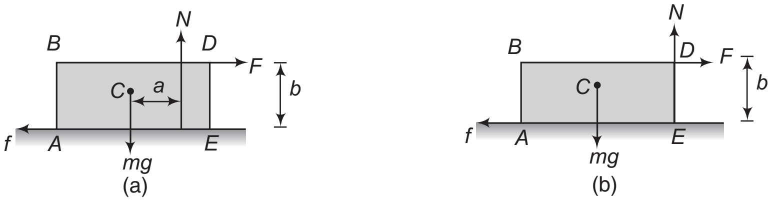
This torque has a tendency to topple the block about point E. To cancel the effect of this unbalanced torque the normal reaction N is shifted towards right a distance 'a' such that, net anticlockwise torque is equal to the net clockwise torque or

$$Fb = (mg)a$$

or

$$a = \frac{Fb}{mg}$$

Now, if F or b (or both) is increased, distance a also increases. But it can not go beyond the right edge of the block. So, in extreme case (beyond which the block will topple down), the normal reaction passes through E as shown in Fig. 12.113 (b).



(a) (b)

Fig. 12.113

Now, if F or b is further increased, the block will topple down. This is why the block having the broader base has less chances of toppling in comparison to a block of smaller base. Because the block of larger base has more margin for the normal reaction to shift. On the similar ground we can see why the rolling is so easy.

Because in this case the normal reaction has zero margin to shift. So even if the body is in translational equilibrium ($F = f$, $N = mg$) an unbalanced torque is left behind and the body starts toppling and here the toppling means motion. Under ideal conditions, the body will start moving by a very small force F tending to zero also.

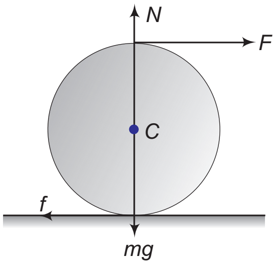


Fig. 12.114

Example 12.32 *A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height* $\frac{3a}{4}$ *above the base. What is the minimum value of F for which the cube begins to tip about an edge?*

Solution In the limiting case normal reaction will pass through O. The cube will tip about O if torque of F exceeds the torque of mg.

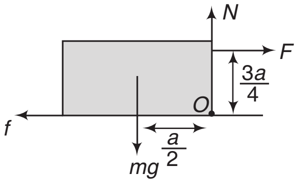


Fig. 12.115

Hence,
$$F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

or
$$F > \frac{2}{3}mg$$

Therefore, minimum value of F is $\frac{2}{3}mg$.

Example 12.33 *A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what condition the cylinder will (a) slide before toppling (b) topple before sliding.*

Solution (a) The cylinder will slide if

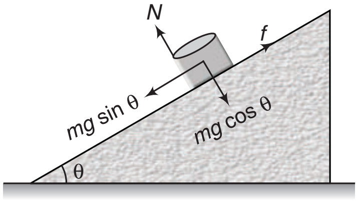


Fig. 12.116

$$mg\sin\theta > \mu mg\cos\theta$$

or
$$\tan\theta > \mu \qquad \text{...(i)}$$

The cylinder will topple if $(mg\sin\theta)\frac{h}{2} > (mg\cos\theta)r$

or
$$\tan\theta > \frac{2r}{h} \qquad \text{...(ii)}$$

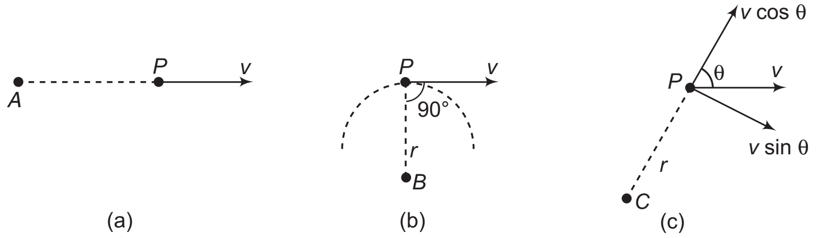
Thus, the condition of sliding is $\tan\theta > \mu$ and condition of toppling is $\tan\theta > \frac{2r}{h}$. Hence, the cylinder will slide before toppling if

$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$

Final Touch Points

1. Whether a particle is in translational motion, rotational motion or in both it merely depends on the reference point with respect to which the motion of the particle is described.



For example: a particle *P* of mass *m* is moving in a straight line as shown in figures (a), (b) and (c).

Refer Fig. (a) With respect to point A, the particle is in pure translational motion. Hence, kinetic energy of the particle can be written as

$$\text{KE} = \frac{1}{2}mv^2$$

Refer Fig. (b) With respect to point *B,* the particle is in pure rotational motion. Hence, the kinetic energy of the particle can be written as

$$\text{KE} = \frac{1}{2}I\omega^2 = \frac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2 = \frac{1}{2}mv^2$$

Refer Fig. (c) With respect to point *C,* the particle can be assumed to be in rotational as well as translational motion. Hence, the kinetic energy of the particle can be written as

$$\begin{aligned}\text{KE} &= \frac{1}{2}m(v\cos\theta)^2 + \frac{1}{2}I\omega^2\\ &= \frac{1}{2}m(v\cos\theta)^2 + \frac{1}{2}(mr^2)\left(\frac{v\sin\theta}{r}\right)^2\\ &= \frac{1}{2}mv^2\end{aligned}$$

Thus, in all the three cases, the kinetic energy of the particle comes out to be the same.

2. In cases where pulley is having some mass and friction is sufficient enough to prevent slipping, the tension on two sides of the pulley will be different and rotational motion of the pulley is also to be considered.

3. Finite angular displacements are not vector quantities, the reason being that they do not obey the law of vector addition. This law asserts that the order in which vectors are added does not affect their sum.

$$A + B = B + A$$

It can be seen applying two successive 90° rotations-one about the *x*-axis, and the other about the *z*-axis to a six-sided dice. In the first case, the *z*-rotation is applied before the *x*-rotation and *vice versa* in the second case. It can be seen that the dice ends up in two completely different states. Clearly, the *z*-rotation plus the *x*-rotation does not equal the *x*-rotation plus the *z*-rotation. This non-commutative algebra cannot be represented by vectors. We conclude that, rotations are not, in general, vector quantities.

However infinitesimal angles do commute under addition, making it possible to treat them as vectors.

4. Rotation plus translation motion of a rigid body is simplified by splitting this motion into two parts.

(i) pure translation motion with the linear velocity and acceleration of the centre of mass.

(ii) pure rotational motion about an axis passing through centre of mass and perpendicular to the plane of motion of the particles.

But this motion may be considered as pure rotational motion about an axis called instantaneous axis of rotation (say *IAOR*). If this *IAOR* is non-inertial, then we cannot apply,

$$\tau_{\text{ext}} = I\alpha$$

about this axis. This is because in the derivation of this equation we use $F = ma$ for each particle. If *IAOR* has an acceleration $\mathbf{a}$, we have to apply a pseudo force $-m\mathbf{a}$ to each particle. These pseudo forces produce a pseudo torque about this axis.

But this equation can be applied about an axis passing through centre of mass even if this is non-inertial. Let us prove this:

Take the origin at the centre of mass. The total torque of the pseudo force is

$$\begin{aligned}\Sigma \mathbf{r}_i \times (-m_i \mathbf{a}_i) &= -(\Sigma m_i \mathbf{r}_i) \times \mathbf{a}\\ &= -M\left(\frac{\Sigma m_i \mathbf{r}_i}{M}\right) \times \mathbf{a}\end{aligned}$$

But $\frac{\Sigma m_i \mathbf{r}_i}{M}$ is the position vector of the centre of mass and that is zero as the centre of mass is at the origin. Hence, the torque of pseudo forces acting on all particles of the rigid body is zero.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on rotational equilibrium about a fixed axis.*

Concept

Given that a rigid body can only rotate about a fixed axis, (i.e. hinged at some point) still it is not rotating or it is at rest or it is in equilibrium.

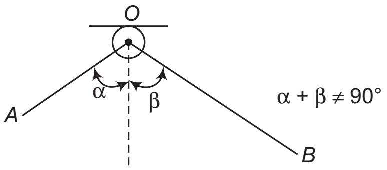
How to Solve?

- Net torque about hinge point should be zero. But torque of hinge force about the same point is already zero.
- Net force on the rigid body is also zero.

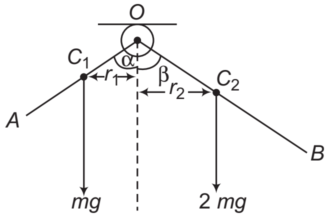
Example 1 *A uniform L shaped rod of mass 3m is hinged at point O. Length OB is two times the length OA. It is in equilibrium.*

Find

(a) relation between α and β (b) net hinge force.



Solution (a) Length OB is two times the length OA. Therefore, mass of OB is $2m$ and that of OA is m and their weight will act at their centres (as the rod is uniform).



If total length is $3l$ then,

$$OA = l \Rightarrow OC_1 = \frac{OA}{2} = \frac{l}{2}$$

and $$r_1 = OC_1 \sin\alpha = \frac{l}{2}\sin\alpha$$

$$OB = 2l \Rightarrow OC_2 = \frac{OB}{2} = l$$

and $$r_2 = OC_2 \sin\beta = l \sin\beta$$

Net torque about $O = 0$

$\Rightarrow$ anticlockwise torque of mg = clockwise torque of $2\,mg$

$$\Rightarrow \quad (mg)r_1 = (2mg)r_2$$

$$\Rightarrow \quad mg\left(\frac{l}{2}\sin\alpha\right) = (2mg)\, l \sin\beta$$

or $\dfrac{\sin\alpha}{\sin\beta} = 4$ is the required relation between α and β.

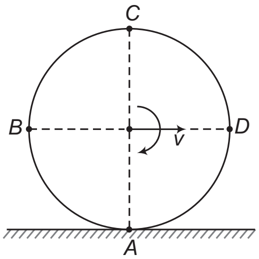
(b) Net force on the rod is also zero. Therefore, hinge force is $3\,mg (= mg + 2mg)$ in upward direction.

Type 2. *To find total kinetic energy of a system of rigid bodies in rotation plus translation.*

Concept

A particle has only translation kinetic energy $\frac{1}{2}mv^2$. But a rigid body may have $\frac{1}{2}mv^2$ and $\frac{1}{2}I\omega^2$. Here, v is velocity of centre of mass, ω is angular speed of the rigid body and I is the moment of inertia about an axis passing through centre of mass and perpendicular to the plane of motion of the body.

Example 2 *A ring of mass 'm' is rolling without slipping with linear speed v as shown in figure. Four particles each of mass 'm' are also attached at points A, B, C and D. Find total kinetic energy of the system.*



Solution Earlier we have learned that in case of pure rolling,

$$\omega = \frac{v}{R},\ v_A = 0,\ v_B = v_D = \sqrt{2}v \quad \text{and} \quad v_C = 2v$$

Now, total kinetic energy = [translational kinetic energy of four particles]+[translational kinetic energy of ring + rotational kinetic energy of ring]

$$\therefore \qquad (\text{KE})_{\text{total}} = \left[\frac{1}{2}m(0)^2 + \frac{1}{2}m(\sqrt{2}\,v)^2 + \frac{1}{2}m(\sqrt{2}\,v)^2 + \frac{1}{2}m(2v)^2\right] + \left[\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right]$$

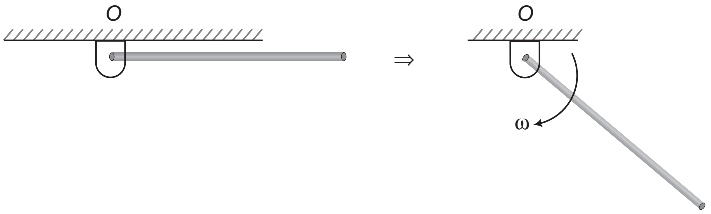
Substituting $I = mR^2$ and $\omega = \frac{v}{R}$

we get, $$(\text{KE})_{\text{total}} = 5mv^2$$ **Ans.**

Type 3. *Energy conservation in pure rotational motion.*

Concept

A rigid body (suppose a rod) is hinged at O as shown in figure. It can have only rotational motion in a vertical plane about a smooth horizontal axis passing through O and perpendicular to plane of paper.



It is released from the horizontal position. As the rod rotates downwards its gravitational potential energy decreases and rotational kinetic energy increases.

How to Solve?

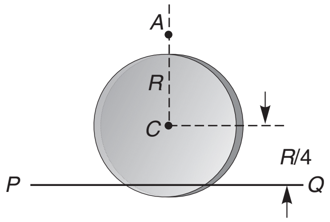
- Put decrease in gravitational potential energy (mgh) equal to increase in rotational kinetic energy $\left(\frac{1}{2}I\omega^2\right)$.

  Here, h is the fall of height of centre of mass of the rigid body. In case of particle, 'h' is decrease in height of the particle. From this equation, we can find the value of ω.
- In pure rotational motion velocity of any point is

$$v = r\omega$$

Here, r is the distance of that point from the axis of rotation.

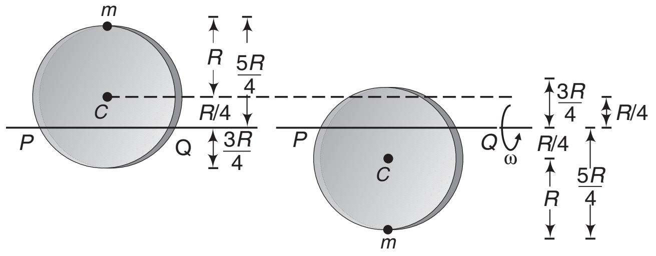
Example 3 *A uniform circular disc has radius R and mass m. A particle, also of mass m, is fixed at a point A on the edge of the disc as shown in the figure. The disc can rotate freely about a horizontal chord PQ that is at a distance R/4 from the centre C of the disc. The line AC is perpendicular to PQ. Initially the disc is held vertical with the point A at its highest position. It is then allowed to fall, so that it starts rotation about PQ. Find the linear speed of the particle as it reaches its lowest position.* **(JEE 1998)**



Solution Initial and final positions are shown below.

Decrease in potential energy of mass

$$= mg\left\{2 \times \frac{5R}{4}\right\} = \frac{5\,mgR}{2}$$



Decrease in potential energy of disc

$$= mg\left\{2 \times \frac{R}{4}\right\} = \frac{mgR}{2}$$

Therefore, total decrease in potential energy of system

$$= \frac{5\,mgR}{2} + \frac{mgR}{2} = 3mgR$$

Gain in kinetic energy of system $= \frac{1}{2}I\omega^2$

where, I = moment of inertia of system (disc + mass) about axis PQ

= moment of inertia of disc + moment of inertia of mass

$$= \left\{\frac{mR^2}{4} + m\left(\frac{R}{4}\right)^2\right\} + m\left(\frac{5R}{4}\right)^2$$

$$I = \frac{15\,mR^2}{8}$$

From conservation of mechanical energy,

Decrease in potential energy = Gain in kinetic energy

$$\therefore \qquad 3\,mgR = \frac{1}{2}\left(\frac{15\,mR^2}{8}\right)\omega^2$$

$$\Rightarrow \qquad \omega = \sqrt{\frac{16g}{5R}}$$

Therefore, linear speed of particle at its lowest point

$$v = r\omega = \left(\frac{5R}{4}\right)$$

$$\omega = \frac{5R}{4}\sqrt{\frac{16g}{5R}}$$

$$\Rightarrow \qquad v = \sqrt{5gR}$$

Ans.

Type 4. *Based on Angular Impulse.*

Concept

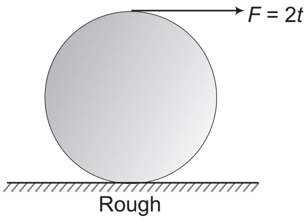
Angular impulse $\text{A.I} = \tau\,\Delta t$

If torque is a function of time, then

$$\text{A.I} = \int \tau\, dt$$

and this angular impulse is equal to the change in angular momentum.

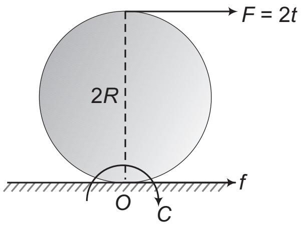
Example 4 *A solid sphere of mass* '*m*' and radius '*R*' is kept over a rough ground. A time varying force $F = 2t$ is acting at the topmost point as shown in figure.



(a) Find angular momentum of the sphere about the bottommost point as a function of time 't'.

(b) Does this result depend on the fact whether the ground is rough or smooth?

Solution (a) Suppose '*f*' is the force of friction acting on the sphere in forward direction as shown in figure.



Taking torque about bottommost point *O*. Torque of friction is already zero, as this force already passes through *O*. Torque of applied force is only there.

$$\therefore \qquad \tau = F \times r_{\perp}$$

$$= (2t)\,(2R) = 4\,RT$$

As, this toque is a function of time.

$$\therefore \text{ Angular impulse} = \int_0^t \tau\, dt = \int_0^t (4Rt)\, dt = 2\,Rt^2$$

This angular impulse is equal to change in angular momentum. Hence, angular momentum at time '*t*' is $2Rt^2$.

(b) This result is independent of the nature of surface (smooth or rough), as the torque of friction about bottommost point is already zero. It does not contribute in angular impulse and therefore in angular momentum.

Type 5. *Role of friction in acceleration pure rolling.*

Concept

Friction has a tendency to stop the relative motion. In case of rolling (rotation + translation) over a stationary ground relative motion is stopped when $v = R\omega$ equation is satisfied at all instants. To satisfy this equation, a should be equal to '$R\alpha$'. Hence, friction has a tendency to satisfy the equation, $a = R\alpha$. Sometimes friction is sufficient and sometimes not.

If friction is sufficient, then accelerated pure rolling will takes place. Otherwise forward or backward slipping occurs.

How to Solve?

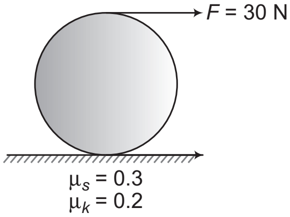
- Find N, $\mu_s N$ and $\mu_k N$ (or μN)
- Find requirement of friction to satisfy the equation $a = R\alpha$.

  For this, apply a general value of f in either forward or backward direction.
- Put $a = R\alpha$

  or
  $$\frac{F_{\text{net}}}{m} = R\left(\frac{\tau_{\text{net}}}{I}\right)$$
  and find the required value of f for accelerated pure rolling.
- If $f \le \mu_s N$ i.e. requirement ≤ availability then pure rolling will take place and $a = R\alpha$. In this case the obtained value of friction in step (iii) will act.
- If $f > \mu_s N$ then either forward or backward slip will take place and $a \ne R\alpha$. But,
  $$a = \frac{F_{\text{net}}}{m} \quad \text{and} \quad \alpha = \frac{\tau_{\text{net}}}{I}$$
  are still applicable.

  In this case, $\mu_k N$ friction will act.
- If required value of friction in step (iii) comes out to be negative then just change the direction of friction which was initially assumed. Otherwise, magnitude wise all calculations are same.
- After calculation, if force of friction comes in forward direction, then if there is slip, it is backward slip and $a < R\alpha$.

Example 5 *A solid sphere of mass 5 kg and radius 1 m is kept over a rough surface as shown in figure. A force $F = 30\text{N}$ is acting at the topmost point.*



(a) Check whether the pure rolling will take place or not.

(b) Find direction and magnitude of friction actually acting on the sphere.

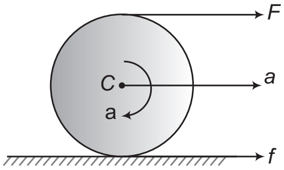
(c) Find linear acceleration 'a' and angular acceleration 'α'.

Take $g = 10\, m/s^2$

Solution (a) Availability of friction.

$$N = mg = 5 \times 10 = 50\text{ N}$$
$$\mu_s N = 0.3 \times 50 = 15\text{ N}$$
$$\mu_k N = 0.2 \times 50 = 10\text{ N}$$

Requirement of friction for accelerated pure rolling (or to satisfy $a = R\alpha$).



Let the friction f acts in forward direction.

$$a = R\alpha$$

$\Rightarrow$
$$\frac{F_{\text{net}}}{m} = R\left(\frac{\tau_{\text{net}}}{I}\right)$$

or
$$\frac{F+f}{m} = R\left[\frac{FR - fR}{\frac{2}{5}mR^2}\right]$$

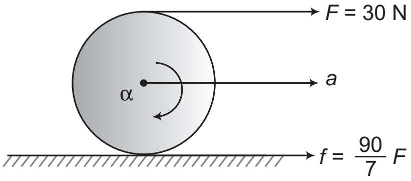
Solving this equation, we get

$$f = \frac{3}{7}F$$
$$= \frac{3}{7}(30) = \frac{90}{7}\text{ N}$$

Since, this value of f is less than $\mu_s N$. Therefore, friction is sufficient for accelerated pure rolling to take place. Hence, pure rolling is taking place.

(b) The above value of f is positive, hence direction of friction is forward and $\frac{90}{7}$ N friction will be acting.

(c) The actual forces acting on the sphere are as under



$$a = \frac{F_{\text{net}}}{m} = \frac{30 + \frac{90}{7}}{5} = 8.57\text{ m/s}^2$$ **Ans.**

Since, $a = R\alpha$

$\therefore$
$$\alpha = \frac{a}{R} = \frac{8.57}{1} = 8.57\text{ rad/s}^2$$ **Ans.**

Example 6 *Repeat all parts of above problem for $F = 40N$.*

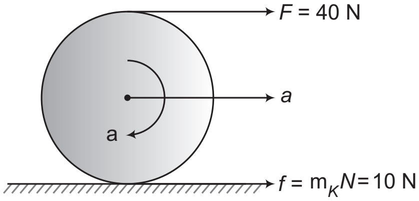
Solution (a) and (b): We have already calculated that, requirement of friction for pure rolling is

$$f = \frac{3}{7}F$$

For,
$$F = 40\text{ N} \Rightarrow f = \frac{120}{7}\text{ N}$$

Now, this value of f is more than $\mu_s N$, so backward slip ($a < R\alpha$) will take place because f is positive so kinetic friction or 10 N will act in forward direction.

(c) The actual forces acting on the sphere are as under :



$$a = \frac{F_{\text{net}}}{m} = \frac{40+10}{5} = 10 \text{ m/s}^2 \qquad \textbf{Ans.}$$

$$\alpha \neq \frac{a}{R}$$

But $$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{FR - fR}{\frac{2}{5}mR^2}$$

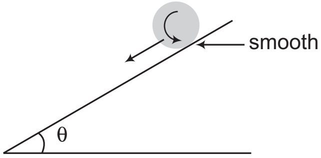
$$= \frac{2.5\,(F-f)}{mR} = \frac{2.5\,(40-10)}{(5)\,(1)}$$

$$= 15 \text{ rad/s}^2. \qquad \textbf{Ans.}$$

Note *We can see that $a < R\alpha$.*

Type 6. *Based on conservation of mechanical energy and no change in rotational kinetic energy.*

Concept



If a body is kept in translational plus rotational motion condition over a smooth inclined plane then two forces acting on the body (weight and normal reaction) pass through centre of mass. So, they cannot provide the torque and therefore angular acceleration. Therefore, its angular speed and rotational kinetic energy remains constant. Same is the case, when the body moves freely under gravity. Here, the only force acting on the body is 'mg' which also passes through centre of mass. When the body moves up, its translational kinetic energy decreases and gravitational potential energy increases but rotational kinetic energy remains constant. Opposite is the case when the body moves down.

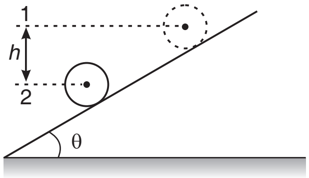
Further, in case of accelerated pure rolling over a stationary rough ground, work done by friction is zero. Hence, mechanical energy will remain constant. In this case,

$$\frac{K_R}{K_T} = 1 \text{ for ring } = \frac{1}{2} \text{ for disc}$$

$$= \frac{2}{5} \text{ for solid sphere etc.}$$

Thus, total mechanical energy ($K_R + K_T \pm mgh$) remains constant in case of accelerated pure rolling, over a smooth ground or under gravity.

Example 7 *A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its centre of mass when its centre of mass has fallen a height h.*



Solution Considering the two shown positions of the cylinder. As it does not slip hence total mechanical energy will be conserved.

Energy at position 1 is $E_1 = mgh$

Energy at position 2 is $E_2 = \frac{1}{2} mv_{COM}^2 + \frac{1}{2} I_{COM} \omega^2$

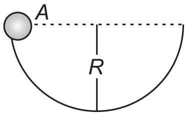
$$\frac{v_{COM}}{r} = \omega \quad \text{and} \quad I_{COM} = \frac{mr^2}{2}$$

$$\Rightarrow \qquad E_2 = \frac{3}{4} mv_{COM}^2$$

From conservation of energy, $E_1 = E_2$ or $mgh = \frac{3}{4} mv_{COM}^2$

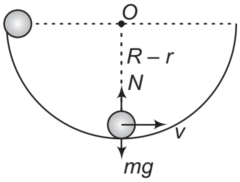
$$\Rightarrow \qquad v_{COM} = \sqrt{\frac{4}{3} gh}$$ **Ans.**

Example 8 *A small solid cylinder of radius r is released coaxially from point A inside the fixed large cylindrical bowl of radius R as shown in figure. If the friction between the small and the large cylinder is sufficient enough to prevent any slipping, then find :*



(a) What fractions of the total energy are translational and rotational, when the small cylinder reaches the bottom of the larger one?

(b) The normal force exerted by the small cylinder on the larger one when it is at the bottom.



Solution (a) $K_{trans} = \frac{1}{2} mv^2$

$$K_{rot} = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{4} mv^2$$

$$K = K_{trans} + K_{rot} = \frac{3}{4} mv^2$$

$$\therefore \qquad \frac{K_{trans}}{K} = \frac{2}{3} \Rightarrow \frac{K_{rot}}{K} = \frac{1}{3}$$

(b) From conservation of energy,

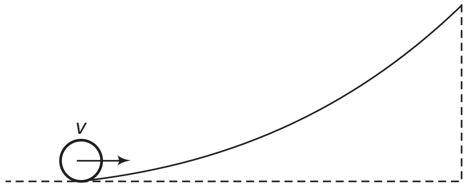
$$mg(R - r) = \frac{3}{4} mv^2$$

$$\therefore \qquad \frac{mv^2}{R - r} = \frac{4}{3} mg$$

Now, $$N - mg = \frac{mv^2}{R - r} = \text{centripetal force} = \frac{4}{3} mg$$

$$\therefore \qquad N = \frac{7}{3} mg$$ **Ans.**

Example 9 *A small object of uniform density rolls up a curved surface with an initial velocity v. It reaches up to a maximum height of* $\frac{3v^2}{4g}$ *with respect to the initial position. The object is* **(JEE 2007)**



(a) *ring* (b) *solid sphere* (c) *hollow sphere* (d) *disc*

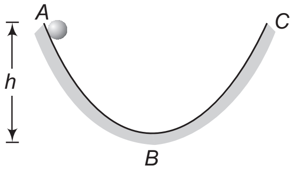
Solution

$$\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mg\left(\frac{3v^2}{4g}\right)$$

$$\therefore \quad I = \frac{1}{2}mR^2$$

$\therefore$ Body is disc. The correct option is (d).

Example 10 *A solid ball rolls down a parabolic path ABC from a height h as shown in figure. Portion AB of the path is rough while BC is smooth. How high will the ball climb in BC ?*



Solution At B, total kinetic energy $= mgh$

Here, m = mass of ball

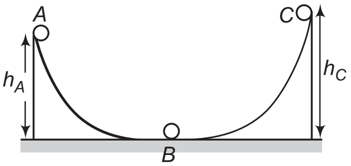
The ratio of rotational to translational kinetic energy would be,

$$\frac{K_R}{K_T} = \frac{2}{5} \Rightarrow \therefore \; K_R = \frac{2}{7}mgh \quad \text{and} \quad K_T = \frac{5}{7}mgh$$

In portion BC, friction is absent. Therefore, rotational kinetic energy will remain constant and translational kinetic energy will convert into potential energy. Hence, if H be the height to which ball climbs in BC, then

$$mgH = K_T \quad \text{or} \quad mgH = \frac{5}{7}mgh \quad \text{or} \quad H = \frac{5}{7}h$$

Example 11 *A ball moves over a fixed track as shown in the figure. From A to B the ball rolls without slipping. If surface BC is frictionless and* K_A, K_B *and* K_C *are kinetic energies of the ball at A, B and C respectively, then* **(JEE 2006)**



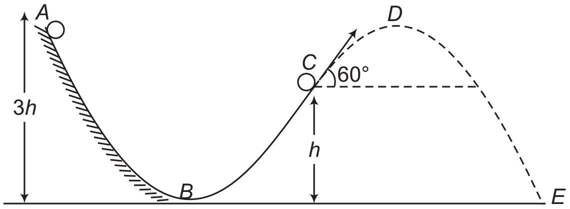
(a) $h_A > h_C; K_B > K_C$ (b) $h_A > h_C; K_C > K_A$

(c) $h_A = h_C; K_B = K_C$ (d) $h_A < h_C; K_B > K_C$

Solution On smooth part BC, due to zero torque, angular velocity and hence the rotational kinetic energy remains constant. While moving from B to C translational kinetic energy converts into gravitational potential energy.

$\therefore$ The correct option is (a)

Example 12 *A small solid sphere of mass 'm' is released from point A. Portion AB is sufficiently rough (to provide accelerated pure rolling), BC is smooth and after C, the ball moves freely under gravity. Find gravitational potential energy (U), rotational kinetic energy* (K_R) *and translational kinetic energy* (K_T) *at points A, B C, D and E.*



Solution AB is sufficiently rough BC is smooth and after C motion is under gravity. So, total mechanical energy $(U + K_R + K_T)$ is always constant. On AB:

$$\frac{K_R}{K_T} = \frac{2}{5} \text{ (for solid sphere)}$$

as accelerated pure rolling is taking place.

After B, rotational kinetic energy will become constant. After C, centre of mass of the ball will follow a projectile motion.

$$\therefore \quad v_D = v_C \cos 60° = \frac{v_C}{2} = \text{half of } v_C$$

$$\therefore \quad (K_T)_D = \frac{1}{4}(K_T)_C$$

At point *A*

$$U = 3mgh$$

$$K_R = 0 \quad \Rightarrow \quad K_T = 0$$

$\therefore$ Total mechanical energy $E = 3mgh$ = constant.

At point *B*

$$U = 0 \quad \Rightarrow \quad E = 3mgh$$

$$\therefore \quad K = E = 3mgh \qquad (K = \text{total K.E})$$

$$\text{But} \quad \frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \quad K_R = \frac{2}{7}K = \frac{6}{7}mgh$$

$$K_T = \frac{5}{7}K = \frac{15}{7}mgh$$

At point *C*

$$U = mgh$$

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$$\therefore \quad K_T = E - U - K_R$$

$$= 3mgh - mgh - \frac{6}{7}mgh = \frac{8}{7}mgh$$

At point *D*

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$$K_T = \frac{1}{4}(K_T)_C = \frac{2}{7}mgh$$

$$\therefore \quad U = E - K_R - K_T = 3mgh - \frac{6}{7}mgh - \frac{2}{7}mgh = \frac{13}{7}mgh$$

At point *E*

$$U = 0$$

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$$\therefore \quad K_T = E - K_R - U = 3\,mgh - \frac{6}{7}mgh - 0 = \frac{15}{7}mgh$$

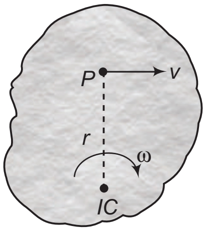
Type 7. *Based on instantaneous axis of rotation (IAOR).*

Concept

We have seen that in case of pure of rolling ($v = R\omega$) over a stationary ground *IAOR* passes through the bottommost point of the rigid body. Following are three more cases where we can locate the position of *IAOR*. In the following cases *IAOR* is written as *IC*.

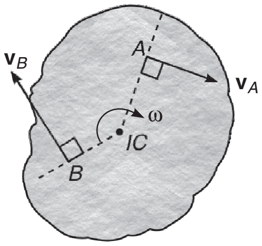
(i) **Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body**

If v and ω are known, the *IC* is located along the line drawn perpendicular to $\mathbf{v}$ at *P*, such that the distance from *P* to *IC* is, $r = \frac{v}{\omega}$. Note that *IC* lie on that side of *P* which causes rotation about the *IC*, which is consistent with the direction of motion caused by ω and $\mathbf{v}$.



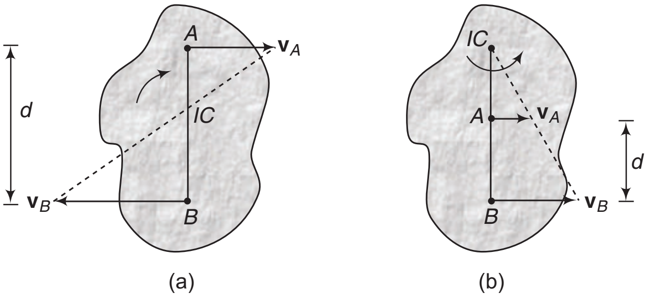
(ii) **Given the lines of action of two non-parallel velocities**

Consider the body shown in figure where the line of action of the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known. Draw perpendiculars at *A* and *B* to these lines of action. The point of intersection of these perpendiculars as shown locates the *IC* at the instant considered.



(iii) **Given the magnitude and direction of two parallel velocities**

When the velocities of points A and B are parallel and have known magnitudes v_A and v_B then the location of the IC is determined by proportional triangles as shown in figure.



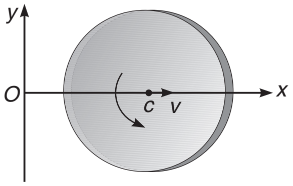
In both the cases, $r_{A,IC} = \dfrac{v_A}{\omega}$ and $r_{B,IC} = \dfrac{v_B}{\omega}$

In Fig. (a) $r_{A,IC} + r_{B,IC} = d$

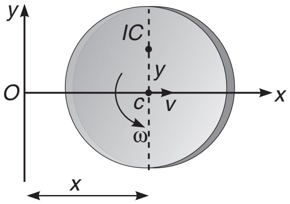
and in Fig. (b) $r_{B,IC} - r_{A,IC} = d$

As a special case, if the body is translating, $v_A = v_B$ and the IC would be located at infinity, in which case $\omega = 0$.

Example 13 *A rotating disc moves in the positive direction of the x-axis. Find the equation y(x) describing the position of the instantaneous axis of rotation if at the initial moment the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counterclockwise with a constant angular acceleration* α. *The initial angular velocity is equal to zero.*



Solution $t = \dfrac{x}{v}$ and $\omega = \alpha t = \dfrac{\alpha x}{v}$



The position of IC will be at a distance

$$y = \frac{v}{\omega} \quad \text{or} \quad y = \frac{v}{\frac{\alpha x}{v}}$$

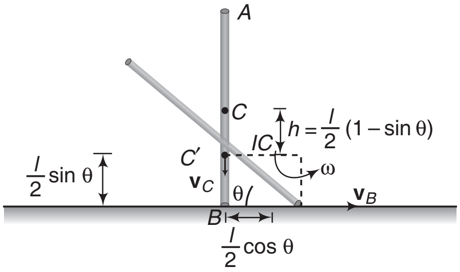
or

$$y = \frac{v^2}{\alpha x} \quad \text{or} \quad xy = \frac{v^2}{\alpha} = \text{constant}$$

This is the desired x-y equation. This equation represents a rectangular hyperbola.

Example 14 *A uniform thin rod of mass m and length l is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with horizontal.*

Solution As the floor is smooth, mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod, hence the centre of mass moves vertically downwards in a straight line. Thus velocities of *COM* and the lower end *B* are in the directions shown in figure. The location of *IC* at this instant can be found by drawing perpendiculars to $\mathbf{v}_C$ and $\mathbf{v}_B$ at respective points. Now, the rod may be assumed to be in pure rotational motion about *IAOR* passing through *IC* with angular speed ω.



Applying conservation of mechanical energy. Decrease in gravitational potential energy of the rod = increase in rotational kinetic energy about IC

∴ $$mgh = \frac{1}{2} I_{IC}\,\omega^2$$

or $$mg\frac{l}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega = \sqrt{\frac{12g(1 - \sin\theta)}{l(1 + 3\cos^2\theta)}}$$

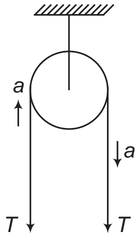
Now, $$|\mathbf{v}_C| = \left(\frac{l}{2}\cos\theta\right)\omega$$

$$= \sqrt{\frac{3gl(1 - \sin\theta)\cos^2\theta}{(1 + 3\cos^2\theta)}}$$ **Ans.**

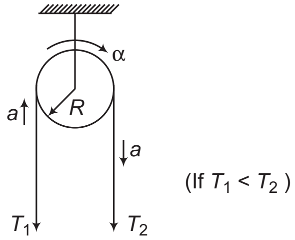
Type 8. *Based on rotational pulleys.*

Concept

If the pulley is not massless and is sufficiently rough then tension on a string passing over it on its both sides will be different. In this case, the string does not slip over the pulley but the pulley also rotates.

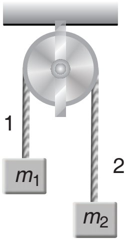


In the above figure, pulley is massless, smooth and stationary. String slips over the pulley.

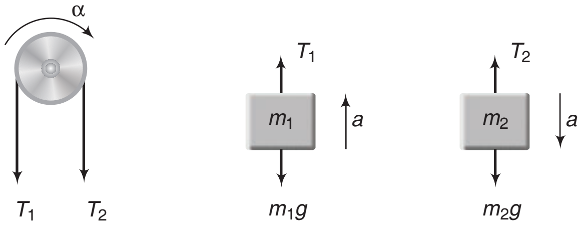


In this figure, pulley is neither massless nor smooth. It rotates with the string. If there is no slip then, $a = R\alpha$.

Example 15 *In the arrangement shown in figure the mass of the uniform solid cylindrical pulley of radius R is equal to m and the masses of two bodies are equal to* m_1 *and* m_2*. The thread slipping and the friction in the axle of the pulley are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions* $\frac{T_1}{T_2}$ *of the vertical sections of the thread in the process of motion.*



Solution Let α = angular acceleration of the pulley and a = linear acceleration of two bodies



Equations of motion are

For mass m_1, $$T_1 - m_1 g = m_1 a \quad \text{...(i)}$$

For mass m_2, $$m_2 g - T_2 = m_2 a \quad \text{...(ii)}$$

For pulley, $$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2} mR^2} \quad \text{...(iii)}$$

For no slipping condition $$a = R\alpha \quad \text{...(iv)}$$

Solving these equations, we get $$\alpha = \frac{2(m_2 - m_1)g}{(2m_1 + 2m_2 + m)R}$$ **Ans.**

and $$\frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$ **Ans.**

Type 9. *When friction converts forward or backward slip into pure rolling.*

Concept

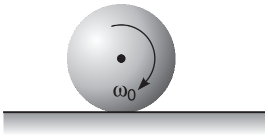
If a body is kept in forward or backward slip condition ($v \neq R\omega$) over a rough horizontal ground, then kinetic friction will act in the opposite direction of slip. This friction provides both linear acceleration 'a' and angular acceleration 'α'. They are in the directions so as the equation $v = R\omega$ (with proper sense of rotation) is satisfied so that pure rolling may start. For example, if initially $v > r\omega$, then 'a' is in the opposite direction of 'v' (to decrease it) and 'α' is in the direction of 'ω' (to increase it).

Once pure rolling starts, relative motion (or slipping) is stopped and friction becomes zero.

During slip, mechanical energy is not conserved. But, some part of mechanical energy is used up in doing work against friction. Once pure rolling starts, mechanical energy becomes constant.

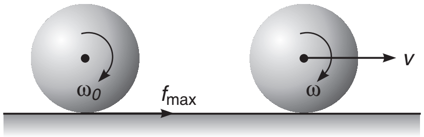
If only one coefficient of friction or μ is given in the question then, instead of kinetic friction, apply μN during the slip.

Example 16 *A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the time t when the slipping stops. In addition, state the linear velocity v and angular velocity ω at the end of slipping.*



Solution Let m be the mass of the sphere.

Since, it is a case of backward slipping, force of friction is in forward direction. Limiting friction will act in this case.



Linear acceleration $a = \frac{f}{m} = \frac{\mu mg}{m} = \mu g$

Angular retardation $\alpha = \frac{\tau}{I} = \frac{f \cdot r}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$

Slipping is ceased when $v = r\omega$

or $$(at) = r(\omega_0 - \alpha t)$$

or $$\mu gt = r\left(\omega_0 - \frac{5}{2}\frac{\mu gt}{r}\right) \quad \text{or} \quad \frac{7}{2}\mu gt = r\omega_0$$

$\therefore$ $$t = \frac{2}{7}\frac{r\omega_0}{\mu g}$$ **Ans.**

$$v = at = \mu gt = \frac{2}{7}r\omega_0$$ **Ans.**

and $$\omega = \frac{v}{r} = \frac{2}{7}\omega_0$$ **Ans.**

Alternate Solution

Net torque on the sphere about the bottommost point is zero as friction is passing through that point. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

$$L_i = L_f$$

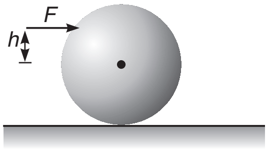
$$\therefore \quad I\omega_0 = I\omega + mrv$$

$$\text{or} \quad \frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

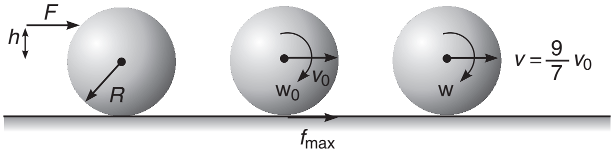
$$\therefore \quad \omega = \frac{2}{7}\omega_0 \quad \text{and} \quad v = r\omega = \frac{2}{7}r\omega_0$$

Ans.

Example 17 *A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in figure. The ball leaves the cue with a speed v_0 and because of its forward english (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$. Show that $h = \frac{4}{5}R$ where R is the radius of the ball.*



Solution Let ω_0 be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let v be the linear speed and ω the angular speed when slipping is ceased.



$$\therefore \quad v = R\omega \quad \text{or} \quad \omega = \frac{v}{R}$$

$$\text{Given,} \quad v = \frac{9}{7}v_0 \qquad \text{...(i)}$$

$$\therefore \quad \omega = \frac{9}{7}\frac{v_0}{R} \qquad \text{...(ii)}$$

Applying, Linear impulse = change in linear momentum

$$\therefore \quad F\,dt = mv_0 \qquad \text{...(iii)}$$

Angular impulse = change in angular momentum

$$\therefore \quad \tau\,dt = I\omega_0 \quad \text{or} \quad (Fh)dt = \frac{2}{5}mR^2\omega_0 \qquad \text{...(iv)}$$

During the slip, angular momentum about bottommost point will remain conserved.

$$\text{i.e.} \quad L_i = L_f$$

$$\text{or} \quad I\omega_0 + mRv_0 = I\omega + mRv$$

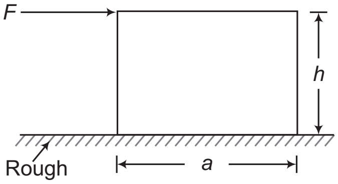
$$\therefore \quad \frac{2}{5}mR^2\omega_0 + mRv_0 = \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \qquad \text{...(v)}$$

Solving Eqs. (iii), (iv) and (v), we get $h = \frac{4}{5}R$ **Proved.**

Type 10. *Based on critical value of μ.*

Concept

If a force is applied on a block as shown in figure and the force is increased then in some cases the block slides before toppling and in other cases it topples before sliding. It mainly depends on:



(i) base length a
(ii) height of point of application of force h
(iii) coefficient of friction μ.

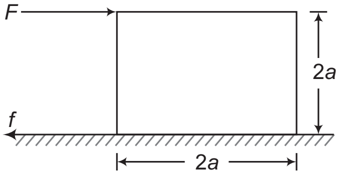
For example, if μ is small then chances of sliding are more. If a is small then chances of toppling are more.

For given values of a and h it only depends on the value of μ. In such problems, there is a critical value μ_{cr}. If given value of $\mu > \mu_{cr}$, then the block topples before sliding and if $\mu < \mu_{cr}$ then the block slides before toppling.

How to Solve?

- Make two conditions:
  Condition of sliding
  Condition of toppling, when the normal reaction (just before toppling) shifts to the right side edge).
  From these, two conditions we can find μ_{cr}.

Example 18 *For the given dimensions shown in figure, find critical value of coefficient of friction μ.*



Solution **Condition of sliding**

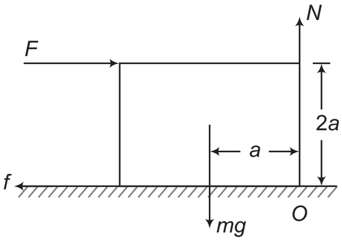
The block will slide if,

$$F > \mu N$$

but $\quad N = mg \qquad$ (m = mass of the block)

$\therefore \quad F > \mu mg \qquad$ ...(i)

Condition of toppling



Block will topple about an axis passing through O and perpendicular to plane of paper if:
clockwise torque of $F >$ anticlockwise torque of mg

$$\therefore \qquad F(2a) > (mg)(a)$$

or
$$F > \frac{1}{2}mg \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that,

$$\mu_{cr} = \frac{1}{2} \qquad \textbf{Ans.}$$

If given value of μ is less than μ_{cr} (say it is $\frac{1}{4}$) then Eq. (i) is,

$$F > \frac{1}{4}mg$$

So, Eq. (i) is satisfied before Eq. (ii). Therefore, the block will slide before toppling.

If given value of μ is greater than μ_{cr}(say it is $\frac{3}{4}$) then Eq. (i) is,

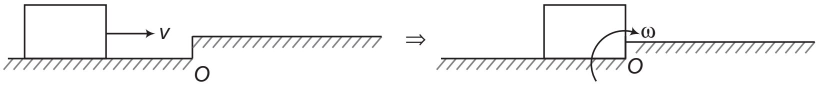
$$F > \frac{3}{4}mg$$

So, Eq. (ii) is satisfied before Eq. (i). Hence, the block will topple before sliding.

Type 11. *When pure translational motion of a rigid body converts into pure rotational motion by a jerk (or linear impulse).*

Concept

A block in pure translational motion (with linear velocity v) meets an obstacle at O. A linear impulse will act on the block at point O. Just after the impact, the block starts rotating about point O (with an angular speed say ω). The value of this 'ω' can be found by conservation of angular momentum about O because during the impact, the angular impulse of the linear impulse about point O will be zero (as $r_\perp = 0$). Just before impact motion is pure translational. So, angular momentum is $mvr\sin\theta$ or $mvr_\perp$. Here, v is the velocity of centre of mass. Just after impact motion is pure rotational. So, angular momentum is $I\omega$, where, I is the moment of inertia passing through O and perpendicular to plane of paper.

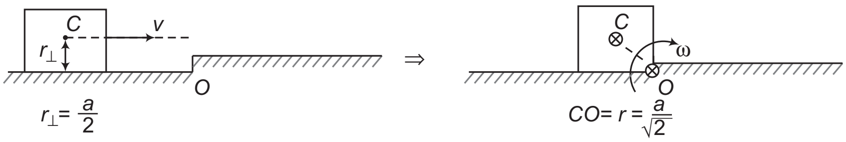


Furthermore, there will be loss of mechanical energy during impact $\left(= E_i - E_f = \frac{1}{2}mv^2 - \frac{1}{2}I\omega^2\right)$. But after impact, the mechanical energy remains constant. As the block moves up, its potential energy increases and rotational kinetic energy decreases.

Example 19 *In the figure shown in the text, if the block is a cube of side 'a'. Find*

(a) ω just after impact
(b) loss of mechanical energy during impact
(c) minimum value of v so as the block overcomes the obstacle and does not turn back.

Solution



(a) From conservation of angular momentum about O,

$$L_i = L_f$$

$$mv_c r_\perp = I_o\omega = (I_c + mr^2)\omega$$

$$\Rightarrow \qquad mv\left(\frac{a}{2}\right) = \left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\omega$$

$$\Rightarrow \qquad \omega = \frac{3}{4}\left(\frac{v}{a}\right)$$ **Ans.**

(b) Loss of mechanical energy,

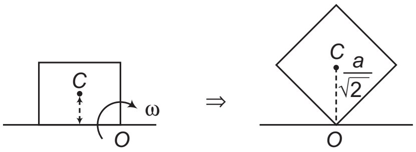
$$= E_i - E_f$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}I_o\omega^2$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}\left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\left(\frac{3}{4}\frac{v}{a}\right)^2$$

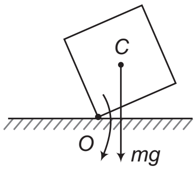
$$= \frac{5}{16}mv^2$$ **Ans.**

(c)



Block overcomes the obstacle at O if centre of mass rises upto a height $\frac{a}{\sqrt{2}}$ as shown in figure (from the initial height $\frac{a}{2}$).

Because after that torque of 'mg' about O will itself rotate the block on other side as shown in figure.



∴ Decrease in rotational kinetic energy = increase in gravitational potential energy

$$\therefore \qquad \frac{1}{2}I_o\omega^2 = mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

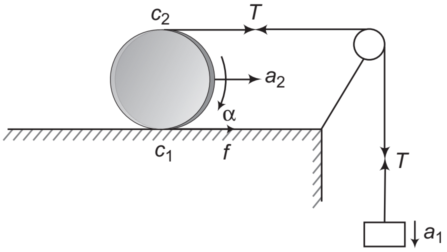
or

$$\frac{1}{2}\left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\left[\frac{3}{4}\frac{v}{a}\right]^2 = mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$\therefore \qquad v = \sqrt{1.1\,g\,a}$$ **Ans.**

Type 12. *To identify number of unknowns and then make equations corresponding to that.*

Concept



A disc and block system are released from rest as shown in figure. Ground is sufficiently rough, so that there is no slip anywhere. We have to find accelerations of both, tension in the string and force of friction.

In such problems, first of all find the number of unknowns:

(i) Block can have only translational motion. So, it has some linear acceleration say a_1

(ii) Disc can have translational as well as rotational motion. So, it has linear acceleration a_2 and angular acceleration 'α'.

(iii) Ground is rough. So, there is one unknown friction 'f'

(iv) One more unknown is tension in the string T.

Therefore, there are total five unknowns in this problem, a_1, a_2, α, T and f.

How to Solve?

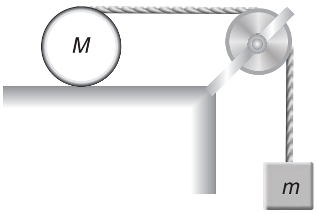
We will make three acceleration equations for a_1, a_2 and α by using the equations

$$a = \frac{F_{\text{net}}}{m} \quad \text{and} \quad \alpha = \frac{\tau_{\text{net}}}{I}$$

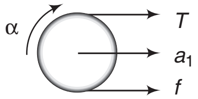
There are two contact equations at c_1 and c_2.

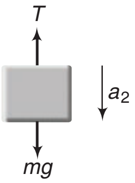
At c_1, disc is in contact with ground and at c_2 with string. At the other end string is connected to the block. So, there should not be any slip at c_1 between disc and ground and at c_2 between disc and string. Following two examples illustrate the method of making these equations.

Example 20 *Consider the arrangement shown in figure. The string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a massless, frictionless pulley to a falling weight. Determine the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.*



Solution Let T be the tension in the string and f the force of (static) friction, between the cylinder and the surface.





a_1 = acceleration of centre of mass of cylinder towards right

a_2 = downward acceleration of block m

α = angular acceleration of cylinder (clockwise)

Acceleration equations

For block, $$mg - T = ma_2 \quad \ldots\text{(i)}$$

For cylinder, $$T + f = Ma_1 \qquad \text{...(ii)}$$

$$\alpha = \frac{(T - f)R}{\frac{1}{2}MR^2} \qquad \text{...(iii)}$$

Contact equations

The string is attached to the mass m at the highest point of the cylinder, hence

$$v_m = v_{\text{COM}} + R\omega$$

Differentiating, we get $$a_2 = a_1 + R\alpha \qquad \text{...(iv)}$$

We also have (for rolling without slipping)

$$a_1 = R\alpha \qquad \text{...(v)}$$

Solving these equations, we get $$a_2 = \frac{8mg}{3M + 8m}$$ **Ans.**

Alternate Solution (Energy Method)

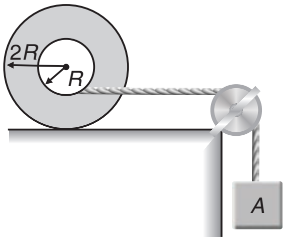
Since, there is no slipping at all contacts mechanical energy of the system will remain conserved.

∴ Decrease in gravitational potential energy of block m in time t = increase in translational kinetic energy of block + increase in rotational as well as translational kinetic energy of cylinder.

∴ $$mgh = \frac{1}{2}mv_2^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_1^2$$

or $$mg\left(\frac{1}{2}a_2t^2\right) = \frac{1}{2}m(a_2t)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2 + \frac{1}{2}M(a_1t)^2 \qquad \text{...(vi)}$$

Solving Eqs. (iv), (v) and (vi), we get the same result.



Example 21 *A thin massless thread is wound on a reel of mass 3 kg and moment of inertia 0.6 kg-m^2. The hub radius is $R = 10$ cm and peripheral radius is $2R = 20$ cm. The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of 1 kg.*

Solution Here, number of unknowns are five:

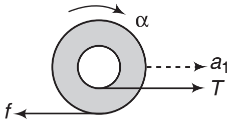
a_1 = acceleration of centre of mass of reel

a_2 = acceleration of 1 kg block

α = angular acceleration of reel (clockwise)

T = tension in the string

and f = force of friction



Acceleration equations :

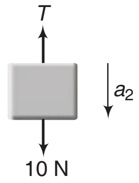
Free body diagram of reel is as shown in figure: (only horizontal forces are shown).

Equations of motion are

$$T - f = 3a_1 \qquad \text{...(i)}$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T.R}{I} = \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \qquad \text{...(ii)}$$

Free body diagram of mass is,



Equation of motion is,

$$10 - T = a_2 \quad \text{...(iii)}$$

Contact equations :

For no slipping condition,

$$a_1 = 2R\alpha \quad \text{or} \quad a_1 = 0.2\,\alpha \quad \text{...(iv)}$$

and $$a_2 = a_1 - R\alpha \quad \text{or} \quad a_2 = a_1 - 0.1\,\alpha \quad \text{...(v)}$$

Solving the above five equations, we get

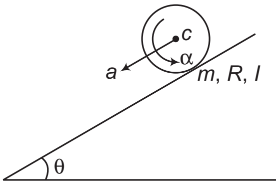
$$a_1 = 0.27 \text{ m/s}^2 \quad \text{and} \quad a_2 = 0.135 \text{ m/s}^2$$ **Ans.**

Type 13. *Energy method of solving problems of accelerated pure rolling.*

Concept

In accelerated pure rolling over a stationary ground work done by friction is zero. So, mechanical energy remains constant. Therefore, some problems of accelerated pure rolling can also be solved by using energy conservation principle.

Example 22 *A body of mass m, radius R and moment of inertia I (about an axis passing through the centre of mass and perpendicular to plane of motion) is released from rest over a sufficiently rough ground (to provide accelerated pure rolling). Find linear acceleration of the body.*



Solution Let linear acceleration is 'a' and angular acceleration 'α'.

For accelerated pure rolling,

$$\alpha = \frac{a}{R}$$

After time t, displacement of centre of mass along the plane, $s = \frac{1}{2}at^2$

$\therefore$ Height fallen by centre of mass

$$h = (s)(\sin\theta) = \frac{1}{2}at^2\sin\theta$$

linear velocity $v = at$

angular velocity $\omega = \alpha\, t = \frac{at}{R}$

From energy conservation principle, decrease in potential energy = increase in translational and rotational kinetic energy.

or $$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Substituting the value we have,

$$mg\left(\frac{1}{2}at^2\sin\theta\right) = \frac{1}{2}m\,(at)^2 + \frac{1}{2}I\left(\frac{at}{R}\right)^2$$

Solving this equation we get, $$a = \frac{g\sin\theta}{1 + \dfrac{I}{mR^2}}$$ **Ans.**

Type 14. *Problems of accelerated pure rolling by finding angular acceleration ‘α’ about the bottommost axis.*

Concept

We have discussed in final touch points that :

$$\alpha = \frac{\tau_{\text{ext}}}{I}$$

can be applied from an inertial frame or about an axis passing through centre of mass (even if it is accelerated).

Same is the case about an axis passing through bottommost axis if accelerated pure rolling is taking place on a stationary ground.

Although the bottommost point is accelerated ($= R\omega^2$, towards centre), yet for symmetrical bodies net torque of pseudo forces on all particles of the rigid body about bottommost axis is zero. So,

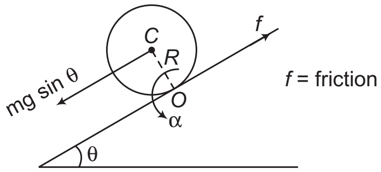
$$\alpha = \frac{\tau_{\text{ext}}}{I}$$

can be applied about this axis also.

Example 23 *In example 22, find linear acceleration ‘a’ of the body by calculating* α *about bottommost axis.*

Solution

$$\alpha = \frac{\tau_0}{I_0} = \frac{(mg\sin\theta)R}{(I_c + MR^2)}$$



But $I_c = I$

$\therefore$

$$\alpha = \frac{mg\,R\sin\theta}{I + mR^2}$$

Now, a_c or $a = R\alpha = \dfrac{mg\,R^2\sin\theta}{I + mR^2}$

or

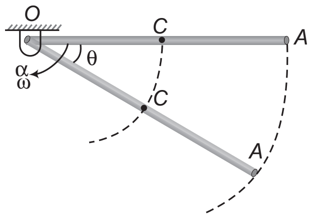
$$a = \frac{g\sin\theta}{1 + \dfrac{I}{mR^2}}$$

Ans.

Type 15. *Based on hinge force.*

Concept

A rod OA is hinged at O. It is released from the horizontal position as shown in figure. We have to find hinge force acting on the rod at a general angle θ.



We can see that motion of the rod is pure rotational about an axis passing through O. As the rod rotates downwards, its gravitational potential energy decreases and rotational kinetic energy increases. If the hinge is smooth, then we can apply the equation.

$$mgh = \frac{1}{2} I \omega^2 \qquad \ldots\text{(i)}$$

Here, $h =$ height fallen by centre of mass C of the rod.

and $$I = I_0$$

From here we can find ω^2.

At the same time, we can see that ω is increasing. So, there is an angular acceleration 'α' about O. Only two forces are acting on the rod, hinge force and weight.

Torque of hinge force about O is zero. Therefore, torque of 'mg' will only provide 'α'. Thus,

$$\alpha = \frac{\tau_{mg}}{I} \qquad \ldots\text{(ii)}$$

Now, only two forces are acting on the rod, hinge force (say $\mathbf{F}$) and weight ($m\mathbf{g}$).

$\therefore$ $$\mathbf{F} + m\mathbf{g} = m\mathbf{a}_{\text{COM}} \text{ or } m\mathbf{a}_{\text{C}}$$

or $$\mathbf{F} = m\mathbf{a}_{\text{c}} - m\mathbf{g} \qquad \ldots\text{(iii)}$$

By finding $\mathbf{a}_{\text{c}}$ we can find the hinge force $\mathbf{F}$ from Eq. (iii).

How to Solve a_c?

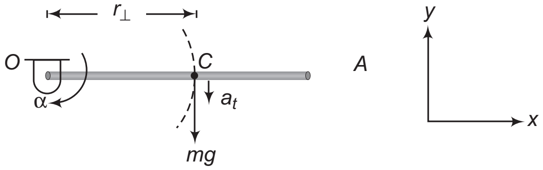
- In the figure, we can see that C is rotating in a circle with centre at O and radius $r = OC = \frac{l}{2}$, where l is the length of rod. In a circular motion, acceleration of a particle has two components.
- (i) radial $a_r = r\omega^2$
- (ii) tangential $a_t = r\alpha$
- ω^2 can be obtained from Eq. (i), α can be obtained from Eq. (ii). Writing all vector quantities in proper vector notations and then substituting in Eq.(iii) we can find the hinge force **F**.

Example 24 *In the figure given in the text if mass of the rod is 'm' then find hinge force.*

(a) Just after the rod is released from the horizontal position.

(b) When the rod becomes vertical.

Solution (a)



Just after the release, $\omega = 0$

$\therefore$ $$a_r = r\omega^2 = 0$$

$$\alpha = \frac{\tau_{\text{mg}}}{I} \qquad \text{(about } O\text{)}$$

$$= \frac{(mg)(r_\perp)}{I_0} = \frac{(mg)\left(\frac{l}{2}\right)}{(ml^2/3)} = \frac{3}{2}\frac{g}{l}$$

$\therefore$ $$a_t = r\alpha = \left(\frac{l}{2}\right)\left(\frac{3}{2}\frac{g}{l}\right) = \frac{3}{4} g$$

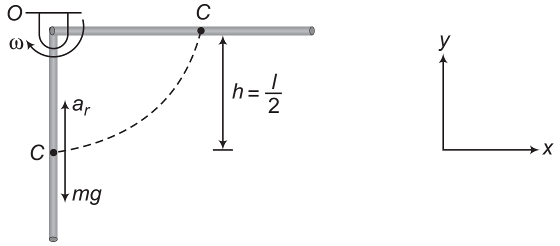
From Eq. (iii), the hinge force is

$$\begin{aligned}\mathbf{F} &= m\,(\mathbf{a}_c - \mathbf{g})\\ &= m\,[a_t\,(-\hat{\mathbf{j}}) - (-g\hat{\mathbf{j}})]\\ &= m\left[-\frac{3}{4}\,g\hat{\mathbf{j}} + g\hat{\mathbf{j}}\right]\\ &= \frac{mg}{4}\,\hat{\mathbf{j}}\end{aligned}$$

Ans.

Therefore, hinge force is $\frac{mg}{4}$, in vertically upward direction.

(b)



When the rod becomes vertical height fallen by centre of mass is $h = \frac{l}{2}$

Therefore, from Eq. (i),

$$\omega^2 = \frac{2mgh}{I} \qquad \left(I = I_0 = \frac{ml^2}{3}\right)$$

$$= \frac{2mg\left(\frac{l}{2}\right)}{(ml^2/3)} = \left(\frac{3g}{l}\right)$$

$$\therefore \qquad a_r = r\omega^2 = \left(\frac{l}{2}\right)\left(\frac{3g}{l}\right) = \frac{3}{2}\,g \qquad \text{(towards } O\text{)}$$

At this moment '*mg*' also passes through *O*. Therefore, its torque about *O* is also zero. So, from Eq. (ii),

$$\alpha = 0 \quad \Rightarrow \quad a_t = r\alpha = 0$$

Now, substituting proper values in Eq. (iii), the hinge force is,

$$\begin{aligned}\mathbf{F} &= m\,(\mathbf{a}_c - \mathbf{g})\\ &= m\,[a_r\hat{\mathbf{j}} - (-\,g\hat{\mathbf{j}})]\\ &= m\left[\left(\frac{3}{2}\,g\right)\hat{\mathbf{j}} + (g)\hat{\mathbf{j}}\right]\\ &= \left(\frac{5}{2}\,mg\right)\hat{\mathbf{j}}\end{aligned}$$

Ans.

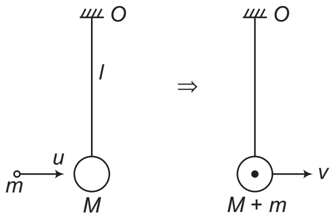
Therefore, hinge force is $\frac{5}{2}mg$ in vertically upward direction.

Type 16. *Collision Problems.*

Concept

There are mainly three types of collisions.

Type 1.



A ball of mass M is suspended by a light string of length 'l'. A bullet of mass 'm' strikes the ball with velocity u and sticks. This type of collision we have already discussed in the previous chapter.

The three important points in this collision are

(i) Velocity of combined mass after collision (say v) can be obtained by conservation of linear momentum or

$$p_f = p_i$$

$$\Rightarrow \quad (M+m)v = mu$$

$$\text{or} \quad v = \frac{mu}{M+m}$$

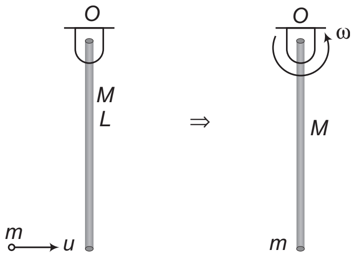
(ii) Mechanical energy is lost only during collision. This loss is given by

$$E_i - E_f \qquad (E = \text{mechanical energy})$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}(M+m)v^2$$

(iii) After collision now the mechanical energy remains constant and the combined mass executes vertical circular motion. If $v \geq \sqrt{5gl}$ circle is completed. If $\sqrt{2gl} < v < \sqrt{5gl}$, string slacks in upper half of the circle and if $0 < v \leq \sqrt{2gl}$, combined mass oscillates in lower half of the circle.

Type 2.



A rod of mass M and length L is hinged at point O. A bullet of mass m moving with velocity u strikes the rod at its bottommost point and sticks. Just after collision the combined system (rod + bullet) starts rotating about the hinge point O with an angular speed ω.

So, the translational motion converts into rotational motion. The three important points in this collision are

(i) At the time of collision, a linear impulse acts on the system at point O from the hinge. Angular impulse of this linear impulse about O is zero (as $r_\perp = 0$). Therefore, angular momentum of the system about O remains constant or

$$L_i = L_f$$

$$\therefore \quad mu\,r_\perp = I\omega$$

$$\text{or} \quad mu\,L = (I_{\text{rod}} + I_{\text{Bullet}})\omega$$

$$= \left(\frac{ML^2}{3} + mL^2\right)\omega$$

From this equation, we can find 'ω'.

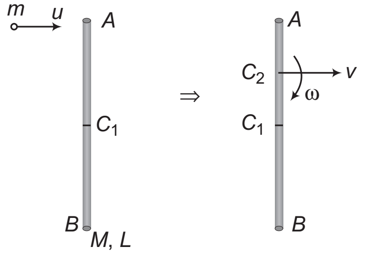
(ii) Mechanical energy is lost only during collision (not after that) and this loss is

$$E_i - E_f = \frac{1}{2}mu^2 - \frac{1}{2}I\omega^2$$

(iii) After the collision, as the system moves upwards its gravitational potential energy increases and rotational kinetic energy decreases. So, we can write :

decrease in rotational kinetic energy = increase in gravitational potential energy.

Type 3.



A rod of mass M and length L is lying on a smooth table. A bullet of mass 'm' and speed 'u' strikes the rod at A and sticks. C_1 is the centre of rod and C_2 is the centre of mass of system (rod + bullet).

Just after collision motion of the combined system is rotation (with angular velocity ω) and translation (with linear velocity v).

Thus, by the collision, translation motion converts into rotation plus translation motion. The three important points in this type of collision are:

(i) System is kept over a smooth horizontal table. So, net linear impulse on the system is zero. Therefore, v can be obtained by conservation of linear momentum or,

$$p_f = p_i$$

$$\Rightarrow \quad (M+m)v = mu$$

$$\therefore \quad v = \frac{mu}{M+m}$$

Since, net linear impulse on the system is zero. Therefore, angular impulse about any point is also zero. Hence, angular momentum of the system can be conserved about any point and ω can be obtained. But normally, we conserve it about point of impact (or A).

$$\therefore \qquad L_f = L_i \qquad \text{(about } A\text{)}$$

$$\text{or} \qquad (m+M)\,v r_\perp \pm I\omega = 0 \quad \Rightarrow \quad L_i = m u r_\perp$$

$$\text{Here,} \qquad = 0 \qquad (\text{as } r_\perp = 0 \text{ from } A)$$

$$I = I_{\text{rod}} + I_{\text{Bullet}}$$

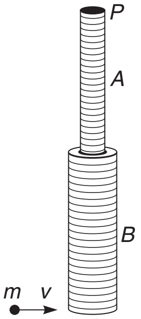
To find $r_\perp$ of v from A first we will have to find position of COM or C_2 of the combined system.

(ii) Mechanical energy is lost only during collision (not after that) and this loss is:

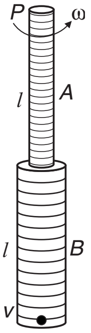
$$E_i - E_f = \left(\frac{1}{2}mu^2\right) - \left[\frac{1}{2}(M+m)v^2 + \frac{1}{2}I\omega^2\right]$$

(iii) Since, the system is kept over a smooth table, v and ω remain constant after the collision.

Example 25 *Two uniform rods A and B of length* 0.6 *m each and of masses* 0.01 *kg and* 0.02 *kg respectively are rigidly joined end to end. The combination is pivoted at the lighter end, P as shown in figure. Such that it can freely rotate about point P in a vertical plane. A small object of mass* 0.05 *kg, moving horizontally, hits the lower end of the combination and sticks to it. What should be the velocity of the object, so that the system could just be raised to the horizontal position?* **(JEE 1994)**



Solution System is free to rotate but not free to translate. During collision, net torque on the system (rod A + rod B + mass m) about point P is zero.



Therefore, angular momentum of system before collision

= angular momentum of system just after collision (about P).

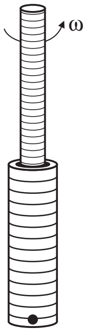
Let ω be the angular velocity of system just after collision, then

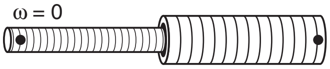
$$L_i = L_f \quad \Rightarrow \quad mv(2l) = I\omega \qquad \text{...(i)}$$

Here, I = moment of inertia of system about P

$$= m\,(2l)^2 + m_A(l^2/3) + m_B\left[\frac{l^2}{12} + \left(\frac{l}{2} + l\right)^2\right]$$

Given, $l = 0.6$ m, $m = 0.05$ kg, $m_A = 0.01$ kg and $m_B = 0.02$ kg.





Substituting the values, we get

$$I = 0.09 \text{ kg-}\text{m}^2$$

Therefore, from Eq. (i)

$$\omega = \frac{2mvl}{I} = \frac{(2)\,(0.05)\,(v)(0.6)}{0.09}$$

$$\omega = 0.67v \qquad \text{...(ii)}$$

Now, after collision, mechanical energy will be conserved.

Therefore, decrease in rotational KE = increase in gravitational PE

or
$$\frac{1}{2}I\omega^2 = mg\,(2l) + m_A\,g\left(\frac{l}{2}\right) + m_B g\,(l + \frac{l}{2})$$

or
$$\omega^2 = \frac{gl\,(4m + m_A + 3\,m_B)}{I}$$

$$= \frac{(9.8)\,(0.6)\,(4 \times 0.05 + 0.01 + 3 \times 0.02)}{0.09}$$

$$= 17.64 \text{ (rad/s)}^2$$

∴
$$\omega = 4.2 \text{ rad/s} \qquad \text{...(iii)}$$

Equating Eqs. (ii) and (iii), we get

$$v = \frac{4.2}{0.67} \text{ m/s} \quad \text{or} \quad v = 6.3 \text{ m/s}$$ **Ans.**

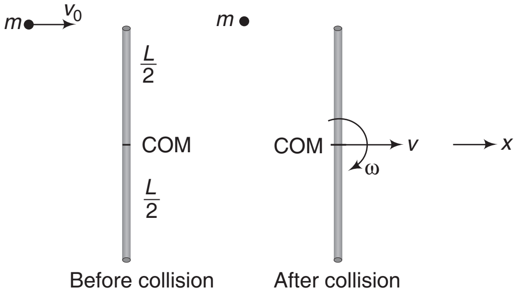
Example 26 *A rod AB of mass M and length L is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end A of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision the particle comes to rest.* **(JEE 2000)**

(a) Find the ratio m/M.

(b) A point P on the rod is at rest immediately after collision. Find the distance AP.

(c) Find the linear speed of the point P a time $\pi L/3v_0$ after the collision.

Solution (a) Suppose velocity of COM of the rod just after collision is v and angular velocity about COM is ω. Applying following three laws:



Before collision After collision

(1) External force on the system (rod + mass) in horizontal plane along x-axis is zero .

∴ Applying conservation of linear momentum in x-direction.

$$mv_0 = Mv \qquad \text{...(i)}$$

(2) Net torque on the system about COM of rod is zero.

$\therefore$ Applying conservation of angular momentum about COM of rod, we get $mv_0\left(\frac{L}{2}\right) = I\omega$

or $$mv_0 \frac{L}{2} = \frac{ML^2}{12}\omega$$

or $$mv_0 = \frac{ML\omega}{6} \qquad \text{...(ii)}$$

(3) Since, the collision is elastic, kinetic energy is also conserved.

$\therefore$ $$\frac{1}{2}mv_0^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

or $$mv_0^2 = Mv^2 + \frac{ML^2}{12}\omega^2 \qquad \text{...(iii)}$$

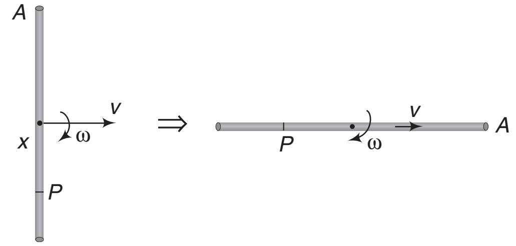
From Eqs. (i), (ii) and (iii), we get the following results

$$\frac{m}{M} = \frac{1}{4}$$ **Ans.**

$$v = \frac{mv_0}{M} \quad \text{and} \quad \omega = \frac{6\,mv_0}{ML}$$

(b) Point P will be at rest if $x\omega = v$

or $$x = \frac{v}{\omega} = \frac{mv_0/M}{6mv_0/ML} \quad \text{or} \quad x = L/6$$



$\therefore$ $$AP = \frac{L}{2} + \frac{L}{6} \quad \text{or} \quad AP = \frac{2}{3}L$$ **Ans.**

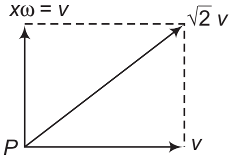
(c) After time $$t = \frac{\pi L}{3\,v_0}$$

angle rotated by rod, $\theta = \omega t = \dfrac{6mv_0}{ML} \cdot \dfrac{\pi L}{3v_0}$

$$= 2\pi\left(\frac{m}{M}\right) = 2\pi\left(\frac{1}{4}\right)$$

$\therefore$ $$\theta = \frac{\pi}{2}$$

Therefore, situation is as shown in figure.



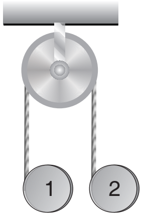
$\therefore$ Resultant velocity of point P will be

$$|\mathbf{v}_P| = \sqrt{2}v = \sqrt{2}\left(\frac{m}{M}\right)v_0$$

$$= \frac{\sqrt{2}}{4}v_0 = \frac{v_0}{2\sqrt{2}}$$

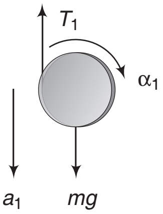
or $$|\mathbf{v}_P| = \frac{v_0}{2\sqrt{2}}$$ **Ans.**

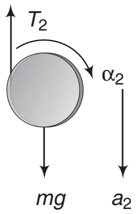
Miscellaneous Examples

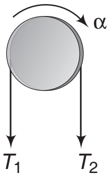


Example 27 *A thread is wound around two discs on either sides. The pulley and the two discs have the same mass and radius. There is no slipping at the pulley and no friction at the hinge. Find out the accelerations of the two discs and the angular acceleration of the pulley.*

Solution Let R be the radius of the discs and T_1 and T_2 be the tensions in the left and right segments of the rope.







Acceleration of disc 1,

$$a_1 = \frac{mg - T_1}{m} \qquad \ldots\text{(i)}$$

Acceleration of disc 2,

$$a_2 = \frac{mg - T_2}{m} \qquad \ldots\text{(ii)}$$

Angular acceleration of disc 1, $$\alpha_1 = \frac{\tau}{I} = \frac{T_1 R}{\frac{1}{2} mR^2} = \frac{2T_1}{mR} \qquad \ldots\text{(iii)}$$

Similarly, angular acceleration of disc 2, $$\alpha_2 = \frac{2T_2}{mR} \qquad \ldots\text{(iv)}$$

Both α_1 and α_2 are clockwise.

Angular acceleration of pulley,

$$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2} mR^2} = \frac{2(T_2 - T_1)}{mR} \qquad \ldots\text{(v)}$$

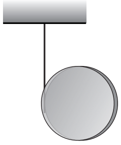
For no slipping, $$R\alpha_1 - a_1 = a_2 - R\alpha_2 = R\alpha \qquad \ldots\text{(vi)}$$

Solving these equations, we get

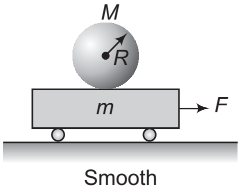
$$\alpha = 0 \qquad \text{and} \qquad a_1 = a_2 = \frac{2g}{3}$$ **Ans.**

Alternate Solution

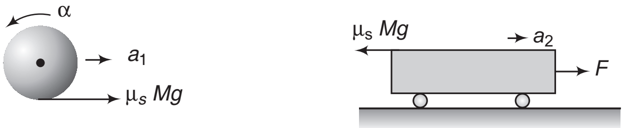
As both the discs are in identical situation, $T_1 = T_2$ and $\alpha = 0$. i.e. each of the discs falls independently and identically. Therefore, this is exactly similar to the problem shown in figure.



Example 28 *Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are* μ_s *and* μ_k *respectively.*



Solution The free body diagrams of the sphere and the plank are as shown below:



Writing equations of motion

For sphere Linear acceleration

$$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \quad \text{...(i)}$$

Angular acceleration

$$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2}$$

$$= \frac{5}{2}\frac{\mu_s g}{R} \quad \text{...(ii)}$$

For plank Linear acceleration

$$a_2 = \frac{F - \mu_s Mg}{m} \quad \text{...(iii)}$$

For no slipping

$$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

Solving the above four equations, we get

$$F = \mu_s g\left(M + \frac{7}{2}m\right)$$

Thus, maximum value of F can be

$$\mu_s g\left(M + \frac{7}{2}m\right) \quad \textbf{Ans.}$$

Example 29 *A uniform disc of radius* r_0 *lies on a smooth horizontal plane. A similar disc spinning with the angular velocity* ω_0 *is carefully lowered onto the first disc. How soon do both discs spin with the same angular-velocity if the friction coefficient between them is equal to* μ*?*

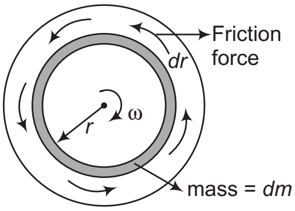
Solution From the law of conservation of angular momentum.

$$I\omega_0 = 2I\omega$$

Here, I = moment of inertia of each disc relative to common rotation axis

$\therefore$ $$\omega = \frac{\omega_0}{2} = \text{steady state angular velocity}$$

The angular velocity of each disc varies due to the torque τ of the friction forces. To calculate τ, let us take an elementary ring with radii r and $r + dr$. The torque of the friction forces acting on the given ring is equal to



$$d\tau = (\text{friction force})\,(r_\perp) = [\mu(dm)g](r)$$

$$= \mu\left(\frac{mg}{\pi r_0^2}\right)(2\pi r dr)r$$

$$= \left(\frac{2\mu mg}{r_0^2}\right) r^2\, dr$$

where, m is the mass of each disc. Integrating this with respect to r between 0 and r_0, we get

$$\tau = \frac{2}{3}\mu mgr_0 = \text{constant}$$

$\therefore$ $$\alpha = \frac{\tau}{I} = \frac{(2\mu mgr_0/3)}{(mr_0^2/2)}$$

$$= \frac{4\mu g}{3r_0} = \text{constant}$$

Now, angular speed of lower disc increases with this α from O to $\frac{\omega_0}{2}$ and α is constant.

$\therefore$ $$\frac{\omega_0}{2} = \alpha t$$

or $$t = \frac{\omega_0}{2\alpha} = \frac{3r_0\omega_0}{8\mu g}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

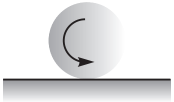
1. Assertion : Moment of inertia of a rigid body about any axis passing through its centre of mass is minimum.

Reason : From theorem of parallel axis,

$$I = I_{cm} + Mr^2$$

2. Assertion : A ball is released on a rough ground in the condition shown in figure. It will start pure rolling after some time towards left side.

Reason : Friction will convert the pure rotational motion of the ball into pure rolling.

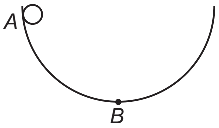


3. Assertion : A solid sphere and a hollow sphere are rolling on ground with same total kinetic energies. If translational kinetic energy of solid sphere is K, then translational kinetic energy of hollow sphere should be greater than K.

Reason : In case of hollow sphere rotational kinetic energy is less than its translational kinetic energy.

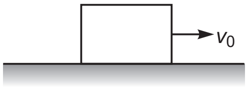
4. Assertion : A small ball is released from rest from point A as shown. If bowl is smooth, than ball will exert more pressure at point B, compared to the situation if bowl is rough.

Reason : Linear velocity and hence, centripetal force in smooth situation is more.



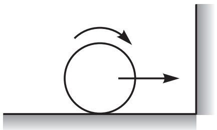
5. Assertion : A cubical block is moving on a rough ground with velocity v_0. During motion net normal reaction on the block from ground will not pass through centre of cube. It will shift towards right.

Reason : It is to keep the block in rotational equilibrium.

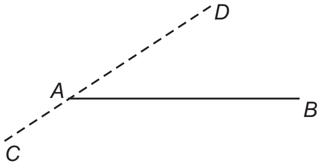


6. Assertion : A ring is rolling without slipping on a rough ground. It strikes elastically with a smooth wall as shown in figure. Ring will stop after some time while travelling in opposite direction.

Reason : After impact net angular momentum about an axis passing through bottommost point and perpendicular to plane of paper is zero.



7. **Assertion :** There is a thin rod *AB* and a dotted line *CD*. All the axes we are talking about are perpendicular to plane of paper. As we take different axes moving from *A* to *D*, moment of inertia of the rod may first decrease then increase.



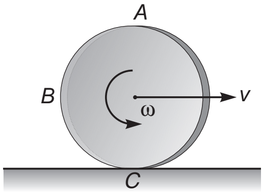
Reason : Theorem of perpendicular axis cannot be applied here.

8. **Assertion :** If linear momentum of a particle is constant, then its angular momentum about any point will also remain constant.

Reason : Linear momentum remains constant, if $\mathbf{F}_{\text{net}} = 0$ and angular momentum remains constant if $\tau_{\text{net}} = 0$.

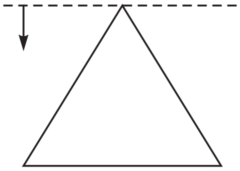
9. **Assertion :** In the figure shown, *A*, *B* and *C* are three points on the circumference of a disc. Let v_A, v_B and v_C are speeds of these three points, then

$$v_C > v_B > v_A$$



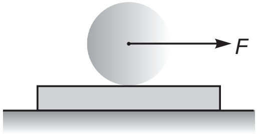
Reason : In case of rotational plus translational motion of a rigid body, net speed of any point (other than centre of mass) is greater than, less than or equal to the speed of centre of mass.

10. **Assertion :** There is a triangular plate as shown. A dotted axis is lying in the plane of slab. As the axis is moved downwards, moment of inertia of slab will first decrease then increase.



Reason : Axis is first moving towards its centre of mass and then it is receding from it.

11. **Assertion :** A horizontal force *F* is applied at the centre of solid sphere placed over a plank. The minimum coefficient of friction between plank and sphere required for pure rolling is μ_1 when plank is kept at rest and μ_2 when plank can move, then $\mu_2 < \mu_1$.



Reason : Work done by frictional force on the sphere in both cases is zero.

Objective Questions

Single Correct Option

1. The moment of inertia of a body does not depend on
(a) mass of the body
(b) the distribution of the mass in the body
(c) the axis of rotation of the body
(d) None of the above

2. The radius of gyration of a disc of radius 25 cm about a centroidal axis perpendicular to disc is
(a) 18 cm (b) 12.5 cm (c) 36 cm (d) 50 cm

3. A shaft initially rotating at 1725 rpm is brought to rest uniformly in 20s. The number of revolutions that the shaft will make during this time is
(a) 1680 (b) 575 (c) 287 (d) 627

4. A man standing on a platform holds weights in his outstretched arms. The system is rotated about a central vertical axis. If the man now pulls the weights inwards close to his body, then
(a) the angular velocity of the system will increase
(b) the angular momentum of the system will remain constant
(c) the kinetic energy of the system will increase
(d) All of the above

5. The moment of inertia of a uniform semicircular disc of mass M and radius r about a line perpendicular to the plane of the disc through the centre is
(a) Mr^2 (b) $\frac{1}{2}Mr^2$ (c) $\frac{1}{4}Mr^2$ (d) $\frac{2}{5}Mr^2$

6. Two bodies A and B made of same material have the moment of inertial in the ratio $I_A : I_B = 16 : 18$. The ratio of the masses $m_A : m_B$ is given by
(a) cannot be obtained (b) 2 : 3
(c) 1 : 1 (d) 4 : 9

7. When a sphere rolls down an inclined plane, then identity the correct statement related to the work done by friction force
(a) The friction force does positive translational work
(b) The friction force does negative rotational work
(c) The net work done by friction is zero
(d) All of the above

8. A circular table rotates about a vertical axis with a constant angular speed ω. A circular pan rests on the turn table (with the centre coinciding with centre of table) and rotates with the table. The bottom of the pan is covered with a uniform small thick layer of ice placed at centre of pan. The ice starts melting. The angular speed of the turn table
(a) remains the same
(b) decreases
(c) increases
(d) may increase or decrease depending on the thickness of ice layer

9. If R is the radius of gyration of a body of mass M and radius r, then the ratio of its rotational to translational kinetic energy in the rolling condition is
(a) $\frac{R^2}{R^2 + r^2}$ (b) $\frac{R^2}{r^2}$ (c) $\frac{r^2}{R^2}$ (d) 1

10. A solid sphere rolls down two different inclined planes of the same height but of different inclinations

(a) in both cases the speeds and time of descend will be same
(b) the speeds will be same but time of descend will be different
(c) the speeds will be different but time of descend will be same
(d) speeds and time of descend both will be different

11. For the same total mass, which of the following will have the largest moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the body

(a) a disc of radius R
(b) a ring of radius R
(c) a square lamina of side $2R$
(d) four rods forming a square of side $2R$

12. A disc and a solid sphere of same mass and radius roll down an inclined plane. The ratio of the friction force acting on the disc and sphere is

(a) $\frac{7}{6}$
(b) $\frac{5}{4}$
(c) $\frac{3}{2}$
(d) depends on angle of inclination

13. A horizontal disc rotates freely with angular velocity ω about a vertical axes through its centre. A ring, having the same mass and radius as the disc, is now gently placed coaxially on the disc. After some time, the two rotate with a common angular velocity. Then

(a) no friction exists between the disc and the ring
(b) the angular momentum of the system is conserved
(c) the final common angular velocity is $\frac{1}{2}\omega$
(d) All of the above

14. A solid homogeneous sphere is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere

(a) total kinetic energy of the sphere is conserved
(b) angular momentum of the sphere about any point on the horizontal surface is conserved
(c) only the rotational kinetic energy about the centre of mass is conserved
(d) None of the above

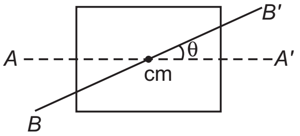
15. A particle of mass $m = 3$ kg moves along a straight line $4y - 3x = 2$ where x and y are in metre, with constant velocity $v = 5\ \text{ms}^{-1}$. The magnitude of angular momentum about the origin is

(a) $12\ \text{kg m}^2\text{s}^{-1}$
(b) $6.0\ \text{kg m}^2\text{s}^{-1}$
(c) $4.5\ \text{kg m}^2\text{s}^{-1}$
(d) $8.0\ \text{kg m}^2\text{s}^{-1}$

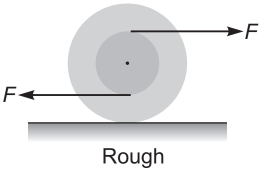
16. A solid sphere rolls without slipping on a rough horizontal floor, moving with a speed v. It makes an elastic collision with a smooth vertical wall. After impact,

(a) it will move with a speed v initially
(b) its motion will be rolling with slipping initially and its rotational motion will stop momentarily at some instant
(c) its motion will be rolling without slipping only after some time
(d) All of the above

17. The figure shows a square plate of uniform mass distribution. AA' and BB' are the two axes lying in the plane of the plate and passing through its centre of mass. If I_0 is the moment of inertia of the plate about AA' then its moment of inertia about the axis BB' is

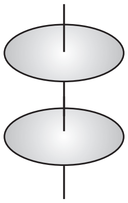


(a) I_0
(b) $I_0 \cos\theta$
(c) $I_0 \cos^2\theta$
(d) None of these



18. A spool is pulled horizontally on rough surface by two equal and opposite forces as shown in the figure. Which of the following statements are correct?

(a) The centre of mass moves towards left
(b) The centre of mass moves towards right
(c) The centre of mass remains stationary
(d) The net torque about the centre of mass of the spool is zero



19. Two identical discs are positioned on a vertical axis as shown in the figure. The bottom disc is rotating at angular velocity ω_0 and has rotational kinetic energy K_0. The top disc is initially at rest. It then falls and sticks to the bottom disc. The change in the rotational kinetic energy of the system is

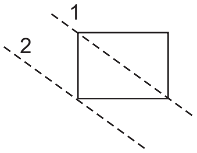
(a) $K_0/2$
(b) $-K_0/2$
(c) $-K_0/4$
(d) $K_0/4$

20. The moment of inertia of hollow sphere (mass M) of inner radius R and outer radius $2R$, having material of uniform density, about a diametric axis is

(a) $31\ MR^2/70$
(b) $43\ MR^2/90$
(c) $19\ MR^2/80$
(d) None of these

21. A rod of uniform cross-section of mass M and length L is hinged about an end to swing freely in a vertical plane. However, its density is non uniform and varies linearly from hinged end to the free end doubling its value. The moment of inertia of the rod, about the rotation axis passing through the hinge point is

(a) $\frac{2ML^2}{9}$
(b) $\frac{3ML^2}{16}$
(c) $\frac{7ML^2}{18}$
(d) None of these

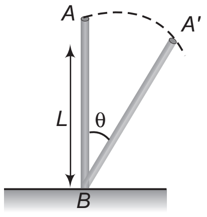


22. Let I_1 and I_2 be the moment of inertia of a uniform square plate about axes shown in the figure. Then, the ratio $I_1 : I_2$ is

(a) $1 : \frac{1}{7}$
(b) $1 : \frac{12}{7}$
(c) $1 : \frac{7}{12}$
(d) $1 : 7$

23. Moment of inertia of a uniform rod of length L and mass M, about an axis passing through $L/4$ from one end and perpendicular to its length is

(a) $\frac{7}{36} ML^2$
(b) $\frac{7}{48} ML^2$
(c) $\frac{11}{48} ML^2$
(d) $\frac{ML^2}{12}$



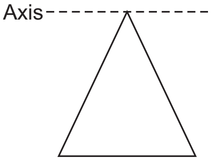
24. A uniform rod of length L is free to rotate in a vertical plane about a fixed horizontal axis through B. The rod begins rotating from rest. The angular velocity ω at angle θ is given as

(a) $\sqrt{\left(\frac{6g}{L}\right)} \sin \frac{\theta}{2}$
(b) $\sqrt{\left(\frac{6g}{L}\right)} \cos \frac{\theta}{2}$
(c) $\sqrt{\left(\frac{6g}{L}\right)} \sin \theta$
(d) $\sqrt{\left(\frac{6g}{L}\right)} \cos \theta$

25. Two particles of masses 1 kg and 2 kg are placed at a distance of 3m. Moment of inertia of the particles about an axis passing through their centre of mass and perpendicular to the line joining them is (in kg-m^2).

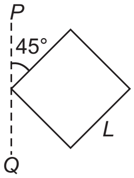
(a) 6
(b) 9
(c) 8
(d) 12

26. Find moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis as shown in figure. The length of each side is L



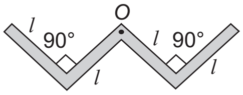
(a) $ML^2/8$
(b) $3\sqrt{3}\,ML^2/8$
(c) $7ML^2/8$
(d) None of these

27. A square is made by joining four rods each of mass M and length L. Its moment of inertia about an axis PQ, in its plane and passing through one of its corner is



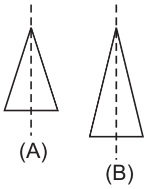
(a) $6\,ML^2$
(b) $\frac{4}{3}ML^2$
(c) $\frac{8}{3}ML^2$
(d) $\frac{10}{3}ML^2$

28. A thin rod of length $4l$, mass $4\,m$ is bent at the points as shown in the figure. What is the moment of inertia of the rod about the axis passing through O and perpendicular to the plane of the paper?



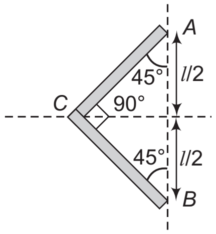
(a) $\frac{ml^2}{3}$
(b) $\frac{10\,ml^2}{3}$
(c) $\frac{ml^2}{12}$
(d) $\frac{ml^2}{24}$

29. The figure shows two cones A and B with the conditions : $h_A < h_B$; $\rho_A > \rho_B$; $R_A = R_B$ $m_A = m_B$. Identify the correct statement about their axis of symmetry.



(a) Both have same moment of inertia
(b) A has greater moment of inertia
(c) B has greater moment of inertia
(d) Nothing can be said

30. Linear mass density of the two rods system, AC and CB is x. Moment of inertia of two rods about an axis passing through AB is



(a) $\frac{xl^3}{4\sqrt{3}}$
(b) $\frac{xl^3}{\sqrt{2}}$
(c) $\frac{xl^3}{4}$
(d) $\frac{xl^3}{6\sqrt{2}}$

Subjective Questions

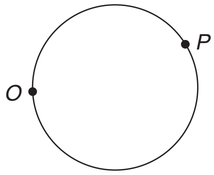
1. If radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?
2. The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius R. Find the distance of the line from the centre.
3. Find the moment of inertia of a uniform square plate of mass M and edge a about one of its diagonals.
4. Moment of inertia of a uniform rod of mass m and length l is $\frac{7}{12}ml^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

5. Two point masses m_1 and m_2 are joined by a weightless rod of length r. Calculate the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the rod.

6. Radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass.

7. A wheel rotates around a stationary axis so that the rotation angle θ varies with time as $\theta = at^2$, where $a = 0.2\ \text{rad/s}^2$. Find the magnitude of net acceleration of the point A at the rim at the moment $t = 2.5\ s$ if the linear velocity of the point A at this moment is $v = 0.65\ \text{m/s}$.

8. Particle P shown in figure is moving in a circle of radius $R = 10\,\text{cm}$ with linear speed $v = 2\,\text{m/s}$. Find the angular speed of particle about point O.



9. A particle of mass m is projected with velocity v at an angle θ with the horizontal. Find its angular momentum about the point of projection when it is at the highest point of its trajectory.

10. Linear mass density (mass/length) of a rod depends on the distance from one end (say A) as $\lambda_x = (\alpha x + \beta)$. Here, α and β are constants. Find the moment of inertia of this rod about an axis passing through A and perpendicular to the rod. Length of the rod is l.

11. When a body rolls, on a stationary ground, the acceleration of the point of contact is always zero. Is this statement true or false?

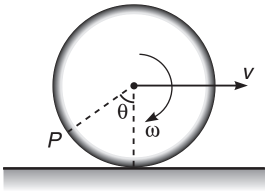
12. A solid sphere of mass m rolls down an inclined plane a height h. Find rotational kinetic energy of the sphere.

13. The topmost and bottommost velocities of a disc are v_1 and $v_2 (< v_1)$ in the same direction. The radius is R. Find the value of angular velocity ω.

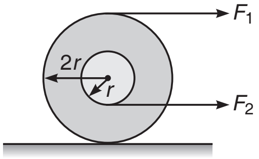
14. A circular lamina of radius a and centre O has a mass per unit area of kx^2, where x is the distance from O and k is a constant. If the mass of the lamina is M, find in terms of M and a, the moment of inertia of the lamina about an axis through O and perpendicular to the lamina.

15. A solid body starts rotating about a stationary axis with an angular acceleration $\alpha = (2.0 \times 10^{-2})\, t\ \text{rad/s}^2$, here, t is in seconds. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\theta = 60°$ with its velocity vector?

16. A ring of radius R rolls on a horizontal ground with linear speed v and angular speed ω. For what value of θ the velocity of point P is in vertical direction. $(v < R\omega)$



17. Two forces F_1 and F_2 are applied on a spool of mass M and moment of inertia I about an axis passing through its centre of mass. Find the ratio $\frac{F_1}{F_2}$, so that the force of friction is zero. Given that $I < 2Mr^2$.



18. A disc is placed on the ground. Friction coefficient is μ. What is the minimum force required to move the disc if it is applied at the topmost point?

19. A cube is resting on an inclined plane. If the angle of inclination is gradually increased, what must be the coefficient of friction between the cube and plane so that,

(a) cube slides before toppling?
(b) cube topples before sliding?

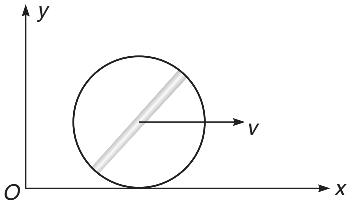
20. A uniform disc of mass 20 kg and radius 0.5 m can turn about a smooth axis through its centre and perpendicular to the disc. A constant torque is applied to the disc for 3 s from rest and the angular velocity at the end of that time is $\frac{240}{\pi}$ rev/min. Find the magnitude of the torque. If the torque is then removed and the disc is brought to rest in t seconds by a constant force of 10 N applied tangentially at a point on the rim of the disc, find t.

21. A uniform disc of mass m and radius R is rotated about an axis passing through its centre and perpendicular to its plane with an angular velocity ω_0. It is placed on a rough horizontal plane with the axis of the disc keeping vertical. Coefficient of friction between the disc and the surface is μ. Find

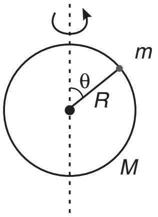
(a) the time when disc stops rotating,
(b) the angle rotated by the disc before stopping.

22. A solid body rotates about a stationary axis according to the law $\theta = at - bt^3$, where $a = 6$ rad/s and $b = 2\ \text{rad/s}^3$. Find the mean values of the angular velocity and acceleration over the time interval between $t = 0$ and the time, when the body comes to rest.

23. A rod of mass m and length $2R$ is fixed along the diameter of a ring of same mass m and radius R as shown in figure. The combined body is rolling without slipping along x-axis. Find the angular momentum about z-axis.



24. The figure shows a thin ring of mass $M = 1$ kg and radius $R = 0.4$ m spinning about a vertical diameter. (Take $I = \frac{1}{2}MR^2$). A small bead of mass $m = 0.2$ kg can slide without friction along the ring. When the bead is at the top of the ring, the angular velocity is 5 rad/s. What is the angular velocity when the bead slips halfway to $\theta = 45°$?



25. A horizontal disc rotating freely about a vertical axis makes 100 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90. Calculate the moment of inertia of disc.

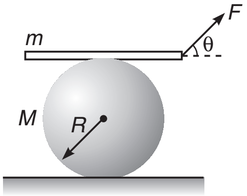
26. A man stands at the centre of a circular platform holding his arms extended horizontally with 4 kg block in each hand. He is set rotating about a vertical axis at 0.5 rev/s. The moment of inertia of the man plus platform is 1.6 kg-m^2, assumed constant. The blocks are 90 cm from the axis of rotation. He now pulls the blocks in toward his body until they are 15 cm from the axis of rotation. Find (a) his new angular velocity and (b) the initial and final kinetic energy of the man and platform. (c) how much work must the man do to pull in the blocks ?

27. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0. Then, by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find :

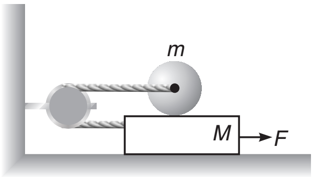
(a) the angular velocity of the system in its final state,

(b) the work performed by the force F.

28. Consider a cylinder of mass M and radius R lying on a rough horizontal plane. It has a plank lying on its top as shown in figure. A force F is applied on the plank such that the plank moves and causes the cylinder to roll. The plank always remains horizontal. There is no slipping at any point of contact. Calculate the acceleration of the cylinder and the frictional forces at the two contacts.



29. Find the acceleration of the cylinder of mass m and radius R and that of plank of mass M placed on smooth surface if pulled with a force F as shown in figure. Given that sufficient friction is present between cylinder and the plank surface to prevent sliding of cylinder.



30. A uniform rod AB of length $2l$ and mass m is rotating in a horizontal plane about a vertical axis through A, with angular velocity ω, when the mid-point of the rod strikes a fixed nail and is brought immediately to rest. Find the impulse exerted by the nail.

31. A uniform rod of length L rests on a frictionless horizontal surface. The rod is pivoted about a fixed frictionless axis at one end. The rod is initially at rest. A bullet travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. The mass of the bullet is one-sixth the mass of the rod.

(a) What is the final angular velocity of the rod ?

(b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision ?

32. A uniform rod AB of mass $3m$ and length $2l$ is lying at rest on a smooth horizontal table with a smooth vertical axis through the end A. A particle of mass $2m$ moves with speed $2u$ across the table and strikes the rod at its mid-point C. If the impact is perfectly elastic. Find the speed of the particle after impact if

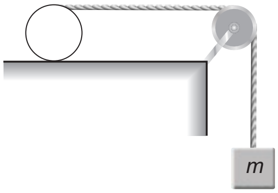
(a) it strikes the rod normally, (b) its path before impact was inclined at 60° to AC.

LEVEL 2

Objective Questions

Single Correct Option

1. In the given figure a ring of mass m is kept on a horizontal surface while a body of equal mass m is attached through a string, which is wounded on the ring. When the system is released, the ring rolls without slipping. Consider the following statement and choose the correct option.



(i) acceleration of the centre of mass of ring is $\frac{2g}{3}$

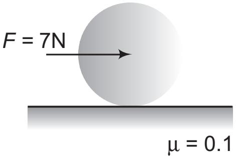
(ii) acceleration of hanging particle is $\frac{4g}{3}$

(iii) frictional force (on the ring) acts in forward direction

(iv) frictional force (on the ring) acts in backward direction

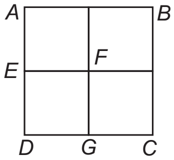
(a) only statements (i) and (ii) are correct
(b) only statements (ii) and (iii) are correct
(c) only statements (iii) and (iv) are correct
(d) None of these

2. A solid sphere of mass 10 kg is placed on a rough surface having coefficient of friction $\mu = 0.1$. A constant force $F = 7$ N is applied along a line passing through the centre of the sphere as shown in the figure. The value of frictional force on the sphere is



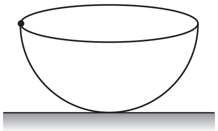
(a) 1 N
(b) 2 N
(c) 3 N
(d) 7 N

3. From a uniform square plate of side a and mass m, a square portion $DEFG$ of side $\frac{a}{2}$ is removed. Then, the moment of inertia of remaining portion about the axis AB is



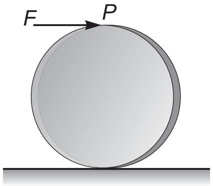
(a) $\frac{7ma^2}{16}$
(b) $\frac{3ma^2}{16}$
(c) $\frac{3ma^2}{4}$
(d) $\frac{9ma^2}{16}$

4. A small solid sphere of mass m and radius r starting from rest from the rim of a fixed hemispherical bowl of radius $R(>>r)$ rolls inside it without sliding. The normal reaction exerted by the sphere on the hemisphere when it reaches the bottom of hemisphere is



(a) $(3/7)\,mg$
(b) $(9/7)\,mg$
(c) $(13/7)\,mg$
(d) $(17/7)\,mg$

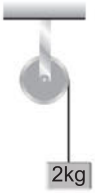
5. A uniform solid cylinder of mass m and radius R is placed on a rough horizontal surface. A horizontal constant force F is applied at the top point P of the cylinder so that it starts pure rolling. The acceleration of the cylinder is



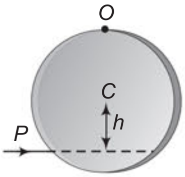
(a) $F/3m$
(b) $2F/3m$
(c) $4F/3m$
(d) $5F/3m$

6. In the above question, the frictional force on the cylinder is
(a) $F/3$ towards right (b) $F/3$ towards left
(c) $2F/3$ towards right (d) $2F/3$ towards left

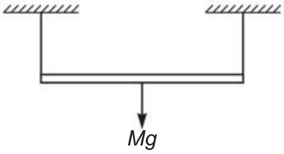
7. A small pulley of radius 20 cm and moment of inertia 0.32 kg-m^2 is used to hang a 2 kg mass with the help of massless string. If the block is released, for no slipping condition acceleration of the block will be
(a) 2 m/s^2 (b) 4 m/s^2
(c) 1 m/s^2 (d) 3 m/s^2



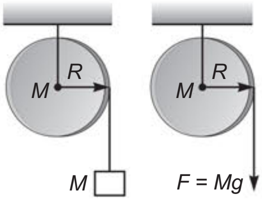
8. A uniform circular disc of radius R is placed on a smooth horizontal surface with its plane horizontal and hinged at circumference through point O as shown. An impulse P is applied at a perpendicular distance h from its centre C. The value of h so that the impulse due to hinge is zero, is
(a) R (b) $R/2$
(c) $R/3$ (d) $R/4$



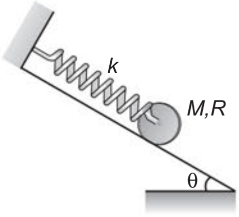
9. A rod is supported horizontally by means of two strings of equal length as shown in figure. If one of the string is cut. Then tension in other string at the same instant will
(a) remain unaffected
(b) increase
(c) decrease
(d) become equal to weight of the rod



10. The figure represents two cases. In first case a block of mass M is attached to a string which is tightly wound on a disc of mass M and radius R. In second case $F = Mg$. Initially, the disc is stationary in each case. If the same length of string is unwound from the disc, then
(a) same amount of work is done on both discs
(b) angular velocities of both the discs are equal
(c) both the discs have unequal angular accelerations
(d) All of the above

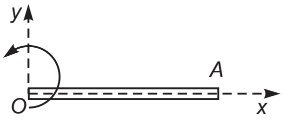


11. A uniform cylinder of mass M and radius R is released from rest on a rough inclined surface of inclination θ with the horizontal as shown in figure. As the cylinder rolls down the inclined surface, the maximum elongation in the spring of stiffness k is
(a) $\frac{3}{4}\frac{Mg\sin\theta}{k}$ (b) $\frac{2\,Mg\sin\theta}{k}$
(c) $\frac{Mg\sin\theta}{k}$ (d) None of these



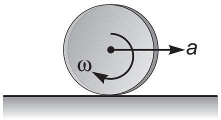
12. A uniform rod of mass m and length l rotates in a horizontal plane with an angular velocity ω about a vertical axis passing through one end. The tension in the rod at a distance x from the axis is
(a) $\frac{1}{2}m\omega^2 x$ (b) $\frac{1}{2}m\omega^2\left(1-\frac{x^2}{l}\right)$ (c) $\frac{1}{2}m\omega^2 l\left(1-\frac{x^2}{l^2}\right)$ (d) $\frac{1}{2}m\omega^2 l\left[1-\frac{x}{l}\right]$

13. A rod of length 1 m rotates in the xy plane about the fixed point O in the anticlockwise sense, as shown in figure with velocity $\omega = a + bt$ where $a = 10\,\text{rad s}^{-1}$ and $b = 5\,\text{rad s}^{-2}$. The velocity and acceleration of the point A at $t = 0$ is



(a) $+10\hat{\mathbf{i}}\ \text{ms}^{-1}$ and $+5\hat{\mathbf{i}}\ \text{ms}^{-2}$
(b) $+10\hat{\mathbf{j}}\ \text{ms}^{-1}$ and $(-100\,\hat{\mathbf{i}} + 5\hat{\mathbf{j}})\ \text{ms}^{-2}$
(c) $-10\hat{\mathbf{j}}\ \text{ms}^{-1}$ and $(100\,\hat{\mathbf{i}} + 5\hat{\mathbf{j}})\ \text{ms}^{-2}$
(d) $-10\hat{\mathbf{j}}\ \text{ms}^{-1}$ and $-5\hat{\mathbf{j}}\ \text{ms}^{-2}$

14. A ring of radius R rolls on a horizontal surface with constant acceleration a of the centre of mass as shown in figure. If ω is the instantaneous angular velocity of the ring, then the net acceleration of the point of contact of the ring with ground is

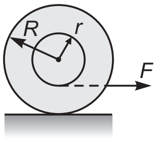


(a) zero
(b) $\omega^2 R$
(c) a
(d) $\sqrt{a^2 + (\omega^2 R)^2}$

15. The density of a rod AB increases linearly from A to B. Its midpoint is O and its centre of mass is at C. Four axes pass through A, B, O and C, all perpendicular to the length of the rod. The moments of inertia of the rod about these axes are I_A, I_B, I_O and I_C respectively. Then

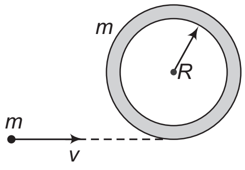
(a) $I_A > I_B$
(b) $I_C < I_B$
(c) $I_O > I_C$
(d) All of these

16. The figure shows a spool placed at rest on a horizontal rough surface. A tightly wound string on the inner cylinder is pulled horizontally with a force F. Identify the correct alternative related to the friction force f acting on the spool



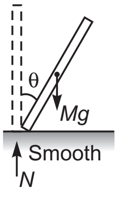
(a) f acts leftwards with $f < F$
(b) f acts leftwards but nothing can be said about its magnitude
(c) $f < F$ but nothing can be said about its magnitude
(d) None of the above

17. A circular ring of mass m and radius R rests flat on a horizontal smooth surface as shown in figure. A particle of mass m, and moving with a velocity v, collides inelastically ($e = 0$) with the ring. The angular velocity with which the system rotates after the particle strikes the ring is



(a) $\dfrac{v}{2R}$
(b) $\dfrac{v}{3R}$
(c) $\dfrac{2v}{3R}$
(d) $\dfrac{3v}{4R}$

18. A stationary uniform rod in the upright position is allowed to fall on a smooth horizontal surface. The figure shows the instantaneous position of the rod. Identify the correct statement.

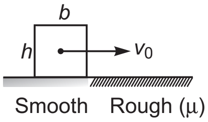


(a) normal reaction N is equal to Mg
(b) N does positive rotational work about the centre of mass
(c) a couple of equal and opposite forces acts on the rod
(d) All of the above

19. A thin uniform rod of mass m and length l is free to rotate about its upper end. When it is at rest. It receives an impulse J at its lowest point, normal to its length. Immediately after impact

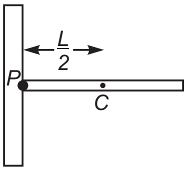
(a) the angular momentum of the rod is Jl
(b) the angular velocity of the rod is $3J/ml$
(c) the kinetic energy of the rod is $3J^2/2m$
(d) All of the above

20. A rectangular block of size $(b \times h)$ moving with velocity v_o enters on a rough surface where the coefficient of friction is μ as shown in figure. Identify the correct statement.



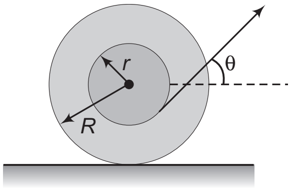
(a) The net torque acting on the block about its COM is $\mu mg \frac{h}{2}$ (clockwise)
(b) The net torque acting on the block about its COM is zero
(c) The net torque acting on the block about its COM is in the anticlockwise sense
(d) None of the above

21. A uniform rod of length L and mass m is free to rotate about a frictionless pivot at one end as shown in figure. The rod is held at rest in the horizontal position and a coin of mass m is placed at the free end. Now the rod is released. The reaction on the coin immediately after the rod starts falling is



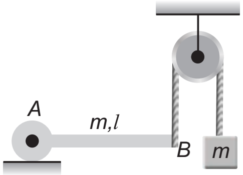
(a) $\frac{3\,mg}{2}$ (b) $2\,mg$
(c) zero (d) $\frac{mg}{2}$

22. A spool is pulled at an angle θ with the horizontal on a rough horizontal surface as shown in the figure. If the spool remains at rest, the angle θ is equal to



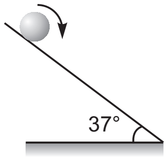
(a) $\cos^{-1}\left(\frac{R}{r}\right)$ (b) $\sin^{-1}\left(\sqrt{1-\frac{r^2}{R^2}}\right)$
(c) $\pi - \cos^{-1}\left(\frac{r}{R}\right)$ (d) $\sin^{-1}\left(\frac{r}{R}\right)$

23. Uniform rod AB is hinged at end A in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Mass of block and rod is same and is equal to m. Then acceleration of block just after release from this position is



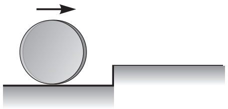
(a) $6\,g/13$ (b) $g/4$
(c) $3g/8$ (d) None of these

24. A cylinder having radius 0.4 m, initially rotating (at $t = 0$) with $\omega_0 = 54$ rad/s is placed on a rough inclined plane with $\theta = 37°$ having friction coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is ($g = 10$ m/s^2)



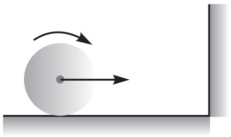
(a) 5.4 s (b) 2.4 s
(c) 1.4 s (d) None of these

25. A disc of mass M and radius R is rolling purely with center's velocity v_0 on a flat horizontal floor when it hits a step in the floor of height $R/4$. The corner of the step is sufficiently rough to prevent any slipping of the disc against itself. What is the velocity of the centre of the disc just after impact?



(a) $4v_0/5$ (b) $4v_0/7$
(c) $5v_0/6$ (d) None of these

26. A solid sphere is rolling purely on a rough horizontal surface (coefficient of kinetic friction = μ) with speed of centre = u. It collides inelastically with a smooth vertical wall at a certain moment, the coefficient of restitution being $\frac{1}{2}$. The sphere will begin pure rolling after a time



(a) $\frac{3u}{7\mu g}$ (b) $\frac{2u}{7\mu g}$ (c) $\frac{3u}{5\mu g}$ (d) $\frac{2u}{5\mu g}$

27. A thin hollow sphere of mass m is completely filled with non viscous liquid of mass m. When the sphere roll-on horizontal ground such that centre moves with velocity v, kinetic energy of the system is equal to

(a) mv^2 (b) $\frac{4}{3}mv^2$ (c) $\frac{4}{5}mv^2$ (d) None of these

28. A solid uniform disc of mass m rolls without slipping down a fixed inclined plank with an acceleration a. The frictional force on the disc due to surface of the plane is

(a) $\frac{1}{4}ma$ (b) $\frac{3}{2}ma$ (c) ma (d) $\frac{1}{2}ma$

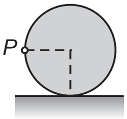
29. A uniform slender rod of mass m and length L is released from rest, with its lower end touching a frictionless horizontal floor. At the initial moment, the rod is inclined at an angle $\theta = 30°$ with the vertical. Then the value of normal reaction from the floor just after release will be

(a) $4mg/7$
(b) $5mg/9$
(c) $2mg/5$
(d) None of these

30. In the above problem, the initial acceleration of the lower end of the rod will be

(a) $g\sqrt{3}/4$
(b) $g\sqrt{3}/5$
(c) $3g\sqrt{3}/7$
(d) None of these

31. A disc of radius R is rolling purely on a flat horizontal surface, with a constant angular velocity. The angle between the velocity and acceleration vectors of point P is

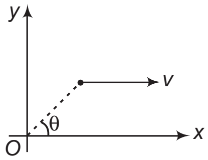


(a) zero
(b) 45°
(c) $\tan^{-1}(2)$
(d) $\tan^{-1}(1/2)$

32. A straight rod AB of mass M and length L is placed on a frictionless horizontal surface. A force having constant magnitude F and a fixed direction starts acting at the end A. The rod is initially perpendicular to the force. The initial acceleration of end B is

(a) zero
(b) $2F/M$
(c) $4F/M$
(d) None of these

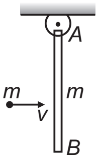
33. A particle moves parallel to x-axis with constant velocity v as shown in the figure. The angular velocity of the particle about the origin O



(a) remains constant
(b) continuously increases
(c) continuously decreases
(d) oscillates

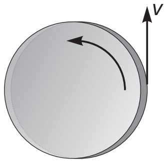
34. A thin uniform rod of mass M and length L is hinged at its upper end, and released from rest from a horizontal position. The tension at a point located at a distance $L/3$ from the hinge point, when the rod becomes vertical, will be

(a) $22\,Mg/27$
(b) $11\,Mg/13$
(c) $6\,Mg/11$
(d) $2\,Mg$



35. A uniform rod AB of length L and mass m is suspended freely at A and hangs vertically at rest when a particle of same mass m is fired horizontally with speed v to strike the rod at its mid point. If the particle is brought to rest after the impact. Then the impulsive reaction at A in horizontal direction is

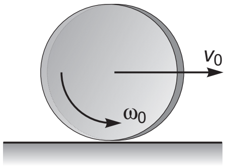
(a) $mv/4$ (b) $mv/2$
(c) mv (d) $2\ mv$



36. A child with mass m is standing at the edge of a merry go round having moment of inertia I, radius R and initial angular velocity ω as shown in the figure. The child jumps off the edge of the merry go round with tangential velocity v with respect to the ground. The new angular velocity of the merry go round is

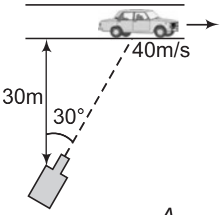
(a) $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$ (b) $\sqrt{\dfrac{(I + mR^2)\omega^2 - mv^2}{I}}$

(c) $\dfrac{I\omega - mvR}{I}$ (d) $\dfrac{(I + mR^2)\,\omega - mvR}{I}$



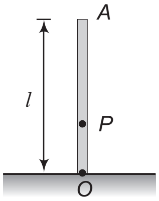
37. A disc of radius R is spun to an angular speed ω_0 about its axis and then imparted a horizontal velocity of magnitude $\frac{\omega_0 R}{4}$. The coefficient of friction is μ. The sense of rotation and direction of linear velocity are shown in the figure. The disc will return to its initial position

(a) if the value of $\mu < 0.5$
(b) irrespective of the value of μ
(c) if the value of $0.5 < \mu < 1$
(d) if $\mu > 1$



38. A racing car is travelling along a straight track at a constant velocity of 40 m/s. A fixed TV camera is recording the event as shown in figure. In order to keep the car in view, in the position shown, the angular velocity of camera should be

(a) 3 rad/s (b) 2 rad/s
(c) 4 rad/s (d) 1 rad/s



39. A uniform rod OA of length l, resting on smooth surface is slightly distributed from its vertical position. P is a point on the rod whose locus is a circle during the subsequent motion of the rod. Then the distance OP is equal to

(a) $l/2$
(b) $l/3$
(c) $l/4$
(d) there is no such point

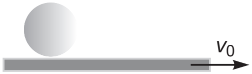
40. In the above question, the velocity of end O when end A hits the ground is

(a) zero
(b) along the horizontal
(c) along the vertical
(d) at some inclination to the ground ($\neq 90°$)

41. In the above question, the velocity of end A at the instant it hits the ground is

(a) $\sqrt{3gl}$ (b) $\sqrt{12gl}$
(c) $\sqrt{6gl}$ (d) None of these

42. A solid sphere of mass m and radius R is gently placed on a conveyer belt moving with constant velocity v_0. If coefficient of friction between belt and sphere is 2/7, the distance traveled by the centre of the sphere before it starts pure rolling is



(a) $\dfrac{v_0^2}{7g}$ (b) $\dfrac{2v_0^2}{49g}$

(c) $\dfrac{2v_0^2}{5g}$ (d) $\dfrac{2v_0^2}{7g}$

More than One Correct Options

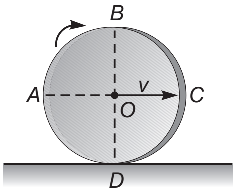
1. A mass m of radius r is rolling horizontally without any slip with a linear speed v. It then rolls up to a height given by $\dfrac{3}{4}\dfrac{v^2}{g}$

(a) the body is identified to be a disc or a solid cylinder
(b) the body is a solid sphere
(c) moment of inertia of the body about instantaneous axis of rotation is $\dfrac{3}{2}mr^2$
(d) moment of inertia of the body about instantaneous axis of rotation is $\dfrac{7}{5}mr^2$

2. Four identical rods each of mass m and length l are joined to form a rigid square frame. The frame lies in the xy plane, with its centre at the origin and the sides parallel to the x and y axes. Its moment of inertia about

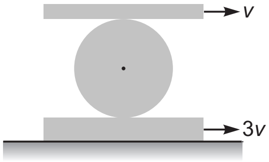
(a) the x-axis is $\dfrac{2}{3}ml^2$
(b) the z-axis is $\dfrac{4}{3}ml^2$
(c) an axis parallel to the z-axis and passing through a corner is $\dfrac{10}{3}ml^2$
(d) one side is $\dfrac{5}{3}ml^2$

3. A uniform circular ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure. Then



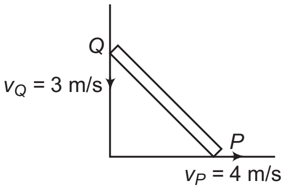
(a) section ABC has greater kinetic energy than section ADC
(b) section BC has greater kinetic energy than section CD
(c) section BC has the same kinetic energy as section DA
(d) the sections CD and DA have the same kinetic energy

4. A cylinder of radius R is to roll without slipping between two planks as shown in the figure. Then



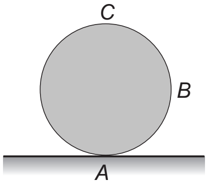
(a) angular velocity of the cylinder is $\dfrac{v}{R}$ counter clockwise
(b) angular velocity of the cylinder is $\dfrac{2v}{R}$ clockwise
(c) velocity of centre of mass of the cylinder is v towards left
(d) velocity of centre of mass of the cylinder is $2v$ towards right

5. A uniform rod of mass $m = 2$ kg and length $l = 0.5$ m is sliding along two mutually perpendicular smooth walls with the two ends P and Q having velocities $v_P = 4$ m/s and $v_Q = 3$ m/s as shown. Then



(a) The angular velocity of rod, $\omega = 10$ rad/s, counter clockwise
(b) The angular velocity of rod, $\omega = 5.0$ rad/s, counter clockwise
(c) The velocity of centre of mass of rod, $v_{\text{cm}} = 2.5$ m/s
(d) The total kinetic energy of rod, $K = \frac{25}{3}$ joule

6. A wheel is rolling without slipping on a horizontal plane with velocity v and acceleration a of centre of mass as shown in figure. Acceleration at



(a) A is vertically upwards
(b) B may be vertically downwards
(c) C cannot be horizontal
(d) a point on the rim may be horizontal leftwards

7. A uniform rod of length l and mass $2\,m$ rests on a smooth horizontal table. A point mass m moving horizontally at right angles to the rod with velocity v collides with one end of the rod and sticks it. Then

(a) angular velocity of the system after collision is $\frac{2}{5}\frac{v}{l}$
(b) angular velocity of the system after collision is $\frac{v}{2l}$
(c) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{3}{10}mv^2$
(d) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{7mv^2}{24}$

8. A non-uniform ball of radius R and radius of gyration about geometric centre $= R/2$, is kept on a frictionless surface. The geometric centre coincides with the centre of mass. The ball is struck horizontally with a sharp impulse $= J$. The point of application of the impulse is at a height h above the surface. Then

(a) the ball with slip on surface for all cases
(b) the ball will roll purely if $h = 5R/4$
(c) the ball will roll purely if $h = 3R/2$
(d) there will be no rotation if $h = R$

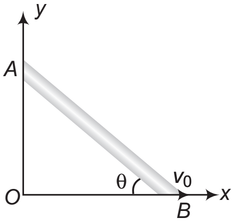
9. A hollow spherical ball is given an initial push, up an incline of inclination angle α. The ball rolls purely. Coefficient of static friction between ball and incline $= \mu$. During its upwards journey

(a) friction acts up along the incline
(b) $\mu_{\min} = (2\tan\alpha)/5$
(c) friction acts down along the incline
(d) $\mu_{\min} = (2\tan\alpha)/7$

10. A uniform disc of mass m and radius R rotates about a fixed vertical axis passing through its centre with angular velocity ω. A particle of same mass m and having velocity of $2\omega R$ towards centre of the disc collides with the disc moving horizontally and sticks to its rim. Then,

(a) the angular velocity of the disc will become $\omega/3$
(b) the angular velocity of the disc will become $5\omega/3$
(c) the impulse on the particle due to disc is $\frac{\sqrt{37}}{3}m\omega R$
(d) the impulse on the particle due to disc is $2m\omega R$

11. The end B of the rod AB which makes angle θ with the floor is being pulled with a constant velocity v_0 as shown. The length of the rod is l.



(a) At $\theta = 37°$ velocity of end A is $\frac{4}{3}v_0$ downwards

(b) At $\theta = 37°$ angular velocity of rod is $\frac{5v_0}{3l}$

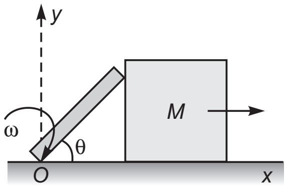
(c) Angular velocity of rod is constant

(d) Velocity of end A is constant

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 4)

A uniform rod of mass m and length l is pivoted at point O. The rod is initially in vertical position and touching a block of mass M which is at rest on a horizontal surface. The rod is given a slight jerk and it starts rotating about point O. This causes the block to move forward as shown. The rod loses contact with the block at $\theta = 30°$. All surfaces are smooth. Now answer the following questions.



1. The value of ratio M/m is

(a) 2 : 3	(b) 3 : 2	(c) 4 : 3	(d) 3 : 4

2. The velocity of block when the rod loses contact with the block is

(a) $\frac{\sqrt{3gl}}{4}$	(b) $\frac{\sqrt{5gl}}{4}$	(c) $\frac{\sqrt{6gl}}{4}$	(d) $\frac{\sqrt{7gl}}{4}$

3. The acceleration of centre of mass of rod, when it loses contact with the block is

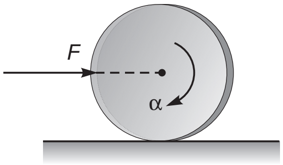
(a) $5g/4$	(b) $5g/2$	(c) $3g/2$	(d) $3g/4$

4. The hinge reaction at O on the rod when it loses contact with the block is

(a) $\frac{3\,mg}{4}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$	(b) $\left(\frac{mg}{4}\right)\hat{\mathbf{j}}$	(c) $\left(\frac{mg}{4}\right)\hat{\mathbf{i}}$	(d) $\frac{mg}{4}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

Passage 2 (Q. Nos. 5 to 7)

Consider a uniform disc of mass m, radius r, rolling without slipping on a rough surface with linear acceleration 'a' and angular acceleration α due to an external force F as shown in the figure. Coefficient of friction is μ



5. The work done by the frictional force at the instant of pure rolling is

(a) $\frac{\mu mgat^2}{2}$	(b) $\mu mgat^2$	(c) $\mu mg\frac{at^2}{\alpha}$	(d) zero

6. The magnitude of frictional force acting on the disc is

(a) ma	(b) μmg	(c) $\frac{ma}{2}$	(d) zero

7. Angular momentum of the disc will be conserved about

(a) centre of mass

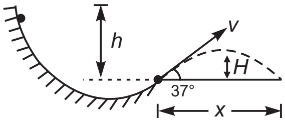
(b) point of contact

(c) a point at a distance $3R/2$ vertically above the point of contact

(d) a point at a distance $4R/3$ vertically above the point of contact

Passage 3 (Q. No. 8 to 10)

A tennis ball, starting from rest, rolls down the hill in the drawing. At the end of the hill the ball becomes airborne, leaving at an angle of 37° with respect to the ground. Treat the ball as a thin-walled spherical shell.



8. The velocity of projection v is

(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{10}{7}gh}$ (c) $\sqrt{\frac{5}{7}gh}$ (d) $\sqrt{\frac{6}{5}gh}$

9. Maximum height reached by ball H above ground is

(a) $\frac{9h}{35}$ (b) $\frac{18h}{35}$ (c) $\frac{18h}{25}$ (d) $\frac{27h}{125}$

10. Range x of the ball is

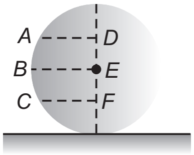
(a) $\frac{144}{125}h$ (b) $\frac{48}{25}h$ (c) $\frac{48}{35}h$ (d) $\frac{24}{7}h$

Match the Columns

1. A solid sphere, a hollow sphere and a disc of same mass and same radius are released from a rough inclined plane. All of them rolls down without slipping. On reaching the bottom of the plane, match the two columns.

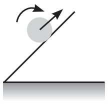
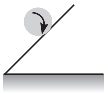
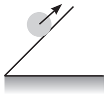
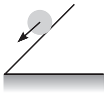
Column I	Column II
(a) time taken to reach the bottom	(p) maximum for solid sphere
(b) total kinetic energy	(q) maximum for hollow sphere
(c) rotational kinetic energy	(r) maximum for disc
(d) translational kinetic energy	(s) same for all

2. A solid sphere is placed on a rough ground as shown. E is the centre of sphere and $DE > EF$. We have to apply a linear impulse either at point A, B or C. Match the following two columns.

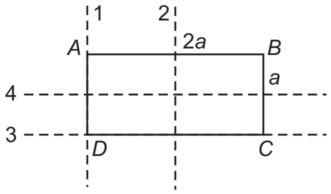


Column I	Column II
(a) Sphere will acquire maximum angular speed it impulse is applied at	(p) A
(b) Sphere will acquire maximum linear speed it impulse is applied at	(q) B
(c) Sphere can roll without slipping if impulse is applied at	(r) C
(d) Sphere can roll with forward slipping if impulse is applied at	(s) at any point A, B or C

3. The inclined surfaces shown in Column I are sufficiently rough. In Column II direction and magnitudes of frictional forces are mentioned. Match the two columns.

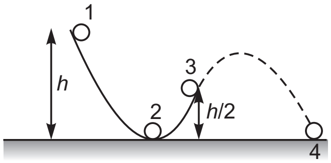
Column I	Column II
(a)  Rolling upwards	(p) upwards
(b)  Kept in rotating position	(q) downwards
(c)  Kept in translational position	(r) maximum friction will act
(d)  Kept in translational position	(s) required value of friction will act

4. A rectangular slab *ABCD* have dimensions $a \times 2a$ as shown in figure. Match the following two columns.



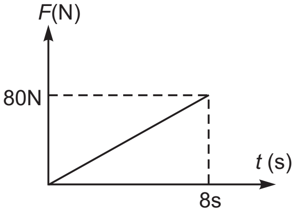
Column I	Column II
(a) Radius of gyration about axis-1	(p) $\frac{a}{\sqrt{12}}$
(b) Radius of gyration about axis-2	(q) $\frac{2a}{\sqrt{3}}$
(c) Radius of gyration about axis-3	(r) $\frac{a}{\sqrt{3}}$
(d) Radius of gyration about axis-4	(s) None

5. A small solid ball rolls down along sufficiently rough surface from 1 to 3 as shown in figure. From point-3 onwards it moves under gravity. Match the following two columns.



Column I	Column II
(a) Rotational kinetic energy of ball at point-2	(p) $\frac{1}{7}mgh$
(b) Translational kinetic energy of ball at point-3	(q) $\frac{2}{7}mgh$
(c) Rotational kinetic energy of ball at point-4	(r) $\frac{5}{7}mgh$
(d) Translational kinetic energy of ball at point-4	(s) None

6. A uniform disc of mass 10 kg, radius 1 m is placed on a rough horizontal surface. The coefficient of friction between the disc and the surface is 0.2. A horizontal time varying force is applied on the centre of the disc whose variation with time is shown in graph.



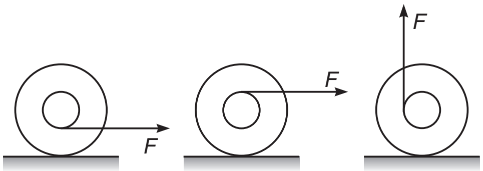
Column I	Column II
(a) Disc rolls without slipping	(p) at $t = 7$ s
(b) Disc rolls with slipping	(q) at $t = 3$ s
(c) Disc starts slipping at	(r) at $t = 4$ s
(d) Friction force is 10 N at	(s) None

7. Match the columns.

Column I	Column II
(a) Moment of inertia of a circular disc of mass M and radius R about a tangent parallel to plane of disc	(p) $\frac{MR^2}{2}$
(b) Moment of inertia of a solid sphere of mass M and radius R about a tangent	(q) $\frac{7}{5}MR^2$
(c) Moment of inertia of a circular disc of mass M and radius R about a tangent perpendicular to plane of disc	(r) $\frac{5}{4}MR^2$
(d) Moment of inertia of a cylinder of mass M and radius R about its axis	(s) $\frac{3}{2}MR^2$

Subjective Questions

1. Figure shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo the string is pulled in the direction shown. In each case there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate ?

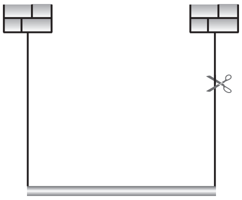


2. A uniform rod of mass m and length l is held horizontally by two vertical strings of negligible mass, as shown in the figure.

(a) Immediately after the right string is cut, what is the linear acceleration of the free end of the rod ?

(b) Of the middle of the rod ?

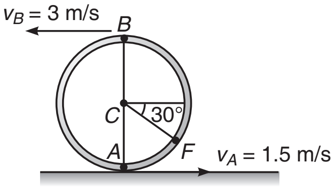
(c) Determine the tension in the left string immediately after the right string is cut.



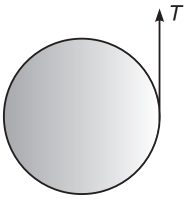
3. A solid disk is rolling without slipping on a level surface at a constant speed of 2.00 m/s. How far can it roll up a 30° ramp before it stops? (Take $g = 9.8\ \text{m/s}^2$)

4. A lawn roller in the form of a thin-walled hollow cylinder of mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

5. Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the centre point C and point F at this instant.



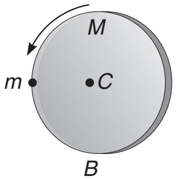
6. A uniform cylinder of mass M and radius R has a string wrapped around it. The string is held fixed and the cylinder falls vertically, as in figure.



(a) Show that the acceleration of the cylinder is downward with magnitude $a = \frac{2g}{3}$.

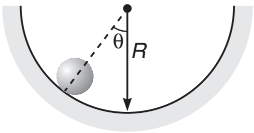
(b) Find the tension in the string.

7. A uniform disc of mass M and radius R is pivoted about the horizontal axis through its centre C. A point mass m is glued to the disc at its rim, as shown in figure. If the system is released from rest, find the angular velocity of the disc when m reaches the bottom point B.



8. A disc of radius R and mass m is projected on to a horizontal floor with a backward spin such that its centre of mass speed is v_0 and angular velocity is ω_0. What must be the minimum value of ω_0 so that the disc eventually returns back ?

9. A ball of mass m and radius r rolls along a circular path of radius R. Its speed at the bottom ($\theta = 0°$) of the path is v_0. Find the force of the path on the ball as a function of θ.



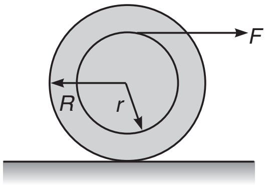
10. A heavy homogeneous cylinder has mass m and radius R. It is accelerated by a force F, which is applied through a rope wound around a light drum of radius r attached to the cylinder (figure). The coefficient of static friction is sufficient for the cylinder to roll without slipping.

(a) Find the friction force.

(b) Find the acceleration a of the centre of the cylinder.

(c) Is it possible to choose r, so that a is greater than $\frac{F}{m}$? How ?

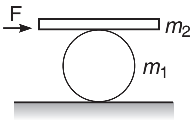
(d) What is the direction of the friction force in the circumstances of part (c) ?



11. A man pushes a cylinder of mass m_1 with the help of a plank of mass m_2 as shown. There is no slipping at any contact. The horizontal component of the force applied by the man is F. Find :

(a) the acceleration of the plank and the centre of mass of the cylinder and

(b) the magnitudes and directions of frictional forces at contact points.

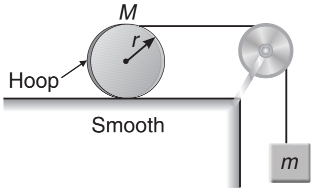


12. For the system shown in figure, $M = 1$ kg, $m = 0.2$ kg, $r = 0.2$ m. Calculate: ($g = 10\text{ m/s}^2$)

(a) the linear acceleration of hoop,

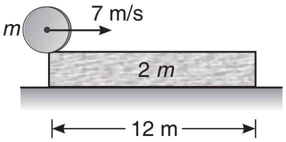
(b) the angular acceleration of the hoop of mass M and

(c) the tension in the rope.

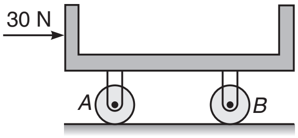


Note *Treat hoop as the ring. Assume no slipping between string and hoop.*

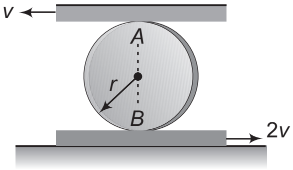
13. A cylinder of mass m is kept on the edge of a plank of mass $2m$ and length 12 m, which in turn is kept on smooth ground. Coefficient of friction between the plank and the cylinder is 0.1. The cylinder is given an impulse, which imparts it a velocity 7 m/s but no angular velocity. Find the time after which the cylinder falls off the plank. ($g = 10\text{ m/s}^2$)



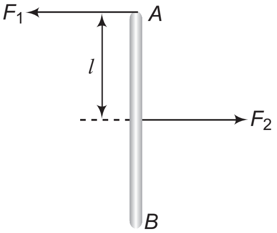
14. The 9 kg cradle is supported as shown by two uniform disks that roll without sliding at all surfaces of contact. The mass of each disk is $m = 6\,\text{kg}$ and the radius of each disk is $r = 80\,\text{mm}$. Knowing that the system is initially at rest, determine the velocity of the cradle after it has moved 250 mm.



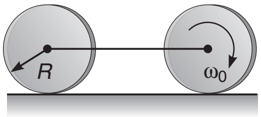
15. The disc of the radius r is confined to roll without slipping at A and B. If the plates have the velocities shown, determine the angular velocity of the disc.



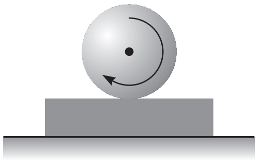
16. A thin uniform rod AB of mass $m = 1\,\text{kg}$ moves transnationally with acceleration $a = 2\,\text{m/s}^2$ due to two antiparallel forces F_1 and F_2. The distance between the points at which these forces are applied is equal to $l = 20\,\text{cm}$. Besides, it is known that $F_2 = 5\,\text{N}$. Find the length of the rod.



17. The assembly of two discs as shown in figure is placed on a rough horizontal surface and the front disc is given an initial angular velocity ω_0. Determine the final linear and angular velocity when both the discs start rolling. It is given that friction is sufficient to sustain rolling in the rear wheel from the starting of motion.



18. A horizontal plank having mass m lies on a smooth horizontal surface. A sphere of same mass and radius r is spined to an angular frequency ω_0 and gently placed on the plank as shown in the figure. If coefficient of friction between the plank and the sphere is μ. Find the distance moved by the plank till the sphere starts pure rolling on the plank. The plank is long enough.



19. A ball rolls without sliding over a rough horizontal floor with velocity $v_0 = 7$ m/s towards a smooth vertical wall. If coefficient of restitution between the wall and the ball is $e = 0.7$. Calculate velocity v of the ball long after the collision.

20. A uniform rod of mass m and length l rests on a smooth horizontal surface. One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result the rod obtains velocity v_0. Find the force with which one-half of the rod will act on the other in the process of motion.

21. A sphere, a disk and a hoop made of homogeneous materials have the same radius (10 cm) and mass (3 kg). They are released from rest at the top of a 30° incline and roll down without slipping through a vertical distance of 2 m. ($g = 9.8\ \text{m/s}^2$)

(a) What are their speeds at the bottom ?
(b) Find the frictional force f in each case
(c) If they start together at $t = 0$, at what time does each reach the bottom ?

22. *ABC* is a triangular framework of three uniform rods each of mass m and length $2l$. It is free to rotate in its own plane about a smooth horizontal axis through A which is perpendicular to *ABC*. If it is released from rest when *AB* is horizontal and C is above *AB*. Find the maximum velocity of C in the subsequent motion.

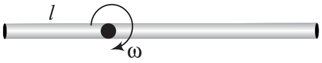
23. A uniform stick of length L and mass M hinged at one end is released from rest at an angle θ_0 with the vertical. Show that when the angle with the vertical is θ, the hinge exerts a force F_r along the stick and F_t perpendicular to the stick given by $F_r = \frac{1}{2} Mg\,(5\cos\theta - 3\cos\theta_0)$ and

$$F_t = \frac{1}{4} Mg\sin\theta$$

24. A uniform rod *AB* of mass $3m$ and length $4l$, which is free to turn in a vertical plane about a smooth horizontal axis through A, is released from rest when horizontal. When the rod first becomes vertical, a point C of the rod, where $AC = 3l$, strikes a fixed peg. Find the linear impulse exerted by the peg on the rod if

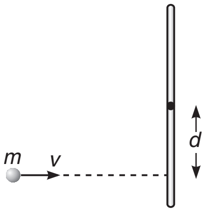
(a) the rod is brought to rest by the peg,
(b) the rod rebounds and next comes to instantaneous rest inclined to the downward vertical at an angle $\frac{\pi}{3}$ radian.

25. A uniform rod of length $4l$ and mass m is free to rotate about a horizontal axis passing through a point distant l from its one end. When the rod is horizontal, its angular velocity is ω as shown in figure. Calculate :

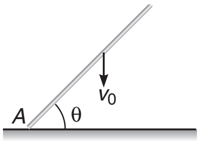


(a) reaction of axis at this instant,
(b) acceleration of centre of mass of the rod at this instant,
(c) reaction of axis and acceleration of centre mass of the rod when rod becomes vertical for the first time,
(d) minimum value of ω , so that centre of rod can complete circular motion.

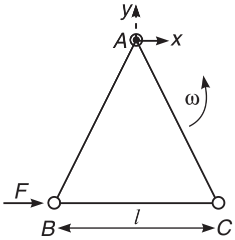
26. A stick of length l lies on horizontal table. It has a mass M and is free to move in any way on the table. A ball of mass m, moving perpendicularly to the stick at a distance d from its centre with speed v collides elastically with it as shown in figure. What quantities are conserved in the collision ? What must be the mass of the ball, so that it remains at rest immediately after collision?



27. A rod of length l forming an angle θ with the horizontal strikes a frictionless floor at A with its centre of mass velocity v_0 and no angular velocity. Assuming that the impact at A is perfectly elastic. Find the angular velocity of the rod immediately after the impact.



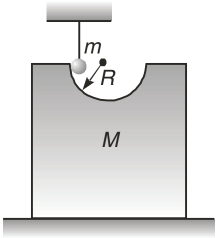
28. Three particles A, B and C, each of mass m, are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side l. This body is placed on a horizontal frictionless table (x-y plane) and is hinged to it at the point A, so that it can move without friction about the vertical axis through A (see figure). The body is set into rotational motion on the table about A with a constant angular velocity ω.



(a) Find the magnitude of the horizontal force exerted by the hinge on the body.

(b) At time T, when the side BC is parallel to the x-axis, a force F is applied on B along BC (as shown). Obtain the x-component and the y-component of the force exerted by the hinge on the body, immediately after time T.

29. A semicircular track of radius $R = 62.5$ cm is cut in a block. Mass of block, having track, is $M = 1$ kg and rests over a smooth horizontal floor. A cylinder of radius $r = 10$ cm and mass $m = 0.5$ kg is hanging by thread such that axes of cylinder and track are in same level and surface of cylinder is in contact with the track as shown in figure. When the thread is burnt, cylinder starts to move down the track. Sufficient friction exists between surface of cylinder and track, so that cylinder does not slip.

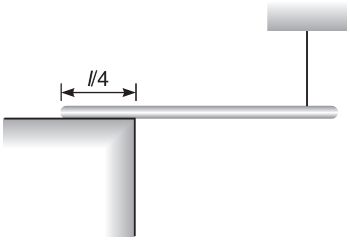


Calculate velocity of axis of cylinder and velocity of the block when it reaches bottom of the track. Also find force applied by block on the floor at that moment. ($g = 10$ m/s^2)

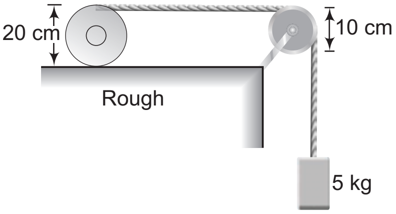
30. A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity. It is placed in contact with a plane inclined at an angle α to the horizontal. If there is a coefficient of friction μ for sliding between the cylinder and plane. Find the distance the cylinder moves up before sliding stops. Also, calculate the maximum distance it travels up the plane. Assume $\mu > \tan \alpha$.

31. Show that if a rod held at angle θ to the horizontal and released, its lower end will not slip if the friction coefficient between rod and ground is greater than $\frac{3\sin\theta\cos\theta}{1+3\sin^2\theta}$.

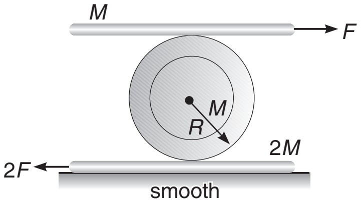
32. One-fourth length of a uniform rod of mass m and length l is placed on a rough horizontal surface and it is held stationary in horizontal position by means of a light thread as shown in the figure. The thread is then burnt and the rod starts rotating about the edge. Find the angle between the rod and the horizontal when it is about to slide on the edge. The coefficient of friction between the rod and the surface is μ.



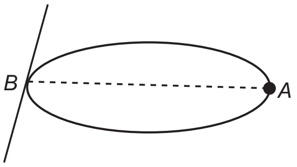
33. In figure the cylinder of mass 10 kg and radius 10 cm has a tape wrapped round it. The pulley weighs 100 N and has a radius 5 cm. When the system is released, the 5 kg mass comes down and the cylinder rolls without slipping. Calculate the acceleration and velocity of the mass as a function of time.



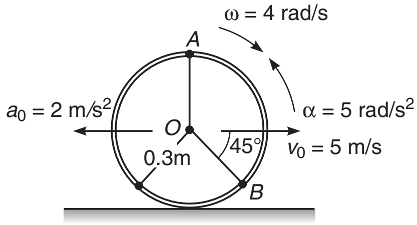
34. A cylinder is sandwiched between two planks. Two constant horizontal forces F and $2F$ are applied on the planks as shown. Determine the acceleration of the centre of mass of cylinder and the top plank, if there is no slipping at the top and bottom of cylinder.



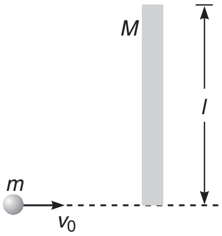
35. A ring of mass m and radius r has a particle of mass m attached to it at a point A. The ring can rotate about a smooth horizontal axis which is tangential to the ring at a point B diametrically opposite to A. The ring is released from rest when AB is horizontal. Find the angular velocity and the angular acceleration of the body when AB has turned through an angle $\frac{\pi}{3}$.



36. A hoop is placed on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular deceleration $\alpha = 5\,\text{rad/s}^2$. Also, its centre has a velocity of $v_0 = 5$ m/s and a deceleration $a_0 = 2\ \text{m/s}^2$. Determine the magnitude of acceleration of point B at this instant.



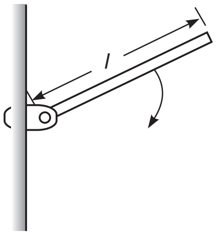
37. A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.



(a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

(b) One point on the plank is at rest immediately after the collision. Where is it?

38. A thin plank of mass M and length l is pivoted at one end. The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



Answers

Introductory Exercise 12.1

1. $\frac{2}{\sqrt{3}}l$ **2.** 55 kg-m^2 **3.** $\frac{5}{3}ml^2$

4. About a diagonal, because the mass is more concentrated about a diagonal

5. (i) $\frac{8}{5}mr^2 + 2ma^2$ (ii) $\frac{8}{5}mr^2 + ma^2$ **6.** (a) $2Ml^2$ (b) $\frac{1}{3}Ml^2$

7. 0.5 kg-m^2 **8.** The one having the smaller density **9.** 0.43 kg-m^2 **10.** $\frac{\pi^2}{3}$

Introductory Exercise 12.2

1. $\frac{\pi}{30}$ rad s^{-1} **2.** $\omega = \frac{v}{2R}$ **3.** $(-\hat{\mathbf{k}})$ rad s^{-1} **4.** $\frac{v}{l}$, perpendicular to paper inwards

Introductory Exercise 12.3

1. $(-2\hat{\mathbf{i}} - 2\hat{\mathbf{k}})$ N-m **2.** 400 N-m (perpendicular to the plane of motion) **3.** 2.71 N-m **4.** $\frac{83}{20}$ N-m

Introductory Exercise 12.4

1. 8π rad-s^{-2}, (40π) rad-s^{-1} **2.** 100 rad **3.** 5 N-m **4.** 800 rad **5.** 0.87 N

6. (a) 4 rad-s^{-1}, -6 rad-s^{-2} (b) -12 rad-s^{-2} **7.** 20 s **8.** (a) 0.01 N-m (b) 10.13 N-m **9.** (a) 36 s (b) $12\sqrt{\frac{2}{k}}$

10. (a) 12.5 rad (b) 127.5 rad **11.** 9 rad, 1.43

Introductory Exercise 12.5

1. $ml^2\omega$ **2.** $4\sqrt{2}$ kg-m^{2}s^{-1} **3.** $\frac{mu^3 \cos\alpha \sin^2\alpha}{2g}$ **4.** No **5.** $-\left(\frac{7}{5}mRv\right)\hat{\mathbf{k}}$ **6.** $\frac{1}{2}mRv$ (clockwise)

Introductory Exercise 12.6

1. $\frac{\omega_0 M}{M + 2m}$ **2.** Duration of day night will increase **3.** True **4.** Increase

Introductory Exercise 12.7

1. $\mathbf{v}_M = \frac{3}{2}v\hat{\mathbf{i}}$, $\mathbf{v}_N = v\hat{\mathbf{i}} - \frac{v}{2}\hat{\mathbf{j}}$, $\mathbf{v}_R = \frac{v}{2}\hat{\mathbf{i}}$, $\mathbf{v}_S = v\hat{\mathbf{i}} + \frac{v}{2}\hat{\mathbf{j}}$

2. $\mathbf{a}_M = (R\omega^2)\hat{\mathbf{j}}$, $\mathbf{a}_N = (R\omega^2)\hat{\mathbf{i}}$, $\mathbf{a}_R = -(R\omega^2)\hat{\mathbf{j}}$, $\mathbf{a}_S = -(R\omega^2)\hat{\mathbf{i}}$

Introductory Exercise 12.8

1. 25 J, 35 J **2.** True **3.** True

Introductory Exercise 12.9

1. $\mathbf{v}_A$ is zero. Rest three velocities are : $|\mathbf{v}_C| = 2v$, $|\mathbf{v}_B| = |\mathbf{v}_D| = \sqrt{2}v$

Introductory Exercise 12.10

1. False **2.** 72 N **3.** (a) $\frac{2}{7\sqrt{3}}$ (b) $\frac{25}{7}$ ms^{-2} (c) $\frac{30}{7}$ ms^{-2} **4.** (a) $g\sin\theta - \mu g\cos\theta$ (b) $\frac{5\mu g\cos\theta}{2R}$ **5.** Leftwards

Introductory Exercise 12.11

1. False **2.** (a) Same in both cases (b) Solid sphere (c) Solid sphere

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (d) **4.** (a) **5.** (a) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (a)
11. (c)

Objective Questions

1. (d) **2.** (a) **3.** (c) **4.** (d) **5.** (b) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (b)
11. (d) **12.** (a) **13.** (b) **14.** (b) **15.** (b) **16.** (d) **17.** (a) **18.** (b) **19.** (b) **20.** (d)
21. (c) **22.** (d) **23.** (b) **24.** (a) **25.** (a) **26.** (d) **27.** (c) **28.** (b) **29.** (a) **30.** (d)

Subjective Questions

1. $6h$ **2.** $\frac{R}{\sqrt{2}}$ **3.** $\frac{Ma^2}{12}$ **4.** $\frac{l}{\sqrt{2}}$ **5.** $I = \mu r^2$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass of two masses.

6. 8 cm **7.** $0.7\,\text{ms}^{-2}$ **8.** $10\,\text{rad-s}^{-1}$ **9.** $\frac{mv^3 \sin^2\theta \cos\theta}{2g}$ **10.** $I = \left(\frac{\alpha l^4}{4} + \frac{\beta l^3}{3}\right)$ **11.** False

12. $\frac{2}{7} mgh$ **13.** $\frac{v_1 - v_2}{2R}$ **14.** $\frac{2}{3} Ma^2$ **15.** 7 s **16.** $\pm \cos^{-1}\left(\frac{v}{R\omega}\right)$ **17.** $\frac{I + 2Mr^2}{4Mr^2 - I}$

18. $\lim\limits_{F \to 0}$ can make the body move **19.** (a) $\mu < 1$ (b) $\mu > 1$

20. $\frac{20}{3}$ N-m, 4 s **21.** (a) $\frac{3\omega_0 R}{4\mu g}$ (b) $\frac{3\omega_0^2 R}{8\mu g}$

22. $\omega_{av} = 4$ rad/s, $\alpha_{av} = -6.0\,\text{rad-s}^{-2}$ **23.** $-\left(\frac{10}{3} mRv\right)\hat{\mathbf{k}}$ **24.** $\frac{25}{6}\,\text{rad-s}^{-1}$ **25.** $7.29 \times 10^{-4}\,\text{kg-m}^2$

26. (a) $14.3\,\text{rad-s}^{-1} = 2.27\,\text{rev-s}^{-1}$ (b) $E_i = 39.9$ J, $E_f = 181$ J (c) 141.1 J

27. (a) $\omega = \left(1 + \frac{2m}{M}\right)\omega_0$ (b) $\frac{1}{2} m\omega_0^2 R^2 \left(1 + \frac{2m}{M}\right)$ **28.** $\frac{4F\cos\theta}{3M + 8m}, \frac{3MF\cos\theta}{3M + 8m}, \frac{MF\cos\theta}{3M + 8m}$

29. $\frac{F}{M + 3m}$ each in opposite directions **30.** $\frac{4}{3} ml\omega$ **31.** (a) $\frac{2v}{9L}$ (b) $\frac{1}{9}$

32. (a) $\frac{2u}{3}$ (b) $\frac{2u}{\sqrt{3}}$

LEVEL 2

Objective Questions

1. (d) **2.** (b) **3.** (b) **4.** (d) **5.** (c) **6.** (a) **7.** (a) **8.** (b) **9.** (c) **10.** (c)
11. (b) **12.** (c) **13.** (b) **14.** (b) **15.** (d) **16.** (a) **17.** (b) **18.** (b) **19.** (d) **20.** (b)
21. (c) **22.** (b) **23.** (c) **24.** (d) **25.** (c) **26.** (a) **27.** (b) **28.** (d) **29.** (a) **30.** (c)
31. (b) **32.** (b) **33.** (c) **34.** (d) **35.** (a) **36.** (d) **37.** (b) **38.** (d) **39.** (c) **40.** (a)
41. (a) **42.** (a)

More than One Correct Options

1. (a,c) **2.** (all) **3.** (a,b,d) **4.** (a,d) **5.** (a,c,d) **6.** (all) **7.** (a,c)
8. (b,d) **9.** (a,b) **10.** (a,c) **11.** (a,b)

Comprehension Based Questions

1. (c) **2.** (a) **3.** (d) **4.** (b) **5.** (d) **6.** (c) **7.** (c) **8.** (d) **9.** (d) **10.** (a)

Match the Columns

1. (a) → q (b) → s (c) → q (d) → p
2. (a) → p (b) → s (c) → p (d) → r
3. (a) → p,s (b) → p,r (c) → q,r (d) → p,r
4. (a) → q (b) → r (c) → r (d) → p
5. (a) → q (b) → s (c) → p (d) → s
6. (a) → q,r (b) → p (c) → s (d) → q
7. (a) → r (b) → q (c) → s (d) → p

Subjective Questions

1. In each case in clockwise direction **2.** (a) $3g/2$, (b) $3g/4$ (c) $mg/4$ **3.** 0.612 m **4.** $\frac{F}{2M}, \frac{F}{2}$

5. 0.75 ms^{-1}, 1.98 ms^{-1} **6.** (b) $\frac{1}{3} Mg$ **7.** $\sqrt{\frac{4mg}{(2m+M)R}}$ **8.** $\frac{2v_0}{R}$

9. $f = \frac{2}{7} mg \sin\theta$, $N = \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R-r)}$

10. (a) $f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F$, assuming f opposite to F (b) $a = \left(\frac{2F}{3mR}\right)(R+r)$ (c) yes, if r is greater than $\frac{1}{2}R$.

(d) f in same direction as F. **11.** (a) $\frac{8F}{3m_1 + 8m_2}$, $\frac{4F}{3m_1 + 8m_2}$

(b) $\frac{3m_1 F}{3m_1 + 8m_2}$ (between plank and cylinder) $\frac{m_1 F}{3m_1 + 8m_2}$ (between cylinder and ground)

12. (a) 1.43 ms^{-2} (b) 7.15 rad-s^{-2} (c) 1.43 N **13.** 2.25 s **14.** 0.745 ms^{-1} (rightwards)

15. $\frac{3}{2}\frac{v}{r}$ (anticlockwise) **16.** 1 m **17.** $\frac{\omega_0 R}{6}; \frac{\omega_0}{6}$ **18.** $S = \frac{2\omega_0^2 r^2}{81\mu g}$ **19.** $v = 1.5$ ms^{-1} **20.** $\frac{9}{2}\frac{mv_0^2}{l}$

21. (a) Sphere, 5.29 ms^{-1}, disk 5.11 ms^{-1}, hoop 4.43 ms^{-1}
(b) Sphere 4.2 N, disk 4.9 N, hoop 7.36 N (c) Sphere, 1.51 s disk 1.56 s hoop 1.81 s

22. $2l\sqrt{\frac{g\sqrt{3}}{l}}$ **24.** (a) $\left(\frac{8}{3}m\right)\sqrt{3gl}$, (b) $\frac{4}{3}m\sqrt{6gl}\,(\sqrt{2}+1)$

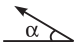
25. (a) $\frac{4}{7} mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$ (b) $\sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$ (c) $\left(\frac{13}{7} mg + ml\omega^2\right), \left(\frac{6g}{7} + l\omega^2\right)$ (d) $\sqrt{\frac{6g}{7l}}$

26. Linear momentum, angular momentum and kinetic energy, $\frac{Ml^2}{12d^2 + l^2}$ **27.** $\omega = \frac{6v_0}{l}\left(\frac{\cos\theta}{1 + 3\cos^2\theta}\right)$

28. (a) $\sqrt{3}\, ml\omega^2$ (b) $F_x = -\frac{F}{4}$, $F_y = \sqrt{3}\, ml\omega^2$ **29.** 2.0 ms^{-1}, 1.5 ms^{-1}, 16.67 N

30. $d_1 = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$, $d_{max} = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha\,(3\mu\cos\alpha - \sin\alpha)}$

32. $\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$ **33.** 3.6 ms^{-2}, $\frac{4gt}{11}$ **34.** $\frac{F}{26M}, \frac{21F}{26M}$ **35.** $\sqrt{\frac{6g\sqrt{3}}{11r}}, \frac{3g}{11r}$ **36.** 6.21 ms^{-2}

37. (b) $\frac{2l}{3}$ from the boy **38.** $\frac{\sqrt{10}}{4} Mg$, 

$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

13

Gravitation

Chapter Contents

13.1 Introduction
13.2 Newton’s Law of Gravitation
13.3 Acceleration Due to Gravity
13.4 Gravitational Field and Field Strength
13.5 Gravitational Potential
13.6 Relation between Gravitational Field and Potential
13.7 Gravitational Potential Energy
13.8 Binding Energy
13.9 Motion of Satellites
13.10 Kepler’s Laws of Planetary Motion

13.1 Introduction

Why are planets, moon and the sun all nearly spherical? Why do some earth satellites circle the earth in 90 minutes, while the moon takes 27 days for the trip? And why don't satellites fall back to earth? The study of gravitation provides the answers for these and many related questions.

Gravitation is one of the four classes of interactions found in nature. These are :

(i) the gravitational force
(ii) the electromagnetic force
(iii) the strong nuclear force (also called the hadronic forces)
(iv) the weak nuclear forces.

Although, of negligible importance in the interactions of elementary particles, gravity is of primary importance in the interactions of large objects. It is gravity that holds the universe together.

In this chapter, we will learn the basic laws that govern gravitational interactions.

13.2 Newton's Law of Gravitation

Along with his three laws of motion, Newton published the law of gravitation in 1687. According to him; *"every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them."*

Thus, the magnitude of the gravitational force F between two particles m_1 and m_2 placed at a distance r is,

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{or} \quad \boxed{F = G\frac{m_1 m_2}{r^2}}$$

Here, G is a universal constant called gravitational constant whose magnitude is,

$$G = 6.67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2$$

The direction of the force F is along the line joining the two particles.

Following three points are important regarding the gravitational force :

(i) Unlike the electrostatic force, it is independent of the medium between the particles.
(ii) It is conservative in nature.
(iii) It expresses the force between two point masses (of negligible volume). However, for external points of spherical bodies the whole mass can be assumed to be concentrated at its centre of mass.

Gravity

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is earth then the gravitation is called 'gravity'. Hence, gravity is the force by which earth attracts a body towards its centre. It is a special case of gravitation.

Extra Points to Remember

- Direct formula $F=\frac{Gm_1m_2}{r^2}$ can be applied under following three conditions:

  (a) To find force between two point masses

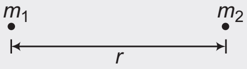


Fig. 13.1

  (b) To find force between two spherical bodies

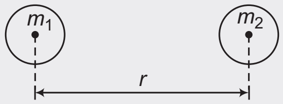


Fig. 13.2

  (c) To find force between a spherical body and a point mass.

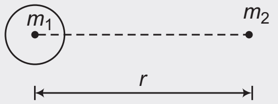


Fig. 13.3

- To find force between a point mass and a rod single integration is required. In this case, we cannot assume whole mass of the rod at its centre to find force between them. Thus,

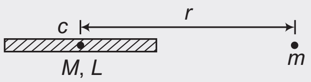


Fig. 13.4

$$F \neq \frac{GMm}{r^2}$$

- To find force between two rods double integration is required but normally, double integration is not asked in physics paper.

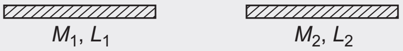


Fig. 13.5

- Two or more than two gravitational forces acting on a body can be added by vector addition method.

Example 13.1 *Three point masses 'm' each are placed at the three vertices of an equilateral triangle of side 'a'. Find net gravitational force on any one point mass.*

Solution We are finding net force on the point mass kept at O.

$$F=\frac{G(m)(m)}{a^2}=\frac{Gm^2}{a^2}$$

Since, the two forces are equal in magnitudes, therefore the resultant force will pass through the centre as shown in figure.

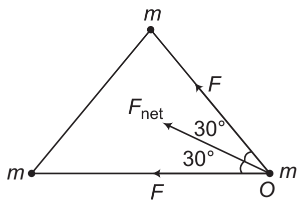


Fig. 13.6

$$F_{\text{net}}=\sqrt{F^2+F^2+2(F)(F)\cos 60^\circ}$$
$$=\sqrt{3}F$$
$$=\frac{\sqrt{3}Gm^2}{a^2}$$

Ans.

Example 13.2 *Four particles each of mass 'm' are placed at the four vertices of a square of side 'a'. Find net force on any one of the particle.*

Solution. We are finding net force on the particle at D.

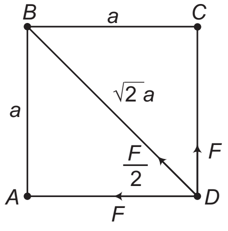


Fig. 13.7

$$F_{DC} = F_{DA} = \frac{G(m)(m)}{a^2} = \frac{Gm^2}{a^2} = F \text{ (say)}$$

$$F_{DB} = \frac{G(m)(m)}{(\sqrt{2}a)^2} = \frac{1}{2}\frac{Gm^2}{a^2} = \frac{F}{2}$$

Now, resultant of F_{DA} and F_{DC} is $\sqrt{2}\,F$ in the direction of DB.

$$\therefore \qquad F_{\text{net}} = \sqrt{2}\,F + \frac{F}{2} = \left(\sqrt{2}+\frac{1}{2}\right)F$$

$$= \left(\sqrt{2}+\frac{1}{2}\right)\frac{Gm^2}{a^2} \quad \text{(towards } DB\text{)}$$

Ans.

Example 13.3 *Six particles each of mass 'm' are placed at six vertices A, B, C, D, E and F of a regular hexagon of side 'a'. A seventh particle of mass 'M' is kept at centre 'O' of the hexagon.*

(a) Find net force on 'M'.

(b) Find net force on 'M' if particle at A is removed.

(c) Find net force on 'M' if particles at A and C are removed.

Solution (a) $F_A = F_B = F_C = F_D = F_E = F_F = \dfrac{GmM}{a^2} = F$ (say)

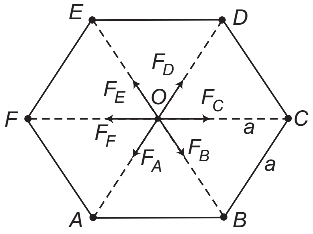


Fig. 13.8

So, net force will be zero, as three pairs of equal and opposite forces are acting on 'M' at O.

(b) When particle at A is removed then F_A is removed. So, there is no force to cancel F_D.

$$\therefore \qquad F_{\text{net}} = F_D = \frac{GMm}{a^2} \quad \text{(towards } D\text{)}$$

Ans.

(c) When particles at A and C are removed then, F_A and F_C are removed. F_B and F_E are still cancelled. So, net force is the resultant of two forces F_D and F_F of equal magnitudes acting at 120°. So, the resultant will pass through the centre or towards E. Magnitude of this resultant is

$$F_{\text{net}} = \sqrt{F^2 + F^2 + 2(F)(F)\cos 120^\circ}$$

$$= F = \frac{GMm}{a^2} \text{ (towards } E\text{)}$$

Ans.

Example 13.4 *Five particles each of mass 'm' are kept at five vertices of a regular pentagon. A sixth particle of mass 'M' is kept at centre of the pentagon 'O'. Distance between 'M' and 'm' is 'a'. Find*

(a) net force on 'M'

(b) magnitude of net force on 'M' if any one particle is removed from one of the vertices.

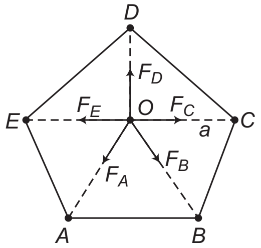


Fig. 13.9

Solution (a) $F_A = F_B = F_C = F_D = F_E = \dfrac{GMm}{a^2} = F$ (say)

Angle between two successive force vectors is $\theta = \dfrac{360°}{5} = 72°$.

When these five force vectors are added as per polygon law of vector addition we get another closed regular polygon as shown below.

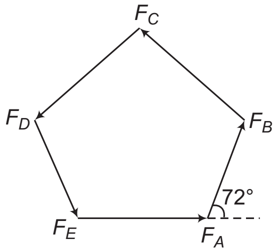


Fig. 13.10

Therefore, net resultant force on '*M*' is zero.

Note *This result is true ($F_{net} = 0$) for any number of particles, provided masses at vertices are equal and polygon is regular.*

(b) When particle at A is removed, then F_A will be removed. So, magnitude of net force will be F as shown below:

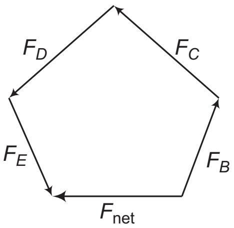


Fig. 13.11

$$\therefore \qquad F_{net} = F = \frac{GMm}{a^2} \qquad \textbf{Ans.}$$

Example 13.5 *A mass m is at a distance a from one end of a uniform rod of length l and mass M. Find the gravitational force on the mass due to the rod.*

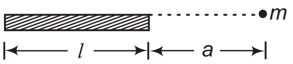


Fig. 13.12

Solution $$dF = \frac{G\,(dM)\,m}{x^2} = \frac{G\left(\frac{M}{l}\cdot dx\right)m}{x^2}$$

$$\therefore \qquad F = \int_{x=a}^{x=(a+l)} dF = \frac{GMm}{a\,(l+a)} \qquad \textbf{Ans.}$$

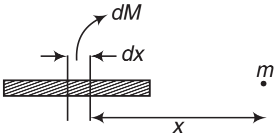


Fig. 13.13

Example 13.6 *A uniform ring of mass m is lying at a distance $\sqrt{3}\,a$ from the centre of a sphere of mass M just over the sphere (where a is the radius of the ring as well as that of the sphere). Find the magnitude of gravitational force between them.*

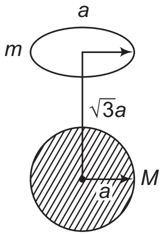


Fig. 13.14

Solution Net force on ring $= \int_{\text{whole ring}} dF \sin 60°$

$$= \int_{\text{whole ring}} \frac{GM\,(dm)}{(2a)^2}\,\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}\,GMm}{8a^2}$$

as $\int_{\text{whole ring}} dm = m$

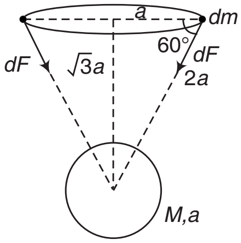


Fig. 13.15

INTRODUCTORY EXERCISE 13.1

1. Three uniform spheres each having a mass *M* and radius *a* are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any one of the spheres due to the other two.
2. Four particles having masses *m*, 2*m*, 3*m* and 4*m* are placed at the four corners of a square of edge *a*. Find the gravitational force acting on a particle of mass *m* placed at the centre.
3. Two particles of masses 1.0 kg and 2.0 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.
4. Three particles *A, B* and *C,* each of mass *m*, are placed in a line with $AB = BC = d$. Find the gravitational force on a fourth particle *P* of same mass, placed at a distance *d* from the particle *B* on the perpendicular bisector of the line *AC.*
5. Spheres of the same material and same radius *r* are touching each other. Show that gravitational force between them is directly proportional to r^4.

13.3 Acceleration due to Gravity

When a body is dropped from a certain height above the ground it begins to fall towards the earth under gravity. The acceleration produced in the body due to gravity is called the acceleration due to gravity. It is denoted by g . Its value close to the earth's surface is 9.8 m/s^2.

Suppose that the mass of the earth is M, its radius is R, then the force of attraction acting on a body of mass m close to the surface of earth is

$$F = \frac{GMm}{R^2}$$

According to Newton's second law, the acceleration due to gravity

$$\boxed{g = \frac{F}{m} = \frac{GM}{R^2}}$$

This expression is independent of m. If two bodies of different masses are allowed to fall freely, they will have the same acceleration, i.e. if they are allowed to fall from the same height, they will reach the earth simultaneously.

Variation in the Value of *g*

The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of earth. Thus, value of g depends on following factors :

Shape of the Earth

The earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius and since

$$g = \frac{GM}{R^2} \quad \text{or} \quad g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles.

Height above the Surface of the Earth

The force of gravity on an object of mass m at a height h above the surface of earth is,

$$F = \frac{GMm}{(R+h)^2}$$

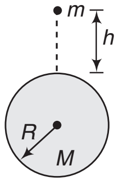


Fig. 13.16

∴ Acceleration due to gravity at this height will be,

$$g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

This can also be written as,
$$g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

or
$$\boxed{g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}} \quad \text{as} \quad \frac{GM}{R^2} = g$$

Thus, $g' < g$

i.e. the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1+\frac{h}{R}\right)^{-2} \quad \text{or} \quad \boxed{g' \approx g\left(1-\frac{2h}{R}\right) \quad \text{if} \quad h << R}$$

Depth below the Surface of the Earth

Let an object of mass m is situated at a depth d below the earth's surface. Its distance from the centre of earth is $(R-d)$. This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. According to Gauss theorem (you will study in class XII) the gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.

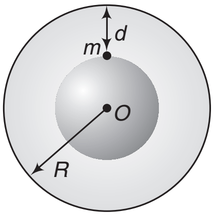


Fig. 13.17

The mass of this sphere is,

$$M' = \left(\frac{M}{4/3\,\pi R^3}\right)\frac{4}{3}\pi\,(R-d)^3$$

or $$M' = \frac{(R-d)^3}{R^3}\,.\,M$$

$$F = \frac{GM'm}{(R-d)^2} = \frac{GMm(R-d)}{R^3}$$

and $$g' = \frac{F}{m}$$

Substituting the values, we get $\boxed{g' = g\left(1-\frac{d}{R}\right)}$ i.e. $g' < g$

Note *We can see from this equation that $g' = 0$ at $d = R$, i.e. acceleration due to gravity is zero at the centre of the earth.*

Thus, the variation in the value of g with r (the distance from the centre of earth) is as shown in Fig. 13.18.

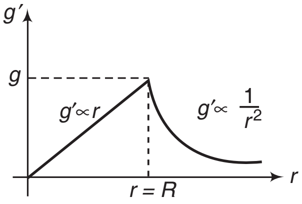


Fig. 13.18

For $r \le R$, $$g' = g\left(1-\frac{d}{R}\right) = \frac{gr}{R}$$

as $$R-d = r \quad \text{or} \quad g' \propto r$$

For $r \ge R$, $$g' = \frac{g}{\left(1+\frac{d}{R}\right)^2} = \frac{gR^2}{r^2}$$

as $$R+d = r \quad \text{or} \quad g' \propto \frac{1}{r^2}$$

Axial Rotation of the Earth

Let us consider a particle P at rest on the surface of the earth, at latitude ϕ. Then the pseudo force acting on the particle is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the centre O of the earth. Thus, the effective acceleration g' is the resultant of g and $r\omega^2$ or

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180° - \phi)}$$

as angle between g and $r\omega^2$ is $(180° - \phi)$.

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2 \cos\phi} \qquad \ldots\text{(i)}$$

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$ rad/s is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq. (i) can be written as

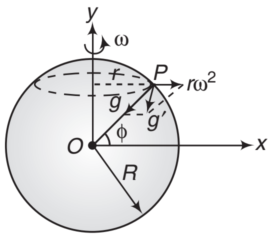


Fig. 13.19

$$g' = (g^2 - 2gR\omega^2\cos^2\phi)^{1/2}$$

$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2}$$

$$\approx g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $$\boxed{g' = g - R\omega^2\cos^2\phi}$$

Following conclusions can be drawn from the above discussion :

(i) The effective value of g is not truly vertical passing through the centre O.

(ii) The effect of centrifugal force due to rotation of earth is to reduce the effective value of g.

(iii) At equator $\phi = 0°$.

Therefore, $g' = g - R\omega^2$

and at poles $\phi = 90°$,

Therefore, $g' = g$

Thus, at equator g' is minimum while at poles g' is maximum.

Example 13.7 *Assuming earth to be a sphere of uniform mass density, how much would a body weigh half way down the centre of the earth, if it weighed 100 N on the surface?*

Solution Given, $mg = 100\text{ N} \Rightarrow g' = g\left(1 - \frac{d}{R}\right)$

$$d = \frac{R}{2} \Rightarrow \frac{d}{R} = \frac{1}{2}$$

$\therefore$ $$g' = g\left(1 - \frac{1}{2}\right) = \frac{g}{2}$$

$\therefore$ $$mg' = \frac{mg}{2} = \frac{100}{2} = 50\text{ N}$$ **Ans.**

Example 13.8 *Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero? Take $g = 10\ m/s^2$ and radius of earth $R = 6400\ km$.*

Solution The weight will become zero, when

$$g' = 0 \quad \text{or} \quad g - R\omega^2 = 0 \qquad \text{(on the equator } g' = g - R\omega^2\text{)}$$

or

$$\omega = \sqrt{\frac{g}{R}}$$

∴

$$\frac{2\pi}{T} = \sqrt{\frac{g}{R}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values,

$$T = \frac{2\pi\sqrt{\dfrac{6400\times 10^3}{10}}}{3600}\ \text{h}$$

or

$$T = 1.4\ \text{h}$$

Ans.

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Example 13.9 *Draw g' versus d and g' versus h graph. Here, 'd' is depth below the surface of earth and h is the height from the surface of earth.*

Solution On the surface of earth,

$$g' = g$$

At depth 'd' below the surface of earth,

$$g' = g\left(1 - \frac{d}{R}\right)$$

i.e. g' decreases linearly with depth.

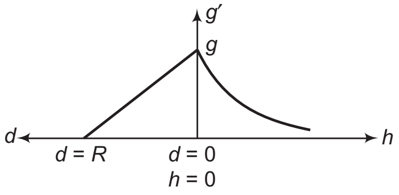


Fig. 13.20

At, $d = R$, $g' = 0$

At height h above the surface of earth.

$$g' = \frac{g}{\left(1 + \dfrac{h}{R}\right)^2}$$

With increase in the value of 'h' value of g' decreases but not linearly. At $h \to \infty$, $g' \to 0$.

The correct graph is as shown in Fig. 13.20.

Example 13.10 *At what depth below the surface of earth, value of acceleration due to gravity is same as the value at height $h=R$, where R is the radius of earth.*

Solution Given that, g' at depth $d=g'$ at height $h=R$

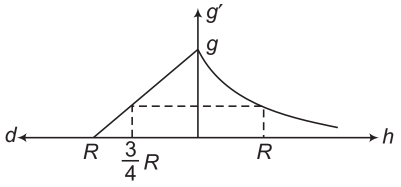


Fig. 13.21

$$\therefore \qquad g\left(1-\frac{d}{R}\right)=\frac{g}{\left(1+\frac{h}{R}\right)^2}$$

Substituting, $h=R$, we get

$$d=\frac{3}{4}R \qquad \textbf{Ans.}$$

From the graph we can see that value of g' at depth $d=\frac{3}{4}R$ and height $h=R$ are same.

INTRODUCTORY EXERCISE 13.2

1. Value of g on the surface of earth is $9.8\ \text{m/s}^2$. Find its value on the surface of a planet whose mass and radius both are two times that of earth.
2. Value of g on the surface of earth is $9.8\ \text{m/s}^2$. Find its value
   (a) at height $h=R$ from the surface,
   (b) at depth $d=\frac{R}{2}$ from the surface. (R = radius of earth)
3. Calculate the distance from the surface of the earth at which the acceleration due to gravity is the same below and above the surface of the earth.
4. Calculate the change in the value of g at altitude 45°. Take radius of earth $R=6.37\times10^3$ km.
5. At what height from the surface of earth will the value of g be reduced by 36% from the value at the surface? Take radius of earth $R = 6400$ km.
6. Determine the speed with which the earth would have to rotate on its axis, so that a person on the equator would weigh $\frac{3}{5}$ th as much as at present. Take $R=6400$ km.
7. A body is weighed by a spring balance to be 1000 N at the north pole. How much will it weight at the equator? Account for the earth's rotation only.
8. At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation?
9. Assuming earth to be spherical, at what height above the north pole, value of g is same as that on the earth's surface at equator?

13.4 Gravitational Field and Field Strength

The space around a mass or system of masses in which any other test mass experiences a gravitational force is called gravitational field. When this test mass is moved from one point to another point, some work is also done by this gravitational force.

Mathematically, gravitational field at any point can be defined by two physical quantities. One is vector quantity, called **gravitational field strength** or **intensity of gravitational field** and it is denoted by **E**. This is related to the gravitational force (a vector quantity) experienced by test mass in gravitational field. We also sometimes called it **gravitational field**.

The other physical quantity is **gravitational potential**. It is represented by V. This is related to the work done (a scalar quantity) by gravitational force in moving the test mass from one point to another point in the gravitational field.

Gravitational Field Strength (E)

The force experienced (both in magnitude and direction) by a unit test mass placed at a point in a gravitational field is called the **gravitational field strength** or **intensity of gravitational field** at that point. Thus,

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{m}}$$

SI unit of **E** is N/kg.

Note *The test mass to find field strength at some point should be infinitesimally small otherwise it will produce its own field and will disturb the original field.*

In Article 13.3, we have seen that acceleration due to gravity **g** is also $\frac{\mathbf{F}}{m}$. Hence, for the earth's gravitational field **g** and **E** are same. The E *versus* r (the distance from the centre of earth) graph are same as that of g' *versus* r graph.

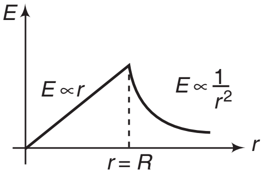


Fig. 13.22

Field due to a Point Mass

Suppose, a point mass M is placed at point O. We want to find the intensity of gravitational field **E** at a point P, a distance r from O. Magnitude of force **F** acting on a particle of mass m placed at P is,

$$F = \frac{GMm}{r^2}$$

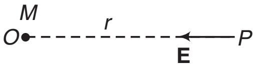


Fig. 13.23

$$\therefore \quad E = \frac{F}{m} \quad \text{or} \quad \boxed{E = \frac{GM}{r^2}}$$

The direction of the force **F** and hence of **E** is from P to O as shown in Fig. 13.23.

Gravitational Field due to a Uniform Solid Sphere

Field at an External Point

A uniform sphere may be treated as a single particle of same mass placed at its centre for calculating the gravitational field at an external point. Thus,

$$\boxed{E(r)=\frac{GM}{r^2}} \quad \text{for} \quad r \geq R \quad \text{or} \quad E(r) \propto \frac{1}{r^2}$$

Here, r is the distance of the point from the centre of the sphere and R the radius of sphere.

Field at an Internal Point

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre itself, it is zero and at surface it is $\frac{GM}{R^2}$, where R is the radius of the sphere. Thus,

$$\boxed{E(r)=\frac{GM}{R^3}r} \quad \text{for} \quad r \leq R \quad \text{or} \quad E(r) \propto r$$

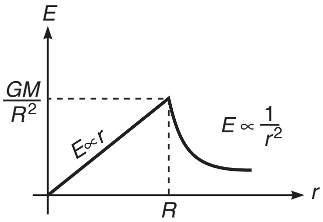


Fig. 13.24

Hence, E *versus* r graph is as shown in Fig. 13.24.

Field due to a Uniform Spherical Shell

At an External Point

For an external point the shell may be treated as a single particle of same mass placed at its centre. Thus, at an external point the gravitational field is given by,

$$\boxed{E(r)=\frac{GM}{r^2}} \quad \text{for} \quad r \geq R$$

at $\quad r = R \quad$ (the surface of shell)

$$E=\frac{GM}{R^2} \quad \text{and otherwise} \quad E \propto \frac{1}{r^2}$$

At an Internal Point

The field inside a uniform spherical shell is zero.

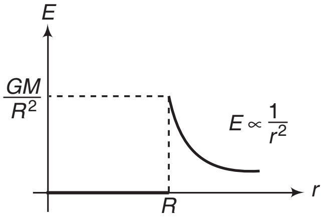


Fig. 13.25

Thus, E *versus* r graph is as shown in Fig. 13.25.

Field due to a Uniform Circular Ring at some Point on its Axis

Field strength at a point P on the axis of a circular ring of radius R and mass M is given by,

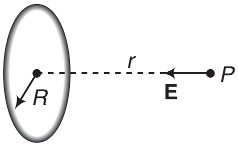


Fig. 13.26

$$\boxed{E(r)=\frac{GMr}{(R^2+r^2)^{3/2}}}$$

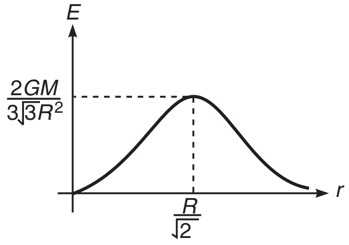


Fig. 13.27

This is directed towards the centre of the ring. It is zero at the centre of the ring and maximum at $r=\frac{R}{\sqrt{2}}$ (can be obtained by putting $\frac{dE}{dr}=0$). Thus, E-r graph is as shown in Fig. 13.27.

The maximum value is $E_{\max}=\frac{2GM}{3\sqrt{3}R^2}$.

13.5 Gravitational Potential

As we have discussed earlier also, this is a scalar quantity related to work done by gravitational force in moving a unit test mass in gravitational field from one point to another point.

Gravitational potential at any point is defined as the negative of work done by gravitational force in moving a unit test mass from infinity (where potential is assumed to be zero) to that point. Thus, potential at P is

$$\boxed{V_P=\frac{-W_{\infty\rightarrow P}}{m}}\qquad\text{(by gravitational force)}$$

It SI unit is J/kg.

Potential due to a Point Mass

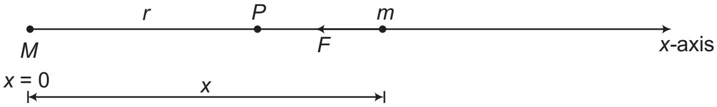


Fig. 13.28

Suppose a point mass 'M' is placed at origin ($x=0$). We wish to find gravitational potential at P, at a distance 'r' from M.

First of all we will calculate the work done by gravitational force in moving a test mass 'm' from infinity to P. Gravitational force on 'm' when it is at a distance 'x' from M is

$$F=-\frac{GMm}{x^2}$$

Here, negative sign implies that this force is towards '*M*' or towards negative *x*-direction. This is a variable force (a function of *x*). Therefore, work done is

$$W = \int_{\infty}^{r} F dx = \int_{\infty}^{r} \left(-\frac{GMm}{x^2} \right) dx \ = \frac{GMm}{r}$$

Now, from the definition of potential,

$$V = -\frac{W}{m} = -\frac{GM}{r}$$

$$\therefore \qquad \boxed{V = -\frac{GM}{r}}$$

Potential due to a Uniform Solid Sphere

Potential at some External Point

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of same mass placed at its centre. Thus,

$$\boxed{V(r) = -\frac{GM}{r}} \qquad r \geq R$$

At the surface, $r = R$ and $V = -\frac{GM}{R}$

Potential at some Internal Point

At some internal point, potential at a distance *r* from the centre is given by,

$$\boxed{V(r) = -\frac{GM}{R^3}(1.5R^2 - 0.5r^2)} \qquad r \leq R$$

At $r = R$, $V = -\frac{GM}{R}$

while at $r = 0$, $V = -\frac{1.5\,GM}{R}$

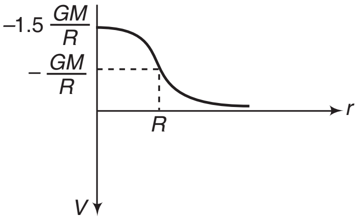


Fig. 13.29

i.e. at the centre of the sphere the potential is 1.5 times the potential at the surface. The variation of *V versus r* graph is as shown in Fig. 13.29.

Potential due to a Uniform Thin Spherical Shell

Potential at an External Point

To calculate the potential at an external point, a uniform spherical shell may be treated as a point mass of same magnitude at its centre. Thus, potential at a distance *r* is given by,

$$\boxed{V(r) = -\frac{GM}{r}} \qquad r \geq R$$

at $r = R$, $V = -\frac{GM}{R}$

Potential at an Internal Point

The potential due to a uniform spherical shell is constant at any point inside the shell and this is equal to $-\frac{GM}{R}$. Thus, V-r graph for a spherical shell is as shown in Fig. 13.30.

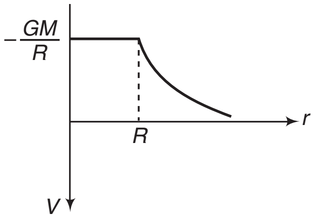


Fig. 13.30

Potential due to a Uniform Ring at some Point on its Axis

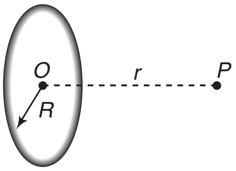


Fig. 13.31

The gravitational potential at a distance r from the centre on the axis of a ring of mass M and radius R is given by,

$$\boxed{V(r) = -\frac{GM}{\sqrt{R^2 + r^2}}} \qquad 0 \le r \le \infty$$

At $r = 0$, $V = -\frac{GM}{R}$, i.e. at the centre of the ring gravitational potential is $-\frac{GM}{R}$.

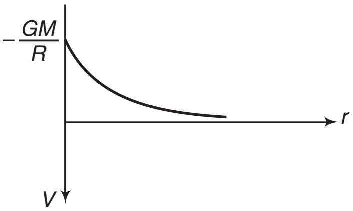


Fig. 13.32

The V-r graph is as shown in Fig. 13.32.

Example 13.11 *Three point masses 'm' each are kept at three vertices of a square of side 'a' as shown in figure. Find gravitational potential and field strength at point O.*

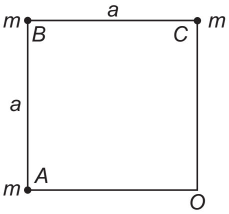


Fig. 13.33

Solution Gravitational potential is a scalar quantity.

$\therefore$ V_O = scalar sum of potentials due to point masses at A, B and C.

$$= -\frac{Gm}{a} - \frac{Gm}{\sqrt{2}a} - \frac{Gm}{a}$$

$$= -\frac{Gm}{a}\left[2 + \frac{1}{\sqrt{2}}\right] \qquad \textbf{Ans.}$$

Gravitational field strength is a vector quantity. So, net field strength at O is the vector sum of three field strengths produced due to the three point masses at A, B and C.

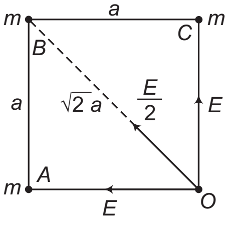


Fig. 13.34

Field strength due to A = field strength due to $C = \frac{Gm}{a^2}$ (towards the point masses)

$= E$ (say)

Field strength due to $B = \frac{Gm}{(\sqrt{2}a)^2} = \frac{Gm}{2a^2} = \frac{E}{2}$

Resultant of E and E is $\sqrt{2}E$ towards B.

$\frac{E}{2}$ is also towards B.

$$\therefore \qquad E_{\text{net}} = \sqrt{2}E + \frac{E}{2} = \left(\sqrt{2} + \frac{1}{2}\right)E$$

$$= \left(\sqrt{2} + \frac{1}{2}\right)\frac{Gm}{a^2} \qquad \text{(towards } B\text{)} \qquad \textbf{Ans.}$$

Example 13.12 *Four point masses each of mass ‘m’ are placed at four vertices A, B, C and D of a regular hexagon of side ‘a’ as shown in figure. Find gravitational potential and field strength at the centre O of the hexagon.*

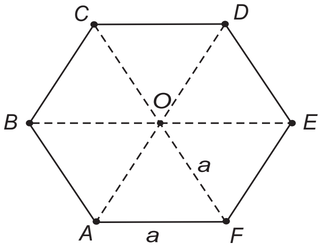


Fig. 13.35

Solution Gravitational potential is a scalar quantity. Therefore,

V_O = scalar sum of gravitational potentials produced by four point masses at A, B, C and D.

$$= -\frac{Gm}{a} - \frac{Gm}{a} - \frac{Gm}{a} - \frac{Gm}{a}$$

$$= -\frac{4Gm}{a}$$ **Ans.**

Gravitational field strength is a vector quantity. So, it is a vector sum of four vectors of equal magnitudes.

$$E = \frac{Gm}{a^2} = E_A = E_B = E_C = E_D$$

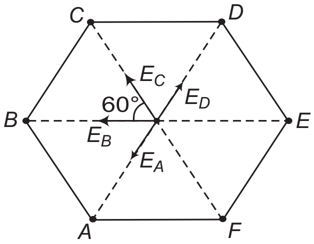


Fig. 13.36

E_A and E_D are cancelled. So, net field strength is a vector sum of E_B and E_C at angle 60°.

∴ $$E_{\text{net}} = \sqrt{E^2 + E^2 + 2(E)(E)\cos 60°} = \sqrt{3}\,E$$

$$= \frac{\sqrt{3}\,Gm}{a^2}$$ **Ans.**

Note *If* $E_1 = E_2 = E$ and θ *be the angle between them then,* $E_{\text{net}} = 2E\cos\left(\frac{\theta}{2}\right)$

This net field strength is along the bisector line of $\angle COB$, away from O, between E_C and E_B.

Example 13.13 *At what distance 'd' from the surface of a solid sphere of radius 'R',*

(a) potential is same as at a distance $\frac{R}{2}$ *from the centre ?*

(b) field strength is same as at a distance $\frac{R}{4}$ *from centre.*

Solution (a) Given, $V_{\text{outside}} = V_{\text{inside}}$

No such point will exist. Because potential at centre is $\frac{-1.5GM}{R}$. Potential at surface is $-\frac{GM}{R}$ and potential at infinity is zero. From centre to surface potential varies between $-\frac{1.5GM}{R}$ and $\frac{-GM}{R}$. From surface to infinity potential varies between $\frac{-GM}{R}$ and zero.

(b) Given, $$E_{\text{inside}} = E_{\text{outside}}$$

$$\therefore \qquad \frac{GM}{R^3}(r_1) = \frac{GM}{r_2^2}$$

Here, r_1 and r_2 are the distances from centre.

$$\therefore \qquad \frac{GM}{R^3}\left(\frac{R}{4}\right) = \frac{GM}{(R+d)^2}$$

Solving this equation, we get

$$d = R$$ **Ans.**

INTRODUCTORY EXERCISE 13.3

1. Two points masses '*m*' each are kept at the two vertices of an equilateral triangle of side '*a*' as shown in figure.

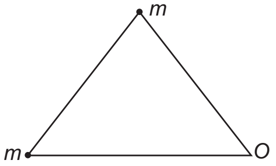


Fig. 13.37

Find gravitational potential and magnitude of field strength at *O*.

2. Five point masses '*m*' each are kept at five vertices of a regular pentagon. Distance of centre of pentagon from any one of the vertices is '*a*'. Find gravitational potential and field strength at centre.
3. In the above problem, if any one point mass is removed then what is gravitational potential and magnitude of field strength at centre?
4. A particle of mass *m* is placed at the centre of a uniform spherical shell of same mass and radius *R*. Find the gravitational potential at a distance $\frac{R}{2}$ from the centre.
5. A particle of mass 20 *g* experiences a gravitational force of 4.0 N along positive *x*-direction. Find the gravitational field at that point.

13.6 Relation between Gravitational Field and Potential

In gravitational field, field strength **E** and potential *V* are different at different points. So, they are functions of position. In cartesian co-ordinate system position of a point can be represented by three variable co-ordinates *x*, *y* and *z*. So, **E** and *V* are functions of three variables x, y and z. But in physics, we normally try to keep least number of variables. So, in one dimension (say along *x*-axis) position of a point can be represented by a single variable co-ordinates *x*.

So, *E* and *V* are functions of only one variable *x*. Similarly, in case of point mass or spherical bodies this single variable is *r*. Where *r* is the distance from point mass or distance from the centre of the spherical body.

Further, **E** and *V* are not two independent functions, but they are related to each other either by differentiation or integration.

Conversion of *V* Function into E Function

To convert *V* function into **E** function, differentiation is required. If there are more than one variables then partial differentiation is done and if there is only one variable then, direct differentiation is done.

More than one variables In this case,

$$\mathbf{E} = -\text{gradient } V = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$$

or $$\boxed{\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]}$$

Here, $\frac{\partial V}{\partial x}$ is called partial differentiation of *V* with respect to *x*. In this differentiation we differentiate *V* with respect to *x*, assuming other two variables *y* and *z* to be constant. Similarly $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$.

Only one variable In this case,

$$E = -\frac{dV}{dx} = -\frac{dV}{dr} \text{ or } E = (-\text{ slope of } V\text{-}x) \text{ or } (-\text{ Slope of } V\text{-}r \text{ graph})$$

Conversion of E Function into *V* Function

In this case, integration will be required. In integration, we will be required value of *V* at some given position (also called limit) to get complete *V* function. Otherwise an unknown constant of integration remains in the expression. One known limit is $V = 0$ at infinity. This limit may not be given in the question.

More than one variables In this case, $\boxed{dV = -\mathbf{E}\cdot d\mathbf{r}}$

or $$\int_a^b dV = -\int_a^b \mathbf{E}\cdot d\mathbf{r}$$

⇒ $$V_b - V_a = -\int_a^b \mathbf{E}\cdot d\mathbf{r}$$

Here, **E** will be given in the question and $d\mathbf{r}$ is a standard vector given by

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

Only one Variable

$$\boxed{\int dV = -\int E\,dr}$$

or $$\int_a^b dV = -\int_a^b E\,dr$$

⇒ $$V_b - V_a = -\int_a^b E\,dr$$

Here, *E* as a function of *r* will be given in the question. We may also write the above equation as

$$\int dV = -\int E\,dx$$

where, *E* is a function of *x*.

Example 13.14 *Gravitational potential in x-y plane varies with x and y coordinates as*

$$V = x^2 y + 2xy$$

Find gravitational field strength **E**.

Solution Gravitational field strength is given by

$$\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial x}\hat{\mathbf{j}}\right]$$

For the given potential function

$$\frac{\partial V}{\partial x} = (2xy + 2y)$$

and $$\frac{\partial V}{\partial y} = x^2 + 2x$$

$\therefore$ $$\mathbf{E} = -[(2xy + 2y)\hat{\mathbf{i}} + (x^2 + 2x)\hat{\mathbf{j}}]$$ **Ans.**

Example 13.15 *Gravitational potential at a distance 'r' from a point mass 'm' is*

$$V = -\frac{Gm}{r}$$

Find gravitational field strength at that point.

Solution $E = -\frac{dV}{dr}$

$$= -\frac{d}{dr}\left(-\frac{Gm}{r}\right) = -\frac{Gm}{r^2} \quad \text{or} \quad |E| = \frac{Gm}{r^2}$$

Negative sign implies that direction of **E** is towards the point mass.

Example 13.16 *Gravitational potential varies along x-axis as shown in figure.*

(a) Plot E versus x graph corresponding to given V-x graph.

(b) A mass of 2 kg is kept at x = 3 m. Find gravitational force on it.

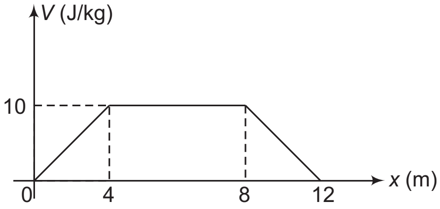


Fig. 13.38

Solution (a) $E = -\frac{dV}{dx} = -$ slope of V-x graph.

From $x = 0$ to $x = 4$ m

$$\text{Slope} = +\frac{10}{4} = +2.5 \text{ N/kg}$$

$\therefore$ $$E = -2.5 \text{ N/kg}$$

From $x = 4$ m to $x = 8$ m

$$\text{Slope} = 0$$

$\therefore$ $$E = 0$$

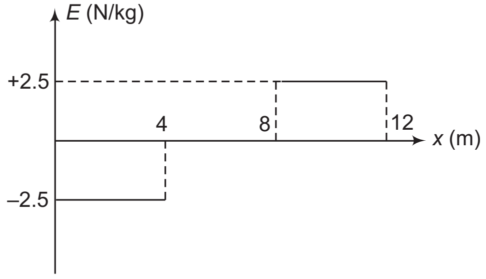


Fig. 13.39

From $x=8$ m to $x=12$ m

$$\text{Slope} = -\frac{10}{4} = -2.5 \text{ N/kg}$$

$\therefore \quad E = +2.5$ N/kg

Therefore, E-x graph is as shown in figure.

(b) At $x = 3$ m, $E = -2.5$ N/kg

$\therefore$ Gravitational force

$$F = mE \qquad \left(\text{as } E = \frac{F}{m}\right)$$

$$= (2)(-2.5) = -5 \text{ N}$$ **Ans.**

Here, negative sign implies that this force is acting towards negative x-direction.

Example 13.17 *Gravitational field in x-y plane is given as*

$$\mathbf{E} = (2x\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}}) \, N/kg$$

Find difference in gravitation potential between two points A and B, where co-ordinates of A and B are (2*m*, 4*m*) *and* (6*m*, 0).

Solution We know that, $dV = -\mathbf{E} \cdot d\mathbf{r}$ …(i)

Here $d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$

$\therefore \quad \mathbf{E} \cdot d\mathbf{r} = (2x\,dx + 3y^2\,dy)$

Now, Eq. (i) can be written as

$$\int_B^A dV = -\int_B^A (2x\,dx + 3y^2\,dy)$$

or

$$V_A - V_B = -[x^2 + y^3]_{(6\text{m},\ 0)}^{(2\text{m},\ 4\text{m})}$$

$$= -[\{(2)^2 + (4)^3\} - (6)^2 + (0)^3]$$

$$= -32 \text{ J/kg}$$ **Ans.**

INTRODUCTORY EXERCISE 13.4

1. The gravitational potential due to a mass distribution is $V = 3x^2y + y^3z$. Find the gravitational field.
2. Gravitational potential at $x = 2$ m is decreasing at a rate of 10 J/kg-m along the positive x-direction. It implies that the magnitude of gravitational field at $x = 2$ m is also 10 N/Kg. Is this statement true or false?
3. The gravitational potential in a region is given by, $V = 20(x + y)$ J/kg. Find the magnitude of the gravitational force on a particle of mass 0.5 kg placed at the origin.
4. The gravitational field in a region is given by $\mathbf{E} = (2\hat{i} + 3\hat{j})$ N/kg.

   Find the work done by the gravitational field when a particle of mass 1 kg is moved on the line $3y + 2x = 5$ from (1m, 1m) to (−2m, 3m).

13.7 Gravitational Potential Energy

The concept of potential energy has already been discussed in the chapter of work, energy and power. The word potential energy is defined only for a conservative force field. There we have discussed that the change in potential energy (dU) of a system corresponding to a conservative force is given by

$$dU = -\mathbf{F}\cdot d\mathbf{r} \quad \text{or} \quad \int_i^f dU = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r} \quad \text{or} \quad U_f - U_i = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, i.e. if we take $r_i = \infty$ (infinite) and $U_i = 0$, then we can write

$$\boxed{U = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = -W}$$

or potential energy of a body or system is negative of work done by the conservative forces in bringing it from infinity to the present position.

Gravitational Potential Energy of a two Particle System

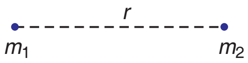


Fig. 13.40

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by,

$$\boxed{U = -\frac{Gm_1m_2}{r}}$$

This is actually the negative of work done in bringing those masses from infinity to a distance r by the gravitational forces between them.

Gravitational Potential Energy for a System of Particles

The gravitational potential energy for a system of particles (say m_1, m_2, m_3 and m_4) is given by

$$\boxed{U = -G\left[\frac{m_4m_3}{r_{43}} + \frac{m_4m_2}{r_{42}} + \frac{m_4m_1}{r_{41}} + \frac{m_3m_2}{r_{32}} + \frac{m_3m_1}{r_{31}} + \frac{m_2m_1}{r_{21}}\right]}$$

Thus, for a n particle system there are $\frac{n(n-1)}{2}$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

Gravitational Potential Energy of a Body on Earth's Surface

The gravitational potential energy of mass m in the gravitational field of mass M at a distance r from it is,

$$U = -\frac{GMm}{r}$$

The earth behaves for all external points as if its mass M were concentrated at its centre. Therefore, a mass m near earth's surface may be considered at a distance R (the radius of earth) from M. Thus, the potential energy of the system will be

$$U = -\frac{GMm}{R}$$

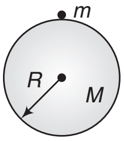


Fig. 13.41

Difference in Potential Energy (ΔU)

Let us find the difference in potential energy in two positions shown in figure. The potential energy when the mass is on the surface of earth (at B) is,

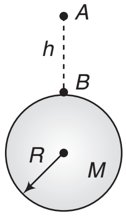


Fig. 13.42

$$U_B = -\frac{GMm}{R}$$

and potential energy when the mass m is at height h above the surface of earth (at A) is,

$$U_A = -\frac{GMm}{R+h} \qquad (U_A > U_B)$$

$$\therefore \qquad \Delta U = U_A - U_B$$

$$= -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$= \frac{GMmh}{R(R+h)}$$

$$= \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} = \frac{mgh}{\left(1+\frac{h}{R}\right)} \qquad \left(\frac{GM}{R^2} = g\right)$$

$$\therefore \qquad \boxed{\Delta U = \frac{mgh}{1+\frac{h}{R}}}$$

For $$h << R, \quad \Delta U \approx mgh$$

Thus, mgh is the difference in potential energy (not the absolute potential energy), for $h << R$.

Extra Points to Remember

- **Maximum height attained by a particle**

  Suppose a particle of mass m is projected vertically upwards with a speed v and we want to find the maximum height h attained by the particle. Then we can use conservation of mechanical energy, i.e. Decrease in kinetic energy = increase in gravitational potential energy of particle.

  $$\therefore \qquad \frac{1}{2}mv^2 = \Delta U$$

  or $$\frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}$$

  Solving this, we get $$\boxed{h = \frac{v^2}{2g - \frac{v^2}{R}}}$$

  From this, we can see that $h \approx \frac{v^2}{2g}$ if v is small.

Example 13.18 *Three masses of 1 kg, 2 kg and 3 kg are placed at the vertices of an equilateral triangle of side 1 m. Find the gravitational potential energy of this system.*
Take $G = 6.67 \times 10^{-11}\ N\text{-}m^2/kg^2$

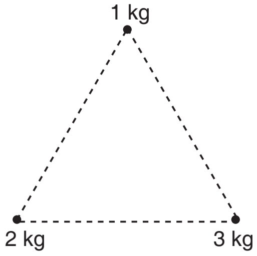


Fig. 13.43

Solution $U = -G\left(\frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right)$

Here, $r_{32} = r_{31} = r_{21} = 1.0$ m, $m_1 = 1$ kg,

$m_2 = 2$ kg

and $m_3 = 3$ kg

Substituting the values, we get

$$U = -(6.67 \times 10^{-11})\left(\frac{3\times 2}{1} + \frac{3\times 1}{1} + \frac{2\times 1}{1}\right)$$

or $U = -7.337 \times 10^{-10}$ J **Ans.**

Example 13.19 *Eight particles of mass 'm' each are placed at the vertices of a cube of side 'a'. Find gravitational potential energy of this system.*

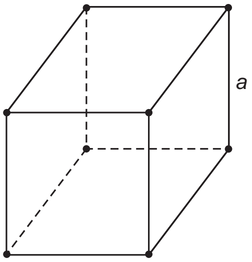


Fig. 13.44

Solution Total particles are $n = 8$

$\therefore$ Total number of pairs are $\frac{n(n-1)}{2}$ or 28.

In 12 pairs, distance between the particles is 'a'.

In 12 pairs, distance is $\sqrt{2}a$ and in remaining 4 pairs distance is $\sqrt{3}a$.

$$\therefore \quad U = 12\left(-\frac{Gmm}{a}\right) + 12\left(\frac{-Gmm}{\sqrt{2}a}\right) + 4\left(\frac{-Gmm}{\sqrt{3}a}\right)$$

$$= -\frac{Gm^2}{a}\left(12 + 6\sqrt{2} + \frac{4}{\sqrt{3}}\right)$$ **Ans.**

Example 13.20 *A particle of mass 'm' is raised from the surface of earth to a height h =2R. Find work done by some external agent in this process. Here, R is the radius of earth and g the acceleration due to gravity on earth's surface.*

Solution No information is given about the change in kinetic energy of the particle. So, assuming change in kinetic energy to be zero.

$$\text{Work done by external agent} = \text{change in potential energy}$$

$$= \Delta U = \frac{mgh}{1+\dfrac{h}{R}}$$

Substituting $h = 2R$, we get, $\quad$ Work done $= \frac{2}{3} mgR$ **Ans.**

Example 13.21 *A particle is projected from the surface of the earth with an initial speed of 4.0 km/s. Find the maximum height attained by the particle. Radius of earth = 6400 km and g = 9.8 m/s².*

Solution The maximum height attained by the particle is,

$$h = \frac{v^2}{2g - \dfrac{v^2}{R}}$$

Substituting the values, we have

$$h = \frac{(4.0 \times 10^3)^2}{2 \times 9.8 - \dfrac{(4.0 \times 10^3)^2}{6.4 \times 10^6}}$$

$$= 9.35 \times 10^5 \text{ m} \quad \text{or} \quad h \approx 935 \text{ km}$$ **Ans.**

INTRODUCTORY EXERCISE 13.5

1. Two particles of masses 20 kg and 10 kg are initially at a distance of 1.0 m. Find the speeds of the particles when the separation between them decreases to 0.5 m, if only gravitational forces are acting.
2. Four particles each of mass m are kept at the four vertices of a square of side 'a'. Find gravitational potential energy of this system.
3. A particle of mass 'm' is raised to a height $h = R$ from the surface of earth. Find increase in potential energy. R = radius of earth. g = acceleration due to gravity on the surface of earth.
4. Show that if a body be projected vertically upward from the surface of the earth so as to reach a height nR above the surface
   (i) the increase in its potential energy is $\left(\frac{n}{n+1}\right) mgR$,
   (ii) the velocity with which it must be projected is $\sqrt{\frac{2ngR}{n+1}}$, where R is the radius of the earth and m the mass of body.
5. A projectile is fired vertically from the earth's surface with an initial speed of 10 km/s. Neglecting air drag, how high above the surface of earth will it go?
6. A particle is fired vertically upwards from earth's surface and it goes upto a maximum height of 6400 km. Find the initial speed of the particle.

13.8 Binding Energy

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.

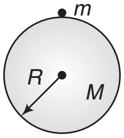


Fig. 13.45

Suppose the mass m is placed on the surface of earth. The radius of the earth is R and its mass is M. Then, the kinetic energy of the particle $K = 0$ and the potential energy is $U = -\frac{GMm}{R}$.

Therefore, the total mechanical energy is,

$$E = K + U = 0 - \frac{GMm}{R}$$

or
$$E = -\frac{GMm}{R}$$

$\therefore$
$$\text{Binding energy} = |E| = \frac{GMm}{R}$$

It is due to this energy, the particle is attached with the earth. If minimum this much energy is given to the particle in any form (normally kinetic) the particle no longer remains attached to the earth. It goes out of the gravitational field of earth.

Escape Velocity

As we discussed above, the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if v_e is the escape velocity of the particle, then

$$\frac{1}{2} m v_e^2 = \frac{GMm}{R} \quad \text{or} \quad v_e = \sqrt{\frac{2GM}{R}}$$

or
$$\boxed{v_e = \sqrt{2gR}} \quad \text{as} \quad g = \frac{GM}{R^2}$$

Substituting the value of g (9.8 m/s^2) and R $(6.4 \times 10^6 \text{ m})$, we get

$$\boxed{v_e \approx 11.2 \text{ km/s}}$$

Thus, the minimum velocity needed to take a particle to infinity from the earth is called the escape velocity. On the surface of earth its value is 11.2 km/s.

Extra Points to Remember

- The value of escape velocity is 11.2 km/s from the surface of earth. From some height above the surface of earth this value will be less than 11.2 km/s.
- Escape velocity is independent of the direction in which it is projected. In the figure shown, body is given 11.2 km/s along three different paths. In each case, it will escape to infinity, but following different paths. For example, along path-1 it will follow a straight line.

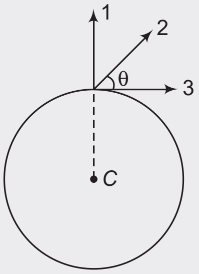


Fig. 13.46

- If velocity of a particle is v_e, then its total mechanical energy is zero. As the particle moves towards infinity its kinetic energy decreases and potential energy increases, but total mechanical energy remains constant. At any point

$$E = K + U = 0 \quad \Rightarrow \quad K = -U$$

  For example, if $K = 100\,\text{J}$ on the surface of earth then $U = -100\,\text{J}$. At some height suppose K becomes 60 J, then U will become $-60\,\text{J}$. At infinity $K = 0$. So U is also zero. Hence, speed at infinity will be zero.
- If velocity of the particle is less than v_e then total mechanical energy is negative and it does not escape to infinity.
- If velocity of the particle is more than v_e then total mechanical energy is positive. Even at infinity some kinetic energy and speed are left in the particle. Although its potential energy becomes zero.

Example 13.22 *Calculate the escape velocity from the surface of moon. The mass of the moon is* 7.4×10^{22} *kg and radius* $= 1.74 \times 10^6$ *m.*

Solution Escape velocity from the surface of moon is $v_e = \sqrt{\dfrac{2GM_m}{R_m}}$

Substituting the values, we have

$$v_e = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^6}}$$

$$= 2.4 \times 10^3 \text{ m/s} = 2.4 \text{ km/s}$$ **Ans.**

Example 13.23 *Kinetic energy of a particle on the surface is* E_0 *and potential energy is* $-\dfrac{E_0}{2}$.

(a) Will the particle escape to infinity ?

(b) At some height its kinetic energy becomes $0.6\,E_0$. *What is potential energy at this height ?*

(c) If the particle escapes to infinity, what is kinetic energy of the particle at infinity?

Solution (a) Total mechanical energy

$$E = K + U = E_0 - \frac{E_0}{2} = 0.5\,E_0$$

Since, E is positive, particle will escape to infinity.

(b) Potential energy $U = E - K = 0.5\,E_0 - 0.6\,E_0 = -0.1\,E_0$ **Ans.**

(c) At infinity, $U = 0$

$\therefore$ $$K = E = 0.5\,E_0$$ **Ans.**

INTRODUCTORY EXERCISE 13.6

1. What is the kinetic energy needed to project a body of mass m from the surface of the earth to infinity? Radius of earth is R and acceleration due to gravity on earth's surface is g.
2. Mass and radius of a planet are two times the values of earth. What is the value of escape velocity from the surface of this planet ?
3. Kinetic energy of a particle on the surface of earth is E_0 and the potential energy is $-2E_0$.
   (a) Will the particle escape to infinity ?
   (b) What is the value of potential energy where speed of the particle becomes zero ?

13.9 Motion of Satellites

Just as the planets revolve around the sun, in the same way few celestial bodies revolve around these planets. These bodies are called 'Satellites'. For example moon is a satellite of earth. Artificial satellites are launched from the earth. Such satellites are used for telecommunication, weather forecast and other applications. The path of these satellites are elliptical with the centre of earth at a focus. However the difference in major and minor axes is so small that they can be treated as nearly circular for not too sophisticated calculations. Let us derive certain characteristics of the motion of satellites by assuming the orbit to be perfectly circular.

Orbital Speed

The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

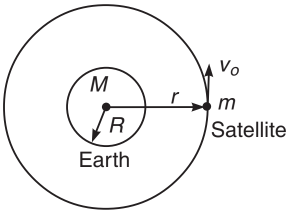


Fig. 13.47

$$\frac{mv_0^2}{r} = \frac{GMm}{r^2}$$

$$\therefore \quad \boxed{v_0 = \sqrt{\frac{GM}{r}}} \quad \text{or} \quad v_0 \propto \frac{1}{\sqrt{r}}$$

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface ($r \approx R$) is,

$$\boxed{v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}}$$

Substituting $v_e = 11.2$ km/s $\Rightarrow$ $\boxed{v_0 = 7.9 \text{ km/s}}$

Period of Revolution

The period of revolution (T) is given by

$$T=\frac{2\pi r}{v_o} \quad \text{or} \quad T=\frac{2\pi r}{\sqrt{\dfrac{GM}{r}}}$$

or $$T=2\pi\sqrt{\frac{r^3}{GM}} \quad \text{or} \quad \boxed{T=2\pi\sqrt{\frac{r^3}{gR^2}}} \qquad (\text{as } GM=gR^2)$$

Energy of Satellite

The potential energy of the system is

$$\boxed{U=-\frac{GMm}{r}}$$

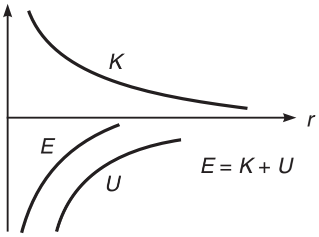


Fig. 13.48

The kinetic energy of the satellite is,

$$K=\frac{1}{2}mv_0^2=\frac{1}{2}m\left(\frac{GM}{r}\right)$$

or $$\boxed{K=\frac{1}{2}\frac{GMm}{r}}$$

The total energy is, $$E=K+U=-\frac{GMm}{2r}$$

or $$\boxed{E=-\frac{GMm}{2r}}$$

This energy is constant and negative, i.e. the system is closed. The farther the satellite from the earth the greater its total energy.

✓ Extra Points to Remember

- $T=2\pi\sqrt{\dfrac{r^3}{gR^2}} \Rightarrow T\propto r^{3/2}$ or $T^2\propto r^3$ (which is also the Kepler's third law)
- Time period of a satellite very close to earth's surface $(r\approx R)$ is,

  $$T=2\pi\sqrt{\frac{R}{g}}$$

  Substituting the values, we get $T\approx 84.6$ min
- Suppose the height of a satellite is such that the time period of the satellite is 24 h and it moves in the same sense as the earth. The satellite will always be overhead a particular place on the equator. As seen from the earth, this satellite will appear to be stationary. Such a satellite is called a geostationary satellite. Putting $T=24$ h in the expression of T, the radius of geostationary satellite comes out to be $r=4.2\times10^4$ km. The height above the surface of earth is about 36000 km.
- The plane of orbit of a satellite always passes through the centre of the earth as force is always towards centre of earth. In case of geostationary satellite it is an equatorial plane (passing through equator)

Example 13.24 *As orbital radius r of a satellite is increased, state which of the following quantities will increase and which will decrease ?*

(i) Orbital speed

(ii) Time period

(iii) Frequency

(iv) Angular speed

(v) Kinetic energy

(vi) Potential energy

(vii) Total mechanical energy

Solution (i) Orbital speed $v_o = \sqrt{\frac{GM}{r}}$ or $v_o \propto \frac{1}{\sqrt{r}}$.

Therefore, orbital speed will decrease.

(ii) $T = 2\pi\sqrt{\frac{r^3}{gR^2}}$ or $T \propto r^{3/2}$

Therefore, time period will increase.

(iii) Frequency, $f = \frac{1}{T}$

Time period is increasing. So, frequency will decrease.

(iv) Angular speed $\omega = \frac{2\pi}{T}$ or $\omega \propto \frac{1}{T}$

Time period is increasing. Hence, angular speed will decrease.

(v) Kinetic energy, $K = \frac{GMm}{2r}$ or $K \propto \frac{1}{r}$.

Therefore, kinetic energy will decrease.

(vi) Potential energy $U = -\frac{GMm}{r}$ or $U \propto -\frac{1}{r}$

Therefore, potential energy will increase.

(vii) Total mechanical energy, $E = -\frac{GMm}{2r}$ or $E \propto -\frac{1}{r}$

So, mechanical energy will also increase.

Example 13.25 *A geostationary satellite is orbiting the earth at a height of* 6R *above the surface of the earth where R is the radius of earth. The time period of another satellite at a distance of* 3.5 *R from the centre of the earth is .... hours.* **(JEE 1987)**

Solution $T \propto r^{\frac{3}{2}}$

$$\therefore \qquad \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} \quad \text{or} \quad T_2 = \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} T_1$$

$$T_2 = \left(\frac{3.5R}{7R}\right)^{\frac{3}{2}}(24)\,\text{h} = 8.48\,\text{h} \quad (T_1 = 24 \text{ h for geostationary satellite})$$

Example 13.26 *A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull. Radius of earth* $= 6400\ km$, $g = 9.8\ m/s^2$.

Solution The speed of the spaceship in a circular orbit close to the earth's surface is given by,

$$v_o = \sqrt{gR}$$

and escape velocity is given by, $v_e = \sqrt{2gR}$

∴ Additional velocity required to escape

$$v_e - v_o = \sqrt{2gR} - \sqrt{gR} = (\sqrt{2} - 1)\sqrt{gR}$$

Substituting the values of g and R, we get

$$v_e - v_o = 3.278 \times 10^3 \text{ m/s}$$ **Ans.**

Example 13.27 *What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of* $2R$? **(JEE 2013, Main)**

(a) $\dfrac{5GmM}{6R}$

(b) $\dfrac{2GmM}{3R}$

(c) $\dfrac{GmM}{2R}$

(d) $\dfrac{GmM}{3R}$

Solution E = Energy of satellite – energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $r = R + 2R = 3R$

Substituting the values in above equation we get, $E = \dfrac{5GMm}{6R}$

The correct answer is (a)

.Example 13.28 *An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.* **(1990, 8M)**

(a) Determine the height of the satellite above the earth's surface.

(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth.

Solution (a) Orbital speed of a satellite at distance r from centre of earth,

$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}} \qquad \text{...(i)}$$

Given, $$v_0 = \frac{v_e}{2} = \frac{\sqrt{2GM/R}}{2} = \sqrt{\frac{GM}{2R}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$h = R = 6400 \text{ km}$$

(b) Decrease in potential energy = increase in kinetic energy

or $$\frac{1}{2}mv^2 = \Delta U$$

$$\therefore \qquad v = \sqrt{\frac{2(\Delta U)}{m}} = \sqrt{\frac{2\left(\frac{mgh}{1 + h/R}\right)}{m}} = \sqrt{gR} \qquad (h = R)$$

$$= \sqrt{9.8 \times 6400 \times 10^3}$$

$$= 7919\,\text{m/s} = 7.9\,\text{km/s}$$ **Ans.**

INTRODUCTORY EXERCISE 13.7

1. Is it possible to have a geostationary satellite which always remains over New Delhi ?
2. Two satellites *A* and *B* revolve around a planet in two coplanar circular orbits in the same sense with radii 10^4 km and 2×10^4 km respectively. Time period of *A* is 28 hours. What is time period of another satellite?
3. Two satellites *A* and *B* of the same mass are orbiting the earth at altitudes *R* and 3*R* respectively, where *R* is the radius of the earth. Taking their orbits to be circular obtain the ratios of their kinetic and potential energies.
4. A satellite of mass 1000 kg is supposed to orbit the earth at a height of 2000 km above the earth's surface. Find (a) its speed in the orbit, (b) its kinetic energy, (c) the potential energy of the earth-satellite system and (d) its time period. Mass of the earth $= 6 \times 10^{24}$ kg.
5. A sky lab of mass 2×10^3 kg is first launched from the surface of earth in a circular orbit of radius 2*R* and then it is shifted from this circular orbit to another circular orbit of radius 3*R*. Calculate the energy required
   (a) to place the lab in the first orbit,
   (b) to shift the lab from first orbit to the second orbit. ($R = 6400$ km, $g = 10\,\text{m/s}^2$)

13.10 Kepler's Laws of Planetary Motion

Kepler discovered three empirical laws that accurately described the motion of the planets. The three laws may be stated as,

(i) Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse. This law is also known as the **law of elliptical orbits** and obviously gives the shape of the orbits of the planets round the sun.

(ii) The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, i.e. its areal velocity (or the area swept out by it per unit time) is constant. This is referred to as the **law of areas** and gives the relationship between the orbital speed of the planet and its distance from the sun.

(iii) The square of the planet's time period is proportional to the cube of the semi-major axis of its orbit. This is known as the **law of time period** and gives the relationship between the size of the orbit of a planet and its time of revolution.

Kepler did not know why the planets move in this way. Three generations later when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be derived. They are consequences of Newton's law of motion and the law of gravitation.

Let us first consider the elliptical orbits described in Kepler's first law. Figure shows the geometry of the ellipse. The longest dimension is the major axis with half length *a.* This half length is called the semi-major axis.

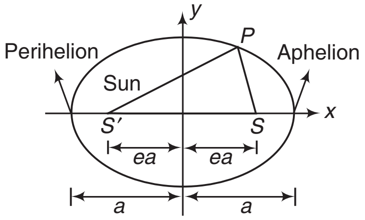


Fig. 13.49

$$SP + S'P = \text{constant}$$

Here, S and S' are the foci and P any point on the ellipse. The sun is at S and planet at P.

The distance of each focus from the centre of ellipse is *ea,* where *e* is the dimensionless number between 0 to 1 called the eccentricity. If $e = 0$, the ellipse is a circle. The actual orbits of the planets are nearly circular, their eccentricities range from 0.007 for Venus to 0.248 for Pluto. For earth $e = 0.017$. The point in the planet's orbit closest to the sun is the perihelion and the point most distant from the sun is aphelion.

Explanation of First Law

Newton was able to show that for a body acted on by an attractive force proportional to $\frac{1}{r^2}$, the only possible closed orbits are a circle or an ellipse. The open orbits must be parabolas or hyperbolas. He also showed that if total energy E is negative the orbit is an ellipse (or circle), if it is zero the orbit is a parabola and if E is positive the orbit is a hyperbola. Further, it was also shown that the orbits under the attractive force $F = \frac{K}{r^n}$, are stable for $n < 3$. Therefore, it follows that circular orbits will be stable for a force varying inversely as the distance or the square of the distance and will be unstable for the inverse cube (or a higher power) law.

Explanation of Second Law

$$PP' = v\,dt$$

$$P'M = (PP')\sin(180° - \theta) = PP'\sin\theta$$

$$= (v\sin\theta)\,dt$$

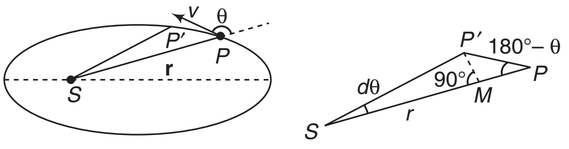


Fig. 13.50

Kepler's second law is shown in figure. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out in this time interval is,

$$\begin{aligned} dA &= \text{area of triangle shown in figure} \\ &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(SP)(P'M) \\ &= \frac{1}{2}(r)\,(v\sin\theta)\,dt \end{aligned}$$

$$\therefore \qquad \text{Areal velocity } \frac{dA}{dt} = \frac{1}{2} rv\sin\theta \qquad \text{...(i)}$$

Now, $rv\sin\theta$ is the magnitude of the vector product $\mathbf{r}\times\mathbf{v}$ which in turn is $\frac{1}{m}$ times the angular momentum $\mathbf{L} = \mathbf{r}\times m\mathbf{v}$ of the planet with respect to the sun. So we have,

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r}\times m\mathbf{v}| = \frac{L}{2m} \qquad \text{...(ii)}$$

or

$$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{constant}}$$

Thus, Kepler's second law, that areal velocity is constant, means that angular momentum is constant. It is easy to see why the angular momentum of the planet must be constant. According to Newton's law the rate of change of **L** equals the torque of the gravitational force **F** acting on the planet,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F} = \tau$$

Here, **r** is the radius vector of planet from the centre of the sun and the force **F** is directed from the planet towards the centre of the sun. So, these vectors always lie along the same line and their vector product $\mathbf{r}\times\mathbf{F}$ is zero. Hence, $\frac{d\mathbf{L}}{dt} = 0$ or $\mathbf{L} = \text{constant}$. Thus, from Eq. (ii) we can see that $\frac{dA}{dt} = \text{constant}$ if $\mathbf{L} = \text{constant}$. **Thus, second law is actually the law of conservation of angular momentum.**

Explanation of Third Law

In Article 13.9 we have already derived Kepler's third law for the particular case of circular orbits ($T^2 \propto r^3$). Newton was able to show that the same relationship holds for an elliptical orbit, with the orbit radius r replaced by semimajor axis a. Thus,

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}}} \qquad \text{(elliptical orbit)}$$

Here, M_s is the mass of the sun.

Extra Points to Remember

- Most of the problems of planetary motion are solved by two conservation laws:
  (i) conservation of angular momentum about centre of the sun and
  (ii) conservation of mechanical (potential + kinetic) energy

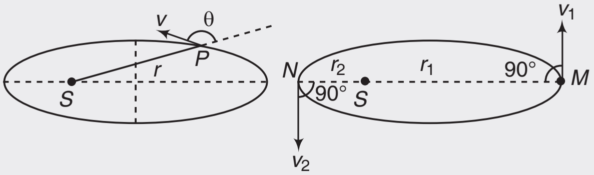


Fig. 13.51

Hence, the following two equations are used in most of the cases,

$$\boxed{\begin{aligned} mvr\sin\theta &= \text{constant} \qquad \ldots\text{(i)} \\ \frac{1}{2}mv^2 - \frac{GMm}{r} &= \text{constant} \qquad \ldots\text{(ii)} \end{aligned}}$$

At aphelion (or M) and perihelion (or N) positions $\theta = 90°$

Hence, Eq. (i) can be written as,

$$mvr\sin 90° = \text{constant}$$

or
$$mvr = \text{constant} \qquad \ldots\text{(iii)}$$

Further, since mass of the planet (m) also remains constant, Eq. (i) can also be written as

$$vr\sin\theta = \text{constant} \qquad \ldots\text{(iv)}$$

or
$$v_1r_1 = v_2r_2 \qquad (\theta = 90°)$$

$$r_1 > r_2$$

∴
$$v_1 < v_2$$

Example 13.29 *Name the physical quantities which remain constant in a planetary motion (in elliptical orbits).*

Solution Angular momentum about centre of sun and mechanical energy.

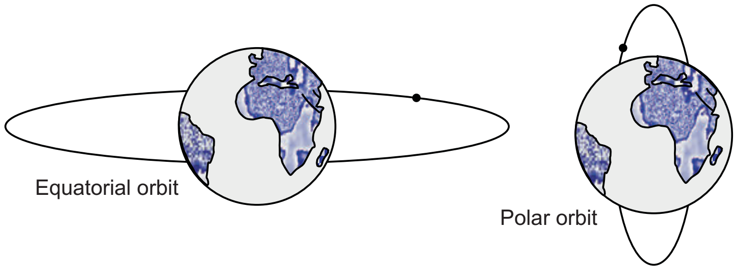
Example 13.30 *Consider a planet moving in an elliptical orbit round the sun. The work done on the planet by the gravitational force of the sun is zero in any small part of the orbit. Is this statement true or false ?*

Solution **False,** only at aphelion and perihelion positions **F** is perpendicular to $d\mathbf{S}$. So, at these two positions work done by gravitational force is zero. At other points angle between **F** and $d\mathbf{S}$ is not 90°. So, work done is not zero. But, being a conservative force work done in a closed path (or in one full rotation) is zero.

Final Touch Points

Polar and Geostationary Satellites

1. **Satellites in low polar orbit** pass over the poles. They orbit between 100 km and 200 km above the Earth's surface, taking around 90 minutes to make each orbit. The earth spins beneath the satellites as it moves, so the satellite can scan the whole surface of the earth. Low orbit polar satellites have uses such as



- Monitoring the weather.
- Observing the earth's surface.
- Military uses including spying.

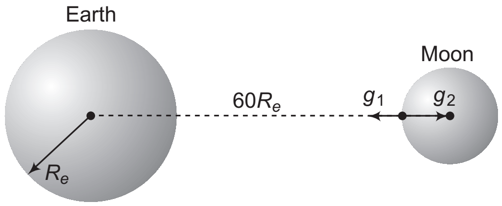
2. **Geostationary satellites** have a different trajectory to polar satellites. They are in orbit above the equator from west to east. The height of their orbit-36,000 km is just the right distance so that it takes them one day (24 hours) to make each orbit. This means that they stay in a fixed position over the earth's surface. A single geostationary satellite is on a line sight with about 40 percent of the earth's surface. Three such satellites, each separated by 120 degrees of longitude, can provide coverage of the entire planet. Geostationary satellites have uses such as:
   - communications- including satellites phones
   - global positioning or GPS.

   Geostationary satellites always appear in the same position when seen from the ground. This is why satellite television dishes can be bolted into one position and do not need to move.

3. Acceleration due to moon's gravity on moon's surface is $\frac{g_e}{6}$ because

$$\frac{M_m}{R_m^2} \approx \frac{1}{6}\frac{M_e}{R_e^2} \qquad \left(g = \frac{GM}{R^2}\right)$$

   While acceleration due to earth's gravity on moon's surface is approximately $\frac{g_e}{(60)^2}$ or $\frac{g_e}{3600}$. This is because distance of moon from the earth's centre is approximately equal to 60 times the radius of earth and $g \propto \frac{1}{r^2}$. In the shown figure



$$g_1 = \frac{g_e}{(60)^2} \quad \text{while} \quad g_2 = \frac{g_e}{6}$$

4. Total energy of a closed system is always negative. For example, energy of planet-sun, satellite-earth or electron-nucleus system is always negative.

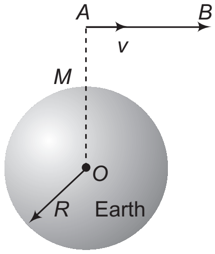
5. If the law of force obeys the inverse square law $\left(F \propto \frac{1}{r^2},\ F = \frac{-dU}{dr}\right)$

$$\boxed{K = \frac{|U|}{2} = |E|}$$

The same is true for electron-nucleus system because there also, the electrostatic force $F_e \propto \frac{1}{r^2}$.

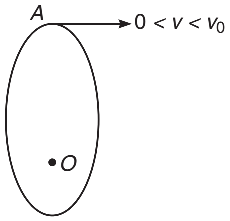
6. Trajectory of a body projected from point *A* in the direction *AB* with different initial velocities :

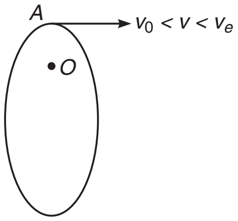
Let a body be projected from point *A* with velocity *v* in the direction *AB*. For different values of *v* the paths are different. Here, are the possible cases.



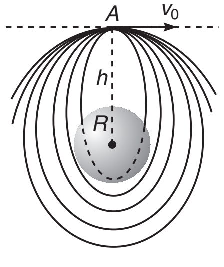
(i) If $v = 0$, path is a straight line from *A* to *M*.
(ii) If $0 < v < v_o$, path is an ellipse with centre *O* of the earth as a focus.
(iii) If $v = v_o$, path is a circle with *O* as the centre.
(iv) If $v_o < v < v_e$, path is again an ellipse with *O* as a focus.
(v) If $v = v_e$, body escapes from the gravitational pull of the earth and path is a parabola
(vi) If $v > v_e$, body again escapes but now the path is a hyperbola.

Here, v_o = orbital speed $\left(\sqrt{\frac{GM}{r}}\right)$ at *A* for a circular orbit and v_e = escape velocity from *A*.





Note 1. *From case (i) to (iv), total mechanical energy is negative. Hence, these are the closed orbits. For case (v), total energy is zero and for case (vi) total energy is positive. In these two cases orbits are open.*
2. *If v is not very large the elliptical orbit will intersect the earth and the body will fall back to earth.*



7. If $F \propto r^n$

then $T^2 \propto (r)^{1-n}$

and if $U \propto r^m$

then $T^2 \propto (r)^{2-m}$ (Applicable only for circular orbits)

8. $T = 2\pi\sqrt{\frac{R}{g}} \approx 84.6$ min comes in following four places in whole physics :

(i) If time period of rotation of earth becomes 84.6 min, effective value of g on equator becomes zero or we feel weightlessness on equator.

(ii) Time period of a satellite close to earth's surface is 84.6 min.

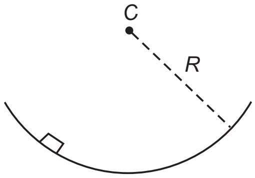
(iii) Time pendulum of a pendulum of infinite length is 84.6 min.

(iv) If a tunnel is dug along any chord of the earth and a particle is released from the surface of earth along this tunnel, then motion of this particle is simple harmonic and time period of this is also 84.6 min.

Note *(a) Points (iii) and (iv) come in the chapter of simple harmonic motion.*

(b) $T = 2\pi\sqrt{\frac{R}{g}}$ is also the time period of small oscillations of a block inside a smooth spherical bowl of radius R.

But this is not 84.6 min because here R is the radius of bowl not the radius of earth.



This expression can be compared with the time period of a pendulum $T = 2\pi\sqrt{\frac{l}{g}}$.

Solved Examples

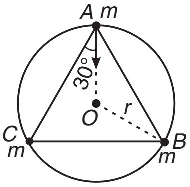
TYPED PROBLEMS

Type 1. *Two or more than two particles rotate in circular motion under their mutual gravitational attraction. We have to find speed of each particle.*

How to Solve?

Find net force on any one particle. It should come towards centre of the circle. This net force provides the necessary centripetal force.

Example 1 *Three particles each of mass m, are located at the vertices of an equilateral triangle of side a. At what speed must they move if they all revolve under the influence of their gravitational force of attraction in a circular orbit circumscribing the triangle while still preserving the equilateral triangle ?*



Solution $\mathbf{F}_A = \mathbf{F}_{AB} + \mathbf{F}_{AC}$

$$= 2\left[\frac{Gm^2}{a^2}\right]\cos 30° = \sqrt{3}\left[\frac{Gm^2}{a^2}\right]$$

$$r = \frac{a}{\sqrt{3}},$$

Now $$\frac{mv^2}{r} = F \quad \text{or} \quad \frac{\sqrt{3}mv^2}{a} = \frac{\sqrt{3}Gm^2}{a^2}$$

∴ $$v = \sqrt{\frac{Gm}{a}}$$ **Ans.**

Example 2 *In the above problem, find total mechanical energy of the system.*

Solution Total mechanical energy,

$$E = U + K$$

where, U = potential energy of 3 identical pairs of masses 'm' each at a distance 'a'

$$= -3\left[\frac{Gmm}{a}\right] = -\frac{3Gm^2}{a}$$

K = kinetic energy of three particles

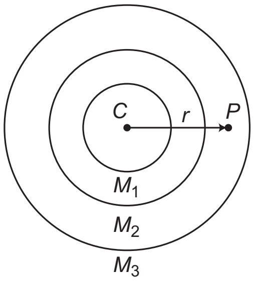
$$= 3\left[\frac{1}{2}mv^2\right] = \frac{3}{2}m\left(\sqrt{\frac{Gm}{a}}\right)^2 = \frac{3}{2}\frac{Gm^2}{a}$$

∴ $$E = U + K = -\frac{3Gm^2}{2a}$$ **Ans.**

Note *Total mechanical energy is negative, as the system is closed.*

Type 2. *Gravitational field and force on a mass due to spherical shells.*

Concept



According to Gauss theorem, net field strength at any point in the above situation is only due to the masses inside an imaginary spherical surface drawn at that point.

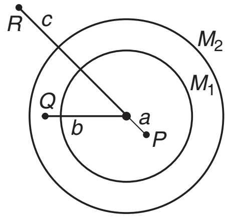
For example, field strength at P is

$$E_P = \frac{G(M_1 + M_2)}{r^2} \quad \text{(towards } C\text{)}$$

If a mass m is kept at P, then force on this mass is

$$F = mE_P = \frac{G(M_1 + M_2)m}{r^2} \quad \text{(towards } C\text{)}$$

Example 3 *Two concentric shells of masses M_1 and M_2 are concentric as shown. Calculate the gravitational force on m due to M_1 and M_2 at points P, Q and R.*



Solution At P, $F = 0$

At Q, $F = \dfrac{GM_1m}{b^2}$

At R, $F = \dfrac{G(M_1 + M_2)m}{c^2}$

Type 3. *To find variation of time period of a satellite in circular orbit with its orbital radius 'r' if gravitational potential energy between two point masses varies as $U \propto r^n$ with negative sign.*

How to Solve?

From the relation between conservative force F and its potential energy U,

$$F = -\frac{dU}{dr}$$

First find the variation of F with r. Then, this force provides the necessary centripetal force. So, find variation of v with r by putting,

$$F = \frac{mv^2}{r}$$

Now, finally time period is given by $T = \dfrac{2\pi r}{v}$. So, find variation of T with r.

Example 4 *Gravitational potential energy between two points masses is*

$$U = -\frac{K\, m_1 m_2}{r^n}$$

where, K is a positive constant. With what power of 'r' time period of a satellite of mass 'm' varies in circular orbit if mass of planet is M ?

Solution $U = -\frac{KMm}{r^n}$

$$F = -\frac{dU}{dr} = \frac{KMmn}{r^{n+1}} \quad \text{or} \quad F \propto r^{-(n+1)}$$

Now, this force provides the necessary centripetal force.

$$\therefore \qquad \frac{mv^2}{r} = \frac{KMm}{r^{n+1}} \quad \text{or} \quad v^2 \propto r^{-n}$$

$$\therefore \qquad v \propto r^{\frac{-n}{2}}$$

Time period is given by

$$T = \frac{2\pi r}{v} \quad \text{or} \quad T \propto \frac{r}{v}$$

$$\text{or} \qquad T \propto \frac{r}{r^{\frac{-n}{2}}} \Rightarrow \therefore \quad T \propto r^{1+\frac{n}{2}}$$

Note *Under normal conditions*

$$U = -\frac{GMm}{r}$$

So, if we compare with the given equation then, $n = 1$.

$$\text{Now,} \qquad T \propto r^{1+\frac{n}{2}} \quad \text{or} \quad T \propto r^{\frac{3}{2}} \qquad (\text{for } n = 1)$$

Example 5 *Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then*

(a) T^2 is proportional to R^2

(b) T^2 is proportional to $R^{7/2}$

(c) T^2 is proportional to $R^{3/2}$

(d) T^2 is proportional to $R^{3.75}$

Solution $\frac{mv^2}{R} \propto R^{-5/2}$

$$\therefore \qquad v \propto R^{-3/4}$$

$$\text{Now,} \qquad T = \frac{2\pi R}{v} \quad \text{or} \quad T^2 \propto \left(\frac{R}{v}\right)^2$$

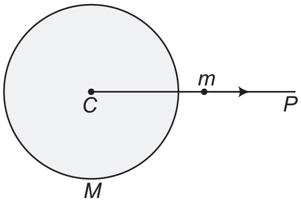
$$\text{or} \qquad T^2 \propto \left(\frac{R}{R^{-3/4}}\right)^2 \quad \text{or} \quad T^2 \propto R^{7/2}$$

$\therefore$ The correct answer is (b).

Type 4. *Based on conservation of mechanical energy.*

Concept

In the figure, $M >> m$, hence only the particle moves along the line CP. The spherical body remains at rest. The problem of any such type can be solved by energy conservation principle or



$$E_i = E_f$$

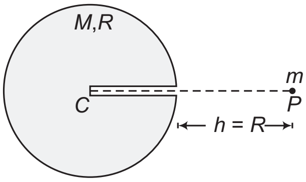
$$\Rightarrow \quad U_i + K_i = U_f + K_f$$

Here, $U = -\dfrac{GMm}{r}$ for external points

and $U = mV$ for internal points of sphere.

where, V is the gravitational potential due to M. In this type, we are using only one conservation law. Therefore, number of unknown should be only one.

Example 6 *There is a smooth tunnel upto centre C of a solid sphere of mass 'M' and radius R. A particle of mass $m\,(<< M)$ is released from point P along the line CP. Find velocity of 'm' while striking at C.*



Solution Using mechanical energy conservation equation.

$$E_C = E_P \quad \Rightarrow \quad K_C + U_C = K_P + U_P$$

$$\Rightarrow \quad \frac{1}{2}mv_C^2 + mV_C = 0 - \frac{GMm}{(R+R)} \qquad \ldots\text{(i)}$$

Here, V_C = potential at C due to mass $M = -\dfrac{3}{2}\dfrac{GM}{R}$

Substituting this value in Eq. (i) and then solving we get,

$$v_C = \sqrt{\frac{2GM}{R}} \qquad \textbf{Ans.}$$

Example 7 *A particle of mass 'm' is projected from the surface of earth with velocity $v = 2v_e$, where v_e is the value of escape velocity from the surface of earth. Find velocity of the particle on reaching to interstellar space (at infinity) in terms of v_e.*

Solution $v_e = \sqrt{\dfrac{2GM}{R}} \Rightarrow \dfrac{GM}{R} = \dfrac{v_e^2}{2}$ ...(i)

Using conservation of mechanical energy at the surface of earth and infinity.

We have,
$$K_f + U_f = K_i + U_i$$

$$\Rightarrow \qquad \frac{1}{2} m v_\infty^2 + 0 = \frac{1}{2} m \,(2v_e)^2 - \frac{GMm}{R} \qquad \text{...(ii)}$$

Substituting the value of $\dfrac{GM}{R} = \dfrac{v_e^2}{2}$

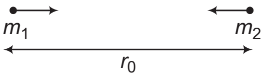
From Eq. (i) in Eq. (ii) we get, $v_\infty = \sqrt{3}\, v_e$

Alternate Method

On the surface of earth speed is two times the escape velocity. So, kinetic energy is four times. One kinetic energy is used in taking it to infinity. So, three kinetic energies are still left at infinity. In terms of speed it is $\sqrt{3} v_e$.

Type 5. *Based on conservation of mechanical energy and linear momentum.*

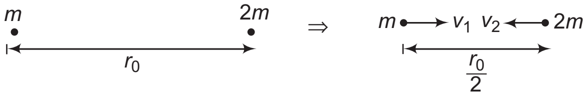
Concept



Two particles (or two spherical bodies) of masses m_1 and m_2 are free to move along the line joining them. This time their masses are comparable. Net force on the two particle system is zero.Therefore, linear momentum of the system may be conserved. Further, gravitational forces are conservative in nature.

Therefore, mechanical energy of the system is also conserved. Since, we are using two conservation laws, therefore number of unknowns are also two.

Example 8 *In the figure shown in the text, $m_1 = m$, $m_2 = 2m$ and initial distance between them is r_0. Find velocities of the masses when separation between them becomes $\dfrac{r_0}{2}$.*



Solution Let their velocities are v_1 and v_2. From conservation of linear momentum.

$$p_i = p_f$$

$$\therefore \qquad 0 = mv_1 - 2mv_2 \qquad \text{...(i)}$$

From conservation of mechanical energy,

$$E_i = E_f$$

$$\text{or} \qquad K_i + U_i = K_f + U_f$$

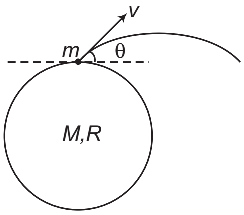
$$\text{or} \qquad 0 - \frac{G(m)\,(2m)}{r_0} = \frac{1}{2} m v_1^2 + \frac{1}{2} \times 2m \times v_2^2 - \frac{G(m)\,(2m)}{(r_2/2)} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_1 = 2\sqrt{\frac{2Gm}{3r_0}}, \; v_2 = \sqrt{\frac{2Gm}{3r_0}} \qquad \textbf{Ans.}$$

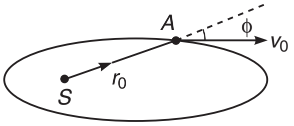
Type 6. *Based on conservation of angular momentum ($L = mvr \sin\theta$) and mechanical energy.*

Concept



In the figure shown, $M >> m$. Hence, the sphere of mass M is at rest and only the particle of mass 'm' will move. This particle is projected with velocity v at some angle θ from the surface as shown. Gravitational force on 'm' is always towards centre of 'M'. So, its torque about centre is always zero and angular momentum of particle about centre may be conserved. Further, the gravitational force, being a conservative force mechanical energy can also be conserved. We are applying two conservation laws, so number of unknown are also two.

Example 9 *Find the maximum and minimum distances of the planet A from the sun S, if at a certain moment of time it was at a distance r_0 and travelling with the velocity v_0, with the angle between the radius vector and velocity vector being equal to ϕ.*



Solution At minimum and maximum distances velocity vector (**v**) makes an angle of 90° with radius vector. Hence, from conservation of angular momentum,

$$mv_0r_0 \sin\phi = mrv \quad \text{...(i)}$$

Here, m is the mass of the planet.

From energy conservation law, it follows that

$$\frac{mv_0^2}{2} - \frac{GMm}{r_0} = \frac{mv^2}{2} - \frac{GMm}{r} \quad \text{...(ii)}$$

Here, M is the mass of the sun.

Solving Eqs. (i) and (ii) for r, we get two values of r, one is $r_{\max}$ and another is $r_{\min}$. So,

$$r_{\max} = \frac{r_0}{2-K}\left(1 + \sqrt{1 - K(2-K)\sin^2\phi}\right)$$

and

$$r_{\min} = \frac{r_0}{2-K}\left(1 - \sqrt{1 - K(2-K)\sin^2\phi}\right)$$

Here,

$$K = \frac{r_0^2 v_0^2}{GM}$$

Example 10 *A projectile of mass m is fired from the surface of the earth at an angle $\alpha = 60°$ from the vertical. The initial speed v_0 is equal to $\sqrt{\frac{GM_e}{R_e}}$. How high does the projectile rise ? Neglect air resistance and the earth's rotation.*

Solution Let v be the speed of the projectile at highest point and $r_{\max}$ its distance from the centre of the earth. Applying conservation of angular momentum and mechanical energy,

$$mv_0 \sin \alpha = mvr_{\max} \sin 90°$$

or
$$mv_0 \sin \alpha = mvr_{\max} \qquad \text{...(i)}$$

$$\frac{1}{2} mv_0^2 - \frac{GM_e m}{R_e} = \frac{1}{2} mv^2 - \frac{GM_e m}{r_{\max}} \qquad \text{...(ii)}$$

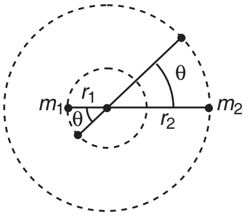
Solving these two equations with the given data we get,

$$r_{\max} = \frac{3R_e}{2}$$

or the maximum height
$$h_{\max} = r_{\max} - R_e = \frac{R_e}{2} \qquad \textbf{Ans.}$$

Type 7. *Based on double star system.*

Concept



In motion of a planet round the sun we have assumed the mass of the sun to be too large in comparison to the mass of the planet. Under such situation the sun remains stationary and the planet revolves round the sun. If however masses of sun and planet are comparable and motion of sun is also to be considered, then both of them revolve around their centre of mass with same angular velocity but different linear speeds in the circles of different radii. The centre of mass remains stationary. This system of two stars is called a double star system.

We use following equations under this condition.

$$m_1 r_1 = m_2 r_2 \qquad \text{...(i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \qquad \text{...(ii)}$$

Solving these two equations, we can find that

$$\omega = \sqrt{\frac{GM}{r^3}} \quad \text{or} \quad T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Here,
$$M = m_1 + m_2 \quad \text{and} \quad r = r_1 + r_2$$

Further, angular momentum of the system about COM

$$L = (I_1 + I_2)\,\omega = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \omega = \mu r^2 \omega$$

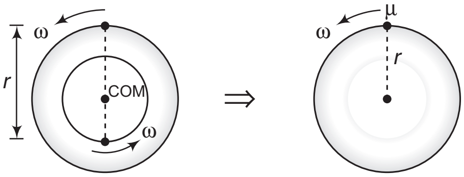
Kinetic energy of system, $K = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \omega^2 = \frac{1}{2} \mu r^2 \omega^2$

and moment of inertia of system,

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 = \mu r^2$$

Here,
$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Thus, the two bodies can be replaced by a single body whose mass is equal to reduced mass. This single body revolve in a circular orbit whose radius is equal to the distance between two bodies and centripetal force of circular motion is equal to force of interaction between two bodies for actual separation.



Example 11 *A planet of mass m_1 revolves round the sun of mass m_2. The distance between the sun and the planet is r. Considering the motion of the sun find the total energy of the system assuming the orbits to be circular.*

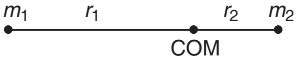
Solution Both the planet and the sun revolve around their centre of mass with same angular velocity (say ω)

$$r = r_1 + r_2 \qquad \text{...(i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{r^2} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$r_1 = r\left(\frac{m_2}{m_1 + m_2}\right)$$



$$r_2 = r\left(\frac{m_1}{m_1 + m_2}\right)$$

and
$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

Now, total energy of the system is

$$E = \text{PE} + \text{KE} \quad \text{or} \quad E = -\frac{Gm_1 m_2}{r} + \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2$$

Substituting the values of r_1, r_2 and ω^2, we get

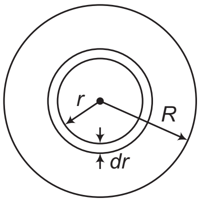
$$E = -\frac{Gm_1 m_2}{2r}$$

Ans.

Type 8. *To find gravitational field strength for spherical mass distribution when mass density is a function of r (not constant), where r is the distance from centre.*

Concept

Mass density of a solid sphere of radius R varies as $\rho = \rho_0 r^2$, where ρ_0 is a positive constant, we have to find variation of E with r.



According to Gauss theorem, the mass inside a spherical surface only contributes in the field strength on the surface of that sphere or at a distance 'r' from the centre

$$E = \frac{Gm_{\text{in}}}{r^2} \qquad \text{...(i)}$$

where, m_{in} is the mass inside that sphere. Here, m_{in} will be obtained by integration as the mass density is not constant.

Let us take a small element of thickness dr at distance 'r'. Volume of this element.

$$dV = (4\pi r^2)dr$$

Mass density at distance r is

$$\rho = \rho_0 r^2$$

Therefore, small mass of this element

$$dm = \rho\, dV$$

or
$$dm = (4\pi\rho_0 r^4)dr$$

$\therefore$
$$m_{\text{in}} = \int_0^r dm = \int_0^r (4\pi\rho_0 r^4)\, dr$$

or
$$m_{\text{in}} = \frac{4\pi\rho_0}{5} r^5 \qquad \text{...(ii)}$$

Substituting this value of m_{in} in Eq. (i) we can find E.

Example 12 *In the problem discussed in the text, find E-r expressions for inside and outside points.*

Solution **Inside points ($r \le R$)**

Directly substituting value of m_{in} from Eq. (ii) in Eq. (i) we have,

$$E = \frac{G}{r^2}\left[\frac{4\pi\rho_0 r^5}{5}\right]$$

or
$$E = \frac{4\pi G\rho_0 r^3}{5}$$

Outside points ($r \ge R$)

Mass is only upto $r = R$. So, substituting $r = R$ in Eq. (ii) we have,

$$m_{\text{in}} = \frac{4\pi\rho_0}{5} R^5$$

Now, substituting this value of m_{in} in Eq. (i), we have

$$E = \frac{G}{r^2}\left[\frac{4\pi\rho_0 R^5}{5}\right]$$

or
$$E = \frac{4\pi\, G\rho_0 R^5}{5r^2}$$

Note *Direction of* **E** *is always towards the centre of the sphere.*

Type 9. *To draw E - r and V - r graphs due to two points masses, along the line joining two masses.*

Concept

(i) Expressions of E and V due to a point mass are

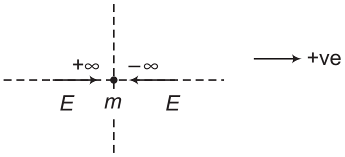
$$E = \frac{Gm}{r^2} \qquad \text{(towards the point mass)}$$

and
$$V = -\frac{Gm}{r}$$

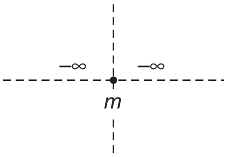
(ii) As $r \to \infty$, E and V both $\to 0$.

(iii) As $r \to 0$, $E \to \infty$ and $V \to -\infty$

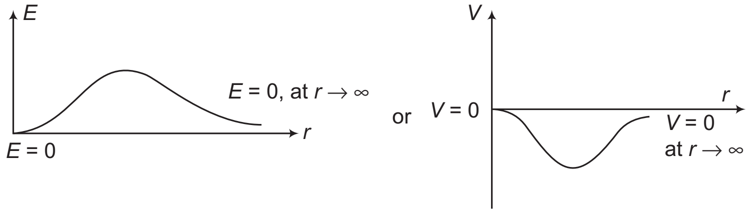
(iv) E is a vector quantity. On two sides of a point mass directions are different. So, on one side if value is $+\infty$ (just over the mass as $r \to 0$) and on its other side value will be $-\infty$.



(v) V is a scalar quantity. On both sides of the mass (as $r \to 0$) value will be $-\infty$.



(vi) Between two zero values, we will get one maximum (or minimum value)

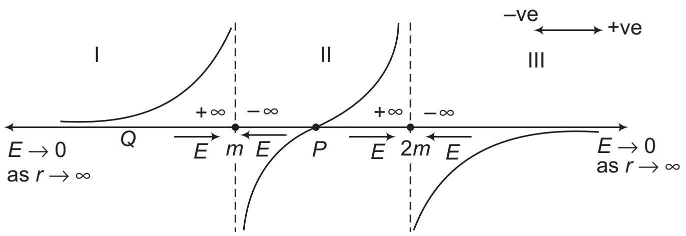


How to Solve?

Just write down the values at $r \to \infty$ and $r \to 0$, then draw the graph with the help of five points discussed above.

Example 13 Two points masses 'm' and $2m$ are kept at certain distance as shown in figure. Draw E - r and V - r graphs along the line joining them corresponding to given mass system.

Solution **E - r Graph**

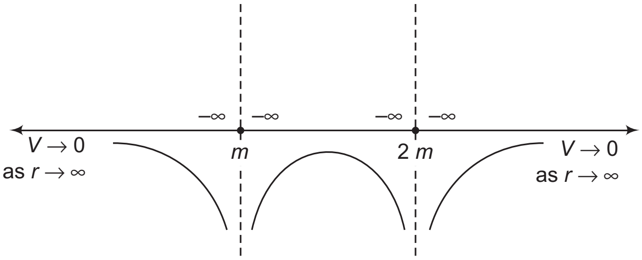


Region I Fields of m and $2m$ both are positive. Hence, net field at any point is also positive.

Region II Field at m is negative and field of $2m$ is positive. So, at some point P (nearer to m) two fields are equal and opposite and net field is zero.

Region III Fields of m and $2m$ both are negative. Hence, net field at any point is also negative.

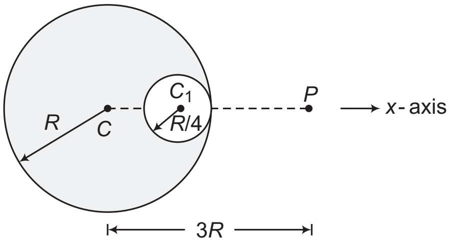
V-r Graph



Gravitational potential is always negative on both sides. So, net potential at any point is always negative. Between two masses potential varies between $-\infty$ and $-\infty$. So, graph is as shown in figure.

Type 10. *To find value of E or V at some point due to a solid sphere having some cavity in it.*

How to Solve?



Suppose a solid sphere of mass M and radius R has a cavity of radius $\frac{R}{4}$ as shown in figure and we wish to find net gravitational field and potential at point P. Then,

$$\mathbf{E}_T = \mathbf{E}_R + \mathbf{E}_C$$

$$\therefore \qquad \mathbf{E}_R = \mathbf{E}_T - \mathbf{E}_C \qquad \ldots\text{(i)}$$

Similarly,
$$V_R = V_T - V_C \qquad \ldots\text{(ii)}$$

Here, R stands for remaining mass, T for total mass and C for cavity.

Example 14 *In the problem discussed in the text, find the values of E and V at P due to the remaining mass.*

Solution Total mass is M of volume $\frac{4}{3}\pi R^3$. Therefore, mass of cavity of volume $\frac{4}{3}\pi\left(\frac{R}{4}\right)^3$ will be $\frac{M}{64}$.

Further,
$$C_1P = 2R + \frac{R}{4} = \frac{9R}{4}$$

Using Eq. (i),
$$\mathbf{E}_R = \frac{GM}{(3R)^2}(-\hat{\mathbf{i}}) - \frac{G\left(\frac{M}{64}\right)}{(9R/4)^2}(-\hat{\mathbf{i}})$$

Note *Field strength is always towards the centre.*

$$\therefore \qquad \mathbf{E}_R = \frac{35}{324}\frac{GM}{R^2}(-\hat{\mathbf{i}})$$ **Ans.**

$\therefore$ Field Strength is $\dfrac{35\,GM}{324\,R^2}$ towards C.

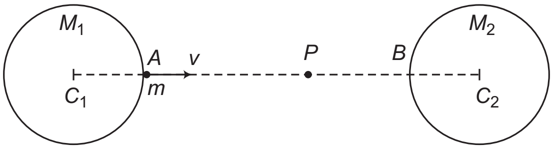
Using Eq. (ii), we have

$$V_R = -\frac{GM}{3R} - \left[\frac{-G\left(\frac{M}{64}\right)}{(9R/4)}\right]$$

$$= -\frac{47}{144}\frac{GM}{R}$$ **Ans.**

Type 11. *To find minimum velocity required to project one particle from the surface of one planet to other planet.*

Concept



Suppose the particle of mass 'm' is projected from the surface of planet of mass M_1 and we wish to project it upto the surface of other planet. In between the two planets, there is a point P, where net field strength is zero. Or, net force on 'm' is zero.

From C_1 to P field strength of M_1 is stronger and net force on 'm' is towards C_1. At point P field strengths of both M_1 and M_2 are equal and opposite. Between C_2 and P field strengths of M_2 is stronger and net force on 'm' is towards C_2.

So, we have to project the particle only upto P. After P, it automatically moves towards the surface of B by the attraction of M_2.

How to Solve?

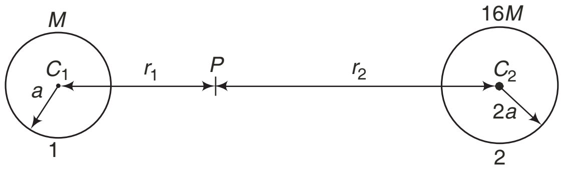
First find the point P, where net gravitational force on 'm' is zero. Then, apply energy conservation equation between A and P where,

v_A = minimum velocity required and v_P is tending to zero.

Note *By applying energy conservation principle we can also find velocity at B, on reaching to the surface of B.*

Example 15 *Distance between the centres of two stars is* $10a$. *The masses of these stars are* M *and* $16M$ *and their radii* a *and* $2a$ *respectively. A body of mass* m *is fired straight from the surface of the larger star towards the surface of the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of* G, M *and* a. **(JEE 1996)**

Solution Let there are two stars 1 and 2 as shown below.

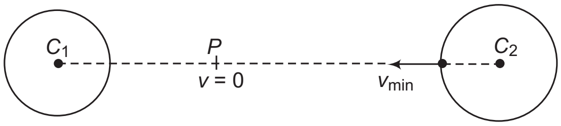


Let P is a point between C_1 and C_2, where gravitational field strength is zero or at P field strength due to star 1 is equal and opposite to the field strength due to star 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G\,(16\,M)}{r_2^2} \quad \text{or} \quad \frac{r_2}{r_1} = 4 \quad \text{also} \quad r_1 + r_2 = 10\,a$$

$$\therefore \qquad r_2 = \left(\frac{4}{4+1}\right)(10\,a) = 8\,a \quad \text{and} \quad r_1 = 2a$$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.



From conservation of mechanical energy $\frac{1}{2}mv^2_{\min}$

= Potential energy of the body at P – Potential energy at the surface of larger star.

$$\therefore \qquad \frac{1}{2}mv^2_{\min} = \left[-\frac{GMm}{r_1} - \frac{16\,GMm}{r_2}\right] - \left[-\frac{GMm}{10a - 2a} - \frac{16\,GMm}{2a}\right]$$

$$= \left[-\frac{GMm}{2a} - \frac{16\,GMm}{8a}\right] - \left[-\frac{GMm}{8a} - \frac{8\,GMm}{a}\right]$$

$$\text{or} \qquad \frac{1}{2}mv^2_{\min} = \left(\frac{45}{8}\right)\frac{GMm}{a} \Rightarrow \therefore\ v_{\min} = \frac{3\sqrt{5}}{2}\left(\sqrt{\frac{GM}{a}}\right)$$ **Ans.**

Type 12. *To find gravitational potential due to two or more than two spherical shells.*

Concept

At any point inside the shell (upto the surface).

$$V = -\frac{GM}{R} = \text{constant}$$

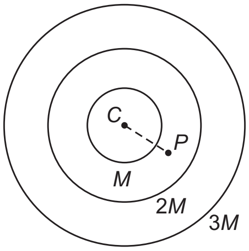
In the denominator, substitute R, radius of the shell.

At any point outside the shell,

$$V = -\frac{GM}{r} \quad \text{or} \quad V \propto -\frac{1}{r}$$

In the denominator, substitute r, actual distance of the point from the centre.

Example 16 *Three spherical shells of masses M, 2M and 3M have radii R, 3R and 4R as shown in figure. Find net potential at point P, where CP = 2R*



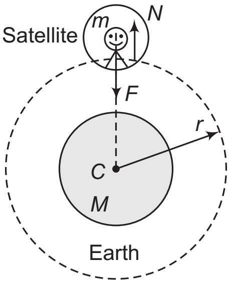
Solution Point P lies outside the innermost shell. So, in the denominator we will substitute actual distance of P from the centre or $r = CP = 2R$.

This point P lies inside the other two shells. So, we will substitute their radii, $3R$ and $4R$ in the denominator.

$$\therefore \qquad V_P = -\frac{GM}{2R} - \frac{G(2M)}{3R} - \frac{G(3M)}{4R} = -\frac{23\,GM}{12\,R} \qquad \textbf{Ans.}$$

Miscellaneous Examples

Example 17 *Explain the reason of weightlessness inside a satellite.*



Solution Feeling of weight is due to the normal reaction from the ground. Inside a satellite, this normal reaction becomes zero, which can be proved as given below.

Orbital speed is given by

$$v = \sqrt{\frac{GM}{r}} \qquad \text{...(i)}$$

Two forces are acting on the person.

(i) F = gravitational force from earth

or $$F = \frac{GMm}{r^2} \qquad \text{...(ii)}$$

(ii) N = normal reaction

Person is also revolving in circular motion with same speed v. So, it needs a centripetal force.

$$\therefore \qquad F - N = \frac{mv^2}{r} \quad \text{or} \quad N = F - \frac{mv^2}{r}$$

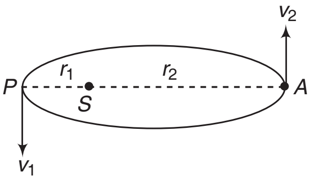
Substituting the values of F and v in Eqs. (i) and (ii) we have,

$$N = \frac{GMm}{r^2} - \frac{m}{r}\left(\sqrt{\frac{GM}{r}}\right)^2 = 0 \qquad \textbf{Hence Proved.}$$

Note *Moon is also an earth's satellite but there we don't feel weightlessness because mass of moon is large and we feel weight due to the gravitational force due to the moon (which is not insignificant like other satellite as its mass is less in comparison to the moon).*

Example 18 *Find the speeds of a planet of mass m in its perihelion and aphelion positions. The semimajor axis of its orbit is a, eccentricity is e and the mass of the sun is M. Also find the total energy of the planet in terms of the given parameters.*

Solution Let v_1 and v_2 be the speeds of the planet at perihelion and aphelion positions.



$$r_1 = a(1-e)$$

and $$r_2 = a(1+e) \qquad \text{...(i)}$$

Applying conservation of angular momentum of the planet at P (perihelion) and A (aphelion)

$$mv_1 r_1 \sin 90° = mv_2 r_2 \sin 90°$$

or $$v_1 r_1 = v_2 r_2 \qquad \text{...(ii)}$$

Applying conservation of mechanical energy in these two positions, we have

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$v_1 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)} \quad \text{and} \quad v_2 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

Further, total energy of the planet

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}m\left[\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\right] - \frac{GMm}{a(1-e)}$$

$$= \frac{GMm}{a(1-e)}\left[\left(\frac{1+e}{2}\right) - 1\right]$$

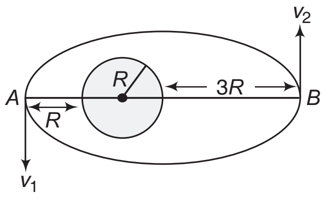
$$= \frac{GMm}{a(1-e)}\left(\frac{e-1}{2}\right) \quad \text{or} \quad E = -\frac{GMm}{2a}$$ **Ans.**

Example 19 *The minimum and maximum distances of a satellite from the centre of the earth are 2R and 4R respectively, where R is the radius of earth and M is the mass of the earth. Find*

(a) its minimum and maximum speeds,

(b) radius of curvature at the point of minimum distance.

Solution (a) Applying conservation of angular momentum



$$mv_1(2R) = mv_2(4R)$$

$$v_1 = 2v_2 \qquad \text{... (i)}$$

From conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{2R} = \frac{1}{2}mv_2^2 - \frac{GMm}{4R} \qquad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 = \sqrt{\frac{GM}{6R}}, \quad v_1 = \sqrt{\frac{2GM}{3R}}$$

(b) If r is the radius of curvature at point A

$$\frac{mv_1^2}{r} = \frac{GMm}{(2R)^2}$$

$$r = \frac{4v_1^2R^2}{GM} = \frac{8R}{3} \qquad \text{(putting value of } v_1\text{)}$$

Example 20 *A planet of mass m revolves in elliptical orbit around the sun of mass M so that its maximum and minimum distances from the sun are equal to r_a and r_p respectively. Find the angular momentum of this planet relative to the sun.*

Solution Using conservation of angular momentum

$$mv_pr_p = mv_ar_a$$

As velocities are perpendicular to the radius vectors at apogee and perigee.

$$\Rightarrow \qquad v_pr_p = v_ar_a$$

Using conservation of energy,

$$-\frac{GMm}{r_p} + \frac{1}{2}mv_p^2 = \frac{-GMm}{r_a} + \frac{1}{2}mv_a^2$$

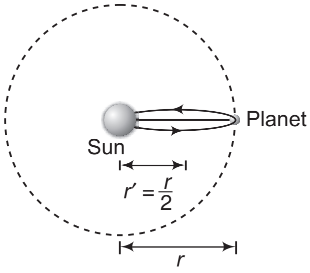
By solving, the above equations,

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_p + r_a)}}$$

$$L = mv_pr_p = m\sqrt{\frac{2GMr_pr_a}{(r_p + r_a)}}$$

Example 21 *If a planet was suddenly stopped in its orbit supposed to be circular, show that it would fall onto the sun in a time $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.*

Solution Consider an imaginary planet moving along a strongly extended flat ellipse, the extreme points of which are located on the planet's orbit and at the centre of the sun. The semi-major axis of the orbit of such a planet would apparently be half the semi-major axis of the planet's orbit. So, the time period of the imaginary planet T' according to Kepler's law will be given by



$$\left(\frac{T'}{T}\right) = \left(\frac{r'}{r}\right)^{\frac{3}{2}}$$

or

$$T' = T\left(\frac{1}{2}\right)^{\frac{3}{2}} \qquad \left(\text{as} \quad r' = \frac{r}{2}\right)$$

∴ Time taken by the planet to fall onto the sun is

$$t = \frac{T'}{2} = \frac{T}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}} \Rightarrow t = \frac{\sqrt{2}}{8}\,T$$

Example 22 *A satellite is revolving round the earth in a circular orbit of radius r and velocity* v_0. *A particle is projected from the satellite in forward direction with relative velocity* $v = (\sqrt{5/4} - 1)\, v_0$. *Calculate its minimum and maximum distances from earth's centre during subsequent motion of the particle.*

Solution $v_o = \sqrt{\frac{GM}{r}}$ = orbital speed of satellite ...(i)

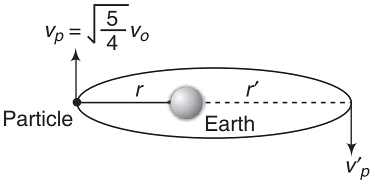
where, M = mass of earth.

Absolute velocity of particle would be

$$v_p = v + v_0 = \sqrt{\frac{5}{4}}\, v_0 = \sqrt{1.25}\, v_0 \qquad \text{...(ii)}$$

Since, v_p lies between orbital velocity and escape velocity, path of the particle would be an ellipse with r being the minimum distance.

Let r' be the maximum distance and v_p' its velocity at that moment.



Then, from conservation of angular momentum and conservation of mechanical energy, we get

$$m v_p r = m v_p' r' \qquad \text{...(iii)}$$

and

$$\frac{1}{2} m v_p^2 - \frac{GMm}{r} = \frac{1}{2} m v_p'^2 - \frac{GMm}{r'} \qquad \text{...(iv)}$$

Solving the above Eqs. (i), (ii), (iii) and (iv), we get

$$r' = \frac{5r}{3} \quad \text{and} \quad r$$

Hence, the maximum and minimum distances are $\frac{5r}{3}$ and r respectively.

Example 23 *An earth satellite is revolving in a circular orbit of radius a with velocity* v_o. *A gun is in the satellite and is aimed directly towards the earth. A bullet is fired from the gun with muzzle velocity* $\frac{v_0}{2}$. *Neglecting resistance offered by cosmic dust and recoil of gun, calculate maximum and minimum distance of bullet from the centre of earth during its subsequent motion.*

Solution Orbital speed of satellite is

$$v_o = \sqrt{\frac{GM}{a}} \qquad \text{...(i)}$$

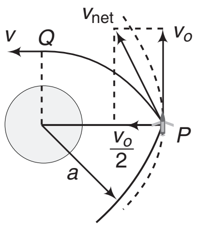
From conservation of angular momentum at P and Q, we have

$$m a v_o = m v r$$

or

$$v = \frac{a v_o}{r} \qquad \text{...(ii)}$$

From conservation of mechanical energy at P and Q, we have



$$\frac{1}{2}m\left(v_o^2+\frac{v_o^2}{4}\right)-\frac{GMm}{a}=\frac{1}{2}mv^2-\frac{GMm}{r}$$

or $$\frac{5}{8}v_o^2-\frac{GM}{a}=\frac{v^2}{2}-\frac{GM}{r}$$

Substituting values of v and v_o from Eqs. (i) and (ii), we get

$$\frac{5}{8}\frac{GM}{a}-\frac{GM}{a}=\frac{a^2}{r^2}\cdot\left(\frac{GM}{2a}\right)-\frac{GM}{r}$$

or $$-\frac{3}{8a}=\frac{a}{2r^2}-\frac{1}{r}$$

or $$-3r^2=4a^2-8ar$$

or $$3r^2-8ar+4a^2=0$$

or $$r=\frac{8a\pm\sqrt{64a^2-48a^2}}{6}$$

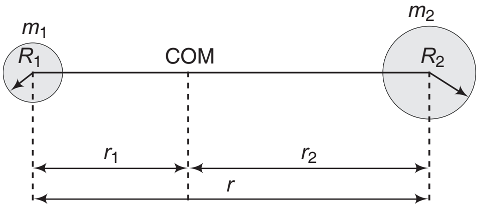
or $$r=\frac{8a\pm 4a}{6}$$

or $$r=2a \text{ and } \frac{2a}{3}$$

Hence, the maximum and minimum distances are $2a$ and $\frac{2a}{3}$ respectively.

Example 24 *Binary stars of comparable masses m_1 and m_2 rotate under the influence of each other's gravity with a time period T. If they are stopped suddenly in their motions, find their relative velocity when they collide with each other. The radii of the stars are R_1 and R_2 respectively. G is the universal constant of gravitation.*

Solution Both the stars rotate about their centre of mass (COM).



For the position of COM, $$\frac{r_1}{m_2}=\frac{r_2}{m_1}=\frac{r_1+r_2}{m_1+m_2}=\frac{r}{m_1+m_2} \qquad (r=r_1+r_2)$$

Also, $$m_1 r_1 \omega^2 = \frac{Gm_1 m_2}{r^2}$$

or $$\omega^2 = \frac{Gm_2}{r_1 r^2} \qquad \left(\omega = \frac{2\pi}{T}\right)$$

But, $$r_1 = \frac{m_2 r}{m_1 + m_2}$$

∴ $$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

or $$r = \left\{\frac{G(m_1 + m_2)}{\omega^2}\right\}^{1/3} \qquad \text{...(i)}$$

Applying conservation of mechanical energy, we have

$$-\frac{Gm_1 m_2}{r} = -\frac{Gm_1 m_2}{(R_1 + R_2)} + \frac{1}{2}\mu v_r^2 \qquad \text{...(ii)}$$

Here, $$\mu = \text{reduced mass}$$
$$= \frac{m_1 m_2}{m_1 + m_2}$$

and v_r = relative velocity between the two stars.

From Eq. (ii), we find that

$$v_r^2 = \frac{2Gm_1 m_2}{\mu}\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$
$$= \frac{2Gm_1 m_2}{\dfrac{m_1 m_2}{m_1 + m_2}}\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$
$$= 2G(m_1 + m_2)\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$

Substituting the value of r from Eq. (i), we get

$$v_r = \sqrt{2G(m_1 + m_2)\left[\frac{1}{R_1 + R_2} - \left\{\frac{4\pi^2}{G(m_1 + m_2)T^2}\right\}^{1/3}\right]}$$

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

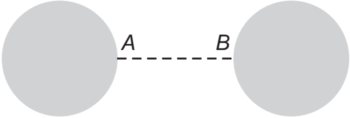
1. **Assertion :** When two masses come closer, their gravitational potential energy decreases.

   Reason : Two masses attract each other.

2. **Assertion :** In moving from centre of a solid sphere to its surface, gravitational potential increases.

   Reason : Gravitational field strength increases.

3. **Assertion :** There are two identical spherical bodies fixed in two positions as shown. While moving from A to B gravitational potential first increases then decreases.



   Reason : At centre point of A and B field strength will be zero.

4. **Assertion :** If we plot potential *versus* x-coordinate graph along the x-axis, then field strength is zero where slope of V-x graph is zero.

   Reason : If potential is function of x-only then

$$E = -\frac{dV}{dx}$$

5. **Assertion :** A particle is projected upwards with speed v and it goes to a height h. If we double the speed then it will move to height $4h$.

   Reason : In case of earth, acceleration due to gravity g varies as

$$g \propto \frac{1}{r^2} \qquad \text{(for } r \geq R\text{)}$$

6. **Assertion :** In planetary motion angular momentum of planet about centre of sun remains constant. But linear momentum of system does not remain constant.

   Reason : Net torque on planet about any point is zero.

7. **Assertion :** Plane of space satellite is always equatorial plane.

   Reason : On the equator value of g is minimum.

8. **Assertion :** On satellites we feel weightlessness. Moon is also a satellite of earth. But we do not feel weightlessness on moon.

   Reason : Mass of moon is considerable.

9. **Assertion :** Plane of geostationary satellites always passes through equator.

   Reason : Geostationary satellites always lies above Moscow.

10. **Assertion :** It we double the circular radius of a satellite, then its potential energy, kinetic energy and total mechanical energy will become half.

    Reason : Orbital speed of a satellite.

    $$v \propto \frac{1}{\sqrt{r}}$$

    where, r is its radius of orbit.

11. **Assertion :** If the radius of earth is decreased keeping its mass constant, effective value of g may increase or decrease at pole.

    Reason : Value of g on the surface of earth is given by $g = \frac{GM}{R^2}$.

Objective Questions

Single Correct Option

1. A satellite orbiting close to the surface of earth does not fall down because the gravitational pull of earth
   (a) is balanced by the gravitational pull of moon
   (b) is balanced by the gravitational pull of sun
   (c) provides the necessary acceleration for its motion along the circular path
   (d) makes it weightless

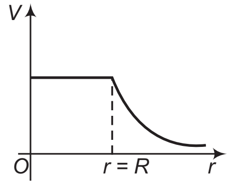
2. For the planet-sun system identify the correct statement.
   (a) the angular momentum of the planet is conserved about any point
   (b) the total energy of the system is conserved
   (c) the momentum of the planet is conserved
   (d) All of the above

3. If the earth stops rotating about its axis, then the magnitude of gravity
   (a) increases everywhere on the surface of earth
   (b) will increase only at the poles
   (c) will not change at the poles
   (d) All of the above

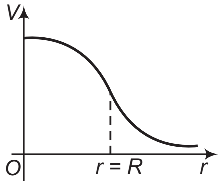
4. For a body to escape from earth, angle from horizontal at which it should be fired is
   (a) 45°
   (b) 0°
   (c) 90°
   (d) any angle

5. The correct variation of gravitational potential V with radius r measured from the centre of earth of radius R is given by

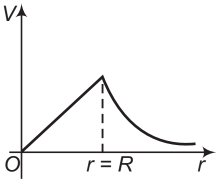
(a)



(b)



(c)



(d) None of these

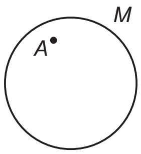
6. The Gauss' theorem for gravitational field may be written as

(a) $\oint \mathbf{g} \cdot d\mathbf{S} = \frac{m}{G}$ (b) $-\oint \mathbf{g} \cdot d\mathbf{S} = 4\pi mG$ (c) $\oint \mathbf{g} \cdot d\mathbf{S} = \frac{m}{4\pi G}$ (d) $-\oint \mathbf{g} \cdot d\mathbf{S} = \frac{m}{G}$

7. In the earth-moon system, if T_1 and T_2 are period of revolution of earth and moon respectively about the centre of mass of the system then

(a) $T_1 > T_2$ (b) $T_1 = T_2$ (c) $T_1 < T_2$ (d) Insufficient data

8. The figure shows a spherical shell of mass M. The point A is not at the centre but away from the centre of the shell. If a particle of mass m is placed at A, then

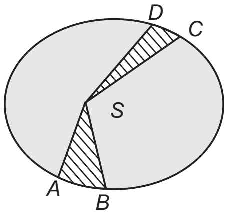


(a) it remains at rest
(b) it experiences a net force towards the centre
(c) it experiences a net force away from the centre
(d) None of the above

9. If the distance between the earth and the sun were reduced to half its present value, then the number of days in one year would have been

(a) 65 (b) 129
(c) 183 (d) 730

10. The figure represents an elliptical orbit of a planet around sun. The planet takes time T_1 to travel from A to B and it takes time T_2 to travel from C to D. If the area CSD is double that of area ASB, then



(a) $T_1 = T_2$
(b) $T_1 = 2T_2$
(c) $T_1 = 0.5T_2$
(d) Data insufficient

11. At what depth from the surface of earth the time period of a simple pendulum is 0.5% more than that on the surface of the Earth? (Radius of earth is 6400 km)

(a) 32 km (b) 64 km (c) 96 km (d) 128 km

12. If M is the mass of the earth and R its radius, the ratio of the gravitational acceleration and the gravitational constant is

(a) $\frac{R^2}{M}$ (b) $\frac{M}{R^2}$ (c) MR^2 (d) $\frac{M}{R}$

13. The height above the surface of earth at which the gravitational field intensity is reduced to 1% of its value on the surface of earth is

(a) $100R_e$ (b) $10R_e$ (c) $99R_e$ (d) $9R_e$

14. For a satellite orbiting close to the surface of earth the period of revolution is 84 min. The time period of another satellite orbiting at a height three times the radius of earth from its surface will be

(a) $(84)\,2\sqrt{2}$ min (b) 8 (84) min

(c) $(84)\,3\sqrt{3}$ min (d) 3 (84) min

15. The angular speed of rotation of earth about its axis at which the weight of man standing on the equator becomes half of his weight at the poles is given by

(a) 0.034 rad s^{-1} (b) 8.75×10^{-4} rad s^{-1}

(c) 1.23×10^{-2} rad s^{-1} (d) 7.65×10^{-7} rad s^{-1}

16. The height from the surface of earth at which the gravitational potential energy of a ball of mass m is half of that at the centre of earth is (where R is the radius of earth)

(a) $\frac{R}{4}$ (b) $\frac{R}{3}$ (c) $\frac{3R}{4}$ (d) $\frac{4R}{3}$

17. A body of mass m is lifted up from the surface of earth to a height three times the radius of the earth R. The change in potential energy of the body is

(a) $3mgR$ (b) $\frac{5}{4}mgR$

(c) $\frac{3}{4}mgR$ (d) $2mgR$

18. A satellite is revolving around earth in its equatorial plane with a period T. If the radius of earth suddenly shrinks to half without change in the mass. Then, the new period of revolution will be

(a) $8T$ (b) $2\sqrt{2}\,T$

(c) $2T$ (d) T

19. A planet has twice the density of earth but the acceleration due to gravity on its surface is exactly the same as that on the surface of earth. Its radius in terms of earth's radius R will be

(a) $R/4$ (b) $R/2$

(c) $R/3$ (d) $R/8$

20. The speed of earth's rotation about its axis is ω. Its speed is increased to x times to make the effective acceleration due to gravity equal to zero at the equator, then x is around ($g = 10$ ms^{-2}; $R = 6400$ km)

(a) 1 (b) 8.5

(c) 17 (d) 34

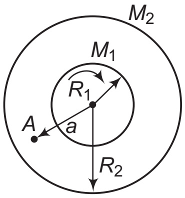
21. A satellite is seen every 6 h over the equator. It is known that it rotates opposite to that of earth's direction. Then, the angular velocity (in radian per hour) of satellite about the centre of earth will be

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

22. For a planet revolving around sun, if a and b are the respective semi-major and semi-minor axes, then the square of its time period is proportional to

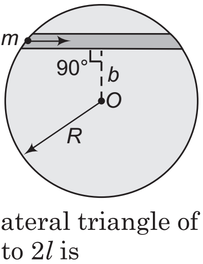
(a) $\left(\frac{a+b}{2}\right)^3$ (b) $\left(\frac{a-b}{2}\right)^3$

(c) b^3 (d) a^3



23. The figure represents two concentric shells of radii R_1 and R_2 and masses M_1 and M_2 respectively. The gravitational field intensity at the point A at distance a ($R_1 < a < R_2$) is

(a) $\dfrac{G(M_1+M_2)}{a^2}$
(b) $\dfrac{GM_1}{a^2}+\dfrac{GM_2}{R_2^2}$
(c) $\dfrac{GM_1}{a^2}$
(d) zero



24. A straight tunnel is dug into the earth as shown in figure at a distance b from its centre. A ball of mass m is dropped from one of its ends. The time it takes to reach the other end is approximately

(a) 42 min
(b) 84 min
(c) $84\left(\dfrac{b}{R}\right)$ min
(d) $42\left(\dfrac{b}{R}\right)$ min

25. Three identical particles each of mass M are placed at the corners of an equilateral triangle of side l. The work done by external force to increase the side of triangle from l to $2l$ is

(a) $-\dfrac{3}{2}\dfrac{GM^2}{l}$
(b) $\dfrac{-3GM^2}{l}$
(c) $\dfrac{3}{2}\dfrac{GM^2}{l}$
(d) $\dfrac{3GM^2}{l}$

26. A particle is thrown vertically upwards from the surface of earth and it reaches to a maximum height equal to the radius of earth. The ratio of the velocity of projection to the escape velocity on the surface of earth is

(a) $\dfrac{1}{\sqrt{2}}$
(b) $\dfrac{1}{2}$
(c) $\dfrac{1}{4}$
(d) $\dfrac{1}{2\sqrt{2}}$

27. The gravitational potential energy of a body at a distance r from the centre of earth is U. Its weight at a distance $2r$ from the centre of earth is

(a) $\dfrac{U}{r}$
(b) $\dfrac{U}{2r}$
(c) $\dfrac{U}{4r}$
(d) $\dfrac{U}{\sqrt{2}r}$

Subjective Questions

1. A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere.

2. What is the fractional decrease in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation h? ($h << R$)

3. Two masses m_1 and m_2 at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart.

4. If a satellite is revolving close to a planet of density ρ with period T, show that the quantity ρT^2 is a universal constant.

5. A satellite is revolving around a planet in a circular orbit. What will happen, if its speed is increased from v_0 to

(a) $\sqrt{1.5}\,v_0$
(b) $2v_0$

6. If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?

7. Two concentric spherical shells have masses m_1, m_2 and radii $R_1, R_2 (R_1 < R_2)$. Calculate the force exerted by this system on a particle of mass m, if it is placed at a distance $\frac{(R_1 + R_2)}{2}$ from the centre.

8. Find the force of attraction on a particle of mass m placed at the centre of a semicircular wire of length L and mass M.

9. A rocket is accelerated to speed $v = 2\sqrt{gR}$ near the earth's surface (R = radius of earth). Show that very far from earth its speed will be $v = \sqrt{2gR}$.

10. Two spheres one of mass M has radius R. Another sphere has mass $4M$ and radius $2R$. The centre to centre distance between them is $12R$. Find the distance from the centre of smaller sphere where

(a) net gravitational field is zero,

(b) net gravitational potential is half the potential on the surface of larger sphere.

11. A uniform solid sphere of mass M and radius a is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance (a) $\frac{3}{2}a$ from the centre, (b) $\frac{5}{2}a$ from the centre.

12. The density inside a solid sphere of radius a is given by $\rho = \rho_0 a/r$, where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centre.

13. Two neutron stars are separated by a distance of 10^{10} m. They each have a mass of 10^{30} kg and a radius of 10^5 m. They are initially at rest with respect to each other.

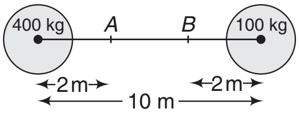
As measured from the rest frame, how fast are they moving when

(a) their separation has decreased to one-half its initial value,

(b) they are about to collide.

14. A mass m is taken to a height R from the surface of the earth and then is given a vertical velocity v. Find the minimum value of v, so that mass never returns to the surface of the earth. (Radius of earth is R and mass of the earth M).

15. In the figure masses 400 kg and 100 kg are fixed.



(a) How much work must be done to move a 1 kg mass from point A to point B ?

(b) What is the minimum kinetic energy with which the 1 kg mass must be projected from A to the right to reach the point B ?

16. Two identical stars of mass M orbit around their centre of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.

(a) Find the gravitational force of one star on the other.

(b) Find the orbital speed of each star and the period of the orbit.

(c) What minimum energy would be required to separate the two stars to infinity ?

17. Consider two satellites A and B of equal mass, moving in the same circular orbit of radius r around the earth but in the opposite sense and therefore a collision occurs.

(a) Find the total mechanical energy $E_A + E_B$ of the two satellite-plus-earth system before collision.

(b) If the collision is completely inelastic, find the total mechanical energy immediately after collision. Describe the subsequent motion of the combined satellite.

18. In a certain binary star system, each star has the same mass as our sun. They revolve about their centre of mass. The distance between them is the same as the distance between earth and the sun. What is their period of revolution in years ?

19. (a) Does it take more energy to get a satellite upto 1500 km above earth than to put it in circular orbit once it is there.

(b) What about 3185 km?

(c) What about 4500 km? (Take $R_e = 6370$ km)

LEVEL 2

Objective Questions

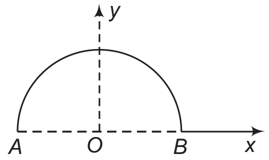
Single Correct Option

1. An artificial satellite of mass m is moving in a circular orbit at a height equal to the radius R of the earth. Suddenly due to internal explosion the satellite breaks into two parts of equal pieces. One part of the satellite stops just after the explosion. The increase in the mechanical energy of the system due to explosion will be
(Given, acceleration due to gravity on the surface of earth is g)

(a) mgR (b) $\frac{mgR}{2}$

(c) $\frac{mgR}{4}$ (d) $\frac{3mgR}{4}$

2. Gravitational field at the centre of a semicircle formed by a thin wire AB of mass M and length l is



(a) $\frac{GM}{l^2}$ along x-axis (b) $\frac{GM}{\pi l^2}$ along y-axis

(c) $\frac{2\pi GM}{l^2}$ along x-axis (d) $\frac{2\pi GM}{l^2}$ along y-axis

3. Four particles, each of mass M, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(a) $\frac{GM}{R}$ (b) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

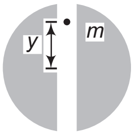
(c) $\sqrt{\frac{GM}{R}(2\sqrt{2}+1)}$ (d) $\sqrt{\frac{GM}{R}\frac{2\sqrt{2}+1}{4}}$

4. A projectile is fired from the surface of earth of radius R with a velocity kv_e (where v_e is the escape velocity from surface of earth and $k < 1$). Neglecting air resistance, the maximum height of rise from the centre of earth is

(a) $\dfrac{R}{k^2 - 1}$ (b) k^2R

(c) $\dfrac{R}{1 - k^2}$ (d) kR

5. Suppose a vertical tunnel is dug along the diameter of earth, which is assumed to be a sphere of uniform mass density ρ. If a body of mass m is thrown in this tunnel, its acceleration at a distance y from the centre is given by



(a) $\dfrac{4\pi}{3} G\rho y m$ (b) $\dfrac{3}{4}\pi\rho y$

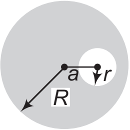
(c) $\dfrac{4}{3}\pi\rho y$ (d) $\dfrac{4}{3}\pi G\rho y$

6. A train of mass m moves with a velocity v on the equator from east to west. If ω is the angular speed of earth about its axis and R is the radius of the earth then the normal reaction acting on the train is

(a) $mg\left[1 - \dfrac{(\omega R - 2v)\,\omega}{g} - \dfrac{v^2}{Rg}\right]$ (b) $mg\left[1 - 2\dfrac{(\omega R - v)\,\omega}{g} - \dfrac{v^2}{Rg}\right]$

(c) $mg\left[1 - \dfrac{(\omega R + 2v)\,\omega}{g} - \dfrac{v^2}{Rg}\right]$ (d) $mg\left[1 - 2\dfrac{(\omega R - v)\,\omega}{g} - \dfrac{v^2}{Rg}\right]$

7. The figure represents a solid uniform sphere of mass M and radius R. A spherical cavity of radius r is at a distance a from the centre of the sphere. The gravitational field inside the cavity is



(a) non-uniform (b) towards the centre of the cavity

(c) directly proportional to a (d) All of these

8. If v_e is the escape velocity for earth when a projectile is fired from the surface of earth. Then, the escape velocity if the same projectile is fired from its centre is

(a) $\sqrt{\dfrac{3}{2}}\,v_e$ (b) $\dfrac{3}{2}\,v_e$

(c) $\sqrt{\dfrac{2}{3}}\,v_e$ (d) $\dfrac{2}{3}\,v_e$

9. If the gravitational field intensity at a point is given by $g = \dfrac{GM}{r^{2.5}}$. Then, the potential at a distance r is

(a) $\dfrac{-2GM}{3r^{1.5}}$ (b) $\dfrac{-GM}{r^{2.5}}$

(c) $\dfrac{2GM}{3r^{1.5}}$ (d) $\dfrac{GM}{r^{3.5}}$

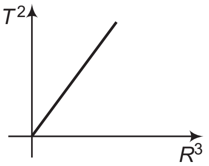
10. Three identical particles each of mass M move along a common circular path of radius R under the mutual interaction of each other. The velocity of each particle is

(a) $\sqrt{\dfrac{GM}{R}\sqrt{\dfrac{2}{3}}}$ (b) $\sqrt{\dfrac{GM}{\sqrt{3}R}}$

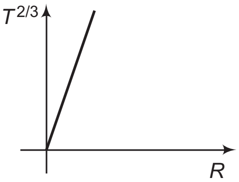
(c) $\sqrt{\dfrac{GM}{3R}}$ (d) $\sqrt{\dfrac{2}{3}\dfrac{GM}{R}}$

11. If T be the period of revolution of a planet revolving around sun in an orbit of mean radius R, then identify the **incorrect** graph.

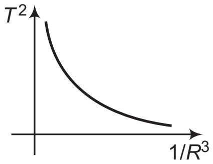
(a)



(b)



(c)

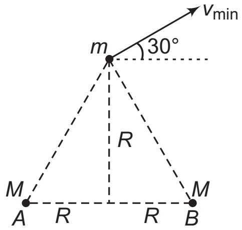


(d) None of these

12. A person brings a mass of 1 kg from infinity to a point A. Initially, the mass was at rest but it moves at a speed of 3 m/s as it reaches A. The work done by the person on the mass is – 5.5 J. The gravitational potential at A is

(a) – 1 J/kg (b) – 4.5 J/kg (c) – 5.5 J/kg (d) – 10 J/kg

13. With what minimum speed should m be projected from point C in presence of two fixed masses M each at A and B as shown in the figure such that mass m should escape the gravitational attraction of A and B?



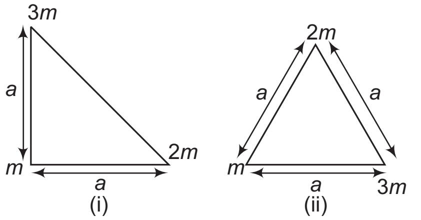
(a) $\sqrt{\frac{2GM}{R}}$

(b) $\sqrt{\frac{2\sqrt{2}GM}{R}}$

(c) $2\sqrt{\frac{GM}{R}}$

(d) $2\sqrt{2}\sqrt{\frac{GM}{R}}$

14. Consider two configurations of a system of three particles of masses m, $2m$ and $3m$. The work done by gravity in changing the configuration of the system from figure (i) to figure (ii) is



(i) (ii)

(a) zero (b) $\frac{6\,Gm^2}{a}\left\{1+\frac{1}{\sqrt{2}}\right\}$ (c) $\frac{6\,Gm^2}{a}\left\{1-\frac{1}{\sqrt{2}}\right\}$ (d) $\frac{6\,Gm^2}{a}\left\{2-\frac{1}{\sqrt{2}}\right\}$

15. A tunnel is dug along the diameter of the earth. There is a particle of mass m at the centre of the tunnel. Find the minimum velocity given to the particle so that is just reaches to the surface of the earth.(R = radius of earth)

(a) $\sqrt{\frac{GM}{R}}$

(b) $\sqrt{\frac{GM}{2R}}$

(c) $\sqrt{\frac{2GM}{R}}$

(d) it will reach with the help of negligible velocity

16. A body is projected horizontally from the surface of the Earth (radius = R) with a velocity equal to n times the escape velocity. Neglect rotational effects of the earth. The maximum height attained by the body from the earth's surface is $R/2$. Then, n must be

(a) $\sqrt{0.6}$ (b) $(\sqrt{3})/2$ (c) $\sqrt{0.4}$ (d) 1/2

17. A tunnel is dug in the earth across one of its diameter. Two masses m and $2m$ are dropped from the two ends of the tunnel. The masses collide and stick each other. They perform SHM, the amplitude of which is (R = radius of earth)

(a) R (b) $R/2$ (c) $R/3$ (d) $2R/3$

18. There are two planets. The ratio of radius of the two planets is k but ratio of acceleration due to gravity of both planets is g. What will be the ratio of their escape velocity?

(a) $(kg)^{1/2}$ (b) $(kg)^{-1/2}$ (c) $(kg)^2$ (d) $(kg)^{-2}$

19. A body of mass 2 kg is moving under the influence of a central force whose potential energy is given by $U = 2r^3$ J. If the body is moving in a circular orbit of 5 m, its energy will be

(a) 625 J (b) 250 J (c) 500 J (d) 125 J

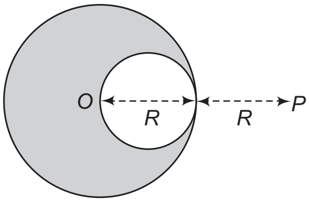
20. A research satellite of mass 200 kg circles the earth in an orbit of average radius $3R/2$, where R is the radius of the earth. Assuming the gravitational pull on the mass of 1 kg on the earth's surface to be 10 N, the pull on the satellite will be

(a) 1212 N (b) 889 N (c) 1280 N (d) 960 N

21. A satellite of mass m revolves around the earth of radius R at a height x from its surface. If g is the acceleration due to gravity on the surface of the earth, the orbital speed of the satellite is

(a) $\sqrt{gx}$ (b) $\sqrt{\dfrac{gR}{R-x}}$ (c) $\sqrt{\dfrac{gR^2}{R-x}}$ (d) $\sqrt{\dfrac{gR^2}{R+x}}$

22. A solid sphere of uniform density and radius R applies a gravitational force of attraction equal to F_1 on a particle placed at P, distance $2R$ from the centre O of the sphere. A spherical cavity of radius $R/2$ is now made in the sphere as shown in figure. The sphere with cavity now applies a gravitational force F_2 on same particle placed at P. The ratio F_2/F_1 will be



(a) 1/2 (b) 7/9
(c) 3 (d) 7

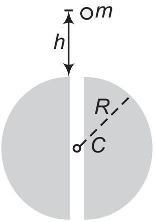
More than One Correct Options

1. Three planets of same density have radii R_1, R_2 and R_3 such that $R_1 = 2R_2 = 3R_3$. The gravitational field at their respective surfaces are g_1, g_2 and g_3 and escape velocities from their surfaces are v_1, v_2 and v_3, then

(a) $g_1/g_2 = 2$ (b) $g_1/g_3 = 3$
(c) $v_1/v_2 = 1/4$ (d) $v_1/v_3 = 3$

2. For a geostationary satellite orbiting around the earth identify the necessary condition.

(a) it must lie in the equatorial plane of earth
(b) its height from the surface of earth must be 36000 km
(c) it period of revolution must be $2\pi\sqrt{\dfrac{R}{g}}$, where R is the radius of earth
(d) its period of revolution must be 24 hrs



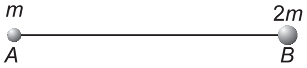
3. A ball of mass m is dropped from a height h equal to the radius of the earth above the tunnel dug through the earth as shown in the figure. Choose the correct options. (Mass of earth = M)

(a) Particle will oscillate through the earth to a height h on both sides

(b) Particle will execute simple harmonic motion

(c) Motion of the particle is periodic

(d) Particle passes the centre of earth with a speed $v = \sqrt{\frac{2GM}{R}}$



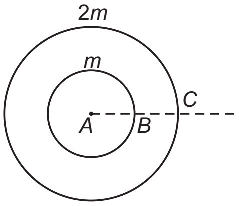
4. Two point masses m and $2m$ are kept at points A and B as shown. E represents magnitude of gravitational field strength and V the gravitational potential. As we move from A to B

(a) E will first decrease then increases

(b) E will first increase then decrease

(c) V will first decrease then increase

(d) V will first increase then decrease



5. Two spherical shells have masses m and $2m$ as shown. Choose the correct options.

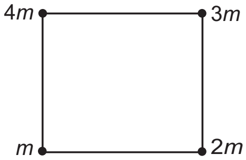
(a) Between A and B gravitational field strength is zero

(b) Between A and B gravitational potential is constant

(c) There will be two points one lying between B and C and other lying between C and infinity where gravitational field strength are same

(d) There will be a point between B and C where gravitational potential will be zero

6. Four point masses are placed at four corners of a square as shown. When positions of m and $2m$ are interchanged



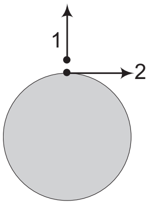
(a) gravitational field strength at centre will increase

(b) gravitational field strength at centre will decrease

(c) gravitational potential at centre will remain unchanged

(d) gravitational potential at centre will decrease

7. Two identical particles 1 and 2 are projected from surface of earth with same velocities in the directions shown in figure.



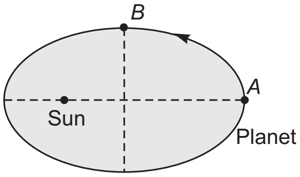
(a) Both the particles will stop momentarily (before striking with ground) at different times

(b) Particle-2 will rise upto lesser height compared to particle-2

(c) Minimum speed of particle-2 is more than that of particle-1

(d) Particle-1 will strike the ground earlier

8. A planet is moving round the sun in an elliptical orbit as shown. As the planet moves from A to B



(a) its kinetic energy will decrease
(b) its potential energy will remain unchanged
(c) its angular momentum about centre of sun will remain unchanged
(d) its speed is minimum at A

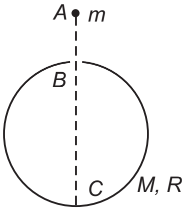
9. A satellite of mass m is just placed over the surface of earth. In this position mechanical energy of satellite is E_1. Now it starts orbiting round the earth in a circular path at height h = radius of earth. In this position, kinetic energy, potential energy and total mechanical energy of satellite are K_2, U_2 and E_2 respectively. Then

(a) $U_2 = \frac{E_1}{2}$ (b) $E_2 = \frac{E_1}{4}$ (c) $K_2 = -E_2$ (d) $K_2 = -\frac{U_2}{2}$

10. A satellite is revolving round the earth in circular orbit
(a) if mass of earth is made four times, keeping other factors constant, orbital speed of satellite will become two times
(b) corresponding to change in part (a), times period of satellite will remain half
(c) when value of G is made two times orbital speed increases and time period decreases
(d) G has no effect on orbital speed and time period

Match the Columns

1. There is a small hole in a spherical shell of mass M and radius R. A particle of mass m is dropped from point A as shown. Match the two columns for the situation shown in figure.



Column I	Column II
(a) Potential energy from A to B	(p) Continuously increases
(b) Potential energy from B to C	(q) Continuously decreases
(c) Speed of particle from B to C	(r) First increases then remains constant
(d) Acceleration of particle from A to C	(s) None of these

2. Five point masses m each are placed at five corners of a regular pentagon. Distance of any corner from centre is r. Match the following two columns.

Column I	Column II
(a) Gravitational field strength at centre	(p) Gm/r^2
(b) Gravitational potential at centre	(q) $4Gm/r$
(c) When one mass is removed gravitational field strength at centre	(r) zero
(d) When one mass is removed gravitational potential at centre	(s) None of these

3. | Potential | on the surface of a solid sphere is x and radius is y. Match the following two columns.

Column I	Column II
(a) Field strength at distance $2y$ from centre	(p) $\frac{x}{2y}$
(b) | Potential | at distance $\frac{y}{2}$ from centre	(q) $\frac{x}{2}$
(c) Field strength at distance $y/2$ from centre	(r) $\frac{x}{4y}$
(d) | Potential | at distance $2y$ from centre	(s) None

4. Match the following two columns.

Column I	Column II
(a) Work done is raising a mass m to a height $h = R$	(p) $\frac{1}{4} mgR$
(b) Kinetic energy of a satellite of mass m at height $h = R$	(q) mgR
(c) Difference in energies of two satellites each of mass m but one at height $h_1 = R$ and another of height $h_2 = 2R$	(r) $\frac{1}{2} mgR$
(d) Kinetic energy required to raise a particle of mass m to a height $h = R$ if projected vertically from surface of earth.	(s) None

5. Match the following two columns.

Column I	Column II
(a) Gravitational field strength is maximum at	(p) $r = 0$
(b) Gravitational field strength is zero at	(q) $r = R$
(c) Gravitational potential is minimum at	(r) $r = \frac{R}{\sqrt{2}}$
(d) Gravitational potential is zero at	(s) None of these

Here, r is distance from centre of a solid sphere or distance from centre of a ring along its axis.

Subjective Questions

1. Three particles of mass m each are placed at the three corners of an equilateral triangle of side a. Find the work which should be done on this system to increase the side of the triangle to $2a$.

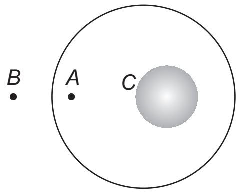
2. A man can jump vertically to a height of 1.5 m on the earth. Calculate the radius of a planet of the same mean density as that of the earth from whose gravitational field he could escape by jumping. Radius of earth is 6.41×10^6 m.

3. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the surface of earth. (Radius of earth = 6400 km)

(a) Determine the height of the satellite above the earth's surface.

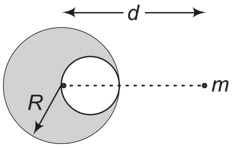
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the earth, find the speed with which it hits the surface of earth.

4. A uniform metal sphere of radius R and mass m is surrounded by a thin uniform spherical shell of same mass and radius $4R$. The centre of the shell C falls on the surface of the inner sphere. Find the gravitational fields at points A and B.

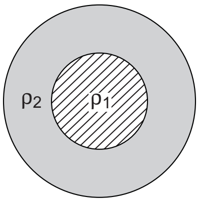


$AC = 3R, \quad BC = 5R$

5. Figure shows a spherical cavity inside a lead sphere. The surface of the cavity passes through the centre of the sphere and touches the right side of the sphere. The mass of the sphere before hollowing was M. With what gravitational force does the hollowed out lead sphere attract a particle of mass m that lies at a distance d from the centre of the lead sphere on the straight line connecting the centres of the spheres and of the cavity.



6. The density of the core of a planet is ρ_1 and that of the outer shell is ρ_2, the radii of the core and that of the planet are R and $2R$ respectively. The acceleration due to gravity at the surface of the planet is same as at a depth R. Find the ratio of $\frac{\rho_1}{\rho_2}$.



7. If a satellite is revolving around a planet of mass M in an elliptical orbit of semi-major axis a. Show that the orbital speed of the satellite when it is at a distance r from the focus will be given by

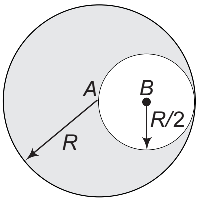
$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

8. A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance $\sqrt{3}a$ from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.

9. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

10. A smooth tunnel is dug along the radius of earth that ends at centre. A ball is released from the surface of earth along tunnel. Coefficient of restitution for collision between soil at centre and ball is 0.5. Calculate the distance travelled by ball just before second collision at centre. Given mass of the earth is M and radius of the earth is R.

11. Inside a fixed sphere of radius R and uniform density ρ, there is spherical cavity of radius $\frac{R}{2}$ such that surface of the cavity passes through the centre of the sphere as shown in figure. A particle of mass m_0 is released from rest at centre B of the cavity. Calculate velocity with which particle strikes the centre A of the sphere. Neglect earth's gravity. Initially sphere and particle are at rest.

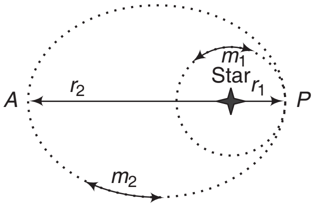


12. A ring of radius $R = 4$ m is made of a highly dense material. Mass of the ring is $m_1 = 5.4 \times 10^9$ kg distributed uniformly over its circumference. A highly dense particle of mass $m_2 = 6 \times 10^8$ kg is placed on the axis of the ring at a distance $x_0 = 3$ m from the centre. Neglecting all other forces, except mutual gravitational interaction of the two. Calculate

(i) displacement of the ring when particle is at the centre of ring, and

(ii) speed of the particle at that instant.

13. Two planets of equal mass orbit a much more massive star (figure). Planet m_1 moves in a circular orbit of radius 1×10^8 km with period 2 yr. Planet m_2 moves in an elliptical orbit with closest distance $r_1 = 1 \times 10^8$ km and farthest distance $r_2 = 1.8 \times 10^8$ km, as shown.



(a) Using the fact that the mean radius of an elliptical orbit is the length of the semi-major axis, find the period of m_2's orbit.

(b) Which planet has the greater speed at point P ? Which has the greater total energy ?

(c) Compare the speed of planet m_2 at P with that at A.

14. In a double star, two stars one of mass m_1 and another of mass m_2, with a separation d, rotate about their common centre of mass. Find

(a) an expression for their time period of revolution.

(b) the ratio of their kinetic energies.

(c) the ratio of their angular momenta about the centre of mass.

(d) the total angular momentum of the system.

(e) the kinetic energy of the system.

Answers

Introductory Exercise 13.1

1. $\frac{\sqrt{3}\,GM^2}{4a^2}$ **2.** $\frac{4\sqrt{2}\,Gm^2}{a^2}$ **3.** $a_1 = 5.3 \times 10^{-10}\ \text{ms}^{-2}$, $a_2 = 2.65 \times 10^{-10}\ \text{ms}^{-2}$ **4.** $\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)\frac{Gm^2}{d^2}$ (along *PB*)

Introductory Exercise 13.2

1. $4.9\ \text{ms}^{-2}$ **2.** (a) $2.45\ \text{ms}^{-2}$ (b) $4.9\ \text{ms}^{-2}$ **3.** $\frac{(\sqrt{5}-1)R}{2}$, where R is the radius of earth **4.** $-0.0168\ \text{ms}^{-2}$

5. 1600 km **6.** 7.8×10^{-4} rad/s **7.** 997 N **8.** 1.237×10^{-3} rad/s, 84.6 min

9. Approximately 10 km

Introductory Exercise 13.3

1. $\frac{-2Gm}{a}, \frac{\sqrt{3}\,Gm}{a^2}$ **2.** $\frac{-5\,Gm}{a}$, zero **3.** $\frac{-4\,Gm}{a}, \frac{Gm}{a^2}$ **4.** $\frac{-3\,Gm}{R}$ **5.** 200 N/kg along $+x$ - direction

Introductory Exercise 13.4

1. $-[6xy\hat{\mathbf{i}} + (3x^2 + 3y^2z)\,\hat{\mathbf{j}} + y^3\,\hat{\mathbf{k}}]$ **2.** False **3.** $10\sqrt{2}$ N **4.** zero

Introductory Exercise 13.5

1. $2.1 \times 10^{-5}\ \text{ms}^{-1}$ and 4.2×10^{-5} m/s **2.** $\frac{-Gm^2}{a}(4+\sqrt{2})$ **3.** $\frac{1}{2}\,mgR$ **5.** 2.51×10^4 km **6.** 7.9 km/s

Introductory Exercise 13.6

1. mgR **2.** 11.2 km/s **3.** (a) No (b) $-E_0$

Introductory Exercise 13.7

1. No **2.** $56\sqrt{2}$ h **3.** 2 : 1, 2 : 1

4. (a) 6.90 km/s (b) 2.38×10^{10} J (c) -4.76×10^{10} J with usual reference (d) $\left(\frac{21}{106}\right)^{3/2}(24)$ h

5. (a) 9.6×10^{10} J (b) 1.07×10^{10} J

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (d) **7.** (d) **8.** (a) **9.** (c) **10.** (b)
11. (d)

Single Correct Option

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (d) **6.** (b) **7.** (b) **8.** (a) **9.** (b) **10.** (c)
11. (b) **12.** (b) **13.** (d) **14.** (b) **15.** (b) **16.** (b) **17.** (c) **18.** (d) **19.** (b) **20.** (c)
21. (c) **22.** (d) **23.** (c) **24.** (a) **25.** (c) **26.** (a) **27.** (c)

Subjective Questions

1. 1.334×10^{-9} J **2.** $\frac{dg}{g} = -2\left(\frac{h}{R}\right)$ **3.** $\sqrt{\frac{2G(m_1 + m_2)}{r}}$

5. (a) Orbit will become elliptical (b) The satellite will escape

6. 6 h **7.** $F = \frac{4Gm_1 m}{(R_1 + R_2)^2}$ **8.** $\frac{2\pi GMm}{L^2}$ **10.** (a) $4R$ (b) $7.65\,R$ and $1.49\,R$

11. (a) $\frac{4GM}{9a^2}$ (towards the centre) (b) $\frac{8GM}{25a^2}$ (towards the centre) **12.** $\frac{\pi G\rho_0 a}{2}$

13. (a) 81.6 km/s (b) 1.8×10^4 kms^{-1} **14.** $v = \sqrt{\frac{GM}{R}}$

15. (a) 7.5×10^{-9} J (b) 8.17×10^{-9} J

16. (a) $F = \frac{GM^2}{4R^2}$ (b) $v = \sqrt{\frac{GM}{4R}}, T = \frac{4\pi R^{3/2}}{\sqrt{GM}}$ (c) $\frac{GM^2}{4R}$ **17.** (a) $\frac{-GMm}{r}$ (b) $\frac{-2GMm}{r}$

18. 0.71 yr **19.** (a) No (b) Same (c) Yes

LEVEL 2

Single Correct Option

1. (c) **2.** (d) **3.** (d) **4.** (c) **5.** (d) **6.** (a) **7.** (c) **8.** (a) **9.** (a) **10.** (b)
11. (d) **12.** (d) **13.** (b) **14.** (c) **15.** (a) **16.** (a) **17.** (c) **18.** (d) **19.** (a) **20.** (b)
21. (d) **22.** (b)

More than One Correct Options

1. (a,b,d) **2.** (a,b,d) **3.** (a,c,d) **4.** (a,d) **5.** (a,b,c) **6.** (a,c) **7.** (b,c,d)
8. (c,d) **9.** (all) **10.** (a,b,c)

Match the Columns

1. (a) → q (b) → s (c) → s (d) → r
2. (a) → r (b) → s (c) → p (d) → s
3. (a) → r (b) → s (c) → p (d) → q
4. (a) → r (b) → p (c) → s (d) → r
5. (a) → q,r (b) → p (c) → p (d) → s

Subjective Questions

1. $\frac{3Gm^2}{2a}$ **2.** 3.1×10^3 m **3.** (a) 6400 km (b) 7.92 kms^{-1} **4.** $\frac{Gm}{16R^2}, \frac{61Gm}{900R^2}$

5. $\frac{GMm}{d^2}\left[1 - \frac{1}{8(1 - R/2d)^2}\right]$ **6.** 7/3 **8.** $\frac{\sqrt{3}GMm}{8a^2}$ **9.** $\frac{3}{2}\sqrt{\frac{5GM}{a}}$ **10.** $d = 2R$

11. $\sqrt{\frac{2}{3}\pi G\rho R^2}$ **12.** (i) 0.3 m (ii) 18 cm/s

13. (a) 3.31 yr (b) m_2 has greater speed and greater total energy (c) $v_P = 1.8v_A$

14. (a) $2\pi\sqrt{\frac{d^3}{G(m_1 + m_2)}}$ (b) $\frac{m_2}{m_1}$ (c) $\frac{m_2}{m_1}$ (d) $\mu\omega d^2$

(e) $\frac{1}{2}\mu\omega^2 d^2$, where μ is the reduced mass and ω the angular velocity.

14

Simple Harmonic Motion

Chapter Contents

14.1 Introduction
14.2 Displacement Equation of SHM
14.3 Time Equation of SHM
14.4 Relation between SHM and Uniform Circular Motion
14.5 Methods of Finding Time Period of a SHM
14.6 Vector Method of Combining Two or More SHM

14.1 Introduction

Following are given some general points regarding motion, periodic motion and simple harmonic motion (SHM):

1. In general, motion of a body (or its path) depends on two factors:
   (i) the nature of force (or acceleration) of the body and
   (ii) its velocity

   For example

   A constant force or constant acceleration always gives a straight line or parabolic path. If initial velocity is zero or parallel (or antiparallel) to constant acceleration then path is straight line. In all other cases, path is a parabola. For small height, acceleration due to gravity $(\mathbf{a}=\mathbf{g})$ is constant. So, path is either straight line or parabola.

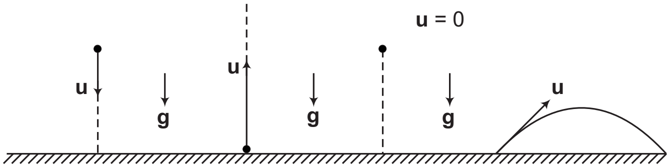


Fig. 14.1

   If force of constant magnitude is acting on a particle and its direction is always perpendicular to velocity, then path is circular motion in which speed is constant. This is also called uniform circular motion.

$$|\mathbf{F}_1|=|\mathbf{F}_2|=|\mathbf{F}_3|=|\mathbf{F}_4|=\frac{mv^2}{R}=\text{centripetal force}$$

$$|\mathbf{v}_1|=|\mathbf{v}_2|=|\mathbf{v}_3|=|\mathbf{v}_4|=v=\text{speed of the particle.}$$

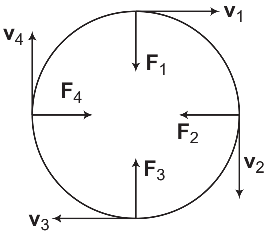


Fig. 14.2

   Now let us consider a particle free to move along x-axis, which is being acted upon by a force given by,

$$F=-kx^n$$

   Here, k is a positive constant.

   Now, following cases are possible depending on the value of n :
   (i) If n is an even integer (0, 2, 4, … etc), force is always along negative x-axis. If the particle is released from any position on the x-axis (except at $x=0$) a force in negative direction of x-axis acts on it and it moves rectilinearly along negative x-axis.
   (ii) If n is an odd integer (1, 3, 5, … etc), force is along negative x-axis for $x>0$, along positive x-axis for $x<0$ and zero for $x=0$. Thus, the particle will oscillate about stable equilibrium position (also called the mean position), $x=0$. The force in this case is called the restoring force. Of these, if $n=1$, i.e. $F=-kx$ the motion is said to be SHM.
2. In every oscillatory motion, there is one mean position (or stable equilibrium position) and two extreme positions.
3. Distance between mean position and the extreme position is called amplitude of oscillation A.

4. Oscillations does not start by itself. Normally the body has to be displaced from the mean position. In this displacement $F=-kx$ type force opposes the motion. So, work has to be done against this force which remains stored in the system in the form of mechanical energy. In the absence of any dissipative forces (like friction or viscous force) this mechanical energy remains constant. While moving from extreme positions to mean position potential energy decreases and kinetic energy increases but total mechanical energy remains constant. Similarly, in moving from mean position to extreme positions potential energy increases and kinetic energy decreases.
5. The more the initial displacement from the mean position, more is the amplitude, more is the initial work done and more is the mechanical energy given for oscillations.
6. Like any other motion, SHM is also a motion. So, basic characteristics of any motion (like $\mathbf{v}=\frac{d\mathbf{s}}{dt}$, $\mathbf{a}=\frac{d\mathbf{v}}{dt}$ etc.) can also be applied to SHM.
7. In most of the motions, displacement of the particle is measured from the starting point ($t=0$). But in SHM displacement is normally measured from the mean position and it is denoted by x (or y).
8. **At mean position**, $x=0$, $F=0$, $a=0$

   $v=$ maximum, kinetic energy $=$ maximum and potential energy is minimum (not necessarily zero)

   At extreme positions, $x=\pm A$, $F=$ maximum, $a=$ maximum, kinetic energy $=0$ and potential energy $=$ maximum.
9. SHM is called simple harmonic motion because its mathematics is simple. Mathematics of periodic motion of higher powers ($n=3, 5, 7$ etc in $F=-kx^n$) is slightly complex. This is the reason, we study only simple harmonic oscillatory motion at this stage.
10. In SHM, we shall discuss $x, A, v, a, F, K, U, E, T, f$ and ω. Here,

    $x=$ displacement from the mean position
    $A=$ amplitude of oscillations
    $v=$ velocity (or speed)
    $a=$ acceleration
    $F=$ force
    $K=$ kinetic energy
    $U=$ potential energy
    $E=$ total mechanical energy
    $T=$ time period of oscillations
    $f=$ frequency of oscillations and
    $\omega=$ angular frequency of oscillations.
11. SHM may be linear or angular. For example, motion of a pendulum for small amplitudes is angular SHM and motion of spring block system is linear SHM.
12. Like any other motion, equations of SHM can be written in terms of its displacement ($v^2=u^2+2as$) or time ($v=u+at$). The only difference is, in most of the equations of SHM 'x' is measured from the mean position.

14.2 Displacement Equations of SHM

In most of the equations of SHM displacement will be represented by x, where $x =$ displacement from the mean position.

Force Equation As we have discussed earlier also,

$$\boxed{F = -Kx}$$

is the force required for a particle to execute SHM. This type of force is obtained when a block is attached with a spring. Therefore, motion of block with a spring is always SHM.

Following are given some important points in this equation :

(i) x varies between $+A$ and $-A$

(ii) F - x graph is a straight line passing through $x = 0$ or the mean position (not necessarily the origin)

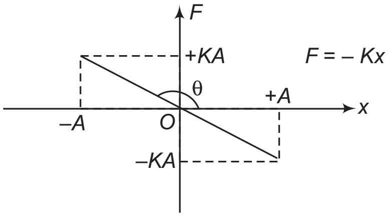


Fig. 14.3

At $x = \pm A$, $F = \mp KA$

(iii) $F = 0$ at $x = 0$ or at the mean position

(iv) $|F|_{\max} = KA$ at $x = \pm A$ or at the extreme positions.

(v) $F \propto -x$

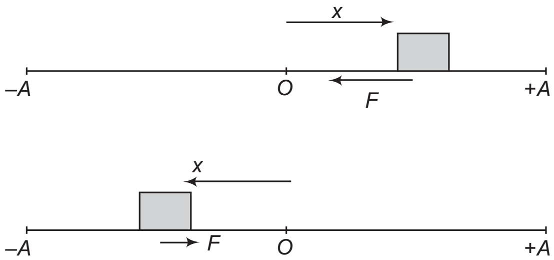


Fig. 14.4

Here, negative sign implies that direction of F and x are always opposite or direction of force is always towards the mean position. Hence, it is always restoring in nature.

Further, magnitude of force is proportional to magnitude of x. It means magnitude of force varies linearly with magnitude of x. If magnitude of x is doubled then magnitude of force will also become two times.

(vi) $\tan\theta =$ slope of F - x graph $= -K$

Acceleration Equation

$$\boxed{a=\frac{F}{m}=-\frac{K}{m}x=-\omega^2 x}$$

Here, $$\boxed{\omega^2=\frac{K}{m} \text{ or } \omega=\sqrt{\frac{K}{m}}}$$

Following are given some important points in above two equations:

(i) $\omega=\sqrt{\frac{K}{m}}$ is called angular frequency of SHM. Later we will see that most of the equations of SHM can be derived from uniform circular motion. Angular frequency of SHM is similar to angular speed ω of uniform circular motion. At both places we can find time period T and frequency f from ω.

$$\omega=\sqrt{\frac{K}{m}}=\frac{2\pi}{T}$$

$$\therefore \qquad \boxed{T=2\pi\sqrt{\frac{m}{K}}} \qquad \text{(in SHM)}$$

(ii) a - x graph is a straight line passing through $x=0$ or the mean position.

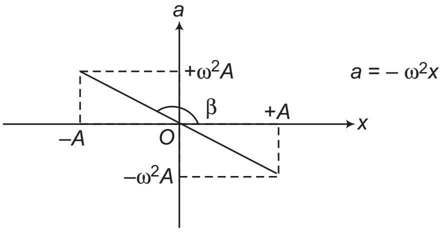


Fig. 14.5

$$\tan\beta=\text{Slope}=-\omega^2 \quad \Rightarrow \quad \text{At } x=\pm A,\ a=\mp\omega^2 A$$

(iii) $a=0$ at $x=0$, or at the mean position.

(iv) $|a|_{\max}=\omega^2 A$ at $x=\pm A$ or at extreme positions.

(v) $a \propto -x$

So, direction of 'a' is always opposite to the direction of x

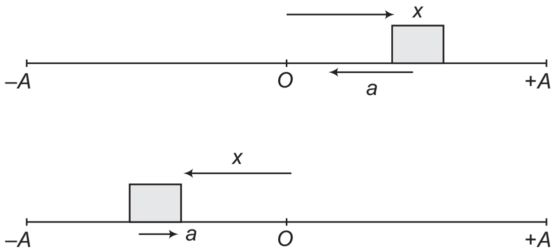


Fig. 14.6

or acceleration is always towards the mean position. Further, magnitude of 'a' is directly proportional to (linear variation) magnitude of 'x'.

Velocity Equation v - x graph in SHM is given by

$$\boxed{v=\pm\omega\sqrt{A^2-x^2}}$$

Note *We will derive this v-x equation in example* 14.7.

Following are given some important points in this equation:

(i) The given equation can be written as

$$\frac{v^2}{\omega^2}+\frac{x^2}{(1)^2}=A^2$$

Therefore, v - x graph is an ellipse as shown below.

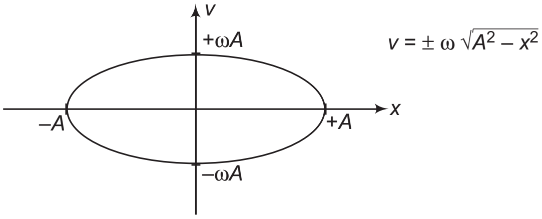


Fig. 14.7

(ii) $v=0$ at $x=\pm A$ or at extreme positions.

(iii) $v=\pm\omega A$ or $|v|_{\max}=\omega A$ at $x=0$, or at the mean position.

(iv) For upper half of the ellipse velocity is positive (when the body is moving from $-A$ to $+A$). For lower half of the ellipse velocity is negative (when the body is moving from $+A$ to $-A$).

(v) At mean position (or $x=0$), velocity is $+\omega A$ when the body is moving towards $+A$ and velocity is $-\omega A$ when the body is moving towards $-A$.

Energy Equations

Total Mechanical Energy

As we have discussed earlier also, at mean position potential energy is minimum (say U_0). This minimum potential energy U_0 may be zero, positive or negative. If not given in the question then we will take is zero. Now, suppose W is the work done in displacing the body from $x=0$ to $x=A$. Then, total mechanical energy is now $E=W+U_0$. Let us start with an example of spring-block system kept at height h, so that the block has already a potential energy, $U_0=mgh$ at mean position or natural length of spring where $F=0$.

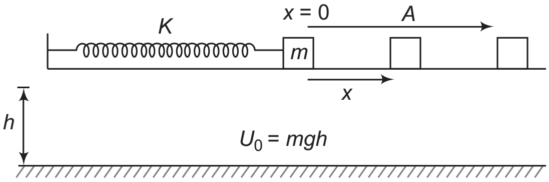


Fig. 14.8

Now, the block is displaced upto $x=A$. Work done (by external agent) in this displacement (against the spring force $-kx$) is $W=\frac{1}{2}KA^2$.

∴ Total mechanical energy $E = U_0 + W$

or $$\boxed{E = U_0 + \frac{1}{2}KA^2 = U_0 + \frac{1}{2}m\omega^2 A^2} \qquad \left(\text{as } \omega^2 = \frac{K}{m} \text{ or } K = m\omega^2\right)$$

Note *(i) In the above discussion, U_0 is positive as the spring block system is kept at some height. This value will be zero if the system is kept on ground. In some cases, it may be negative also.*

(ii) In the absence of any dissipative forces, this total mechanical energy remains constant at every point between +A and –A.

At extreme positions, kinetic energy is zero. So, total mechanical energy is in the form of potential energy.

$$E = U = U_0 + \frac{1}{2}KA^2$$

or $$U_0 + \frac{1}{2}m\omega^2 A^2$$

At mean position, potential energy is minimum or U_0 and the kinetic energy is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2 A^2$.

Thus, U_0 is present at every point. This is initial work done $W\left(= \frac{1}{2}KA^2 \text{ or } \frac{1}{2}m\omega^2 A^2\right)$

which changes between potential and kinetic energies. This is also called energy of oscillation. At mean position, it completely converts into kinetic energy and at extreme positions it converts into potential energy.

Potential Energy From Fig. 14.8, we can see that, at a general displacement x (from the mean position) the potential energy is

$$U = U_0 + \text{Spring potential energy} \quad \text{or} \quad \boxed{U = U_0 + \frac{1}{2}Kx^2 = U_0 + \frac{1}{2}m\omega^2 x^2}$$

Kinetic Energy Here, let us call it T as we are using K for spring constant of spring.

At a general displacement x,

$$T = E - U$$

$$= \left(U_0 + \frac{1}{2}KA^2\right) - \left(U_0 + \frac{1}{2}Kx^2\right)$$

∴ $$T = \frac{1}{2}K(A^2 - x^2) = \frac{1}{2}m\omega^2 (A^2 - x^2)$$

Note *At some places kinetic energy may be represented by K and force constant by k.*

Alternate Method

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m[\pm \omega\sqrt{A^2 - x^2}]$$

$$= \frac{1}{2}m\omega^2 (A^2 - x^2)$$

Graphs The three equations of total mechanical energy, potential energy and kinetic energy are

$$E = U_0 + \frac{1}{2} K A^2 = U_0 + \frac{1}{2} m\omega^2 A^2$$

$$U = U_0 + \frac{1}{2} K x^2 = U_0 + \frac{1}{2} m\omega^2 x^2$$

$$T = \frac{1}{2} K (A^2 - x^2) = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

E is a constant function. U - x and T - x graphs are parabolas as shown in the Fig.14.9.

At mean position ($x = 0$)

$$E = U_0 + \frac{1}{2} KA^2$$

$$U = U_0 = \text{minimum value and}$$

$$T = \frac{1}{2} KA^2 = \text{maximum value.}$$

At extreme positions ($x = \pm A$)

$$E = U_0 + \frac{1}{2} KA^2$$

$$U = U_0 + \frac{1}{2} KA^2 = \text{maximum value}$$

and $T = 0$

The three graphs are as shown in figure.

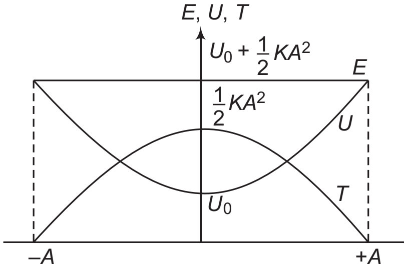


Fig. 14.9

Example 14.1 *Describe the motion of a particle acted upon by a force*

(i) $F = -2(x - 2)^3$

(ii) $F = -2(x - 2)^2$

(iii) $F = -2(x - 2)$

Solution (i) $F = -2(x - 2)^3$

$$F = 0 \quad \text{at} \quad x = 2$$

Force is along negative x-direction for $x > 2$ and it is along positive x-direction for $x < 2$. Thus, the motion of the particle is oscillatory (but not simple harmonic) about $x = 2$.

(ii) $F = 0$ for $x = 2$, but force is always along negative x-direction for any value of x except at $x = 2$. Thus, the motion of the particle is rectilinear along negative x-direction provided it is not kept at rest at $x = 2$.

(iii) Let, us take $x - 2 = X$, then the given force can be written as,

$$F = -2X$$

This is the equation of SHM. Hence, the particle oscillates simple harmonically about $X = 0$ or $x = 2$.

Example 14.2 *Maximum acceleration of a particle in SHM is 16 cm/s^2 and maximum velocity is 8 cm/s. Find time period and amplitude of oscillations.*

Solution

$$a_{\max} = \omega^2 A = 16\,\text{cm/s} \qquad \ldots\text{(i)}$$

$$u_{\max} = \omega A = 8\,\text{cm/s} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$A = 4\,\text{cm} \qquad \textbf{Ans.}$$

and

$$\omega = 2\,\text{rad/s}$$

Now,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = (\pi)\,\text{s}$$

or

$$T \approx 3.14\,\text{s} \qquad \textbf{Ans.}$$

Example 14.3 *F - x equation of a body in SHM is*

$$F + 4x = 0$$

Here, F is in newton and x in metre. Mass of the body is 1 kg. Find time period of oscillations.

Solution The given equation can be written as

$$F = -4x$$

Comparing this equation with the standard equation of SHM or

$$F = -Kx$$

we get

$$K = 4\,\text{N/m}$$

Now,

$$T = 2\pi\sqrt{\frac{m}{K}}$$

$$= 2\pi\sqrt{\frac{1}{4}} = (\pi)\text{s} \qquad \textbf{Ans.}$$

Example 14.4 *A linear harmonic oscillator has a total mechanical energy of* 200 *J. Potential energy of it at mean position is* 50 *J. Find*

(i) the maximum kinetic energy,
(ii) the minimum potential energy,
(iii) the potential energy at extreme positions.

Solution At mean position, potential energy is minimum and kinetic energy is maximum. Hence,

$$U_{\min} = 50\,\text{J} \qquad \text{(at mean position)}$$

and

$$K_{\max} = E - U_{\min} = 200 - 50$$

$$= 150\,\text{J} \qquad \text{(at mean position)}$$

At extreme positions kinetic energy is zero and potential energy is maximum

∴

$$U_{\max} = E$$

$$= 200\,\text{J} \qquad \text{(at extreme position)}$$

Example 14.5 *A particle executes SHM.*

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude?

(b) At what value of displacement are the kinetic and potential energies equal?

Solution Here, let us take minimum potential energy at mean position

$$U_0 = 0$$

We know that $E_{\text{total}} = \frac{1}{2} m\omega^2 A^2$

$$\text{KE} = \frac{1}{2} m\omega^2 (A^2 - x^2) \quad \text{and} \quad U = \frac{1}{2} m\omega^2 x^2$$

(a) When $x = \frac{A}{2}$

$$\text{KE} = \frac{1}{2} m\omega^2 \left(\frac{3A^2}{4}\right)$$

$\Rightarrow$ $\frac{\text{KE}}{E_{\text{total}}} = \frac{3}{4}$ **Ans.**

At $x = \frac{A}{2}$, $U = \frac{1}{2} m\omega^2 \left(\frac{A^2}{4}\right)$

$\Rightarrow$ $\frac{\text{PE}}{E_{\text{total}}} = \frac{1}{4}$ **Ans.**

(b) Since, $K = U$

$$\frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} m\omega^2 x^2$$

or $2x^2 = A^2$ or $x = \frac{A}{\sqrt{2}} = 0.707A$ **Ans.**

Example 14.6 *The potential energy of a particle oscillating along x-axis is given as*

$$U = 20 + (x - 2)^2$$

Here, U is in joules and x in metres. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Solution (a) $F = -\frac{dU}{dx} = -2(x - 2)$

By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$

The motion of the particle is simple harmonic.

(b) The mean position of the particle is $X = 0$ or $x - 2 = 0$, which gives $x = 2$ m.

(c) Maximum kinetic energy of the particle is,

$$K_{\max} = E - U_{\min} = 36 - 20 = 16 \text{ J}$$

Note *$U_{\min}$ is 20 J at mean position or at $x = 2$ m.*

INTRODUCTORY EXERCISE 14.1

1. a - x equation of a particle in SHM is $a + 4x = 0$

   Here, a is in cm/s^2 and x in cm. Find time period in seconds.

2. At $x = \frac{A}{4}$, what fraction of the mechanical energy is potential? What fraction is kinetic? Assume potential energy to be zero at mean position.

3. A cart of mass 2.00 kg is attached to the end of a horizontal spring with force constant $k = 150$ N/m. The cart is displaced 15.0 cm from its equilibrium position and released. What are (a) the amplitude (b) the period (c) the frequency (d) the mechanical energy (e) the maximum velocity of the cart? Neglect friction.

4. A 0.5 kg body performs simple harmonic motion with a frequency of 2 Hz and an amplitude of 8 mm. Find the maximum velocity of the body, its maximum acceleration and the maximum restoring force to which the body is subjected.

5. Can we use the equation $v = u + at$ in SHM or not ?

14.3 Time Equations of SHM

In one dimensional motion, if acceleration as a function of time is given, then by integrating a - t equation we can make v - t equation. By further integrating v - t equation we can make s - t or x - t equation.

Meaning of General Solution

In one dimensional motion, if $a =$ constant then general solution of v - t equation is (which can be obtained by integrating a - t equation)

$$v = u + at$$

In this equation, u and a are constants but v and t are variables. In different problems, values of these constants u and a are different but basic nature of all v - t equations are same. Here, the basic nature is linear equation in v and t or straight line graph between v and t.

Standard Differential Equation of SHM

In article 14.2, we have seen that a - x equation of SHM is

$$a = -\omega^2 x$$

This equation can also be written as

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \qquad \ldots\text{(i)}$$

This is standard second order differential equation of SHM. Solving this differential equation we can find x - t equation. But normally we do not solve a differential equation in physics at this stage. So, we can directly write general solution of Eq. (i). The general solution of Eq. (i) is

$$x = A\sin(\omega t \pm \phi) \quad \text{or} \quad x = A\cos(\omega t \pm \phi)$$

In these equations x and t are variables. But A, ω and ϕ are constants. In different problems, values of these constants are different but basic nature of all equations will be same. Here, the basic nature is sinusoidal (sine or cosine) x - t equation and x - t graph.

By differentiating x - t equation we can also make other equations. For example, if

$$x = A\sin(\omega t \pm \phi)$$

Then
$$v = \frac{dx}{dt} = \omega A\cos(\omega t \pm \phi)$$

and,
$$a = \frac{dv}{dt} = -\omega^2 A\sin(\omega t \pm \phi) = -\omega^2 x \qquad [\text{as } A\sin(\omega t \pm \phi) = x]$$

Time Period (T) and Frequency (f)

In the above equations, time period of oscillations is

$$T = \frac{2\pi}{\omega}$$

and frequency of oscillation is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Phase Angle at Time t

In any trigonometric function its angle is very important. The value of function varies with this angle. For example, suppose we have a trigonometric function.

$$y = \sin\theta \text{ then, } y = 0 \text{ for } \theta = 0°, \; y = \frac{1}{2} \text{ for } \theta = 30°$$

So, value of y depends on the angle θ. This angle is called the phase angle. In x - t, v - t and a - t equations of SHM we have seen above that the phase angle is same, $(\omega t \pm \phi)$. This may be called the phase angle of x - t, v - t and a - t equations at time t. With the help of this angle values of x, v and a can be obtained at any time t. Here, x varies between $+A$ and $-A$, v varies between $+\omega A$ and $-\omega A$ and a varies between $+\omega^2 A$ and $-\omega^2 A$. If $(\omega t \pm \phi)$ is 30° at some instant, then according to above equations values of x, v and a are $+\frac{A}{2}$, $+\frac{\sqrt{3}\,\omega A}{2}$ and $-\frac{\omega^2 A}{2}$.

This condition is shown in following figure:

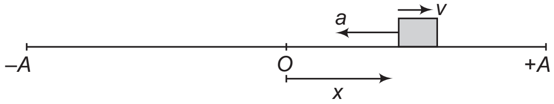


Fig. 14.10

$$x = +\frac{A}{2}, \; v = \frac{+\sqrt{3}\,\omega A}{2} \text{ and } a = \frac{-\omega^2 A}{2}$$

Initial Phase Angle

In the phase angle $\omega t \pm \phi$ if we substitute $t = 0$, then the angle is $\pm\phi$. This is called initial phase angle. With the help of initial phase angle we can determine initial values of x, v and a.

Dependence of Initial Phase Angle

Initial phase angle depends on following three factors :

(i) Whether we write sine equation or cosine equation.

(ii) Initial value of x (at $t = 0$)

(iii) Initial direction of velocity.

For example, in first figure the body starts

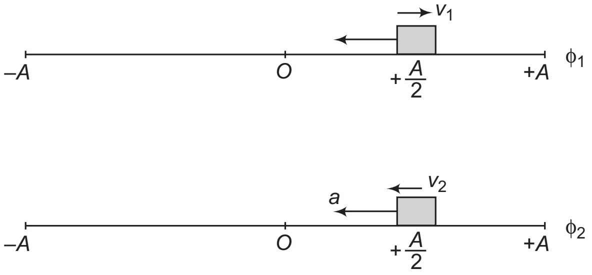


Fig. 14.11

($t = 0$, $u \neq 0$) from $x = +\frac{A}{2}$ with velocity $v_1 = \frac{\sqrt{3}\,\omega A}{2}$ and the initial phase angle is suppose ϕ_1 (corresponding to sine equation)

In second figure, body starts from the same point $x = +\frac{A}{2}$ but direction of velocity is opposite or $v_2 = -\frac{\sqrt{3}\,\omega A}{2}$ and the initial phase angle is suppose ϕ_2 (corresponding to sine equation). Then

$$\phi_1 \neq \phi_2$$

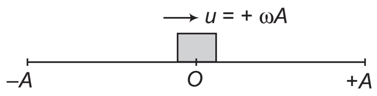
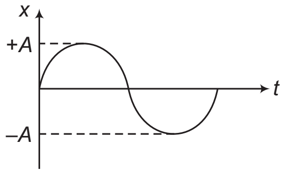
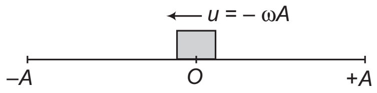
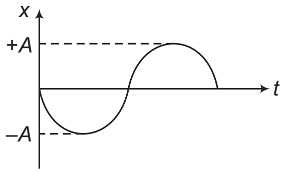
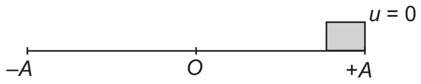
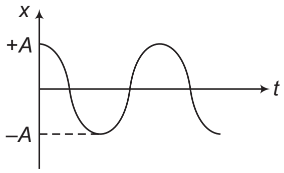
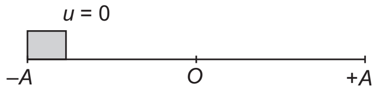
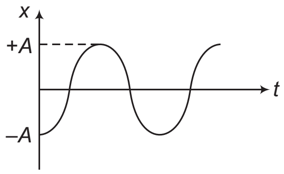
Similarly, if we wish to write cosine equations and their initial phase angles are suppose ϕ_1' and ϕ_2', then $\phi_1 \neq \phi_1'$ and $\phi_2 \neq \phi_2'$

Four Frequently used x - t Equations

Between $+A$ and $-A$ there may be infinite number of points from where the body may start its journey ($t = 0$ but u is zero only if, it starts from $+A$ or $-A$).

Further at every point (except at $+A$ and $-A$) two directions (positive x or negative x) of initial velocity can exist. So, infinite number of initial phase angles are possible or infinite number of x - t equations can be written. But following four initial conditions ($t = 0$) and corresponding four x - t equations are frequently used.

Table 14.1

Initial Conditions at $t=0$	x - t equation	x - t graph
	$x = A\sin\omega t$	
	$x = -A\sin\omega t$	
	$x = A\cos\omega t$	
	$x = -A\cos\omega t$	

Six Time Equations of SHM and their Graphs

Six time equations in SHM are x - t, v - t, a - t, E - t, U - t and T - t ($T =$ kinetic energy). If one equation is given other five equations can be made. Let us start with x - t equation :

$$\boxed{x = A\sin\omega t} \qquad \ldots\text{(i)}$$

According to this equation the body starts from mean positive, moving towards positive x-direction with initial velocity, $u = +\omega A$.

Now,
$$v = \frac{dx}{dt} = \frac{d}{dt}(A\sin\omega t) = \omega A\cos\omega t$$

or
$$\boxed{v = \omega A\cos\omega t} \qquad \ldots\text{(ii)}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(\omega A\cos\omega t) = -\omega^2 A\sin\omega t$$

or
$$\boxed{a = -\omega^2 A\sin\omega t} \qquad \ldots\text{(iii)}$$

$$\boxed{E = U_0 + \frac{1}{2}KA^2 = U_0 + \frac{1}{2}m\omega^2 A^2} \qquad \ldots\text{(iv)}$$

$$U = U_0 + \frac{1}{2}Kx^2 = U_0 + \frac{1}{2}KA^2 \sin^2 \omega t$$

$$= U_0 + \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t$$

or $$\boxed{U = U_0 + \frac{1}{2}KA^2 \sin^2 \omega t = U_0 + \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t} \quad \ldots\text{(v)}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega A \cos \omega t)^2$$

$$= \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t = \frac{1}{2}KA^2 \cos^2 \omega t$$

or $$\boxed{T \text{ or KE} = \frac{1}{2}KA^2 \cos^2 \omega t = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t} \quad \ldots\text{(vi)}$$

From the above six equations, we can draw following conclusions :

(i) $x - t$, $v - t$ and $a - t$ are sine or cosine functions of same ω. So, x, v and a oscillate sinusoidally with same time period $T = \frac{2\pi}{\omega}$.

(ii) U - t and T - t are $\sin^2$ and $\cos^2$ functions of time. But oscillation frequency of $\sin^2$ or $\cos^2$ functions is double of the oscillation frequency of sine and cosine functions. So, U and T oscillate with double the frequency of oscillations of x, v and a.

Minimum value of U is U_0 and maximum value of U is $U_0 + \frac{1}{2}KA^2$ or $U_0 + \frac{1}{2}m\omega^2 A^2$. Therefore, U oscillates between U_0 and $U_0 + \frac{1}{2}KA^2$. Similarly, minimum value of T is zero and the maximum value is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2 A^2$.

Hence, T oscillates between 0 and $\frac{1}{2}KA^2$.

(iii) E does not oscillate because it is constant.

(iv) At $t = 0$

$x = 0$ (starts from mean position)
$v = \omega A$ (maximum velocity)
$a = 0$
$E = U_0 + \frac{1}{2}KA^2$
$U = U_0$ (minimum value)

and T or KE $= \frac{1}{2}KA^2$

or $\frac{1}{2}m\omega^2 A^2$ (maximum value)

(v) Average value of $\cos^2$ function in one time period is $\frac{1}{2}$. Therefore, from Eq. (vi) we can see that maximum value of kinetic energy is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2\ A^2$. But average value of kinetic energy in one time period is $\frac{1}{4}KA^2$ or $\frac{1}{4}m\omega^2 A^2$.

(vi) The six graphs are as shown below.

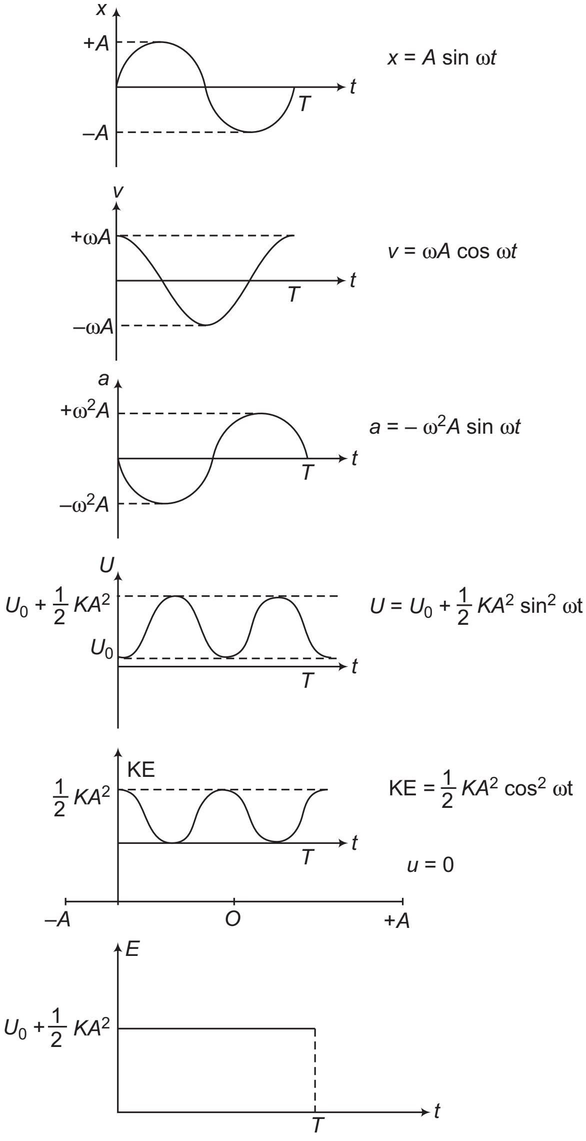


Fig. 14.12

In the first five graphs we can see that in time $T(=2\pi/\omega)$: x, v and a oscillate once but U and KE oscillate twice.

Note *(i) If the first equation, $x = A\sin\omega t$ is changed then other five equations (and their graphs also) will change. But the basic nature will remain same. Here the basic nature is that x, v and a oscillate sinusoidally with same time period and frequency. E does not oscillate. U and KE oscillate with double frequency.*

(ii) Phase difference between x - t and v - t equations or between v - t and a - t equations is $\frac{\pi}{2}$ or 90°. Phase difference between x - t and a - t equations is π or 180°.

List of Formulae of Articles 14.2 and 14.3 (with $U_0 = 0$)

Table 14.2

S.No.	Name of the equation	Expression of the equation	Remarks
1.	Displacement-time	$x = A\cos(\omega t + \phi)$	x varies between $+A$ and $-A$
2.	Velocity - time $\left(v = \frac{dx}{dt}\right)$	$v = -A\omega\sin(\omega t + \phi)$	v varies between $+A\omega$ and $-A\omega$
3.	Acceleration - time $\left(a = \frac{dv}{dt}\right)$	$a = -A\omega^2\cos(\omega t + \phi)$	a varies between $+A\omega^2$ and $-A\omega^2$
4.	Kinetic energy - time $\left(\text{KE} = \frac{1}{2}mv^2\right)$	$\text{KE} = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$	KE varies between 0 and $\frac{1}{2}mA^2\omega^2$
5.	Potential energy - time $\left(U = \frac{1}{2}m\omega^2x^2\right)$	$\text{KE} = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$	U varies between $\frac{1}{2}mA^2\omega^2$ and 0
6.	Total energy - time ($E = \text{KE} + U$)	$E = \frac{1}{2}m\omega^2A^2$	E is constant
7.	Velocity - displacement	$v = \pm\omega\sqrt{A^2 - x^2}$	$v = 0$ at $x = \pm A$ and at $x = 0$ $v = \pm A\omega$
8.	Acceleration - displacement	$a = -\omega^2x$	$a = 0$ at $x = 0$ $a = \pm\omega^2A$ at $x = \mp A$
9.	Kinetic energy - displacement	$\text{KE} = \frac{1}{2}m\omega^2(A^2 - x^2)$	$\text{KE} = 0$ at $x = \pm A$, $\text{KE} = \frac{1}{2}m\omega^2A^2$ at $x = 0$
10.	Potential energy - displacement	$U = \frac{1}{2}m\omega^2x^2$	$U = 0$ at $x = 0$, $U = \frac{1}{2}m\omega^2A^2$ at $x = \pm A$
11.	Total energy - displacement	$E = \frac{1}{2}m\omega^2A^2$	E is constant

Example 14.7 *From the time equations of SHM, prove the relation,*

$$v \pm \omega\sqrt{A^2 - x^2}$$

Solution Let, $x = A\sin\omega t$

Then,
$$\sin\omega t = \frac{x}{A}$$

Now,
$$v = \frac{dx}{dt} = \omega A\cos\omega t$$
$$= \omega A\sqrt{1 - \sin^2\omega t}$$

$$= \omega A\sqrt{1-\left(\frac{x}{A}\right)^2}$$

∴ $$v = \omega A\sqrt{\frac{A^2 - x^2}{A^2}}$$

or $$v = \pm\omega\sqrt{A^2 - x^2}$$ **Hence Proved.**

Example 14.8 *If a SHM is represented by the equation* $x = 10 \sin\left(\pi t + \frac{\pi}{6}\right)$ *in SI units, then determine its amplitude, time period and maximum velocity* $v_{\max}$?

Solution Comparing the above equation with

$$x = A \sin(\omega t + \phi), \text{ we get}$$

$$A = 10 \text{ m}$$ **Ans.**

$$\omega = (\pi)\frac{\text{rad}}{\text{s}} \quad \text{and} \quad \phi = \frac{\pi}{6}$$

∵ $$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = 2 \text{s}$$ **Ans.**

$$v_{\max} = \omega A = (10\pi) \text{ m/s}$$ **Ans.**

Example 14.9 *A particle executes SHM with a time period of* 4 *s. Find the time taken by the particle to go directly from its mean position to half of its amplitude.*

Solution If the particle starts from the mean position in positive direction, then:

$$x = A \sin(\omega t)$$

or $$\frac{A}{2} = A \sin(\omega t)$$

or $$\frac{1}{2} = \sin(\omega t)$$

$$\omega t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$t = \frac{\pi}{6\omega} = \frac{\pi T}{6(2\pi)}$$

as $$\omega = \frac{2\pi}{T}$$

⇒ $$t = \frac{T}{12} = \frac{4}{12} = \frac{1}{3} \text{s}$$ **Ans.**

Example 14.10 *A particle executes simple harmonic motion about the point $x = 0$. At time $t = 0$, it has displacement $x = 2$ cm and zero velocity. If the frequency of motion is $0.25\ \text{s}^{-1}$, find (a) the period, (b) angular frequency, (c) the amplitude, (d) maximum speed, (e) the displacement from the mean position at $t = 3$ s and (f) the velocity at $t = 3$ s.*

Solution (a) Period $T = \dfrac{1}{f}$

$$= \frac{1}{0.25\ \text{s}^{-1}} = 4\ \text{s}$$ **Ans.**

(b) Angular frequency $\omega = \dfrac{2\pi}{T}$

$$= \frac{2\pi}{4} = \frac{\pi}{2}\ \text{rad/s}$$

$$= 1.57\ \text{rad/s}$$ **Ans.**

(c) Amplitude is the maximum displacement from mean position. Hence, $A = 2 - 0 = 2$ cm.

(d) Maximum speed $v_{\max} = A\,\omega$

$$= 2 \cdot \frac{\pi}{2} = \pi\ \text{cm/s}$$

$$= 3.14\ \text{cm/s}$$ **Ans.**

(e) At $t = 0$, particle starts from extreme position.

$\therefore$ $$x = A\cos\omega t$$

$$= 2\cos\left(\frac{\pi}{2}t\right)$$

At $$t = 3\ \text{sec}$$

$$x = 0$$ **Ans.**

(f) Velocity at $x = 0$, is $v_{\max}$ i.e. 3.14 cm/s. **Ans.**

Example 14.11 *Find the period of the function,*

$$y = \sin\omega t + \sin 2\omega t + \sin 3\omega t$$

Solution The given function can be written as,

$$y = y_1 + y_2 + y_3$$

Here, $$y_1 = \sin\omega t,\ T_1 = \frac{2\pi}{\omega}$$

$$y_2 = \sin 2\omega t,\ T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

and $$y_3 = \sin 3\omega t,\ T_3 = \frac{2\pi}{3\omega}$$

$\therefore$ $$T_1 = 2T_2 \quad \text{and} \quad T_1 = 3T_3$$

So, the time period of the given function is T_1 or $\frac{2\pi}{\omega}$. **Ans.**

Because in time $T = \frac{2\pi}{\omega}$, first function completes one oscillation, the second function two oscillations and the third three.

INTRODUCTORY EXERCISE 14.2

1. A 2.0 kg particle undergoes SHM according to $x = 1.5 \sin\left(\frac{\pi t}{4} + \frac{\pi}{6}\right)$ (in SI units)
   (a) What is the total mechanical energy of the particle?
   (b) What is the shortest time required for the particle to move from $x = 0.5$ m to $x = -0.75$ m?
2. Given that the equation of motion of a mass is $x = 0.20 \sin(3.0t)$ m. Find the velocity and acceleration of the mass when the object is 5 cm from its equilibrium position. Repeat for $x = 0$.
3. A particle executes simple harmonic motion of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = \frac{A}{2}$ and it moves along the positive x-direction. Find the phase constant δ, if the equation is written as $x = A \sin(\omega t + \delta)$.
4. An object of mass 0.8 kg is attached to one end of a spring and the system is set into simple harmonic motion. The displacement x of the object as a function of time is shown in the figure. With the aid of the data, determine
   (a) the amplitude A of the motion,
   (b) the angular frequency ω,
   (c) the spring constant K,
   (d) the speed of the object at $t = 1.0$ s and
   (e) the magnitude of the object's acceleration at $t = 1.0$ s.

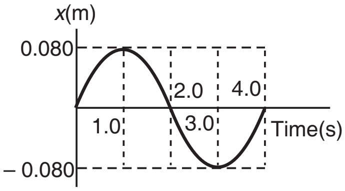


Fig. 14.13

5. The equation of motion of a particle started at $t = 0$ is given by $x = 5 \sin\left(20t + \frac{\pi}{3}\right)$, where x is in cm and t in sec. When does the particle (a) first come to rest, (b) first have zero acceleration, (c) first have maximum speed.
6. Describe the motion corresponding to x - t equation, $x = 10 - 4 \cos \omega t$.

14.4 Relation between Simple Harmonic Motion and Uniform Circular Motion

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P oscillates between B and C. The angle that the radius OQ makes with the x-axis is, $\theta = \omega t + \phi$. Here, ϕ is the angle made by the radius OQ with the x-axis at time $t = 0$. Further,

$$OP = OQ \cos\theta$$

or

$$x = A \cos(\omega t + \phi)$$

This is standard $x-t$ equation of SHM. Hence, P executes SHM. That is

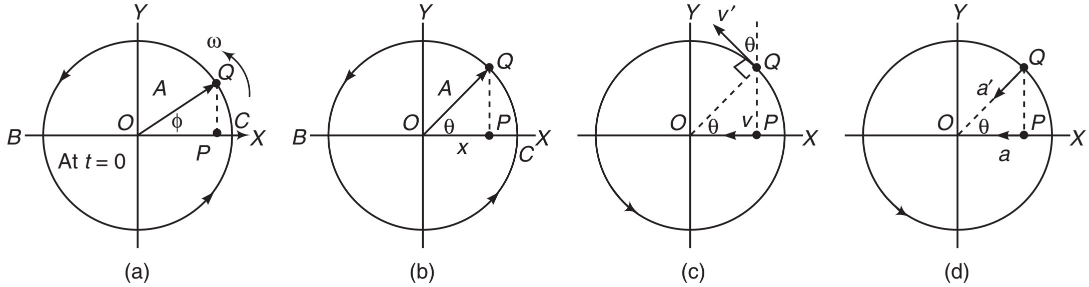


Fig. 14.14 Relation between SHM and uniform circular motion. (b) Position, (c) velocity and (d) acceleration

When a particle moves with uniform circular motion, its projection on a diameter moves with SHM.

The velocity of Q is perpendicular to OQ and has a magnitude of velocity $v'=\omega A$. The component of v' along the x-axis is,

$$v=-v'\sin\theta \quad \text{or} \quad v=-\omega A\sin(\omega t+\phi)$$

which is also the velocity of P. The acceleration of Q is centripetal and has a magnitude, $a'=\omega^2 A$.

The component of a' along the x-axis is

$$a=-a'\cos\theta \quad \text{or} \quad a=-\omega^2 A\cos(\omega t+\phi)$$

Which again coincides with the acceleration of P.

Note *v = velocity of SHM of particle P* $=\frac{dx}{dt}=\frac{d}{dt}[A\cos(\omega t+\phi)]=-\omega A\sin(\omega t+\phi)$

a = acceleration of SHM of particle $=\frac{dv}{dt}=-\omega^2 A\cos(\omega t+\phi)$

14.5 Methods of Finding Time Period of a Simple Harmonic Motion

If F-x or a-x equations are known then we can easily find the time period of SHM. For example, F-x equation is of type $F=-Kx$

and time period in this case is $T=2\pi\sqrt{\frac{m}{K}}$

Similarly, a-x equation is of type

$$a=-\omega^2 x$$

$$\therefore \quad \omega=\sqrt{\left|\frac{a}{x}\right|}=\frac{2\pi}{T}$$

$$\text{or} \quad T=2\pi\sqrt{\left|\frac{x}{a}\right|}=2\pi\sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$$

In angular SHM, we will be required $\tau-\theta$ or $\alpha-\theta$ equations.

Now, there are following two methods of finding F-x and a-x equations (or $\tau-\theta$ and $\alpha-\theta$ equations).

Restoring Force or Torque Method

The following steps are usually followed in this method:

Step 1 Find the stable equilibrium position which usually is also known as the mean position. Net force or torque on the particle in this position is zero. Potential energy is minimum.

Step 2 Displace the body from its mean position by a small displacement x (in case of a linear SHM) or θ (in case of an angular SHM).

Step 3 Find net force or torque in this displaced position.

Step 4 Show that this force or torque has a tendency to bring the body back to its mean position and magnitude of force or torque is proportional to displacement, i.e.

$$F \propto -x \quad \text{or} \quad F = -Kx \qquad \ldots\text{(i)}$$

or

$$\tau \propto -\theta \quad \text{or} \quad \tau = -K\theta \qquad \ldots\text{(ii)}$$

This force or torque is also known as restoring force or restoring torque.

Step 5 Find linear acceleration by dividing Eq. (i) by mass m or angular acceleration by dividing Eq. (ii) by moment of inertia I. Hence,

$$a = -\frac{K}{m}.x = -\omega^2 x \quad \text{or} \quad \alpha = -\frac{K}{I}\theta = -\omega^2\theta$$

Step 6 Finally,

$$\omega = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

or

$$\frac{2\pi}{T} = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$\therefore$

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

Energy Method

Repeat steps-1 and step-2 as in method-1. Find the total mechanical energy (E) in the displaced position. Since, mechanical energy in SHM remains constant.

$$\frac{dE}{dt} = 0$$

By differentiating the energy equation with respect to time and substituting $\frac{dx}{dt} = v$, $\frac{d\theta}{dt} = \omega$, $\frac{dv}{dt} = a$, and $\frac{d\omega}{dt} = \alpha$ we come to step-5 or we directly get $F - x$ and $a - x$ equations (or $\tau - \theta$ or $x - \theta$ equation). The remaining procedure is same.

Note *(i) E usually consists of following terms :*

(a) Gravitational PE (b) Elastic PE (c) Electrostatic PE

(d) Rotational KE and (e) Translational KE

(ii) For gravitational PE, choose the reference point ($h = 0$) at mean position.

Now, let us take few examples of finding time period (T) of certain simple harmonic motions.

The Simple Pendulum

An example of SHM is the motion of a pendulum. A simple pendulum is defined as a particle of mass m suspended from a point O by a string of length l and of negligible mass.

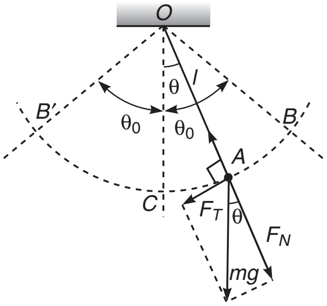


Fig. 14.15

When the particle is pulled aside to position B, so that the string makes an angle θ_0 with the vertical OC and then released, the pendulum will oscillate between B and the symmetric position B'. The oscillatory motion is due to the tangential component F_T of the weight mg of the particle. This force F_T is maximum at B and B' and zero at C. Thus, we can write,

$$F_T = -mg \sin\theta$$

Here, minus sign appears because it is opposite to the displacement.

$$x = CA$$

$$\therefore \quad ma_T = -mg\sin\theta \qquad \ldots\text{(i)}$$

Here, $$a_T = l\alpha \qquad \left(\text{where, } \alpha = \frac{d^2\theta}{dt^2}\right)$$

and $$\sin\theta \approx \theta \qquad \text{for small oscillations}$$

$$\therefore \quad ml\alpha = -mg\theta$$

Since, α is proportional to $-\theta$. Hence, motion is simple harmonic.

or $$\alpha = -\left(\frac{g}{l}\right)\theta \quad \text{or} \quad \left|\frac{\theta}{\alpha}\right| = \frac{l}{g}$$

$$\therefore \quad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note *that the period is independent of the mass of the pendulum.*

Energy Method

Let us derive the same expression by energy method. Suppose ω be the angular velocity of particle at angular displacement θ about point O. Then, total mechanical energy of particle in position A is,

$$E = \frac{1}{2}I\omega^2 + mg(h_A - h_C)$$

or $$E = \frac{1}{2}(ml^2)\,\omega^2 + mgl(1-\cos\theta)$$

E is constant, therefore, $\frac{dE}{dt} = 0$ or $0 = ml^2\omega\left(\frac{d\omega}{dt}\right) + mgl\sin\theta\left(\frac{d\theta}{dt}\right)$

Putting $\frac{d\theta}{dt} = \omega$, $\frac{d\omega}{dt} = \alpha$ and $\sin\theta \approx \theta$, we get the same expression

$$\alpha = -\left(\frac{g}{l}\right)\theta$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Spring-block System

Suppose a mass m is attached to the free end of a massless spring of spring constant k, with its other end fixed to a rigid support.

If the mass be displaced through a distance x, as shown, a linear restoring force,

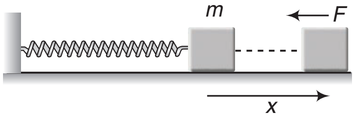


Fig. 14.16

$$F = -kx \qquad \text{...(i)}$$

starts acting on the mass, tending to bring it back into its original position. The negative sign simply indicates that it is directed oppositely to the displacement of the mass.

Eq. (i) can be written as,

$$ma = -kx \qquad \text{...(ii)}$$

$$\text{or} \qquad \left|\frac{x}{a}\right| = \frac{m}{k}$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Energy Method

The time period of the spring-block system can also be obtained by the energy method. Let v be the speed of the mass in displaced position. Then total mechanical energy of the spring-block system is.

$$E = \text{kinetic energy of the block + elastic potential energy}$$

$$\text{or} \qquad E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Since, E is constant

$$\frac{dE}{dt} = 0 \quad \text{or} \quad 0 = mv\left(\frac{dv}{dt}\right) + kx\left(\frac{dx}{dt}\right)$$

Substituting, $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = a$

We have,

$$ma = -kx$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\frac{m}{k}}$$

Example 14.12 *F-x equation of a body of mass 2 kg in SHM is*

$$F + 8x = 0$$

Here, F is in newton and x in metre. Find time period of oscillations.

Solution The given equation can be written as, $F = -8x$

Comparing with the standard equation of SHM, $F = -Kx$ we have,

$$K = 8\,\text{N/m}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{K}}$$

$$= 2\pi\sqrt{\frac{2}{8}}$$

$$= (\pi)\,\text{s} = 3.14\,\text{s} \qquad \textbf{Ans.}$$

Example 14.13 *Acceleration of a particle in SHM at displacement $x = 10$ cm (from the mean position is $a = -2.5$ cm/s^2). Find time period of oscillations.*

Solution Time period is given by

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

Substituting the values we have,

$$T = 2\pi\sqrt{\frac{10}{2.5}} = (4\pi)\,\text{s} \qquad \textbf{Ans.}$$

Example 14.14 *Find time period of a vertical spring-block system by both methods.*

Solution **Force Method**

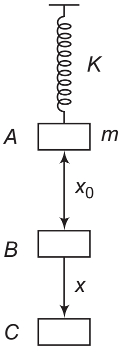


Fig. 14.17

Position *A* In this position, spring is in its natural length. But net force on block is not zero. Its *mg* is downwards.

Position B This is equilibrium position. Net force on block is zero. Spring force kx_0 in upward direction is equal to mg force in downward direction.

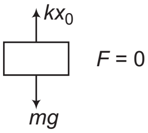


Fig. 14.18

Position C This is displaced position.

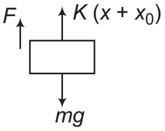


Fig. 14.19

Net restoring force is upward direction or towards the mean position is

$$F = -[k(x + x_0) - mg] = -kx \quad (\text{as } kx_0 = mg)$$

Since, $F = -kx$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Energy Method Taking $h = 0$, at the mean position B, total mechanical energy in displaced position C is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k(x + x_0)^2 - mg\,x$$

As $E = \text{constant}$ $\frac{dE}{dt} = 0$

or $$\frac{1}{2}m(2v)\frac{dv}{dt} + \frac{1}{2}k[2(x + x_0)]\frac{dx}{dt} - mg\frac{dx}{dt} = 0$$

Substituting, $\frac{dx}{dt} = v$, $\frac{dv}{dt} = a$, $kx_0 = mg$ and $ma = F$, the above equation also converts into,

$$F = -kx$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 14.15 *A block with a mass of 3.00 kg is suspended from an ideal spring having negligible mass and stretches the spring by 0.2 m.*

(a) What is the force constant of the spring?

(b) What is the period of oscillation of the block if it is pulled down and released?

Solution (a) In equilibrium,

$$kl = mg \qquad (l = \text{extension in spring})$$

$\therefore$ $$k = \frac{mg}{l} \qquad \ldots\text{(i)}$$

Substituting the proper values, we have

$$k = \frac{(3.00)(9.8)}{0.2}$$

$$= 147 \text{ N/m}$$ **Ans.**

(b) From Eq. (i) $$\frac{m}{k} = \frac{l}{g}$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$

$$= 2\pi\sqrt{\frac{0.2}{9.8}} = 0.897\,\text{s}$$ **Ans.**

INTRODUCTORY EXERCISE 14.3

1. a - x equation of a body in SHM is $a + 16\,x = 0$. Here, x is in cm and a in cm/s^2. Find time period of oscillations.
2. A mass M, attached to a spring, oscillates with a period of 2 s. If the mass is increased by 4 kg, the time period increases by one second. Assuming that Hooke's law is obeyed, find the initial mass M.
3. Three masses of 500 g, 300 g and 100 g are suspended at the end of a spring as shown and are in equilibrium. When the 500 g mass is removed suddenly, the system oscillates with a period of 2 s. When the 300 g mass is also removed, it will oscillate with period T. Find the value of T.

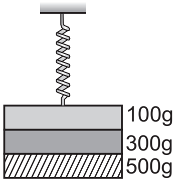


Fig. 14.20

4. A particle executes simple harmonic motion. Its instantaneous acceleration is given by $a = -px$, where p is a positive constant and x is the displacement from the mean position. Find angular frequency of oscillations.
5. A mass M is suspended from a spring of negligible mass. The spring is pulled a little then released, so that the mass executes simple harmonic motion of time period T. If the mass is increased by m, the time period becomes $\frac{5T}{3}$. Find the ratio of m/M.
6. The length of a simple pendulum is decreased by 21%. Find the percentage change in its time period.
7. A particle executes SHM on a straight line path. The amplitude of oscillation is 2 cm. When the displacement of the particle from the mean position is 1 cm, the numerical value of magnitude of acceleration is equal to the numerical value of magnitude of velocity. Find the frequency of SHM (in Hz).

14.6 Vector Method of Combining Two or More Simple Harmonic Motions

A simple harmonic motion is produced when a force (called restoring force) proportional to the displacement acts on a particle. If a particle is acted upon by two such forces the resultant motion of the particle is a combination of two simple harmonic motions. Suppose the two individual motions are represented by,

$$x_1 = A_1 \sin \omega t$$

and $$x_2 = A_2 \sin (\omega t + \phi)$$

Both the simple harmonic motions have same angular frequency ω.

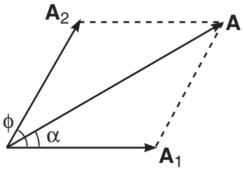


Fig. 14.21

The resultant displacement of the particle is given by,

$$\begin{aligned} x &= x_1 + x_2 \\ &= A_1 \sin \omega t + A_2 \sin (\omega t + \phi) \\ &= A \sin (\omega t + \alpha) \end{aligned}$$

Here, $$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos \phi}$$

and $$\tan \alpha = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}$$

Thus, we can see that this is similar to the vector addition. The same method of vector addition can be applied to the combination of more than two simple harmonic motions.

Example 14.16 *Find the displacement equation of the simple harmonic motion obtained by combining the motions.*

$$x_1 = 2 \sin \omega t, \quad x_2 = 4 \sin \left(\omega t + \frac{\pi}{6}\right) \quad \textit{and} \quad x_3 = 6 \sin \left(\omega t + \frac{\pi}{3}\right)$$

Solution The resultant equation is,

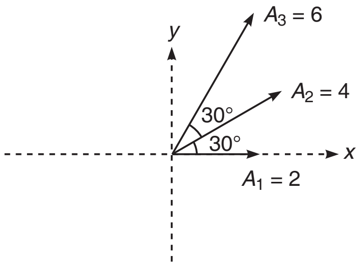


Fig. 11.22

$$x = A \sin(\omega t + \phi)$$

$$\Sigma A_x = 2 + 4\cos 30° + 6\cos 60° = 8.46$$

and $$\Sigma A_y = 4\sin 30° + 6\cos 30° = 7.2$$

$\therefore$ $$A = \sqrt{(\Sigma A_x)^2 + (\Sigma A_y)^2}$$

$$= \sqrt{(8.46)^2 + (7.2)^2}$$

$$= 11.25$$

and $$\tan\phi = \frac{\Sigma A_y}{\Sigma A_x}$$

$$= \frac{7.2}{8.46} = 0.85$$

or $$\phi = \tan^{-1}(0.85)$$

$$= 40.4°$$

Thus, the displacement equation of the combined motion is,

$$x = 11.25 \sin(\omega t + \phi)$$

where, $$\phi = 40.4°$$ **Ans.**

INTRODUCTORY EXERCISE 14.4

1. A particle is subjected to two simple harmonic motions of the same frequency and direction. The amplitude of the first motion is 4.0 cm and that of the second is 3.0 cm. Find the resultant amplitude if the phase difference between the two motions is
   (a) 0° (b) 60° (c) 90° (d) 180°
2. A particle is subjected to two simple harmonic motions.
   $$x_1 = 4.0 \sin(100\pi t) \quad \text{and} \quad x_2 = 3.0 \sin\left(100\pi t + \frac{\pi}{3}\right)$$
   Find
   (a) the displacement at $t = 0$
   (b) the maximum speed of the particle and
   (c) the maximum acceleration of the particle.
3. Three simple harmonic motions of equal amplitudes A and equal time periods in the same direction combine. The phase of the second motion is 60° ahead of the first and the phase of the third motion is 60° ahead of the second. Find the amplitude of the resultant motion.
4. A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions. Find the phase difference between the individual motions.

Final Touch Points

1. Simple Pendulum

(i) Time period $T = 2\pi\sqrt{\frac{l}{g}}$ is applicable only for small oscillations.

(ii) If the time period of a simple pendulum is 2 s, it is called **seconds pendulum.** Length of this pendulum is 1 m.

(iii) If length of the pendulum is large, g no longer remains vertical but will be directed towards the centre of the earth and expression for time period is given by,

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l}+\frac{1}{R}\right)}}$$

Here, R is the radius of earth. From this expression we can see that,

(a) if $l \ll R$, $\frac{1}{l} >> \frac{1}{R}$ and $T = 2\pi\sqrt{\frac{l}{g}}$

(b) as $l \to \infty, \frac{1}{l} \to 0$ and $T = 2\pi\sqrt{\frac{R}{g}}$ and substituting the value of R and g, we get $T = 84.6$ min.

(iv) Time period of a simple pendulum depends on acceleration due to gravity 'g' $\left(\text{as } T \propto \frac{1}{\sqrt{g}}\right)$ so take $|\mathbf{g}_{\text{eff}}|$ in $T = 2\pi\sqrt{\frac{l}{g}}$. Following two cases are possible :

(a) If a simple pendulum is in a carriage which is accelerating with acceleration **a**, then

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{a}$$

e.g. if the acceleration **a** is upwards, then

$$|\mathbf{g}_{\text{eff}}| = g + a \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{g+a}}$$

If the acceleration **a** is downwards, then (with $g > a$)

$$|\mathbf{g}_{\text{eff}}| = g - a \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{g-a}}$$

If the acceleration **a** is in horizontal direction, then

$$|\mathbf{g}_{\text{eff}}| = \sqrt{a^2 + g^2}$$

In a freely falling lift $g_{\text{eff}} = 0$ and $T = \infty$ i.e. the pendulum will not oscillate.

(b) If in addition to gravity one additional force **F**, (e.g. electrostatic force $\mathbf{F}_e$) is also acting on the bob, then in that case,

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \frac{\mathbf{F}}{m}$$

Here, m is the mass of the bob.

(v) Due to change in temperature, length of pendulum and so the time period will change. If $\Delta\theta$ is the increase in temperature then,

$$l' = l(1 + \alpha\Delta\theta) \quad \text{or} \quad \frac{l'}{l} = 1 + \alpha\Delta\theta$$

$$\therefore \qquad \frac{T'}{T} = \sqrt{\frac{l'}{l}} = (1 + \alpha\Delta\theta)^{1/2} \approx \left(1 + \frac{1}{2}\alpha\Delta\theta\right)$$

∴ $$\frac{T'}{T} - 1 \approx \frac{1}{2}\alpha\Delta\theta$$

or $$\frac{T' - T}{T} = \frac{1}{2}\alpha\Delta\theta$$

or $$\Delta T = \frac{1}{2}T\alpha\Delta\theta$$

Note *In case of a pendulum clock, time is lost if T increases and gained if T decreases. Time lost or gained in time t is given by,*

$$\Delta t = \frac{\Delta T}{T'} . t$$

e.g. if $T = 2$ s, $T' = 3$ s, then $\Delta T = 1$ s

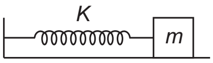
∴ *Time lost by the clock in 1 h.*

$$\Delta t = \frac{1}{3} \times 3600 = 1200 \text{ s}$$

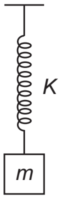
(vi) Above the surface of earth, blow the surface of earth, in moving from pole to equator and on moon value of g decreases, therefore time period of pendulum will increase.

2. Spring-block System

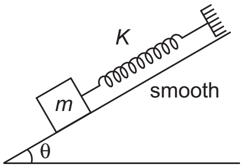
(i)



(a)



(b)



(c)

In all three figures shown above, restoring force in displaced position is $F = -kx$. Therefore, time period is,

$$T = 2\pi\sqrt{\frac{m}{K}}$$

The only difference is, their mean positions are different. In the first figure, mean position is at the natural length of spring. In the second figure, mean position is obtained after an extension of $x_0 = \frac{mg}{K}$ (as $kx_0 = mg$) and in the third figure after an extension of $x_0 = \frac{mg\sin\theta}{K}$.

(ii) In case of a vertical spring-block system, time period can also be written as,

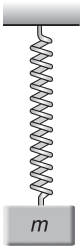
$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Here, l = extension in the spring when the mass m is suspended from the spring. This can be seen as under :

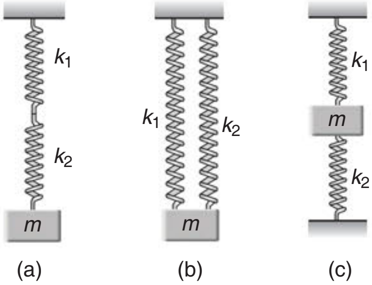
$$kl = mg \qquad \text{(in equilibrium position)}$$

∴ $$\frac{m}{k} = \frac{l}{g}$$

or $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$



(iii) **Equivalent force constant (k)** If a spring block system is constructed by using two springs and a mass, the following three situations are possible:



Refer Fig. (a)

In this case,

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$$

or $$k=\frac{k_1k_2}{k_1+k_2}$$

∴ $$T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$$

Fig. (b) and (c)

In both the cases,

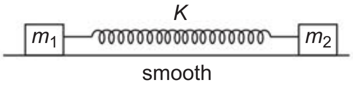
$$k=k_1+k_2$$

or $$T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m}{k_1+k_2}}$$

(iv) If spring has a mass m_s and a mass m is suspended from it, then time period is given by,

$$T=2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$$

(v) If the spring connected with two masses m_1 and m_2 is compressed or elongated by x_0 and then left for oscillations then both blocks execute SHM with same time period (and therefore same frequency) but different amplitudes.



This time period is

$$T=2\pi\sqrt{\frac{\mu}{K}}$$

Here, μ is called their reduced mass given by

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}\text{ or }\mu=\frac{m_1m_2}{m_1+m_2}$$

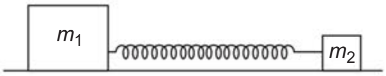
Their amplitudes of oscillations are in inverse ratio of their masses or

$$A\propto\frac{1}{m}\quad\Rightarrow\quad\frac{A_1}{A_2}=\frac{m_2}{m_1}$$

⇒ $$A_1=\left(\frac{m_2}{m_1+m_2}\right)x_0\quad\text{and}\quad A_2=\left(\frac{m_1}{m_1+m_2}\right)x_0$$

Note *If $m_1>>m_2$, then $\frac{1}{m_1}<<\frac{1}{m_2}$*

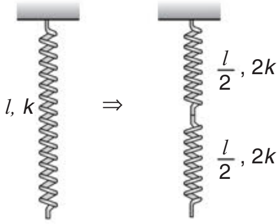
∴ $$\frac{1}{\mu}\approx\frac{1}{m_2}\quad\text{or}\quad\mu\approx m_2\quad\text{and}\quad T=2\pi\sqrt{\frac{m_2}{K}}$$



So, in this case we can understand that m_1 is almost stationary. Only m_2 will oscillate.

(vi) The force constant (k) of a spring is inversely proportional to the length of the spring. i.e.

$$k \propto \frac{1}{\text{length of spring}}$$

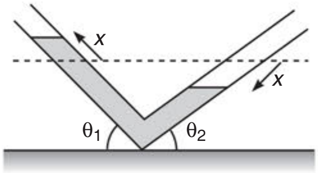


This can be visualized as:

A spring of length l and spring constant k can be supposed to be made up by two springs in series, of length $\frac{l}{2}$ and force constant $2k$. In series,

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2} = \frac{(2k)(2k)}{2k + 2k} = k$$

3. Oscillations of a fluid column



Initially the level of liquid in both the columns is same. The area of cross-section of the tube is uniform. If the liquid is depressed by x in one limb, it will rise by x along the length of the tube is the other limb. Here, the restoring force is provided by the hydrostatic pressure difference.

$$\therefore \qquad F = -(\Delta P)A = -(h_1 + h_2)\rho g A$$

$$= -\rho g A(\sin\theta_1 + \sin\theta_2)\,x$$

Let, m be the mass of the liquid in the tube. Then,

$$ma = -\rho g A(\sin\theta_1 + \sin\theta_2)\,x$$

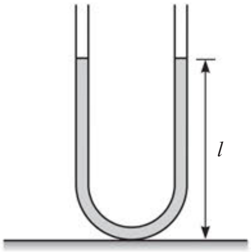
Since, F or a is proportional to $-x$, the motion of the liquid column is simple harmonic in nature, time period of which is given by,

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or

$$\boxed{T = 2\pi\sqrt{\frac{m}{\rho g A(\sin\theta_1 + \sin\theta_2)}}}$$

Note *For a U-tube, if the liquid is filled to a height* l, $\theta_1 = 90° = \theta_2$ *and* $m = 2(lA\rho)$



So, $$T = 2\pi\sqrt{\frac{l}{g}}$$

Thus, we see that the expression $T = 2\pi\sqrt{\frac{l}{g}}$ *comes in picture at three places.*

(i) Time period of a simple pendulum for small oscillations.
(ii) Time period of a spring-block system in vertical position.
(iii) Time period of a liquid column in a U-tube filled to a height l.

But l has different meanings at different places.

4. Lissajous figures Suppose two forces act on a particle, the first alone would produce a simple harmonic motion in x-direction given by,

$$x = a\sin\omega t$$

and the second would produce a simple harmonic motion in y-direction given by,

$$y = b\sin(\omega t + \phi)$$

The amplitudes a and b may be different and their phases differ by ϕ. The frequencies of the two simple harmonic motions are assumed to be equal. The resultant motion of the particle is a combination of the two simple harmonic motions.

Depending on the value of ϕ and relation between a and b, the particle follows different paths. These different paths are called Lissajous figures. Given below are few special cases :

***Case* 1 (When $\phi = 0°$)** When the phase difference between two simple harmonic motions is 0°, i.e.

$$x = a \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{a} \qquad \text{...(i)}$$

$$y = b \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{y}{b} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x}{a} = \frac{y}{b} \quad \text{or} \quad y = \left(\frac{b}{a}\right) x$$

which is equation of a straight line with slope $\frac{b}{a}\cdot$ Thus, the path of the particle is a straight line. As a special case $y = x$ if $a = b$ or slope is 1.

***Case* 2 $\left(\text{when } \phi = \frac{\pi}{2}\right)$** When the phase difference is $\frac{\pi}{2}$ i.e.

$$x = a \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{a} \qquad \text{...(iii)}$$

$$y = b \sin \left(\omega t + \frac{\pi}{2}\right) = b \cos \omega t$$

$$\Rightarrow \qquad \cos \omega t = \frac{y}{b} \qquad \text{...(iv)}$$

Squaring and adding, Eqs. (iii) and (iv), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an **ellipse.** Again as a special case, the above equation reduces to,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$\text{or} \qquad x^2 + y^2 = a^2 \qquad (\text{for } a = b)$$

This is an equation of a **circle.**

5. Free and damped oscillations

We know that in reality, a spring won't oscillate forever with constant amplitude. These constant amplitude oscillations (which will really occur in vacuum) are called **"free oscillations"**. Frictional forces will diminish the amplitude of oscillation until eventually the system comes to rest.

We will now add frictional forces to the mass and spring. Imagine that the mass was put in a viscous liquid. To incorporate friction, we can just say that there is a frictional force that's proportional to the velocity of the mass. This is a pretty good approximation for a body moving at a low velocity in air, or in a liquid. So we say the frictional force $f_r = -bv$. The constant b depends on the kind of liquid and the shape of the mass. The negative sign, just says that the force is in the opposite direction to the body's motion. Let's add this frictional force in to the equation $F_{\text{net}} = ma$

$$-kx - bv = ma \qquad \text{...(i)}$$

In terms of derivatives,

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \qquad \text{...(ii)}$$

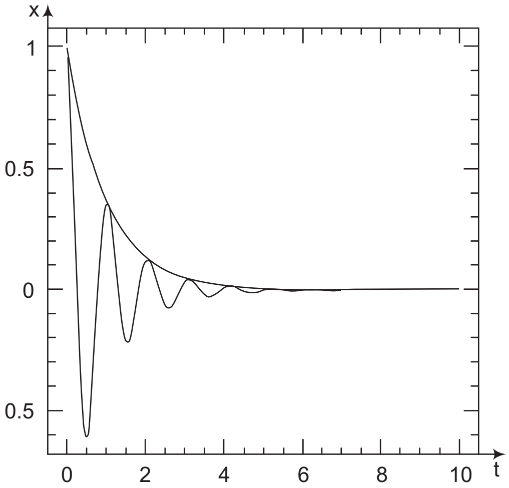
This is a differential equation. In the solution of this type of differential equation instead of the amplitude being constant, it is decaying with time.

$$\boxed{A(t) = A_0 e^{-\alpha t}}$$

Here, α is a positive constant.

So, $\boxed{x(t) = A(t)\cos(\omega t + \delta) = A_0 e^{-\alpha t}\cos(\omega t + \delta)}$

Here's a plot of an example of such a function.



These types of oscillations which eventually come to end are called **"damped oscillations".** There are further three types of damped oscillations, namely

(i) Under damped
(ii) Critically damped and
(iii) Over damped oscillations

But their detailed discussion is out of our syllabus at this stage.

Natural Frequency (or Characteristic Frequency)

That is the frequency at which a system would oscillate by itself if displaced. The natural frequency of a spring-mass system is $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$, where k is the spring constant and m is the mass of the object attached to the spring.

Forced Oscillations

A periodic force at a given frequency (called driving frequency f_d) is applied to an oscillating system of natural frequency f_0. At the beginning (transient stage), there is a mixture of two kinds of oscillations, one has the frequency f_0 and the other has f_d. The former will gradually die out because of the damping effect. Eventually (at the steady state) the system settles down with oscillation at the frequency of the driving force (f_d).

Resonance

When the driving frequency is at the same frequency as the natural frequency of the oscillator, the amplitude of oscillation is at its greatest. When this happens the energy of the oscillator becomes a maximum. This is called a condition of resonance.

Resonance and its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance ; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. Inexpensive loudspeakers often have an unwanted boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone. Resonance also occurs in electric circuits ; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step ; the frequency of their steps was close to the natural vibration frequency of the bridge and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge.

Nearly everyone has seen the film of the collapse of the Tacoma Narrows suspension bridge in 1940.

Solved Examples

TYPED PROBLEMS

Type 1. *According to given x - t equation we have to find initial conditions (at $t = 0$).*

Concept

The method discussed here is a general method which can be applied for any type of motion.

How to Solve?

Differentiate the given x - t equation and find v - t equation. Now, put $t = 0$ in both x - t and v - t equations.

Example 1 *x - t equation of a particle executing SHM is*

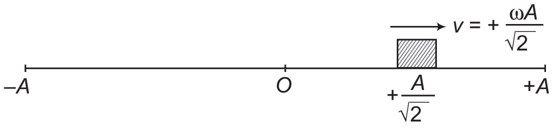
$$x = A\cos(\omega t - 45°)$$

Find the point from where particle starts its journey and the direction of its initial velocity.

Solution Given,
$$x = A\cos(\omega t - 45°) \qquad \text{...(i)}$$
$$v = \frac{dx}{dt} = -\omega A\sin(\omega t - 45°) \qquad \text{...(ii)}$$

Putting $t = 0$ in both equations we have,

$$x = +\frac{A}{\sqrt{2}} \quad \text{and} \quad v = +\frac{\omega A}{\sqrt{2}}$$



At $t = 0$

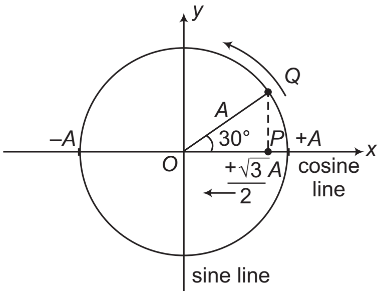
So, the particle starts from $x = +\frac{A}{\sqrt{2}}$ with velocity in positive direction.

Note *By further differentiating v - t equation, we can make a - t equation and by putting $t = 0$, we can also find initial value of acceleration.*

Type 2. *First Use of reference circle (to make x - t equation corresponding to given initial conditions).*

How to Solve?

For the given SHM particle P on the diameter we have to find its circular motion particle Q on the circle according to given initial conditions. The following points will help you in finding the particle Q.



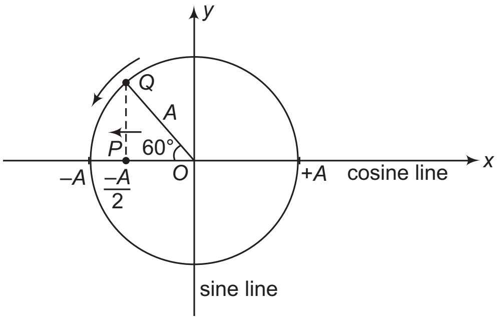
- Particle Q can rotate, either clockwise or anticlockwise. But in all problems, we will take it anticlockwise.
- We can draw the perpendicular on any diameter but in all problems we will draw the perpendicular lying along x-axis.
- If initial velocity of SHM particle P is negative then Q lies in upper half of the reference circle. Because, when Q moves in upper half of the circle (anticlockwise) P moves from $+A$ to $-A$ along negative x-direction. Similarly, if P has an initial velocity in positive direction then Q is in lower half of the circle.
- If initial value of x is positive then Q lies to the right of O (between 0 and $+A$) and if x is negative, then Q lies to the left of O (between 0 and $-A$).
- If initial value of x is suppose $+\frac{\sqrt{3}}{2}A$ and velocity is negative then circular particle Q is in the position as shown in figure.

  This particle is 120° ahead (in anticlockwise direction) or 240° behind (in clockwise direction) from sine line or it is 30° ahead or 330° behind the cosine line. Hence, the x-t equation corresponding to given initial conditions may be

$$\begin{aligned} x &= A\sin(\omega t + 120°) \\ &= A\sin(\omega t - 240°) \\ &= A\cos(\omega t + 30°) \\ &= A\cos(\omega t - 330°) \end{aligned}$$

Example 2 *A particle in SHM starts its journey (at $t=0$) from $x=-\frac{A}{2}$ in negative direction. Write x-t equation corresponding to given conditions. Angular frequency of oscillations is ω.*

Solution For the given initial conditions, particle Q on reference circle is as shown in figure. This is 210° ahead or 150° behind the sine line or 120° ahead or 240° behind the cosine line. Therefore, the x-t equation can be written as



$$x = A\sin(\omega t + 210°) = A\sin(\omega t - 150°) = A\cos(\omega t + 120°) = A\cos(\omega t - 240°)$$

Type 3. *Second use of reference circle (to find the shortest time in moving the particle from one point to another point between $+A$ and $-A$).*

Concept

Linear speed of particle in SHM is not constant (more near mean positive and less near mean extreme positions)

So, we cannot apply

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

but the angular speed of its corresponding particle on reference circle is constant.

Therefore, $$t = \frac{\theta}{\omega} \quad (\theta \text{ in radian})$$

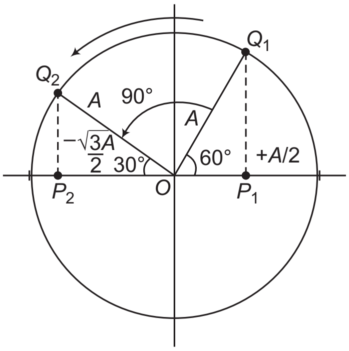
can be applied for that particle. The following table will help you in solving such problems.

Angle rotated in radian	Angle rotated in degrees	Time taken
2π	360°	T
π	180°	$T/2$
$\pi/3$	120°	$T/3$
$\pi/2$	90°	$T/4$
$\pi/4$	45°	$T/8$
$\pi/6$	30°	$T/12$

Example 3 *In terms of time period of oscillations T, find the shortest time in moving a particle from* $+\frac{A}{2}$ *to* $-\frac{\sqrt{3}}{2}$.

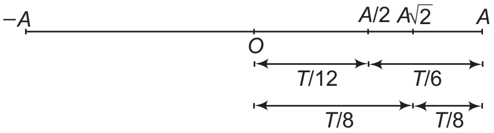
Solution As shown in figure, as the SHM particle P moves from P_1 to P_2, its corresponding particle Q on reference circle rotates from Q_1 to Q_2 an angle of 90°. From the above table, we can see that, in rotating an angle of 90° the time taken is $\frac{T}{4}$.

$$t = \frac{T}{4}$$ **Ans.**



Note *Some, standard results which are frequently asked are given in following table:*

From	To	Time taken
O	$A/2$	$T/12$
$A/2$	A	$T/6$
O	$A/\sqrt{2}$	$T/8$
$A/\sqrt{2}$	A	$T/8$

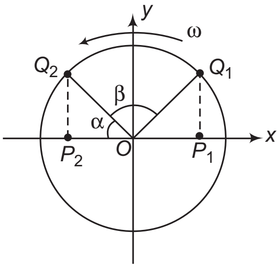


Near the mean position, speed of particle is more. Therefore, it covers first half of the distance (from 0 to $A/2$) in a shorter time compared to the second half (from $A/2$ to A).

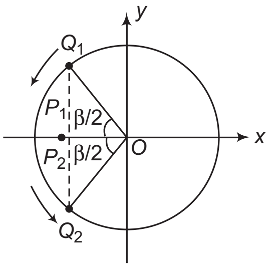
Type 4. *Third Use of reference circle (finding time of collision of two particles executing SHM along same straight line with same ω and A but different initial conditions).*

How to Solve?

Corresponding to initial conditions of SHM particles P_1 and P_2, first find their particles Q_1 and Q_2 on the reference circle.



Now, we can see that the particles Q_1 and Q_2 will never collide as they are rotating in the same direction (anticlockwise) with same ω and T. But the perpendiculars from Q_1 and Q_2 on our diameter along x-axis will collide when this diameter becomes the bisector line of the constant angle β (between OQ_1 and OQ_2) as shown below.



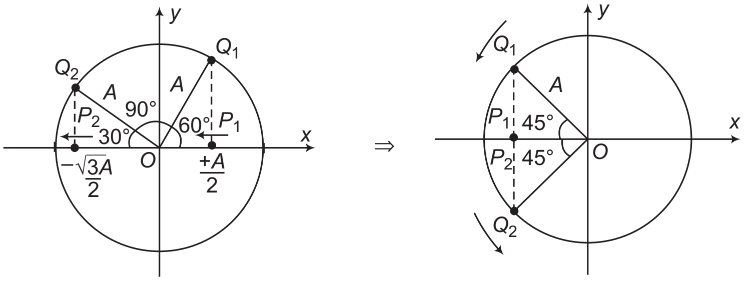
Now, either of the two particles has rotated an angle.

$$\theta = \alpha + (\beta/2)$$

∴ Time taken is
$$t = \frac{\theta}{\omega}$$

Example 4 *Two SHM particles* P_1 *and* P_2 *start from* $+\frac{A}{2}$ *and* $-\frac{\sqrt{3}A}{2}$, *both in negative directions. Find the time (in terms of T) when they collide. Both particles have same* ω, *A and T and they execute SHM along the same line.*

Solution



In the first figure, we have shown two particles Q_1 and Q_2 corresponding to given initial conditions of P_1 and P_2. In the second figure, we have shown the moment when the SHM particles are colliding at $x = -A\cos 45° = -\frac{A}{\sqrt{2}}$.

We can see that either of the particles Q_1 and Q_2 has rotated an angle $\theta = 75°$ or $\frac{5\pi}{12}$. So, the time taken is

$$t = \frac{\theta}{\omega} = \frac{(5\pi/12)}{(2\pi/T)} = \left(\frac{5}{24}\right)T$$

Note *In the above problem, the constant angle between* OQ_1 *and* OQ_2 *is 90°. So, the constant phase difference between two SHM particles corresponding to given initial conditions is also 90°.*

Type 5. *Spring Block Systems.*

Concept

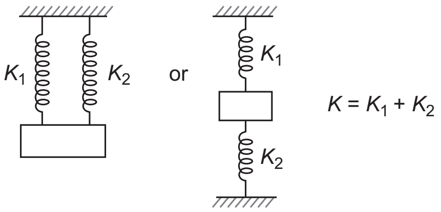
There are a lot of problems based on spring-block system. The main concepts are:

(i) $T = 2\pi\sqrt{\frac{m}{K}}$ and $\omega = \frac{2\pi}{T} = \sqrt{\frac{K}{m}}$

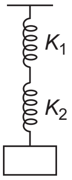
(ii) Force constant of spring,

$$K \propto \frac{1}{\text{Length of spring}}$$

(iii)



(iv)



$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\text{or } K = \frac{K_1 K_2}{K_1 + K_2}$$

Let us take some examples based on the above concepts.

Example 5 *A mass is suspended separately by two springs and the time periods in the two cases are T_1 and T_2. Now the same mass is connected in parallel $(K = K_1 + K_2)$ with the springs and the time is suppose T_P. Similarly time period in series is T_S, then find the relation between T_1, T_2 and T_P in the first case and T_1, T_2 and T_S in the second case.*

Solution $T = 2\pi\sqrt{\frac{m}{K}}$

$\therefore$ $$T = \frac{\alpha}{\sqrt{K}} \qquad \text{(where, } 2\pi\sqrt{m} = \alpha = \text{constant)}$$

$\Rightarrow$ $$K = \frac{\alpha^2}{T^2} \qquad \text{...(i)}$$

or $$\frac{1}{K} = \frac{T^2}{\alpha^2} \qquad \text{...(ii)}$$

First case In parallel,

$$K = K_1 + K_2$$

Using Eq. (i), $$\frac{\alpha^2}{T_P^2} = \frac{\alpha^2}{T_1^2} + \frac{\alpha^2}{T^2}$$

$\therefore$ $$T_P^{-2} = T_1^{-2} + T_2^{-2}$$ **Ans.**

This is the desired relation.

Second case In series, $$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}$$

Now, using Eq. (ii) we have, $$\frac{T_s^2}{\alpha^2} = \frac{T_1^2}{\alpha^2} + \frac{T_2^2}{\alpha^2}$$

$\therefore$ $$T_s^2 = T_1^2 + T_2^2$$ **Ans.**

This is the desired relation in this case.

Example 6 *Time period of a block with a spring is T_0. Now, the spring is cut in two parts in the ratio 2 : 3. Now find the time period of same block with the smaller part of the spring.*

Solution $T = 2\pi\sqrt{\frac{m}{K}}$ $\Rightarrow$ $$T \propto \frac{1}{\sqrt{K}}$$

But $$K \propto \frac{1}{l} \qquad (l = \text{length of spring})$$

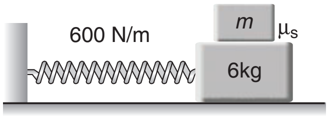
$\therefore$ $$T \propto \sqrt{l} \qquad \text{...(i)}$$

Now, suppose total length of the spring is $5l$ and it is cut in two parts of lengths $2l$ and $3l$. Eq. (i) can be written as

$$\frac{T_{2l}}{T_{5l}}=\sqrt{\frac{2l}{5l}}=\sqrt{\frac{2}{5}}$$

$\therefore$
$$T_{2l}=\left(\sqrt{\frac{2}{5}}\right)T_{5l}=\left(\sqrt{\frac{2}{5}}\right)T_0$$ **Ans.**

Example 7 *With the assumption of no slipping, determine the mass m of the block which must be placed on the top of a 6 kg cart in order that the system period is 0.75 s. What is the minimum coefficient of static friction* μ_s *for which the block will not slip relative to the cart if the cart is displaced 50 mm from the equilibrium position and released? Take* $(g=9.8\ m/s^2)$.



Solution (a)
$$T=2\pi\sqrt{\frac{m+6}{600}} \qquad \left(T=2\pi\sqrt{\frac{m}{K}}\right)$$

or
$$0.75=2\pi\sqrt{\frac{m+6}{600}}$$

$\therefore$
$$m=\frac{(0.75)^2\times 600}{(2\pi)^2}-6$$
$$=2.55\text{ kg}$$ **Ans.**

(b) Maximum acceleration of SHM is,

$$a_{\max}=\omega^2 A \qquad (A=\text{amplitude})$$

i.e. maximum force on mass *'m'* is $m\omega^2A$ which is being provided by the force of friction between the mass and the cart. Therefore,

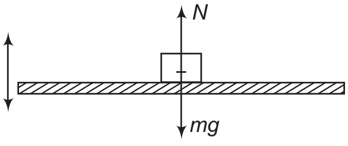
$$\mu_s mg\geq m\omega^2A \quad \text{or} \quad \mu_s\geq\frac{\omega^2A}{g}$$

or
$$\mu_s\geq\left(\frac{2\pi}{T}\right)^2\cdot\frac{A}{g} \quad \text{or} \quad \mu_s\geq\left(\frac{2\pi}{0.75}\right)^2\left(\frac{0.05}{9.8}\right) \qquad (A=50\text{ mm})$$

or
$$\mu_s\geq 0.358$$

Thus, the minimum value of μ_s should be 0.358. **Ans.**

Example 8 *A block is kept over a horizontal platform, executing vertical SHM of angular frequency* ω. *Find maximum amplitude of oscillations, so that the block does not leave contact with the platform.*



Solution At mean position, acceleration, $a = 0$

$$\Rightarrow \qquad N = mg \qquad (N = \text{normal reaction})$$

Below the mean position, acceleration is upwards (towards the mean position)

$$\therefore \qquad N - mg = ma \quad \text{or} \quad N = m(g + a)$$

Above the mean position, acceleration is downwards (towards the mean position)

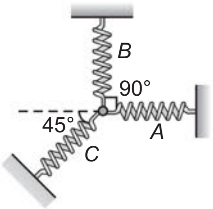
$$\therefore \qquad mg - N = ma \quad \text{or} \quad N = m\,(g - a)$$

In this case, N decreases as 'a' increases. At the extreme position acceleration is maximum $(=\omega^2 A)$, so N is minimum. When the block leaves contact with the plank N becomes zero.

$$\therefore \qquad 0 = m(g - \omega^2 A) \quad \text{or} \quad A = \frac{g}{\omega^2}$$

Therefore, the maximum value of amplitude A for not leaving the contact is $\frac{g}{\omega^2}$.

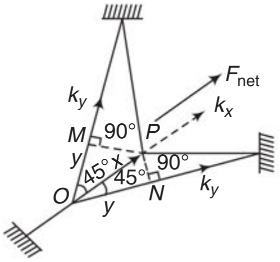
Example 9 *A particle of mass m is attached with three springs A, B and C of equal force constants k as shown in figure. The particle is pushed slightly against the spring C and released. Find the time period of oscillation.*



Solution $OP = x$

$$\angle POM = \angle PON \approx 45°$$

$$y = x \cos 45° = \frac{x}{\sqrt{2}}$$



Net restoring force,

$$F_{\text{net}} = -\,[kx + 2ky \cos 45°\,]$$

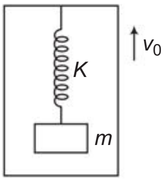
$$= -\,(2k)\,x \qquad \left(\text{as } y = \frac{x}{\sqrt{2}}\right)$$

$$\therefore \qquad k_e = 2\,k$$

Now,

$$T = 2\pi\sqrt{\frac{m}{k_e}} = 2\pi\sqrt{\frac{m}{2k}}$$

Example 10 *A spring block system is kept inside a lift moving with a constant velocity v_0 as shown in figure. Block is in equilibrium and at rest with respect to lift. The lift is suddenly stopped at time $t = 0$. Taking upward direction as the positive direction, write x-t equation of the block.*



Solution Lift is suddenly stopped but block will not stop instantly. It will have same velocity v_0 at $t = 0$. But this is the velocity at its mean position (as it was at rest w.r.t lift). So, this is also its maximum velocity of SHM.

$$\therefore \qquad v_0 = v_{\max} = \omega A \;\Rightarrow\; A = \frac{v_0}{\omega} \qquad \left(\text{where, } \omega = \sqrt{\frac{K}{m}}\right)$$

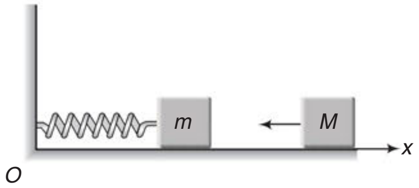
Now, at $t = 0$, block is at mean position and has a velocity v_0 in upward or positive direction. Therefore, the x-t equation is

$$x = A \sin \omega t \qquad \textbf{Ans.}$$

where,

$$A = \frac{v_0}{\omega} \text{ and } \omega = \sqrt{\frac{K}{m}}$$

Example 11 *One end of an ideal spring is fixed to a wall at origin O and axis of spring is parallel to x-axis. A block of mass* $m = 1$ *kg is attached to free end of the spring and it is performing SHM. Equation of position of the block in co-ordinate system shown in figure is* $x = 10 + 3\sin(10t)$. *Here, t is in second and x in cm. Another block of mass* $M = 3$ *kg, moving towards the origin with velocity 30 cm/s collides with the block performing SHM at* $t = 0$ *and gets stuck to it. Calculate*



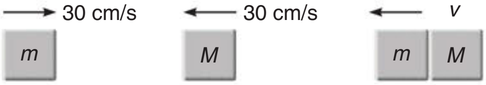
(a) new amplitude of oscillations,

(b) new equation for position of the combined body,

(c) loss of energy during collision. Neglect friction.

Solution (a) Initially, $\omega^2 = \frac{k}{m}$

$$\therefore \qquad k = m\omega^2 = (1)\,(10)^2 = 100 \text{ N/m}$$



At $t = 0$, block of mass m is at mean position ($x = 10$ cm) and moving towards positive x-direction with velocity $A\omega$ or 30 cm/s.

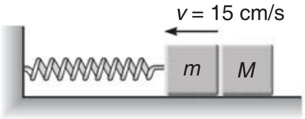
From conservation of linear momentum,

$$(M + m)v = M(30) - m(30)$$

Substituting the values, we have

$v =$ velocity of combined mass just after collision = 15 cm/s or 0.15 m/s

From conservation of mechanical energy,



$$\frac{1}{2}(M + m)v^2 = \frac{1}{2}k(A')^2$$

Here, A' = new amplitude of oscillation of combined mass

$$\therefore \qquad A' = \left(\sqrt{\frac{M + m}{k}}\right)v = \left(\frac{4}{100}\right)^{1/2}(0.15) = 0.03 \text{ m}$$

or $\qquad A' = 3$ cm **Ans.**

(b) New angular frequency $\omega' = \sqrt{\frac{k}{M + m}} = \sqrt{\frac{100}{4}} = 5$ rad/s

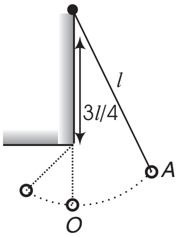
$$\therefore \qquad x' = 10 - 3\sin 5t$$ **Ans.**

(c) Loss of mechanical energy $E_f - E_i$

$$= \frac{1}{2}(1)(0.3)^2 + \frac{1}{2}(3)(0.3)^2 - \frac{1}{2}(4)(0.15)^2$$

$$= 0.135 \text{ J}$$ **Ans.**

Type 6. *Time period of a periodic motion which is not completely simple harmonic.*

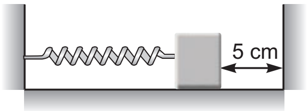


Example 12 *A pendulum has a period T for small oscillations. An obstacle is placed directly beneath the pivot, so that only the lowest one-quarter of the string can follow the pendulum bob when it swings to the left of its resting position. The pendulum is released from rest at a certain point. How long will it take to return to that point again ? In answering this question, you may assume that the angle between the moving string and the vertical stays small throughout the motion.*

Solution Half the oscillation is completed with length l and rest half with length $\frac{l}{4}$.

$$T = \frac{T_1}{2} + \frac{T_2}{2} = \frac{1}{2}\left[2\pi\sqrt{\frac{l}{g}} + 2\pi\sqrt{\frac{\frac{l}{4}}{g}}\right]$$

$$= \frac{3}{4}\left[2\pi\sqrt{\frac{l}{g}}\right] = \frac{3}{4}T, \text{ where } T = 2\pi\sqrt{\frac{l}{g}} \qquad \textbf{Ans.}$$



Example 13 *A block of mass 100 g attached to a spring of stiffness 100 N/m is lying on a frictionless floor as shown. The block is moved to compress the spring by 10 cm and released. If the collision with the wall is elastic then find the time period of oscillations.*

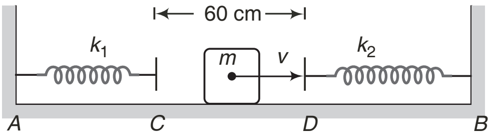
Solution The given distance on RHS 5 cm is $\frac{A}{2}$ and from 0 to $\frac{A}{2}$ time taken is $\frac{T}{12}$.

$$T = t_{\text{LHS}} + t_{\text{RHS}} = \frac{T}{2} + 2\left(\frac{T}{12}\right)$$

$$= \frac{2}{3}T = \frac{2}{3}(2\pi)\sqrt{\frac{m}{k}}$$

$$= \frac{4\pi}{3}\sqrt{\frac{0.1}{100}} = 0.133 \text{ s} \qquad \textbf{Ans.}$$

Example 14 *Two light springs of force constants k_1 and k_2 and a block of mass m are in one line AB on a smooth horizontal table such that one end of each spring is fixed on rigid supports and the other end is free as shown in the figure. The distance CD between the free ends of the spring is 60 cm. If the block moves along AB with a velocity 120 cm/s in between the springs, calculate the period of oscillation of the block. (Take $k_1 = 1.8$ N/m, $k_2 = 3.2$ N/m, $m = 200$ g)* **(JEE 1985)**



Solution Between C and D block will move with constant speed of 120 cm/s. Therefore, period of oscillation will be (starting from C).

$$T = t_{CD} + \frac{T_2}{2} + t_{DC} + \frac{T_1}{2}$$

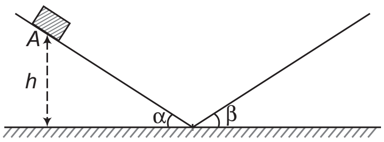
Here, $$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \text{ and } T_2 = 2\pi\sqrt{\frac{m}{k_2}}$$

and $$t_{CD} = t_{DC} = \frac{60}{120} = 0.5 \text{ s}$$

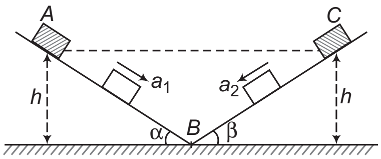
$\therefore$ $$T = 0.5 + \frac{2\pi}{2}\sqrt{\frac{0.2}{3.2}} + 0.5 + \frac{2\pi}{2}\sqrt{\frac{0.2}{1.8}}$$

$$T = 2.82 \text{ s}$$ **Ans.**

Example 15 *A block is released from point A as shown in figure. All surfaces are smooth and there is no loss of mechanical energy anywhere. Find the time period of oscillations of block.*



Solution As there is no loss of mechanical energy, the block rises upto the same height 'h' on other side also.



$$AB = \frac{h}{\sin\alpha}, \; BC = \frac{h}{\sin\beta}$$

$$a_1 = g\sin\alpha, \; a_2 = g\sin\beta$$

$$u_A = u_C = 0$$

Using $$t = \sqrt{\frac{2s}{a}}$$ (with $u = 0$)

$\therefore$ Time period of oscillations,

$$T = 2t_{AB} + 2t_{CB} = 2(t_{AB} + t_{CB})$$

$$= 2\left[\sqrt{\frac{2 \times \frac{h}{\sin\alpha}}{g\sin\alpha}} + \sqrt{\frac{2 \times \frac{h}{\sin\beta}}{g\sin\beta}}\right]$$

$$= 2\sqrt{\frac{2h}{g}}\,[\text{cosec}\,\alpha + \text{cosec}\,\beta]$$ **Ans.**

Type 7. *When point of suspension of the pendulum has an acceleration* **a**.

Concept

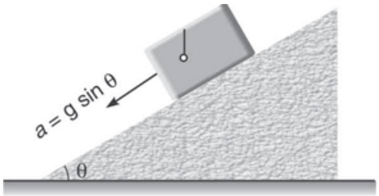
As we have discussed in the theory, time period in this case is

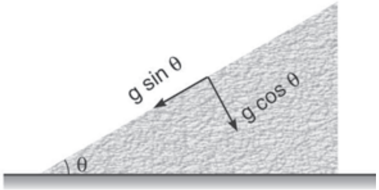
$$T = 2\pi\sqrt{\frac{l}{g_e}}$$

where, $$\mathbf{g}_e = \mathbf{g} - \mathbf{a} \text{ or } \mathbf{g} + (-\mathbf{a}) \text{ and } g_e = |\mathbf{g}_e|$$

Example 16 *A simple pendulum of length l is suspended from the ceiling of a cart which is sliding without friction on an inclined plane of inclination θ. What will be the time period of the pendulum?*

Solution Here, point of suspension has an acceleration. $a = g \sin\theta$ (down the plane). Further, $\mathbf{g}$ can be resolved into two components $g\sin\theta$ (along the plane) and $g\cos\theta$ (perpendicular to plane).





$$\therefore \qquad \mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{a} = \mathbf{g} + (-\mathbf{a})$$

Resultant of $\mathbf{g}$ and $-\mathbf{a}$ will be $g\cos\theta$.

$$\therefore \qquad g_{\text{eff}} = g\cos\theta \qquad \text{(perpendicular to plane)}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{l}{|\mathbf{g}_{\text{eff}}|}} = 2\pi\sqrt{\frac{l}{g\cos\theta}} \qquad \textbf{Ans.}$$

Note *If* $\theta = 0°$, $T = 2\pi\sqrt{\frac{l}{g}}$ *which is quiet obvious.*

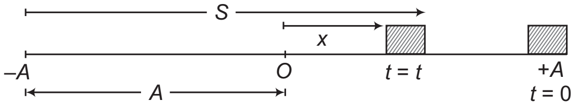
Type 8. *When displacement is not measured from the mean position.*

Concept

In most of the x-t equations of SHM, x is measured from the mean position. In this type of problems, we have to write S-t equation. But S here is not measured from the mean position. The method can be best explained by the following example:

Example 17 *A block in SHM starts from $+A$ position. Write S-t equation of the block, if S is measured from $-A$.*

Solution



When the block starts from $+A$ and x at any time 't' is measured from the mean position, then

$$x = A\cos\omega t$$

In the figure, we can see that

$$S = A + x \quad \text{or} \quad S = A + A\cos\omega t \quad \text{or} \quad S = A(1+\cos\omega t) \qquad \textbf{Ans.}$$

Check From the equation we can see that,

$$S = 2A \text{ at } t = 0$$

$$S = A \text{ at } t = \frac{T}{4} \text{ and } \omega = \frac{2\pi}{T} \Rightarrow \omega t = \frac{\pi}{2}$$

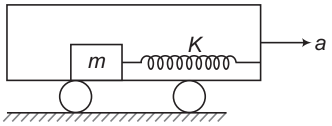
$$S = 0 \text{ at } T = \frac{T}{2} \text{ and } \omega = \frac{2\pi}{T} \Rightarrow \omega t = \pi$$

Type 9. *Based on the concept of pseudo force.*

Concept

If a block is observed from an accelerating or non-inertial frame of reference, a pseudo force ($= -m\mathbf{a}$) has to be applied on the block. Here, 'm' is the mass of the block and $\mathbf{a}$ the acceleration of frame of reference.

Example 18 *A spring block system is kept inside the smooth surface of a trolley as shown in figure.*



At $t = 0$, trolley is given an acceleration 'a' in the direction shown in figure.

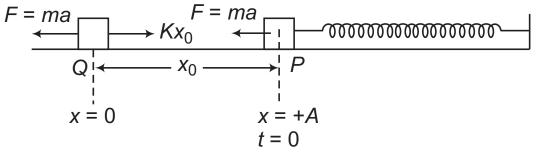
Write S - t equation of the block

(a) with respect to trolley.

(b) with respect to ground.

Note *S has to be measured from the starting point (at $t = 0$) not from the mean position of SHM with respect to trolley.*

Solution (a) Trolley is an accelerating or non-inertial frame. So, a pseudo force $F = ma$ has to be applied on the block towards left.



Here, P or natural length of the spring is not the mean position (of SHM of the block with respect to trolley).

Let mean position (where $\mathbf{F}_{\text{net}} = 0$) is obtained after an extension of x_0. Then

$$ma = kx_0 \Rightarrow x_0 = \frac{ma}{k}$$

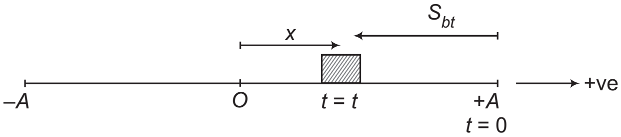
This x_0 is also the amplitude of SHM of block with respect to trolley.

$\therefore$ $$A = x_0 = \frac{ma}{k}$$

Further, at $t = 0$, block is at $x = +A$.

Therefore, x - t equation of block is, $\quad x = A\cos\omega t \quad$ (where, $\omega = \sqrt{\frac{k}{m}}$)

But in this equation x is measured from the mean position and we have to measure S from the starting point ($= +A$)



From the figure we can see that,

S_{bt} = displacement of block with respect to trolley.

$$= -(A - x)$$

$$= -(A - A\cos\omega t)$$

$$S_{bt} = (A\cos\omega t - A)$$ **Ans.**

(b) Displacement of trolley at time 't' is

$$S_t = \frac{1}{2}at^2$$

Now $$S_{bt} = S_b - S_t$$

∴ $$S_b = S_{bt} + S_t$$

or S_b = displacement of block with respect to ground.

or $$S_b = (A\cos\omega t - A) + \frac{1}{2}at^2$$ **Ans.**

Type 10. *Distance travelled in a given time interval.*

Concept

Linear speed of particle in SHM is not constant. Therefore, we cannot apply,

distance = speed × time

How to Solve?

From the given x-t equation, find the starting point, amplitude and time period of oscillations.

Example 19 *x-t equation of a particle in SHM is*

$$x = (4\,cm)\cos\left(\frac{\pi}{2}t\right)$$

Here, t is in seconds. Find the distance travelled by the particle in first three seconds.

Solution $A = 4$ cm

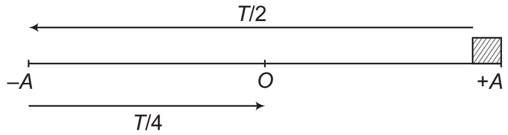
$$\omega = \left(\frac{\pi}{2}\right)\text{rad/s}$$

∴ $$T = \frac{2\pi}{T} = 4\,\text{sec}$$

The given equation is $x = A\cos\omega t$.

Therefore, the particle starts from $x = +A$.

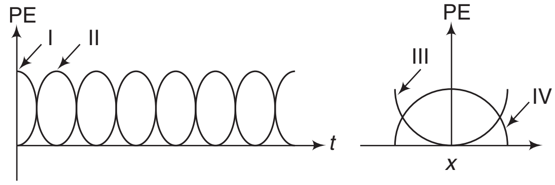
The given time $t = 3$ sec is $\frac{3T}{4}$. So, the particle moves from $+A$ to $-A$ and then from $-A$ to O.



So, distance travelled in the given time interval is $3A$ or 12 cm. **Ans.**

Type 11. *Graph Based problems.*

Example 20 *For a particle executing SHM, the displacement x is given by $x = A\cos\omega t$. Identify the graph which represents the variation of potential energy (PE) as a function of time t and displacement x.* **(JEE 2003)**

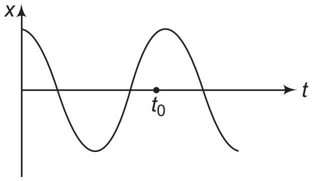


(a) I, III *(b)* II, IV *(c)* II, III *(d)* I, IV

Solution Potential energy is minimum (in this case zero) at mean position ($x = 0$) and maximum at extreme positions ($x = + A$).

At time $t = 0$, $x = A$. Hence, PE should be maximum. Therefore, graph I is correct. Further in graph III, PE is minimum at $x = 0$. Hence, this is also correct.

Example 21 *x - t graph of a particle in SHM is*



At time $t = t_0$, what are the signs of v and a of the particle?

Solution $v = \frac{dx}{dt}$ = slope of x - t graph.

At time $t = t_0$, slope is positive. So, velocity is positive.

$$a \propto -x$$

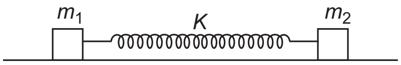
At time $t = t_0$, x is positive. Hence, acceleration is negative because 'a' and 'x' are always of opposite signs.

Note *At time t_0, velocity is positive and acceleration is negative. So, speed of the particle at this instant is decreasing or the particle is moving towards the extreme position.*

Type 12. *Based on two body oscillator system.*

Concept

As discussed in the theory, if the spring is compressed or elongated by x_0 and released then both blocks execute SHM with same time period T (and ω also) but different amplitudes in the inverse ratio of their masses.



$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

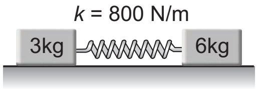
Here, $\mu = \text{reduced mass} = \dfrac{m_1 m_2}{m_1 + m_2}$

$$A \propto \frac{1}{m} \quad \text{or} \quad \frac{A_1}{A_2} = \frac{m_2}{m_1}$$

$\therefore$ $A_1 = \left(\dfrac{m_2}{m_1 + m_2}\right) x_0$

and $A_2 = \left(\dfrac{m_1}{m_1 + m_2}\right) x_0$

Example 22 *The system shown in the figure can move on a smooth surface. They are initially compressed by 6 cm and then released, then choose the correct options.*



(a) *The system performs, SHM with time period* $\dfrac{\pi}{10}$ *s*

(b) *The block of mass 3 kg perform SHM with amplitude 4 cm*

(c) *The block of mass 6 kg will have maximum momentum of 2.40 kg-m/s*

(d) *The time periods of two blocks are in the ratio of* $1:\sqrt{2}$

Solution μ = Reduced mass

$$= \frac{m_1\, m_2}{m_1 + m_2} = 2 \text{ kg}$$

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

$$= 2\pi\sqrt{\frac{2}{800}} = \frac{\pi}{10} \text{ s} = T_3 = T_6$$

$$A \propto \frac{1}{m}$$

$\therefore$ $\dfrac{A_3}{A_6} = \dfrac{6}{3} = \dfrac{2}{1}$

$\therefore$ $A_3 = 4 \text{ cm} \quad \text{and} \quad A_6 = 2 \text{ cm}$

$$(P_6)_{\max} = m_6\, (v_6)_{\max}$$

$$= m_6\, (\omega A_6)$$

$$= (6)\left(\frac{2\pi}{\pi/10}\right)(2 \times 10^{-2})$$

$$= 2.4 \text{ kg-m/s}$$

$\therefore$The correct options are (a, b, c)

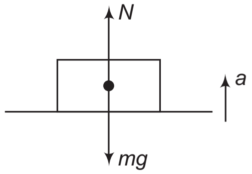
Type 13. *Change in variable force method for finding time period of SHM.*

Concept

(i) Two types of forces may act on a body: constant and variable.

Weight of a body (for small heights) and the upthrust force on a solid when it is completely immersed in a liquid are constant forces.

Spring force, upthrust force when the body is partially immersed in a liquid, tension and normal reaction are variable forces. Tension and normal reaction seem to be constant forces but actually they are variable forces.



In the figure shown,

$N = mg$ if $a = 0$

$N = m(g + a)$ if a is upwards and

$N = m(g - a)$ if a is downwards.

So, normal reaction N is not a constant force.

(ii) In equilibrium, summation of all constant and variable forces is zero.

(iii) When displaced from the mean position, constant forces do not contribute in the net restoring force (therefore in time period). They can only change the mean position.

So, just forget the constant forces. Only change in variable force becomes the net restoring force.

$$\therefore \qquad F_{\text{net}} = -|\Delta \mathbf{F}| = \text{net restoring force.}$$

where, $\mathbf{F}$ is the variable force.

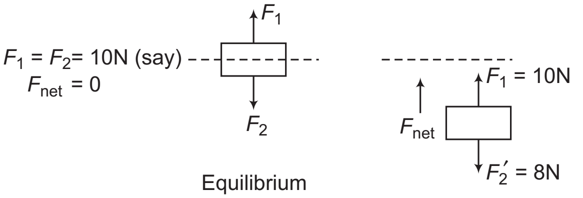
(iv) If two or more than two variable forces are acting on a body and both either increase or decrease (when displaced from the mean position), then

$$\mathbf{F}_{\text{net}} = -[|\Delta \mathbf{F}_1| + |\Delta \mathbf{F}_2|]$$

If one variable force increases and the other decreases then,

$$\mathbf{F}_{\text{net}} = -[|\Delta \mathbf{F}_1| \sim |\Delta \mathbf{F}_2|]$$

Let us understand this concept with an example.



Equilibrium position

Displaced position

In the above figures, F_1 is a constant force and F_2 is a variable force. In equilibrium,

$$F_1 = F_2 = 10\,\text{N (say)}$$

or

$$F_{\text{net}} = 0$$

When displaced from the mean position, F_1 being a constant force remains 10 N.

But, suppose F_2 becomes $F_2' = 8\,\text{N}$.

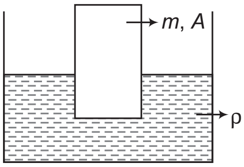
Then, net force is $2\,\text{N} (= F_1 - F_2')$ in upward direction or towards the mean position. We can see that this net force is also equal to change (here decrease) in variable force F_2.

How to Solve?

- Consider a mean or equilibrium position of the body.
- Displace the body from the mean position and find the change(s) in variable force(s) and calculate the net restoring force by the method discussed above. This net restoring force should be of type

$$F = -kx \Rightarrow \text{Now, } T = 2\pi\sqrt{\frac{m}{k}}$$

Example 23 *A plank of mass 'm' and area of cross-section A is floating in a non-viscous liquid of density ρ. When displaced slightly from the mean position, it starts oscillating. Prove that oscillations are simple harmonic and find its time period.*



Solution Two forces are acting on the plank: weight and upthrust. Weight is a constant force. So, forget it. Upthrust (if partially immersed) is a variable force. When displaced downwards by a distance x, then

$$\begin{aligned}\text{Net restoring force } F_{\text{net}} &= -\,[\text{change or increase in variable force upthrust}]\\ &= -\,[(\text{extra immersed volume})(\text{density of liquid})(g)]\\ &= -\,[(Ax)(\rho)(g)]\end{aligned}$$

This force is of type $F = -kx$.

So, motion is simple harmonic, where $k = \rho Ag$

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{k}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{\rho Ag}} \qquad \textbf{Ans.}$$

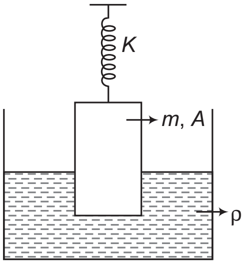
Note (i) *Upthrust force is given by*

$$F = (\text{immersed volume of solid})\,(\text{density of liquid})\,(g)$$
$$= V_i \rho g \Rightarrow \therefore \quad \Delta F = (\Delta V_i)\rho g \qquad [\text{as } \rho \text{ and } g \text{ are constant}]$$

(ii) *In the above problem, upthrust is just like a spring force of force constant:*

$$k = \rho Ag$$

Example 24 *A plank of mass 'm' and area of cross-section A is floating as shown in figure. When slightly displaced from mean position, plank starts oscillations. Find time period of these oscillations.*



Solution Three forces are acting on the plank :

(i) weight (ii) spring force

(iii) upthrust

Weight is a constant force. So, it will not contribute in the time period. Spring force and upthrust are variable forces. When displaced downwards from the mean position, both forces increase.

So, their force constants are additive

$$\therefore \qquad k_e = k + \rho A g$$

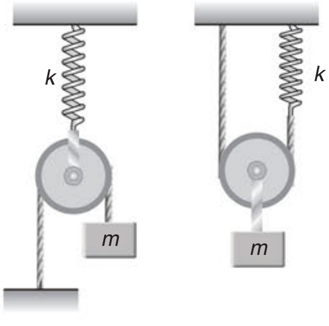
$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{k_e}}$$

$$= 2\pi\sqrt{\frac{m}{k+\rho A g}} \qquad \textbf{Ans.}$$

Note *If the plank oscillates with fully immersed position, then upthrust also becomes a constant force. So, the only varibale force remains kx. Therefore,*

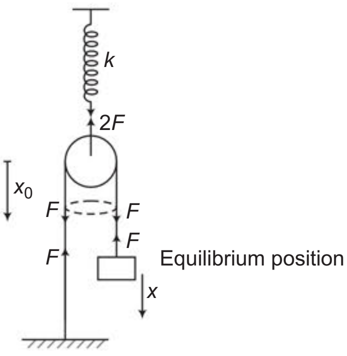
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 25 *Figure shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down from its equilibrium position and released, find the period of its vertical oscillation in cases (a) and (b).*



(a) (b)

Solution (a)



Two forces are acting on the block :

(i) weight

(ii) tension

Weight is a constant force which does not contribute in time period. Tension is a variable force. Let tension increases by F when displaced from the equilibrium position by x. This increase in tension is the change in variable force. So, this is also the net restoring force. The distribution of this increase in tension F is as shown in figure.

Spring gets an extra stretching force $2F$. Let x_0 is the extra extension in the spring by this extra force $2F$. Then,

$$2F = kx_0$$

or
$$x_0 = \frac{2F}{k} \qquad \ldots\text{(i)}$$

Now, we can see that by this extra extension of spring, pulley will also come down by the same distance x_0 and

$$x = 2x_0 = 2\left(\frac{2F}{k}\right)$$

$\Rightarrow$
$$F = \frac{k}{4}x$$

But F is also the net restoring force on block. So, we can write

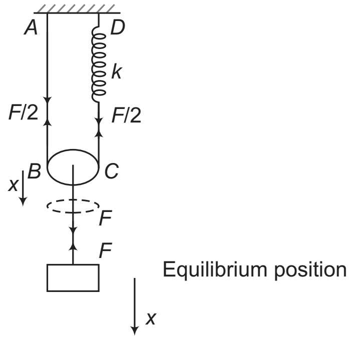
$$F = -\left(\frac{k}{4}\right)x$$

$\therefore$
$$T = 2\pi\sqrt{\frac{m}{k/4}} \qquad (\text{as } k_e = k/4)$$

$$= 2\pi\sqrt{\frac{4m}{k}}$$

Ans.

(b) Using the same concepts as discussed above,



$$\frac{F}{2} = kx_0$$

or
$$x_0 = \frac{F}{2k}$$

But this time,

x_0 = extra extension on spring.

= extra increase in length $ABCD$

= $2x$

or
$$\frac{F}{2k} = 2x$$

or
$$F = 4kx$$

Due to the restoring nature of F we can write,

$$F = -(4k)x$$

$\therefore$
$$T = 2\pi\sqrt{\frac{m}{4k}} \qquad (\text{as } k_e = 4k)$$

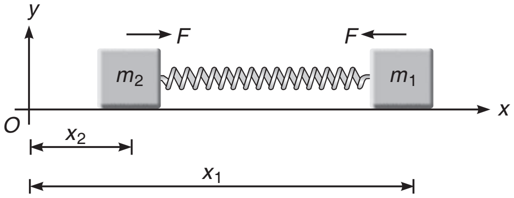
Miscellaneous Examples

Example 29 *For a two body oscillator system, prove the relation,*

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ *= reduced mass*

Solution A system of two bodies connected by a spring so that both are free to oscillate simple harmonically along the length of the spring constitutes a two body harmonic oscillator.



Suppose, two masses m_1 and m_2 are connected by a horizontal massless spring of force constant k, so as to be free to oscillate along the length of the spring on a frictionless horizontal surface.

Let l_0 be the natural length of the spring and let x_1 and x_2 be the coordinates of the two masses at any instant of time. Then,

Extension of the spring $$x = (x_1 - x_2) - l_0 \quad \text{...(i)}$$

For $x > 0$, the spring force $F = kx$ acts on the two masses in the directions shown in above figure.

Thus, we can write

$$m_1 \frac{d^2x_1}{dt^2} = -kx \quad \text{...(ii)}$$

$$m_2 \frac{d^2x_2}{dt^2} = kx \quad \text{...(iii)}$$

Multiplying Eq. (ii) by m_2 and Eq. (iii) by m_1 and subtracting the latter from the former, we have

$$m_1m_2 \frac{d^2x_1}{dt^2} - m_1m_2 \frac{d^2x_2}{dt^2} = -(m_2 + m_1)\,kx$$

or $$m_1m_2 \frac{d^2(x_1 - x_2)}{dt^2} = -kx\,(m_1 + m_2)$$

or $$\left(\frac{m_1m_2}{m_1 + m_2}\right) \frac{d^2}{dt^2}(x_1 - x_2) = -kx \quad \text{...(iv)}$$

Differentiating Eq. (i), twice with respect to time, we have

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_1 - x_2)$$

Also, $$\frac{m_1 m_2}{m_1 + m_2} = \mu = \text{ reduced mass of the two blocks}$$

Substituting these values in Eq. (iv), we have

$$\mu \frac{d^2x}{dt^2} = -kx$$

or $$\mu a_r = -kx \qquad \left(\text{Here, } a_r = \frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = \text{Relative acceleration}\right)$$

This, is the standard differential equation of SHM. Time period of which is

$$T = 2\pi\sqrt{\left|\frac{x}{a_r}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{\mu}{k}}$$

Example 30 *Two particles move parallel to x-axis about the origin with the same amplitude and frequency. At a certain instant they are found at distance* $\frac{A}{3}$ *from the origin on opposite sides but their velocities are found to be in the same direction. What is the phase difference between the two ?*

Solution Let equations of two SHM be

$$x_1 = A \sin \omega t \qquad \dots \text{(i)}$$

$$x_2 = A \sin (\omega t + \phi) \qquad \dots \text{(ii)}$$

Give that $$\frac{A}{3} = A \sin \omega t$$

and $$-\frac{A}{3} = A \sin (\omega t + \phi)$$

Which gives $$\sin \omega t = \frac{1}{3} \qquad \dots \text{(iii)}$$

$$\sin (\omega t + \phi) = -\frac{1}{3} \qquad \dots \text{(iv)}$$

From Eq. (iv),

$$\sin \omega t \cos \phi + \cos \omega t \sin \phi = -\frac{1}{3}$$

$\Rightarrow$ $$\frac{1}{3}\cos \phi + \sqrt{1 - \frac{1}{9}} \sin \phi = -\frac{1}{3}$$

Solving this equation, we get

or $$\cos \phi = -1, \frac{7}{9}$$

$\Rightarrow$ $$\phi = \pi \quad \text{or} \quad \cos^{-1}\left(\frac{7}{9}\right)$$

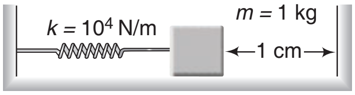
Differentiating Eqs. (i) and (ii), we obtain

$$v_1 = A\omega \cos \omega t \quad \text{and} \quad v_2 = A\omega \cos (\omega t + \phi)$$

If we put $\phi = \pi$, we find v_1 and v_2 are of opposite signs. Hence, $\phi = \pi$ is not acceptable.

$\therefore$ $$\phi = \cos^{-1}\left(\frac{7}{9}\right)$$ **Ans.**

Example 31 *For the arrangement shown in figure, the spring is initially compressed by* 3 *cm. When the spring is released the block collides with the wall and rebounds to compress the spring again.*



(a) *If the coefficient of restitution is* $\frac{1}{\sqrt{2}}$, *find the maximum compression in the spring after collision.*

(b) *If the time starts at the instant when spring is released, find the minimum time after which the block becomes stationary.*

Solution (a) Velocity of the block just before collision,

$$\frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$$

or
$$v_0 = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

Here, $x_0 = 0.03$ m, $x = 0.01$ m, $k = 10^4$ N/m, $m = 1$ kg

$\therefore$
$$v_0 = 2\sqrt{2} \text{ m/s}$$

After collision,
$$v = ev_0 = \frac{1}{\sqrt{2}} 2\sqrt{2} = 2 \text{ m/s}$$

Maximum compression in the spring is

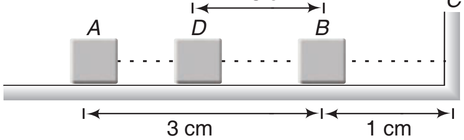
$$\frac{1}{2}kx_m^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

or
$$x_m = \sqrt{x^2 + \frac{m}{k}v^2}$$

$$= \sqrt{(0.01)^2 + \frac{(1)(2)^2}{10^4}} \text{ m} = 2.23 \text{ cm}$$ **Ans.**

(b) In the case of spring-mass system, since the time period is independent of the amplitude of oscillation.

$$\text{Time} = t_{AB} + t_{BC} + t_{CB} + t_{BD}$$
$$= \frac{T_0}{4} + \left(\frac{T_0}{2\pi}\right)\sin^{-1}\left(\frac{1}{3}\right) + \left(\frac{T_0}{2\pi}\right)\sin^{-1}\left(\frac{1}{2.23}\right) + \frac{T_0}{4}$$

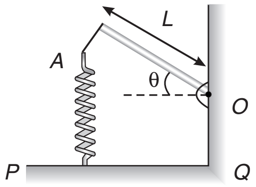


Here,
$$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

Substituting the values, we get

$$\text{Total time} = \sqrt{\frac{m}{k}}\left[\pi + \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{1}{2.23}\right)\right]$$ **Ans.**

Example 32 *A long uniform rod of length L and mass M is free to rotate in a horizontal plane about a vertical axis through its one end ‘O’. A spring of force constant k is connected between one end of the rod and PQ. When the rod is in equilibrium it is parallel to PQ.*



(a) *What is the period of small oscillation that result when the rod is rotated slightly and released?*

(b) *What will be the maximum speed of the displaced end of the rod, if the amplitude of motion is θ_0?*

Solution (a) Restoring torque about ‘O’ due to elastic force of the spring

$$\tau = -FL = -kyL \qquad (F = ky)$$

$$\tau = -kL^2\theta \qquad (\text{as } y = L\theta)$$

$$\tau = I\alpha = \frac{1}{3}ML^2\frac{d^2\theta}{dt^2}$$

$$\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} = -kL^2\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$\omega = \sqrt{\frac{3k}{M}}$$

$$\Rightarrow \qquad T = 2\pi\sqrt{\frac{M}{3k}}$$ **Ans.**

(b) In angular SHM, maximum angular velocity

$$\left(\frac{d\theta}{dt}\right)_{\max} = \theta_0\,\omega = \theta_0\sqrt{\frac{3k}{M}}$$

$$v = r\left(\frac{d\theta}{dt}\right)$$

So ,

$$v_{\max} = L\left(\frac{d\theta}{dt}\right)_{\max} = L\theta_0\sqrt{\frac{3k}{M}}$$ **Ans.**

Example 33 *A block with a mass of 2 kg hangs without vibrating at the end of a spring of spring constant 500 N/m, which is attached to the ceiling of an elevator. The elevator is moving upwards with an acceleration $\frac{g}{3}$. At time $t = 0$, the acceleration suddenly ceases.*

(a) *What is the angular frequency of oscillation of the block after the acceleration ceases?*

(b) *By what amount is the spring stretched during the time when the elevator is accelerating?*

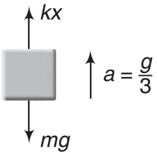
(c) *What is the amplitude of oscillation and initial phase angle observed by a rider in the elevator in the equation, $x = A\sin(\omega t + \phi)$? Take the upward direction to be positive. Take $g = 10.0\ m/s^2$.*

Solution (a) Angular frequency

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = \sqrt{\frac{500}{2}}$$

or $$\omega = 15.81 \text{ rad/s}$$ **Ans.**

(b) Equation of motion of the block (while elevator is accelerating) is,



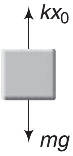
$$kx - mg = ma = m\frac{g}{3}$$

$\therefore$ $$x = \frac{4mg}{3k}$$

$$= \frac{(4)(2)(10)}{(3)(500)} = 0.053 \text{ m}$$

or $$x = 5.3 \text{ cm}$$ **Ans.**

(c) (i) In equilibrium when the elevator has zero acceleration, the equation of motion is,



$$kx_0 = mg$$

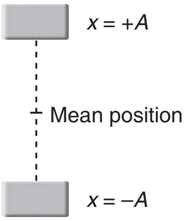
or $$x_0 = \frac{mg}{k} = \frac{(2)(10)}{500}$$

$$= 0.04 \text{ m} = 4 \text{ cm}$$

$\therefore$ Amplitude $$A = x - x_0 = 5.3 - 4.0$$

$$= 1.3 \text{ cm}$$ **Ans.**

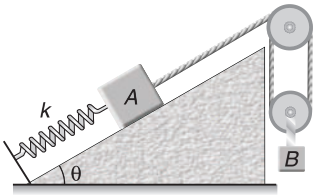
(ii) At time $t = 0$, block is at $x = -A$. Therefore, substituting $x = -A$ and $t = 0$ in equation,



$$x = A \sin(\omega t + \phi)$$

We get initial phase $$\phi = \frac{3\pi}{2}$$ **Ans.**

Example 34 *Calculate the angular frequency of the system shown in figure. Friction is absent everywhere and the threads, spring and pulleys are massless. Given that* $m_A = m_B = m$.



Solution Let x_0 be the extension in the spring in equilibrium. Then, equilibrium of A and B give,

$$T = kx_0 + mg\sin\theta \qquad \text{...(i)}$$

and
$$2T = mg \qquad \text{...(ii)}$$

Here, T is the tension in the string. Now, suppose A is further displaced by a distance x from its mean position and v be its speed at this moment. Then, B lowers by $\frac{x}{2}$ and speed of B at this instant will be $\frac{v}{2}$. Total energy of the system in this position will be,

$$E = \frac{1}{2}k(x + x_0)^2 + \frac{1}{2}m_A v^2 + \frac{1}{2}m_B\left(\frac{v}{2}\right)^2 + m_A g h_A - m_B g\, h_B$$

or
$$E = \frac{1}{2}k(x + x_0)^2 + \frac{1}{2}mv^2 + \frac{1}{8}mv^2 + mgx\sin\theta - mg\frac{x}{2}$$

or
$$E = \frac{1}{2}k(x + x_0)^2 + \frac{5}{8}mv^2 + mgx\sin\theta - mg\frac{x}{2}$$

Since, E is constant,
$$\frac{dE}{dt} = 0$$

or
$$0 = k(x + x_0)\frac{dx}{dt} + \frac{5}{4}mv\left(\frac{dv}{dt}\right) + mg(\sin\theta)\left(\frac{dx}{dt}\right) - \frac{mg}{2}\left(\frac{dx}{dt}\right)$$

Substituting,
$$\frac{dx}{dt} = v \Rightarrow \frac{dv}{dt} = a$$

and
$$kx_0 + mg\sin\theta = \frac{mg}{2} \qquad \text{[From Eqs. (i) and (ii)]}$$

We get,
$$\frac{5}{4}m\,a = -kx$$

Since,
$$a \propto -x$$

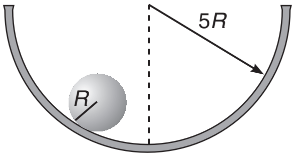
Motion is simple harmonic, time period of which is,

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

$$= 2\pi\sqrt{\frac{5m}{4k}}$$

∴
$$\omega = \frac{2\pi}{T} = \sqrt{\frac{4k}{5m}} \qquad \textbf{Ans.}$$

Example 35 *A solid sphere (radius = R) rolls without slipping in a cylindrical through (radius = 5R). Find the time period of small oscillations.*



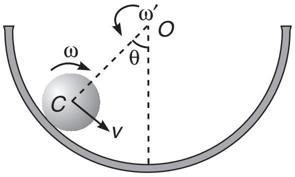
Solution For pure rolling to take place,

$$v = R\omega$$

ω' = angular velocity of COM of sphere C about O

$$= \frac{v}{4R} = \frac{R\omega}{4R} = \frac{\omega}{4}$$

$$\therefore \quad \frac{d\omega'}{dt} = \frac{1}{4}\frac{d\omega}{dt}$$



or $$\alpha' = \frac{\alpha}{4}$$

$$\alpha = \frac{a}{R} \text{ for pure rolling}$$

where, $$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

$$= \frac{5g\sin\theta}{7}$$

as, $$I = \frac{2}{5}mR^2$$

$$\therefore \quad \alpha' = \frac{5g\sin\theta}{28R}$$

For small θ, $\sin\theta \approx \theta$, being restoring in nature,

$$\alpha' = -\frac{5g}{28R}\theta$$

$$\therefore \quad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha'}\right|}$$

$$= 2\pi\sqrt{\frac{28R}{5g}}$$ **Ans.**

Example 36 *Consider the earth as a uniform sphere of mass M and radius R. Imagine a straight smooth tunnel made through the earth which connects any two points on its surface. Show that the motion of a particle of mass m along this tunnel under the action of gravitation would be simple harmonic. Hence, determine the time that a particle would take to go from one end to the other through the tunnel.*

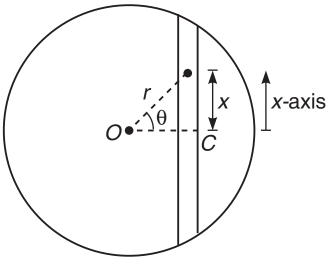
Solution Suppose at some instant, the particle is at radial distance r from centre of earth O. Since, the particle is constrained to move along the tunnel, we define its position as distance x from C. Hence, equation of motion of the particle is,

$$ma_x = F_x$$

The gravitational force on mass m at distance r is,

$$F = \frac{GMmr}{R^3} \qquad \text{(towards } O\text{)}$$

Therefore,



$$\begin{aligned} F_x &= -F \sin\theta \\ &= -\frac{GMmr}{R^3}\left(\frac{x}{r}\right) \\ &= -\frac{GMm}{R^3}\cdot x \end{aligned}$$

Since, $F_x \propto -x$, motion is simple harmonic in nature. Further,

$$ma_x = -\frac{GMm}{R^3}\cdot x$$

or

$$a_x = -\frac{GM}{R^3}\cdot x$$

∴ Time period of oscillation is,

$$\begin{aligned} T &= 2\pi\sqrt{\left|\frac{x}{a_x}\right|} \\ &= 2\pi\sqrt{\frac{R^3}{GM}} \end{aligned}$$

The time taken by particle to go from one end to the other is $\frac{T}{2}$.

∴

$$\begin{aligned} t &= \frac{T}{2} \\ &= \pi\sqrt{\frac{R^3}{GM}} \end{aligned}$$

Ans.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

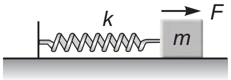
(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : In $x = A\cos\omega t$, x is the displacement measured from extreme position.
Reason : In the above equation $x = A$ at time $t = 0$.

2. Assertion : A particle is under SHM along the x-axis. Its mean position is $x = 2$, amplitude is $A = 2$ and angular frequency ω. At $t = 0$, particle is at origin, then x-co-ordinate *versus* time equation of the particle will be $x = -2\cos\omega t + 2$.

Reason : At $t = 0$, particle is at rest.

3. Assertion : A spring block system is kept over a smooth surface as shown in figure. If a constant horizontal force F is applied on the block it will start oscillating simple harmonically.



Reason : Time period of oscillation is less than $2\pi\sqrt{\dfrac{m}{k}}$.

4. Assertion : Time taken by a particle in SHM to move from $x = A$ to $x = \dfrac{\sqrt{3}A}{2}$ is same as the time taken by the particle to move from $x = \dfrac{\sqrt{3}A}{2}$ to $x = \dfrac{A}{2}$.

Reason : Corresponding angles rotated in the reference circle are same in the given time intervals.

5. Assertion : Path of a particle in SHM is always a straight line.

Reason : All straight line motions are not simple harmonic.

6. Assertion : In spring block system if length of spring and mass of block both are halved, then angular frequency of oscillations will remain unchanged.

Reason : Angular frequency is given by $\omega = \sqrt{\dfrac{k}{m}}$

7. Assertion : All small oscillations are simple harmonic in nature.

Reason : Oscillations of spring block system are always simple harmonic whether amplitude is small or large.

8. Assertion : In $x = A\cos\omega t$, the dot product of acceleration and velocity is positive for time interval $0 < t < \frac{\pi}{2\omega}$.

Reason : Angle between them is 0°.

9. Assertion : For a given simple harmonic motion displacement (from the mean position) and acceleration have a constant ratio.

Reason : $T = 2\pi\sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$

10. Assertion : We can call circular motion also as simple harmonic motion.

Reason : Angular velocity in uniform circular motion and angular frequency in simple harmonic motion have the same meanings.

Objective Questions

Single Correct Option

1. A simple harmonic oscillation has an amplitude A and time period T. The time required to travel from $x = A$ to $x = \frac{A}{2}$ is

(a) $\frac{T}{6}$ (b) $\frac{T}{4}$ (c) $\frac{T}{3}$ (d) $\frac{T}{12}$

2. The potential energy of a particle executing SHM varies sinusoidally with frequency f. The frequency of oscillation of the particle will be

(a) $\frac{f}{2}$ (b) $\frac{f}{\sqrt{2}}$ (c) f (d) $2f$

3. For a particle undergoing simple harmonic motion, the velocity is plotted against displacement. The curve will be

(a) a straight line (b) a parabola

(c) a circle (d) an ellipse

4. A simple pendulum is made of bob which is a hollow sphere full of sand suspended by means of a wire. If all the sand is drained out, the period of the pendulum will

(a) increase (b) decrease

(c) remain same (d) become erratic

5. Two simple harmonic motions are given by $y_1 = a\sin\left[\left(\frac{\pi}{2}\right)t + \phi\right]$ and $y_2 = b\sin\left[\left(\frac{2\pi}{3}\right)t + \phi\right]$. The phase difference between these after 1 s is

(a) zero (b) $\pi/2$

(c) $\pi/4$ (d) $\pi/6$

6. A particle starts performing simple harmonic motion. Its amplitude is A. At one time its speed is half that of the maximum speed. At this moment the displacement is

(a) $\frac{\sqrt{2}A}{3}$ (b) $\frac{\sqrt{3}A}{2}$

(c) $\frac{2A}{\sqrt{3}}$ (d) $\frac{3A}{\sqrt{2}}$

7. Which of the following is not simple harmonic function?

(a) $y = a \sin 2\omega t + b \cos^2 \omega t$
(b) $y = a \sin \omega t + b \cos 2\omega t$
(c) $y = 1 - 2 \sin^2 \omega t$
(d) $y = (\sqrt{a^2 + b^2}) \sin \omega t \cos \omega t$

8. The displacement of a particle varies according to the relation $y = 4(\cos \pi t + \sin \pi t)$. The amplitude of the particle is

(a) 8 units
(b) 2 units
(c) 4 units
(d) $4\sqrt{2}$ units

9. Two pendulums X and Y of time periods 4 s and 4.2 s are made to vibrate simultaneously. They are initially in same phase. After how many vibrations of X, they will be in the same phase again?

(a) 30
(b) 25
(c) 21
(d) 26

10. A mass M is suspended from a massless spring. An additional mass m stretches the spring further by a distance x. The combined mass will oscillate with a period

(a) $2\pi \sqrt{\left\{\frac{(M + m)\,x}{mg}\right\}}$
(b) $2\pi \sqrt{\left\{\frac{mg}{(M + m)\,x}\right\}}$
(c) $2\pi \sqrt{\left\{\frac{(M + m)}{mgx}\right\}}$
(d) $\frac{\pi}{2} \sqrt{\left\{\frac{mg}{(M + m)\,x}\right\}}$

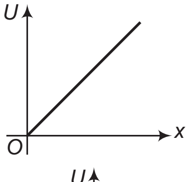
11. Two bodies P and Q of equal masses are suspended from two separate massless springs of force constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal. The ratio of the amplitude of P to that of Q is

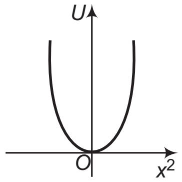
(a) $\sqrt{\frac{k_1}{k_2}}$
(b) $\frac{k_1}{k_2}$
(c) $\sqrt{\frac{k_2}{k_1}}$
(d) $\frac{k_2}{k_1}$

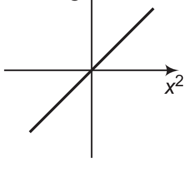
12. A disc of radius R is pivoted at its rim. The period for small oscillations about an axis perpendicular to the plane of disc is

(a) $2\pi\sqrt{\frac{R}{g}}$
(b) $2\pi\sqrt{\frac{2R}{g}}$
(c) $2\pi\sqrt{\frac{2R}{3g}}$
(d) $2\pi\sqrt{\frac{3R}{2g}}$

13. Identify the correct variation of potential energy U as a function of displacement x from mean position (or x^2) of a harmonic oscillator (U at mean position = 0)

(a)


(b)


(c)


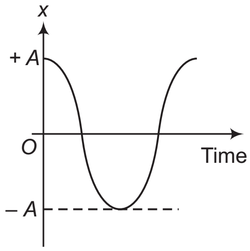
(d) None of these

14. If the length of a simple pendulum is equal to the radius of the earth, its time period will be

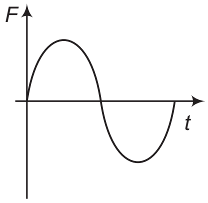
(a) $2\pi\sqrt{R/g}$ (b) $2\pi\sqrt{R/2g}$

(c) $2\pi\sqrt{2R/g}$ (d) infinite

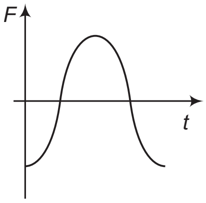
15. The displacement-time (x-t) graph of a particle executing simple harmonic motion is shown in figure. The correct variation of net force F acting on the particle as a function of time is



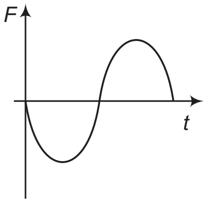
(a)



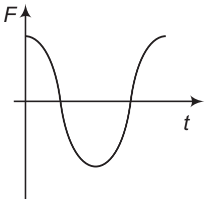
(b)



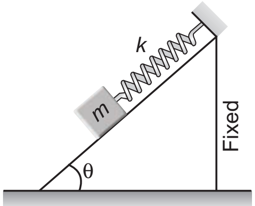
(c)



(d)



16. In the figure shown the time period and the amplitude respectively, when m is left from rest when spring is relaxed are (the inclined plane is smooth)



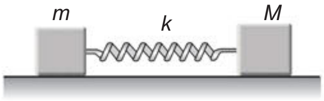
(a) $2\pi\sqrt{\frac{m}{k}}, \frac{mg\sin\theta}{k}$ (b) $2\pi\sqrt{\frac{m\sin\theta}{k}}, \frac{2\,mg\sin\theta}{k}$

(c) $2\pi\sqrt{\frac{m}{k}}, \frac{mg\cos\theta}{k}$ (d) None of these

17. The equation of motion of a particle of mass 1 g is $\frac{d^2x}{dt^2} + \pi^2 x = 0$, where x is displacement (in m) from mean position. The frequency of oscillation is (in Hz)

(a) 1/2 (b) 2 (c) $5\sqrt{10}$ (d) $1/5\sqrt{10}$

18. The spring as shown in figure is kept in a stretched position with extension x when the system is released. Assuming the horizontal surface to be frictionless, the frequency of oscillation is



(a) $\frac{1}{2\pi}\sqrt{\left[\frac{k(M+m)}{Mm}\right]}$ (b) $\frac{1}{2\pi}\sqrt{\left[\frac{mM}{k(M+m)}\right]}$

(c) $\frac{1}{2\pi}\sqrt{\left[\frac{kM}{m+M}\right]}$ (d) $\frac{1}{2\pi}\sqrt{\left[\frac{km}{M+m}\right]}$

19. The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet. It is a second's pendulum on earth?

(a) $\sqrt{2}$ s (b) $2\sqrt{2}$ s (c) $\frac{1}{\sqrt{2}}$ s (d) $\frac{1}{2\sqrt{2}}$ s

20. The resultant amplitude due to superposition of three simple harmonic motions $x_1 = 3\sin\omega t$, $x_2 = 5\sin(\omega t + 37°)$ and $x_3 = -15\cos\omega t$ is

(a) 18 (b) 10 (c) 12 (d) None of these

21. Two SHMs $s_1 = a\sin\omega t$ and $s_2 = b\sin\omega t$ are superimposed on a particle. The s_1 and s_2 are along the directions which makes 37° to each other

(a) the particle will perform SHM
(b) the path of particle is straight line
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

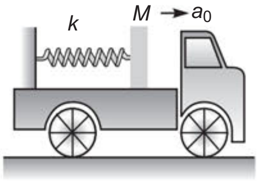
22. The amplitude of a particle executing SHM about O is 10 cm. Then

(a) when the KE is 0.64 times of its maximum KE, its displacement is 6 cm from O
(b) its speed is half the maximum speed when its displacement is half the maximum displacement
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

23. A particle is attached to a vertical spring and is pulled down a distance 4 cm below its equilibrium and is released from rest. The initial upward acceleration is 0.5 ms^{-2}. The angular frequency of oscillation is

(a) 3.53 rad/s (b) 0.28 rad/s
(c) 1.25 rad/s (d) 0.08 rad/s

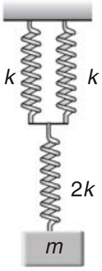
24. A block of mass 1 kg is kept on smooth floor of a truck. One end of a spring of force constant 100 N/m is attached to the block and other end is attached to the body of truck as shown in the figure. At $t = 0$, truck begins to move with constant acceleration 2 m/s^2. The amplitude of oscillation of block relative to the floor of truck is



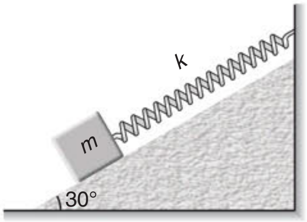
(a) 0.06 m (b) 0.02 m (c) 0.04 m (d) 0.03 m

Subjective Questions

1. Find the period of oscillation of the system shown in figure.



2. A block of mass 0.2 kg is attached to a massless spring of force constant 80 N/m as shown in figure. Find the period of oscillation. Take $g = 10$ m/s^2. Neglect friction.

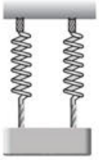


3. A body of weight 27 N hangs on a long spring of such stiffness that an extra force of 9 N stretches the spring by 0.05 m. If the body is pulled downward and released, what is the period ?

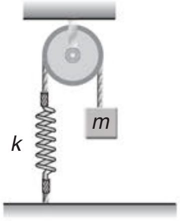
4. A clock with an iron pendulum keeps correct time at 20°C. How much time will it lose or gain in a day if the temperature changes to 40°C. Thermal coefficient of linear expansion $\alpha = 0.000012$ per°C.

5. A 50 g mass hangs at the end of a massless spring. When 20 g more are added to the end of the spring, it stretches 7.0 cm more. (a) Find the spring constant. (b) If the 20 g are now removed, what will be the period of the motion ?

6. An object suspended from a spring exhibits oscillations of period T. Now, the spring is cut in half and the two halves are used to support the same object, as shown in figure. Show that the new period of oscillation is $T/2$.



7. The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.



8. A simple pendulum with a solid metal bob has a period T. What will be the period of the same pendulum if it is made to oscillate in a non-viscous liquid of density one-tenth of the metal of the bob?

9. A particle moves under the force $F(x) = (x^2 - 6x)$ N, where x is in metres. For small displacements from the origin what is the force constant in the simple harmonic motion approximation?

10. The initial position and velocity of a body moving in SHM with period $T = 0.25$ s are $x = 5.0$ cm and $v = 218$ cm/s. What are the amplitude and phase constant of the motion?

11. A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Write down the equation of motion of this particle when the initial phase of oscillation is 45°.

12. Potential energy of a particle in SHM along x-axis is given by

$$U = 10 + (x - 2)^2$$

Here, U is in joule and x in metre. Total mechanical energy of the particle is 26 J. Mass of the particle is 2 kg. Find

(a) angular frequency of SHM,

(b) potential energy and kinetic energy at mean position and extreme position,

(c) amplitude of oscillation,

(d) x-coordinates between which particle oscillates.

13. A simple pendulum is taken at a place where its separation from the earth's surface is equal to the radius of the earth. Calculate the time period of small oscillations if the length of the string is 1.0 m. Take $g = \pi^2$ m/s^2 at the surface of the earth.

14. A solid cylinder of mass $M = 10$ kg and cross-sectional area $A = 20$ cm^2 is suspended by a spring of force constant $k = 100$ N/m and hangs partially immersed in water. Calculate the period of small oscillations of the cylinder.

15. A simple pendulum of length l and mass m is suspended in a car that is moving with constant speed v around a circle of radius r. Find the period of oscillation and equilibrium position of the pendulum.

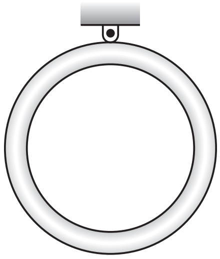
16. A body of mass 0.10 kg is attached to a vertical massless spring with force constant 4.0×10^3 N/m. The body is displaced 10.0 cm from its equilibrium position and released. How much time elapses as the body moves from a point 8.0 cm on one side of the equilibrium position to a point 6.0 cm on the same side of the equilibrium position?

17. A body of mass 200 g is in equilibrium at $x = 0$ under the influence of a force $F(x) = (-100x + 10x^2)$ N.

(a) If the body is displaced a small distance from equilibrium, what is the period of its oscillations?

(b) If the amplitude is 4.0 cm, by how much do we error in assuming that $F(x) = -kx$ at the end points of the motion.

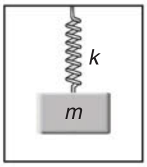
18. A ring of radius r is suspended from a point on its circumference. Determine its angular frequency of small oscillations.



19. A spring mass system is hanging from the ceiling of an elevator in equilibrium. The elevator suddenly starts accelerating upwards with acceleration a. Find

(a) the frequency and

(b) the amplitude of the resulting SHM.



20. A body makes angular simple harmonic motion of amplitude $\pi / 10$ rad and time period 0.05 s. If the body is at a displacement $\theta = \pi / 10$ rad at $t = 0$, write the equation giving angular displacement as a function of time.

21. A particle executes simple harmonic motion of period 16 s. Two seconds later after it passes through the centre of oscillation its velocity is found to be 2 m/s. Find the amplitude.

22. A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force?

Hint: Electrostatic force is given by $\mathbf{F} = q\mathbf{E}$

23. Find the period of oscillation of a pendulum of length l if its point of suspension is

(a) moving vertically up with acceleration a.

(b) moving vertically down with acceleration a ($< g$).

(c) falling freely under gravity

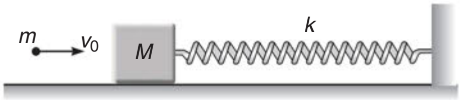
(d) moving horizontally with acceleration a.

24. A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position a lump of putty with mass m is dropped vertically on the block from a very small height and sticks to it.

(a) Find the new amplitude and period.

(b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

25. A bullet of mass m strikes a block of mass M. The bullet remains embedded in the block. Find the amplitude of the resulting SHM.



26. An annular ring of internal and outer radii r and R respectively oscillates in a vertical plane about a horizontal axis perpendicular to its plane and passing through a point on its outer edge. Calculate its time period and show that the length of an equivalent simple pendulum is $\frac{3R}{2}$ as $r \to 0$ and $2R$ as $r \to R$.

27. A body of mass 200 g oscillates about a horizontal axis at a distance of 20 cm from its centre of gravity. If the length of the equivalent simple pendulum is 35 cm, find its moment of inertia about the point of suspension.

28. Show that the period of oscillation of simple pendulum at depth h below earth's surface is inversely proportional to $\sqrt{R - h}$, where R is the radius of earth. Find out the time period of a second pendulum at a depth $R/2$ from the earth's surface ?

29. The period of a particle in SHM is 8 s. At $t = 0$ it is in its equilibrium position.

(a) Compare the distance travelled in the first 4 s and the second 4 s.

(b) Compare the distance travelled in the first 2 s and the second 2 s.

30. (a) The motion of the particle in simple harmonic motion is given by $x = a \sin \omega t$. If its speed is u, when the displacement is x_1 and speed is v, when the displacement is x_2, show that the amplitude of the motion is

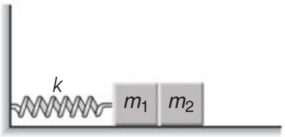
$$A = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{1/2}$$

(b) A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values x_1 and x_2, the corresponding values of velocity are u_1 and u_2, show that the period is

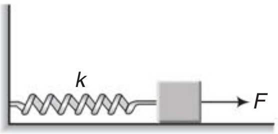
$$T = 2\pi \left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$$

31. Show that the combined spring energy and gravitational energy for a mass m hanging from a light spring of force constant k can be expressed as $U_0 + \frac{1}{2} ky^2$, where y is the distance above or below the equilibrium position and U_0 is constant.

32. The masses in figure slide on a frictionless table. m_1 but not m_2, is fastened to the spring. If now m_1 and m_2 are pushed to the left, so that the spring is compressed a distance d, what will be the amplitude of the oscillation of m_1 after the spring system is released ?



33. The spring shown in figure is unstretched when a man starts pulling on the cord. The mass of the block is M. If the man exerts a constant force F, find

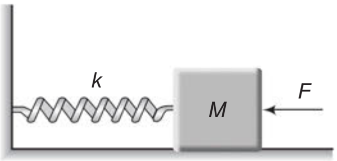


(a) the amplitude and the time period of the motion of the block,

(b) the energy stored in the spring when the block passes through the equilibrium position and

(c) the kinetic energy of the block at this position.

34. In figure, $k = 100$ N/m, $M = 1$ kg and $F = 10$ N.



(a) Find the compression of the spring in the equilibrium position.

(b) A sharp blow by some external agent imparts a speed of 2 m/s to the block towards left. Find the sum of the potential energy of the spring and the kinetic energy of the block at this instant.

(c) Find the time period of the resulting simple harmonic motion.

(d) Find the amplitude.

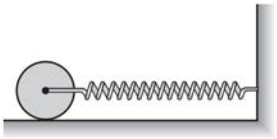
(e) Write the potential energy of the spring when the block is at the left extreme.

(f) Write the potential energy of the spring when the block is at the right extreme.

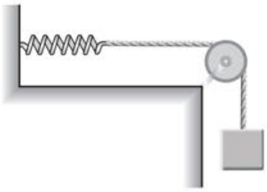
The answers of (b), (e) and (f) are different. Explain why this does not violate the principle of conservation of energy?

35. Pendulum A is a physical pendulum made from a thin, rigid and uniform rod whose length is d. One end of this rod is attached to the ceiling by a frictionless hinge, so that the rod is free to swing back and forth. Pendulum B is a simple pendulum whose length is also d. Obtain the ratio $\frac{T_A}{T_B}$ of their periods for small angle oscillations.

36. A solid cylinder of mass m is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. (See the accompanying figure.) Show that the centre of mass of the cylinder executes simple harmonic motion with a period $T = 2\pi\sqrt{\frac{3m}{2k}}$, if displaced from mean position.



37. A cord is attached between a 0.50 kg block and a spring with force constant $k = 20$ N/m. The other end of the spring is attached to the wall and the cord is placed over a pulley ($I = 0.60\,MR^2$) of mass 5.0 kg and radius 0.50 m. (See the accompanying figure). Assuming no slipping occurs, what is the frequency of the oscillations when the body is set into motion?



38. Two linear SHM of equal amplitudes A and frequencies ω and 2ω are impressed on a particle along x and y-axes respectively. If the initial phase difference between them is $\pi/2$. Find the resultant path followed by the particle.

39. A particle is subjected to two simple harmonic motions given by

$$x_1 = 2.0\sin(100\,\pi t) \quad \text{and} \quad x_2 = 2.0\sin(120\,\pi\, t + \pi/3)$$

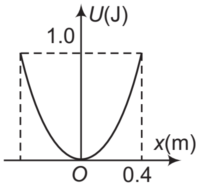
where, x is in cm and t in second. Find the displacement of the particle at

(a) $t = 0.0125$, (b) $t = 0.025$.

LEVEL 2

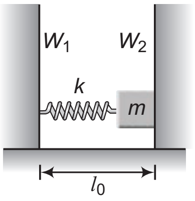
Objective Questions

Single Correct Option



1. A particle of mass 2 kg moves in simple harmonic motion and its potential energy U varies with position x as shown. The period of oscillation of the particle is

(a) $\frac{2\pi}{5}$ s
(b) $\frac{2\sqrt{2}\pi}{5}$ s
(c) $\frac{\sqrt{2}\pi}{5}$ s
(d) $\frac{4\pi}{5}$ s



2. In the figure shown, a spring mass system is placed on a horizontal smooth surface in between two vertical rigid walls W_1 and W_2. One end of spring is fixed with wall W_1 and other end is attached with mass m which is free to move. Initially, spring is tension free and having natural length l_0. Mass m is compressed through a distance a and released. Taking the collision between wall W_2 and mass m as elastic and K as spring constant, the average force exerted by mass m on wall W_2 in one oscillation of block is

(a) $\frac{2aK}{\pi}$
(b) $\frac{2ma}{\pi}$
(c) $\frac{aK}{\pi}$
(d) $\frac{2aK}{m}$

3. Two simple harmonic motions are represented by the following equations $y_1 = 40 \sin \omega t$ and $y_2 = 10 (\sin \omega t + c \cos \omega t)$. If their displacement amplitudes are equal, then the value of c (in appropriate units) is

(a) $\sqrt{13}$
(b) $\sqrt{15}$
(c) $\sqrt{17}$
(d) 4

4. A particle executes simple harmonic motion with frequency 2.5 Hz and amplitude 2 m. The speed of the particle 0.3 s after crossing, the equilibrium position is

(a) zero
(b) 2π m/s
(c) 4π m/s
(d) π m/s

5. A particle oscillates simple harmonically with a period of 16 s. Two second after crossing the equilibrium position its velocity becomes 1 m/s. The amplitude is

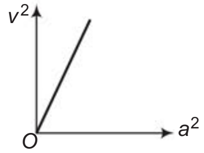
(a) $\frac{\pi}{4}$ m
(b) $\frac{8\sqrt{2}}{\pi}$ m
(c) $\frac{8}{\pi}$ m
(d) $\frac{4\sqrt{2}}{\pi}$ m

6. A seconds pendulum is suspended from the ceiling of a trolley moving horizontally with an acceleration of 4 m/s^2. Its period of oscillation is

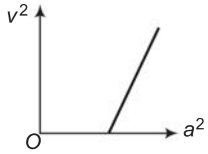
(a) 1.90 s
(b) 1.70 s
(c) 2.30 s
(d) 1.40 s

7. A particle is performing a linear simple harmonic motion. If the instantaneous acceleration and velocity of the particle are a and v respectively, identify the graph which correctly represents the relation between a and v.

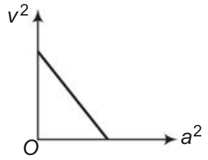
(a)



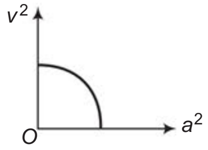
(b)



(c)



(d)



8. In a vertical U-tube a column of mercury oscillates simple harmonically. If the tube contains 1 kg of mercury and 1 cm of mercury column weighs 20 g, then the period of oscillation is

(a) 1 s
(b) 2 s
(c) $\sqrt{2}$ s
(d) Insufficient data

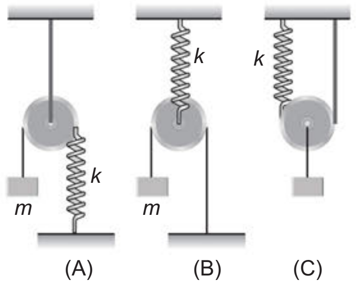
9. A solid cube of side a and density ρ_0 floats on the surface of a liquid of density ρ. If the cube is slightly pushed downward, then it oscillates simple harmonically with a period of

(a) $2\pi\sqrt{\frac{\rho_0}{\rho}\frac{a}{g}}$
(b) $2\pi\sqrt{\frac{\rho}{\rho_0}\frac{a}{g}}$
(c) $2\pi\sqrt{\frac{a}{\left(1-\frac{\rho}{\rho_0}\right)g}}$
(d) $2\pi\sqrt{\frac{a}{\left(1+\frac{\rho}{\rho_0}\right)g}}$

10. A uniform stick of length l is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance d from the centre of mass. The time period of small oscillations has a minimum value when d/l is

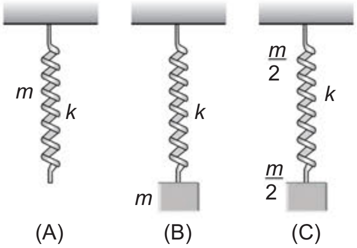
(a) $\frac{1}{\sqrt{2}}$
(b) $\frac{1}{\sqrt{12}}$
(c) $\frac{1}{\sqrt{3}}$
(d) $\frac{1}{\sqrt{6}}$

11. Three arrangements of spring-mass system are shown in figures (A), (B) and (C). If T_1, T_2 and T_3 represent the respective periods of oscillation, then correct relation is



(a) $T_1 > T_2 > T_3$
(b) $T_3 > T_2 > T_1$
(c) $T_2 > T_1 > T_3$
(d) $T_2 > T_3 > T_1$

12. Three arrangements are shown in figure.



(a) A spring of mass m and stiffness k

(b) A block of mass m attached to massless spring of stiffness k

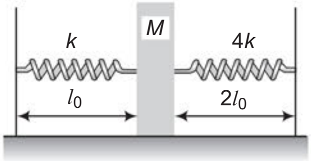
(c) A block of mass $\frac{m}{2}$ attached to a spring of mass $\frac{m}{2}$ and stiffness k

If T_1, T_2 and T_3 represent the period of oscillation in the three cases respectively, then identify the correct relation.

(a) $T_1 < T_2 < T_3$ (b) $T_1 < T_3 < T_2$

(c) $T_1 > T_3 > T_2$ (d) $T_3 < T_1 < T_2$

13. A block of mass M is kept on a smooth surface and touches the two springs as shown in the figure but not attached to the springs. Initially springs are in their natural length. Now, the block is shifted $(l_0/2)$ from the given position in such a way that it compresses a spring and released. The time-period of oscillation of mass will be



(a) $\frac{\pi}{2}\sqrt{\frac{M}{k}}$ (b) $2\pi\sqrt{\frac{M}{5k}}$

(c) $\frac{3\pi}{2}\sqrt{\frac{M}{k}}$ (d) $\pi\sqrt{\frac{M}{2k}}$

14. A particle moving on x-axis has potential energy $U = 2 - 20x + 5x^2$ joule along x-axis. The particle is released at $x = -3$. The maximum value of x will be (x is in metre)

(a) 5 m (b) 3 m

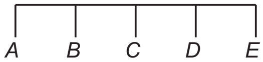
(c) 7 m (d) 8 m

15. A block of mass m, when attached to a uniform ideal spring with force constant k and free length L executes SHM. The spring is then cut in two pieces, one with free length $n\,L$ and other with free length $(1 - n)\,L$. The block is also divided in the same fraction. The smaller part of the block attached to longer part of the spring executes SHM with frequency f_1. The bigger part of the block attached to smaller part of the spring executes SHM with frequency f_2. The ratio f_1/f_2 is

(a) 1 (b) $\frac{n}{1-n}$

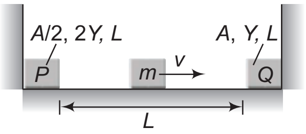
(c) $\frac{1+n}{n}$ (d) $\frac{n}{1+n}$

16. A body performs simple harmonic oscillations along the straight line $ABCDE$ with C as the midpoint of AE. Its kinetic energies at B and D are each one fourth of its maximum value. If $AE = 2R$, the distance between B and D is



(a) $\frac{\sqrt{3}}{2}R$ (b) $\frac{R}{\sqrt{2}}$ (c) $\sqrt{3}\,R$ (d) $\sqrt{2}R$

17. In the given figure, two elastic rods P and Q are rigidly joined to end supports. A small mass m is moving with velocity v between the rods. All collisions are assumed to be elastic and the surface is given to be smooth. The time period of small mass m will be (A = area of cross section, Y = Young's modulus, L = length of each rod)



(a) $\frac{2L}{v} + 2\pi\sqrt{\frac{mL}{AY}}$ (b) $\frac{2L}{v} + 2\pi\sqrt{\frac{2mL}{AY}}$

(c) $\frac{2L}{v} + \pi\sqrt{\frac{mL}{AY}}$ (d) $\frac{2L}{v}$

18. A particle executes SHM of period 1.2 s and amplitude 8 cm. Find the time it takes to travel 3 cm from the positive extremity of its oscillation. [$\cos^{-1}(5/8) = 0.9$ rad]

(a) 0.28 s (b) 0.32 s (c) 0.17 s (d) 0.42 s

19. A wire frame in the shape of an equilateral triangle is hinged at one vertex so that it can swing freely in a vertical plane, with the plane of the triangle always remaining vertical. The side of the frame is $1/\sqrt{3}$ m. The time period in seconds of small oscillations of the frame will be ($g = 10\ \text{m/s}^2$)

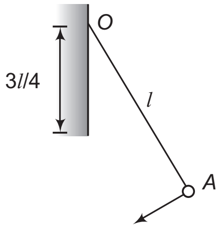
(a) $\pi/\sqrt{2}$ (b) $\pi/\sqrt{3}$

(c) $\pi/\sqrt{6}$ (d) $\pi/\sqrt{5}$

20. A particle moves along the x-axis according to $x = A\,[1 + \sin\omega t]$. What distance does is travel in time interval from $t = 0$ to $t = 2.5\pi/\omega$?

(a) $4\,A$ (b) $6\,A$ (c) $5\,A$ (d) $3\,A$

21. A small bob attached to a light inextensible thread of length l has a periodic time T when allowed to vibrate as a simple pendulum. The thread is now suspended from a fixed end O of a vertical rigid rod of length $3l/4$. If now the pendulum performs periodic oscillations in this arrangement, the periodic time will be

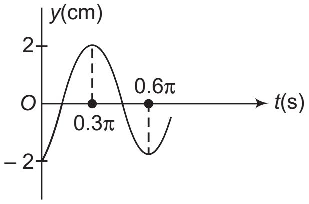


(a) $3T/4$ (b) $4T/5$ (c) $2T/3$ (d) $5T/6$

22. A stone is swinging in a horizontal circle of diameter 0.8 m at 30 rev/min. A distant light causes a shadow of the stone on a nearly wall. The amplitude and period of the SHM for the shadow of the stone are

(a) 0.4 m, 4s
(b) 0.2 m, 2s
(c) 0.4 m, 2s
(d) 0.8 m, 2s

23. Part of SHM is graphed in the figure. Here, y is displacement from mean position. The correct equation describing the SHM is



(a) $y = 4\cos(0.6\,t)$
(b) $y = 2\sin\left(\frac{10}{3}t - \frac{\pi}{2}\right)$
(c) $y = 2\sin\left(\frac{\pi}{2} - \frac{10}{3}t\right)$
(d) $y = 2\cos\left(0.6\,t + \frac{\pi}{2}\right)$

24. A particle performs SHM with a period T and amplitude a. The mean velocity of particle over the time interval during which it travels a distance $a/2$ from the extreme position is

(a) $6\,a/T$
(b) $2\,a/T$
(c) $3\,a/T$
(d) $a/2T$

25. A man of mass 60 kg is standing on a platform executing SHM in the vertical plane. The displacement from the mean position varies as $y = 0.5\sin(2\pi ft)$. The value of f, for which the man will feel weightlessness at the highest point, is (y in metre)

(a) $g/4\pi$
(b) $4\pi g$
(c) $\frac{\sqrt{2g}}{2\pi}$
(d) $2\pi\sqrt{2g}$

26. A particle performs SHM on a straight line with time period T and amplitude A. The average speed of the particle between two successive instants, when potential energy and kinetic energy become same is

(a) $\frac{A}{T}$
(b) $\frac{4\sqrt{2}A}{T}$
(c) $\frac{2A}{T}$
(d) $\frac{2\sqrt{2}A}{T}$

27. The time taken by a particle performing SHM to pass from point A to B where its velocities are same is 2 s. After another 2 s it returns to B. The ratio of distance OB to its amplitude (where O is the mean position) is

(a) $1:\sqrt{2}$
(b) $(\sqrt{2}-1):1$
(c) $1:2$
(d) $1:2\sqrt{2}$

28. A particle is executing SHM according to the equation $x = A\cos\omega t$. Average speed of the particle during the interval $0 \le t \le \frac{\pi}{6\omega}$ is

(a) $\frac{\sqrt{3}A\omega}{2}$
(b) $\frac{\sqrt{3}A\omega}{4}$
(c) $\frac{3\,A\omega}{\pi}$
(d) $\frac{3\,A\omega}{\pi}(2-\sqrt{3})$

More than One Correct Options

1. A simple pendulum with a bob of mass m is suspended from the roof of a car moving with horizontal acceleration a

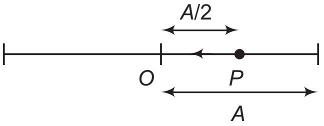
(a) The string makes an angle of $\tan^{-1}(a/g)$ with the vertical

(b) The string makes an angle of $\sin^{-1}\left(\frac{a}{g}\right)$ with the vertical

(c) The tension in the string is $m\sqrt{a^2+g^2}$

(d) The tension in the string is $m\sqrt{g^2-a^2}$

2. A particle starts from a point P at a distance of $A/2$ from the mean position O and travels towards left as shown in the figure. If the time period of SHM, executed about O is T and amplitude A then the equation of the motion of particle is



(a) $x = A\sin\left(\frac{2\pi}{T}t+\frac{\pi}{6}\right)$

(b) $x = A\sin\left(\frac{2\pi}{T}t+\frac{5\pi}{6}\right)$

(c) $x = A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{6}\right)$

(d) $x = A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{3}\right)$

3. A spring has natural length 40 cm and spring constant 500 N/m. A block of mass 1 kg is attached at one end of the spring and other end of the spring is attached to a ceiling. The block is released from the position, where the spring has length 45 cm

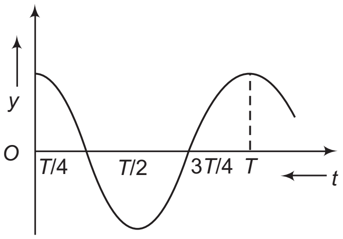
(a) the block will perform SHM of amplitude 5 cm

(b) the block will have maximum velocity $30\sqrt{5}$ cm/s

(c) the block will have maximum acceleration 15 m/s^2

(d) the minimum elastic potential energy of the spring will be zero

4. The displacement-time graph of a particle executing SHM is shown in figure. Which of the following statements is/are true?



(a) The velocity is maximum at $t = T/2$

(b) The acceleration is maximum at $t = T$

(c) The force is zero at $t = 3T/4$

(d) The kinetic energy equals the total oscillation energy at $t = T/2$

5. For a particle executing SHM, x = displacement from mean position, v = velocity and a = acceleration at any instant, then

(a) v-x graph is a circle

(b) v-x graph is an ellipse

(c) a-x graph is a straight line

(d) a-x graph is a circle

6. The acceleration of a particle is $a = -100x + 50$. It is released from $x = 2$. Here, a and x are in SI units

(a) the particle will perform SHM of amplitude 2 m
(b) the particle will perform SHM of amplitude 1.5 m
(c) the particle will perform SHM of time period 0.63 s
(d) the particle will have a maximum velocity of 15 m/s

7. Two particles are performing SHM in same phase. It means that

(a) the two particles must have same distance from the mean position simultaneously
(b) two particles may have same distance from the mean position simultaneously
(c) the two particles must have maximum speed simultaneously
(d) the two particles may have maximum speed simultaneously

8. A particle moves along y-axis according to the equation

y (in cm) $= 3\sin 100\pi t + 8\sin^2 50\pi t - 6$

(a) the particle performs SHM
(b) the amplitude of the particle's oscillation is 5 cm
(c) the mean position of the particle is at $y = -2$ cm
(d) the particle does not perform SHM

Comprehension Based Questions

Passage (Q Nos. 1 to 2)

A 2 kg block hangs without vibrating at the bottom end of a spring with a force constant of 400 N/m. The top end of the spring is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of 5 m/s^2 when the acceleration suddenly ceases at time $t = 0$ and the car moves upward with constant speed ($g = 10\ m/s^2$)

1. What is the angular frequency of oscillation of the block after the acceleration ceases?

(a) $10\sqrt{2}$ rad/s
(b) 20 rad/s
(c) $20\sqrt{2}$ rad/s
(d) 32 rad/s

2. The amplitude of the oscillation is

(a) 7.5 cm
(b) 5 cm
(c) 2.5 cm
(d) 1 cm

Match the Columns

1. For the x-t equation of a particle in SHM along x-axis, match the following two columns.

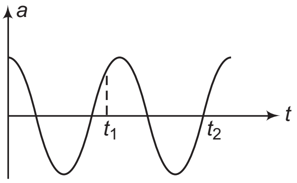
$$x = 2 + 2\cos \omega t$$

Column I	Column II
(a) Mean position	(p) $x = 0$
(b) Extreme position	(q) $x = 2$
(c) Maximum potential energy at	(r) $x = 4$
(d) Zero potential energy at	(s) Can't tell

2. Potential energy of a particle at mean position is 4 J and at extreme position is 20 J. Given that amplitude of oscillation is A. Match the following two columns.

Column I	Column II
(a) Potential energy at $x = \frac{A}{2}$	(p) 18 J
(b) Kinetic energy at $x = \frac{A}{4}$	(q) 16 J
(c) Kinetic energy at $x = 0$	(r) 8 J
(d) Kinetic energy at $x = \frac{A}{2}$	(s) None

3. Acceleration-time graph of a particle in SHM is as shown in figure. Match the following two columns.



Column I	Column II
(a) Displacement of particle at t_1	(p) zero
(b) Displacement of particle at t_2	(q) positive
(c) Velocity of particle at t_1	(r) negative
(d) Velocity of particle at t_2	(s) maximum

4. Mass of a particle is 2 kg. Its displacement-time equation in SHM is

$$x = 2 \sin (4\pi t) \qquad \text{(SI Units)}$$

Match the following two columns for 1 second time interval.

Column I	Column II
(a) Speed becomes 30 m/s	(p) two times
(b) Velocity becomes + 10 m/s	(q) four times
(c) Kinetic energy becomes 400 J	(r) one time
(d) Acceleration becomes– 100 m/s^2	(s) None

5. x-t equation of a particle in SHM is, $x = 4 + 6 \sin \pi t$. Match the following tables corresponding to time taken in moving from

Column I	Column II
(a) $x = 10$ m to $x = 4$ m	(p) $\frac{1}{3}$ second
(b) $x = 10$ m to $x = 7$ m	(q) $\frac{1}{2}$ second
(c) $x = 7$ m to $x = 1$ m	(r) 1 second
(d) $x = 10$ m to $x = -2$ m	(s) None

Subjective Questions

1. A 1 kg block is executing simple harmonic motion of amplitude 0.1 m on a smooth horizontal surface under the restoring force of a spring of spring constant 100 N/m. A block of mass 3 kg is gently placed on it at the instant it passes through the mean position. Assuming that the two blocks move together. Find the frequency and the amplitude of the motion.

2. Two particles are in SHM along same line. Time period of each is T and amplitude is A. After how much time will they collide if at time $t = 0$. (a) first particle is at $x_1 = +\frac{A}{2}$ and moving towards positive x-axis and second particle is at $x_2 = -\frac{A}{\sqrt{2}}$ and moving towards negative x-axis, (b) rest information are same as mentioned in part (a) except that particle first is also moving towards negative x-axis.

3. A particle that hangs from a spring oscillates with an angular frequency of 2 rad/s. The spring particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.5 m/s. The car then stops suddenly. (a) With what amplitude does the particle oscillate ? (b) What is the equation of motion for the particle ? (Choose upward as the positive direction)

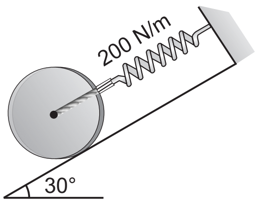
4. A 2 kg mass is attached to a spring of force constant 600 N/m and rests on a smooth horizontal surface. A second mass of 1 kg slides along the surface toward the first at 6 m/s.
   (a) Find the amplitude of oscillation if the masses make a perfectly inelastic collision and remain together on the spring. What is the period of oscillation ?
   (b) Find the amplitude and period of oscillation if the collision is perfectly elastic.
   (c) For each case, write down the position x as a function of time t for the mass attached to the spring, assuming that the collision occurs at time $t = 0$. What is the impulse given to the 2 kg mass in each case?

5. A block of mass 4 kg hangs from a spring of force constant $k = 400$ N/m. The block is pulled down 15 cm below equilibrium and released. How long does it take the block to go from 12 cm below equilibrium (on the way up) to 9 cm above equilibrium?

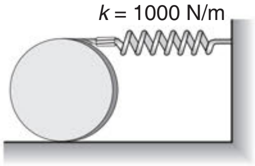
6. A plank with a body of mass m placed on it starts moving straight up according to the law $y = a(1 - \cos \omega t)$, where y is the displacement from the initial position, $\omega = 11$ rad/s. Find
   (a) The time dependence of the force that the body exerts on the plank.
   (b) The minimum amplitude of oscillation of the plank at which the body starts falling behind the plank.

7. A particle of mass m free to move in the x-y plane is subjected to a force whose components are $F_x = -kx$ and $F_y = -ky$, where k is a constant. The particle is released when $t = 0$ at the point (2, 3). Prove that the subsequent motion is simple harmonic along the straight line $2y - 3x = 0$.

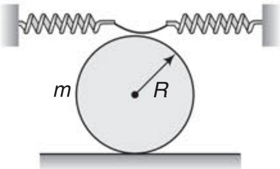
8. Determine the natural frequency of vibration of the 100 N disk. Assume the disk does not slip on the inclined surface.



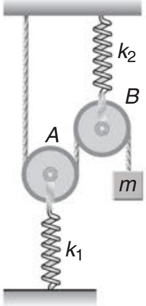
9. The disk has a weight of 100 N and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counterclockwise 0.4 rad, determine the equation which describes its oscillatory motion when it is released.



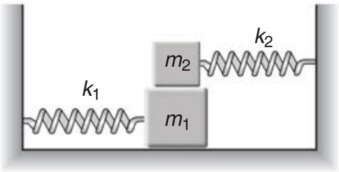
10. A solid uniform cylinder of mass m performs small oscillations due to the action of two springs of stiffness k each (figure). Find the period of these oscillations in the absence of sliding.



11. A block of mass m is attached to one end of a light inextensible string passing over a smooth light pulley B and under another smooth light pulley A as shown in the figure. The other end of the string is fixed to a ceiling. A and B are held by springs of spring constants k_1 and k_2. Find angular frequency of small oscillations of the system.

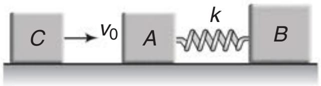


12. In the shown arrangement, both the springs are in their natural lengths. The coefficient of friction between m_2 and m_1 is μ. There is no friction between m_1 and the surface. If the blocks are displaced slightly, they together perform simple harmonic motion. Obtain



(a) Frequency of such oscillations.
(b) The condition if the frictional force on block m_2 is to act in the direction of its displacement from mean position.
(c) If the condition obtained in (b) is met, what can be maximum amplitude of their oscillations?

13. Two blocks A and B of masses $m_1 = 3$ kg and $m_2 = 6$ kg respectively are connected with each other by a spring of force constant $k = 200$ N/m as shown in figure. Blocks are pulled away from each other by $x_0 = 3$ cm and then released. When spring is in its natural length and blocks are moving towards each other, another block C of mass $m = 3$ kg moving with velocity $v_0 = 0.4$ m/s (towards right) collides with A and gets stuck to it. Neglecting friction, calculate



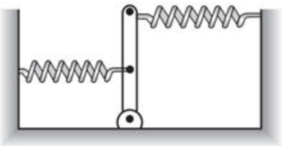
(a) velocities v_1 and v_2 of the blocks A and B respectively just before collision and their angular frequency.

(b) velocity of centre of mass of the system, after collision,

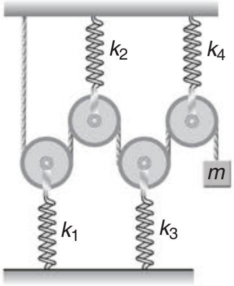
(c) amplitude of oscillations of combined body,

(d) loss of energy during collision.

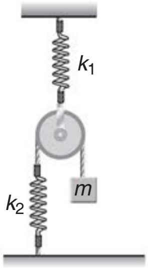
14. A rod of length l and mass m, pivoted at one end, is held by a spring at its mid-point and a spring at far end. The springs have spring constant k. Find the frequency of small oscillations about the equilibrium position.



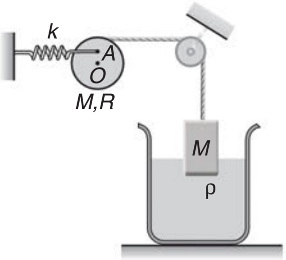
15. In the arrangement shown in figure, pulleys are light and springs are ideal. k_1, k_2, k_3 and k_4 are force constants of the springs. Calculate period of small vertical oscillations of block of mass m.



16. A light pulley is suspended at the lower end of a spring of constant k_1, as shown in figure. An inextensible string passes over the pulley. At one end of string a mass m is suspended, the other end of the string is attached to another spring of constant k_2. The other ends of both the springs are attached to rigid supports, as shown. Neglecting masses of springs and any friction, find the time period of small oscillations of mass m about equilibrium position.

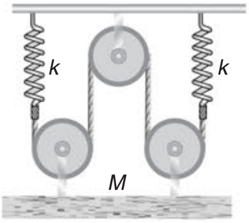


17. Figure shows a solid uniform cylinder of radius R and mass M, which is free to rotate about a fixed horizontal axis O and passes through centre of the cylinder. One end of an ideal spring of force constant k is fixed and the other end is hinged to the cylinder at A. Distance OA is equal to $\frac{R}{2}$. An inextensible thread is wrapped round the cylinder and passes over a smooth, small pulley. A block of equal mass M and having cross sectional area A is suspended from free end of the thread. The block is partially immersed in a non-viscous liquid of density ρ.



If in equilibrium, spring is horizontal and line OA is vertical, calculate frequency of small oscillations of the system.

18. Find the natural frequency of the system shown in figure. The pulleys are smooth and massless.



Answers

Introductory Exercise 14.1

1. π **2.** $\frac{1}{16}, \frac{15}{16}$ **3.** (a) 15.0 cm (b) 0.726 s (c) 1.38 Hz (d) 1.69 J (e) 1.30 m/s

4. 0.101 m/s, 1.264 m/s^2, 0.632 N **5.** No

Introductory Exercise 14.2

1. (a) 1.39 J (b) 1.1 s **2.** ± 0.58 m/s, – 0.45 ms^2, ± 0.60 m/s, zero **3.** $\frac{\pi}{6}$

4. (a) 0.08 m (b) 1.57 rad/s (c) 1.97 N/m (d) Zero (e) 0.197 m/s^2

5. (a) $\frac{\pi}{120}$ sec (b) $\frac{\pi}{30}$ sec (c) $\frac{\pi}{30}$ sec **6.** See the hints.

Introductory Exercise 14.3

1. $\left(\frac{\pi}{2}\right)$ sec **2.** 3.2 kg **3.** 1 sec **4.** $\sqrt{p}$ **5.** $\frac{16}{9}$ **6.** 11 % **7.** $\frac{\sqrt{3}}{2\pi}$

Introductory Exercise 14.4

1. (a) 7.0 cm (b) 6.1 cm (c) 5.0 cm (d) 1.0 cm

2. (a) 2.6 unit (b) 1917unit (c) 6.0×10^5unit **3.** $2A$ **4.** $\frac{2\pi}{3}$

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (d) **7.** (d) **8.** (a) **9.** (a) **10.** (d)

Objective Questions

1. (a) **2.** (a) **3.** (d) **4.** (c) **5.** (d) **6.** (b) **7.** (b) **8.** (d) **9.** (c) **10.** (a)
11. (c) **12.** (d) **13** (c) **14.** (b) **15.** (b) **16.** (a) **17.** (a) **18.** (a) **19.** (b) **20.** (d)
21. (c) **22.** (a) **23.** (a) **24.** (b)

Subjective Questions

1. $T = 2\pi\sqrt{\frac{m}{k}}$ **2.** 0.314 s **3.** 0.78 s **4.** The clock will lose 10.37s **5.** (a) 2.8 N/m (b) 0.84 s

7. $T = 2\pi\sqrt{\frac{m}{k}}$ **8.** $\left(\sqrt{\frac{10}{9}}\right)T$ **9.** 6.0 N/m **10.** 10.0 cm, $\frac{\pi}{6}$ rad **11.** $y = (0.1 \text{ m}) \sin\left[(4\text{s}^{-1})t + \frac{\pi}{4}\right]$

12. (a) 1 rad/s (b) $U_{\text{mean}} = 10$ J, $K_{\text{mean}} = 16$ J, $U_{\text{extreme}} = 26$ J, $K_{\text{extreme}} = 0$ (c) 4 m (d) $x = 6$ m and $x = -2$ m

13. 4 s **14.** 1.8 s

15. $2\pi\sqrt{\frac{l}{\left\{g^2 + \left(\frac{v^2}{r}\right)^2\right\}^{1/2}}}$ and inclined to the vertical at an angle $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$ away from the centre

16. 1.4×10^{-3} s **17.** (a) 0.28 s (b) 0.4% **18.** $\sqrt{\frac{g}{2r}}$ **19.**(a) $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (b) $\frac{ma}{k}$ **20.** $\theta = \left(\frac{\pi}{10}\text{ rad}\right)\cos[(40\pi\text{s}^{-1})t]$

21. 7.2 m **22.** $2\pi\sqrt{\dfrac{l}{g-\dfrac{qE}{m}}}$ **23.** (a) $2\pi\sqrt{\dfrac{l}{(g+a)}}$ (b) $2\pi\sqrt{\dfrac{l}{(g-a)}}$ (c) Infinite (d) $2\pi\sqrt{\dfrac{l}{(g^2+a^2)^{1/2}}}$

24. (a) $A_2 = A_1\sqrt{\dfrac{M}{M+m}}$, $T = 2\pi\sqrt{\dfrac{M+m}{k}}$ (b) $A_2 = A_1$, $T = 2\pi\sqrt{\dfrac{M+m}{k}}$ **25.** $\dfrac{mv_0}{\sqrt{k(M+m)}}$

26. $T = 2\pi\sqrt{\dfrac{\dfrac{3R}{2}+\dfrac{r^2}{2R}}{g}}$ **27.** 1.4×10^5 g-cm^2 **28.** $2\sqrt{2}$ s **29.** (a) equal (b) equal

32. $A = d\sqrt{\dfrac{m_1}{m_1+m_2}}$ **33.** (a) $\dfrac{F}{k}$, $2\pi\sqrt{\dfrac{M}{k}}$ (b) $\dfrac{F^2}{2k}$ (c) $\dfrac{F^2}{2k}$

34. (a) 10 cm (b) 2.5 J (c) $\dfrac{\pi}{5}$ sec (d) 20 cm (e) 4.5 J (f) 0.5 J **35.** 0.816 **37.** 0.38 Hz

38. Parabola, $y = A\left(1-\dfrac{2x^2}{A^2}\right)$ **39.** (a) –2.41 cm (b) 0.27 cm

LEVEL 2

Single Correct Option

1. (d) **2.** (a) **3.** (b) **4.** (a) **5.** (b) **6.** (a) **7.** (c) **8.** (a) **9.** (a) **10.** (b)
11. (c) **12.** (b) **13.** (c) **14.** (c) **15.** (a) **16.** (c) **17.** (a) **18.** (c) **19.** (d) **20.** (c)
21. (a) **22.** (c) **23.** (b) **24.** (c) **25.** (c) **26.** (b) **27.** (a) **28.** (d)

More than One Correct Options

1. (a,c) **2.** (b,d) **3.** (b,c,d) **4.** (b, c) **5.** (b,c) **6.** (b,c, d) **7.** (b,c) **8.** (a, b, c)

Comprehension Based Questions

1. (a) **2.** (c)

Match the Columns

1. (a) → q (b) → p,r (c) → p,r (d) → s
2. (a) → r (b) → s (c) → q (d) → s
3. (a) → r (b) → p (c) → r (d) → r,s
4. (a) → s (b) → q (c) → s (d) → q
5. (a) → q (b) → p (c) → p (d) → r

Subjective Questions

1. 0.8 Hz, 0.05 m **2.** (a) $\dfrac{19}{48}T$ (b) $\dfrac{11}{48}T$ **3.** (a) $A = 0.75$ m (b) $x = -0.75\sin 2t$

4. (a) 14.1 cm, 0.44 s (b) 23 cm, 0.36 s (c) $x = \pm$ (14.1cm) $\sin(10\sqrt{2}\,t)$, $x = \pm$ (23 cm) $\sin(10\sqrt{3}\,t)$, 4 N-s, 8 N-s

5. $\dfrac{\pi}{20}$ s = 0.157 s **6.** (a) $N = m(g + a\omega^2\cos\omega t)$ (b) 8.1 cm **8.** 0.56 Hz **9.** $\theta = 0.4\cos(16.16\,t)$

10. $T = \dfrac{\pi}{2}\sqrt{\dfrac{3m}{k}}$ **11.** $\sqrt{\dfrac{k_1k_2}{4m(k_1+k_2)}}$ **12.** (a) $\dfrac{1}{2\pi}\sqrt{\dfrac{k_1+k_2}{m_1+m_2}}$ (b) $\dfrac{k_1}{k_2} < \dfrac{m_1}{m_2}$ (c) $\dfrac{\mu(m_1+m_2)m_2g}{m_1k_2-m_2k_1}$

13. (a) 0.2 m/s, 0.1 m/s, 10 rad/s (b) 0.1 m/s (towards right) (c) 4.8 cm (d) 0.03 J

14. $\dfrac{1}{2\pi}\sqrt{\dfrac{15k}{4m}}$ **15.** $T = 4\pi\sqrt{m\left(\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dfrac{|}{k_3}+\dfrac{1}{k_4}\right)}$ **16.** $T = 2\pi\sqrt{\dfrac{m(4k_2+k_1)}{k_1k_2}}$

17. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k+4A\rho g}{6M}}$ **18.** $\dfrac{1}{\pi}\sqrt{\dfrac{2k}{M}}$

15

Elasticity

Chapter Contents

15.1 Introduction
15.2 Elasticity
15.3 Stress and Strain
15.4 Hooke’s Law and Modulus of Elasticity
15.5 The Stress-Strain Curve
15.6 Potential Energy Stored in a Stretched Wire
15.7 Thermal Stresses of Strain

15.1 Introduction

The properties of material under the action of external deforming forces are very essential, for an engineer, to enable him, in designing him all types of structures and machines.

Whenever a load is attached to a thin hanging wire it elongates and the load moves downwards (sometimes through a negligible distance). The amount by which the wire elongates depends upon the amount of load and the nature of wire material. Cohesive force, between the molecules of the hanging wire offer resistance against the deformation, and the force of resistance increases with the deformation. The process of deformation stops when the force of resistance is equal to the external force (i.e. the load attached). Sometimes the force of resistance offered by the molecules is less than the external force. In such a case, the deformation continues until the wire breaks.

Thus, we may conclude that if some external deforming force is applied to a body it has two effects on it, namely :

(i) deformation of the body,

(ii) internal resistance (restoring) forces are developed.

15.2 Elasticity

As we have already discussed that whenever a single force (or a system of forces) acts on a body it undergoes some deformation and the molecules offer some resistance to the deformation. When the external force is removed, the force of resistance also vanishes and the body returns back to its original shape. But it is only possible if the deformation is within a certain limit. Such a limit is called elastic limit. This property of materials of returning back to their original position is called the elasticity.

A body is said to be perfectly elastic if it returns back completely to its original shape and size after removing the external force. If a body remains in the deformed state and does not even partially regain its original shape after the removal of the deforming forces, it is called a perfectly inelastic or plastic body. Quite often, when the external forces are removed, the body partially regains the original shape. Such bodies are partially elastic. If the force acting on the body is increased and the deformation exceeds the elastic limit, the body loses to some extent, its property of elasticity. In this case, the body will not return to its original shape and size even after removal of the external force. Some deformation is left permanently.

15.3 Stress and Strain

Stress

When an external force is applied to a body then at each cross-section of the body an internal restoring force is developed which tends to restore the body to its original state. The internal restoring force per unit area of cross-section of the deformed body is called stress. It is usually denoted by σ (sigma).

Thus,
$$\boxed{\text{Stress } (\sigma) = \frac{\text{Restoring force}}{\text{Area}}}$$

Note *In equilibrium, when further deformation stops, the restoring force is equal to the external force (or the suspended load from a hanging wire). Therefore, the stress is also sometime called the external deforming force per unit area.*

Strain

When the size or shape of a body is changed under an external force, the body is said to be strained. The change occurred in the unit size of the body is called strain. Usually, it is denoted by ε. Thus,

$$\boxed{\varepsilon = \frac{\Delta x}{x}}$$

Here, Δx is the change (may be in length, volume etc.) and x the original value of the quantity in which change has occurred. For example, when the length of a suspended wire increases under an applied load, the value of strain is,

$$\boxed{\varepsilon = \frac{\Delta l}{l}}$$

15.4 Hooke's Law and the Modulus of Elasticity

According to Hooke's law,

"For small deformation, the stress in a body is proportional to the corresponding strain." i.e.

$$\text{stress} \propto \text{strain} \quad \text{or} \quad \boxed{\text{stress} = (E)\,(\text{strain})}$$

Here, $E = \frac{\text{stress}}{\text{strain}}$ is a constant called the modulus of elasticity. Now, depending upon the nature of deforming force applied on the body, stress, strain and hence modulus of elasticity are classified in following three types:

Young's Modulus of Elasticity (Y)

When a wire is acted upon by two equal and opposite forces in the direction of its length, the length of the body is changed. The change in length per unit length $\left(\frac{\Delta l}{l}\right)$ is called the **longitudinal strain** and the restoring force (which is equal to the applied force in equilibrium) per unit area of cross section of the wire is called the **longitudinal stress**.

For small change in the length of the wire, the ratio of the longitudinal stress to the corresponding strain is called the Young's modulus of elasticity (Y) of the wire. Thus,

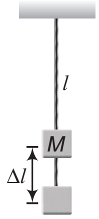


Fig. 15.1

$$Y = \frac{F/A}{\Delta l/l} \quad \text{or} \quad \boxed{Y = \frac{Fl}{A\Delta l}}$$

Let there be a wire of length l and radius r. Its one end is clamped to a rigid support and a mass M is attached at the other end. Then,

$$F = Mg \quad \text{and} \quad A = \pi r^2$$

Substituting in above equation, we have

$$\boxed{Y = \frac{Mgl}{(\pi r^2)\Delta l}}$$

Bulk Modulus of Elasticity *(B)*

When a uniform pressure (normal force) is applied all over the surface of a body, the volume of the body changes. The change in volume per unit volume of the body is called the '**volume strain**' and the normal force acting per unit area of the surface (pressure) is called the normal stress or **volume stress**. For small strains, the ratio of the volume stress to the volume strain is called the 'bulk modulus' of the material of the body. It is denoted by *B*. Then,

$$\boxed{B = \frac{-p}{\Delta V/V} \quad \text{or} \quad \frac{-\Delta p}{(\Delta V/V)}}$$

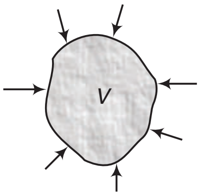


Fig. 15.2

Here, negative sign implies that, when the pressure increases volume decreases and *vice-versa*.

Compressibility

The reciprocal of the bulk modulus of the material of a body is called the 'compressibility' of that material. Thus,

$$\boxed{\text{Compressibility} = \frac{1}{B}}$$

Modulus of Rigidity (η)

When a body is acted upon by an external force tangential to a surface of the body, the opposite surface being kept fixed, it suffers a change in shape, its volume remaining unchanged. Then, the body is said to be sheared.

The ratio of the displacement of a layer in the direction of the tangential force and the distance of the layer from the fixed surface is called the **shearing strain** and the tangential force acting per unit area of the surface is called the "**shearing stress**".

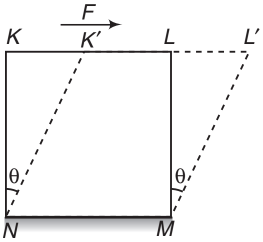


Fig. 15.3

For small strain the ratio of the shearing stress to the shearing strain is called the "modulus of rigidity" of the material of the body. It is denoted by η.

Thus, $$\eta = \frac{F/A}{KK'/KN}$$

Here, $$\frac{KK'}{KN} = \tan\theta \approx \theta$$

∴ $$\eta = \frac{F/A}{\theta} \quad \text{or} \quad \boxed{\eta = \frac{F}{A\theta}}$$

Extra Points to Remember

- If a spring is stretched or compressed by an amount Δl, the restoring force produced in it is,

$$F_s = k\,\Delta l \qquad \text{...(i)}$$

Here, k = force constant of spring

Similarly, if a wire is stretched by an amount Δl, the restoring force produced in it is,

$$\boxed{F = \left(\frac{YA}{l}\right)\Delta l} \qquad \text{...(ii)}$$

as, $Y = \dfrac{F/A}{\Delta l/l}$

Comparing Eqs. (i) and (ii), we can see that force constant of a wire is,

$$k = \frac{YA}{l} \qquad \text{...(iii)}$$

i.e. a wire is just like a spring of force constant $\dfrac{YA}{l}$. So, all formulae which we use in case of a spring can be applied to a wire also.

From Eq. (iii), we may also conclude that force constant of a spring is inversely proportional to the length of the spring l or

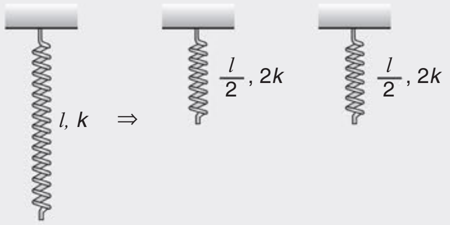


Fig. 15.4

$$\boxed{k \propto \frac{1}{l}}$$

i.e. if a spring is cut into two equal pieces its force constant is doubled.

Example 15.1 *Determine the elongation of the steel bar* $1\,m$ *long and* $1.5\ cm^2$ *cross-sectional area when subjected to a pull of* $1.5 \times 10^4\ N$.
(Take $Y = 2.0 \times 10^{11}\ N/m^2$*)*

Solution $Y = \dfrac{F/A}{\Delta l/l}$

$$\therefore \qquad \Delta l = \frac{Fl}{AY}$$

Substituting the values,

$$\Delta l = \frac{(1.5 \times 10^4)(1.0)}{(1.5 \times 10^{-4})(2.0 \times 10^{11})}$$

$$= 0.5 \times 10^{-3}\ \text{m}$$

or $\qquad \Delta l = 0.5\ \text{mm}$ **Ans.**

Example 15.2 *The bulk modulus of water is* $2.3 \times 10^9\ N/m^2$.

(a) Find its compressibility in the units atm^{-1}.

(b) How much pressure in atmospheres is needed to compress a sample of water by 0.1% ?

Solution Here, $B = 2.3 \times 10^9\ \text{N/m}^2$

$$= \frac{2.3 \times 10^9}{1.01 \times 10^5}$$

$$= 2.27 \times 10^4\ \text{atm}$$

(a) Compressibility $= \frac{1}{B} = \frac{1}{2.27 \times 10^4} = 4.4 \times 10^{-5}\ \text{atm}^{-1}$ **Ans.**

(b) Here, $\frac{\Delta V}{V} = -\,0.1\% = -\,0.001$

Required increase in pressure,

$$\Delta p = B \times \left(-\frac{\Delta V}{V}\right)$$

$$= 2.27 \times 10^4 \times 0.001$$

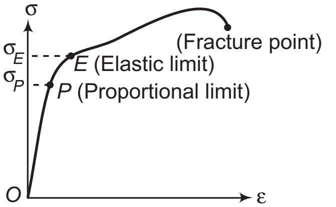
$$= 22.7\ \text{atm}$$ **Ans.**

INTRODUCTORY EXERCISE 15.1

1. Two wires *A* and *B* of same dimensions are stretched by same amount of force. Young's modulus of *A* is twice that of *B*. Which wire will get more elongation?
2. A rod 100 cm long and of 2 cm × 2 cm cross-section is subjected to a pull of 1000 kg force. If the modulus of elasticity of the material is $2.0 \times 10^6\ \text{kg/cm}^2$, determine the elongation of the rod.
3. A cast iron column has internal diameter of 200 mm. What should be the minimum external diameter so that it may carry a load of 1.6 MN without the stress exceeding $90\ \text{N/mm}^2$?
4. Find the dimensions of stress, strain and modulus of elasticity.

15.5 The Stress-Strain Curve

A plot of longitudinal stress (either tensile or compressive) *versus* longitudingal strain for a typical solid is shown in figure. The strain is directly proportional to the applied stress for values of stress upto σ_P. In this linear region, the material returns to its original size when the stress is removed. Point *P* is known as the proportional limit of the solid. For stresses between σ_P and σ_E, where point *E* is called the elastic limit, the material also returns to its original size.



The stress-strain curve for a typical solid

Fig. 15.5

However, notice that stress and strain are not proportional in this region. For deformations beyond the elastic limit, the material does not return to its original size when the stress is removed, it is permanently distorted. Finally, further stretching beyond the elastic limit leads to the eventual fracture of the solid. The proportionality constant for linear region or the slope of stress-strain curve in this curve is called the Young's modulus of elasticity *Y*.

15.6 Potential Energy Stored in a Stretched Wire

When a wire is stretched, work is done against the inter atomic forces. This work is stored in the wire in the form of elastic potential energy. Suppose on applying a force F on a wire of length l, the increase in length is Δl. The area of cross-section of the wire is A. The potential energy stored in the wire should be,

$$\boxed{U = \frac{1}{2} k (\Delta l)^2}$$

Here, $$k = \frac{YA}{l}$$

∴ $$U = \frac{1}{2} \frac{YA}{l} (\Delta l)^2$$

Elastic potential energy per unit volume of the wire (also called energy density) is,

$$u = \frac{U}{\text{volume}} \quad \text{or} \quad u = \frac{\frac{1}{2}\frac{YA}{l}(\Delta l)^2}{Al}$$

or $$u = \frac{1}{2}\left(\frac{\Delta l}{l}\right)\left(Y \cdot \frac{\Delta l}{l}\right) \quad \text{or} \quad u = \frac{1}{2}(\text{strain})\,(Y \times \text{strain})$$

or $$\boxed{u = \frac{1}{2}(\text{strain}) \times (\text{stress})}$$

15.7 Thermal Stresses or Strains

Whenever there is some increase or decrease in the temperature of the body, it causes the body to expand or contract. If the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

Consider a rod AB fixed at two supports as shown in figure.

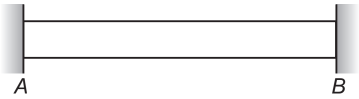


Fig. 15.6

Let l = length of rod

A = area of cross-section of the rod

Y = Young's modulus of elasticity of the rod

and α = thermal coefficient of linear expansion of the rod

Let the temperature of the rod is increased by an amount t. The length of the rod would had increased by an amount Δl, if it were not fixed at two supports. Here

$$\Delta l = l \alpha t$$

But the rod is fixed at the supports. Hence a compressive strain will be produced in the rod. Because at the increased temperature, the natural length of the rod is $l + \Delta l$, while being fixed at two supports its actual length is l. Hence, thermal strain

$$\varepsilon = \frac{\Delta l}{l + \Delta l} \approx \frac{\Delta l}{l} = \frac{l \alpha t}{l} = \alpha t$$

or $$\boxed{\varepsilon = \alpha t}$$

Therefore, thermal stress $$\sigma = Y\varepsilon \qquad \text{(stress} = Y \times \text{strain)}$$

or $$\boxed{\sigma = Y\alpha t}$$

or force on the supports,

$$\boxed{F = \sigma A = YA\alpha t}$$

This force F is in the direction shown below.

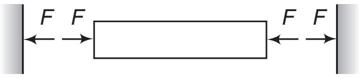


Fig. 15.7

Example 15.3 *A steel wire* 4.0 *m in length is stretched through* 2.0 *mm. The cross-sectional area of the wire is* 2.0 mm^2. *If Young's modulus of steel is* 2.0×10^{11} N/m^2. *Find*

(a) the energy density of wire,

(b) the elastic potential energy stored in the wire.

Solution Here, $l = 4.0$ m, $\Delta l = 2 \times 10^{-3}$ m, $A = 2.0 \times 10^{-6}$ m^2, $Y = 2.0 \times 10^{11}$ N/m^2

(a) The energy density of stretched wire

$$U = \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times Y \times (\text{strain})^2$$

$$= \frac{1}{2} \times 2.0 \times 10^{11} \times \left(\frac{(2 \times 10^{-3})}{4}\right)^2$$

$$= 0.25 \times 10^5 = 2.5 \times 10^4 \text{ J/m}^3$$ **Ans.**

(b) Elastic potential energy = energy density × volume

$$= 2.5 \times 10^4 \times (2.0 \times 10^{-6}) \times 4.0 \text{ J}$$

$$= 20 \times 10^{-2} = 0.20 \text{ J}$$ **Ans.**

Example 15.4 *A rubber cord has a cross-sectional area* 1 mm^2 *and total unstretched length* 10.0 *cm. It is stretched to* 12.0 *cm and then released to project a missile of mass* 5.0 *g. Taking Young's modulus Y for rubber as* 5.0×10^8 N/m^2. *Calculate the velocity of projection.*

Solution Equivalent force constant of rubber cord

$$k = \frac{YA}{l} = \frac{(5.0 \times 10^8)(1.0 \times 10^{-6})}{(0.1)} = 5.0 \times 10^3 \text{ N/m}$$

Now, from conservation of mechanical energy,

elastic potential energy of cord = kinetic energy of missile

$$\therefore \qquad \frac{1}{2} k (\Delta l)^2 = \frac{1}{2} mv^2 \quad \Rightarrow \quad \therefore \quad v = \left(\sqrt{\frac{k}{m}}\right) \Delta l$$

$$= \left(\sqrt{\frac{5.0 \times 10^3}{5.0 \times 10^{-3}}}\right)(12.0 - 10.0) \times 10^{-2} = 20 \text{ m/s}$$

Ans.

Note *Following assumptions have been made in this problem :*

(*i*) *k has been assumed constant, even though it depends on the length* (*l*).

(*ii*) *The whole of the elastic potential energy is converting into kinetic energy of missile.*

INTRODUCTORY EXERCISE 15.2

1. Find the dimensions of energy density.
2. (a) A wire 4 m long and 0.3 mm in diameter is stretched by a force of 100 N. If extension in the wire is 0.3 mm, calculate the potential energy stored in the wire.

   (b) Find the work done in stretching a wire of cross-section 1 mm^2 and length 2 m through 0.1 mm. Young's modulus for the material of wire is 2.0×10^{11} N/m^2.

Extra Points to Remember

- Modulus of elasticity E (whether, it is Y, B or η) is given by

$$E = \frac{\text{stress}}{\text{strain}}$$

Following conclusions can be made from the above expression :

(i) $E \propto$ stress (for same strain), i.e. if we want the equal amount of strain in two different materials, the one which needs more stress is having more E.

(ii) $E \propto \frac{1}{\text{strain}}$ (for same stress), i.e. if the same amount of stress is applied on two different materials, the one having the less strain is having more E. Rather we can say that the material which offers more resistance to the external forces is having greater value of E. So, we can see that modulus of elasticity of steel is more than that of rubber or

$$\boxed{E_{\text{steel}} > E_{\text{rubber}}}$$

(iii) E = stress for unit strain $\left(\frac{\Delta x}{x} = 1 \text{ or } \Delta x = x\right)$, i.e. suppose the length of a wire is 2 m, then the Young's modulus of elasticity (Y) is the stress applied on the wire to stretch the wire by the same length of 2 m.

- The material which has smaller value of Y is more ductile, i.e. it offers less resistance in framing it into a wire. Similarly, the material having the smaller value of B is more malleable. Thus, for making wire we will more concentrate on Y.

- A solid will have all the three modulii of elasticity Y, B and η. But in case of a liquid or a gas only B can be defined, as a liquid or a gas cannot be framed into a wire or no shear force can be applied on them.
- For a liquid or a gas,

$$\boxed{B = \left(\frac{-dp}{dV/V}\right) \quad \text{or} \quad \left(-\frac{\Delta p}{\Delta V/V}\right)}$$

So, instead of p we are more interested in change in pressure dp or Δp.

- In case of a gas, bulk modulus is process dependent and is given by,

$$\boxed{B = xp}$$

in the process pV^x = constant

For example, for $x = 1$, or pV = constant (isothermal process), $B = p$.

i.e., isothermal bulk modulus of a gas (denoted by B_T) is equal to the pressure of the gas at that instant of time or

$$\boxed{B_T = p}$$

Similarly, for $x = \gamma = \dfrac{C_p}{C_V}$ or pV^γ = constant (adiabatic process), $B = \gamma p$.

i.e., adiabatic bulk modulus of a gas (denoted by B_s) is equal to γ times the pressure of the gas at that instant of time or $B_s = \gamma p$

In general for a gas, $B \propto p$ in all processes.

Physically this can be explained as under:

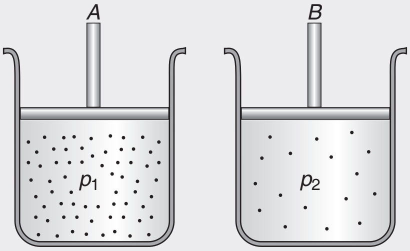


Fig. 15.8

Suppose we have two containers A and B. Some gas is filled in both the containers. But the pressure in A is more than the pressure in B, i.e. $p_1 > p_2$

So, bulk modulus of A should be more than the bulk modulus of B, or $B_1 > B_2$

and this is quiet obvious, because it is more difficult to compress the gas in chamber A, i.e. it provides more resistance to the external forces. And as we have discussed earlier also, the modulus of elasticity is more for a material which offers more resistance to external forces.

- When a pressure is applied on a substance, its density is changed. The change in density can be calculated as under :

$$\rho = \frac{\text{mass}}{\text{volume}} \qquad (\rho = \text{density})$$

or

$$\rho \propto \frac{1}{V} \qquad (\text{mass} = \text{constant})$$

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV}$$

or

$$\rho' = \rho\left(\frac{V}{V + dV}\right)$$

$$= \rho\left(\frac{V}{V-(dp/B)V}\right) \qquad \left(\text{as } B = -\frac{dp}{dV/V}\right)$$

$$\rho' = \frac{\rho}{1-\frac{dp}{B}}$$

From this expression, we can see that ρ′ increases as pressure is increased (*dp* is positive) and *vice-versa*.

We can also write the above equation as

$$\rho' = \rho\left(1-\frac{dp}{B}\right)^{-1}$$

or $$\rho' \approx \rho\left(1+\frac{dp}{B}\right) \qquad (\text{if } dp << B)$$

or $$\rho'-\rho = \Delta\rho = \rho\frac{(dp)}{B}$$

∴ $$\Delta\rho = \rho\left(\frac{\Delta p}{B}\right)$$

- In the figure shown

Work done by gravity is

$$W = (Mg)\Delta l \qquad \ldots\text{(i)}$$

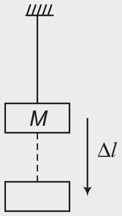


Fig. 15.9

But potential energy stored in the stretched wire is

$$U = \left(\frac{\text{energy}}{\text{volume}} \text{ or energy density}\right)(\text{volume})$$

$$= \left[\frac{1}{2}(\text{stress}) \times (\text{strain})\right][\text{volume}]$$

$$= \frac{1}{2}\left(\frac{Mg}{A}\right)\left(\frac{\Delta l}{l}\right)(Al)$$

or $$U = \frac{1}{2}(Mg)(\Delta l) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we can see that half of the work done by gravity is stored as potential energy in stretched wire and rest half (or 50%) is dissipated in the form of heat, sound etc. during stretching.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on change in length of a wire.*

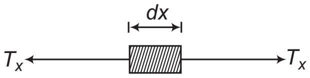
Concept

We have discussed in article 15.4 that change in length is given by $\Delta l = \dfrac{Fl}{AY}$

Here, F is the internal restoring force in the wire. So, this is also the tension in the wire. Therefore, we can also write the above equation as

$$\Delta l = \frac{Tl}{AY}$$

Now, the concept is, if tension is uniform then this equation can be applied directly to find the change in length. If tension is non-uniform then we will find change in length by integration.



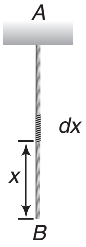
In that case, $$\Delta l = \int dl = \int \frac{T_x \cdot dx}{AY}$$

Here, T_x is the tension at some intermediate point x and

$dl = \dfrac{T_x\,dx}{AY}$ is the change in length in a small element dx, due to the tension T_x at this point.

Note *To find T_x at some intermediate point (if it is non-uniform) students can refer the chapter 'Laws of motion'.*

Example 1 *A bar of mass m and length l is hanging from point A as shown in figure. Find the increase in its length due to its own weight. The Young's modulus of elasticity of the wire is Y and area of cross-section of the wire is A.*



Solution Consider a small section dx of the bar at a distance x from B.

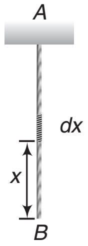
Tension in the bar at this point is $T_x = W_x = \left(\dfrac{mg}{l}\right) x$

Elongation in section dx will be $dl = \dfrac{T_x dx}{AY} = \left(\dfrac{mg}{lAY}\right) x\,dx$

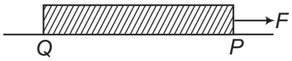
Total elongation in the bar can be obtained by integrating this expression from $x = 0$ to $x = l$.

$\therefore$ $$\Delta l = \int_{x=0}^{x=l} dl = \left(\frac{mg}{lAY}\right)\int_0^l x\,dx$$

or $$\Delta l = \frac{mgl}{2AY}$$ **Ans.**



Example 2 *A rod PQ of mass m, area of cross section A, length l and Young's modulus of elasticity Y is lying on a smooth table as shown in figure. A force F is applied at P. Find*



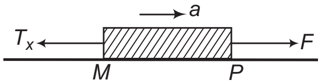
(a) tension at a distance x from end P,

(b) longitudinal stress at this point,

(c) total change in length and

(d) total strain in the rod.

Solution (a) Acceleration of the rod, $a = \frac{F}{m}$



$$F - T_x = (m_{PM})\,a = \left(\frac{m}{l}x\right)\left(\frac{F}{m}\right)$$

$$\therefore \quad T_x = F\left(1 - \frac{x}{l}\right) \qquad \textbf{Ans.}$$

(b) Stress $$\sigma = \frac{F}{A} = \frac{T_x}{A} = \frac{F}{A}\left(1 - \frac{x}{l}\right) \qquad \textbf{Ans.}$$

(c) Change in length $$\Delta l = \int_0^l \frac{T_x dx}{AY}$$

$$= \int_0^l \frac{F\left(1 - \frac{x}{l}\right) dx}{AY}$$

$$\Delta l = \frac{Fl}{2AY} \qquad \textbf{Ans.}$$

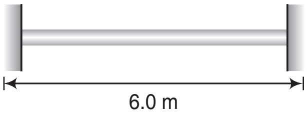
(d) Strain $$= \frac{\Delta l}{l} = \frac{F}{2AY} \qquad \textbf{Ans.}$$

Type 2. *Based on thermal stresses.*

Concept

As discussed in article 15.7, if temperature of a rod is increased or decreased, it has a tendency to expand or contract. If rod is not fixed from its two ends and it is free to expand or contract then no stresses will be developed in it. If rod is fixed from its two ends then it means rod is not allowed to expand or contract. So stresses are developed. If temperature is decreased, then its natural length is less but it is fixed from two ends. So, we can assume that it has been stretched by Δl from its natural length $l - \Delta l (\approx l)$. So, tensile stresses will be developed. Similarly, compressive stresses are developed if temperature is increased.

Example 3 *A steel rod of length 6.0 m and diameter 20 mm is fixed between two rigid supports. Determine the stress in the rod, when the temperature increases by $80^\circ C$ if*



(a) the ends do not yield *(b) the ends yield by 1 mm.*

Take $Y = 2.0 \times 10^6 \ kg/cm^2$ and $\alpha = 12 \times 10^{-6}$ per°C.

Solution Given, length of the rod $l = 6\text{ m} = 600\text{ cm}$

Diameter of the rod $d = 20\text{ mm} = 2\text{ cm}$

Increase in temperature $t = 80°\text{C}$

Young's modulus

$$Y = 2.0 \times 10^6\ \text{kg/cm}^2$$

and thermal coefficient of linear expansion $\alpha = 12 \times 10^{-6}$ per°C

(a) When the ends do not yield

Let, $\sigma_1 =$ stress in the rod

Using the relation $\sigma = \alpha t Y$

$$\therefore \quad \sigma_1 = (12 \times 10^{-6})(80)(2 \times 10^6)$$

$$= 1920\ \text{kg/cm}^2$$ **Ans.**

(b) When the ends yield by 1 mm.

Increase in length due to increase in temperature $\Delta l = l\alpha t$

of this 1 mm or 0.1 cm is allowed to expand. Therefore, net compression in the rod

$$\Delta l_{\text{net}} = (l\alpha t - 0.1)$$

or compressive strain in the rod,

$$\varepsilon = \frac{\Delta l_{\text{net}}}{l} = \left(\alpha t - \frac{0.1}{l}\right)$$

$$\therefore \quad \text{stress } \sigma_2 = Y\varepsilon = Y\left(\alpha t - \frac{0.1}{l}\right)$$

Substituting the values,

$$\sigma_2 = 2 \times 10^6 \left(12 \times 10^{-6} \times 80 - \frac{0.1}{600}\right)$$

$$= 1587\ \text{kg/cm}^2$$ **Ans.**

Note *For more examples of thermal stresses, students can refer the topic thermal expansion.*

Type 3. *Based on breaking stress.*

Concept

Every material has a limit of maximum stress which can be applied to this. This is called breaking stress. Material breaks beyond this.

Now, the maximum force which can be applied across it depends on its area of cross-section.

$$\sigma_m = \frac{F_m}{A} \qquad (\sigma_m = \text{breaking stress})$$

$$\therefore \quad \text{Maximum force } F_m = (\sigma_m)A$$

Example 4 *If the elastic limit of copper is* $1.5 \times 10^8\ N/m^2$, *determine the minimum diameter a copper wire can have under a load of* 10.0 *kg, if its elastic limit is not to be exceeded.*

Solution $\dfrac{F}{A_{\min}} = \sigma_m$

$$\therefore \quad \frac{F}{\left(\dfrac{\pi d_{\min}^2}{4}\right)} = \sigma_m \quad \text{or} \quad d_{\min} = \sqrt{\frac{4F}{\pi\,\sigma_m}}$$

$$= \sqrt{\frac{4 \times 10 \times 9.8}{3.14 \times 1.5 \times 10^8}} = 9.1 \times 10^{-4}\ \text{m} = 0.91\ \text{mm} \qquad \textbf{Ans.}$$

Example 5 *Find the greatest length of steel wire that can hang vertically without breaking. Breaking stress of steel* = $8.0 \times 10^8\ N/m^2$. *Density of steel* = $8.0 \times 10^3\ kg/m^3$. *Take* $g = 10\ m/s^2$.

Solution Let l be the length of the wire that can hang vertically without breaking. Then, the stretching force on it is equal to its own weight. If therefore, A is the area of cross-section and ρ the density, then

$$\text{Maximum stress } (\sigma_m) = \frac{\text{weight}}{A} \qquad \left(\text{Stress} = \frac{\text{force}}{\text{area}}\right)$$

or
$$\sigma_m = \frac{(Al\rho)g}{A}$$

$$\therefore \quad l = \frac{\sigma_m}{\rho g}$$

Substituting the values
$$l = \frac{8.0 \times 10^8}{(8.0 \times 10^3)(10)} = 10^4\ \text{m} \qquad \textbf{Ans.}$$

Type 4. *Based on change in density due to change in pressure.*

Concept

As discussed in extra touch points, if extra pressure Δp is applied on a body its volume decreases without change in its mass. So, its density increases. If $\Delta p << B$, where B is the bulk modulus, then increase in density is given by $\Delta\rho = \dfrac{\rho\Delta p}{B}$

Example 6 *Bulk modulus of water is* $2.3 \times 10^9\ N/m^2$. *Taking average density of water* $\rho = 10^3\ kg/m^3$, *find increase in density at a depth of* 1*km. Take* $g = 10\ m/s^2$

Solution Pressure increases with depth of a liquid. At a depth 'h' below the water surface increase in pressure is given by $\Delta p = \rho g h$

Using the equation,
$$\Delta\rho = \rho\frac{\Delta p}{B}$$

we get
$$\Delta\rho = \frac{\rho\,(\rho g h)}{B} = \frac{\rho^2 g h}{B}$$

Substituting the values we have,

$$\Delta\rho = \frac{(10^3)^2(10)\,(10^3)}{2.3 \times 10^9} = 4.33\ \text{kg/m}^3 \qquad \textbf{Ans.}$$

Miscellaneous Examples

Example 7 *The pressure of a medium is changed from* 1.01×10^5 *Pa to* 1.165×10^5 *Pa and change in volume is 10% keeping temperature constant. The bulk modulus of the medium is*

(a) 204.8×10^5 *Pa* *(b)* 102.4×10^5 *Pa* *(c)* 51.2×10^5 *Pa* *(d)* 1.55×10^5 *Pa*

Solution From the definition of bulk modulus, $B=\dfrac{-\Delta p}{(\Delta V/V)}$

Substituting the values, we have

$$B=\frac{(1.165-1.01)\times10^5}{(10/100)}=1.55\times10^5\ \text{Pa}$$

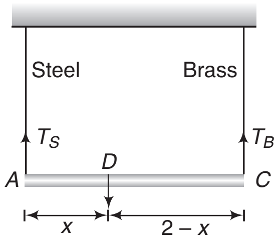
Therefore, the correct option is (d).

Example 8 *A light rod of length* 2.00 *m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section* $10^{-3}\ m^2$ *and the other is of brass of cross-section* $2\times10^{-3}\ m^2$. *Find out the position along the rod at which a weight may be hung to produce,*

(a) equal stresses in both wires *(b) equal strains on both wires.*

Young's modulus for steel is $2\times10^{11}\ N/m^2$ *and for brass is* $10^{11}\ N/m^2$.

Solution (a) Given, stress in steel = stress in brass



$$\therefore \quad \frac{T_S}{A_S}=\frac{T_B}{A_B}$$

$$\therefore \quad \frac{T_S}{T_B}=\frac{A_S}{A_B}$$

$$=\frac{10^{-3}}{2\times10^{-3}}=\frac{1}{2} \quad \ldots\text{(i)}$$

As the system is in equilibrium, taking moments about D, we have

$$T_S\cdot x=T_B(2-x)$$

$$\therefore \quad \frac{T_S}{T_B}=\frac{2-x}{x} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get $x=1.33\,\text{m}$ **Ans.**

(b) Strain $=\dfrac{\text{Stress}}{Y}$

Given, strain in steel = strain in brass

$$\therefore \quad \frac{T_S/A_S}{Y_S}=\frac{T_B/A_B}{Y_B}$$

$$\therefore \quad \frac{T_S}{T_B}=\frac{A_SY_S}{A_BY_B}=\frac{(1\times10^{-3})(2\times10^{11})}{(2\times10^{-3})(10^{11})}=1 \quad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we have $x=1.0$ m **Ans.**

Example 9 *A sphere of radius* $0.1\,m$ *and mass* 8π *kg is attached to the lower end of a steel wire of length* 5.0 *m and diameter* 10^{-3} *m. The wire is suspended from* 5.22 *m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is* $1.994 \times 10^{11}\,N/m^2$.

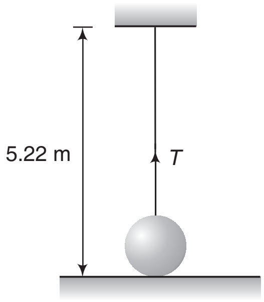
Solution Let Δl be the extension of wire when the sphere is at mean position. Then, we have

$$l + \Delta l + 2r = 5.22$$

or

$$\begin{aligned} \Delta l &= 5.22 - l - 2r \\ &= 5.22 - 5 - 2 \times 0.1 \\ &= 0.02 \text{ m} \end{aligned}$$

Let T be the tension in the wire at mean position during oscillations, then



$$Y = \frac{T/A}{\Delta l/l}$$

$$\therefore \quad T = \frac{YA\Delta l}{l} = \frac{Y\pi r^2 \Delta l}{l}$$

Substituting the values, we have

$$T = \frac{(1.994 \times 10^{11}) \times \pi \times (0.5 \times 10^{-3})^2 \times 0.02}{5}$$

$$= 626.43 \text{ N}$$

The equation of motion at mean position is,

$$T - mg = \frac{mv^2}{R} \qquad \text{...(i)}$$

Here, $R = 5.22 - r = 5.22 - 0.1 = 5.12$ m

and $m = 8\pi$ kg $= 25.13$ kg

Substituting the proper values in Eq. (i), we have

$$(626.43) - (25.13 \times 9.8) = \frac{(25.13)\,v^2}{5.12}$$

Solving this equation, we get

$$v = 8.8 \text{ m/s}$$ **Ans.**

Example 10 *A thin ring of radius R is made of a material of density ρ and Young's modulus Y. If the ring is rotated about its centre in its own plane with angular velocity ω, find the small increase in its radius.*

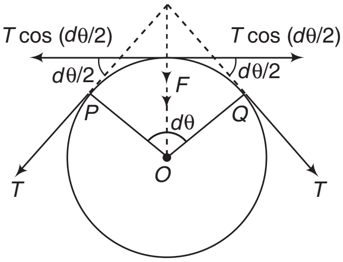
Solution Consider an element PQ of length dl. Let T be the tension and A the area of cross-section of the wire.

$$\text{Mass of element } dm = \text{volume} \times \text{density}$$
$$= A(dl)\rho$$

The component of T, towards the centre provides the necessary centripetal force to portion PQ.

$$\therefore \qquad F = 2T\sin\left(\frac{d\theta}{2}\right) = (dm)R\omega^2 \qquad \text{...(i)}$$

For small angles $$\sin\frac{d\theta}{2} \approx \frac{d\theta}{2} = \frac{(dl/R)}{2}$$



or $$d\theta = \frac{dl}{R}$$

Substituting in Eq. (i), we have

$$T \cdot \frac{dl}{R} = A(dl)\rho R\omega^2$$

or $$T = A\rho\omega^2R^2$$

Let ΔR be the increase in radius.

$$\text{Longitudinal strain} = \frac{\Delta l}{l} = \frac{\Delta(2\pi R)}{2\pi R} = \frac{\Delta R}{R}$$

Now, $$Y = \frac{T/A}{\Delta R/R}$$

$$\therefore \qquad \Delta R = \frac{TR}{AY}$$

$$= \frac{(A\rho\omega^2R^2)R}{AY}$$

or $$\Delta R = \frac{\rho\omega^2R^3}{Y}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Steel is more elastic than rubber.

Reason : For same strain, steel requires more stress to be produced in it.

2. Assertion : If pressure is increased, bulk modulus of gases will increase.

Reason : With increase in pressure, temperature of gas also increases.

3. Assertion : From the relation $Y = \frac{Fl}{A\Delta l}$, we can say that, if length of a wire is doubled, its Young's modulus of elasticity will also becomes two times.

Reason : Modulus of elasticity is a material property.

4. Assertion : Bulk modulus of elasticity can be defined for all three states of matter, solid liquid and gas.

Reason : Young's modulus is not defined for liquids and gases.

5. Assertion : Every wire is like a spring, whose spring constant, $K \propto \frac{1}{l}$ where l is length of wire.

Reason : It follows from the relation

$$K = \frac{YA}{l}$$

6. Assertion : Ratio of stress and strain is always constant for a substance.

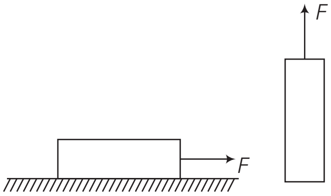
Reason : This ratio is called modulus of elasticity.

7. Assertion : Ratio of isothermal bulk modulus and adiabatic bulk modulus for a monoatomic gas at a given pressure is $\frac{3}{5}$.

Reason : This ratio is equal to

$$\gamma = \frac{C_p}{C_V}.$$

8. **Assertion:** A uniform elastic rod lying on smooth horizontal surface is pulled by a constant horizontal force of magnitude F as shown in the figure. Another identical elastic rod is pulled vertically upwards by a constant vertical force of magnitude F as shown in the figure. The extensions in both the rods will be same.



Reason: In a uniform elastic rod, the extension depends only on forces acting at the ends of rods.

9. **Assertion:** Identical springs of steel and copper are equally stretched. More work will be done on the steel spring.

Reason: Steel is more elastic than copper.

Objective Questions

Single Correct Option

1. The bulk modulus for an incompressible liquid is

(a) zero	(b) unity
(c) infinity	(d) between 0 and 1

2. The Young's modulus of a wire of length (L) and radius (r) is Y. If the length is reduced to $\frac{L}{2}$ and radius $\frac{r}{2}$, then its Young's modulus will be

(a) $\frac{Y}{2}$	(b) Y
(c) $2Y$	(d) $4Y$

3. The maximum load that a wire can sustain is W. If the wire is cut to half its value, the maximum load it can sustain is

(a) W	(b) $\frac{W}{2}$
(c) $\frac{W}{4}$	(d) $2W$

4. Identify the case when an elastic metal rod does not undergo elongation
   (a) it is pulled with a constant acceleration on a smooth horizontal surface
   (b) it is pulled with constant velocity on a rough horizontal surface
   (c) it is allowed to fall freely
   (d) All of the above

5. Vessel of 1×10^{-3} m^3 volume contains an oil. If a pressure of 1.2×10^5 N/m^2 is applied on it, then volume decreases by 0.3×10^{-3} m^3. The bulk modulus of oil is

(a) 6×10^{10} N/m^2	(b) 4×10^5 N/m^2
(c) 2×10^7 N/m^2	(d) 1×10^6 N/m^2

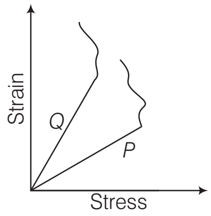
6. A load of 4 kg is suspended from a ceiling through a steel wire of length 20 m and radius 2 mm. It is found that the length of the wire increases by 0.031 mm, as equilibrium is achieved. If $g = 3.1 \times \pi \text{ ms}^{-2}$, the value of Young's modulus of the material of the wire (in Nm^{-2}) is

(a) 2×10^{12}
(b) 4×10^{11}
(c) 2×10^{11}
(d) 0.02×10^{9}

7. A wire of length 1m and radius 1mm is subjected to a load. The extension is x. The wire is melted and then drawn into a wire of square cross-section of side 2 mm. Its extension under the same load will be

(a) $\dfrac{\pi^2 x}{8}$
(b) $\dfrac{\pi^2 x}{16}$
(c) $\dfrac{\pi^2 x}{2}$
(d) $\dfrac{x}{2\pi}$

8. Figure shows the stress-strain curve of two metals P and Q. From the graph, it can be concluded that



(a) P has greater Young's modulus and lesser ductility
(b) Q has greater Young's modulus and lesser ductility
(c) P has greater Young's modulus and greater ductility
(d) Q has greater Young's modulus and greater ductility

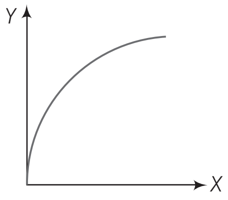
9. The bulk modulus of water is $2.0 \times 10^9 \text{ N/m}^2$. The pressure required to increase the density of water by 0.1% is

(a) $2.0 \times 10^3 \text{ N/m}^2$
(b) $2.0 \times 10^6 \text{ N/m}^2$
(c) $2.0 \times 10^5 \text{ N/m}^2$
(d) $2.0 \times 10^7 \text{ N/m}^2$

10. If the work done in stretching a wire by 1 mm is 2 J, then work necessary for stretching another wire of same material but with double radius of cross-section and half of the length by 1 mm is

(a) 8 J
(b) 16 J
(c) 4 J
(d) 32 J

11. The graph shows the behaviour of a steel wire in the region for which the wire obeys Hooke's law. The graph is a parabola. The variables X and Y-axes, respectively can be [stress (σ), strain (ε) and elastic potential energy (U)]



(a) U, σ
(b) U, ε
(c) σ, ε
(d) None of these

12. Depth of sea is maximum at Mariana Trench in West Pacific Ocean. Trench has a maximum depth of about 11 km. At bottom of trench water column above it exerts 1000 atm pressure. Percentage change in density of sea water at such depth will be around

(Given, $B = 2 \times 10^9$ Nm^{-2} and $p_{\text{atm}} = 1 \times 10^5$ Nm^{-2})

(a) about 5% (b) about 10%
(c) about 3% (d) about 7%

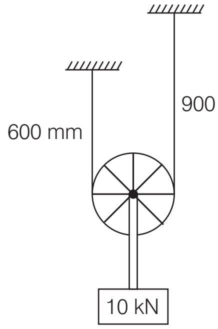
13. One end of a horizontal thick copper wire of length $2L$ and radius $2R$ is welded to an end of another horizontal thin copper wire of length L and radius R. When the arrangement is stretched by applying forces at two ends, the ratio of the elongation in the thin wire to that in the thick wire is

(a) 0.25 (b) 0.50
(c) 2.00 (d) 4.00

14. An air filled balloon is at a depth of 1km below the water level in an ocean. The normal stress of the balloon (in Pa) is
(Given, $\rho_{\text{water}} = 10^3$ kgm^{-3}, $g = 9.8$ ms^{-2} and $p_{\text{atm}} = 10^5$ Pa)

(a) 10^6 (b) 9.9×10^5
(c) 9.9×10^7 (d) 9.9×10^6

15. A load of 10 kN is supported from a pulley, which in turn is supported by a rope of cross-sectional area 10^3 mm^2 and modulus of elasticity 10^3 Nmm^{-2} as shown in the figure. Neglecting friction at the pulley, then downward deflection of the load (in mm) is



(a) 3.75 (b) 4.25 (c) 2.75 (d) 4.00

16. A body of mass 3.14 kg is suspended from one end of a wire of length 10 m. The radius of cross-section of the wire is changing uniformly from 5×10^{-4} m at the top (i.e. point of suspension) to 9.8×10^{-4} m at the bottom. Young's modulus of elasticity is 2×10^{11} N/m^2. The change in length of the wire is

(a) 4×10^{-3} m (b) 3×10^{-3} m
(c) 10^{-3} m (d) 2×10^{-3} m

Subjective Questions

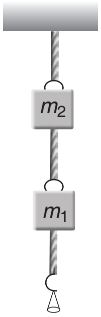
1. What is the density of lead under a pressure of 2.0×10^8 N/m^2, if the bulk modulus of lead is 8.0×10^9 N/m^2 and initially the density of lead is 11.4 g/cm^3?

2. A cylindrical steel wire of 3 m length is to stretch no more than 0.2 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire ?
$Y_{\text{steel}} = 2.1 \times 10^{11}$ N/m^2

3. The elastic limit of a steel cable is $3.0 \times 10^8 \text{ N/m}^2$ and the cross-section area is 4 cm^2. Find the maximum upward acceleration that can be given to a 900 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

4. Find the increment in the length of a steel wire of length 5 m and radius 6 mm under its own weight. Density of steel = 8000 kg/m^3 and Young's modulus of steel = $2 \times 10^{11} \text{ N/m}^2$. What is the energy stored in the wire ? (Take $g = 9.8 \text{ m/s}^2$)

5. Two wires shown in figure are made of the same material which has a breaking stress of $8 \times 10^8 \text{ N/m}^2$. The area of cross-section of the upper wire is 0.006 cm^2 and that of the lower wire is 0.003 cm^2. The mass $m_1 = 10 \text{ kg}$, $m_2 = 20$ kg and the hanger is light. Find the maximum load that can be put on the hanger without breaking a wire. Which wire will break first if the load is increased ? (Take $g = 10 \text{ m/s}^2$)



6. A steel wire and a copper wire of equal length and equal cross-sectional area are joined end to end and the combination is subjected to a tension. Find the ratio of

(a) the stresses developed in the two wires,
(b) the strains developed. (Y of steel $= 2 \times 10^{11} \text{ N/m}^2$ and Y of copper $= 1.3 \times 10^{11} \text{ N/m}^2$)

7. Calculate the approximate change in density of water in a lake at a depth of 400 m below the surface. The density of water at the surface is 1030 kg/m^3 and bulk modulus of water is $2 \times 10^9 \text{ N/m}^2$.

8. In taking a solid ball of rubber from the surface to the bottom of a lake of 180 m depth, reduction in the volume of the ball is 0.1%. The density of water of the lake is $1 \times 10^3 \text{ kg/m}^3$. Determine the value of the bulk modulus of elasticity of rubber. ($g = 9.8 \text{ m/s}^2$)

LEVEL 2

Single Correct Option

1. A wire elongates by l units, when a load w is suspended from it. If the wire gets over a pulley (equally on both the sides) and two weights w each are hung at the two ends, the elongation of wire (in units) will be

(a) zero
(b) $\frac{l}{2}$
(c) l
(d) $2l$

2. Maximum stress that can be applied to a wire which supports on elevator is σ. Mass of elevator is m and it is moved upwards with an acceleration of $g/2$. Minimum diameter of wire (Neglecting weight of wire) must be

(a) $\sqrt{\dfrac{2mg}{\pi\sigma}}$ (b) $\sqrt{\dfrac{3mg}{2\pi\sigma}}$

(c) $\sqrt{\dfrac{5mg}{2\pi\sigma}}$ (d) $\sqrt{\dfrac{6mg}{\pi\sigma}}$

3. A bob of mass 10 kg is attached to a wire 0.3 m long. Its breaking stress is 4.8×10^7 N/m^2. The area of cross-section of the wire is 10^{-6} m^2. What is the maximum angular velocity with which it can be rotated in a horizontal circle?

(a) 8 rad/s (b) 4 rad/s

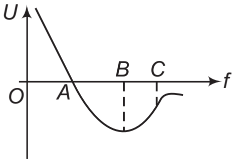
(c) 2 rad/s (d) 1 rad/s

4. A uniform steel rod of cross-sectional area A and length L is suspended so that it hangs vertically. The stress at the middle point of the rod is

(a) $\frac{1}{2}\rho g L$ (b) $\frac{1}{4}\rho g L$

(c) $\rho g L$ (d) None of these

5. The potential energy U of diatomic molecules as a function of separation r is shown in figure. Identify the correct statement.



(a) The atoms are in equilibrium if $r = OA$

(b) The force is repulsive only if r lies between A and B

(c) The force is attractive if r lies between A and B

(d) The atoms are in equilibrium if $r = OB$

6. The length of a steel wire is l_1 when the stretching force is T_1 and l_2 when the stretching force is T_2. The natural length of the wire is

(a) $\dfrac{l_1T_1 + l_2T_2}{T_1 + T_2}$ (b) $\dfrac{l_2T_1 + l_1T_2}{T_1 + T_2}$

(c) $\dfrac{l_2T_1 - l_1T_2}{T_1 - T_2}$ (d) $\dfrac{l_1T_1 - l_2T_2}{T_1 - T_2}$

7. A mass m is suspended from a wire. Change in length of the wire is Δl. Now the same wire is stretched to double its length and the same mass is suspended from the wire. The change in length in this case will become (it is assumed that elongation in the wire is within the proportional limit)

(a) Δl (b) $2\Delta l$ (c) $4\Delta l$ (d) $8\Delta l$

8. A uniform metal rod fixed at its ends of 2 mm^2 cross-section is cooled from 40°C to 20°C. The coefficient of the linear expansion of the rod is 12×10^{-6} per degree celsius and its Young's modulus of elasticity is 10^{11} N/m^2. The energy stored per unit volume of the rod is

(a) 2880 J/m^3 (b) 1500 J/m^3

(c) 5760 J/m^3 (d) 1440 J/m^3

9. A rod of length 1000 mm and coefficient of linear expansion $\alpha = 10^{-4}$ per degree celsius is placed in horizontal smooth surface symmetrically between fixed walls separated by 1001 mm. The Young's modulus of rod is 10^{11} N/m^2. If the temperature is increased by 20°C, then the stress developed in the rod is (in N/m^2)

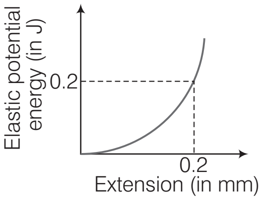
(a) 10^5 (b) 10^8
(c) 10^7 (d) 10^6

10. A uniform elastic plank moves due to a constant force F_0 applied at one end whose area is S. The Young's modulus of the plank is Y. The strain produced in the direction of force is

(a) $\frac{F_0}{2SY}$ (b) $\frac{F_0}{SY}$ (c) $\frac{2F_0}{SY}$ (d) $\frac{\sqrt{2}F_0}{SY}$

More than One Correct Options

1. Figure shows the graph of elastic potential energy (U) stored *versus* extension, for a steel wire ($Y = 2 \times 10^{11}$ Pa) of volume 200 cc. If area of cross-section A and original length L, then



(a) $A = 10^{-4}$ m^2 (b) $A = 10^{-3}$ m^2 (c) $L = 1.5$ m (d) $L = 2$ m

2. A metal wire of length L, area of cross-section A and Young's modulus Y is stretched by a variable force F such that F is always slightly greater than the elastic forces of resistance in the wire. When the elongation of the wire is l

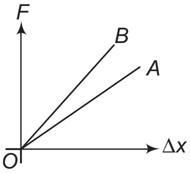
(a) the work done by F is $\frac{YAl^2}{2L}$

(b) the work done by F is $\frac{YAl^2}{L}$

(c) the elastic potential energy stored in the wire is $\frac{YAl^2}{2L}$

(d) the elastic potential energy stored in the wire is $\frac{YAl^2}{4L}$

3. Two wires A and B of same length are made of same material. The figure represents the load F *versus* extension Δx graph for the two wires. Then



(a) The cross sectional area of A is greater than that of B
(b) The elasticity of B is greater than that of A
(c) The cross-sectional area of B is greater than that of A
(d) The elasticity of A is greater than that of B

4. A body of mass M is attached to the lower end of a metal wire, whose upper end is fixed. The elongation of the wire is l.

(a) Loss in gravitational potential energy of M is Mgl

(b) The elastic potential energy stored in the wire is Mgl

(c) The elastic potential energy stored in the wire is $\frac{1}{2}Mgl$

(d) Heat produced is $\frac{1}{2}Mgl$

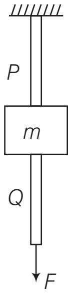
5. Two light wires P and Q shown in the figure are made of same material and have radii r_P and r_Q, respectively. The block between them has a mass m. When the force $F = \frac{mg}{3}$, then one of the wires breaks. Choose the correct option(s).

(a) P breaks, if $r_P = r_Q$

(b) P breaks, if $r_P < 2r_Q$

(c) Either P or Q may break, if $r_P = 2r_Q$

(d) To predict, which wire will break, the lengths of P and Q must be known



6. Two wires A and B have equal lengths and are made of the same material, but diameter of wire A is twice that of wire B. Then, for a given load,

(a) the extension of B will be four times that of A

(b) the extensions of A and B will be equal

(c) the strain in B is four times that in A

(d) the strains in A and B will be equal

7. A light rod of length 2 m is suspended from a ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 10^{-3} m^2 and the other is of brass of cross-section 2×10^{-3} m^2. x is the distance from steel wire end, at which a weight may be hung. $Y_{\text{steel}} = 2 \times 10^{11}$ Pa and $Y_{\text{brass}} = 10^{11}$ Pa

Which of the following statement(s) is/are correct?

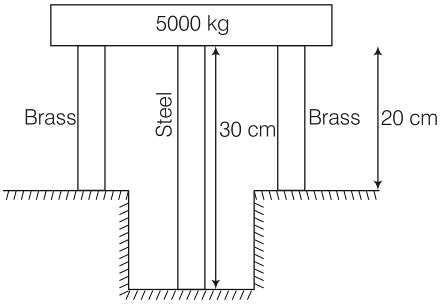
(a) $x = 1.2$ m, if the strains of both the wires are to be equal

(b) $x = 1.42$ m, if the stresses of both the wires are to be equal

(c) $x = 1$ m, if the strains of both the wires are to be equal

(d) $x = 1.33$ m, if the stresses of both the wires are to be equal

8. A steel rod of cross-sectional area 16 cm^2 and two brass rods each of cross-sectional area 10 cm^2 together support a load of 5000 kg as shown in the figure. (Given, $Y_{\text{steel}} = 2 \times 10^6$ kg cm^{-2} and $Y_{\text{brass}} = 10^6$ kg cm^{-2}). Choose the correct option(s).



(a) Stress in brass rod = 121 kg cm^{-2}

(b) Stress in steel rod = 161 kg cm^{-2}

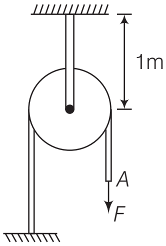
(c) Stress in brass rod = 141 kg cm^{-2}

(d) Stress in steel rod = 141 kg cm^{-2}

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 3)

The axle of a pulley of mass 1 kg is attached to the end of an elastic string of length 1 m, cross-sectional area 10^{-3} m^2 *and Young's modulus* 2×10^5 Nm^{-2}, *whose other end is fixed to the ceiling. A rope of negligible mass is placed on the pulley such that its left end is fixed to the ground and its right end is hanging freely, from the pulley which is at rest in equilibrium. Now, the free end A of the rope is subjected to pulling with constant force* $F = 10\,N$. *Friction between the rope and the pulley can be neglected.* (*Given,* $g = 10\ ms^{-2}$)



1. The elongation of the string before applying force is
(a) 50 cm (b) 5 cm (c) 0.5 cm (d) 0.05 cm

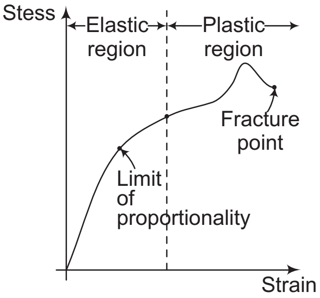
2. The greatest elongation of the string is
(a) 35 cm (b) 30 cm (c) 25 cm (d) 20 cm

3. The maximum displacement of point A after applying F is
(a) 70 cm (b) 60 cm (c) 40 cm (d) 50 cm

Passage 2 (Q. Nos. 4 to 6)

On gradual loading, stress-strain relationship for a metal wire is as follows.

Within proportionality limit, stress $\propto$ *strain or,* $\dfrac{Stress}{Strain}$ = *a constant for the material of wire.*



4. Two wires of same material have length and radius (L, r) and $\left(2L, \dfrac{r}{2}\right)$. The ratio of their Young's modulii is
(a) 1 : 2 (b) 2 : 3
(c) 2 : 1 (d) 1: 1

5. Just on crossing the yield region, the material will have
(a) increased and breaking stress (b) reduced and breaking stress
(c) constant stress (d) None of these

6. If $\frac{\text{Stress}}{\text{Strain}}$ is x in elastic region and y in the region of yield, then

(a) $x > y$ (b) $x = y$ (c) $x < y$ (d) $x = 2y$

Match the Columns

1. Match the following two columns. (dimension wise)

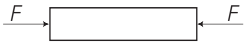
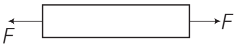
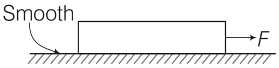
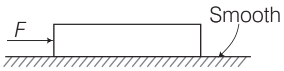
Column I	Column II
(a) Stress	(p) coefficient of friction
(b) Strain	(q) relative density
(c) Modulus of elasticity	(r) energy density
(d) Force constant of a wire	(s) None

2. A wire of length l, area of cross section A and Young's modulus of elasticity Y is stretched by a longitudinal force F. The change in length is Δl. Match the following two columns.

Note *In column I, corresponding to every option, other factors remain constant.*

Column I	Column II
(a) F is increased	(p) Δl will increase
(b) l is increased	(q) stress will increase
(c) A is increased	(r) Δl will decrease
(d) Y is increased	(s) stress will decrease

3. In Column I, a uniform bar of uniform cross-sectional area under the application of forces is shown in the figure and in Column II, some effects/phenomena are given. Match the two columns.

Column I	Column II
(a) 	(p) Uniform stresses are developed in the rod.
(b) 	(q) Non-uniform stresses are developed in the rod.
(c) 	(r) Compressive stresses are developed in the rod.
(d) 	(s) Tensile stresses are developed in the rod.

Subjective Questions

1. A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in a cylindrical container. A massless piston of area A (the area of container is also A) floats on the surface of the liquid. When a mass M is placed on the piston to compress the liquid, fractional change in radius of the sphere is $\frac{Mg}{\alpha AB}$. Find the value of α.

2. A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its cross-sectional area is $4.9 \times 10^{-7}\ \text{m}^2$. If the mass is pulled a little in the vertically downward direction and released, it performs SHM with angular frequency 140 rad s^{-1}. If the Young's modulus of the material of the wire is $p \times 10^9\ \text{Nm}^{-2}$, find the value of p.

3. A wire having a length L and cross-sectional area A is suspended at one of its ends from a ceiling. Density and Young's modulus of material of the wire are ρ and Y, respectively. Its strain energy due to its own weight is $\dfrac{\rho^2 g^2 A L^3}{\alpha Y}$. Find the value of α.

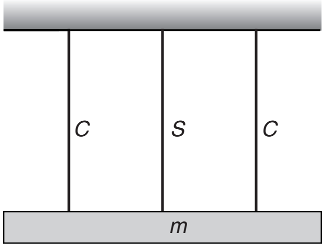
4. A wire of length 3 m, diameter 0.4 mm and Young's modulus $8 \times 10^{10}\ \text{N/m}^2$ is suspended from a point and supports a heavy cylinder of volume $10^{-3}\ \text{m}^3$ at its lower end. Find the decrease in length when the metal cylinder is immersed in a liquid of density 800 kg/m^3.

5. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire of length 5 m and diameter 4 mm which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor. Calculate the velocity of the ball at its lowest position ($Y_{\text{steel}} = 2 \times 10^{11}\ \text{N/m}^2$).

6. A uniform ring of radius R and made up of a wire of cross-sectional radius r is rotated about its axis with a frequency f. If density of the wire is ρ and Young's modulus is Y. Find the fractional change in radius of the ring.

7. A 6 kg weight is fastened to the end of a steel wire of unstretched length 60 cm. It is whirled in a vertical circle and has an angular velocity of 2 rev/s at the bottom of the circle. The area of cross-section of the wire is 0.05 cm^2. Calculate the elongation of the wire when the weight is at the lowest point of the path. Young's modulus of steel = $2 \times 10^{11}\ \text{N/m}^2$.

8. A homogeneous block with a mass m hangs on three vertical wires of equal length arranged symmetrically. Find the tension of the wires if the middle wire is of steel and the other two are of copper. All the wires have the same cross-section. Consider the modulus of elasticity of steel to be double than that of copper.



9. A uniform copper bar of density ρ, length L, cross-sectional area S and Young's modulus Y is moving horizontally on a frictionless surface with constant acceleration a_0. Find
(a) the stress at the centre of the wire,
(b) total elongation of the wire.

10. A 5 m long cylindrical steel wire with radius 2×10^{-3} m is suspended vertically from a rigid support and carries a bob of mass 100 kg at the other end. If the bob gets snapped, calculate the change in temperature of the wire ignoring radiation losses. (Take $g = 10\ \text{m/s}^2$)
(For the steel wire, Young's modulus = $2.1 \times 10^{11}\ \text{N/m}^2$; Density = 7860 kg/m^3;
Specific heat = 420 J/kg-°C).

Answers

Introductory Exercise 15.1

1. Wire *B* **2.** 0.0125 cm **3.** 250.2 mm **4.** $[ML^{-1}T^{-2}]$, $[M^0L^0T^0]$, $[ML^{-1}T^{-2}]$

Introductory Exercise 15.2

1. $[ML^{-1}T^{-2}]$ **2.** (a) 0.015 J (b) 5.0×10^{-4} J

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (c) **3.** (d) **4.** (b) **5.** (a) **6.** (d) **7.** (c) **8.** (c) **9.** (a)

Objective Questions

1. (c) **2.** (b) **3.** (a) **4.** (c) **5.** (b) **6.** (a) **7.** (b) **8.** (a) **9.** (b) **10.** (b)
11. (b) **12.** (a) **13** (c) **14.** (d) **15.** (a) **16.** (c)

Subjective Questions

1. 11.69 g/cm^3 **2.** 1.91mm **3.** 34.64 m/s^2 **4.** $\Delta l = 4.9 \times 10^{-6}$m, $U = 5.43 \times 10^{-5}$J

5. Load = 14 kg, lower string **6.** (a) 1 (b) Ratio = $\frac{13}{20}$ **7.** 2.0 kg/m^3 **8.** $B = 1.76 \times 10^9$ N/m^2

LEVEL 2

Single Correct Option

1. (c) **2.** (d) **3.** (b) **4.** (a) **5.** (d) **6.** (c) **7.** (c) **8.** (a) **9.** (b) **10.** (a)

More than One Correct Options

1. (a, d) **2.** (a, c) **3.** (c) **4.** (a, c, d) **5.** (a, b,c) **6.** (a, c) **7.** (c, d) **8.** (a, b)

Comprehension Based Questions

1. (b) **2.** (c) **3.** (d) **4.** (d) **5.** (b) **6.** (a)

Match the Columns

1.(a) → r (b) → p, q (c) → r (d) → s
2.(a) → p, q (b) → p (c) → r, s (d) → r
3.(a) → p, r (b) → p, s (c) → q, s (d) → q, r

Subjective Questions

1. 3 **2.** 4 **3.** 6 **4.** $\Delta l = 2.34 \times 10^{-3}$ m **5.** $v = 31.23$ m/s **6.** $\frac{4\pi^2 f^2 pR^2}{Y}$ **7.** 3.8×10^{-4} m

8. $T_c = \frac{mg}{4}$, $T_s = 2T_c$ **9.** (a) $\frac{1}{2}L\rho a_0$ (b) $\frac{1}{2}\frac{\rho a_0 L^2}{Y}$ **10.** 4.568×10^{-3}°C

16

Fluid Mechanics

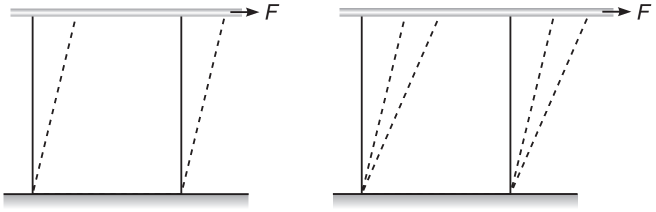
Chapter Contents

16.1 Definition of a Fluid
16.2 Density of a Liquid
16.3 Pressure in a Fluid
16.4 Pressure Difference in Accelerating Fluids
16.5 Archimedes' Principle
16.6 Flow of Fluids
16.7 Application based on Bernoulli's Equation
16.8 Viscosity
16.9 Surface Tension
16.10 Capillary Rise or Fall

16.1 Definition of a Fluid

Fluid mechanics deals with the behaviour of fluids at rest and in motion. A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be.

Thus, fluids comprise the liquid and gas (or vapour) phases of the physical forms in which matter exists. The distinction between a fluid and the solid state of matter is clear if you compare fluid and solid behaviour. A solid deforms when a shear stress is applied but it does not continue to increase with time. However, if a shear stress is applied to a fluid, the deformation continues to increase as long as the stress is applied. We may alternatively define a fluid as a substance that **cannot sustain a shear stress when at rest.**



(a) Solid (b) Fluid

Fig. 16.1 Behaviour of a solid and a fluid, under the action of a constant shear force

Ideal Fluid

An ideal liquid is incompressible and non-viscous in nature. An incompressible liquid means the density of the liquid is constant, it is independent of the variations in pressure. A non-viscous liquid means that, parts of the liquid in contact do not exert any tangential force on each other. Thus, there is no friction between the adjacent layers of a liquid. The force by one part of the liquid on the other part is **perpendicular to the surface of contact.**

16.2 Density of a Liquid

Density (ρ) of any substance is defined as the mass per unit volume or

$$\rho = \frac{\text{mass}}{\text{volume}} \quad \text{or} \quad \boxed{\rho = \frac{m}{V}}$$

Relative Density (RD)

In case of a liquid, sometimes an another term **Relative Density** (RD) is defined. It is the ratio of density of the substance to the density of water at 4°C. Hence,

$$\boxed{\text{RD} = \frac{\text{Density of substance}}{\text{Density of water at } 4^\circ\text{C}}}$$

RD is a pure ratio. So, it has no units. It is also sometimes referred as specific gravity. Density of water at 4°C in CGS is 1 g/cm^3. Therefore, numerically the RD and density of substance (in CGS) are equal. In SI units, the density of water at 4°C is 1000 kg/m^3.

Density of a Mixture of two or more Liquids

Here, we have two cases:

Case 1 Suppose two liquids of densities ρ_1 and ρ_2 having masses m_1 and m_2 are mixed together. Then, the density of the mixture will be

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{(m_1 + m_2)}{(V_1 + V_2)}$$

$$= \frac{(m_1 + m_2)}{\left(\dfrac{m_1}{\rho_1} + \dfrac{m_2}{\rho_2}\right)}$$

If $m_1 = m_2$, then

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

Case 2 If two liquids of densities ρ_1 and ρ_2 having volumes V_1 and V_2 are mixed, then the density of the mixture is,

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{m_1 + m_2}{V_1 + V_2}$$

$$= \frac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2}$$

If $V_1 = V_2$, then

$$\rho = \frac{\rho_1 + \rho_2}{2}$$

Effect of Temperature on Density

As the temperature of a liquid is increased, the mass remains the same while the volume is increased and hence, the density of the liquid decreases $\left(\text{as } \rho \propto \dfrac{1}{V}\right)$. Thus,

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV} = \frac{V}{V + V\gamma\Delta\theta}$$

or

$$\frac{\rho'}{\rho} = \frac{1}{1 + \gamma\Delta\theta}$$

Here, $\gamma =$ thermal coefficient of volume expansion

and $\Delta\theta =$ rise in temperature

$\therefore$

$$\boxed{\rho' = \frac{\rho}{1 + \gamma\Delta\theta}}$$

We can also write

$$\rho' = \rho(1 + \gamma\Delta\theta)^{-1}$$

or

$$\boxed{\rho' \approx \rho(1 - \gamma\Delta\theta)} \qquad (\text{if } \gamma\Delta\theta << 1)$$

Effect of Pressure on Density

As pressure is increased, volume decreases and hence density will increase. Thus,

$$\rho \propto \frac{1}{V}$$

$$\therefore \qquad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V+dV} = \frac{V}{V - \left(\frac{dp}{B}\right)V}$$

or

$$\frac{\rho'}{\rho} = \frac{1}{1 - \frac{dp}{B}}$$

Here, $dp =$ change in pressure

and $B =$ bulk modulus of elasticity of the liquid

Therefore,

$$\boxed{\rho' = \frac{\rho}{1 - \frac{dp}{B}}}$$

$$\Rightarrow \qquad \rho' = \rho\left(1 - \frac{dp}{B}\right)^{-1}$$

or

$$\rho' \approx \rho\left(1 + \frac{dp}{B}\right) \qquad (\text{if } dp << B)$$

Further,

$$\boxed{\rho' - \rho = \Delta\rho = \frac{\rho(dp)}{B} \text{ or } \frac{\rho(\Delta p}{B}}$$

Example 16.1 *Relative density of an oil is* 0.8. *Find the absolute density of oil in CGS and SI units.*

Solution Density of oil (in CGS) = (RD) g/cm^3

$$= 0.8 \text{ g/cm}^3$$

$$= 800 \text{ kg/m}^3$$

Ans.

16.3 Pressure in a Fluid

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid.While the fluid as a whole is at rest, the molecules that makes up the fluid are in motion, the force exerted by the fluid is due to molecules colliding with their surroundings.

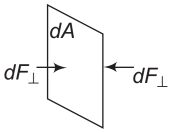


Fig. 16.2

If we think of an imaginary surface within the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface, otherwise the surface would accelerate and the fluid would not remain at rest.

Consider a small surface of area dA centered on a point on the fluid, the normal force exerted by the fluid on each side is $dF_\perp$. The pressure p is defined at that point as the normal force per unit area, i.e.

$$\boxed{p = \frac{dF_\perp}{dA}}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_\perp}{A}$$

where, $F_\perp$ is the normal force on one side of the surface. The SI unit of pressure is pascal, where

$$1 \text{ pascal} = 1 \text{ Pa} = 1.0 \text{ N/m}^2$$

One unit used principally in meterology is the Bar which is equal to 10^5 Pa.

$$1 \text{ Bar} = 10^5 \text{ Pa}$$

Atmospheric Pressure (p_0)

It is pressure of the earth's atmosphere. This changes with weather and elevation. Normal atmospheric pressure at sea level (an average value) is 1.013×10^5 Pa. Thus,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

Note *Fluid pressure acts perpendicular to any surface in the fluid no matter how that surface is oriented. Hence, pressure has no intrinsic direction of its own, it is a scalar. By contrast, force is a vector with a definite direction.*

Absolute Pressure and Gauge Pressure

The excess pressure above atmospheric pressure is usually called gauge pressure and the total pressure is called absolute pressure. Thus,

$$\boxed{\text{Gauge pressure} = \text{absolute pressure} - \text{atmospheric pressure}}$$

Absolute pressure is always greater than or equal to zero. While gauge pressure can be negative also.

Variation in Pressure with Depth

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. But often the fluid's weight is not negligible and under such condition pressure increases with increasing depth below the surface.

Let us now derive a general relation between the pressure p at any point in a fluid at rest and the elevation y of that point. We will assume that the density ρ and the acceleration due to gravity g are the same throughout the fluid. If the fluid is in equilibrium, every volume element is in equilibrium.

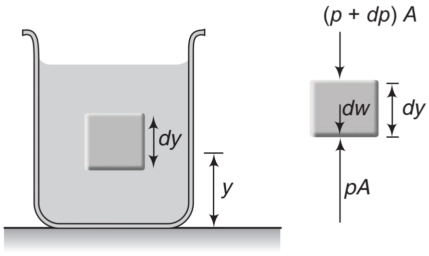


Fig. 16.3

Consider a thin element of fluid with height dy. The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The weight of the fluid element is

$$dw = (\text{volume})\ (\text{density})\ (g) = (A\ dy)(\rho)(g) \quad \text{or} \quad dw = \rho g A dy$$

What are the other forces in y-direction on this fluid element? Call the pressure at the bottom surface p, the total y-component of upward force is pA. The pressure at the top surface is $p + dp$ and the total y-component of downward force on the top surface is $(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force including the weight and the forces at the bottom and top surfaces must be zero.

$$\Sigma F_y = 0$$

$$\therefore \quad pA - (p + dp)A - \rho g A\, dy = 0$$

or
$$\boxed{\frac{dp}{dy} = -\rho g} \qquad \ldots\text{(i)}$$

This equation shows that when y increases, p decreases, i.e. as we move upward in the fluid, pressure decreases.

If p_1 and p_2 be the pressures at elevations y_1 and y_2 and if ρ and g are constant, then integrating Eq. (i), we get

$$\int_{p_1}^{p_2} dp = -\rho g \int_{y_1}^{y_2} dy$$

or
$$\boxed{p_2 - p_1 = -\rho g (y_2 - y_1)} \qquad \ldots\text{(ii)}$$

It is often convenient to express Eq. (ii) in terms of the depth below the surface of a fluid. Take point 1 at depth h below the surface of fluid and let p represents pressure at this point. Take point 2 at the surface of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is,

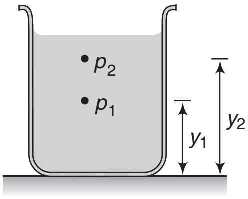


Fig. 16.4

$$h = y_2 - y_1$$

and Eq. (ii) becomes
$$p_0 - p = -\rho g (y_2 - y_1) = -\rho g h$$

$$\therefore \quad \boxed{p = p_0 + \rho g h} \qquad \ldots\text{(iii)}$$

Thus, pressure increases linearly with depth, if ρ and g are uniform. A graph between p and h is shown below.

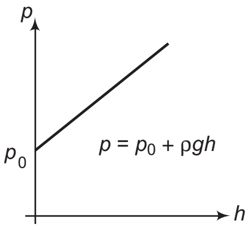


Fig. 16.5

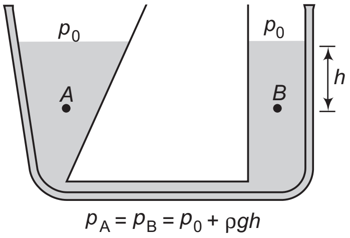


Fig. 16.6

Further, the pressure is the same at any two points at the same level in the fluid. The shape of the container does not matter.

Pascal's Law

Pascal's law or the **principle of transmission of fluid-pressure** is a principle in fluid mechanics that states that pressure exerted anywhere in a confined incompressible fluid is transmitted equally in all directions throughout the fluid.

The simplest instance of this is stepping on a balloon; the balloon bulges out on all sides under the foot and not just on one side. This is precisely what Pascal's law is all about – the air which is the fluid in this case, was confined by the balloon and you applied pressure with your foot causing it to get displaced uniformly.

A well known application of Pascal's law is the hydraulic lift used to support or lift heavy objects. It is schematically illustrated in figure.

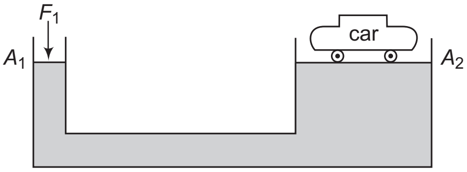


Fig. 16.7

A piston with small cross section area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = \frac{F_1}{A_1}$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

or

$$F_2 = \frac{A_2}{A_1} . F_1$$

Now, since $A_2 > A_1$, therefore, $F_2 > F_1$. Thus, hydraulic lift is a force multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators and hydraulic brakes all use this principle.

Extra Points to Remember

- At same point on a fluid pressure is same in all directions. In the figure,

$$p_1 = p_2 = p_3 = p_4$$

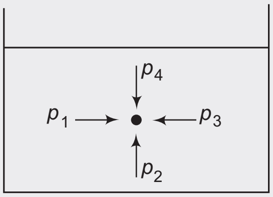


Fig. 16.8

- Forces acting on a fluid in equilibrium have to be perpendicular to its surface. Because it cannot sustain the shear stress.

- In the **same liquid** pressure will be same at all points at the same level (provided their speeds are same).

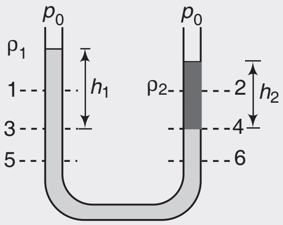


Fig. 16.9

For example, in the figure $p_1 \neq p_2$, $p_3 = p_4$ and $p_5 = p_6$

Further, $p_3 = p_4$

$\therefore$ $P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$

or $\rho_1 h_1 = \rho_2 h_2$ or $\boxed{h \propto \frac{1}{\rho}}$

- **Barometer** It is a device used to measure atmospheric pressure.

  In principle, any liquid can be used to fill the barometer, but mercury is the substance of choice because its great density makes possible an instrument of reasonable size.

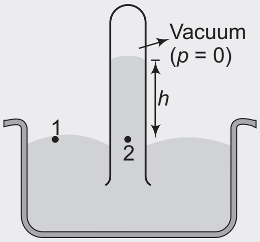


Fig. 16.10

$$p_1 = p_2$$

Here, p_1 = atmospheric pressure (p_0)

and $p_2 = 0 + \rho g h = \rho g h$

Here, ρ = density of mercury

$\therefore$ $p_0 = \rho g h$

Thus, the mercury barometer reads the atmospheric pressure (p_0) directly from the height of the mercury column. For example, if the height of mercury in a barometer is 760 mm, then atmospheric pressure will be

$$p_0 = \rho g h = (13.6 \times 10^3)(9.8)(0.760) = 1.01 \times 10^5 \text{ N/m}^2$$

- **Manometer** It is a device used to measure the pressure of a gas inside a container.

  The U-shaped tube often contains mercury.

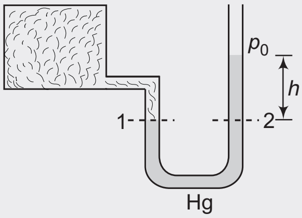


Fig. 16.11

$$\boxed{p_1 = p_2}$$

Here, p_1 = pressure of the gas in the container (p)

and p_2 = atmospheric pressure (p_0) + $\rho g h$

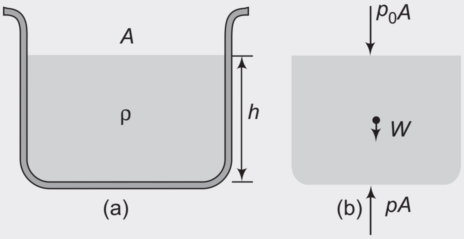
$\therefore$ $\boxed{p = p_0 + h\rho g}$

This can also be written as $\boxed{p - p_0 = \text{gauge pressure} = h\rho g}$

Here, ρ is the density of the liquid used in U-tube.

Thus, by measuring h we can find absolute (or gauge) pressure in the vessel.

- **Free body diagram of a liquid** The free body diagram of the liquid (showing the vertical forces only) is shown in Fig. 16.12 (b). For the equilibrium of liquid,



(a) (b)

Fig. 16.12

net downward force = net upward force

$\therefore$ $p_0 A + W = pA$ here, $W = \rho g h A$ $\therefore$ $p = p_0 + \rho g h$

Example 16.2 *For the arrangement shown in the figure, what is the density of oil ?*

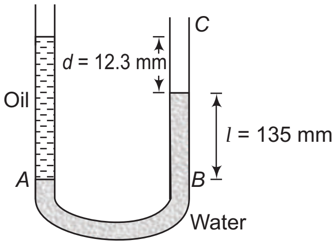


Fig. 16.13

Solution $p_B = p_A$

∴ $p_0 + \rho_w gl = p_0 + \rho_{\text{oil}} (l + d) g$

⇒ $\rho_{\text{oil}} = \dfrac{\rho_w l}{l + d} = \dfrac{(1000)(135)}{(135 + 12.3)} = 916 \text{ kg/m}^3$ **Ans.**

Example 16.3 *A U-tube of uniform cross-sectional area and open to the atmosphere is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in figure with $h_2 = 1.0$ cm, determine the value of h_1.*

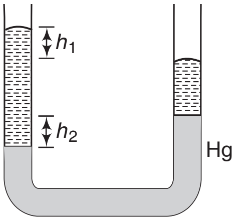


Fig. 16.14

Solution $p_1 = p_2$

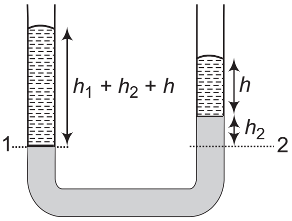


Fig. 16.15

∴ $p_0 + \rho_w g (h_1 + h_2 + h) = p_0 + \rho_w gh + \rho_{\text{Hg}} gh_2$

∴ $$h_1 = \frac{(\rho_{\text{Hg}} - \rho_w) h_2}{\rho_w}$$

$$= \frac{(13.6 - 1)(1.0)}{1}$$

$$= 12.6 \text{ cm}$$ **Ans.**

Example 16.4 *A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm.*

(a) What is the gauge pressure at the water mercury interface ?

(b) Calculate the vertical distance h from the top of the mercury in the right hand arm of the tube to the top of the water in the left-hand arm.

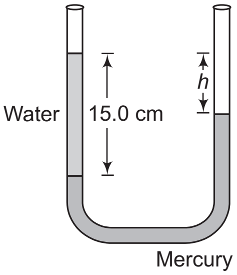


Fig. 16.16

Solution (a) Gauge pressure $= \rho_w g h_w$

$$= (10^3)(9.8)(0.15)$$

$$= 1470 \text{ N/m}^2$$ **Ans.**

(b) Let us calculate pressure or two sides at the level of water and mercury interface.

$$p_0 + \rho_w g\, h_w = p_0 + \rho_{\text{Hg}}\, g h_{\text{Hg}}$$

$$\rho_w h_w = \rho_{\text{Hg}}\, h_{\text{Hg}}$$

$$\therefore \quad (1)(15) = (13.6)(15 - h)$$

$$h = 13.9 \text{ cm}$$ **Ans.**

Example 16.5 *For the system shown in figure, the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm^2. The piston on the right, at S, has cross-sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil ($\rho = 0.78$ g/cm^3), what is the force F required to hold the system in equilibrium?*

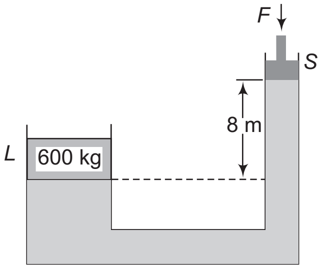


Fig. 16.17

Solution Let A_1 = area of cross-section on LHS and

A_2 = area of cross-section on RHS

Equating the pressure on two sides of the dotted line.

Then, $$\frac{F}{A_2} + \rho g h = \frac{Mg}{A_1} \qquad (M = 600 \text{ kg})$$

$$\therefore \quad F = A_2 \left[\frac{Mg}{A_1} - \rho g h\right]$$

$$= (25 \times 10^{-4}) \left[\frac{600 \times 9.8}{800 \times 10^{-4}} - 780 \times 9.8 \times 8\right]$$

$$= 31 \text{ N}$$ **Ans.**

INTRODUCTORY EXERCISE 16.1

1. Water and oil are poured into the two limbs of a U-tube containing mercury. The interfaces of the mercury and the liquids are at the same height in both limbs.

   Determine the height of the water column h_1 if that of the oil $h_2 = 20$ cm. The density of the oil is 0.9.

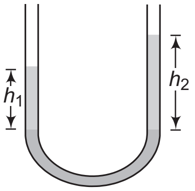


Fig. 16.18

2. The liquids shown in figure in the two arms are mercury (specific gravity = 13.6) and water. If the difference of heights of the mercury columns is 2 cm, find the height h of the water column.

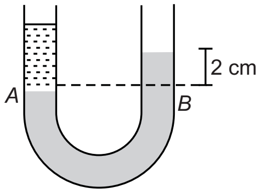


Fig. 16.19

3. The heights of mercury surfaces in the two arms of the manometer shown in Fig.16.20 are 2 cm and 8 cm. Atmospheric pressure is $1.01 \times 10^5 \text{Nm}^{-2}$.

   Find (a) the pressure of the gas in the cylinder and (b) the pressure of mercury at the bottom of the U tube.

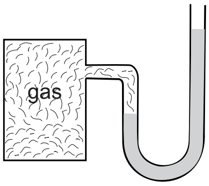


Fig. 16.20

4. A cylinderical vessel containing a liquid is closed by a smooth piston of mass m as shown in the Fig. 16.21. The area of cross section of the piston is A. If the atmospheric pressure is p_0, find the pressure of the liquid just below the piston.

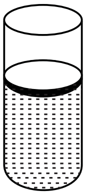


Fig. 16.21

5. The area of cross section of the two arms of a hydraulic press are 1 cm^2 and 10 cm^2 respectively (Fig.16.22). A force of 5 N is applied on the water in the thinner arm. What force should be applied on the water in the thicker arm so that the water may remain in equilibrium?

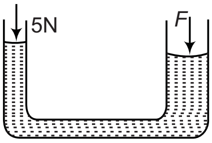


Fig. 16.22

6. The area of cross section of the wider tube shown in Fig. (16.23) is 900 cm^2. If the body standing on the position weighs 45 kg, find the difference in the levels of water in the two tubes.

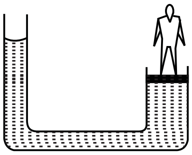


Fig. 16.23

16.4 Pressure Difference in Accelerating Fluids

If fluids are at rest then pressure does not change in horizontal direction.

It changes only in vertical direction.

At a height difference 'h', difference in pressure (or change in pressure) is

Pressure increases with depth. So, $\Delta p = +\rho gh$ in moving downwards and $\Delta p = -\rho gh$ in moving upwards.

If fluids are accelerated then pressure changes in both horizontal and vertical directions.

In Horizontal Direction

(i) Pressure decreases in the direction of acceleration.

(ii) At a horizontal distance x change in pressure is

(a = horizontal acceleration)

Take $\Delta p = +\rho ax$, while moving in opposite direction of acceleration. Because pressure increases in the opposite direction of acceleration and take $\Delta P = -\rho ax$ while moving in the direction of acceleration.

In Vertical Direction

Instead of 'g' in the equation, $\Delta p = \pm \rho gh$ take effective value of g or g_e. Thus,

Here, $g_e = g$ if vertical acceleration of fluid is zero.

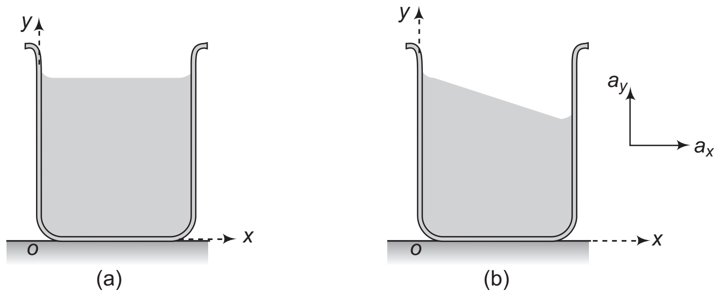
$g_e = g + a$ if vertical acceleration of fluid 'a' is upwards and

$g_e = g - a$ if vertical acceleration 'a' is downwards.

In equation form, all above statements can be explained as below.

Consider a liquid kept at rest in a beaker as shown in Fig. 16.24 (a). In this case, we know that pressure do not change in horizontal direction (x-direction), it decreases upwards along y-direction. So, we can write the equations,

$$\boxed{\frac{dp}{dx} = 0 \qquad \text{and} \qquad \frac{dp}{dy} = -\rho g} \qquad \text{...(i)}$$



(a) (b)

Fig. 16.24

But, suppose the beaker is accelerated and it has components of acceleration a_x and a_y in x and y directions respectively, then the pressure decreases along both x and y-directions. The above equations in that case become,

$$\boxed{\begin{aligned}\frac{dp}{dx} &= -\rho a_x \\ \text{and} \quad \frac{dp}{dy} &= -\rho(g + a_y)\end{aligned}} \qquad \text{...(ii)}$$

These equations can be derived as under :

Consider a beaker filled with some liquid of density ρ accelerating upwards with an acceleration a_y along positive y-direction. Let us draw the free body diagram of a small element of fluid of area A and length dy as shown in figure.

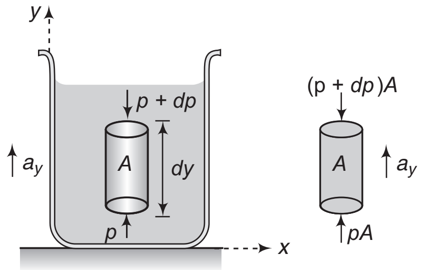


Fig. 16.25

Equation of motion for this fluid element is,

$$pA - W - (p + dp)A = (\text{mass})(a_y) \quad \text{or} \quad -W - (dp)A = (A\rho\, dy)(a_y)$$

$$\text{or} \quad -(A\rho g\, dy) - (dp)A = (A\rho\, dy)(a_y) \quad \text{or} \quad \frac{dp}{dy} = -\rho(g + a_y)$$

Similarly, if the beaker moves along positive x-direction with an acceleration a_x, the equation of motion for the fluid element shown in figure is,

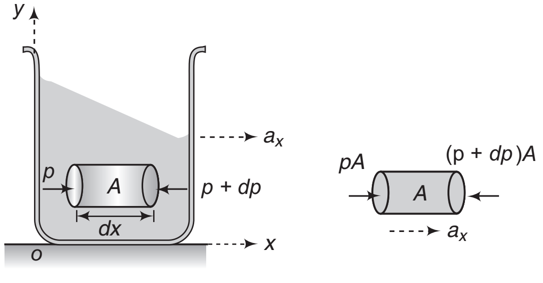


Fig. 16.26

$$pA - (p + dp)A = (\text{mass})(a_x)$$

$$\text{or} \qquad -(dp)A = (A\rho\, dx)a_x$$

$$\text{or} \qquad \frac{dp}{dx} = -\rho a_x$$

Note *(i)* $\frac{dp}{dx} = -\rho a_x$, $\Delta p = -\rho a_x x$ = *pressure difference in a horizontal distance x.*

Here, negative sign implies that pressure decreases with x or in the direction of acceleration.

(ii) In Fig. 16.26, pressure on left hand side of the fluid element should be more than the pressure on right hand side of this element. This is because this element is accelerated only due to this pressure difference.

(iii) $\frac{dp}{dy} = -\rho (g + a_y)$

$\Rightarrow$ $\Delta p = -\rho (g + a_y)\, y$

or $\Delta p = -\rho g_e h$

Here, negative sign implies that pressure decreases with 'y' or in moving upwards.

Free Surface of a Liquid Accelerated in Horizontal Direction

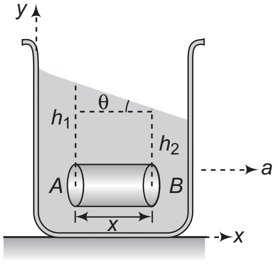


Fig. 16.27

Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration '*a*'. Let *A* and *B* be two points in the liquid at a separation *x* in the same horizontal line. As we have seen in this case

$$\frac{dp}{dx} = -\rho a$$

or $$dp = -\rho a\, dx$$

Integrating this with proper limits, we get

$$p_A - p_B = \rho a x \qquad \ldots\text{(iii)}$$

Further, $$p_A = p_0 + \rho g h_1$$

and $$p_B = p_0 + \rho g h_2$$

Substituting in Eq. (iii), we get

$$\rho g (h_1 - h_2) = \rho a x$$

$\therefore$ $$\frac{h_1 - h_2}{x} = \frac{a}{g} = \tan\theta$$

$\therefore$ $$\boxed{\tan\theta = \frac{a}{g}}$$

Pressure Difference in Rotating Fluids

In a rotating fluid (also accelerating) pressure increases in moving away from the rotational axis. At a distance '*x*' from the rotational axis, pressure difference is

$$\boxed{\Delta p = \pm \frac{\rho \omega^2 x^2}{2}}$$

Take $\Delta p = +\frac{\rho \omega^2 x^2}{2}$ in moving away from the rotational axis, as pressure increases in this direction

and take $\Delta p = -\frac{\rho \omega^2 x^2}{2}$ in moving towards the rotational axis.

Proof

Suppose that liquid of density ρ kept inside a tube of area of cross-section A is rotating with an angular velocity 'ω' as shown.

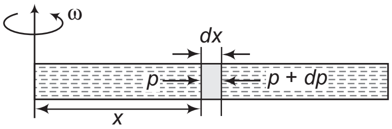


Fig. 16.28

Consider a small element of length 'dx' at a distance x from the axis of rotation. Mass of this element is,

$$dm = (\text{density})\,(\text{volume})$$

or $$dm = (\rho\, A dx)$$

This element is rotating in a circle of radius 'x'. So, this is accelerated towards centre with a centripetal acceleration

$$a = x\omega^2 \qquad (\text{as } a = R\omega^2)$$

To provide this acceleration, pressure on right hand side of the element should be more.

∴ $$(p+dp)A-(p)A = (dm)\,a = (\rho\, Adx)\,(x\omega^2)$$

or $$dp = (\rho\, x\,\omega^2)\,dx$$

∴ $$\Delta p = \int_0^x (\rho\omega^2)\,xdx$$

or $$\Delta p = \frac{\rho\,\omega^2 x^2}{2}$$ **Hence proved**.

Example 16.6 *A closed container shown in figure is filled with water* $(\rho = 10^3\,kg/m^3)$

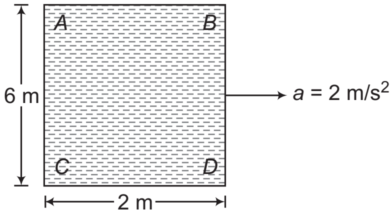


Fig. 16.29

This is accelerated in horizontal direction with an acceleration, $a = 2\,m/s^2$. *Find (a)* $p_C - p_D$ *and (b)* $p_A - p_D$

Solution (a) In horizontal direction, pressure decreases in the direction of acceleration.

Thus, $$p_C > p_D$$

or $$p_C - p_D = +\rho\, ax$$

Substituting the values, we have

$$p_C - p_D = (10^3)(2)(2)$$

or $$p_C - p_D = 4.0 \times 10^3 \text{ N/m}^2$$ **Ans.**

(b) In vertical direction, pressure increases with depth.

$\therefore$ $$p_C > p_A$$

or $$p_A - p_C = -\rho g h \qquad (g_e = g)$$

$$= -(10^3)(10)(6)$$

$$= -60 \times 10^3 \text{ N/m}^2$$

Now, $$p_A - p_D = (p_A - p_C) + (p_C - p_D)$$

$$= (-60 \times 10^3) + (4.0 \times 10^3)$$

$$= -56 \times 10^3 \text{ N/m}^2$$

$$= -5.6 \times 10^4 \text{ N/m}^2$$ **Ans.**

Note *Container is closed. So, nowhere inside the container pressure is atmospheric pressure p_0.*

Example 16.7 *A closed tube is filled with*

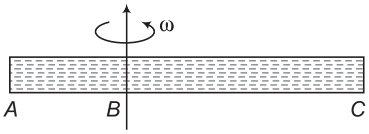


Fig. 16.30

$$AB = 2 \text{ m}$$
$$BC = 4 \text{ cm}$$

water ($\rho = 10^3$ *kg/m*^{3}). *It is rotating about an axis shown in figure with an angular velocity* $\omega = 2$ *rad/s. Find,* $p_A - p_C$.

Solution Pressure decreases in moving towards the axis of rotation and increases in moving away from the axis $\left(\Delta p = \pm \frac{\rho \omega^2 x^2}{2}\right)$

$\therefore$ $$p_A > p_B \text{ and } p_B < p_C$$

$$p_A - p_C = (p_A - p_B) + (p_B - p_C)$$

$$= \left(+\frac{\rho \omega^2 x_1^2}{2}\right) + \left(\frac{-\rho \omega^2 x_2^2}{2}\right)$$

Here, $$x_1 = AB = 2 \text{ m}$$

and $$x_2 = BC = 4 \text{ m}$$

Substituting the values we have,

$$p_A - p_C = \frac{(10^3)(2)^2\,(2)^2}{2} - \frac{(10^3)(2)^2\,(4)^2}{2}$$

$$= -2.4 \times 10^3\ \text{N/m}^2$$ **Ans.**

Example 16.8 *A liquid of density* ρ *is in a bucket that spins with angular velocity 'ω' as shown in figure. Prove that the free surface of the liquid has a parabolic shape. Find equation of this.*

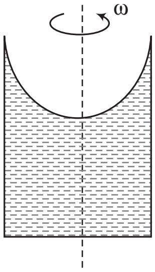


Fig. 16.31

Solution In the figure shown, suppose the coordinates of point M are (x, y) with respect to the coordinate axes.

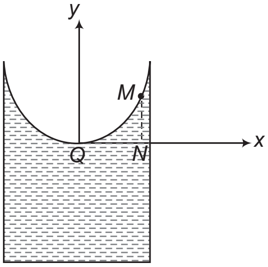


Fig. 16.32

Then, $MN = y,\ NQ = x$

Points M and Q are open to atmosphere.

$\therefore$ $p_M = p_Q =$ atmospheric pressure p_0

Now, $p_M - p_N = -\rho g y$

$\Rightarrow$ $p_0 - p_N = -\rho g y$ ...(i)

$$p_N - p_Q = \frac{\rho\omega^2 x^2}{2}$$

$\therefore$ $$p_N - p_0 = \frac{\rho\omega^2 x^2}{2}$$ ...(ii)

Adding Eqs. (i) and (ii), we have

$$0 = -\rho g y + \frac{\rho \omega^2 x^2}{2}$$

$$y = \frac{\omega^2 x^2}{2g}$$

This is the required equation of free surface of the liquid and we can see that this is an equation of a parabola.

INTRODUCTORY EXERCISE 16.2

1. In example 16.6, which point has the maximum pressure and which has the minimum pressure.
2. A cubical closed vessel of side 5 m filled with a liquid is accelerated with an acceleration a. Find the value of a so that pressure at mid point M of AC is equal to pressure at N.

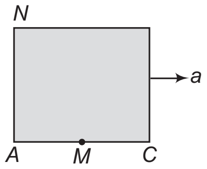


Fig. 16.33

3. Water $(\rho = 10^3\ \text{kg/m}^3)$ is filled in tube AB as shown in figure. $\omega = 10$ rad/s. Tube is open at end A. Atmospheric pressure is $p_0 = 10^5\ \text{N/m}^2$. Find absolute pressure at end B.

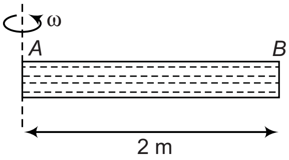


Fig. 16.34

16.5 Archimedes' Principle

If a heavy object is immersed in water, it seems to weightless than when it is in air. This is because the water exerts an upward force called **buoyant force.** It is equal to the weight of the fluid displaced by the body.

A body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.

This result is known as **Archimedes' principle.**

Thus, the magnitude of buoyant force (F) is given by,

$$\boxed{F = V_i \rho_L g}$$

Here, V_i = immersed volume of solid, ρ_L = density of liquid

and g = acceleration due to gravity

Proof

Consider an arbitrarily shaped body of volume V placed in a container filled with a fluid of density ρ_L. The body is shown completely immersed, but complete immersion is not essential to the proof. To begin with, imagine the situation before the body was immersed. The region now occupied by the body was filled with fluid, whose weight was $V\rho_L g$. Because the fluid as a whole was in hydrostatic equilibrium, the net upwards force (due to difference in pressure at different depths) on the fluid in that region was equal to the weight of the fluid occupying that region.

Now, consider what happens when the body has displaced the fluid. The pressure at every point on the surface of the body is unchanged from the value at the same location when the body was not present. This is because the pressure at any point depends only on the depth of that point below the fluid surface. Hence, the net force exerted by the surrounding fluid on the body is exactly the same as that exerted on the region before the body was present. But we know the latter to be $V\rho_L g$, the weight of the displaced fluid. Hence, this must also be the buoyant force exerted on the body. Archimedes' principle is thus proved.

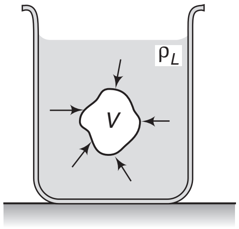


Fig. 16.35

Law of Floatation

Consider an object of volume V and density ρ_S floating in a liquid of density ρ_L. Let V_i be the volume of object immersed in the liquid.

For equilibrium of object,

$$\text{Weight} = \text{Upthrust}$$

$$\therefore \quad V\rho_S g = V_i \rho_L g$$

$$\therefore \quad \frac{V_i}{V} = \frac{\rho_S}{\rho_L} \qquad \ldots\text{(i)}$$

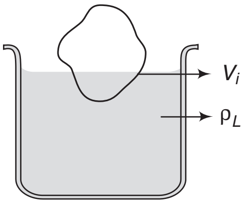


Fig. 16.36

This is the fraction of volume immersed in liquid.

Percentage of volume immersed in liquid $= \dfrac{V_i}{V} \times 100 = \dfrac{\rho_S}{\rho_L} \times 100$

Three possibilities may now arise:

(i) If $\rho_S < \rho_L$, only fraction of body will be immersed in the liquid. This fraction will be given by the above equation.

(ii) If $\rho_S = \rho_L$, the whole of the rigid body will be immersed in the liquid. Hence, the body remains floating in the liquid wherever it is left.

(iii) If $\rho_S > \rho_L$, the body will sink.

Buoyant Force in Accelerating Fluids

Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with an acceleration **a**. The buoyant force F in this case becomes,

$$\boxed{F = V\rho_L g_{\text{eff}}} \qquad (V = \text{immersed volume of solid or } V_i)$$

Here, $$g_{\text{eff}} = |\mathbf{g} - \mathbf{a}|$$

For example, if the lift has an upward acceleration a, the value of g_{eff} is $g + a$ and if it has a downward acceleration a, the g_{eff} is $g - a$. In a freely falling lift g_{eff} is zero (as $a = g$) and hence, net buoyant force is zero. This is why, in a freely falling vessel filled with some liquid, the air bubbles do not rise up (which otherwise move up due to buoyant force). The above result can be derived as follows.

Suppose a body is dipped inside a liquid of density ρ_L in an elevator moving up with an acceleration a. As was done earlier also, replace the body into the liquid by the same liquid of equal volume. The replaced liquid is at rest with respect to the elevator. Thus, this replaced liquid is also moving up with an acceleration a together with the rest of the liquid.

The forces acting on the replaced liquid are,

(i) the buoyant force F and

(ii) the weight mg of the substituted liquid.

From Newton's second law,

$$F - mg = ma \quad \text{or} \quad F = m(g + a)$$

Here, $m = V\rho_L$

$$\therefore \qquad F = V\rho_L (g + a) = V\rho_L g_{\text{eff}}$$

where $$g_{\text{eff}} = g + a$$

Extra Points to Remember

- In weight of a solid, take total volume of solid, density of solid and g. In upthrust (or buoyant force) take immersed volume of solid, density of liquid and g_{eff}. Thus,

$$\boxed{\begin{aligned} W &= V\rho_s g \\ F &= V_i \rho_L g_{\text{eff}} \end{aligned}}$$

- Upthrust force also makes a pair of equal and opposite forces. On solid it is upwards and on liquid, it is downwards.

Example 16.9 *Density of ice is* $900\ kg/m^3$. *A piece of ice is floating in water of density* $1000\ kg/m^3$. *Find the fraction of volume of the piece of ice outside the water.*

Solution Let V be the total volume and V_i the volume of ice piece immersed in water. For equilibrium of ice piece,

$$\text{weight} = \text{upthrust}$$

$$\therefore \qquad V\rho_i g = V_i \rho_w g$$

Here, $$\rho_i = \text{density of ice} = 900\ \text{kg/m}^3$$

and $$\rho_w = \text{density of water} = 1000\ \text{kg/m}^3$$

Substituting in above equation, we get

$$\frac{V_i}{V} = \frac{900}{1000} = 0.9$$

i.e. the fraction of volume outside the water,

$$f = 1 - 0.9 = 0.1$$ **Ans.**

Example 16.10 *A metallic sphere floats in an immiscible mixture of water* $(\rho_w = 10^3\ kg/m^3)$ *and a liquid* $(\rho_L = 13.5 \times 10^3\ kg/m^3)$ *such that its* $\frac{4}{5}$ *th volume is in water and* $\frac{1}{5}$ *th volume in the liquid. Find the density of metal.*

Solution Total upthrust = weight of metal sphere

$$\therefore \qquad \left(\frac{4}{5}V\right)(10^3)g + \left(\frac{1}{5}V\right)(13.5 \times 10^5)g = (V)(\rho_{\text{metal}})(g)$$

$$\therefore \qquad \rho_{\text{metal}} = 3.5 \times 10^3\ \text{kg/m}^3$$ **Ans.**

Example 16.11 *A block of mass 1 kg and density* $0.8\ g/cm^3$ *is held stationary with the help of a string as shown in figure. The tank is accelerating vertically upwards with an acceleration* $a = 1.0\ m/s^2$. *Find*

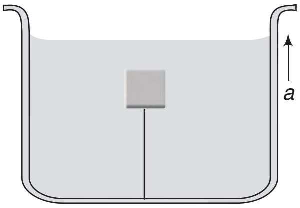


Fig. 16.37

(a) the tension in the string,
(b) if the string is now cut find the acceleration of block.
(Take $g = 10\ m/s^2$ *and density of water =* $10^3\ kg/m^3$*).*

Solution (a) Free body diagram of the block is shown in Fig. 16.38.

In the figure,

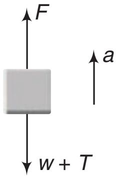


Fig. 16.38

$$F = \text{upthrust force}$$

$$= V\rho_\omega(g + a)$$

$$= \left(\frac{\text{mass of block}}{\text{density of block}}\right)\rho_\omega\ (g + a)$$

$$= \left(\frac{1}{800}\right)(1000)(10 + 1) = 13.75\ \text{N}$$

$$w = mg = 10\ \text{N}$$

Equation of motion of the block is,

$$F - T - w = ma$$

$$\therefore \quad 13.75 - T - 10 = 1 \times 1$$

$$\therefore \quad T = 2.75 \text{ N}$$ **Ans.**

(b) When the string is cut, $T = 0$

$$\therefore \quad a = \frac{F - w}{m}$$

$$= \frac{13.75 - 10}{1}$$

$$= 3.75 \text{ m/s}^2$$ **Ans.**

Example 16.12 *The tension in a string holding a solid block below the surface of a liquid (of density greater than that of solid) as shown in figure is T_0 when the system is at rest. What will be the tension in the string if the system has an upward acceleration a ?*

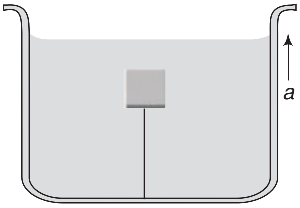


Fig. 16.39

Solution Let m be the mass of block.

Initially for the equilibrium of block,

$$F = T_0 + mg \qquad \text{...(i)}$$

Here, F is the upthrust on the block.

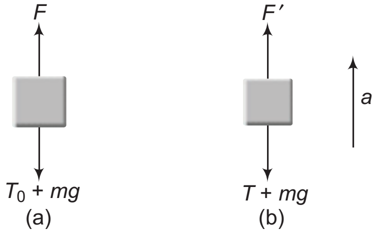


Fig. 16.40

When the lift is accelerated upwards, g_{eff} becomes $g + a$ instead of g. Hence,

$$F' = F\left(\frac{g + a}{g}\right) \qquad \text{...(ii)}$$

From Newton's second law,

$$F' - T - mg = ma \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$T = T_0\left(1 + \frac{a}{g}\right) \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 16.3

1. A block of material has a density ρ_1 and floats three-fourth submerged in a liquid of unknown density. Show that the density ρ_2 of the unknown liquid is given by $\rho_2 = \frac{4}{3}\rho_1$.
2. A block of wood weighing 71.2 N and of specific gravity 0.75 is tied by a string to the bottom of a tank of water in order to have the block totally immersed. What is the tension in the string?
3. A beaker when partly filled with water has total mass 20.00 g. If a piece of metal with density $3.00\ \text{g/cm}^3$ and volume $1.00\ \text{cm}^3$ is suspended by a thin string, so that it is submerged in the water but does not rest on the bottom of the beaker, how much does the beaker then appear to weigh if it is resting on a scale?
4. A small block of wood of density $0.4 \times 10^3\ \text{kg/m}^3$ is submerged in water at a depth of 2.9 m. Find
   (a) the acceleration of the block towards the surface when the block is released and
   (b) the time for the block to reach the surface. Ignore viscosity.

16.6 Flow of Fluids

Steady Flow

If the velocity of fluid particles at any point does not vary with time, the flow is said to be steady. Steady flow is also called streamlined or laminar flow. The velocity at different points may be different. Hence, in the figure,

$$\mathbf{v}_1 = \text{constant},\ \mathbf{v}_2 = \text{constant},\ \mathbf{v}_3 = \text{constant but } \mathbf{v}_1 \neq \mathbf{v}_2 \neq \mathbf{v}_3$$

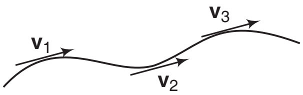


Fig 16.41

Principle of Continuity

It states that, when an incompressible and non-viscous liquid flows in a stream lined motion through a tube of non-uniform cross-section, then the product of the area of cross section and the velocity of flow is same at every point in the tube.

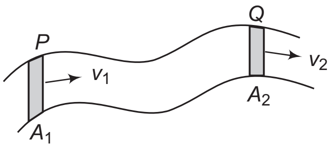


Fig. 16.42

Thus, $\boxed{A_1 v_1 = A_2 v_2}$ or $Av = \text{constant}$ or $v \propto \frac{1}{A}$

This is basically the law of conservation of mass in fluid dynamics.

Proof

Let us consider two cross sections P and Q of area A_1 and A_2 of a tube through which a fluid is flowing. Let v_1 and v_2 be the speeds at these two cross sections. Then, being an incompressible fluid, mass of fluid going through P in a time interval Δt = mass of fluid passing through Q in the same interval of time Δt.

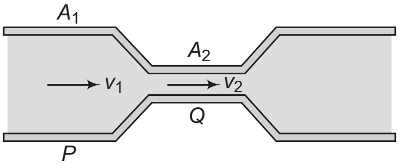


Fig. 16.43

$\therefore$ $$A_1 v_1 \rho \Delta t = A_2 v_2 \rho \, \Delta t$$

or $$A_1 v_1 = A_2 v_2$$ **Proved.**

Therefore, the velocity of the liquid is smaller in the wider parts of the tube and larger in the narrower parts.

or $$v_2 > v_1 \quad \text{as} \quad A_2 < A_1$$

Note *The product Av is the volume flow rate* $\frac{dV}{dt}$*, the rate at which volume crosses a section of the tube.*

Hence,

$$\boxed{\frac{dV}{dt} = \text{volume flow rate} = Av}$$

The mass flow rate is the mass flow per unit time through a cross-section. This is equal to density (ρ) times the volume flow rate $\frac{dV}{dt}$.

We can generalize the continuity equation for the case in which the fluid is not incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2 then,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

So, this is the continuity equation for a compressible fluid.

Bernoulli's Equation

Bernoulli's equation relates the pressure, flow speed and height for flow of an ideal (incompressible and non-viscous) fluid. The pressure of a fluid depends on height as in the static situation and it also depends on the speed of flow.

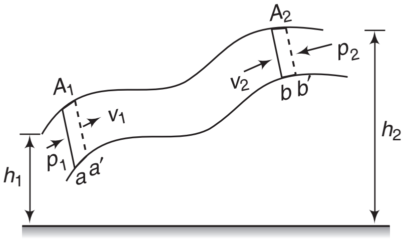


Fig. 16.44

The dependence of pressure on speed can be understood from the continuity equation. When an incompressible fluid flows along a tube with varying cross section, its speed must change, and so, an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure must be different in regions of different cross section.

When a horizontal flow tube narrows and a fluid element speeds up, it must be moving towards a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes additional pressure difference.

To derive Bernoulli's equation, we apply the work-energy theorem to the fluid in a section of the fluid element. Consider the element of fluid that at some initial time lies between two cross sections *a* and *b*. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval, the fluid that is initially at *a* moves to a' a distance $aa' = ds_1 = v_1\ dt$ and the fluid that is initially at *b* moves to b' a distance $bb' = ds_2 = v_2\ dt$. The cross-section areas at the two ends are A_1 and A_2 as shown. The fluid is incompressible, hence, by the continuity equation, the volume of fluid dV passing through any cross-section during time dt is the same.

That is, $$dV = A_1 ds_1 = A_2 ds_2$$

Work Done on the Fluid Element

Let us calculate the work done on this fluid element during time interval *dt*. The pressure at the two ends are p_1 and p_2, the force on the cross section at *a* is $p_1 A_1$ and the force at *b* is $p_2 A_2$. The net work done dW on the element by the surrounding fluid during this displacement is,

$$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = (p_1 - p_2) dV \qquad \text{...(i)}$$

The second term is negative, because the force at *b* opposes the displacement of the fluid.

This work dW is due to forces other than the conservative force of gravity, so it equals the change in total mechanical energy (kinetic plus potential). The mechanical energy for the fluid between sections *a* and *b* does not change.

Change in Potential Energy

At the beginning of *dt* the potential energy for the mass between *a* and a' is $dmgh_1 = \rho dVgh_1$. At the end of *dt* the potential energy for the mass between *b* and b' is $dmgh_2 = \rho dVgh_2$. The net change in potential energy dU during *dt* is,

$$dU = \rho(dV)g(h_2 - h_1) \qquad \text{...(ii)}$$

Change in Kinetic Energy

At the beginning of *dt* the fluid between *a* and a' has volume $A_1 ds_1$, mass $\rho A_1 ds_1$ and kinetic energy $\frac{1}{2}\rho(A_1 ds_1)v_1^2$. At the end of *dt* the fluid between *b* and b' has kinetic energy $\frac{1}{2}\rho(A_2 ds_2)v_2^2$. The net change in kinetic energy dK during time *dt* is,

$$dK = \frac{1}{2}\rho(dV)(v_2^2 - v_1^2) \qquad \text{...(iii)}$$

Combining Eqs. (i), (ii) and (iii) in the energy equation,

$$dW = dK + dU$$

We obtain,

$$(p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho(dV)g(h_2 - h_1)$$

or
$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \qquad \ldots\text{(iv)}$$

This is Bernoulli's equation. It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We can also express Eq. (iv) in a more convenient form as,

$$\boxed{p_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2} \qquad \text{Bernoulli's equation}$$

The subscripts 1 and 2 refer to any two points along the flow tube, so we can also write

$$\boxed{p + \rho g h + \frac{1}{2}\rho v^2 = \text{constant}}$$

✓ Extra Points to Remember

- In Bernoulli's equation, there are three terms; p, $\frac{1}{2}\rho v^2$ and $\rho g h$. Under following three cases, this equation reduces to a two term Bernoulli.

Case 1 If all points are open to atmosphere then pressure at every point may be assumed to be constant $(= p_0)$ and the Bernoulli equation can be written as,

$$\frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

At greater heights 'h', speed 'v' will be less as ρ and g are constants.

Case 2 If the liquid is passing through a pipe of uniform cross-section, then from continuity equation (Av=constant), speed v is same at all points. Therefore, the Bernoulli equation becomes,

$$p + \rho g h = \text{constant}$$

or
$$p_1 + \rho g h_1 = p_2 + \rho g h_2$$

or
$$p_1 - p_2 = \rho g (h_2 - h_1)$$

This is the pressure relation we have already derived for a fluid at rest or pressure decreases with height of liquid and increases with depth of liquid.

Case 3 If a liquid is flowing in a horizontal pipe, then height 'h' of the liquid at every point may be assumed to be constant. So, the two term Bernoulli becomes,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

From this equation, we may conclude that pressure decreases at a point where speed increases.

Example 16.13 *Water is flowing through a horizontal tube of non-uniform cross section. At a place, the radius of the tube is* 1.0 *cm and the velocity of water is* 2 *m/s. What will be the velocity of water where the radius of the pipe is* 2.0 *cm*?

Solution Using equation of continuity, $A_1 v_1 = A_2 v_2$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

or $$v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right) v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$$

Substituting the values, we get

$$v_2 = \left(\frac{1.0 \times 10^{-2}}{2.0 \times 10^{-2}}\right)^2 (2)$$

or $$v_2 = 0.5 \text{ m/s}$$ **Ans.**

Example 16.14 *Calculate the rate of flow of glycerine of density* 1.25×10^3 *kg/m*^{3} *through the conical section of a pipe, if the radii of its ends are* 0.1 *m and* 0.04 *m and the pressure drop across its length is* 10 *N/m*^{2}.

Solution From continuity equation,

$$A_1 v_1 = A_2 v_2$$

or $$\frac{v_1}{v_2} = \frac{A_2}{A_1} = \frac{\pi r_2^2}{\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$$

$$= \left(\frac{0.04}{0.1}\right)^2 = \frac{4}{25} \qquad \text{...(i)}$$

From Bernoulli's equation,

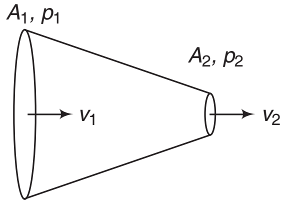


Fig. 16.45

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \text{ or } v_2^2 - v_1^2 = \frac{2(p_1 - p_2)}{\rho}$$

or $$v_2^2 - v_1^2 = \frac{2 \times 10}{1.25 \times 10^3} = 1.6 \times 10^{-2} \text{ m}^2/\text{s}^2 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $v_2 \approx 0.128$ m/s

∴ Rate of volume flow through the tube

$$Q = A_2 v_2 = (\pi r_2^2) v_2$$

$$= \pi (0.04)^2 (0.128)$$

$$= 6.43 \times 10^{-4} \text{ m}^3/\text{s}$$ **Ans.**

Example 16.15 *Water is flowing smoothly through a closed-pipe system. At one point the speed of the water is 3.0 m/s, while at another point 1.0 m higher the speed is 4.0 m/s. If the pressure is 20 kPa at the lower point, what is the pressure at the upper point ? What would the pressure at the upper point be if the water were to stop flowing and the pressure at the lower point were 18 kPa ?*

Solution (i)
$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

$$(20 \times 10^3) + \frac{1}{2} \times 10^3 \times (3)^2 + 0 = p_2 + \frac{1}{2} \times 10^3 \times (4)^2 + 10^3 \times 10 \times 1$$

$$\therefore \quad p_2 = 6.5 \times 10^3 \text{ N/m}^2$$

$$= 6.5 \text{ kPa}$$ **Ans.**

(ii) Again applying the same equation, we have

$$(18 \times 10^3) + 0 + 0 = p_2 + 0 + (10^3)(10)(1)$$

$$\Rightarrow \quad p_2 = 8 \times 10^3 \text{ N/m}^2$$

$$= 8 \text{ kPa}$$ **Ans.**

INTRODUCTORY EXERCISE 16.4

1. Water flows through a tube shown in figure. The areas of cross section at A and B are 1 cm^2 and 0.5 cm^2 respectively. The height difference between A and B is 5 cm. If the speed of water at A is 10 cms^{-1}, find (a) the speed at B and (b) the difference in pressures at A and B.

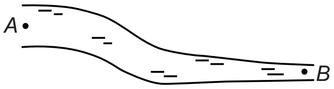


Fig. 16.46

2. Water flows through a horizontal tube of variable cross section as shown in figure. The area of cross section at A and B are 4 mm^2 and 2 mm^2 respectively. If 1 cc of water enters per second through A, find (a) the speed of water at A, (b) the speed of water at B and (c) the pressure difference $p_A - p_B$.

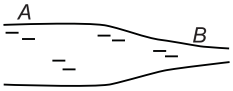


Fig. 16.47

3. Water from a tap emerges vertically downwards with an initial speed of 1.0 m/s. The cross-sectional area of tap is 10^{-4} m^2. Assume that the pressure is constant throughout the stream of water and that the flow is steady, the cross-sectional area of stream 0.15 m below the tap is **(JEE 1998)**

(a) 5.0×10^{-4} m^2 (b) 1.0×10^{-4} m^2

(c) 5.0×10^{-5} m^2 (d) 2.0×10^{-5} m^2

4. A horizontal pipeline carries water in a streamline flow. At a point along the pipe, where the cross-sectional area is 10 cm^2, the water velocity is 1 ms^{-1} and the pressure is 2000 Pa. The pressure of water at another point where the cross-sectional area is 5 cm^2, is ......Pa. (Density of water = 10^3 kg-m^3) **(JEE 1994)**

16.7 Application Based on Bernaulli's Equation

Atomizer or Spray Gun

Fig. 16.48 shows the essential parts of spray gun. When the piston is pressed, the air rushes out of the horizontal tube *B* decreasing the pressure to p_2 which is less than the pressure p_1 in the container. As a result, the liquid rises up in the vertical tube *A*. When it collides with the high speed air in tube *B*, it breaks up into a fine spray. Filter pumps, bunsen burner and sprayers used for perfumes or to spray insecticides work on the same principle.

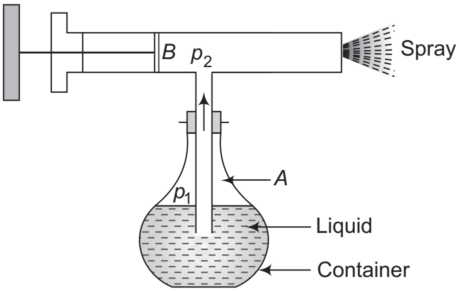


Fig. 16.48

Principle of Lifting of an Aircraft

We all know that airplanes fly. This is a result of a lift force acting on the wings of the aircraft. Lift can be generated when an asymmetric object moves through a fluid. The asymmetry of the object requires the fluid particles to travel different distances along different paths as they pass around the object. The molecules will either speed up or slowdown in order to remain even with the fluid molecules on the other side. For example, in Fig. 16.49, the air particles passing along the top of the body must travel a greater distance than those travelling below.

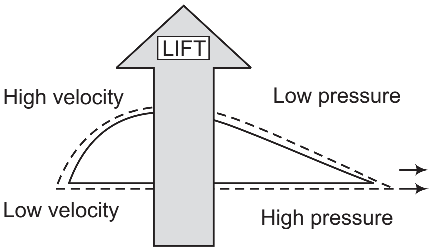


Fig. 16.49. The principle of lift of an aircraft

In order to arrive at the other side of the foil at the same time, these particles must travel more quickly than those travelling below the foil. The difference in the velocities of the fluid molecules results in a pressure difference according to the Bernoulli principle: high velocities are associated with relatively low pressure regions, while low velocities are associated with relatively high pressure regions. This pressure difference produces a force in upward direction.

Magnus Effect and Spinning of a Ball

Spinning objects can also generate force. When an object spins in a fluid medium, the fluid boundary layer spins along with the object. This results in a difference in velocity on two sides. A pressure difference is created from one side of the object to the other and the object curves towards the area of low pressure. This influence of rotation of an object on its flight path is termed the magnus effect. This magnus force changes the path of a rotating cricket ball. This is called spinning of ball.

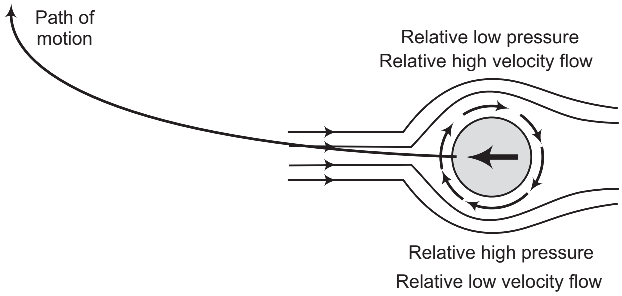


Fig. 16.50. The magnus effect

Blowing Off the Roof during Wind Storm

During certain wind storm or cyclone, the roofs of some houses are blown off without damaging the other parts of the house. The high wind blowing over the roof creates a low pressure p_2 in accordance with Bernoulli's principle. The pressure p_1 below the roof is equal to the atmospheric pressure which is larger than p_2. The difference of pressure $(p_1 - p_2)$ causes an upward thrust and the roof is lifted up. Once the roof is lifted up, it is blown off with the wind.

Venturimeter

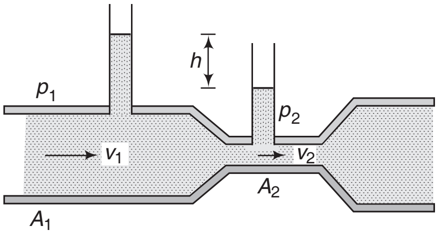


Fig. 16.51

Figure shows a venturimeter used to measure flow speed in a pipe of non-uniform cross-section. We apply Bernoulli's equation to the wide and narrow parts of the pipe, with $h_1 = h_2$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

From the continuity equation $v_2 = \dfrac{A_1 v_1}{A_2}$

Substituting and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right) \qquad \ldots\text{(i)}$$

Because A_1 is greater than A_2, v_2 is greater than v_1 and hence the pressure p_2 is less than p_1. A net force to the right accelerates the fluid as it enters the narrow part of the tube (called throat) and a net force to the left slows as it leaves.

The pressure difference is also equal to ρgh, where h is the difference in liquid level in the two tubes. Substituting in Eq. (i), we get

$$\boxed{v_1 = \sqrt{\dfrac{2gh}{\left(\dfrac{A_1}{A_2}\right)^2 - 1}}}$$

Note *The discharge or volume flow rate can be obtained as,*

$$\frac{dV}{dt} = A_1 v_1 = A_1 \sqrt{\frac{2gh}{\left(\dfrac{A_1}{A_2}\right)^2 - 1}}$$

Speed of Efflux

Suppose, the surface of a liquid in a tank is at a height h from the orifice O on its sides, through which the liquid issues out with velocity v. The speed of the liquid coming out is called the speed of efflux. If the dimensions of the tank be sufficiently large, the velocity of the liquid at its surface may be taken to be zero. Applying Bernoulli's equation at the surface and just outside the orifice.

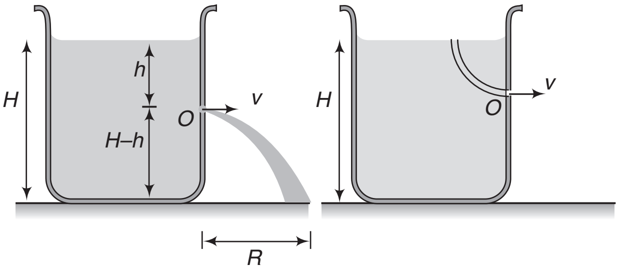


Fig. 16.52

$$\frac{1}{2}\rho v^2 + \rho gh + p = \text{constant}$$

with $h = 0$ at the orifice, we have

$$\rho gh + p_0 = \frac{1}{2}\rho v^2 + p_0 \quad \text{or} \quad \boxed{v = \sqrt{2gh}}$$

Evangelista Torricelli show that this velocity is the same as the liquid will attain in falling freely through the vertical height (h) from the surface to the orifice. This is known as **Torricelli's theorem** and may be stated as, "*The velocity of efflux of a liquid issuing out of an orifice is the same as it would attain if allowed to fall freely through the vertical height between the liquid surface and orifice.*"

Range (*R*)

Let us find the range R on the ground.

Considering the vertical motion of the liquid,

$$(H - h) = \frac{1}{2} gt^2 \quad \text{or} \quad \boxed{t = \sqrt{\frac{2(H-h)}{g}}}$$

Now, considering the horizontal motion,

$$R = vt$$

or $$R = (\sqrt{2gh})\left(\sqrt{\frac{2(H-h)}{g}}\right)$$

or $$\boxed{R = 2\sqrt{h(H-h)}}$$

From the expression of R, following conclusions can be drawn,

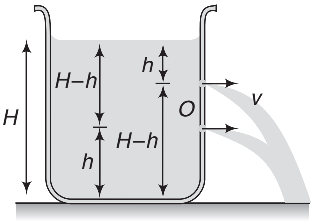


Fig. 16.53

(i) $R_h = R_{H-h}$ as $R_h = 2\sqrt{h(H-h)}$ and $R_{H-h} = 2\sqrt{(H-h)h}$

This is shown in Fig. 16.53.

(ii) R is maximum at $h = \frac{H}{2}$ and $R_{\max} = H$.

Proof $$R^2 = 4(Hh - h^2)$$

For R to be maximum, $\frac{dR^2}{dh} = 0$ or $H - 2h = 0$ or $h = \frac{H}{2}$

That is, R is maximum at $h = \frac{H}{2}$

and $$R_{\max} = 2\sqrt{\frac{H}{2}\left(H - \frac{H}{2}\right)} = H$$ **Hence Proved.**

Time Taken to Empty a Tank

We are here interested in finding the time required to empty a tank if a hole is made at the bottom of the tank.

Consider a tank filled with a liquid of density ρ upto a height H. A small hole of area of cross section a is made at the bottom of the tank. The area of cross-section of the tank is A.

Let at some instant of time the level of liquid in the tank is y. Velocity of efflux at this instant of time would be,

$$v = \sqrt{2gy}$$

Now, at this instant volume of liquid coming out of the hole per second is $\left(\frac{dV_1}{dt}\right)$.

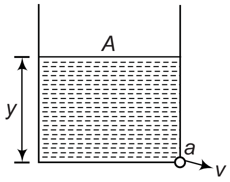


Fig. 16.54

Volume of liquid coming down in the tank per second is $\left(\frac{dV_2}{dt}\right)$.

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$$\therefore \qquad av = A\left(-\frac{dy}{dt}\right)$$

$$\therefore \qquad a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right)$$

or
$$\int_0^t dt = -\frac{A}{a\sqrt{2g}}\int_H^0 y^{-1/2}\,dy \qquad \text{...(i)}$$

$$\therefore \qquad t = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_0^H$$

$$\therefore \qquad \boxed{t = \frac{A}{a}\sqrt{\frac{2H}{g}}}$$

Example 16.16 *Water flows through a horizontal tube as shown in figure . If the difference of heights of water column in the vertical tubes is 2 cm and the areas of cross-section at A and B are* 4cm^2 *and* $2\,\text{cm}^2$ *respectively. Find the rate of flow of water across any section.*

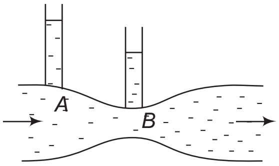


Fig. 16.55

Solution Applying Bernoulli's equation at A and B

$$p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B \qquad (h_A = h_B)$$

or
$$\frac{1}{2}\rho v_B^2 - \frac{1}{2}\rho v_A^2 = \rho g\ (h_A - h_B)$$

or
$$v_B^2 - v_A^2 = 2g\,(h_A - h_B)$$
$$= 2 \times 10 \times 0.02$$

or
$$v_B^2 - v_A^2 = 0.4\ \text{m}^2/\text{s}^2 \qquad \text{...(i)}$$

$$v_A A_A = v_B A_B$$

or
$$4v_A = 2v_B$$

$$\therefore \qquad v_B = 2v_A \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_A = 0.363\ \text{m/s}$$

Volume flow rate $= v_A A_A$

$$= (0.365)\,(4 \times 10^{-4})$$
$$= 1.46 \times 10^{-4}\ \text{m}^3/\text{s} = 146\,\text{cm}^3/\text{s}$$ **Ans.**

Example 16.17 *Water flows through the tube as shown in figure. The areas of cross-section of the wide and the narrow portions of the tube are* $5\ cm^2$ *and* $2\,cm^2$ *respectively. The rate of flow of water through the tube is* $500\ cm^3/s$*. Find the difference of mercury levels in the U-tube.*

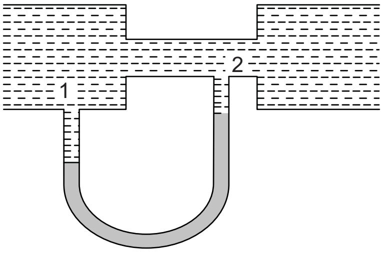


Fig. 16.56

Solution Applying continuity equation

$$A_1v_1 = A_2v_2 = \text{rate of flow of water}$$

$$5v_1 = 2v_2 = 500\,\text{cm}^3/\text{s}$$

$$\therefore \qquad v_1 = 100\ \text{cm/s} = 1.0\ \text{m/s}$$

$$v_2 = 250\,\text{cm/s} = 2.5\ \text{m/s}$$

Now, applying Bernoulli's equation at 1 and 2

$$p_1 + \frac{1}{2}\rho_w v_1^2 + \rho_w g h_1 = p_2 + \frac{1}{2}\rho_w v_2^2 + \rho_w g h_2 \qquad (h_1 = h_2)$$

$$\Rightarrow \qquad \frac{1}{2}\rho_w (v_2^2 - v_1^2) = p_1 - p_2 = \rho_{\text{Hg}} g h_{\text{Hg}}$$

$$\text{or} \qquad \frac{1}{2}\rho_w\ (v_2^2 - v_1^2) = \rho_{\text{Hg}}\ g\ h_{\text{Hg}}$$

$$\therefore \qquad h_{\text{Hg}} = \frac{\rho_w\ (v_2^2 - v_1^2)}{2\ \rho_{\text{Hg}}\ g}$$

$$= \frac{10^3\ (6.25 - 1)}{2 \times 13.6 \times 10^3 \times 9.8}$$

$$= 0.0196\,\text{m}$$

$$= 1.96\,\text{cm}$$ **Ans**

Example 16.18 *A tank is filled with a liquid upto a height H. A small hole is made at the bottom of this tank. Let* t_1 *be the time taken to empty first half of the tank and* t_2 *the time taken to empty rest half of the tank. Then find* $\frac{t_1}{t_2}$.

Solution Substituting the proper limits in Eq. (i), derived in the theory, we have

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}} \int_H^{H/2} y^{-1/2}\ dy$$

or $$t_1 = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_{H/2}^{H}$$

or $$t_1 = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$

or $$t_1 = \frac{A}{a}\sqrt{\frac{H}{g}}\,(\sqrt{2} - 1) \qquad \text{...(ii)}$$

Similarly, $$\int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}}\int_{H/2}^{0} y^{-1/2}\,dy$$

or $$t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\frac{t_1}{t_2} = \sqrt{2} - 1$$

or $$\frac{t_1}{t_2} = 0.414$$ **Ans.**

Note *From the above example we can see that* $t_1 < t_2$. *This is because initially the pressure is high and the liquid comes out with greater speed.*

INTRODUCTORY EXERCISE 16.5

1. There is a small hole at the bottom of tank filled with water. If total pressure at the bottom is 3 atm (1 atm $= 10^5$ Nm^{-2}), then find the velocity of water flowing from hole.

2. Liquid is filled in a container upto a height of H. A small hole is made at the bottom of the tank. Time taken to empty from H to $\frac{H}{3}$ is t_0. Find the time taken to empty the tank from $\frac{H}{3}$ to zero.

3. Water is filled in a cylindrical container to a height of 3 m. The ratio of the cross-sectional area of the orifice and the beaker is 0.1. The square of the speed of the liquid coming out from the orifice is ($g = 10\,\text{m/s}^2$) **(JEE 2005)**

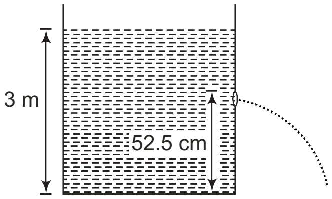


Fig. 16.57

(a) $50\,\text{m}^2/\text{s}^2$ (b) $50.5\,\text{m}^2/\text{s}^2$

(c) $51\,\text{m}^2/\text{s}^2$ (d) $52\,\text{m}^2/\text{s}^2$

16.8 Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to the other.

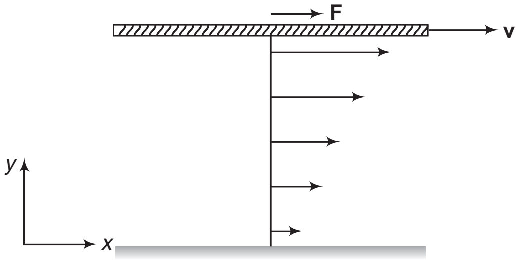


Fig. 16.58

The simplest example of viscous flow is motion of a fluid between two parallel plates.

The bottom plate is stationary and the top plate moves with constant velocity **v**. The fluid in contact with each surface has same velocity at that surface. The flow speeds of intermediate layers of fluid increase uniformly from bottom to top, as shown by arrows. So, the fluid layers slide smoothly over one another.

According to Newton, the frictional force F (or viscous force) between two layers depends upon the following factors,

(i) Force F is directly proportional to the area (A) of the layers in contact, i.e.

$$F \propto A$$

(ii) Force F is directly proportional to the velocity gradient $\left(\frac{dv}{dy}\right)$ between the layers. Combining these two, we have

$$F \propto A\frac{dv}{dy} \quad \text{or} \quad \boxed{F = -\eta A\frac{dv}{dy}}$$

Here, η is constant of proportionality and is called **coefficient of viscosity.** Its value depends on the nature of the fluid. The negative sign in the above equation shows that the direction of viscous force F is opposite to the direction of relative velocity of the layer.

The SI unit of η is N-s/m^2. It is also called decapoise or pascal second. Thus,

$$1 \text{ decapoise} = 1 \text{ N-s /m}^2 = 1 \text{ Pa-s} = 10 \text{ poise}$$

The SI unit of viscosity is sometimes referred to as the poiseuille (symbol Pl).

Thus, $1 \text{ Pl} = 1 \text{ N-s/m}^2$

Dimensions of η are $[\text{ML}^{-1}\text{T}^{-1}]$.

Coefficient of viscosity of water at 10°C is $\eta = 1.3 \times 10^{-3}$ N-s/m^2. Experiments show that coefficient of viscosity of a liquid decreases as its temperature rises.

Stoke's Law and Terminal Velocity

When an object moves through a fluid, it experiences a viscous force which acts in opposite direction of its velocity. The mathematics of the viscous force for an irregular object is difficult, we will consider here only the case of a small sphere moving through a fluid.

The formula for the viscous force on a sphere was first derived by the English physicist G. Stokes in 1843. According to him, a spherical object of radius r moving at velocity v experiences a viscous force given by

$$\boxed{F = 6\pi\eta r v} \qquad (\eta = \text{coefficient of viscosity})$$

This law is called **Stoke's law.**

Terminal Velocity (v_T)

Consider a small sphere falling from rest through a large column of viscous fluid. The forces acting on the sphere are,

(i) Weight w of the sphere acting vertically downwards

(ii) Upthrust F_t acting vertically upwards

(iii) Viscous force F_v acting vertically upwards, i.e. in a direction opposite to velocity of the sphere.

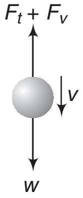


Fig. 16.59

Initially, $$F_v = 0 \qquad (\text{as } v = 0)$$

and $$w > F_t$$

and the sphere accelerates downwards. As the velocity of the sphere increases, F_v increases. Eventually a stage in reached when

$$w = F_t + F_v \qquad \ldots(\text{i})$$

After this net force on the sphere is zero and it moves downwards with a constant velocity called **terminal velocity** (v_T).

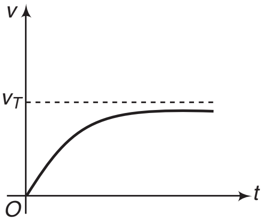


Fig. 16.60

Substituting proper values in Eq. (i) we have,

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_T \qquad \ldots(\text{ii})$$

Here, ρ = density of sphere, σ = density of fluid

and η = coefficient of viscosity of fluid

From Eq. (ii), we get $$\boxed{v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}}$$

Figure shows the variation of the velocity v of the sphere with time.

Extra Points to Remember

- Terminal velocity $v_T \propto r^2$
- If the fluid is air, then its density σ is negligible compared to density of sphere. So, in that case upthrust will be zero and terminal velocity will be

$$v_T = \frac{2}{9}\frac{r^2 \rho g}{\eta}$$

In this case, weight is equal to the viscous force when terminal velocity is attained.

- If density of fluid is greater than density of sphere ($\sigma > \rho$) then terminal velocity comes out to be negative.

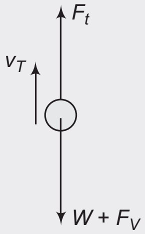


Fig. 16.61

So, in this case terminal velocity is upwards. In the beginning upthrust is greater than the weight. Viscous force in this case will be downwards.

$$v_T = \frac{2}{9}\frac{r^2(\sigma - \rho)g}{\eta}$$

When terminal velocity is attained,

$$F_t = W + F_v$$

This is the reason why, air bubbles rise up in water.

Example 16.19 *A plate of area 2 m^2 is made to move horizontally with a speed of 2 m/s by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.*

Solution Velocity gradient $= \dfrac{\Delta v}{\Delta y} = \dfrac{2-0}{1-0} = 2\,\dfrac{\text{m/s}}{\text{m}}$

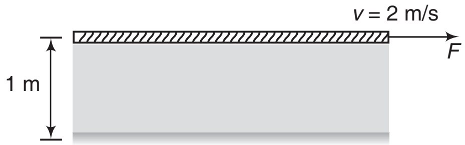


Fig. 16.62

From, Newton's law of viscous force,

$$|F| = \eta A \frac{\Delta v}{\Delta y}$$

$$= (0.01 \times 10^{-1})(2)(2)$$

$$= 4 \times 10^{-3} \text{ N.}$$

So, to keep the plate moving, a force of 4×10^{-3} N must be applied. **Ans.**

Example 16.20 *Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of* 1 *m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?*

Solution $$v_T \propto r^2 \qquad \text{...(i)}$$

Let r be the radius of small rain drops and R the radius of large drop.

Equating the volumes, we have

$$\frac{4}{3}\pi R^3 = 2\left(\frac{4}{3}\pi r^3\right)$$

$$\therefore \quad R = (2)^{1/3}.r \quad \text{or} \quad \frac{R}{r} = (2)^{1/3}$$

$$\therefore \quad \frac{v_T{}'}{v_T} = \left(\frac{R}{r}\right)^2 = (2)^{2/3}$$

$$\therefore \quad v_T{}' = (2)^{2/3}v_T = (2)^{2/3}(1.0)\ \text{m/s}$$

$$= 1.587\ \text{m/s}$$ **Ans.**

Example 16.21 *With what terminal velocity will an air bubble* 0.8 *mm in diameter rise in a liquid of viscosity* 0.15 *N-s/m*^{2} *and specific gravity* 0.9 *? Density of air is* 1.293 *kg/m*^{3}.

Solution The terminal velocity of the bubble is given by,

$$v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$$

Here, $r = 0.4 \times 10^{-3}$ m, $\sigma = 0.9 \times 10^3$ kg/m^3, $\rho = 1.293$ kg/m^3, $\eta = 0.15$ N-s/m^2

and $g = 9.8$ m/s^2

Substituting the values, we have

$$v_T = \frac{2}{9} \times \frac{(0.4 \times 10^{-3})^2(1.293 - 0.9 \times 10^3) \times 9.8}{0.15}$$

$$= -0.0021\ \text{m/s}$$

or $$v_T = -\ 0.21\,\text{cm}$$ **Ans.**

Note *Here, negative sign implies that the bubble will rise up.*

Example 16.22 *A spherical ball of radius* 3.0×10^{-4} *m and density* 10^4 *kg/m*^{3} *falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, find h. Viscosity of water is* 9.8×10^{-6} *N-s/m*^{2}.

Solution Before entering the water the velocity of ball is $\sqrt{2gh}$. If after entering the water this velocity does not change then this value should be equal to the terminal velocity. Therefore,

$$\sqrt{2gh} = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$$

$$\therefore \qquad h = \frac{\left\{\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}\right\}^2}{2g}$$

$$= \frac{2}{81} \times \frac{r^4(\rho-\sigma)^2 g}{\eta^2}$$

$$= \frac{2}{81} \times \frac{(3\times 10^{-4})^4 (10^4 - 10^3)^2 \times 9.8}{(9.8\times 10^{-6})^2}$$

$$= 1.65 \times 10^3 \text{ m}$$ **Ans.**

Example 16.23 *A small sphere falls from rest in a viscous liquid. Due to friction, heat is produced. Find the relation between the rate of production of heat and the radius of the sphere at terminal velocity.* **(JEE 2004)**

Solution Terminal velocity $v_T = \frac{2r^2 g}{9\eta}(\rho_S - \rho_L)$

and viscous force $F = 6\pi\eta r v_T$

Rate of production of heat (power) : as viscous force is the only dissipative force. Hence,

$$\frac{dQ}{dt} = Fv_T = (6\pi\eta r v_T)(v_T)$$

$$= 6\pi\eta r v_T^2$$

$$= 6\pi\eta r\left\{\frac{2}{9}\frac{r^2 g}{\eta}(\rho_S - \rho_L)\right\}^2$$

$$= \frac{8\pi g^2}{27\eta}(\rho_S - \rho_L)^2 r^5 \quad \text{or} \quad \frac{dQ}{dt} \propto r^5$$ **Ans.**

INTRODUCTORY EXERCISE 16.6

1. A typical riverborne silt particle has a radius of 20 μm and a density of 2×10^3 kg/m^3. The viscosity of water is 1.0 mPl. Find the terminal speed with which such a particle will settle to the bottom of a motionless volume of water.
2. Two equal drops of water are falling through air with a steady velocity v. If the drops coalesced, what will be the new velocity ?
3. A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. If the river is 1 m deep and the water in contact with the bed is stationary, find the tangential force needed to keep the plate moving. Coefficient of viscosity of water at the temperature of the river $= 10^{-2}$ poise.
4. The velocity of water in a river is 18 km/h near the surface. If the river is 5 m deep, find the shearing stress between the horizontal layers of water. The coefficient of viscosity of water $= 10^{-2}$ poise.

16.9 Surface Tension

A needle can made to float on a water surface if it is placed there carefully. The forces that support the needle are not buoyant forces but are due to surface tension. The surface of a liquid behaves like a membrane under tension. The molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid.But a surface molecule has a net force towards inside of the liquid. Thus, the liquid tends to minimize its surface area, just as a stretched membrane does.

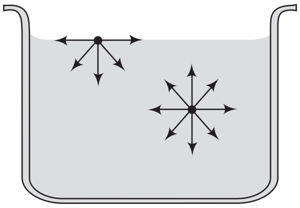


Fig. 16.63

Freely falling raindrops are spherical because a sphere has a smaller surface area for a given volume than any other shape. Hence, the surface tension can be defined as the *property of a liquid at rest by virtue of which its free surface behaves like a stretched membrane under tension and tries to occupy as small area as possible.*

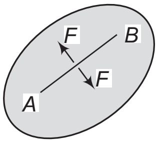


Fig. 16.64

Let an imaginary line *AB* be drawn in any direction in a liquid surface. The surface on either side of this line exerts a pulling force on the surface on the other side. This force is at right angles to the line *AB*. The magnitude of this force per unit length of *AB* is taken as a measure of the surface tension of the liquid. Thus, if *F* be the total force acting on either side of the line *AB* of length *L,* then the surface tension is given by,

$$T = \frac{F}{L}$$

Hence, the surface tension of a liquid is defined as the force per unit length in the plane of the liquid surface, acting at right angles on either side of an imaginary line drawn on that surface.

Examples of Surface Tension

Example 1 *Take a ring of wire and dip it in a soap solution. When the ring is taken out, a soap film is formed. Place a loop of thread gently on the soap film. Now, prick a hole inside the loop. The thread is radially pulled by the film surface outside and it takes a circular shape.*

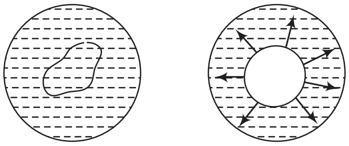


Fig. 16.65

Reason Before the pricking, there were surfaces both inside and outside the thread loop. Surfaces on both sides pull it equally and the net force is zero. Once the surface inside was punctured, the outside surface pulled the thread to take the circular shape so that area outside the loop becomes minimum (because for given perimeter area of circle is maximum).

Example 2 *A piece of wire is bent into a U-shape and a second piece of wire slides on the arms of the U. When the apparatus is dipped into a soap solution and removed, a liquid film is formed. The film exerts a surface tension force on the slider and if the frame is kept in a horizontal position, the slider quickly slides towards the closing arm of the frame. If the frame is kept vertical, one can have some weight to keep it in equilibrium. This shows that the soap surface in contact with the slider pulls it parallel to the surface.*

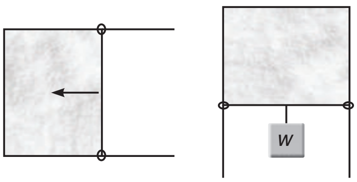


Fig. 16.66

Example 3 ***Needle supported on water surface*** *Take a greased needle of steel on a piece of blotting paper and place it gently over the water surface. Blotting paper soaks water and soon sinks down but the needle keeps floating. The floating needle causes a little depression. The forces F and F due to surface tension of the curved surface are inclined as shown in Fig.*16.67. *The vertical components of these two forces support the weight of the needle.*

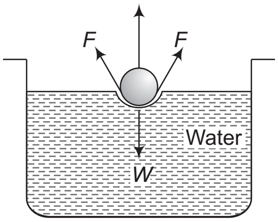


Fig. 16.67

Example 4 *Small mercury droplets are spherical and larger ones tend to flatten.*

Reason Small mercury droplets are spherical because the forces of surface tension tend to reduce their area to a minimum value and a sphere has minimum surface area for a given volume.

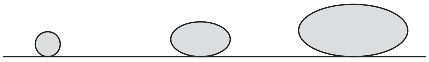


Fig. 16.68

Larger drops of mercury are flattened due to the large gravitational force acting on them. Here the shape is such that the sum of the gravitational potential energy and the surface potential energy must be minimum. Hence the centre of gravity moves down as low as possible. This explains flattening of the larger drops.

Example 5 *The hair of a painting brush cling together when taken out of water.*

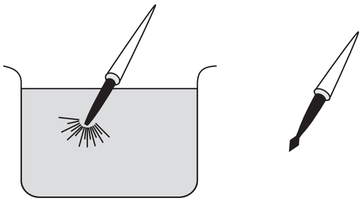


Fig. 16.69

Reason This is because the water films formed on them tend to contract to minimum area.

Note *The surface tension of a particular liquid usually decreases as temperature increases. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers. This requires increasing the surface area of the water ,which is difficult to do because of surface tension. Hence, hot water and soapy water is better for washing.*

Surface Energy

When the surface area of a liquid is increased, the molecules from the interior rise to the surface. This requires work against force of attraction of the molecules just below the surface. This work is stored in the form of potential energy. Thus, the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called 'surface energy'. The surface energy is related to the surface tension as discussed below.

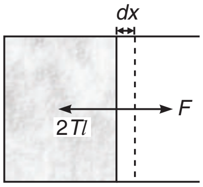


Fig. 16.70

Let a liquid film be formed on a wire frame and a straight wire of length *l* can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If *T* be the surface tension of the solution, each surface will pull the wire parallel to itself with a force *Tl*. Thus, net force on the wire due to both the surfaces is 2*Tl*. One has to apply an external force *F* equal and opposite to it to keep the wire in equilibrium.

Thus,

$$F = 2Tl$$

Now, suppose the wire is moved through a small distance *dx*, the work done by the force is,

$$dW = F\ dx = (2Tl)\ dx$$

But $(2l)\ (dx)$ is the total increase in area of both the surfaces of the film. Let it be dA. Then,

$$dW = T\ dA$$

or
$$\boxed{T = \frac{dW}{dA} \quad \text{or} \quad \frac{\Delta W}{\Delta A}}$$

Thus, the surface tension *T* can also be defined as the work done in increasing the surface area by unity.

Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface.

∴
$$\boxed{T = \frac{dU}{dA} \quad \text{or} \quad \frac{\Delta U}{\Delta A}} \qquad (\text{as } dW = dU)$$

Thus, the surface tension of a liquid is equal to the surface energy per unit surface area.

Excess Pressure inside a Bubble or Liquid Drop

Surface tension causes a pressure difference between the inside and outside of a soap bubble or a liquid drop.

Excess Pressure Inside Soap Bubble

A soap bubble consists of two spherical surface films with a thin layer of liquid between them. Because of surface tension, the film tend to contract in an attempt to minimize their surface area. But as the bubble contracts, it compresses the inside air, eventually increasing the interior pressure to a level that prevents further contraction.

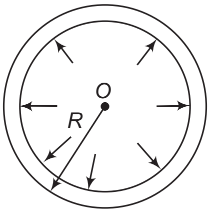


Fig. 16.71

We can derive an expression for the excess pressure inside a bubble in terms of its radius R and the surface tension T of the liquid.

Each half of the soap bubble is in equilibrium. The lower half is shown in figure. The forces at the flat circular surface where this half joins the upper half are

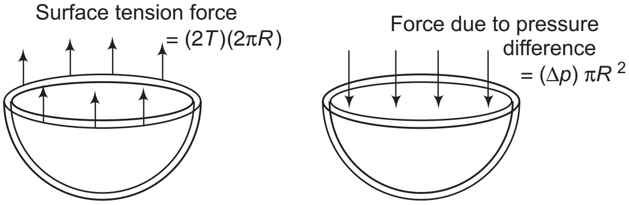


Fig. 16.72

(a) The upward force of surface tension. The total surface tension force for each surface (inner and outer) is $T(2\pi R)$, for a total of $(2T)(2\pi R)$.

(b) downward force due to pressure difference.

The magnitude of this force is $(\Delta p)(\pi R^2)$. In equilibrium these two forces have equal magnitude.

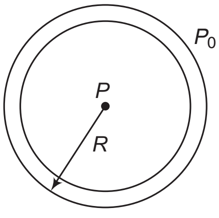


Fig. 16.73

$$\therefore \quad (2T)(2\pi R) = (\Delta P)(\pi R^2)$$

or
$$\boxed{\Delta p = \frac{4T}{R}}$$

Note *Suppose, the pressure inside the air bubble is P, then*

$$p - p_0 = \frac{4T}{R}$$

Excess Pressure Inside a Liquid Drop

A liquid drop has only one surface film. Hence, the surface tension force is $T(2\pi R)$, half that for a soap bubble. Thus, in equilibrium,

$$T(2\pi R) = \Delta p(\pi R^2)$$

or
$$\boxed{\Delta p = \frac{2T}{R}}$$

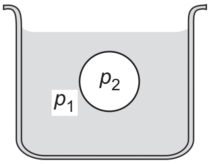


Fig. 16.74

Note (i) *If we have an air bubble inside a liquid, a single surface is formed. There is air on the concave side and liquid on the convex side. The pressure in the concave side (that is in the air) is greater than the pressure in the convex side (that is in the liquid) by an amount* $\frac{2T}{R}$.

$$\therefore \qquad \boxed{p_2 - p_1 = \frac{2T}{R}}$$

The above expression has been written by assuming P_1 *to be constant from all sides of the bubble. For small size bubbles this can be assumed.*

(ii) *From the above discussion, we can make a general statement. The pressure on the concave side of a spherical liquid surface is greater than the convex side by* $\frac{2T}{R}$.

Extra Points to Remember

- **Radius of curvature of a curve :**

  To describe the shape of a curved surface or interface, it is necessary to know the radii of curvature to a curve at some point. Consider the curve *AB* as shown in figure:

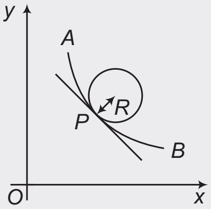


Fig. 16.75

  Let *P* be a point on this curve. The radius of curvature *R* of *AB* at *P* is defined as the radius of the circle which is tangent to the curve at point *P*.

- **Principal radii of curvature of a surface :**

  The spherical and cylindrical surfaces are rather simple cases for mathematical treatment. In many other cases however, the shapes are more complicated.

  Let us now consider a curved surface. At each point on a given surface, two radii of curvature (which are denoted by r_1 and r_2) are required to describe the shape.

  If we want to determine these radii at any point (say *P*), the normal to the surface at this point is drawn and a plane is constructed through the surface containing the normal. This will intersect the surface in a plane curve. The radius of curvature of the curve at point *P* is denoted by r_1. An infinite number of such planes can be constructed each of which intersects the surface at *P*. For each of these planes, a radius of curvature can be obtained.

  If we construct a second plane through the surface, containing the normal and perpendicular to the first plane, the second line of intersection and hence the second radius of curvature at point *P* (i.e., r_2) is obtained. These two radii define the curvature at *P* completely. It can be shown that $(1/r_1 + 1/r_2)$ called mean radius of curvature of the surface is constant, which is independent of the choice of the planes.

  An infinite set of such pairs of radii is possible. For standardization, the first plane is rotated around the normal until the radius of curvature in that plane reaches minimum. The other radius of curvature is therefore maximum. These are the principal radii of curvature (denoted by R_1 and R_2).

- **Young-Laplace Equation :**

  There exists a difference in pressure across a curved surface which is a consequence of surface tension. The pressure is greater on the concave side. The Laplace equation relates the pressure difference to the shape of the Young surface.

This difference in pressure is given by

$$\boxed{\Delta p = T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}$$

The simplified forms of spherical, cylindrical and planar surfaces are given below

For a spherical surface:

$$R_1 = R_2 = R \text{ (say), therefore } \Delta p = \frac{2T}{R}$$

For a cylindrical surface:

$$R_1 = R \text{ and } R_2 = \infty \text{, therefore } \Delta p = \frac{T}{R}$$

For a planer surface:

$$R_1 = R_2 = \infty \text{, therefore } \Delta p = 0$$

- As we know that molecules of a liquid reach to its surface after struggling with the net inward force acting on them from other liquid molecules. So, we can say that the surface molecules have some extra energy compared to inner molecules and every system has a tendency to keep its energy minimum. So, a liquid also has a tendency to keep its energy minimum by putting least number of molecules on the surface or by making its surface area minimum. This property of a liquid is called surface tension.
- Surface tension is a property of a liquid. It does not depend on the surface area.

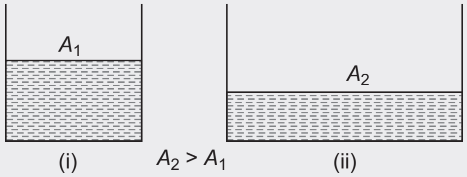


Fig. 16.76

For example, both containers have the same liquid under same conditions. $A_2 > A_1$ but surface tension will be same in both cases.

Example 16.24 *How much work will be done in increasing the diameter of a soap bubble from* 2 *cm to* 5 *cm ? Surface tension of soap solution is* 3.0×10^{-2} *N/m*.

Solution Soap bubble has two surfaces. Hence,

$$W = T\,\Delta A \qquad \left(T = \frac{\Delta W}{\Delta A}\right)$$

Here, $$\Delta A = 2[4\pi\{(2.5\times10^{-2})^2 - (1.0\times10^{-2})^2\}]$$

$$= 1.32\times10^{-2}\ \text{m}^2$$

$\therefore$ $$W = (3.0\times10^{-2})(1.32\times10^{-2})\ \text{J}$$

$$= 3.96\times10^{-4}\ \text{J}$$ **Ans.**

Example 16.25 *Calculate the energy released when* 1000 *small water drops each of same radius* 10^{-7} *m coalesce to form one large drop. The surface tension of water is* 7.0×10^{-2} *N/m.*

Solution Let r be the radius of smaller drops and R of bigger one. Equating the initial and final volumes, we have

$$\frac{4}{3}\pi R^3 = (1000)\left(\frac{4}{3}\pi r^3\right)$$

or $$R = 10r = (10)(10^{-7})\text{ m}$$

or $$R = 10^{-6}\text{ m}$$

Further, the water drops have only one free surface. Therefore,

$$\begin{aligned}\Delta A &= 4\pi R^2 - (1000)(4\pi r^2)\\ &= 4\pi\,[(10^{-6})^2 - (10^3)(10^{-7})^2]\\ &= -36\pi(10^{-12})\text{ m}^2\end{aligned}$$

Here, negative sign implies that surface area is decreasing. Hence, energy released in the process.

$$\begin{aligned}U = T|\Delta A| &= (7\times 10^{-2})(36\pi\times 10^{-12})\text{ J}\\ &= 7.9\times 10^{-12}\text{ J}\end{aligned}$$ **Ans.**

Example 16.26 *What should be the pressure inside a small air bubble of* 0.1 *mm radius situated just below the water surface. Surface tension of water* $= 7.2\times 10^{-2}$ *N/m and atmospheric pressure* $= 1.013\times 10^5$ *N/m*^{2}.

Solution Surface tension of water $T = 7.2\times 10^{-2}$ N/m

Radius of air bubble $R = 0.1\text{ mm} = 10^{-4}$ m

The excess pressure inside the air bubble is given by,

$$p_2 - p_1 = \frac{2T}{R}$$

∴ Pressure inside the air bubble,

$$p_2 = p_1 + \frac{2T}{R}$$

Substituting the values, we have

$$\begin{aligned}p_2 &= (1.013\times 10^5) + \frac{(2\times 7.2\times 10^{-2})}{10^{-4}}\\ &= 1.027\times 10^5\text{ N/m}^2\end{aligned}$$ **Ans.**

Example 16.27 *Two separate air bubbles (radii* 0.004 *m and* 0.002 *m) formed of the same liquid (surface tension* 0.07 *N/m) come together to form a double bubble. Find the radius and the sense of curvature of the internal film surface common to both the bubbles.*

Solution $p_1 = p_0 + \frac{4T}{r_1} \Rightarrow p_2 = p_0 + \frac{4T}{r_2}$

$$r_2 < r_1$$

∴ $$p_2 > p_1$$

i.e. pressure inside the smaller bubble will be more. The excess pressure

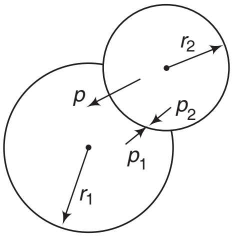


Fig. 16.77

$$p = p_2 - p_1 = 4T\left(\frac{r_1 - r_2}{r_1 r_2}\right) \qquad \ldots\text{(i)}$$

This excess pressure acts from concave to convex side, the interface will be concave towards smaller bubble and convex towards larger bubble. Let R be the radius of interface then,

$$p = \frac{4T}{R} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$R = \frac{r_1 r_2}{r_1 - r_2} = \frac{(0.004)(0.002)}{(0.004 - 0.002)}$$

$$= 0.004 \text{ m}$$ **Ans**

Example 16.28 *Under isothermal condition two soap bubbles of radii r_1 and r_2 coalesce to form a single bubble of radius r. The external pressure is p_0. Find the surface tension of the soap in terms of the given parameters.*

Solution As mass of the air is conserved,

$$\therefore \qquad n_1 + n_2 = n \qquad \text{(as } pV = nRT\text{)}$$

$$\therefore \qquad \frac{p_1 V_1}{RT_1} + \frac{p_2 V_2}{RT_2} = \frac{pV}{RT}$$

As temperature is constant,

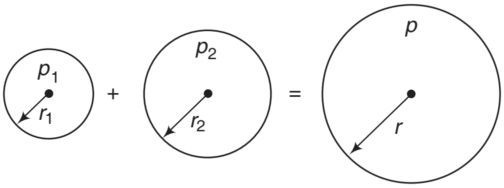


Fig. 16.78

$$T_1 = T_2 = T$$

$$\therefore \qquad p_1 V_1 + p_2 V_2 = pV$$

$$\therefore \qquad \left(p_0 + \frac{4S}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(p_0 + \frac{4S}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right) = \left(p_0 + \frac{4S}{r}\right)\left(\frac{4}{3}\pi r^3\right)$$

Solving, this we get

$$S = \frac{p_0 (r^3 - r_1^3 - r_2^3)}{4(r_1^2 + r_2^2 - r^2)} \qquad \textbf{Ans.}$$

Note *To avoid confusion with the temperature, surface tension here is represented by S.*

Example 16.29 *Assume that a drop of liquid evaporates by decrease in its surface energy, so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, density of liquid is ρ and L is its latent heat of vaporization* **(2013 Main)**

(a) $\frac{\rho L}{T}$ (b) $\sqrt{\frac{T}{\rho L}}$ (c) $\frac{T}{\rho L}$ (d) $\frac{2T}{\rho L}$

Solution Decrease in surface energy = heat required in vaporization.

$$\therefore \qquad T(dS) = L(dm)$$

$$\therefore \qquad T(2)(4\pi r)\,dr = L(4\pi r^2\,dr)\rho$$

$$\therefore \qquad r = \frac{2T}{\rho L}$$

The correct options is (d).

INTRODUCTORY EXERCISE 16.7

1. A mercury drop of radius 1 cm is sprayed into 10^6 droplets of equal size. Calculate the increase in surface energy if surface tension of mercury is 35×10^{-3} N/m.
2. A water film is made between two straight parallel wires of length 10 cm each and at a distance of 0.5 cm from each other. If the distance between the wires is increased by 1 mm, how much work will be done? Surface tension of water $= 7.2 \times 10^{-2}$ N/m
3. A soap bubble of radius R has been formed at normal temperature and pressure under isothermal conditions. Compute the work done. The surface tension of soap solution is T.
4. A small air bubble of radius 'r' is at a depth 'h' below the water surface (density of water $= \rho$). Surface tension of water is T, atmospheric pressure is p_0. Find pressure inside the air bubble for the condition $r << h$.

16.10 Capillary Rise or Fall

Cohesive and Adhesive Forces

The force of attraction between the molecules of the same substance is called cohesion.

In case of solids, the force of cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence, liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Examples

(i) Two drops of liquid coalesce into one when brought in mutual contact because of the cohesive force.

(ii) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(iii) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

The force of attraction between molecules of different substance is called adhesion.

Examples

(i) Adhesive force enables us to write on the black board with a chalk.

(ii) Due to force of adhesion, water wets the glass plate.

(iii) Fevicol and gum are used in gluing two surfaces together because of adhesive force.

Contact Angle

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved. When a glass plate is immersed in water, the surface near the plate becomes concave. On the other hand, if glass plate is immersed in mercury, the surface is depressed near the plate.

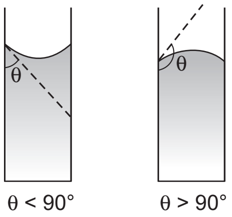


Fig. 16.79

The angle between the tangent planes at the solid surface and the liquid at the contact is called the contact angle. In this case the tangent plane to the solid surface is to be drawn towards the liquid and the tangent plane to the liquid is to be drawn away from the solid.

Those liquids which wet the walls of the container (say in case of water and glass) have meniscus concave upwards and their values of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass is about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.

Shape of Liquid Meniscus

When the adhesive force (P) between solid and liquid molecules is more than the cohesive force (Q) between liquid-liquid molecules (as with water and glass), shape of the meniscus is concave and the angle of contact θ is less than 90°. In this case, the liquid wets or adheres to the solid surface. The resultant (R) of P and Q passes through the solid.

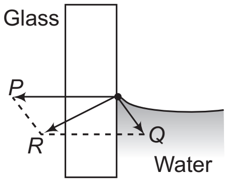


Fig. 16.80

On the other hand when $P < Q$ (as with glass and mercury), shape of the meniscus is convex and the angle of contact $\theta > 90°$. The resultant (R) of P and Q in this case passes through the liquid.

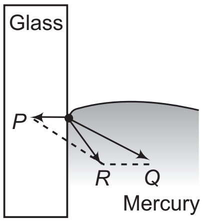


Fig. 16.81

Let us now see why the liquid surface bends near the contact with a solid. A liquid in equilibrium can not sustain tangential stress. The resultant force on any small part of the surface layer must be perpendicular to the surface at that point. Basically three forces are acting on a small part of the liquid surface near its contact with solid. These forces are,

(i) P, attraction due to the molecule of the solid surface near it

(ii) Q, attraction due to liquid molecules near this part and

(iii) The weight w of the part considered.

We have considered very small part, so weight of that part can be ignored for better understanding. As we have seen in the last figures, to make the resultant (R) of P and Q perpendicular to the liquid surface the surface becomes curved (convex or concave).

Capillarity

Surface tension causes elevation or depression of the liquid in a narrow tube. This effect is called capillarity. When a glass capillary tube (A tube of very small diameter is called a capillary tube) open at both ends is dipped vertically in water the water in the tube will rise above the level of water in the vessel as shown in figure (a). In case of mercury, the liquid is depressed in the tube below the level of mercury in the vessel as shown in figure (b).

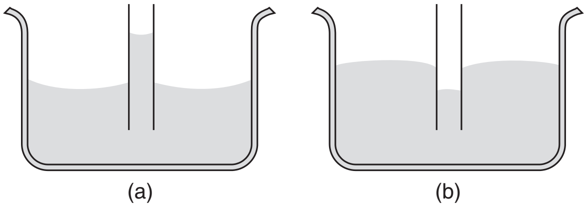


Fig. 16.82

When the contact angle is less than 90° the liquid rises in the tube. For a non-wetting liquid angle of contact is greater than 90° and the surface is depressed, pulled down by the surface tension forces.

Explanation

When a capillary tube is dipped in water, the water meniscus inside the tube is concave. The pressure just below the meniscus is less than the pressure just above it by $\frac{2T}{R}$, where T is the surface tension of water and R is the radius of curvature of the meniscus. The pressure on the surface of water is p_0, the atmospheric pressure. The pressure just below the plane surface of water outside the tube is also

p_0, but that just below the meniscus inside the tube is $p_0 - \frac{2T}{R}$. We know that pressure at all points in the same level of water must be the same. Therefore, to make up the deficiency of pressure $\frac{2T}{R}$ below the meniscus water begins to flow from outside into the tube. The rising of water in the capillary stops at a certain height *h.* In this position, the pressure of water column of height *h* becomes equal to $\frac{2T}{R}$, i.e.

$$h\rho g = \frac{2T}{R}$$

or
$$h = \frac{2T}{R\rho g}$$

If *r* is the radius of the capillary tube and θ the angle of contact, then

$$R = \frac{r}{\cos\theta}$$

$$\therefore \qquad \boxed{h = \frac{2T\cos\theta}{r\rho g}}$$

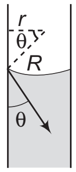


Fig. 16.83

Alternative Proof for the Formula of Capillary Rise

As we have already seen, when the contact angle is less than 90°, the total surface tension force just balances the extra weight of the liquid in the tube.

The water meniscus in the tube is along a circle of circumference $2\pi r$ which is in contact with the glass. Due to the surface tension of water, a force equal to *T* per unit length acts at all points of the circle. If the angle of contact is θ, then this force is directed inward at an angle θ from the wall of the tube.

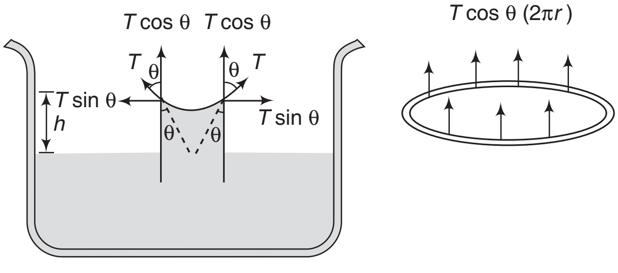


Fig. 16.84

In accordance with Newton's third law, the tube exerts an equal and opposite force *T* per unit length on the circumference of the water meniscus. This force which is directed outward, can be resolved into two components $T\cos\theta$ per unit length acting vertically upward and $T\sin\theta$ per unit length acting horizontally outward. Considering the entire circumference $2\pi r$, for each horizontal component $T\sin\theta$ there is an equal and opposite component and the two neutralise each other. The vertical components being in the same direction are added up to give a total upward force $(2\pi r)(T\cos\theta)$. It is this force which supports the weight of the water column so raised.

Thus,

$$(T\cos\theta)(2\pi r) = \text{Weight of the liquid column.}$$
$$= (\pi r^2 \rho g h) \qquad \ldots\text{(i)}$$

$$\therefore \qquad h = \frac{2T\cos\theta}{r\rho g}$$

The result has following notable features :

(i) If the contact angle θ is greater than 90°, the term $\cos\theta$ is negative and hence, h is negative. The expression then gives the depression of the liquid in the tube.

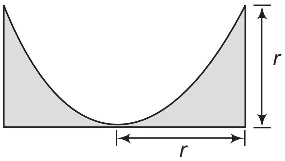


Fig. 16.85

(ii) The **correction** due to weight of the liquid contained in the meniscus can be made for contact angle $\theta = 0°$. The meniscus is then hemispherical. The volume of the shaded part is

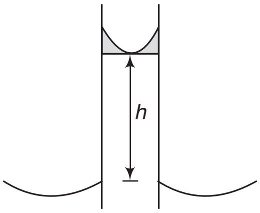


Fig. 16.86

$$V = (\pi r^2)(r) - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^3$$

The weight of this shaded part is $\frac{1}{3}\pi r^3 \rho g$.

Therefore, we can write Eq. (i) as,

$$(T\cos 0°)(2\pi r) = \pi r^2 \rho g h + \frac{1}{3}\pi r^3 \rho g$$

or

$$\boxed{h = \frac{2T}{r\rho g} - \frac{r}{3}}$$

(iii) Suppose a capillary tube is held vertically in a liquid which has a concave meniscus, then capillary rise is given by,

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2T}{R\rho g} \qquad \left(\text{as } R = \frac{r}{\cos\theta}\right)$$

or

$$hR = \frac{2T}{\rho g} \qquad \ldots\text{(ii)}$$

When the length of the tube is greater than h, the liquid rises in the tube, so as to satisfy the above relation. But if the length of the tube is insufficient (i.e. less than h) say h', the liquid does not emerge in the form of a fountain from the upper end (because it will violate the law of conservation of energy) but the angle made by the liquid surface and hence, the R changes in such a way that the force $2\pi rT\cos\theta$ equals the weight of the liquid raised. Thus,

$$2\pi rT\cos\theta' = \pi r^2\rho gh'$$

$$h' = \frac{2T\cos\theta'}{r\rho g}$$

or

$$h' = \frac{2T}{R'\rho g}$$

or

$$h'R' = \frac{2T}{\rho g} \qquad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we have

$$hR = h'R' = \frac{2T}{\rho g}$$

Practical Applications of Capillarity

1. The oil in a lamp rises in the wick by capillary action.
2. The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
3. If one end of the towel dips into a bucket of water and the other end hangs over the bucket, the towel soon becomes wet throughout, due to capillary action.
4. Ink is absorbed by the blotter due to capillary action.

Example 16.30 *A capillary tube whose inside radius is* 0.5 *mm is dipped in water having surface tension* 7.0×10^{-2} *N/m. To what height is the water raised above the normal water level? Angle of contact of water with glass is* 0°. *Density of water is* 10^3 *kg/m*^{3} *and* $g = 9.8$ *m/s*^{2}.

Solution $h = \frac{2T\cos\theta}{r\rho g}$

Substituting the proper values, we have

$$h = \frac{(2)(7.0\times10^{-2})\cos 0^\circ}{(0.5\times10^{-3})(10^3)(9.8)}$$

$$= 2.86\times10^{-2}\ \text{m} = 2.86\ \text{cm}$$ **Ans.**

Example 16.31 *A glass tube of radius* 0.4 *mm is dipped vertically in water. Find upto what height the water will rise in the capillary ? If the tube in inclined at an angle of* 60° *with the vertical, how much length of the capillary is occupied by water ? Surface tension of water* = 7.0×10^{-2} *N/m, density of water* = 10^3 *kg/m*^{3}.

Solution For glass-water, angle of contact $\theta = 0°$.

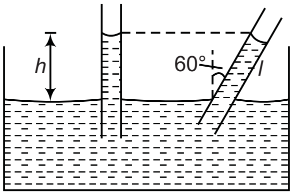


Fig. 16.87

Now,
$$h = \frac{2T\cos\theta}{r\rho g}$$

$$= \frac{(2)(7.0\times 10^{-2})\cos 0°}{(0.4\times 10^{-3})(10^3)(9.8)}$$

$$= 3.57\times 10^{-2}\ \text{m} = 3.57\,\text{cm}$$ **Ans.**

$$l = \frac{h}{\cos 60°} = \frac{3.57}{\frac{1}{2}} = 7.14\,\text{cm}$$ **Ans.**

Example 16.32 *Mercury has an angle of contact of* $120°$ *with glass. A narrow tube of radius* 1.0 *mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside. Surface tension of mercury at the temperature of the experiment is* 0.5 *N/m and density of mercury is* 13.6×10^3 *kg/m*^{3}. *(Take* $g = 9.8\ m/s^2$*).*

Solution $h = \dfrac{2T\cos\theta}{r\rho g}$

Substituting the values, we get
$$h = \frac{2\times 0.5\times \cos 120°}{10^{-3}\times 13.6\times 10^3\times 9.8}$$

$$= -3.75\times 10^{-3}\ \text{m}$$

or
$$h = -3.75\ \text{mm}$$ **Ans.**

Note *Here, negative sign implies that mercury suffers capillary depression.*

Example 16.33 *If a* 5 *cm long capillary tube with* 0.1 *mm internal diameter open at both ends is slightly dipped in water having surface tension* 75 *dyne cm*$^{-1}$, *state whether (a) water will rise half way in the capillary, (b) Water will rise up to the upper end of capillary and (c) water will overflow out of the upper end of capillary? Explain your answer.*

Solution Given that surface tension of water, $T = 75$ dyne/cm

Radius $r = \dfrac{0.1}{2}\,\text{mm} = 0.05\,\text{mm} = 0.005\,\text{cm}$

Density $\rho = 1\,\text{gm/cm}^3$, angle of contact, $\theta = 0°$

Let h be the height to which water rises in the capillary tube. Then

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2\times 75\times\cos 0^\circ}{0.005\times 1\times 981}\text{ cm}$$

$$= 30.58\text{ cm}$$

But length of capillary tube is $h' = 5$ cm

(a) Because $h > \frac{h'}{2}$ therefore the first possibility does not exist.

(b) Because the tube is of insufficient length therefore the water will rise upto the upper end of the tube.

(c) The water will not overflow of the upper end of the capillary. It will rise only up to the upper end of the capillary.

The liquid meniscus will adjust its radius of curvature R' in such a way that

$$R'h' = Rh \qquad \left[\because hR = \frac{2T}{\rho g} = \text{constant}\right]$$

where, R is the radius of curvature that the liquid meniscus would possess if the capillary tube were of sufficient length.

$$\therefore \qquad R' = \frac{Rh}{h'} = \frac{rh}{h'} = \frac{0.005\times 30.58}{5} = 0.0306\text{ cm} \qquad \left[\because R = \frac{r}{\cos\theta} = \frac{r}{\cos 0^\circ} = r\right]$$

INTRODUCTORY EXERCISE 16.8

1. Water rises in a capillary tube to a height of 2.0 cm. In another capillary tube whose radius is one third of it, how much the water will rise?

2. Water rises up in a glass capillary upto a height of 9.0 cm, while mercury falls down by 3.4 cm in the same capillary. Assume angles of contact for water glass and mercury glass 0° and 135° respectively. Determine the ratio of surface tension of mercury and water ($\cos 135^\circ = -0.71$).

3. A tube of insufficient length is immersed in water (surface tension $= 0.7$ N/m) with 1 cm of it projecting vertically upwards outside the water. What is the radius of meniscus ?
Given, radius of tube $= 1$ mm.

4. A capillary tube is dipped in a liquid. Let pressures at points A, B and C be p_A, p_B and p_C respectively, then

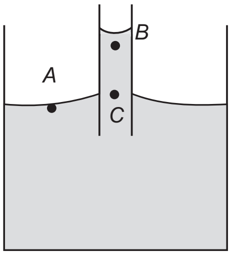


Fig. 16.88

(a) $p_A = p_B = p_C$ (b) $p_A = p_B < p_C$ (c) $p_A = p_C < p_B$ (d) $p_A = p_C > p_B$

Final Touch Points

1. **Application on Surface Tension**
   - (i) When the detergent materials are added to liquids, the angle of contact decreases and hence the wettability increases. On the other hand, when water proofing material is added to a fabric it increases the angle of contact, making the fabric water-repellant.
   - (ii) Surface tension of all lubricating oils and paints is kept low so that they spread over a large area.
   - (iii) Oil spreads over the surface of water because the surface tension of oil is less than the surface tension of cold water.

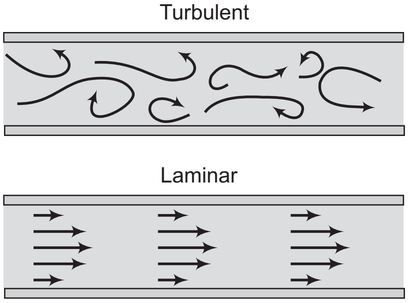
2. **Effect of Temperature and Impurities on Surface Tension**

   The surface tension of a liquid decreases with the rise in temperature and *vice-versa*. Surface tension becomes zero at a critical temperature. It is for this reason that hot soup tastes better. Surface tension of a liquid changes appreciably with addition of impurities. For example, surface tension of water increases with addition of highly soluble substances like Nacl, $ZnSO_4$ etc. On the other hand surface tension of water gets reduced with addition of sparingly soluble substances like phenol, soap etc.

3. **Laminar and Turbulent Flow, Reynolds Number**

   When a liquid flowing in a pipe is observed carefully, it will be seen that the pattern of flow becomes more disturbed as the velocity of flow increases. Perhaps this phenomenon is more commonly seen in a river or stream. When the flow is slow the pattern is smooth, but when the flow is fast, eddies develop and swirl in all directions.

   At the low velocities, flow is calm. This is called "laminar flow".



   In a series of experiments, **Reynolds** showed this by injecting a thin stream of dye into the fluid and finding that it ran in a smooth stream in the direction of the flow at low speeds. As the velocity of flow increased, he found that the smooth line of dye was broken up, at high velocities, the dye was rapidly mixed into the disturbed flow of the surrounding fluid. This is called **"turbulent flow".**

   After many experiments **Reynolds** saw that the expression

$$\frac{\rho u d}{\eta}$$

   where, ρ = density, u = mean velocity, d = diameter and η = viscosity would help in predicting the change in flow type. If the value is less than about 2000 then flow is laminar, if greater than 4000 then turbulent and in between these two in the transition zone.

   This value is known as the Reynolds number, R_e.

$$R_e = \frac{\rho u d}{\eta}$$

   Laminar flow: $R_e < 2000$, Transitional flow: $2000 < R_e < 4000$, Turbulent flow: $R_e > 4000$

   SI Units of Reynolds Number

$$\rho = \text{kg/m}^3,\ u = \text{m/s},\ d = m,\ \eta = \text{Ns/m}^2 = \text{kg/ms}$$

$$R_e = \frac{\rho u d}{\eta} = \frac{(\text{kg/m}^3)(\text{m/s})(\text{m})}{(\text{kg/ms})} = 1$$

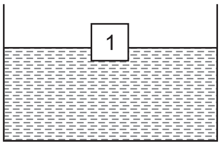
   i.e. it has no units. A quantity that has no units is known as a non-dimensional (or dimensionless) quantity. Thus, the Reynolds number, R_e is a non-dimensional number.

Solved Examples

TYPED PROBLEMS

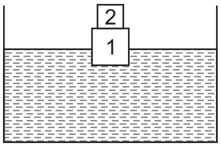
Type 1. *Based on Law of Floatation.*

Concept



(i)

⇓



(ii)

Whenever a block floats in a liquid (density of block should be less than density of liquid), there is one single equation.

Weight of solid = Upthrust on solid

$$\therefore \qquad V\rho_s g = V_i \rho_l g$$

Here, V is total volume of solid and V_i is immersed volume of solid.

$$\therefore \qquad \boxed{\frac{V_i}{V} = f_i = \frac{\rho_s}{\rho_l}}$$

Here, f_1 is immersed fraction of volume of solid.

Further, in figure (i) immersed volume is less and in figure (ii) immersed volume is more. Upthrust on immersed volume in figure (i) is equal to weight of block-1 and upthrust on immersed volume in figure (ii) is equal to weight of both the blocks 1 and 2. In other words, we can also say that upthrust on extra immersed volume in figure (ii) is equal to extra weight. In equation form, we can write as

$$m_1 g = (V_i)_1 \rho_l g \quad \text{and} \quad (m_1 + m_2) g = (V_i)_2 \rho_l g$$

$$\text{or} \qquad m_2 g = [(V_i)_2 - (V_i)_1] \rho_l g \quad \text{or} \quad (\Delta m) g \quad \text{or} \quad \Delta w = (\Delta V_i) \rho_l g$$

Note *If fluid is accelerated, then in the expression of upthrust g is replaced be g_{eff}.*

Example 1 *A block of wood floats in a bucket of water placed in a lift. Will the block sink more or less if the lift starts accelerating up?*

Solution Under nromal conditions, fraction of volume immersed under floating condition is

$$f_1 = \frac{\rho_s}{\rho_l} \qquad \text{...(i)}$$

If the lift starts accelerating up, then

$$\text{upthrust} - \text{weight} = ma$$

$$\therefore \qquad V_i \rho_l (g + a) - V\rho_s g = V\rho_s a$$

Solving this equation, we get

$$\frac{V_i}{V} = f_2 = \frac{\rho_s}{\rho_l} \qquad \text{...(ii)}$$

$$f_1 = f_2$$

So, the block neither sinks less nor more.

Note *From this example, we can make a general concept that by the acceleration of container, fraction (or percentage of volume immersed) does not change.*

Example 2 *A raft of wood (density $=600 kg/m^3$) of mass 120 kg floats in water. How much weight can be put on the raft to make it just sink?*

Solution Weight of raft + external weight = upthrust on 100% volume of raft

$$\therefore \qquad (120 + m)\, g = \left(\frac{120}{600}\right) \times 10^3 \times g$$

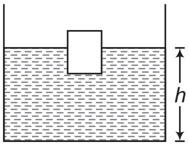
$$\therefore \qquad m = 80 \text{ kg}$$ **Ans.**

Note *Here immersed fraction used in upthrust is the total volume of solid* $\left(= \frac{m}{\rho} = \frac{120}{600} m^3\right)$.

Type 2. *Level Problems.*

Concept

Let us first note down the following four results:



(i) If ice is floating in water, then after melting of ice the level 'h' remains unchanged.

(ii) In floating to floating condition level 'h' remains unchanged.

For example Suppose some wooden blocks are floating in water kept inside a boat. If these wooden blocks are thrown into the water, then level will remain unchanged. This is because under both the conditions wooden blocks are floating.

(iii) In the floating to sink condition, level 'h' decreases.

For example In the above case if wooden blocks are replaced by stone pieces then level will fall. Because, initially stone pieces were floating (with the help of boat) but eventually those pieces will sink.

(iv) A solid is floating in a liquid of density ρ_1. After sometime, the solid melts and density (of solid) after melting (in liquid state) is suppose ρ_2, then there are following three cases :

Case 1 If $\rho_2 = \rho_1$, then level remains unchanged. Note that this is also result (i).

Case 2 If $\rho_2 < \rho_1$, then level will increase.

Case 3 If $\rho_2 > \rho_1$, then level will decrease.

Example 3 *A piece of ice is floating in a glass vessel filled with water. Then prove that level of water in the vessel remains unchanged after melting of ice .*

Note *This is result (i).*

Solution Let m be the mass of ice piece floating in water.

In equilibrium, weight of ice piece = upthrust

$$\text{or} \qquad mg = V_i \rho_w g$$

$$\text{or} \qquad V_i = \frac{m}{\rho_w} \qquad \text{...(i)}$$

Here, V_i is the volume of ice piece immersed in water.

When the ice melts, let V be the volume of water formed by m mass of ice. Then,

$$V = \frac{m}{\rho_w} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we see that

$$V_i = V$$

Hence, the level will not change. **Ans.**

Example 4 *A piece of ice having a stone frozen in it floats in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?*

Note *This is result (iii).*

Solution Let, m_1 = mass of ice, m_2 = mass of stone

ρ_S = density of stone and ρ_w = density of water

In equilibrium, when the piece of ice floats in water,

weight of (ice + stone) = upthrust

or $$(m_1 + m_2)g = V_i\rho_w g$$

$\therefore$ $$V_i = \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w} \qquad \text{...(i)}$$

Here, V_i = Volume of ice immersed

When the ice melts, m_1 mass of ice converts into water and stone of mass m_2 is completely submerged.

Volume of water formed by m_1 mass of ice,

$$V_1 = \frac{m_1}{\rho_w}$$

Volume of stone (which is also equal to the volume of water displaced)

$$V_2 = \frac{m_2}{\rho_S}$$

Since, $$\rho_S > \rho_w$$

Therefore, $$V_1 + V_2 < V_i$$

or the level of water will decrease. **Ans.**

Example 5 *A solid floats in a liquid of different material. Carry out an analysis to see whether the level of liquid in the container will rise or fall when the solid melts.*

Note *This is result (iv).*

Solution Let M = Mass of the floating solid

ρ_2 = density of liquid formed by the melting of the solid

ρ_1 = density of the liquid in which the solid is floating

The mass of liquid displaced by the solid is M. Hence, the volume of liquid displaced is $\frac{M}{\rho_1}$.

When the solid melts, the volume occupied by it is $\frac{M}{\rho_2}$. Hence, the level of liquid in container will rise or fall according as

$$\frac{M}{\rho_2} > \text{ or } < \frac{M}{\rho_1} \quad \text{i.e.} \quad \rho_2 < \text{ or } > \rho_1$$

There will be no change in the level if $\rho_1 = \rho_2$. In case of ice, floating in water $\rho_1 = \rho_2$ and hence, the level of water remains unchanged when ice melts.

Type 3. *Based on apparent weight or change in weight of a solid inside a liquid.*

Concept

If density of a solid is greater than density of liquid then it cannot float, it will sink. In this case, we hold the solid inside the liquid. Upthrust acts on its 100% volume and the solid is felt lighter. The equations are as under.

$w_{\text{app}} = w_{\text{actual}} -$ Upthrust on 100% volume or change in weight $\Delta w =$ Upthrust $= V \rho_l g$ ...(i)

From Eq. (i) we come across following two results:

(i) If the liquid in which solid is immersed, is water, then

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \text{Relative density of body (RD)}$$

This can be shown as under

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \frac{\text{Weight in air}}{\text{Upthrust in water}} = \frac{V\rho_S g}{V\rho_w g}$$

$$= \frac{\rho_S}{\rho_w} = \text{RD}$$

Hence proved.

(ii) Change in weight Δw is directly proportional to density of liquid or relative density of liquid. Thus,

$$\Delta w \propto \rho_l \text{ or } (\text{RD})_l \quad \Rightarrow \quad \frac{\Delta w_1}{\Delta w_2} = \frac{(\rho_l)_l}{(\rho_l)_2} = \frac{(\text{RD})_{l_1}}{(\text{RD})_{l_2}}$$

Note *In this case, 100% volume of solid remains immersed in the liquid, So V and g are same in two liquids in Eq. (i).*

Example 6 *A metallic sphere weighs 210 g in air, 180 g in water and 120 g in an unknown liquid. Find the density of metal and of liquid.*

Solution Relative density of metal

$$= \frac{\text{weight in air}}{\text{change in weight of water}} = \frac{210}{210 - 180} = 7$$

$\therefore$ Density of metal = 7 g/cm^2

Change in weight in a liquid = upthrust in liquid

$$= (V_{\text{solid}})(\rho_{\text{liquid}})\,g$$

or $$\Delta w \propto \rho_{\text{liquid}}$$

$\therefore$ $$\frac{\Delta w_l}{\Delta w_w} = \frac{\rho_l}{\rho_w}$$

or
$$\rho_l = \frac{\Delta w_l}{\Delta w_w}\rho_w$$
$$= \left(\frac{210-120}{210-180}\right)(1)\ \text{gm/cm}^3$$
$$= 3\ \text{g/cm}^3$$ **Ans.**

Example 7 *An ornament weighing 50 g in air weighs only 46 g is water. Assuming that some copper is mixed with gold to prepare the ornament. Find the amount of copper in it. Specific gravity of gold is 20 and that of copper is 10.*

Solution Let m be the mass of the copper in ornament. Then, mass of gold in it is $(50 - m)$.

Volume of copper $$V_1 = \frac{m}{10} \qquad \left(\text{volume} = \frac{\text{mass}}{\text{density}}\right)$$

and volume of gold $$V_2 = \frac{50-m}{20}$$

When immersed in water ($\rho_w = 1\ \text{g/cm}^3$)

Decrease in weight = upthrust

$\therefore$ $$(50-46)g = (V_1 + V_2)\rho_w g$$

or $$4 = \frac{m}{10} + \frac{50-m}{20} \quad \text{or} \quad 80 = 2m + 50 - m$$

$$m = 30\ \text{g}$$ **Ans.**

Example 8 *An iron casting containing a number of cavities weighs 6000 N in air and 4000 N in water. What is the volume of the cavities in the casting? Density of iron is $7.87\ g/cm^3$. Take $g = 9.8\ m/s^2$ and density of water $= 10^3\ kg/m^3$.*

Solution Let v be the volume of cavities and V the volume of solid iron. Then,

$$V = \frac{\text{mass}}{\text{density}} = \left(\frac{6000/9.8}{7.87 \times 10^3}\right) = 0.078\ \text{m}^3$$

Further, decrease in weight = upthrust

$\therefore$ $$(6000 - 4000) = (V + v)\rho_w g$$

or $$2000 = (0.078 + v) \times 10^3 \times 9.8$$

or $$0.078 + v \approx 0.2$$

$\therefore$ $$v = 0.12\ \text{m}^3$$ **Ans.**

Type 4. *Based on pressure equation.*

Concept

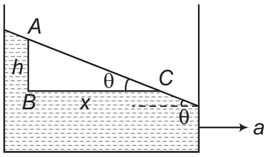
(i) If we wish to find pressure difference between two points A and B, then start from A, write pressure p_A and reach upto point B. Keep on writing increase or decrease in pressure and finally write pressure equal to p_B.

(ii) If fluids are at rest then there is no change in pressure in horizontal direction. In vertical direction write $(+\rho gh)$ while moving in downward direction and $(-\rho gh)$ while moving in upward direction (ρ = density of liquid).

(iii) If the fluid is accelerated, write $(+\rho ax)$ in horizontal direction opposite to horizontal acceleration 'a' and $(-\rho ax)$ in the same direction of acceleration. Here, x is the horizontal distance. In vertical direction, write $(\pm\rho g_e h)$. Here, g_e is the effective value of acceleration due to gravity.

(iv) If the fluid is rotating, write $\left(+\rho\frac{\omega^2x^2}{2}\right)$ while moving away from the axis and $\left(-\frac{\rho\omega^2x^2}{2}\right)$ in moving towards the axis.

Example 9 *A liquid kept in a container has a horizontal acceleration 'a' as shown in figure.*



Using the pressure equation along the path ABC find the angle θ.

Solution Writing pressure equation along the path ABC we have

$$p_A + \rho gh - \rho ax = p_C \qquad (\rho = \text{density of liquid})$$

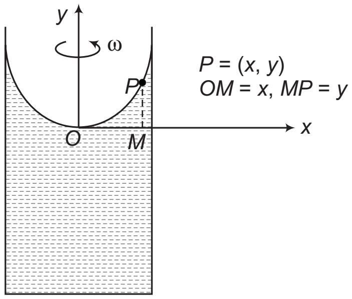
Points A and C are open to atmosphere.

$$\therefore \qquad p_A = p_C = p_0 = \text{atmosphere pressure}$$

$$\Rightarrow \qquad \rho gh - \rho ax = 0 \quad \text{or} \quad \frac{h}{x} = \frac{a}{g}$$

$$\Rightarrow \qquad \tan\theta = \frac{a}{g} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a}{g}\right)$$ **Ans.**

Example 10 *A liquid of density 'ρ' is rotated with an angular speed 'ω' as shown in figure. Using the pressure equation concept find the equation of free surface of the liquid.*



Solution Writing pressure equation along the path PMO.

$$p_P + \rho gy - \frac{\rho\omega^2x^2}{2} = p_O$$

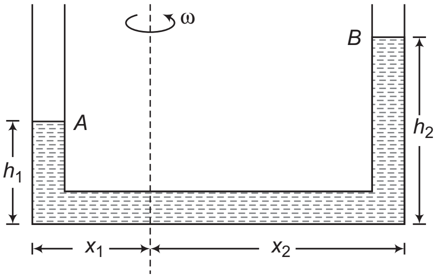
Points P and O are open to atmosphere.

So, pressures at these points are same (= atmospheric pressure). Therefore, the above equation becomes,

$$\rho gy - \frac{\rho \omega^2 x^2}{2} = 0 \quad \text{or} \quad y = \left(\frac{\omega^2}{2g}\right) x^2$$

This is the required parabolic equation of free surface of liquid.

Example 11



A liquid of density ρ is rotated by an angular speed ω as shown in figure, Using the concept of pressure equation, find a relation between h_1, h_2, x_1 and x_2.

Solution Writing pressure equation between points A and B, we have

$$p_A + \rho g h_1 - \frac{\rho \omega^2 x_1^2}{2} + \frac{\rho \omega^2 x_2^2}{2} - \rho g h_2 = p_B$$

Points A and B are open to atmosphere.

Therefore,

$$p_A = p_B = p_0 = \text{atmospheric pressure.}$$

Substituting the values in above equation we have,

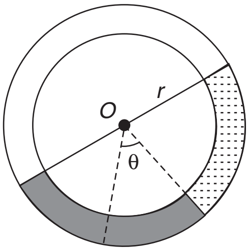
$$\rho g h_1 - \frac{\rho \omega^2 x_1^2}{2} + \frac{\rho \omega^2 x_2^2}{2} - \rho g h_2 = 0$$

On simplifying we get,

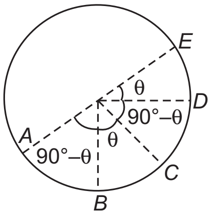
$$(h_2 - h_1) = \frac{\omega^2}{2g}\left(x_2^2 - x_1^2\right)$$

This is the desired relation between h_1, h_2, x_1 and x_2.

Example 12 *A small uniform tube is bent into a circle of radius r whose plane is vertical. Equal volumes of two fluids whose densities are* ρ *and* σ(ρ > σ) *fill half the circle. Find the angle that the radius passing through the interface makes with the vertical.*



Solution $h_{AB} = r - r\cos(90° - \theta) = r - r\sin\theta$



$$h_{BC} = r - r\cos\theta$$
$$h_{CD} = r\sin(90° - \theta) = r\cos\theta$$
$$h_{DE} = r\sin\theta$$

Writing pressure equation between points A and E we have

$$p_A + (r - r\sin\theta)\rho g - (r - r\cos\theta)\rho g - (r\cos\theta)(\sigma)\,g - (r\sin\theta)\,\sigma\,g = p_E$$

But $p_A = p_E = p_{\text{gas}}$

Solving this equation, we get

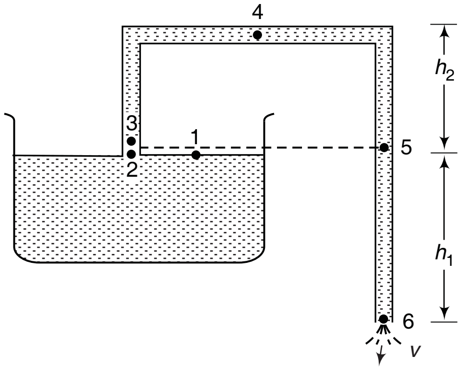
$$\tan\theta = \left(\frac{\rho - \sigma}{\rho - \sigma}\right)$$ **Ans.**

Type 5. *Based on concept of syphon.*

Concept

In the figure shown, $v_1 \approx v_2 = 0$

$$p_1 = p_2 = p_6 = p_0$$



Note *Point 3 is just above point 2.* $v_2 = 0$ *but* $v_3 = v$

If area of cross-section of pipe is uniform, then from continuity equation,

$$v_3 = v_4 = v_5 = v_6 = v \text{ (say)}$$

Applying Bernoulli's equation at 1 (or 2), 3, 4, 5 and 6, we have

$$p_0 + 0 + 0 = p_3 + \frac{1}{2}\rho v^2 + 0 = p_4 + \frac{1}{2}\rho v^2 + \rho g h_2$$
$$= p_5 + \frac{1}{2}\rho v^2 + 0 = p_0 + \frac{1}{2}\rho v^2 - \rho g h_1$$

From this equation, following conclusions can be made:

(i) $p_1 = p_2 = p_6 = p_0$

(ii) $p_3 = p_5 < p_0$

(iii) $v_1 = v_2 = 0$

(vi) $v_3 = v_4 = v_5 = v_6 = v$

(v) $v = \sqrt{2gh_1}$ so, h_1 should be greater than zero

(vi) $p_4 = p_0 - \rho g (h_1 + h_2)$

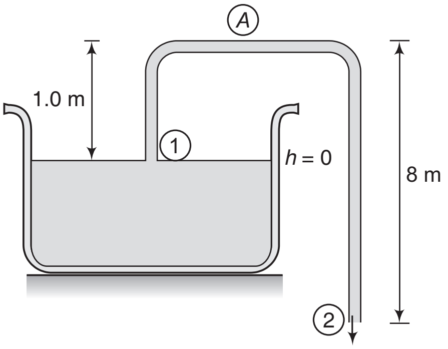
From the last equation we can see that p_4 decreases as $(h_1 + h_2)$ increases. Minimum value of pressure is at 4 or p_4 and this minimum value is zero and this will occur at,

$$0 = p_0 - \rho g (h_1 + h_2)_{\max}$$

Thus, $(h_1 + h_2)_{\max} = \dfrac{p_0}{\rho g}$ and simultaneously $h_1 > 0$

Thus, syphon will work when $h_1 > 0$ and $(h_1 + h_2) < \dfrac{p_0}{\rho g}$.

Example 13 *The U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface. The tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine the speed of the free jet and the minimum absolute pressure of the water in the bend. Given atmospheric pressure* $= 1.01 \times 10^5 N/m^2$, $g = 9.8\ m/s^2$ *and density of water* $= 10^3\ kg/m^3$.



Solution (a) Applying Bernoulli's equation between points (1) and (2)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since, area of reservoir >> area of pipe

$$v_1 \approx 0,$$

also $p_1 = p_2 =$ atmospheric pressure

So,
$$v_2 = \sqrt{2g(h_1 - h_2)}$$
$$= \sqrt{2 \times 9.8 \times 7}$$
$$= 11.7 \text{ m/s}$$

Ans.

(b) The minimum pressure in the bend will be at A. Therefore, applying Bernoulli's equation between (1) and (A)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A$$

Again, $v_1 \approx 0$ and from continuity equation $v_A = v_2$

or
$$p_A = p_1 + \rho g(h_1 - h_A) - \frac{1}{2}\rho v_2^2$$

Therefore, substituting the values, we have

$$p_A = (1.01 \times 10^5) + (1000)(9.8)(-1) - \frac{1}{2} \times (1000)(11.7)^2$$
$$= 2.27 \times 10^4 \text{ N/m}^2$$ **Ans.**

Type 6. *Based on pressure force and its torque.*

Concept

$$p = \frac{F}{A} \quad \Rightarrow \quad F = pA$$

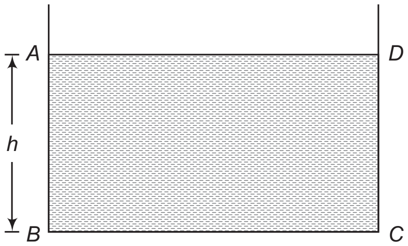
Let us call this force as the pressure force. Now, this force is calculated on a surface. If surface is horizontal then pressure is uniform at all points.

So, $F = pA$ can be applied directly. For calculation of torque, point of application of force is required. In the above case, point of application of force may be assumed at geometrical centre of the surface.

If the surface is vertical or inclined, pressure is non-uniform (it increases with depth) so pressure force and its torque can be obtained by integration. After finding force and torque by integration, we can also find point of application of this force by the relation.

$$r_\perp = \frac{\tau}{F} \qquad (\text{as } \tau = F \times r_\perp)$$

Example 14 *A liquid of density ρ is filled upto a height of 'h' in a container as shown in figure. Base of the container is a square of side L. Ignoring the atmospheric pressure find*



(a) pressure force F_1 on its base.

(b) torque of force F_1 about an axis passing through C and perpendicular to plane of paper.

(c) pressure force F_2 on the vertical side wall DC.

(d) torque of force F_2 about the same axis mentioned in part (b).

(e) point of application of force F_2.

Solution (a) Base is horizontal. So, we can directly apply

$$F_1 = pA = (\rho gh)(L)(L)$$

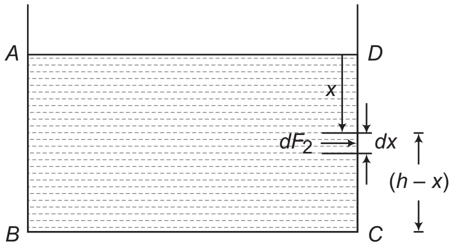
$$= \rho ghL^2 \qquad \textbf{Ans.}$$

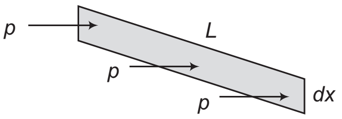
(b) Point of application of F_1 is at geometrical centre of base or at a perpendicular distance $\frac{L}{2}$ from the axis mentioned in the question.

$$\therefore \qquad \tau_{F_1} = F_1 \times r_\perp = (\rho ghL^2)\left(\frac{L}{2}\right)$$

$$= \frac{1}{2}\rho gh L^3 \qquad \textbf{Ans.}$$

(c) and (d) : Side wall is vertical. Pressure is non-uniform. So, force and torque both will be obtained by integration.





Pressure at depth x,

$$p = \rho gx$$

Area of small element shown in figure is

$$dA = L\,(dx)$$

$$\therefore \qquad dF_2 = (p)(dA) = (\rho gx)(L\,dx)$$

Perpendicular distance of this small force dF_2 from the axis mentioned in the question is $(h - x)$. Therefore, small torque of force dF_2 is

$$d\tau = (dF_2)(h - x)$$

$$= (\rho gx)(h - x)(L\,dx)$$

Now, $$F_2 = \int_{x=0}^{x=h} dF_2$$

and $$\tau_{F_2} = \int_{x=0}^{x=h} d\tau$$

Substituting the values and then integrating, we get

$$F_2 = \frac{\rho gLh^2}{2}$$

and $$\tau_{F_2} = \frac{\rho gLh^3}{6}$$

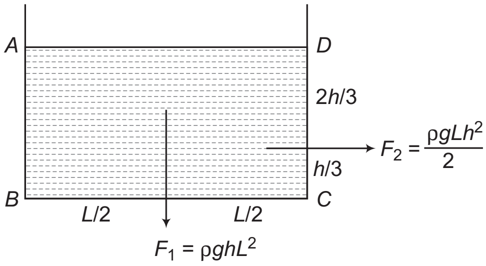
(e) For point of application of F_2, we can apply

$$r_\perp = \frac{\tau_{F_2}}{F_2} \qquad \text{(from } C\text{)}$$

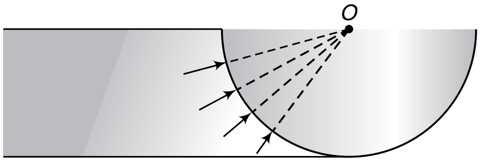
$$= \frac{(\rho gLh^3/6)}{(\rho gh^2L/2)} = \frac{h}{3}$$

Note that point of application of F_2 is below the centre, as pressure is not uniform. It is increasing with depth.

F_1 and F_2 and their points of application are shown in figure below.



Note



In the figure shown, torque of hydrostatic force about point O, the centre of a semicylindrical (or hemispherical) gate is zero as the hydrostatic force at all points passes through point O.

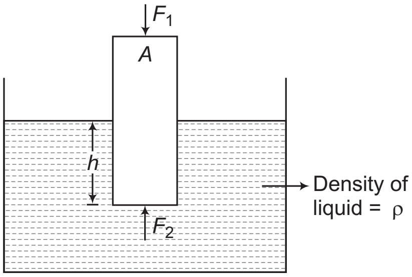
Type 7. *Based on the concept of pressure force and upthrust.*

Concept

(i) Pressure in a fluid (gas or a liquid) increases with depth. At a height (or depth) difference of h change in pressure is $\Delta p = \pm \rho g h$.

Here, ρ is the density of fluid. Now, density of air is almost negligible, so for small height differences it is almost constant $(= p_0)$. Therefore, value of atmospheric pressure is almost same everywhere. But this is not the case with a liquid, whose density is not negligible.

(ii) If a solid is floating in a liquid then net pressure force (including $p_0 A$) on this solid is upwards. This force is called upthrust and this is numerically equal to $V_i \rho_l g$ and in equilibrium this is equal to weight of the solid.

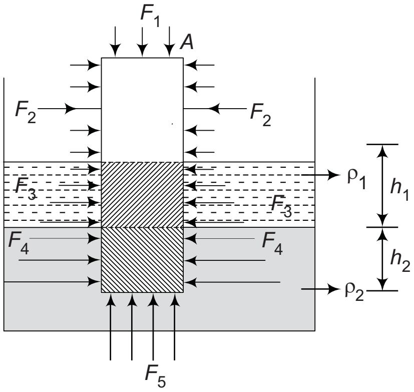


A plank of area of cross-section A is floating in a liquid of density ρ as shown in figure.

Net upward pressure force $= F_2 - F_1 = p_2 A - p_1 A$

$$= (p_0 + \rho gh) - p_0 A$$
$$= (hA)\rho g$$
$$= V_i \rho g \qquad (hA = \text{immersed volume})$$
$$= \text{upthrust}$$
$$= \text{weight of plank in equilibrium}$$

(iii) In some cases when a plank floats in two or more than two liquids then net force by a liquid on a plank is zero but immersed volume of plank on this liquid contributes in the upthrust.



A plank of area of cross-section A is floating in two liquids. h_1 height of the plank is immersed in first liquid and h_2 height in second liquid. Atmospheric pressures is constant at small height differences. So F_1 and F_2 forces shown in the figure can be obtained directly (= pressure × area). Inside the liquids pressure increases with depth. So, F_3 and F_4 will be obtained by integration as they are acting on vertical surfaces.

F_5 is acting on a horizontal surface so this force can also be obtained directly.

In the above figure

Net force on the plank by atmosphere $= F_1 = p_0 A$

Net force on the plank by liquid -1 = 0

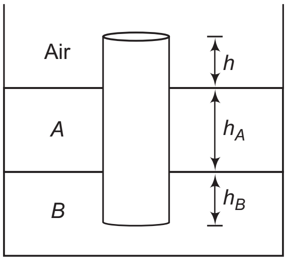
Net force on the plank by liquid - 2 $= F_5 = pA = (p_0 + \rho_1 gh_1 + \rho_2 gh_2) A$

Net upthrust pressure force on the plank $= F_5 - F_1$

$$= (h_1 A)\rho_1 g + (h_2 A)\rho_2 g$$
$$= (V_i)_1 \rho_1 g + (V_i)_2 \rho_2 g \qquad (V_i = \text{immersed volume})$$
$$= U_1 + U_2 \qquad (U = \text{upthrust})$$
$$= U_{\text{total}}$$
$$= \text{Weight of plank in equilibrium.}$$

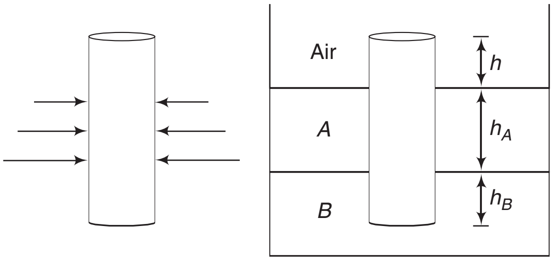
In this case, we can see that net force on plank by liquid-1 is zero, but immersed volume of plank in liquid-1 contributes in the upthrust $(= U_1)$.

Example 15 *A uniform solid cylinder of density* $0.8\ g/cm^3$ *floats in equilibrium in a combination of two non-mixing liquids A and B with its axis vertical. The densities of the liquids A and B are* $0.7\ g/cm^3$ *and* $1.2\ g/cm^3$*, respectively. The height of liquid A is* $h_A = 1.2$ *cm. The length of the part of the cylinder immersed in liquid B is* $h_B = 0.8$ *cm.* **(JEE 2002)**



(a) *Find the total force exerted by liquid A on the cylinder.*
(b) *Find h, the length of the part of the cylinder in air.*
(c) *The cylinder is depressed in such a way that its top surface is just below the upper surface of liquid A and is then released. Find the acceleration of the cylinder immediately after it is released.*

Solution (a) Liquid A is applying the hydrostatic force on cylinder uniformly from all the sides. So, net force is zero.



(b) In equilibrium

Weight of cylinder = Net upthrust on the cylinder

Let s be the area of cross-section of the cylinder, then

$$\text{weight} = (s)\,(h + h_A + h_B)\,\rho_{\text{cylinder}}\,g$$

and upthrust on the cylinder

$$= \text{upthrust due to liquid } A + \text{ upthrust due to liquid } B$$
$$= sh_A\rho_A g + sh_B\rho_B g$$

Equating these two,

$$s\,(h + h_A + h_B)\,\rho_{\text{cylinder}}\,g = sg(h_A\rho_A + h_B\rho_B)$$

or
$$(h + h_A + h_B)\,\rho_{\text{cylinder}} = h_A\rho_A + h_B\rho_B$$

Substituting,

$$h_A = 1.2 \text{ cm}, h_B = 0.8 \text{ cm and } \rho_A = 0.7 \text{ g/cm}^3$$
$$\rho_B = 1.2 \text{ g/cm}^3 \text{ and } \rho_{\text{cylinder}} = 0.8 \text{ g/cm}^3$$

In the above equation, we get $h = 0.25$ cm **Ans.**

(c) $$\text{Net upward force} = \text{extra upthrust} = sh\rho_B g$$

$\therefore$ $$\text{Net acceleration } a = \frac{\text{Force}}{\text{mass of cylinder}}$$

or $$a = \frac{sh\rho_B g}{s(h + h_A + h_B)\,\rho_{\text{cylinder}}}$$

or $$a = \frac{h\rho_B g}{(h + h_A + h_B)\,\rho_{\text{cylinder}}}$$

Substituting the values of h, h_A h_B, ρ_B and ρ_{cylinder},

we get, $$a = \frac{g}{6} \qquad \text{(upwards)}$$

Type 8. *Based on surface tension force.*

Concept

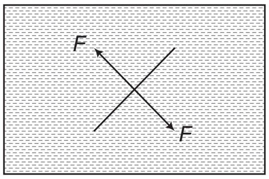
Surface tension is given by

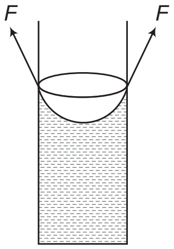
$$T = \frac{F}{l}$$

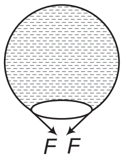
$\therefore$ Surface tension force, $F = Tl$

This force acts only on a line on free surface of the liquid. This force has a tendency to decrease the free surface of the liquid.

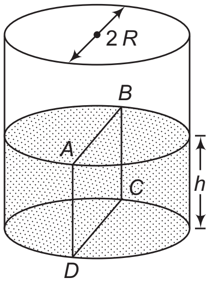
Following are given some figures for the direction of this force.







Example 16 Water is filled up to a height h in a beaker of radius R as shown in the figure. The density of water is ρ, the surface tension of water is T and the atmospheric pressure is p_0. Consider a vertical section $ABCD$ of the water column through a diameter of the beaker. The force on water on one side of this section by water on the other side of this section has magnitude **(JEE 2007)**



(a) $|2p_0Rh + \pi R^2\rho gh - 2RT|$

(b) $|2p_0Rh + R\rho gh^2 - 2RT|$

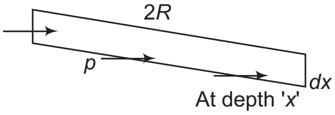
(c) $|p_0\pi R^2 + R\rho gh^2 - 2RT|$

(d) $|p_0\pi R^2 + R\rho gh^2 + 2RT|$

Solution Force from right hand side liquid on left hand side liquid.

(i) Due to surface tension force $= 2RT$ (towards right)

(ii) Due to liquid pressure force



$$= \int_{x=0}^{x=h} (p_0 + \rho g x)(2R \cdot dx)$$

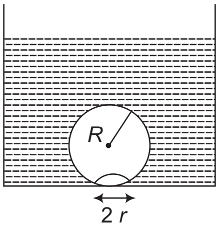
$$= (2p_0Rh + R\rho gh^2) \qquad \text{(towards left)}$$

$\therefore$ Net force is $|2p_0Rh + R\rho gh^2 - 2RT|$

$\therefore$ The correct option is (b).

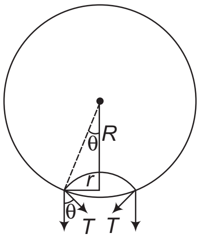
Example 17 *On heating water, bubbles beings formed at the bottom of the vessel detach and rise. Take the bubbles to be spheres of radius R and making a circular contact of radius r with the bottom of the vessel. If r <<R and the surface tension of water is T, value of r just before bubbles detach is (density of water is ρ_w)*

(JEE 2012 Main)



(a) $R^2\sqrt{\dfrac{\rho_w g}{3T}}$ (b) $R^2\sqrt{\dfrac{\rho_w g}{6T}}$ (c) $R^2\sqrt{\dfrac{\rho_w g}{T}}$ (d) $R^2\sqrt{\dfrac{3\rho_w g}{T}}$

Solution The bubble will detach if,



Buoyant force ≥ Surface tension force

$$\frac{4}{3}\pi R^3 \rho_w g \geq \int T \times dl \sin\theta$$

$$(\rho_w)\left(\frac{4}{3}\pi R^3\right) g \geq (T)(2\pi r)\sin\theta$$

Here, $\sin\theta = \dfrac{r}{R}$

Solving, $r = \sqrt{\dfrac{2\rho_w R^4 g}{3T}} = R^2\sqrt{\dfrac{2\rho_w g}{3T}}$

No option matches with the correct answer.

Passage (Ex. 18 to 20)

When liquid medicine of density ρ is to be put in the eye, it is done with the help of a dropper. As the bulb on the top of the dropper is pressed, a drop forms at the opening of the dropper. We wish to estimate the size of the drop.

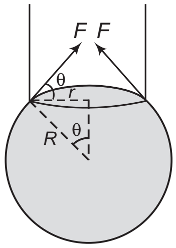
We first assume that the drop formed at the opening is spherical because that requires a minimum increase in its surface energy. To determine the size, we calculate the net vertical force due to the surface tension T when the radius of the drop is R. When this force becomes smaller than the weight of the drop, the drop gets detached from the dropper.

Example 18 *If the radius of the opening of the dropper is r, the vertical force due to the surface tension on the drop of radius R (assuming r << R) is*

(a) $2\pi rT$ (b) $2\pi RT$

(c) $\frac{2\pi r^2 T}{R}$ (d) $\frac{2\pi R^2 T}{r}$

Solution Vertical force due to surface tension,



$$F_v = F\sin\theta$$
$$= (T)(2\pi r)(r/R)$$
$$= \frac{2\pi r^2 T}{R}$$

∴ Correct option is (c).

Example 19 *If* $r = 5\times10^{-4}\ m, \rho = 10^3\ kg\ m^{-3}, g = 10\ ms^{-2}$, $T = 0.11\ Nm^{-1}$, *the radius of the drop when it detaches from the dropper is approximately*

(a) $1.4\times10^{-3}\ m$ (b) $3.3\times10^{-3}\ m$

(c) $2.0\times10^{-3}\ m$ (d) $4.1\times10^{-3}\ m$

Solution $\frac{2\pi r^2 T}{R} = mg = \frac{4}{3}\pi R^3\cdot\rho\cdot g$

$$\therefore \quad R^4 = \frac{3r^2T}{2\rho g} = \frac{3\times(5\times10^{-4})^2(0.11)}{2\times10^3\times10}$$
$$= 4.125\times10^{-12}\ \text{m}^4$$
$$\therefore \quad R = 1.425\times10^{-3}\text{m}$$
$$\approx 1.4\times10^{-3}\text{m}$$

∴ Correct option is (a).

Example 20 *After the drop detaches, its surface energy is*

(a) $1.4\times10^{-6}\ J$ (b) $2.7\times10^{-6}\ J$ (c) $5.4\times10^{-6}\ J$ (d) $8.1\times10^{-9}\ J$

Solution Surface energy,

$$E = (4\pi R^2)\,T$$
$$= (4\pi)(1.4\times10^{-3})^2(0.11)$$
$$= 2.7\times10^{-6}\text{J}$$

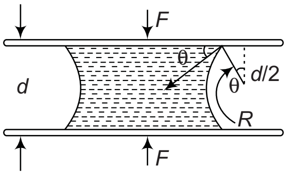
∴ Correct option is (b).

Type 9. *Based on force between two glass plates.*

Concept

When a small drop of water is placed between two glass plates put face to face, it forms a thin cylindrical film which is concave outward along its boundary. In article 16.9, we have seen that pressure difference on two sides of a cylindrical surface is

$$\Delta p = \frac{T}{R} \qquad \text{...(i)}$$



If d be the distance between the two plates and θ the angle of contact for water and glass, then, from the figure, $\cos\theta = \frac{\frac{1}{2}d}{R}$ or $\frac{1}{R} = \frac{2\cos\theta}{d}$.

Substituting for $\frac{1}{R}$ in Eq. (i), we get $\Delta p = \frac{2T}{d}\cos\theta$.

Angle θ can be taken zero for water and glass, i.e. $\cos\theta = 1$. Thus, the upper plate is pressed downward by the atmospheric pressure minus $\frac{2T}{d}$. Hence, the resultant downward pressure acting on the upper plate is $\frac{2T}{d}$. If A be the area of plate wetted by the film, the resultant force F pressing the upper plate downward is given by

$$F = \text{resultant pressure} \times \text{area} = \frac{2TA}{d}.$$

Example 21 *A drop of water of volume* $0.05\ cm^3$ *is pressed between two glass plates, as a consequence of which, it spreads and occupies an area of* $40\ cm^2$. *If the surface tension of water is 70 dyne/cm, find the normal force required to separate out the two glass plates in newton.*

Solution We have discussed above,

$$F = \frac{2AT}{d} = \frac{2A^2T}{Ad}$$

But, $Ad = \text{volume}$

$$\therefore \qquad F = \frac{2A^2T}{V}$$

Substituting the values we get,

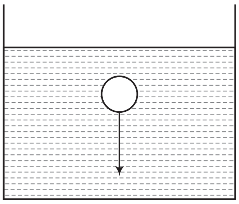
$$F = \frac{2 \times (40 \times 10^{-4})^2 \times (70 \times 10^{-3})}{0.05 \times 10^{-6}}$$

$$= 45\ \text{N}$$

Ans.

Miscellaneous Examples

Example 22 *A ball of volume V and density* ρ_1 *is moved downwards by a distance 'd' in liquid of density* ρ_2*. Find total change in potential energy of the system.*



Solution Decrease in potential energy of the ball.

$$= m_1 gh \qquad (m_1 = \text{mass of ball})$$

$$= (V\rho_1)\, gd$$

or
$$\Delta U_1 = -V\rho_1\, gd$$

When V volume of solid comes down, then it is replaced by V volume of liquid.

$\therefore$ Increase in potential energy of liquid :

$$= m_2 gh \qquad (m_2 = \text{mass of liquid of volume } V)$$

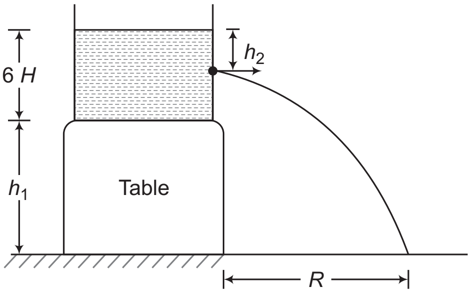
$$= (V\rho_2)\, gd$$

$\therefore$
$$\Delta U_2 = +V\rho_2 gd$$

Total change in potential energy,

$$\Delta U = \Delta U_1 + \Delta U_2 = V(\rho_2 - \rho_1)\, gd \qquad \textbf{Ans.}$$

Example 23 *In the figure shown find the value of* h_2 *for maximum range R if*



(a) $h_1 = 4H$ (b) $h_1 = 8H$

Also find the value of this maximum range in both cases.

Solution (a) $h_1 + 6H = 4H + 6H = 10H = h$ (say)

Maximum range will be obtained from

$$h_2 = \frac{h}{2} = 5H \qquad \textbf{Ans.}$$

and this maximum range will be,

$$R_{\max} = h = 10\,H \qquad \textbf{Ans.}$$

(b) In this case,

$$h = h_1 + 6H = 8H + 6H = 14H$$

$\frac{h}{2}$ or $7H$ point lies on the table. So, maximum range will be obtained from the bottommost point of the liquid container or $h_2 = 6H$ **Ans.**

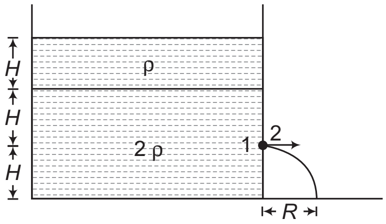
and this maximum range will be

$$R = 2\sqrt{h_2(h - h_2)}$$

$$= 2\sqrt{6H \times 8H} = 8\sqrt{3}H \qquad \textbf{Ans.}$$

Note *From top to centre range first increases and then from centre to bottom range then decreases. Therefore, in the second case range will be maximum from bottommost point of the liquid container.*

Example 24 *In the figure shown,*



find the range R.

Solution Applying Bernoulli's equation at points 1 and 2.

$$p_1 + \frac{1}{2}(2\rho)\,v_1^2 + (2\rho)\,gh_1 = p_2 + \frac{1}{2}(2\rho)\,v_2^2 + (2\rho)\,gh_2$$

Here, $v_1 \approx 0,\ v_2 = v,\ h_1 = h_2,\ p_2 = p_0$

and $p_1 = p_0 + \rho\,gH + 2\rho gH$

$$= p_0 + 3\rho gH$$

Substituting in the above equation we have,

$$p_0 + 3\rho gH = p_0 + \rho\,v^2$$

$\therefore$ $v = \sqrt{3\,gH}$

Now, this velocity is horizontal. So, time taken by the liquid to fall to the ground is free fall time or

$$t = \sqrt{\frac{2h}{g}} \qquad \text{(where, } h = H\text{)}$$

or

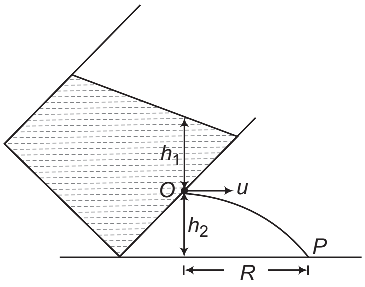
$$t = \sqrt{\frac{2H}{g}}$$

$\therefore$

$$R = vt = (\sqrt{3gH})\left(\sqrt{\frac{2H}{g}}\right)$$

$$= \sqrt{6}H \qquad \textbf{Ans.}$$

Example 25 *In the figure shown*



find v, t_{op} *and* R.

Solution $v = \sqrt{2 \times g \times \text{distance of orifice from top surface of liquid}}$ **Ans.**

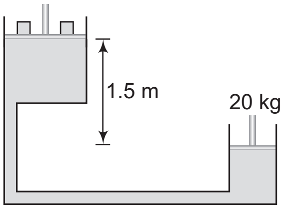
$$= \sqrt{2gh_1}$$

$$t_{op} = \text{free fall time} \qquad \text{(as } v \text{ is horizontal)}$$

$$= \sqrt{\frac{2h_2}{g}} \qquad \text{(where, } h_2 = \text{free fall height)}$$

$$R = v\,t_{op} = (\sqrt{2gh_1})\left(\sqrt{\frac{2h_2}{g}}\right) = 2\sqrt{h_1 h_2}$$ **Ans.**

Example 26 *Figure shows a hydraulic press with the larger piston of diameter 35 cm at a height of 1.5 m relative to the smaller piston of diameter 10 cm. The mass on the smaller piston is 20 kg. What is the force exerted on the load by the larger piston? The density of oil in the press is 750 kg/m*^{3}. *(Take* $g = 9.8\ m/s^2$*)*



Solution Pressure on the smaller piston $= \dfrac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2}$ N/m^2

Pressure on the larger piston $= \dfrac{F}{\pi \times (17.5 \times 10^{-2})^2}$ N/m^2

The difference between the two pressures $= h\rho g$

where $h = 1.5$ m

and $\rho = 750$ kg/m^3

Thus, $$\frac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2} - \frac{F}{\pi \times (17.5 \times 10^{-2})^2} = 1.5 \times 750 \times 9.8$$

which gives, $F = 1.3 \times 10^3$ N **Ans.**

Note *Atmospheric pressure is common to both pistons and has been ignored.*

Example 27 *A glass full of water upto a height of* 10 *cm has a bottom of area* 10 cm^2*, top of area* 30 cm^2 *and volume* 1 *litre.*

(a) Find the force exerted by the water on the bottom.

(b) Find the resultant force exerted by the sides of the glass on the water.

(c) If the glass is covered by a jar and the air inside the jar is completely pumped out, what will be the answers to parts (a) and (b).

(d) If a glass of different shape is used, provided the height, the bottom area, the top area and the volume are unchanged, will the answers to parts (a) and (b) change.

Take $g = 10\ m/s^2$*, density of water* $= 10^3\ kg/m^3$ *and atmospheric pressure* $= 1.01 \times 10^5\ N/m^2$

Solution (a) Force exerted by the water on the bottom

$$F_1 = (p_0 + \rho g h)A_1 \quad \text{...(i)}$$

Here, p_0 = atmospheric pressure $= 1.01 \times 10^5$ N/m^2

ρ = density of water $= 10^3$ k/gm^3

$g = 10$ m/s^2, $h = 10$ cm $= 0.1$ m

and A_1 = area of base $= 10$ cm^2 $= 10^{-3}$ m^2

Substituting in Eq. (i), we get

$$F_1 = (1.01 \times 10^5 + 10^3 \times 10 \times 0.1) \times 10^{-3}$$

or $F_1 = 102$ N (downwards) **Ans.**

(b) Force exerted by atmosphere on water

$$F_2 = (p_0)A_2$$

Here, A_2 = area of top $= 30$ cm^2 $= 3 \times 10^{-3}$ m^2

∴ $F_2 = (1.01 \times 10^5)(3 \times 10^{-3})$

$= 303$ N (downwards)

Force exerted by bottom on the water

$$F_3 = -F_1 \quad \text{or} \quad F_3 = 102 \text{ N (upwards)}$$

weight of water W = (volume)(density)(g) $= (10^{-3})(10^3)(10)$

$= 10$ N (downwards)

Let F be the force exerted by side walls on the water (upwards). Then, for equilibrium of water

Net upward force = net downward force

or $F + F_3 = F_2 + W$

∴ $F = F_2 + W - F_3 = 303 + 10 - 102$

or $F = 211$ N (upwards) **Ans.**

(c) If the air inside the jar is completely pumped out,

$F_1 = (\rho g h)A_1$ (as $p_0 = 0$)

$= (10^3)(10)(0.1)(10^{-3}) = 1$ N (downwards) **Ans.**

In this case, $F_2 = 0$

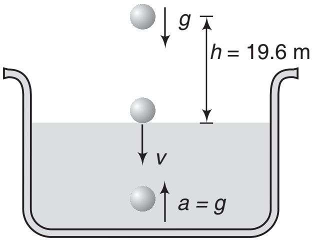
and $F_3 = 1$ N (upwards)

∴ $F = F_2 + W - F_3 = 0 + 10 - 1 = 9$ N (upwards) **Ans.**

(d) No, the answer will remain the same. Because the answers depend upon p_0, ρ, g, h, A_1 and A_2

Example 28 *A solid ball of density half that of water falls freely under gravity from a height of* 19.6 *m and then enters water. Upto what depth will the ball go. How much time will it take to come again to the water surface ? Neglect air resistance and viscosity effects in water. (Take* $g = 9.8\ m/s^2$*)*

Solution $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 19.6} = 19.6\ \text{m/s}$



Let ρ be the density of ball and 2ρ the density of water. Net retardation inside the water,

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$$

$$= \frac{V(2\rho)g - V(\rho)(g)}{V(\rho)}$$

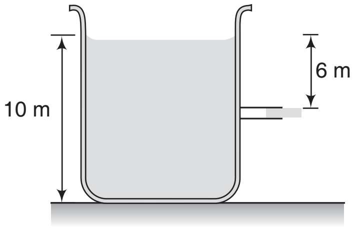
$$= g = 9.8\ \text{m/s}^2$$

Hence, the ball will go upto the same depth 19.6 m below the water surface. **Ans.**

Further, time taken by the ball to come back to water surface is,

$$t = 2\left(\frac{v}{a}\right) = 2\left(\frac{19.6}{9.8}\right) = 4\ \text{s}$$ **Ans.**

Example 29 *A fresh water on a reservoir is* 10 *m deep. A horizontal pipe* 4.0 *cm in diameter passes through the reservoir* 6.0 *m below the water surface as shown in figure. A plug secures the pipe opening.*



(a) Find the friction force between the plug and pipe wall.

(b) The plug is removed. What volume of water flows out of the pipe in 1 *h? Assume area of reservoir to be too large.*

Solution (a) Force of friction

= pressure difference on the sides of the plug × area of cross section of the plug

$= (\rho g h) A$

$= (10^3)(9.8)(6.0)(\pi)(2 \times 10^{-2})^2$

$= 73.9\ \text{N}$ **Ans.**

(b) Assuming the area of the reservoir to be too large,

Velocity of efflux

$$v = \sqrt{2gh} = \text{constant}$$

$$\therefore \qquad v = \sqrt{2 \times 9.8 \times 6} = 10.84 \text{ m/s}$$

Volume of water coming out per sec,

$$\frac{dV}{dt} = Av$$

$$= \pi (2 \times 10^{-2})^2 (10.84)$$

$$= 1.36 \times 10^{-2} \text{ m}^3/\text{s}$$

∴ The volume of water flowing through the pipe in 1 h

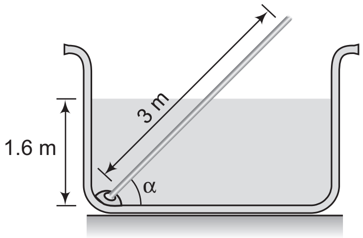
$$V = \left(\frac{dV}{dt}\right) t$$

$$= (1.36 \times 10^{-2})(3600)$$

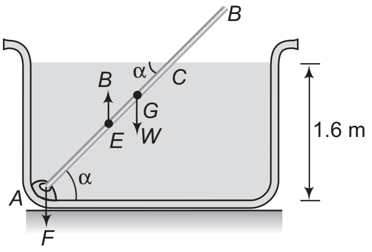
$$= 48.96 \text{ m}^3$$

Ans.

Example 30 *A wooden rod weighing 25 N is mounted on a hinge below the free surface of water as shown. The rod is 3 m long and uniform in cross section and the support is 1.6 m below the free surface. At what angle α rod is in equilibrium? The cross-section of the rod is $9.5 \times 10^{-4}\ m^2$ in area. Density of water is $1000\ kg/m^3$. Assume buoyancy to act at centre of immersion. $g = 9.8\ m/s^2$. Also find the reaction on the hinge in this position.*



Solution Let G be the mid-point of AB and E the mid point of AC (i.e. the centre of buoyancy)



$$AC = 1.6 \operatorname{cosec} \alpha$$

$$\text{Volume of } AC = (1.6 \times 9.5 \times 10^{-4}) \operatorname{cosec} \alpha$$

Weight of water displaced by AC

$$= (1.6 \times 9.5 \times 10^{-4} \times 10^3 \times 9.8) \operatorname{cosec} \alpha$$

$$= 14.896 \operatorname{cosec} \alpha$$

Hence, the buoyant force is 14.896 cosec α acting vertically upwards at E. While the weight of the rod is 25 N acting vertically downwards at G. Taking moments about A,

$$(14.896 \operatorname{cosec}\alpha)\,(AE\cos\alpha) = (25)(AG\cos\alpha)$$

or $$(14.896 \operatorname{cosec}\alpha)\left(\frac{1.6 \operatorname{cosec}\alpha}{2}\right) = 25 \times \frac{3}{2}$$

or $$\sin^2\alpha = 0.32$$

∴ $$\sin\alpha = 0.56$$

or $$\alpha = 34.3°$$ **Ans.**

Further, let F be the reaction at hinge in vertically downward direction. Then, considering the translatory equilibrium of rod in vertical direction we have,

$$F + \text{weight of the rod} = \text{upthrust}$$

∴ $$F = \text{upthrust} - \text{weight of the rod}$$

$$= 14.896 \operatorname{cosec}(34.3°) - 25$$

$$= 26.6 - 25$$

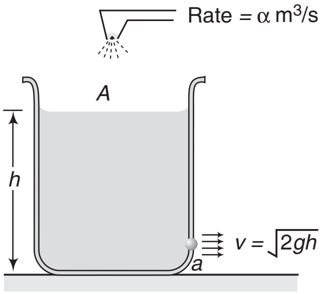
∴ $$F = 1.6\text{ N} \quad \text{(downwards)}$$ **Ans.**

Example 31 *A cylindrical tank of base area A has a small hole of area a at the bottom. At time $t = 0$, a tap starts to supply water into the tank at a constant rate $\alpha\ m^3/s$.*

(a) what is the maximum level of water $h_{\max}$ in the tank?

(b) find the time when level of water becomes h ($< h_{\max}$).

Solution (a) Level will be maximum when



Rate of inflow of water = rate of outflow of water

i.e. $$\alpha = av$$

or $$\alpha = a\sqrt{2gh_{\max}}$$

⇒ $$h_{\max} = \frac{\alpha^2}{2ga^2}$$ **Ans.**

(b) Let at time t, the level of water be h. Then,

$$A\left(\frac{dh}{dt}\right) = \alpha - a\sqrt{2gh} \quad \text{or} \quad \int_0^h \frac{dh}{\alpha - a\sqrt{2gh}} = \int_0^t \frac{dt}{A}$$

Solving this, we get

$$t = \frac{A}{ag}\left[\frac{\alpha}{a}\ln\left\{\frac{\alpha - a\sqrt{2gh}}{\alpha}\right\} - \sqrt{2gh}\right]$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Pressure is a vector quantity.

Reason : Pressure $P = \frac{F}{A}$. Here F, the force is a vector quantity.

2. Assertion : Surface tension $\left(T = \frac{F}{l}\right)$ is not a vector quantity.

Reason : Direction of force is specified.

3. Assertion : At depth h below the water surface pressure is p. Then at depth $2h$ pressure will be $2p$. (Ignore density variation).

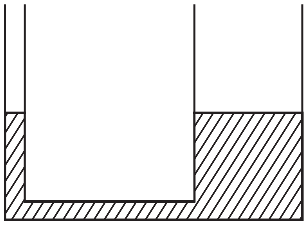
Reason : With depth pressure increases linearly.

4. Assertion : Weight of solid in air is w and in water is $\frac{2w}{3}$. Then relative density of solid is 3.0.

Reason : Relative density of any solid is given by

$$\text{RD} = \frac{\text{Weight in air}}{\text{Change in weight in water}}$$

5. Assertion : Water is filled in a U-tube of different cross-sectional area on two sides as shown in figure. Now equal amount of oil $(RD = 0.5)$ is poured on two sides. Level of water on both sides will remain unchanged.

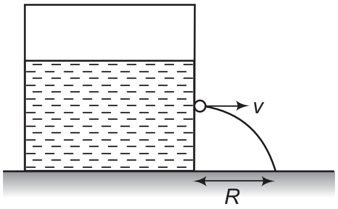


Reason : Same weight of oil poured on two sides will produce different pressures.

6. Assertion : An ideal fluid is flowing through a pipe. Speed of fluid particles is more at places where pressure is low.

Reason : Bernoulli's theorem can be derived from work-energy theorem.

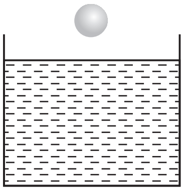
7. **Assertion :** In the figure shown v and R will increase if pressure above the liquid surface inside the chamber is increased.



Reason : Value of v or R is independent of density of liquid.

8. **Assertion :** A ball is dropped from a certain height above the free surface of an ideal fluid. When the ball enters the liquid it may accelerate or retard.

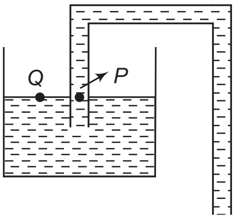
Reason : Ball accelerates or retards it all depends on the density of ball and the density of liquid.



9. **Assertion :** On moon, barometer height will be six times compared to the height on earth.

Reason : Value of g on moon's surface is $\frac{1}{6}$ the value of g on earth's surface.

10. **Assertion :** In the siphon shown in figure, pressure at P is equal to atmospheric pressure.



Reason : Pressure at Q is atmospheric pressure and points P and Q are at same levels.

11. **Assertion :** Force of buoyancy due to atmosphere on a small body is almost zero (or negligible).

Reason : If a body is completely submerged in a fluid, then buoyant force is zero.

Objective Questions

Single Correct Option

1. When a sphere falling in a viscous fluid attains a terminal velocity, then
   (a) the net force acting on the sphere is zero
   (b) the drag force balances the buoyant force
   (c) the drag force balances the weight of the sphere
   (d) the buoyant force balances the weight and drag force

2. Which one of the following represents the correct dimensions of the quantity : $x = \frac{\eta}{\rho}$, where $\eta =$ coefficient of viscosity and $\rho =$ the density of a liquid?

(a) $[ML^{-2}T^{-1}]$ (b) $[ML^{-4}T^{-2}]$

(c) $[ML^{-5}T^{-2}]$ (d) $[M^0L^2T^{-1}]$

3. Viscosity of liquids

(a) increases with increase in temperature

(b) is independent of temperature

(c) decreases with decrease in temperature

(d) decreases with increase in temperature

4. At critical temperature, the surface tension of a liquid

(a) is zero

(b) is infinity

(c) is same as that at any other temperature

(d) cannot be determined

5. A liquid will not wet the surface of a solid if the angle of contact is

(a) 0° (b) 45°

(c) 60° (d) >90°

6. The lower end of a capillary tube touches a liquid whose angle of contact is 110°,the liquid

(a) rises into the tube

(b) falls in the tube

(c) may rise or fall inside

(d) neither rises nor falls inside the tube

7. Two water droplets combine to form a large drop. In this process energy is

(a) liberated

(b) absorbed

(c) neither liberated nor absorbed

(d) sometimes liberated and sometimes absorbed

8. A number of small drops of mercury adiabatically coalesce to form a single drop. The temperature of the drop will

(a) increase (b) remain same

(c) decrease (d) depend on size

9. Two soap bubbles in vacuum of radius 3 cm and 4 cm coalesce to form a single bubble under isothermal conditions. Then the radius of bigger bubble is

(a) 7 cm (b) $\frac{12}{7}$ cm

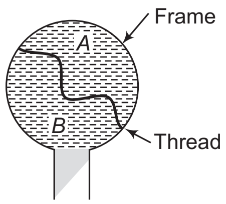
(c) 12 cm (d) 5 cm

10. A small ball (mass m) falling under gravity in a viscous medium experiences a drag force proportional to the instantaneous speed u such that $F_{\text{drag}} = ku$. Then the terminal speed of ball within viscous medium is

(a) $\frac{k}{mg}$ (b) $\frac{mg}{k}$

(c) $\sqrt{\frac{mg}{k}}$ (d) None of these

11. A thread is tied slightly loose to a wire frame as in figure and the frame is dipped into a soap solution and taken out. The frame is completely covered with the film. When the portion *A* is punctured with a pin, the thread



(a) becomes concave towards *A*
(b) becomes convex towards *A*
(c) either (a) or (b) depending on the size of *A* with respect to *B*
(d) remains in the initial position

12. A steel ball of mass m falls in a viscous liquid with terminal velocity v, then the steel ball of mass 8 m will fall in the same liquid with terminal velocity

(a) v (b) $4v$ (c) $8v$ (d) $16\sqrt{2}v$

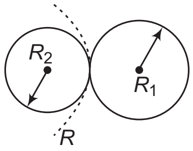
13. A liquid flows between two parallel plates along the x-axis. The difference between the velocity of two layers separated by the distance dy is dv. If A is the area of each plate, then Newton's law of viscosity may be written as

(a) $F = -\eta A\dfrac{dv}{dx}$ (b) $F = +\eta A\dfrac{dv}{dx}$
(c) $F = -\eta A\dfrac{dv}{dy}$ (d) $F = +\eta A\dfrac{dv}{dy}$

14. The work done to split a liquid drop of radius R into N identical drops is (take σ as the surface tension of the liquid)

(a) $4\pi R^2(N^{1/3}-1)\,\sigma$ (b) $4\pi R^2 N\sigma$
(c) $4\pi R^2(N^{1/2}-1)$ (d) None of these

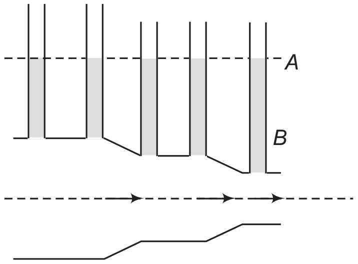
15. Two soap bubbles of different radii R_1 and $R_2(< R_1)$ coalesce to form an interface of radius R as shown in figure. The correct value of R is



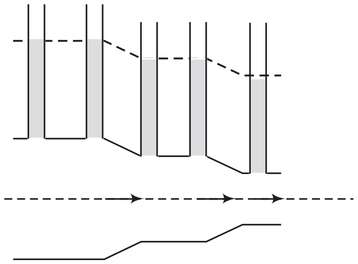
(a) $R = R_1 - R_2$ (b) $R = \dfrac{R_1 + R_2}{2}$
(c) $\dfrac{1}{R} = \dfrac{1}{R_2} - \dfrac{1}{R_1}$ (d) $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

16. A viscous liquid flows through a horizontal pipe of varying cross-sectional area. Identify the option which correctly represents the variation of height of rise of liquid in each vertical tube

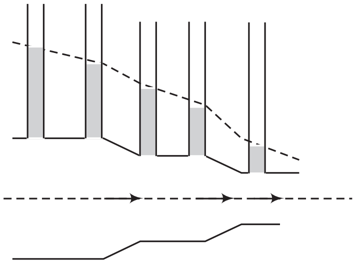
(a)



(b)



(c)



(d) None of these

17. The terminal velocity of a rain drop is 30 cm/s. If the viscosity of air is 1.8×10^{-5} Nsm^{-2}. The radius of rain drop is

(a) 1 μm
(b) 0.5 mm
(c) 0.05 mm
(d) 1 mm

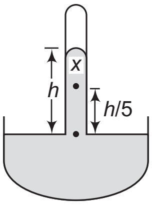
18. If a capillary tube is dipped and the liquid levels inside and outside the tube are same, then the angle of contact is

(a) zero
(b) 90°
(c) 45°
(d) Cannot be obtained

19. Uniform speed of 2 cm diameter ball is 20 cm/s in a viscous liquid. Then, the speed of 1 cm diameter ball in the same liquid is

(a) 5 cms^{-1}
(b) 10 cms^{-1}
(c) 40 cms^{-1}
(d) 80 cms^{-1}

20. The height of mercury barometer is h when the atmospheric pressure is 10^5 Pa.
The pressure at x in the shown diagram is



(a) 10^5 Pa
(b) 0.8×10^5 Pa
(c) 0.2×10^5 Pa
(d) 120×10^5 Pa

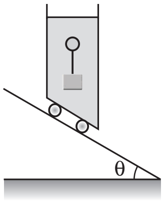
21. A body floats in water with its one-third volume above the surface. The same body floats in a liquid with one-third volume immersed. The density of the liquid is

(a) 9 times more than that of water
(b) 2 times more than that of water
(c) 3 times more than that of water
(d) 1.5 times more than that of water

22. A piece of ice is floating in a beaker containing thick sugar solution of water. As the ice melts, the total level of the liquid

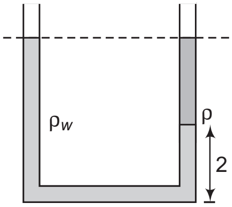
(a) increases
(b) decreases
(c) remains unchanged
(d) insufficient data

23. A body floats in completely immersed condition in water as shown in figure. As the whole system is allowed to slide down freely along the inclined surface, the magnitude of buoyant force



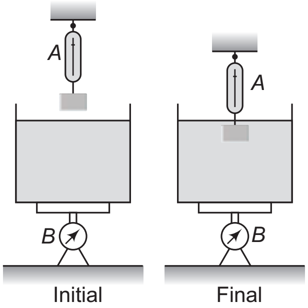
(a) remains unchanged
(b) increases
(c) decreases
(d) becomes zero

24. The figure represents a U-tube of uniform cross-section filled with two immiscible liquids. One is water with density ρ_w and the other liquid is of density ρ. The liquid interface lies 2 cm above the base. The relation between ρ and ρ_w is



(a) $\rho = \rho_w$ (b) $\rho = 1.02\,\rho_w$ (c) $\rho = 1.2\,\rho_w$ (d) None of these

25. For the arrangement shown in figure, initially the balance A and B reads F_1 and F_2 respectively and $F_1 > F_2$. Finally when the block is immersed in the liquid then the readings of balance A and B are f_1 and f_2 respectively. Identify the statement which is **not always** (where, F is some force) **correct** statement.



(a) $f_1 > f_2$
(b) $F_1 + F > F_2 + F$
(c) $f_1 + f_2 = F_1 + F_2$
(d) None of these

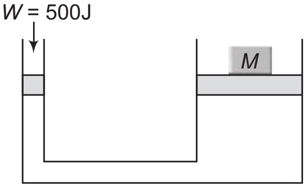
26. When a tap is closed, the manometer attached to the pipe reads 3.5×10^5 Nm^{-2}. When the tap is opened, the reading of manometer falls to 3.0×10^5 Nm^{-2}. The velocity of water in the pipe is

(a) 0.1 ms^{-1}
(b) 1 ms^{-1}
(c) 5 ms^{-1}
(d) 10 ms^{-1}

27. A balloon of mass M descends with an acceleration a_0. The mass that must be thrown out in order to give the balloon an equal upward acceleration will be

(a) $\dfrac{Ma_0}{g}$ (b) $\dfrac{2Ma_0}{g}$ (c) $\dfrac{2Ma_0}{g + a_0}$ (d) $\dfrac{M\,(g + a_0)}{a_0}$

28. The hydraulic press shown in the figure is used to raise the mass M through a height of 0.5 cm by performing 500 J of work at the small piston. The diameter of the large piston is 10 cm, while that of the smaller one is 2 cm. The mass M is



(a) 100 kg (b) 10^6 kg (c) 10^3 kg (d) None of these

29. When equal volumes of two substances are mixed, the specific gravity of the mixture is 4. When equal weights of the same substances are mixed, the specific gravity of the mixture is 3. The specific gravities of the two substances could be

(a) 6 and 2
(b) 3 and 4
(c) 2.5 and 3.5
(d) 5 and 3

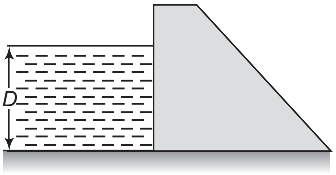
30. A block of ice of total area A and thickness 0.5 m is floating in water. In order to just support a man of mass 100 kg, the area A should be (the specific gravity of ice is 0.9)

(a) 2.2 m^2
(b) 1.0 m^2
(c) 0.5 m^2
(d) None of these

31. A piece of gold ($\rho = 19.3\ \mathrm{g/cm^3}$) has a cavity in it. It weights 38.2 g in air and 36.2 g in water. The volume of the cavity in gold is

(a) 0.2 cm^3
(b) 0.04 cm^3
(c) 0.02 cm^3
(d) 0.01 cm^3

32. Water stands at a depth D behind the vertical upstream face of a dam as shown in the figure. The force exerted on the dam by water per unit width is

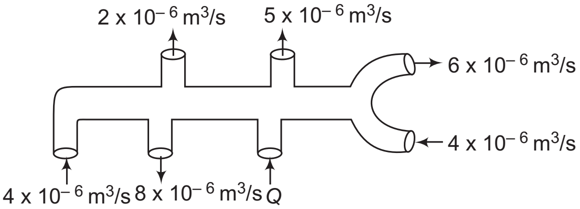


(a) $\frac{1}{3}\rho g D^2$
(b) $\frac{1}{2}\rho g D^2$
(c) $\frac{1}{3}\rho g D$
(d) $\frac{1}{2}(\rho g D)^2$

33. The volume of a liquid flowing per second out of an orifice at the bottom of a tank does not depend upon

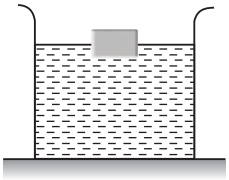
(a) the height of the liquid above the orifice
(b) the acceleration due to gravity
(c) the density of the liquid
(d) the area of the orifice

34. The pipe shows the volume flow rate of an ideal liquid at certain time and its direction. What is the value of Q in $\mathrm{m^3/s}$? (Assume steady state and equal area of cross section at each opening)



(a) 10×10^{-6}
(b) 11×10^{-6}
(c) 13×10^{-6}
(d) 18×10^{-6}

35. A uniform cube of mass M is floating on the surface of a liquid with three fourth of its volume immersed in the liquid (density = ρ). The length of the side of the cube is equal to



(a) $(4M/3\rho)^{2/3}$ (b) $(M/3\rho)^{2/3}$ (c) $(M/4\rho)^{2/3}$ (d) None of these

36. Water rises to a height of 10 cm in a certain capillary tube. An another identical tube when dipped in mercury the level of mercury is depressed by 3.42 cm. Density of mercury is 13.6 g/cc. The angle of contact for water in contact with glass is 0° and mercury in contact with glass is 135°. The ratio of surface tension of water to that of Hg is

(a) 1 : 3 (b) 1 : 4 (c) 1 : 5.5 (d) 1 : 6.5

37. A capillary glass tube records a rise of 20 cm when dipped in water. When the area of cross-section of the tube is reduced to half of the former value, water will rise to a height of

(a) $10\sqrt{2}$ cm (b) 10 cm (c) 20 cm (d) $20\sqrt{2}$ cm

38. A cylindrical vessel open at the top is 20 cm high and 10 cm in diameter. A circular hole of cross sectional area 1 cm^2 is cut at the centre of the bottom of the vessel. Water flows from a tube above it into the vessel at the rate of 10^2 cm^3/s. The height of water in the vessel under steady state is (Take $g = 10\ \text{m/s}^2$)

(a) 20 cm (b) 15 cm (c) 10 cm (d) 5 cm

39. A horizontal pipeline carries water in a streamline flow. At a point along the tube where the cross sectional area is 10^{-2} m^2, the water velocity is 2 m/s and the pressure is 8000 Pa. The pressure of water at another point where cross sectional area is 0.5×10^{-2} m^2 is

(a) 4000 Pa (b) 1000 Pa (c) 2000 Pa (d) 3000 Pa

40. Eight spherical rain drops of the same mass and radius are falling down with a terminal speed of 6 cms^{-1}. If they coalesce to form one big drop, what will be its terminal speed? Neglect the buoyancy due to air

(a) 1.5 cms^{-1} (b) 6 cms^{-1} (c) 24 cms^{-1} (d) 32 cms^{-1}

41. In a surface tension experiment with a capillary tube water rises upto 0.1 m. If the same experiment is repeated in an artificial satellite, which is revolving around the earth; water will rise in the capillary tube upto a height of

(a) 0.1 m (b) 0.2 m (c) 0.98 m (d) full length of tube

42. Two unequal soap bubbles are formed one on each side of a tube closed in the middle by a tap. What happens when the tap is opened to put the two bubbles in communication ?

(a) No air passes in any direction as the pressures are the same on two sides of the tap

(b) Larger bubble shrinks and smaller bubble increases in size till they become equal in size

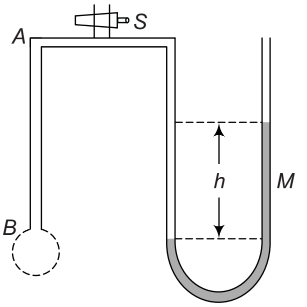
(c) Smaller bubble gradually collapses and the bigger one increases in size

(d) None of the above

43. A capillary tube of radius R is immersed in water and water rises in it to a height h. Mass of water in capillary tube is M. If the radius of the tube is doubled, mass of water that will rise in the capillary tube will be

(a) $2M$ (b) M (c) $\frac{M}{2}$ (d) $4M$

44. A tube of fine bore AB is connected to a manometer M as shown. The stop cock S controls the flow of air. AB is dipped into a liquid whose surface tension is T. On opening the stop cock for a while, a bubble is formed at B and the manometer level is recorded, showing a difference h in the levels in the two arms, if p be the density of manometer liquid and r the radius of curvature of the bubble, then the surface tension T of the liquid is given by



(a) $\rho\, hrg$ (b) $2\rho hgr$ (c) $4\rho hrg$ (d) $\frac{rh\rho g}{4}$

45. A vessel whose bottom has round holes with a diameter of $d = 0.1$ mm is filled with water. The maximum height of the water level h at which the water does not flow out, will be (The water does not wet the bottom of the vessel). [ST of water = 70 dyne/cm]

(a) $h = 24.0$ cm (b) $h = 25.0$ cm (c) $h = 26.0$ cm (d) $h = 28.0$ cm

46. A large number of liquid drops each of radius 'a' coalesce to form a single spherical drop of radius b. The energy released in the process is converted into kinetic energy of the big drop formed. The speed of big drop will be

(a) $\sqrt{\frac{6T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$ (b) $\sqrt{\frac{4T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$

(c) $\sqrt{\frac{8T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$ (d) $\sqrt{\frac{5T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$

47. A glass capillary tube (closed from top) of inner diameter 0.28 mm is lowered vertically into water in a vessel. The pressure in the capillary tube so that water level in the tube is same as that in the vessel in $\mathrm{N/m^2}$ is (surface tension of water $= 0.7\,\mathrm{N/m}$ and atmospheric pressure $= 10^5\,\mathrm{N/m^2}$)

(a) 10^3 (b) 99×10^3

(c) 100×10^3 (d) 101×10^3

48. A thin wire is bent in the form of a ring of diameter 3.0 cm. The ring is placed horizontally on the surface of soap solution and then raised up slowly. Upward force necessary to break the vertical film formed between the ring and the solution is

(a) $6\pi T$ dyne (b) $2\pi T$ dyne

(c) $4\pi T$ dyne (d) $3\pi T$ dyne

49. One end of a glass capillary tube with a radius $r = 0.05$ cm is immersed into water to a depth of $h = 2$ cm. Excess pressure required to blow an air bubble out of the lower end of the tube will be (S.T of water = 70 dyne/cm). Take $g = 980\ \mathrm{cm/s^2}$

(a) 2840 dyne/cm^2 (b) 5840 dyne/cm^2

(c) 7840 dyne/cm^2 (d) 4760 dyne/cm^2

Subjective Questions

Note *Question numbers 16, 17 and 29 are based on flow of viscous liquids. Engineering aspirants can skip those problems.*

1. A body of weight w_1 when floats in water displaces an amount of water w_2. Then $w_1 < w_2$. Is this statement true or false?

2. A weightless balloon is filled with water. What will be its apparent weight when weighed in water?

3. Two vessels A and B have same base area. Equal volumes of a liquid are poured in the two vessels to different heights h_A and $h_B (> h_A)$. In which vessel, the force on the base of vessel will be more?

4. The work done in blowing a bubble of volume V is W, then what is the work done in blowing a soap bubble of volume $2V$?

5. A man is sitting in a boat which is floating in a pond. If the man drinks some water from the pond, how will the level of water in the pond change?

6. A metal ball weighs 0.096 N . When suspended in water it has an apparent weight of 0.071 N. Find the density of the metal.

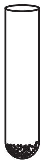
7. A block of wood has a mass of 25 g. When a 5 g metal piece with a volume of 2 cm^3 is attached to the bottom of the block, the wood barely floats in water. What is the volume V of the wood?

8. What is the minimum volume of a block of wood (density = $850\,\text{kg/m}^3$) if it is to hold a 50 kg woman entirely above the water when she stands on it ?

9. A cubical block of ice floating in water has to support a metal piece weighing 0.5 kg. What can be the minimum edge of the block so that it does not sink in water? Specific gravity of ice = 0.9.

10. When a cube of wood floats in water, 60% of its volume is submerged. When the same cube floats in an unknown fluid 85% of its volume is submerged. Find the densities of wood and the unknown fluid.

11. A glass tube of radius 0.8 cm floats vertical in water, as shown in figure. What mass of lead pellets would cause the tube to sink a further 3 cm?

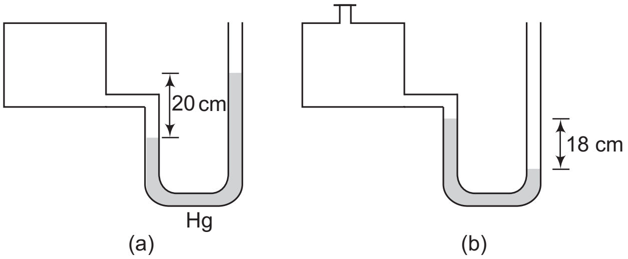


12. A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross section of the piston carrying the load is $425\,\text{cm}^2$. What maximum pressure would the smaller piston have to bear?

13. A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the relative density of sprit?

14. In the above question, if 15.0 cm of water and spirit each are further, poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms?
(Relative density of mercury = 13.6)

15. A manometer reads the pressure of a gas in an enclosure as shown in figure (a) When some of the gas is removed by a pump, the manometer reads as in (b). The liquid used in the manometers is mercury and the atmospheric pressure is 76 cm of mercury.



(i) Give the absolute and gauge pressure of the gas in the enclosure for cases (a) and (b) in units of cm of mercury.

(ii) How would the levels change in case (b) if 13.6 cm of water are poured into the right limb of the manometer?

16. Water at 20°C is flowing in a pipe of radius 20.0 cm. The viscosity of water at 20°C is 1.005 centipoise. If the water's speed in the centre of the pipe is 3.00 m/s, what is water's speed:

(a) 10.0 cm from the centre of the pipe (half way between the centre and the walls)

(b) at the walls of the pipe?

17. Water at 20°C is flowing in a horizontal pipe that is 20.0 m long. The flow is laminar and the water completely fills the pipe. A pump maintains a gauge pressure of 1400 Pa, at a large tank at one end of the pipe. The other end of the pipe is open to the air, The viscosity of water at 20°C is 1.005 poise.

(a) If the pipe has diameter 8.0 cm, what is the volume flow rate?

(b) What gauge pressure must the pump provide to achieve the same volume flow rate for a pipe with a diameter of 4.0 cm?

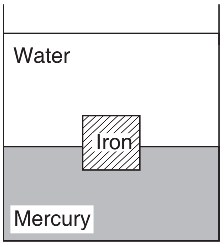
(c) For pipe in part (a) and the same gauge pressure maintained by the pump, what does the volume flow rate become if the water is at a temperature of 60°C (the viscosity of water at 60°C is 0.469 poise)?

18. An irregular piece of metal weighs 10.00 g in air and 8.00 g when submerged in water.

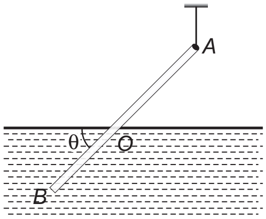
(a) Find the volume of the metal and its density.

(b) If the same piece of metal weighs 8.50 g when immersed in a particular oil, what is the density of the oil ?

19. A tank contains water on top of mercury. A cube of iron, 60 mm along each edge, is sitting upright in equilibrium in the liquids. Find how much of it is in each liquid. The densities of iron and mercury are $7.7 \times 10^3\ \text{kg/m}^3$ and $13.6 \times 10^3\ \text{kg/m}^3$ respectively.



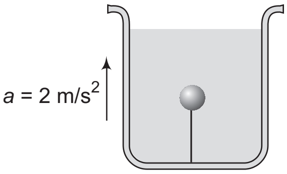
20. A uniform rod AB, 4 m long and weighing 12 kg, is supported at end A, with a 6 kg lead weight at B. The rod floats as shown in figure with one-half of its length submerged. The buoyant force on the lead mass is negligible as it is of negligible volume. Find the tension in the cord and the total volume of the rod.



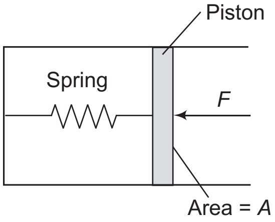
21. A solid sphere of mass $m = 2$ kg and density $\rho = 500\,\text{kg/m}^3$ is held stationary relative to a tank filled with water. The tank is accelerating upward with acceleration 2 m/s^2. Calculate

(a) Tension in the thread connected between the sphere and the bottom of the tank.

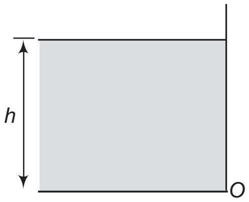
(b) If the thread snaps, calculate the acceleration of sphere with respect to the tank.
(Density of water = 1000 kg/m^3, $g = 10\,\text{m/s}^2$)



22. The pressure gauge shown in figure has a spring for which $k = 60$ N/m and the area of the piston is 0.50 cm^2. Its right end is connected to a closed container of gas at a gauge pressure of 30 kPa. How far will the spring be compressed if the region containing the spring is (a) in vacuum and (b) open to the atmosphere ? Atmospheric pressure is 101 kPa.



23. Water stands at a depth h behind the vertical face of a dam. It exerts a resultant horizontal force on the dam tending to slide it along its foundation and a torque tending to overturn the dam about the point O. Find



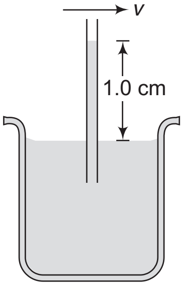
(a) horizontal force, (b) torque about O, (c) the height at which the resultant force would have to act to produce the same torque, l = cross-sectional length and ρ = density of water.

24. Mercury is poured into a U-tube in which the cross-sectional area of the left-hand limb is three times smaller than that of the right one. The level of the mercury in the narrow limb is a distance $l = 30\,\text{cm}$ from the upper end of the tube. How much will the mercury level rise in the right-hand limb if the left one is filled to the top with water ?

25. A water barrel stands on a table of height h. If a small hole is punched in the side of the barrel at its base, it is found that the resultant stream of water strikes the ground at a horizontal distance R from the barrel. What is the depth of water in the barrel ?

26. A pump is designed as a horizontal cylinder with a piston of area A and an outlet orifice of area a arranged near the cylinder axis. Find the velocity of out flow of the liquid from the pump if the piston moves with a constant velocity under the action of a constant force F. The density of the liquid is ρ.

27. When air of density $1.3\ \text{kg/m}^3$ flows across the top of the tube shown in the accompanying figure, water rises in the tube to a height of 1.0 cm. What is the speed of the air ?



28. The area of cross-section of a large tank is $0.5\ \text{m}^2$. It has an opening near the bottom having area of cross-section $1\ \text{cm}^2$. A load of 20 kg is applied on the water at the top. Find the velocity of the water coming out of the opening at the time when the height of water level is 50 cm above the bottom. (Take $g = 10\ \text{m/s}^2$)

29. What is the pressure drop (in mm Hg) in the blood as it passes through a capillary 1 mm long and $2\,\mu\text{m}$ in radius if the speed of the blood through the centre of the capillary is 0.66 mm/s ? (The viscosity of whole blood is 4×10^{-3} Pl).

30. A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.06×10^{-2} N/m. To what length has the capillary to be immersed so that the liquid levels inside and outside the capillary become the same ? What will happen to the water levels inside the capillary if the seal is now broken ?

31. A film of water is formed between two straight parallel wires each 10 cm long and at a separation 0.5 cm. Calculate the work required to increase 1 mm distance between them. Surface tension of water = 72×10^{-3} N/m.

32. A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4}m. If the height of the liquid in the narrow tube is 0.2 m more than that in the wide tube, calculate the true pressure difference. Density of liquid $= 10^3\ \text{kg/m}^3$, surface tension $= 7.2 \times 10^{-3}$ N/m and $g = 9.8\ \text{m/s}^2$.

33. A liquid of specific gravity 1.5 is observed to rise 3.0 cm in a capillary tube of diameter 0.50 mm and the liquid wets the surface of the tube. Calculate the excess pressure inside a spherical bubble of 1.0 cm diameter blown from the same liquid. Angle of contact = 0°.

34. A glass U-tube is such that the diameter of one limb is 3.0 mm and that of the other is 6.0 mm. The tube is inverted vertically with the open ends below the surface of water in a beaker. What is the difference between the heights to which water rises in the two limbs? Surface tension of water is $0.07\,\text{Nm}^{-1}$. Assume that the angle of contact between water and glass is 0°.

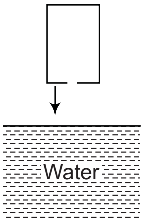
35. A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100 cm is 0.1 mm, calculate its depth where its radius is 0.126 mm, given that the surface tension of mercury is 567 dyne/cm. Assume that the atmospheric pressure is 76 cm of mercury.

36. If a number of little droplets of water, each of radius r, coalesce to form a single drop of radius R, show that the rise in temperature will be given be

$$\frac{3T}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

where, T is the surface tension of water and J is the mechanical equivalent of heat.

37. An empty container has a circular hole of radius r at its bottom. The container is pushed into water very slowly as shown. To what depth the lower surface of container (from surface of water) can be pushed into water such that water does not flow into the container ?

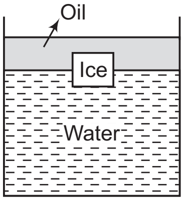


LEVEL 2

Objective Questions

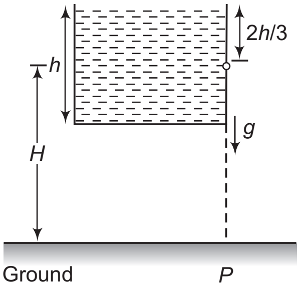
Single Correct Option

1. An ice cube is floating in water above which a layer of lighter oil is poured. As the ice melts completely, the level of interface and the upper most level of oil will respectively



(a) rise and fall
(b) fall and rise
(c) not change and no change
(d) not change and fall

2. An open vessel full of water is falling freely under gravity. There is a small hole in one face of the vessel as shown in the figure. The water which comes out from the hole at the instant when hole is at height H above the ground, strikes the ground at a distance of x from P.



Which of the following is correct for the situation described?

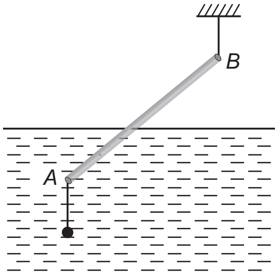
(a) The value of x is $2\sqrt{\dfrac{2hH}{3}}$

(b) The value of x is $\sqrt{\dfrac{4hH}{3}}$

(c) The value of x can't be computed from information provided

(d) The question is irrevalent as no water comes out from the hole

3. A uniform rod AB, 12 m long weighing 24 kg, is supported at end B by a flexible light string and a lead weight (of very small size) of 12 kg attached at end A.



The rod floats in water with one-half of its length submerged. For this situation, mark out the correct statement.
[Take $g = 10 \text{ m/s}^2$, density of water = 1000 kg/m^3]

(a) The tension in the string is 36 g

(b) The tension in the string is 12 g

(c) The volume of the rod is $6.4 \times 10^{-2} \text{ m}^3$

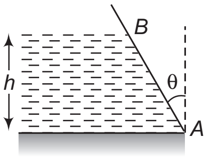
(d) The point of application of the buoyancy force is passing through C (centre of mass of rod)

4. A water hose pipe of cross-sectional area 5 cm^2 is used to fill a tank of 120 L. It has been observed that it takes 2 min to fill the tank. Now, a nozzle with an opening of cross-sectional area 1 cm^2 is attached to the hose. The nozzle is held so that water is projected horizontally from a point 1 m above the ground. The horizontal distance over which the water can be projected is (Take $g = 10 \text{ m/s}^2$)

(a) 3 m (b) 8 m

(c) 4.47 m (d) 8.64 m

5. The height of water in a vessel is h. The vessel wall of width b is at an angle θ to the vertical. The net force exerted by the water on the wall is

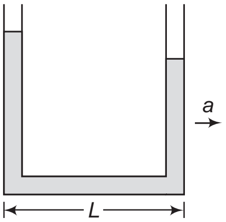


(a) $\frac{1}{3}\rho bh^2 g\cos\theta$ (b) $\frac{1}{2}bh^2\rho g$ (c) $\frac{1}{2}\rho bh^2 g\sec\theta$ (d) zero

6. A body of density ρ is dropped from rest from a height h into a lake of density σ $(\sigma > \rho)$. The maximum depth the body sinks inside the liquid is (neglect viscous effect of liquid)

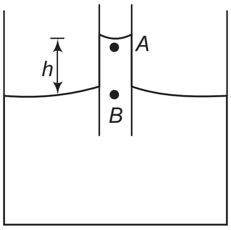
(a) $\frac{h\rho}{\sigma-\rho}$ (b) $\frac{h\sigma}{\sigma-\rho}$ (c) $\frac{h\rho}{\sigma}$ (d) $\frac{h\sigma}{\rho}$

7. A liquid stands at the plane level in the U-tube when at rest. If areas of cross-section of both the limbs are equal, what will be the difference in heights h of the liquid in the two limbs of U-tube, when the system is given an acceleration a in horizontal direction towards right as shown?



(a) $\frac{Lg}{a}$ (b) $\frac{La}{g}$ (c) $\frac{Lg^2}{a^2}$ (d) zero

8. A liquid of density ρ and surface tension σ rises in a capillary tube of inner radius R. The angle of contact between the liquid and the glass is θ. The point A lies just below the meniscus in the tube and the point B lies at the outside level of liquid in the beaker as shown in figure. The pressure at A is



(a) $p_B - \rho gh$ (b) $p_B - \frac{2\sigma\cos\theta}{R}$ (c) $p_{\text{atm}} - \frac{2\sigma\cos\theta}{R}$ (d) All of these

9. A large open tank has two holes in the wall. One is a square hole of side L at a depth h from the top and the other is a circular hole of radius R at a depth $4h$ from the top. When the tank is completely filled with water, quantities of water flowing out per second from both holes are the same. Then R is equal to

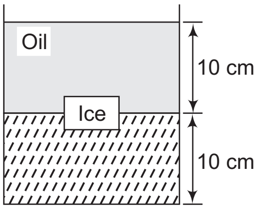
(a) $\frac{L}{\sqrt{2\pi}}$ (b) $2\pi L$ (c) L (d) $\frac{L}{2\pi}$

10. Two identical cylindrical vessels with their bases at the same level each, contain a liquid of density ρ. The area of either base is A but in one vessel the liquid height is h_1 and in the other liquid height is $h_2 (h_2 < h_1)$. If the two vessels are connected, the work done by gravity in equalizing the levels is

(a) $\frac{1}{2}(h_1 - h_2)^2 A\rho g$ (b) $\frac{1}{2}(h_1 + h_2) A\rho g$

(c) $\frac{1}{2}(h_1^2 - h_2^2) A\rho g$ (d) $\frac{1}{4}(h_1 - h_2)^2 A\rho g$

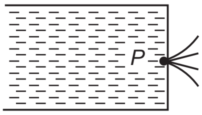
11. A cubical block of side 10 cm floats at the interface of an oil and water as shown in the figure. The density of oil is 0.6 g cm^{-3} and the lower face of ice cube is 2 cm below the interface. The pressure above that of the atmosphere at the lower face of the block is



(a) 200 Pa (b) 620 Pa

(c) 900 Pa (d) 800 Pa

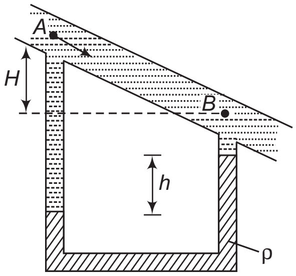
12. A leakage begins in water tank at position P as shown in the figure. The initial gauge pressure (pressure above that of the atmosphere) at P was 5×10^5 N/m^2. If the density of water is 1000 kg/m^3 the initial velocity with which water gushes out is approximately



(a) 3.2 ms^{-1} (b) 32 ms^{-1}

(c) 28 ms^{-1} (d) 2.8 ms^{-1}

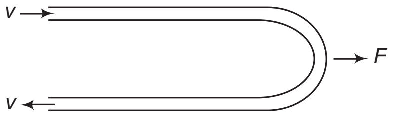
13. The figure shows a pipe of uniform cross-section inclined in a vertical plane. A U-tube manometer is connected between the points A and B. If the liquid of density ρ_0 flows with velocity v_0 in the pipe. Then the reading h of the manometer is



(a) $h = 0$ (b) $h = \frac{v_0^2}{2g}$

(c) $h = \frac{\rho_0}{\rho}\left(\frac{v_0^2}{2g}\right)$ (d) $h = \frac{\rho_0 H}{\rho - \rho_0}$

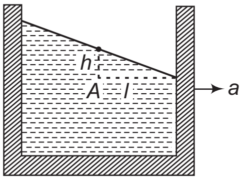
14. A horizontal tube of uniform cross-sectional area A is bent in the form of U as shown in figure. If the liquid of density ρ enters and leaves the tube with velocity v, then the external force F required to hold the bend stationary is



(a) $F = 0$ (b) $\rho A v^2$

(c) $2\rho A v^2$ (d) $\frac{1}{2}\rho A v^2$

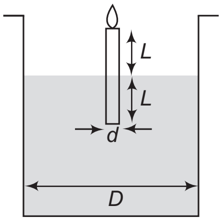
15. A rectangular container moves with an acceleration a along the positive direction as shown in figure. The pressure at the point A in excess of the atmospheric pressure p_0 is (take ρ as the density of liquid)



(a) $\rho g h$ (b) $\rho a l$

(c) $\rho\ (gh + al)$ (d) Both (a) and (b)

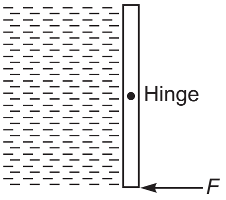
16. A candle of diameter d is floating on a liquid in a cylindrical container of diameter D $(D >> d)$ as shown in figure. It it is burning at the rate of 2 cm/h. Then, the top of the candle will



(a) remain at the same height (b) fall at the rate of 1 cm/h

(c) fall at the rate of 2 cm/h (d) go up at the rate of 1 cm/h

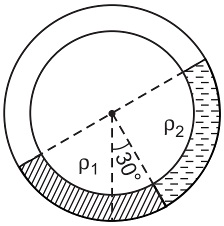
17. A square gate of size 1 m × 1 m is hinged at its mid point. A fluid of density ρ fills the space to the left of the gate. The force F required to hold the gate stationary is



(a) $\frac{\rho g}{3}$ (b) $\frac{1}{2}\rho g$

(c) $\frac{\rho g}{6}$ (d) None of these

18. A thin uniform circular tube is kept in a vertical plane. Equal volumes of two immiscible liquids whose densities are ρ_1 and ρ_2 fill half of the tube as shown. In equilibrium the radius passing through the interface makes an angle of 30° with vertical. The ratio of densities (ρ_1/ρ_2) is equal to

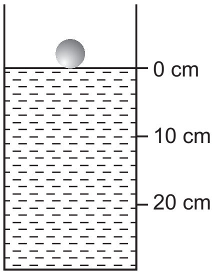


(a) $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$ (b) $\dfrac{\sqrt{3}+1}{2+\sqrt{3}}$ (c) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ (d) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

19. A plate moves normally with the speed v_1 towards a horizontal jet of water of uniform area of cross-section. The jet discharges water at the rate of volume V per second at a speed of v_2. The density of water is ρ. Assume that water splashes along the surface of the plate ar right angles to the original motion. The magnitude of the force acting on the plate due to the jet of water is

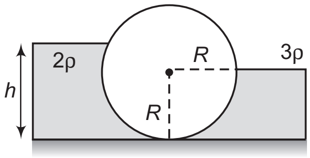
(a) $\rho V v_1$ (b) $\rho\left(\dfrac{V}{v_2}\right)(v_1+v_2)^2$ (c) $\dfrac{\rho V}{v_1+v_2}(v_1)^2$ (d) $\rho V(v_1+v_2)$

20. A spherical ball of density ρ and radius 0.003 m is dropped into a tube containing a viscous fluid up to the 0 cm mark as shown in the figure. Viscosity of the fluid = 1.26 N-s/m^2 and its density $\rho_L = \dfrac{\rho}{2} = 1260\ \text{kg/m}^3$. Assume that the ball reaches a terminal speed at 10 cm mark. The time taken by the ball to travel the distance between the 10 cm and 20 cm mark is ($g = 10\ \text{m/s}^2$)



(a) 2 s (b) 1 s
(c) 0.5 s (d) 5 s

21. In the figure shown, the heavy cylinder (radius R) resting on a smooth surface separates two liquids of densities 2ρ and 3ρ. The height h for the equilibrium of cylinder must be



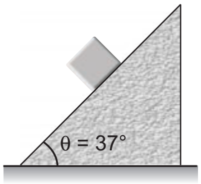
(a) $3R/2$ (b) $R\sqrt{\dfrac{3}{2}}$

(c) $R\sqrt{2}$ (d) None of these

22. A U-tube having horizontal arm of length 20 cm, has uniform cross-sectional area = 1 cm^2. It is filled with water of volume 60 cc. What volume of a liquid of density 4 g/cc should be poured from one side into the U-tube so that no water is left in the horizontal arm of the tube?

(a) 60 cc (b) 45 cc (c) 50 cc (d) 35 cc

23. A cubical block of side a and density ρ slides over a fixed inclined plane with constant velocity v. There is a thin film of viscous fluid of thickness t between the plane and the block. Then the coefficient of viscosity of the film will be



(a) $\frac{3\rho agt}{5v}$ (b) $\frac{4\rho agt}{5v}$ (c) $\frac{\rho agt}{v}$ (d) None of these

24. A spring balance reads 10 kg when a bucket of water is suspended from it. What will be the reading of the balance when an iron piece of mass 7.2 kg suspended by a string is immersed with half its volume inside the water in the bucket? Relative density of iron is 7.2.

(a) 10 kg (b) 10.5 kg (c) 13.6 kg (d) 17.2 kg

25. Three points A, B and C on a steady flow of a non-viscous and incompressible fluid are observed. The pressure, velocity and height of the points A, B and C are (2, 3, 1), (1, 2, 2) and (4, 1, 2) respectively. Density of the fluid is 1 kgm^{-3} and all other parameters are given in SI units. Then which of the following is correct? ($g = 10$ ms^{-2}).

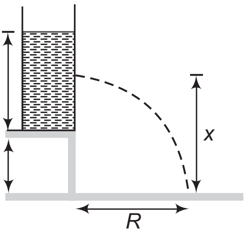
(a) Points A and B lie on the same stream line
(b) Points B and C lie on the same stream line
(c) Points C and A lie on the same stream line
(d) None of the above

26. A body of density ρ is dropped from rest from height h (from the surface of water) into a lake of density of water σ ($\sigma > \rho$). Neglecting all dissipative effects, the acceleration of body while it is in the lake is

(a) $g\left(\frac{\sigma}{\rho}-1\right)$ upwards (b) $g\left(\frac{\sigma}{\rho}-1\right)$ downwards

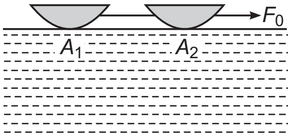
(c) $g\left(\frac{\sigma}{\rho}\right)$ upwards (d) $g\left(\frac{\sigma}{\rho}\right)$ downwards

27. A tank is filled up to a height $2H$ with a liquid and is placed on a platform of height H from the ground. The distance x from the ground where a small hole is punched to get the maximum range R is



(a) H (b) 1.25 H (c) 1.5 H (d) 2 H

28. Two boats of base areas A_1 and A_2, connected by a string are being pulled by an external force F_0. The viscosity of water is η and depth of the water body is H. When the system attains a constant speed, the tension in the thread will be



(a) $F_0\left(\frac{A_1}{A_2}\right)$ (b) $F_0\frac{A_2}{(A_1+A_2)}$ (c) $F_0\frac{A_1}{(A_1+A_2)}$ (d) $F_0\left(\frac{A_2}{A_1}\right)$

29. A U-tube is partially filled with water. Oil which does not mix with water is next poured into one side, until water rises by 25 cm on the other side. If the density of oil is 0.8 g/cm^3, the oil level will stand higher than the water level by

(a) 6.25 cm (b) 12.50 cm (c) 31.75 cm (d) 25 cm

30. There is a horizontal film of soap solution. On it a thread is placed in the form of a loop. The film is punctured inside the loop and the thread becomes a circular loop of radius R. If the surface tension of the soap solution be T, then the tension in the thread will be

(a) $\pi R^2/T$ (b) π^2R^2T

(c) $2\pi RT$ (d) $2RT$

31. A thin metal disc of radius r floats on a liquid surface and bends the surface downwards along the perimeter making an angle θ with vertical edge of the disc. If the disc displaces a weight of liquid w and surface tension of liquid is T, then the weight of metal disc is

(a) $2\pi rT + w$ (b) $2\pi rT\cos\theta - w$

(c) $2\pi rT\cos\theta + w$ (d) $w - 2\pi rT\cos\theta$

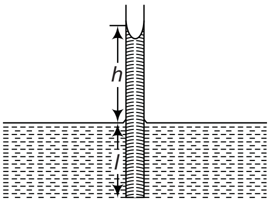
32. The radii of the two columns is U-tube are r_1, and $r_2 (> r_1)$. When a liquid of density ρ (angle of contact is 0°) is filled in it, the level difference of liquid in two arms is h. The surface tension of liquid is

(g = acceleration due to gravity)

(a) $\frac{\rho g h r_1 r_2}{2(r_2 - r_1)}$ (b) $\frac{\rho gh(r_2 - r_1)}{2r_1r_2}$

(c) $\frac{2(r_2 - r_1)}{\rho gh\, r_1r_2}$ (d) $\frac{\rho gh}{2(r_2 - r_1)}$

33. Water rises to a height h in a capillary tube lowered vertically into water to a depth l as shown in the figure. The lower end of the tube is now closed, the tube is then taken out of the water and opened again. The length of the water column remaining in the tube will be



(a) $2h$ if $l \ge h$ and $l + h$ if $l \le h$

(b) h if $l \ge h$ and $l + h$ if $l \le h$

(c) $4h$ if $l \ge h$ and $l - h$ if $l \le h$

(d) $\frac{h}{2}$ if $l \ge h$ and $l + h$ if $l \le h$

34. Two parallel glass plates are dipped partly in the liquid of density 'd' keeping them vertical. If the distance between the plates is 'x', Surface tension for liquid is T and angle of contact is θ then rise of liquid between the plates due to capillary will be

(a) $\frac{T\cos\theta}{xd}$ (b) $\frac{2T\cos\theta}{xdg}$ (c) $\frac{2T}{x\,dg\cos\theta}$ (d) $\frac{T\cos\theta}{x\,dg}$

35. Two identical spherical soap bubbles collapses. If V is the consequent change in volume of the contained air, S is the change in the total surface area and T is the surface tension of the soap solution, then (if p_0 is atmospheric pressure and assume temperature to remain same in all the bubbles).

(a) $3p_0V + 4ST = 0$
(b) $4p_0V + 3ST = 0$
(c) $p_0V + 4TS = 0$
(d) $4p_0V + ST = 0$

36. A cylinder with a movable piston contains air under a pressure p_1 and a soap bubble of radius 'r'. The pressure p_2 to which the air should be compressed by slowly pushing the piston into the cylinder for the soap bubble to reduce its size by half will be (The surface tension is σ, and the temperature T is maintained constant).

(a) $\left[8p_1 + \dfrac{24\sigma}{r}\right]$
(b) $\left[4p_1 + \dfrac{24\sigma}{r}\right]$
(c) $\left[2p_1 + \dfrac{24\sigma}{r}\right]$
(d) $\left[2p_1 + \dfrac{24\sigma}{r}\right]$

37. A thin metal ring of internal radius 8 cm and external radius 9 cm is supported horizontally from the pan of a balance so that it comes in contact with water in a glass vessel, If is found that an extra weight of 7.48 g is required to pull the ring out of water. The surface tension of water is ($g = 10\ \text{m/s}^2$)

(a) $80 \times 10^{-3}\ \text{N/m}$
(b) $25 \times 10^{-3}\ \text{N/m}$
(c) $45 \times 10^{-3}\ \text{N/m}$
(d) $70 \times 10^{-3}\ \text{N/m}$

More than One Correct Options

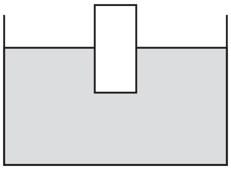
1. A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. River is 1 m deep and the water in contact with the bed is stationary. Then choose correct statement(s).
(coefficient of viscosity of water = 10^{-3} N-s/m^2)

(a) velocity gradient is $2\ \text{s}^{-1}$
(b) velocity gradient is $1\ \text{s}^{-1}$
(c) force required to keep the plate moving with constant speed is 0.02 N
(d) force required to keep the plate moving with constant speed is 0.01 N

2. Choose the correct options.

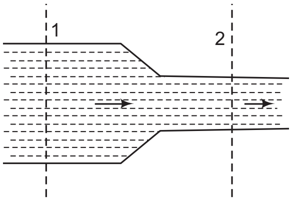
(a) Viscosity of liquids increases with temperature
(b) Viscosity of gases increases with temperature
(c) Surface tension of liquids decreases with temperature
(d) For angle of contact $\theta = 0°$, liquid neither rises nor falls on capillary

3. A plank is floating in a non-viscous liquid as shown. Choose the correct options.



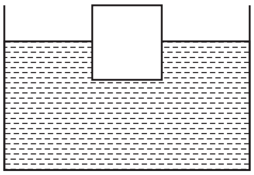
(a) Equilibrium of plank is stable in vertical direction
(b) For small oscillations of plank in vertical direction motion is simple harmonic
(c) Even if oscillations are large, motion is simple harmonic till it is not fully immersed
(d) On vertical displacement motion is periodic but not simple harmonic

4. A non-viscous incompressible liquid is flowing from a horizontal pipe of non-uniform cross section as shown. Choose the correct options.



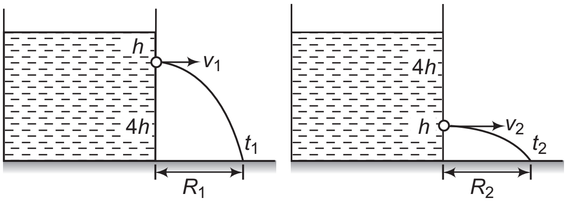
(a) speed of liquid at section-2 is more
(b) volume of liquid flowing per second from section-2 is more
(c) mass of liquid flowing per second at both the sections is same
(d) pressure at section-2 is less

5. A plank is floating in a liquid as shown. Fraction f of its volume is immersed. Choose the correct options.



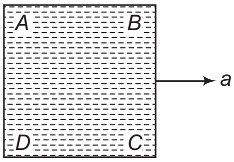
(a) If the system is taken to a place where atmospheric pressure is more, f will increase
(b) In above condition f will remain unchanged
(c) If temperature is increased and expansion of only liquid is considered f will increase
(d) If temperature is increased and expansion of only plank is considered f will decrease

6. In two figures,



(a) $v_1/v_2 = \frac{1}{2}$
(b) $t_1/t_2 = 2$
(c) $R_1/R_2 = 1$
(d) $v_1/v_2 = \frac{1}{4}$

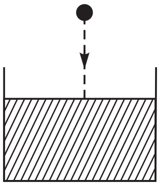
7. A liquid is filled in a container as shown in figure. Container is accelerated towards right. There are four points A, B, C and D in the liquid. Choose the correct options.



(a) $p_A > p_B$
(b) $p_C > p_A$
(c) $p_D > p_B$
(d) $p_A > p_C$

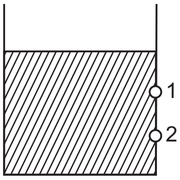
8. A ball of density ρ is dropped from a height on the surface of a non-viscous liquid of density 2ρ. Choose the correct options.

(a) Motion of ball is periodic but not simple harmonic
(b) Acceleration of ball in air and in liquid are equal
(c) Magnitude of upthrust in the liquid is two times the weight of ball
(d) Net force on ball in air and in liquid are equal and opposite

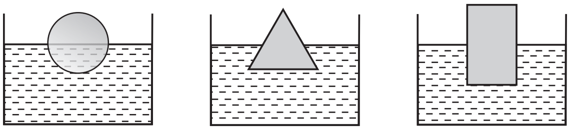


9. Two holes 1 and 2 are made at depths h and $16h$ respectively. Both the holes are circular but radius of hole-1 is two times.

(a) Initially equal volumes of liquid will flow from both the holes in unit time
(b) Initially more volume of liquid will flow from hole-2 per unit time
(c) After some time more volume of liquid will flow from hole-1 per unit time
(d) After some time more volume of liquid will flow from hole-2 per unit time



10. A solid sphere, a cone and a cylinder are floating in water. All have same mass, density and radius. Let f_1, f_2 and f_3 are the fraction of their volumes inside the water and h_1, h_2 and h_3 are the depths inside water. Then

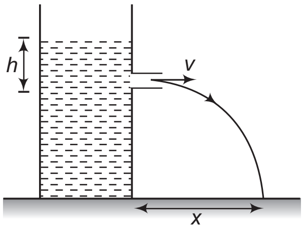


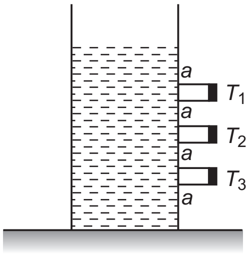
(a) $f_1 = f_2 = f_3$
(b) $f_3 > f_2 > f_1$
(c) $h_3 < h_1$
(d) $h_3 < h_2$

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 3)

The spouting can is something used to demonstrate the variation of pressure with depth. When the corks are removed from the tubes in the side of the can, water flows out with a speed that depends on the depth. In a certain can, three tubes T_1, T_2 and T_3 are set at equal distances 'a' above the base of the can. When water contained in this can is allowed to come out of the tubes, the distances on the horizontal surface are measured as x_1, x_2 and x_3.





1. Speed of efflux is

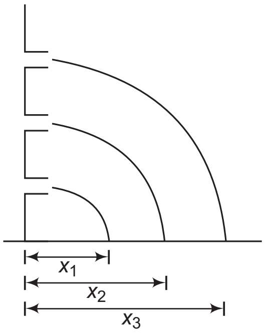
(a) $\sqrt{3gh}$
(b) $\sqrt{2gh}$
(c) $\sqrt{gh}$
(d) $\frac{1}{2}\sqrt{2gh}$

2. Distance x_3 is given by

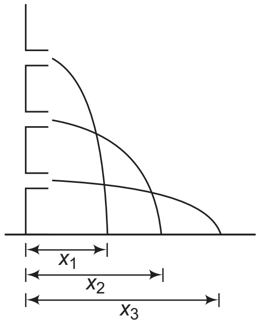
(a) $\sqrt{3}\,a$
(b) $\sqrt{2}\,a$
(c) $\frac{1}{2}\sqrt{3}\,a$
(d) $2\sqrt{3}\,a$

3. The correct sketch is

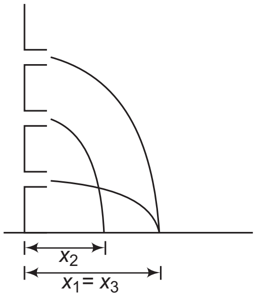
(a)



(b)



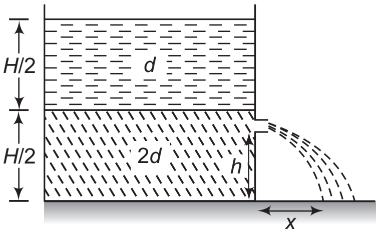
(c)



(d) None of these

Passage 2 (Q. Nos. 4 to 7)

A container of large uniform cross-sectional area A resting on a horizontal surface, holds two immiscible, non-viscous and incompressible liquids of densities d and $2d$ *each of height* $H/2$ *as shown in the figure. The lower density liquid is open to the atmosphere having pressure* P_0. *A homogeneous solid cylinder of length* $L (L < H/2)$ *and cross-sectional area* $A/5$ *is immersed such that it floats with its axis vertical at the liquid-liquid interface with length* $L/4$ *in the denser liquid.*



The cylinder is then removed and the original arrangement is restored. A tiny hole of area $s (s << A)$ *is punched on the vertical side of the container at a height* $h (h < H/2)$. *As a result of this, liquid starts flowing out of the hole with a range* x *on the horizontal surface.*

4. The density D of the material of the floating cylinder is

(a) $5d/4$ (b) $3d/4$ (c) $4d/5$ (d) $4d/3$

5. The total pressure with cylinder, at the bottom of the container is

(a) $p_0 + \frac{(6L + H)}{4} dg$ (b) $p_0 + \frac{(L + 6H)}{4} dg$

(c) $p_0 + \frac{(L + 3H)}{4} dg$ (d) $p_0 + \frac{(L + 2H)}{4} dg$

6. The initial speed of efflux without cylinder is

(a) $v=\sqrt{\frac{g}{3}[3H+4h]}$ (b) $v=\sqrt{\frac{g}{2}[4H-3h]}$

(c) $v=\sqrt{\frac{g}{2}[3H-4h]}$ (d) None of these

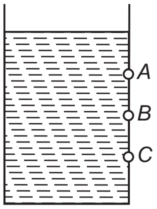
7. The initial value of x is

(a) $\sqrt{(3H+4h)\,h}$ (b) $\sqrt{(3h+4H)h}$

(c) $\sqrt{(3H-4h)\,h}$ (d) $\sqrt{(3H-3h)\,h}$

Match the Columns

1. Three holes A, B and C are made at depths 1m, 2 m and 5 m as shown. Total height of liquid in the container is 8 m. Let v is the speed with which liquid comes out of the hole and R the range on ground. Match the following two columns.

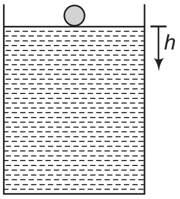


Column I	Column II
(a) v is maximum for	(p) hole A
(b) v is minimum for	(q) hole B
(c) R is maximum for	(r) hole C
(d) R is minimum for	(s) will depend on density of liquid

2. Match the following two columns.

Column I	Column II
(a) When temperature is increased	(p) Upthrust on a floating solid of constant volume will increase
(b) When density of liquid is increased	(q) Upthrust on a floating solid of constant volume will decrease
(c) When density of solid is increased	(r) Viscosity of gas will decrease
(d) When atmospheric pressure is increased	(s) None

3. A ball of density ρ is released from the surface of a liquid whose density varies with depth h as, $\rho_l = \alpha\, h$. Here α is a positive constant. Match the following two columns. (liquid is ideal)

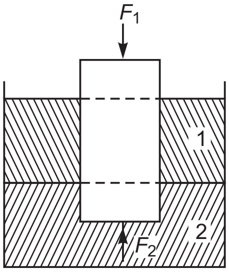


Column I	Column II
(a) Upthrust on ball	(p) will continuously decrease
(b) Speed of ball	(q) will continuously increase
(c) Net force on ball	(r) first increase then decrease
(d) Gravitational potential energy of ball	(s) first decrease then increase

4. Match the following two columns.

Column I	Column II
(a) Surface tension	(p) $[ML^{-1}T^{-2}]$
(b) Coefficient of viscosity	(q) $[L^3T^{-1}]$
(c) Energy density	(r) $[MT^{-2}]$
(d) Volume flow rate	(s) $[ML^{-1}T^{-1}]$

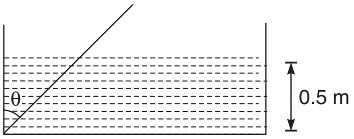
5. A cylinder of weight W is floating in two liquids as shown in figure. Net force on cylinder from top is F_1 and force on cylinder from the bottom is F_2. Match the following two columns.

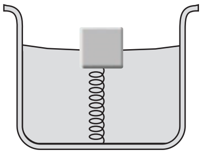


Column I	Column II
(a) Net force on cylinder from liquid-1	(p) Zero
(b) $F_2 - F_1$	(q) W
(c) Net force on cylinder from liquid-2	(r) Net upthrust
(d) Net force on cylinder from atmosphere	(s) None of these

Subjective Questions

1. A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank as shown in figure. The tank is filled with water upto a height of 0.5 m. The specific gravity of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position. (Exclude the case θ = 0)

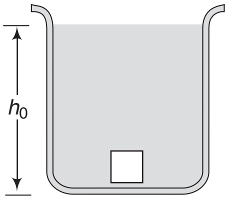




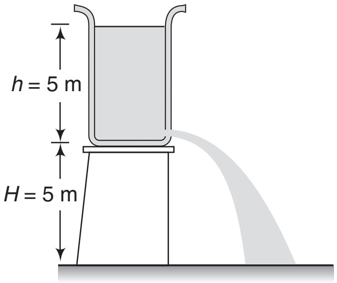
2. A cubical block of wood of edge 3 cm floats in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. Find the maximum weight that can be put on the block without wetting the new weight. Density of wood = 800 kg/m^3 and spring constant of the spring = 50 N/m. (Take $g = 10\ \text{m/s}^2$)

3. Figure shows a container having liquid of variable density. The density of liquid varies as $\rho = \rho_0\left(4 - \dfrac{3h}{h_0}\right)$. Here, h_0 and ρ_0 are constants and h is measured from bottom of the container.

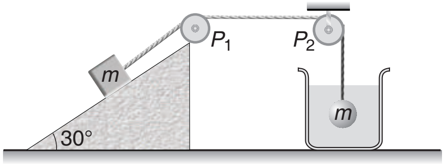
A solid block of small dimensions whose density is $\dfrac{5}{2}\rho_0$ and mass m is released from bottom of the tank. Prove that the block will execute simple harmonic motion. Find the frequency of oscillation.



4. A cylindrical tank 1 m in radius rests on a platform 5 m high. Initially the tank is filled with water to a height of 5 m. A plug whose area is $10^{-4}\ \text{m}^2$ is removed from an orifice on the side of the tank at the bottom. Calculate (a) initial speed with which the water flows from the orifice, (b) initial speed with which water strikes the ground, (c) time taken to empty the tank to half its original value. ($g = 10\ \text{m/s}^2$)

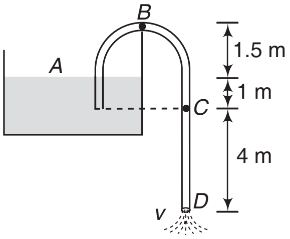


5. A block of mass m is kept over a fixed smooth wedge. Block is attached to a sphere of same mass through fixed massless pullies P_1 and P_2. Sphere is dipped inside the water as shown. If specific gravity of material of sphere is 2. Find the acceleration of sphere.



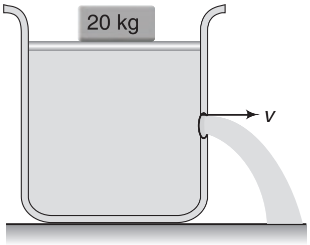
6. A cubic body floats on mercury with 0.25 fraction of its volume below the surface. What fraction of the volume of the body will be immersed in the mercury if a layer of water poured on top of the mercury covers the body completely ?

7. A siphon tube is discharging a liquid of specific gravity 0.9 from a reservoir as shown in the figure.

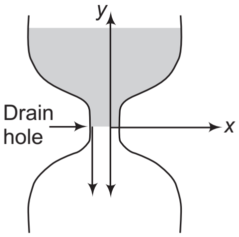


(a) Find the velocity of the liquid through the siphon.
(b) Find the pressure at the highest point *B*.
(c) Find the pressure at point *C*.

8. A long cylindrical tank of cross-sectional area $0.5\,\text{m}^2$ is filled with water. It has a small hole at a height 50 cm from the bottom. A movable piston of cross-sectional area almost equal to $0.5\,\text{m}^2$ is fitted on the top of the tank such that it can slide in the tank freely. A load of 20 kg is applied on the top of the water by piston, as shown in the figure. Calculate the speed of the water jet with which it hits the surface when piston is 1 m above the bottom. (Ignore the mass of the piston).



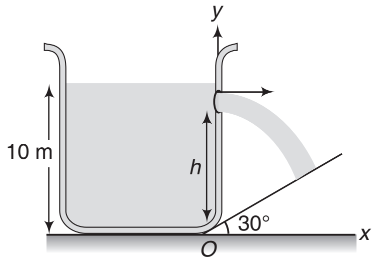
9. The shape of an ancient water clock jug is such that water level descends at a constant rate at all times. If the water level falls by 4 cm every hour, determine the shape of the jar, i.e. specify *x* as a function of *y*. The radius of drain hole is 2 mm and can be assumed to be very small compared to *x*.



10. A spring is attached to the bottom of an empty swimming pool, with the axis of the spring oriented vertically. An 8.00 kg block of wood ($\rho = 840\,\text{kg/m}^3$) is fixed to the top of the spring and compresses it. Then the pool is filled with water, completely covering the block. The spring is now observed to be stretched twice as much as it had been compressed. Determine the percentage of the block's total volume that is hollow. Ignore any air in the hollow space.

11. A rectangular tank of height 10 m filled with water, is placed near the bottom of a plane inclined at an angle 30° with horizontal. At height *h* from bottom a small hole is made (as shown in figure) such that the stream coming out from hole, strikes the inclined plane normally.

Calculate h.



12. A ball of density d is dropped onto a horizontal solid surface. It bounces elastically from the surface and returns to its original position in a time t_1. Next, the ball is released and it falls through the same height before striking the surface of a liquid of density d_L.

(a) If $d < d_L$, obtain an expression (in terms of d, t_1 and d_L) for the time t_2 the ball takes to come back to the position from which it was released.

(b) Is the motion of the ball simple harmonic ?

(c) If $d = d_L$, how does the speed of the ball depend on its depth inside the liquid ? Neglect all frictional and other dissipative forces. Assume the depth of the liquid to be large.

13. There is an air bubble of radius 1.0 mm in a liquid of surface tension 0.075 N/m and density 1000 kg/m^3. The bubble is at a depth of 10 cm below the free surface. By what amount is the pressure inside the bubble greater than the atmospheric pressure ? (Take $g = 9.8$ m/s^2)

14. A metal sphere of radius 1 mm and mass 50 mg falls vertically in glycerine. Find

(a) the viscous force exerted by the glycerine on the sphere when the speed of the sphere is 1 cm/s,

(b) the hydrostatic force exerted by the glycerine on the sphere and (c) the terminal velocity with which the sphere will move down without acceleration. Density of glycerine = 1260 kg/m^3 and its coefficient of viscosity at room temperature = 8.0 poise.

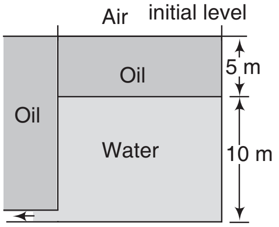
15. A wire forming a loop is dipped into soap solution and taken out, so that a film of soap solution is formed. A loop of 6.28 cm long thread is gently put on the film and the film is pricked with a needle inside the loop. The thread loop takes the shape of a circle. Find the tension in the thread. Surface tension of soap solution = 0.030 N/m.

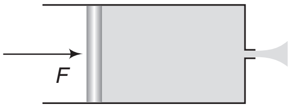
16. A cylindrical vessel is filled with water upto a height of 1 m. The cross-sectional area of the orifice at the bottom is (1/400) that of the vessel.

(a) What is the time required to empty the tank through the orifice at the bottom?

(b) What is the time required for the same amount of water to flow out if the water level in tank is maintained always at a height of 1 m from orifice?

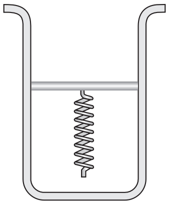
17. A tank having a small circular hole contains oil on top of water. It is immersed in a large tank of the same oil. Water flows through the hole. What is the velocity of this flow initially? When the flow stops, what would be the position of the oil-water interface in the tank from the bottom. The specific gravity of oil is 0.5.

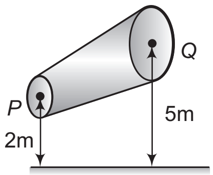




18. What work should be done in order to squeeze all water from a horizontally located cylinder (figure) during the time t by means of a constant force acting on the piston? The volume of water in the cylinder is equal to V, the cross-sectional area of the orifice is s, with s being considerably less than the piston area. The friction and viscosity are negligibly small. Density of water is ρ.

19. A cylinder is fitted with a piston, beneath which is a spring, as in the figure. The cylinder is open at the top. Friction is absent. The spring constant of the spring is 3600 N/m. The piston has a negligible mass and a radius of 0.025 m. (a) When air beneath the piston is completely pumped out, how much does the atmospheric pressure cause the spring to compress? (b) How much work does the atmospheric pressure do in compressing the spring?





20. A non-viscous liquid of constant density 1000 kg/m^3 flows in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in the figure. The area of cross-section of the tube at two points P and Q at heights of 2 m and 5 m are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point P is 1 m/s. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P to Q. Take $g = 9.8 \text{ m/s}^2$.

21. A glass plate of length 10 cm, breadth 1.54 cm and thickness 0.20 cm weighs 8.2 gm in air. It is held vertically with the long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water = 73 dyne per cm, $g = 980 \text{ cm/sec}^2$.

22. Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water at the temperature of the experiment is 7.3×10^{-2} Nm^{-1}. Take the angle of contact to be zero, and density of water to be $1.0 \times 10^3 \text{ kg/m}^3$ ($g = 9.8 \text{ ms}^{-2}$).

23. Two identical soap bubbles each of radius r and of the same surface tension T combine to form a new soap bubble of radius R. The two bubbles contain air at the same temperature. If the atmospheric pressure is p_0 then find the surface tension T of the soap solution in terms of p_0, r and R. Assume process is isothermal.

Note *Students are advised to attempt question numbers 24 and 25 after studying the chapter of electrostatics in class XII.*

24. A soap bubble of radius r and surface tension 'T' is given a potential of' V volt. Show that the new radius 'R' of the bubble is related to its initial radius by equation ,

$$p_0[R^3 - r^3] + 4T\,[R^2 - r^2] - \varepsilon_0 V^2 R/2 = 0$$

where, p_0 is the atmosphere pressure.

25. If the radius and surface tension of a spherical soap bubble are 'R' and 'T' respectively, then show that the charge required to double its radius would be, $8\pi R\,[\varepsilon_0 R\,[7p_0 R + 12T]]^{1/2}$, where p_0 is the atmospheric pressure.

Answers

Introductory Exercise 16.1

1. 18 cm **2.** 27 cm **3.** (a) 1.09×10^5 N/m^2 (b) 1.12×10^5N/m^2 **4.** $p = p_0 + \dfrac{mg}{A}$ **5.** 50 N **6.** 50 cm

Introductory Exercise 16.2

1. Maximum at C and minimum at B **2.** 2g **3.** 3×10^5 N / m^2

Introductory Exercise 16.3

2. 23.7 N **3.** 0.206 N **4.** (a) 14.7 m / s^2 (b) 0.63 s

Introductory Exercise 16.4

1. (a) 20 cm/s (b) 485 N / m^2 **2.** (a) 25 cm/s (b) 50 cm/s (c) 94 N / m^2 **3.** (c) **4.** 500

Introductory Exercise 16.5

1. 20 m/s **2.** $\left(\dfrac{t_0}{\sqrt{3} - 1}\right)$ **3.** (a)

Introductory Exercise 16.6

1. 0.882 mm/s **2.** $v' = (2)^{2/3} v$ **3.** $F = 0.02$ N **4.** 10^{-3} N / m^2

Introductory Exercise 16.7

1. 4.35×10^{-3}J **2.** 1.44×10^{-5} J **3.** $8\pi R^2 T$ **4.** $p = p_0 + h\rho g + \dfrac{2T}{r}$

Introductory Exercise 16.8

1. 6.0 cm **2.** $\dfrac{T_m}{T_w} = 7.23$ **3.** 1.4 mm **4.** (d)

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (d) **4.** (a) **5.** (d) **6.** (d) **7.** (b) **8.** (a) **9.** (d) **10.** (d)
11. (c)

Single Correct Option

1. (a) **2.** (d) **3.** (d) **4.** (a) **5.** (d) **6.** (b) **7.** (a) **8.** (a) **9.** (d) **10.** (d)
11. (a) **12.** (b) **13.** (c) **14.** (a) **15.** (c) **16.** (c) **17.** (c) **18.** (b) **19.** (a) **20.** (b)
21. (b) **22.** (a) **23.** (c) **24.** (a) **25.** (a) **26.** (d) **27.** (c) **28.** (d) **29.** (a) **30.** (d)
31. (c) **32.** (b) **33.** (c) **34.** (c) **35.** (d) **36.** (d) **37.** (d) **38.** (d) **39.** (c) **40.** (c)
41. (d) **42.** (c) **43.** (a) **44.** (d) **45.** (d) **46.** (a) **47.** (d) **48.** (a) **49.** (d)

Subjective Questions

1. False **2.** Zero **3.** Vessel B **4.** $2^{2/3}W$
5. It will remain same **6.** 3840 kg/m^3 **7.** 28 cm^3 **8.** 0.33 m^3
9. 17 cm **10.** 0.6 gm/cm^3, 0.705 gm/cm^3 **11.** 6.03 g
12. 6.92×10^5 Pa **13.** 0.8
14. Mercury will rise in the arm containing sprit. The difference in level is 0.221 cm

15. (i) Absolute pressure = 96 cm of Hg, Gauge pressure = 20 cm of Hg for (a),
absolute pressure = 58 cm of Hg, gauge pressure = – 18 cm of Hg for (b)
(ii) Mercury would rise in the left limb such that the difference in the levels in the two limbs becomes 19 cm.

16. (a) 2.25 m/s (b) zero **17.** (a) 7.0×10^{-2} m^3/s (b) 2.24×10^4 Pa (c) 1.5×10^{-1} m^3/s

18. (a) $2\times 10^{-6} m^3$, 5000 kg/m^3 (b) 750 kg/m^3 **19.** 32 mm in mercury and 28 mm in water

20. $T = 20$ N, $V = 32\times 10^{-3} m^3$ **21.** (a) 24 N (b) 12 m/s^2

22. (a) 10.9 cm (b) 2.5 cm 23. (a) $\frac{\rho g/h^2}{2}$ (b) $\frac{\rho g/h^3}{6}$ (c) $\frac{h}{3}$ **24.** 0.58 cm **25.** $\frac{R^2}{4h}$

26. $v = \sqrt{\frac{2F}{A\rho}}$ **27.** 12.4 m/s **28.** $v_B \approx 3.28$ m/s **29.** 19.5 mm of Hg **30.** $h = 1$ cm, if seal is broken water will rise in the capillary.

31. 14.4×10^{-6} J **32.** 1860 N/m^2 **33.** 440 $dyne/cm^2$ **34.** 4.76 mm

35. 9.48 cm **37.** $\frac{2T}{\rho g r}$

LEVEL 2

Single Correct Option

1. (a) **2.** (d) **3.** (c) **4.** (c) **5.** (c) **6.** (a) **7.** (b) **8.** (d) **9.** (a) **10.** (d)
11. (d) **12.** (b) **13.** (a) **14.** (c) **15.** (b) **16.** (b) **17.** (c) **18.** (d) **19.** (d) **20.** (d)
21. (b) **22.** (d) **23.** (a) **24.** (b) **25.** (d) **26.** (a) **27.** (c) **28.** (c) **29.** (b) **30.** (d)
31. (c) **32.** (a) **33.** (a) **34.** (b) **35.** (a) **36.** (a) **37.** (d)

More than One Correct Options

1. (a,c) **2.** (b,c) **3.** (a,b,c) **4.** (a,c,d) **5.** (b,c,d) **6.** (a,b,c) **7.** (a,c)
8. (a,c,d) **9.** (a,d) **10.** (a,c,d)

Comprehension Based Questions

1. (b) **2.** (d) **3.** (d) **4.** (a) **5.** (b) **6.** (c) **7.** (c)

Match the Columns

1. (a) → r , (b) → p, (c) → r, (d) → p
2. (a) → s, (b) → s, (c) → p, (d) → s
3. (a) → r, (b) → r, (c) → s, (d) → s
4. (a) → r, (b) → s, (c) → p, (d) → q
5. (a) → p, (b) → q,r, (c) → s, (d) → s

Subjective Questions

1. 45° **2.** 0.354 N **3.** $\frac{1}{2\pi}\sqrt{\frac{6g}{5h_0}}$ **4.** (a) 10 m/s (b) 14.1 m/s (c) 9200 s

5. zero **6.** 0.19 **7.** (a) 9.9 m/s (b) 4.36×10^4 Pa (c) 6.6×10^4 Pa **8.** 4.51 m/s

9. $y = 0.4x^4$ **10.** 60.41% **11.** 8.33 m

12. (a) $\frac{t_1 d_L}{d_L - d}$ (b) No (c) The ball will continue to move with constant velocity $v = \frac{g t_1}{2}$ inside the liquid.

13. 1130 Pa **14.** (a) 1.5×10^{-4} N (b) 5.2×10^{-5} N (c) 32.5 m/s **15.** 3.0×10^{-4} N

16. (a) 3 min (b) 1.5 min **17.** 9.8 m/s, 5.0 m **18.** $\frac{1}{2}\frac{\rho V^3}{s^2 t^2}$ **19.** (a) 5.5 cm (b) 5.445 J

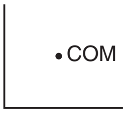
20. 29025 J/m^3, 29400 J/m^3 **21.** 8.1796 g **22.** 5 mm **23.** $T = \frac{p_0(2r^3 - R^3)}{4(R^2 - 2r^2)}$

Hints & Solutions

11. Centre of Mass, Linear Momentum and Collision

INTRODUCTORY EXERCISE 11.1

2.

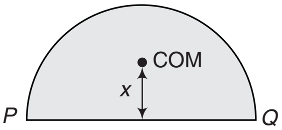


The COM of L-shaped rod shown in above figure is lying outside the body.

4. $x_{\text{CM}} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$

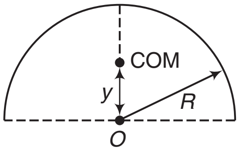
$x_1 = x_2 = 0 \;\Rightarrow\; x_{\text{CM}} = 0$

5.



$x < r/2$, as more mass is concentrated near the line PQ.

6.



All particles of semicircular ring lie at a distance R from origin O. But its COM distance y from O is less than R.

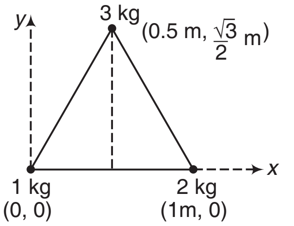
7.

$$x_{\text{CM}} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1)(0) + (2)(1) + (3)(0.5)}{1 + 2 + 3} = \frac{7}{12}\text{ m}$$

$$y_{\text{CM}} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1)(0) + (2)(0) + (3)\left(\dfrac{\sqrt{3}}{2}\right)}{1 + 2 + 3}$$



$$= \frac{\sqrt{3}}{4}\text{ m}$$

$$\therefore \quad d = \sqrt{x_{\text{CM}}^2 + x_{\text{CM}}^2}$$

$$= \frac{\sqrt{19}}{6}\text{ m} \qquad \textbf{Ans.}$$

8.

$$y_{\text{CM}} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$= \frac{(\pi b^2/2)(4b/3\pi) - (\pi a^2/2)(4a/3\pi)}{(\pi b^2/2) - (\pi a^2/2)}$$

$$= \frac{4}{3\pi}\left[\frac{b^3 - a^3}{b^2 - a^2}\right]$$

$$= \frac{4}{3\pi}\left\{\frac{a^2 + ab + b^2}{a + b}\right\} \qquad \textbf{Ans.}$$

9.

$$x_{\text{CM}} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{(4a^2)(a) - (a^2)(3a/2)}{4a^2 - a^2} = \frac{5a}{6} \qquad \textbf{Ans.}$$

$$y_{\text{CM}} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$= \frac{(4a^2)(a) - (a^2)(3a/2)}{4a^2 - a^2}$$

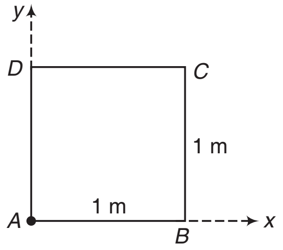
$$= \frac{5a}{6} \qquad \textbf{Ans.}$$

10.

$$x_{\text{CM}} = \frac{m_Ax_A + m_Bx_B + m_Cx_C + m_Dx_D}{m_A + m_B + m_C + m_D}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$= 0.5\text{ m}$$



$$y_{\text{CM}} = \frac{m_Ay_A + m_By_B + m_Cy_C + m_Dy_D}{m_A + m_B + m_C + m_D}$$

$$= \frac{(1)(0) + 2(0) + 3(1) + 4(1)}{1 + 2 + 3 + 4} = 0.7\text{ m}$$

$$\therefore d^2 = x_{\text{CM}}^2 + y_{\text{CM}}^2$$

$$= 0.74\text{ m}^2 \qquad \textbf{Ans.}$$

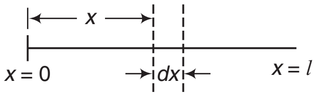
11. $d = \dfrac{A_1 d_1 + A_2 d_2}{A_1 + A_2}$

Here, d = distance of centre of mass from O.

$$\therefore \quad d = \frac{(a^2)(0) + \left(\dfrac{\pi}{4} a^2\right)(a)}{a^2 + (\pi/4)\,a^2}$$

$$= \left(\frac{\pi}{\pi + 4}\right) a$$

12.



Let A = area of cross section of rod.

$$x_{CM} = \frac{\int_0^l x\,dm}{\int_0^l dm} = \frac{\int_0^l (x)\,(A dx)\,\rho_0 \dfrac{x^2}{l^2}}{\int_0^l (A dx)\,\rho_0 \dfrac{x^2}{l^2}}$$

$$= \frac{3l}{4} \qquad \textbf{Ans.}$$

13. $x_{CM} = \dfrac{\int_0^L x\,dm}{\int_0^L dm} = \dfrac{\int_0^L (X)\,(Ax dx)}{\int_0^L (Ax dx)} = \dfrac{2}{3} L$ **Ans.**

INTRODUCTORY EXERCISE 11.2

1. $$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{m_1 (x_i + vt)_1 + m_2 (x_i + vt)_2}{m_1 + m_2}$$

$$= \frac{(1)(10 - 6 \times 2) + (2)(12 + 4 \times 2)}{1 + 2}$$

$$= 12.67 \text{ m} \qquad \textbf{Ans.}$$

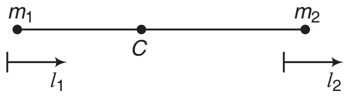
2. $$S_{CM} = \frac{m_1 S_1 + m_2 S_2}{m_1 + m_2}$$

$$= \frac{m_1 (v_1 t) + m_2 (v_2 t)}{m_1 + m_2}$$

$$= \frac{(1)(2)(1) + (2)(-1)(1)}{1 + 2}$$

$$= 0$$

3.



Displacement of centre of mass,

$$S_{CM} = \frac{m_1 l_1 + m_2 l_2}{m_1 + m_2} \qquad \textbf{Ans.}$$

4. (a) $$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\Rightarrow \quad 3 = \frac{m_1(0) + (0.10)(12)}{m_1 + 0.1}$$

Solving this equation we get,

$$m_1 = 0.3 \text{ kg} \qquad \textbf{Ans.}$$

(b) $$\mathbf{P}_{CM} = m_{CM} \mathbf{v}_{CM}$$

$$= (0.1 + 0.3)(6\hat{\mathbf{j}})$$

$$= (2.4\hat{\mathbf{j}}) \text{ kg-m/s} \qquad \textbf{Ans.}$$

(c) $$\mathbf{P}_{CM} = \mathbf{P}_1 + \mathbf{P}_2$$

$$\therefore \quad (2.4\hat{\mathbf{j}}) = (0.3)\mathbf{v}_1 + (0.1)(0)$$

$$\therefore \quad \mathbf{v}_1 = (8\hat{\mathbf{j}}) \text{ m/s} \qquad \textbf{Ans.}$$

5. First stone gets 300 ms journey time and second stone gets 200 ms journey time.

(a) $$d_{CM} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

$$= \frac{m_1\left(\dfrac{1}{2} g t_1^2\right) + m_2\left(\dfrac{1}{2} g t_2^2\right)}{m_1 + m_2}$$

$$= \frac{(m)\left(\dfrac{1}{2} \times 10 \times 0.3 \times 0.3\right) + 2m\left(\dfrac{1}{2} \times 10 \times 0.2 \times 0.2\right)}{m + 2m}$$

$$= 0.283 \text{ m}$$

$$\approx 28 \text{ cm} \qquad \textbf{Ans.}$$

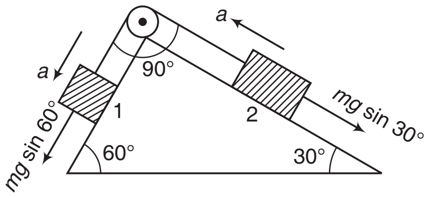
(b) $$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m_1 (g t_1) + m_2 (g t_2)}{m_1 + m_2}$$

$$= \frac{(m)(10 \times 0.3) + (2m)(10 \times 0.2)}{m + 2m}$$

$$= 2.3 \text{ m/s} \qquad \textbf{Ans.}$$

6. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}}$



$$= \frac{mg \sin 60° - mg \sin 30°}{2m}$$

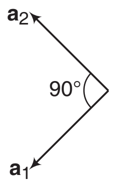
$$a = \left(\frac{\sqrt{3}-1}{4}\right) g$$

$$|\mathbf{a}_1| = |\mathbf{a}_2| = a$$

$$\therefore \quad |\mathbf{a}_1 + \mathbf{a}_2| = \sqrt{2}\, a \text{ at } 90°$$

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

$$= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2) \qquad (\text{as } m_1 = m_2)$$



$$\therefore \quad |\mathbf{a}_{\text{COM}}| = \frac{|\mathbf{a}_1 + \mathbf{a}_2|}{2}$$

$$= \frac{\sqrt{2}a}{2}$$

$$= \frac{a}{\sqrt{2}} = \left(\frac{\sqrt{3}-1}{4\sqrt{2}}\right) g \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.3

1. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad (20)(20\hat{\mathbf{i}}) + 30(20\hat{\mathbf{j}}) + 40(20\hat{\mathbf{k}})$$

$$= (20)(0) + 30(10\hat{\mathbf{i}} + 20\hat{\mathbf{k}}) + 40\mathbf{v}$$

Solving this equation we get,

$$\mathbf{v} = (2.5\hat{\mathbf{i}} + 15\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \text{ cm/s} \qquad \textbf{Ans.}$$

2. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = (25)(5) + (10)(v)$$

$$\text{or} \quad v = -12.5 \text{ m/s}$$

Negative sign implies that this velocity is opposite to the direction of velocity of boy. Two velocities are in opposite directions.

Hence, $|v_r|$ = Magnitude of relative velocity

$$= 12.5 + 5.0 = 17.5 \text{ m/s}$$

3. $p = \sqrt{2km}$

$$\text{or} \quad p \propto \sqrt{m}$$

$$\therefore \quad \frac{p_1}{p_2} = \sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

4. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = (4)(1.4 \times 10^7) + (234)v$$

$$\therefore \quad v = -2.4 \times 10^5 \text{m/s}$$

Here, negative sign implies that this velocity is opposite to the direction of velocity of alpha particle.

5. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = 50(1.8) + (6 \times 10^{24})v$$

$$\therefore \quad v = -1.5 \times 10^{-23} \text{ m/s}$$

Negative sign implies the opposite direction of the velocity of man.

6. $m_1v_1 = m_2v_2$ (in opposite direction)

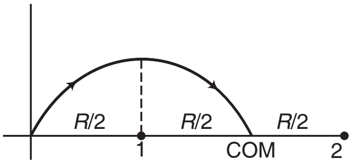
$$\therefore \quad (60)(3) = (20)\, v_2$$

$$\therefore \quad v_2 = 9 \text{ m/s} \qquad \textbf{Ans.}$$

$$\text{KE} = \frac{1}{2} \times 60 \times (3)^2 + \frac{1}{2} \times 20 \times (9)^2$$

$$= 1080 \text{J} = 1.08 \text{ kJ} \qquad \textbf{Ans.}$$

7. Path of COM will remain unchanged



$$x_2 = \frac{3R}{2} = \frac{3}{2}\left(\frac{u^2 \sin 2\theta}{g}\right)$$

$$= \frac{3}{2} \frac{(20)^2 \sin 90°}{10} = 60 \text{m} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.4

1. Thrust force = mg

$$\therefore \quad v_r\left(-\frac{dm}{dt}\right) = mg$$

$$\therefore \quad \left(-\frac{dm}{dt}\right) = \frac{mg}{v_r}$$

$$= \frac{(200)(9.8)}{1.6 \times 10^3} = 1.225 \text{ kg/s}$$

(i) In this case, fuel will be finished in 90 s.

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 9.8 \times 90 + (1.6 \times 10^3) \ln\left(\frac{200}{20}\right)$$

$$= 2.8 \text{ km/s} \qquad \textbf{Ans.}$$

(ii) In this case, fuel is finished in 9 s.

$$\therefore \quad v = 0 - 9.8 \times 10 + 1.6 \times 10^3 \ln\left(\frac{200}{20}\right)$$

$$= 3.6 \text{ km/s}$$ **Ans.**

2. After 1 min total 600 kg gases will be burnt.

$$v = u - gt + V_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 10 \times 60 + (2000)\ln\left(\frac{1000}{400}\right)$$

$$= 1232.6 \text{ m/s}$$

3. $\left(-\frac{dm}{dt}\right) = \mu$

$$F_{\text{net}} = \text{thrust force} - \text{weight}$$

$$\therefore \quad ma = v_r\left(-\frac{dm}{dt}\right) - mg$$

or $$(m_0 - \mu t)\frac{d^2x}{dt^2} = u(\mu) - (m_0 - \mu t)g$$

4. $\left(-\frac{dm}{dt}\right) = \frac{m_0}{3}$

At $t = 1$, mass will remain,

$$m = m_0 - \frac{m_0}{3} = \frac{2}{3}m_0$$

Now using the equation,

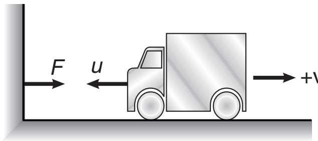
$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - g(1) + u \ln\left(\frac{m_0}{2/3 m_0}\right)$$

$$= u \ln\left(\frac{3}{2}\right) - g$$ **Ans.**

INTRODUCTORY EXERCISE 11.5

1. Using impulse = change in linear momentum



We have, $F \cdot t = mv_f - mv_i = m(v_f - v_i)$

or $F(2) = 2 \times 10^3 [0 - (-4)]$

or $2F = 8 \times 10^3$ or $F = 4 \times 10^3$ N

2. Using $\mathbf{J} = m(\mathbf{v}_f - \mathbf{v}_i)$

$$-3m\,\hat{\mathbf{i}} = m[\mathbf{v}_f - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})]$$

or $\mathbf{v}_f = -3\hat{\mathbf{i}} + (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$

or $\mathbf{v}_f = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

Note *The velocity component in the direction of $\hat{\mathbf{j}}$ is unchanged. This is because there is no impulse component in this direction.*

3. Impulse = area under F - t graph

$$= \Delta p = m(v_f - v_i)$$

$$\therefore \quad v_f = \frac{\text{Area}}{m} \qquad (\text{as } v_i = 0)$$

$$= \frac{1}{2} \times \frac{(20 \times 10^3)(8 + 16)}{12000}$$

$$= 200 \text{ m/s}$$ **Ans.**

4. (a) $v^2 = u^2 - 2as$

$$\therefore \quad a = \frac{u^2}{2s} \qquad (v = 0)$$

$$= \frac{(100)^2}{2 \times 0.06} = 8.3 \times 10^4 \text{ m/s}^2$$

$$v = u - at$$

$$0 = u - at$$

or $$t = \frac{u}{a} = \frac{100}{8.3 \times 10^4}$$

$$= 1.2 \times 10^{-3} \text{ s}$$ **Ans.**

(b) Impulse $= |\Delta p| = m v_i$

$$= (5 \times 10^{-3})(100)$$

$$= 0.5 \text{ N-s}$$ **Ans.**

(c) Impulse $= (F_{\text{av}})\,t$

$$\therefore \quad F_{\text{av}} = \frac{\text{Impulse}}{t}$$

$$= \frac{0.5}{1.2 \times 10^{-3}}$$

$$= 417 \text{ N}$$ **Ans.**

INTRODUCTORY EXERCISE 11.6

1. At maximum extension their velocities are same. This common velocity is given by

$$v = \frac{\text{Total momentum}}{\text{Total mass}}$$

$$= \frac{2 \times 6 - 1 \times 3}{3 + 6} = 1 \text{ m/s}$$

Now, $E_i = E_f$

$$\therefore \quad \frac{1}{2} \times 6 \times (2)^2 + \frac{1}{2} \times 3 \times (1)^2 = \frac{1}{2} \times 9\,(1)^2 + \frac{1}{2} \times 200 \times x_m^2$$

Solving we get, $x_m = 0.3$ m

$$= 30 \text{ cm}$$ **Ans.**

2. Substituting the values in above result

$$\frac{K_i - K_f}{K_i} = \frac{4\,m_1 m_2}{(m_1 + m_2)^2}$$

$$= \frac{4 \times m \times 2m}{(m + 2m)^2}$$

$$= \frac{8}{9} \qquad \textbf{Ans.}$$

3. $$\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i}$$

$$= \frac{1 - \frac{1}{2} m_1 v_1'^{\,2}}{\frac{1}{2} m_1 v_1^2} = 1 - \left(\frac{v_1'}{v_1}\right)^2$$

$$= 1 - \left[\frac{(m_1 - m_2)}{(m_1 + m_2)} v_1 / v_1\right]^2$$

$$= \frac{4\,m_1 m_2}{(m_1 + m_2)^2} \qquad \textbf{Ans.}$$

4. $$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

We can see that $v_1' = v_2$ only if $m_1 = m_2$

Similarly $v_2' = v_1$ if $m_1 = m_2$.

5. In elastic collision between two equal masses, velocities are interchanged.

6. $$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

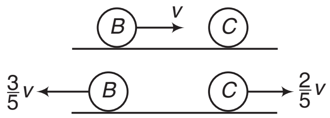
$$= \left(\frac{1-2}{1+2}\right)(+4) + \left(\frac{2 \times 2}{1+2}\right)(-6)$$

$$= -\frac{28}{3}\ \text{m/s} \qquad \textbf{Ans.}$$

$$v_2' = \left(\frac{2-1}{1+2}\right)(-6) + \left(\frac{2 \times 1}{1+2}\right)(+\,4)$$

$$= \frac{2}{3}\ \text{m/s} \qquad \textbf{Ans.}$$

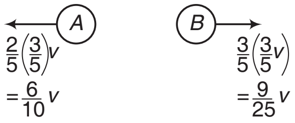
7. **After first collision with $B + C$**



$$v_B' = \left(\frac{m - 4m}{m + 4m}\right) v = -\frac{3}{5} v$$

$$v_C' = \left(\frac{2 \times m}{m + 4m}\right) v = \frac{2}{5} v$$

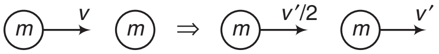
After second collision with $B + A$



Now since $\frac{9}{25} v < \frac{2}{5} v$ or v_C

So, B will not collide with C further.

Hence total collisions are only 2.

8. 

$$p_i = p_f$$

$$\therefore \quad mv = m\frac{v'}{2} + mv'$$

$$\therefore \quad v' = \frac{2}{3} v$$

Now, $$e = \frac{RVOS}{RVOA} = \frac{v' - v'/2}{v}$$

$$= \frac{v'/2}{v} = \frac{(1/3\,v)}{v} = \frac{1}{3} \qquad \textbf{Ans.}$$

9. (a) $p_i = p_f$

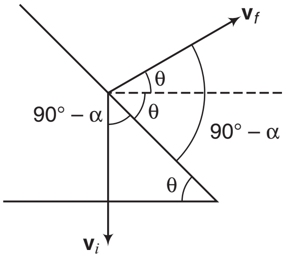
$$\therefore \quad mv = MV$$

$$\Rightarrow \quad M = \frac{mv}{V}$$

(b) $$e = \frac{RVOS}{RVOA} = \frac{V}{v}$$

10. Component parallel to wall will remain unchanged and component perpendicular to wall will become e times but in opposite direction.

11. Initial velocity vector is making an angle of 90° – α with plane. Therefore final velocity vector will also make an angle 90° – α with inclined plane.



$\therefore$ Angle of $\mathbf{v}_f$ with horizontal

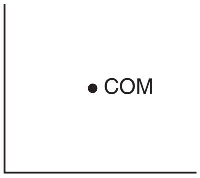
$$\theta = (90° - \alpha) - \alpha$$

$$\theta = (90° - 2\alpha) \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

1. 

In the above case, centre of mass lies outside the body.

2. $a_{\text{CM}} = \dfrac{F}{3m} = \text{constant}$

$\therefore \quad v_{\text{CM}} = a_{\text{CM}}t = \dfrac{F}{3m}\,t$

or $\quad v_{\text{CM}} \propto t$

3. To conserve linear momentum, forces can act on a system but their vector sum should be zero.

4. Gases inside the rocket are pushed backwards.

5. Net vector sum of all internal forces = 0. So, they cannot change the linear momentum.

6. $p_1 = p_2$

$\therefore \quad \sqrt{2K_1 m_1} = \sqrt{2K_2 m_2}$

or $\quad \dfrac{K_1}{K_2} = \dfrac{m_2}{m_1}$

$\Rightarrow \quad K \propto \dfrac{1}{m}$

7. Of the two masses A and B, mass A is moving downwards. Therefore net force on the system is vertically downwards and momentum is not conserved in vertical direction.

8. $\Delta p_1 + \Delta p_2 = 0$

or $\quad \Delta p_1 = -\Delta p_2$

10. Energy can be given in the form of potential energy without giving the momentum.

11. Centre of mass remains stationary.

12. $e = \dfrac{RVOS}{RVOA}$

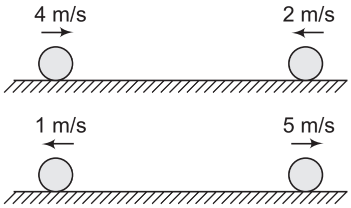
For elastic collision $e = 1$

$\therefore \quad RVOS = RVOA$

Before collision they approach towards each other and after collision they recede from each other.

13. Centre of mass remains stationary.

15. Relative velocity of separation is equal to the relative velocity of approach. One of the case is shown below.



Before collision, relative speed is 2 m/s and relative velocity of approach is 6 m/s.
After collision, relative speed is 4 m/s and relative velocity of separation is 6 m/s.

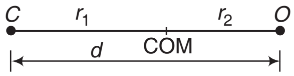
Single Correct Option

10. Impulse = $F_{\text{av}}\ (\Delta t) = |\Delta P|$

$$\therefore \quad F_{\text{av}} = \frac{|\Delta P|}{\Delta t} = \frac{m\,(v_i - v_f)}{\Delta t}$$

$$= \frac{(5)\,(65-15)}{2}$$

$$= 125 \text{ N}$$ **Ans.**

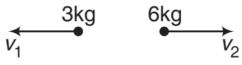
11. 

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$\therefore \quad r_1 = \left(\frac{m_2}{m_1 + m_2}\right) d$$

$$= \left(\frac{16}{12+16}\right)(1.2 \times 10^{-10})$$

$$= 0.68 \times 10^{-10}\text{m}$$ **Ans.**

12. 

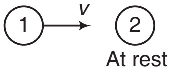
$$3v_1 = 6v^2$$

$$v_2 = \frac{3v_1}{6}$$

$$= \frac{3 \times 16}{6} = 8 \text{ m/s}$$

$$(\text{KE})_{6\text{ kg}} = \frac{1}{2} \times 6 \times (8)^2$$

$$= 192 \text{ J}$$ **Ans.**

13. 

$$m_1 >> m_2$$

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_1 + \left(\frac{2m_1}{m_1 + m_2}\right) v_2$$

Substituting $m_2 \approx 0$ and $v_1 = 0$ we get,

$$v_2 \approx 2v$$

14. Velocity of dropped out mass is also 6 m/s. So its relative velocity is zero and no thrust force will act. Therefore final speed remains 6 m/s.

15. $F = \frac{\Delta p}{\Delta t} = n\,(mv)$

Here n = number of bullets fired per second

$$\Rightarrow \qquad n = \frac{F}{mv}$$

$$= \frac{144}{(40 \times 10^{-3})\,(1200)} = 3 \qquad \textbf{Ans.}$$

16. $|\Delta \mathbf{p}| = |\,\text{Impulse}\,| = |\mathbf{F} \cdot \Delta t|$

Here $\qquad \Delta t = T = \frac{2v \sin 45^\circ}{g} = \frac{\sqrt{2}\,v}{g}$

and $\qquad |\,\mathbf{F}\,| = mg$

$\therefore \qquad |\Delta \mathbf{p}| = (mg)\left(\frac{\sqrt{2}\,v}{g}\right)$

$$= \sqrt{2}\,mv \qquad \textbf{Ans.}$$

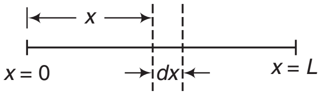
17. Velocity of ball just before collision with ground,

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 4.9} = 9.8 \text{ m/s}$$

Velocity just after collision,

$$u = ev = \left(\frac{3}{4}\right)(9.8) \text{ m/s}$$

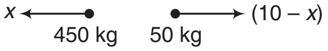
$$T = \frac{2u}{g} = \frac{2 \times (3/4) \times 9.8}{9.8} = \frac{3}{2} \text{ s}$$

18. 

dm = (Mass per unit length) (Length)

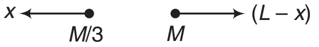
$$= \left(\frac{kx^2}{L}\right) dx \quad \Rightarrow \quad X_{\text{COM}} = \frac{\int_0^L x\,dm}{\int_0^L dm}$$

$$= \frac{\int_0^L (x)\left(\frac{kx^2}{L}\right) dx}{\int_0^L \left(\frac{kx^2}{L}\right) dx} = \frac{3}{4} L \qquad \textbf{Ans.}$$

19. 

$$450\,x = 50\,(10 - x)$$

$\therefore \qquad x = 1 \text{ m} \qquad$ **Ans.**

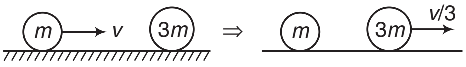
20. 

$$\frac{M}{3}\,x = M\,(L - x)$$

$$x = \frac{3}{4}\,L$$

$\therefore$ Displacement of man relative to ground

$$= L - x = \frac{L}{4} \qquad \textbf{Ans.}$$

21. 

Before collision   After collision

From conservation of momentum we can see that velocity of heavier mass after collision becomes $\frac{v}{3}$.

Now, coefficient of restitution,

$$e = \frac{RVOS}{RVOA} = \frac{v/3}{v} = \frac{1}{3} \qquad \textbf{Ans.}$$

22. In elastic collision between equal masses velocities are interchanged. Therefore change in momentum in any one particle is mu.

Now, $\qquad |\Delta \mathbf{p}| = |\,\text{Impulse}\,|$

$\qquad = \text{area under } F\text{-}t \text{ graph}$

$\therefore \qquad mu = \frac{1}{2} \times t_0 \times F_0$

$\therefore \qquad F_0 = \frac{2\,mu}{t_0} \qquad$ **Ans.**

23. Acceleration of centre of mass $= \frac{F}{2m}$

$$a_{\text{CM}} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}$$

$\therefore \qquad \frac{F}{2m} = \frac{(m)\,(a_0) + (m)\,a_2}{2m}$

$\therefore \qquad a_2 = \frac{F}{m} - a_0 \qquad$ **Ans.**

24. Impulse $= \Delta\,\mathbf{p} = m\,(\mathbf{v}_f - \mathbf{v}_i)$

$$= m\left[\left(-\frac{3}{4}\,v_0 \cos 53^\circ \hat{\mathbf{i}} + \frac{3}{4}\,v_0 \sin 53^\circ \hat{\mathbf{j}}\right) - (v_0 \cos 37^\circ \hat{\mathbf{i}} + v_0 \sin 37^\circ \hat{\mathbf{j}}\,)\right]$$

$$= -\frac{5}{4}\,mv_0 \hat{\mathbf{i}} \qquad \textbf{Ans.}$$

25. $h_n = (e^{2n})\, hi$ ...(i)

Given that

$$(64 \text{ cm}) = (e^2)\,(100 \text{ cm})$$

$$\therefore \qquad e = 0.8$$

Substituting in Eq. (i) we have

$$h_n = (0.8)^{2n}\,(1\text{m})$$

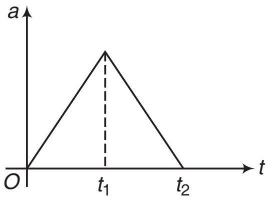
$$= (0.8)^{2n}$$ **Ans.**

26. $\mathbf{p}_i = \mathbf{p}_f$

$\therefore$ $(500)\,(\hat{\mathbf{i}}) = (25)\,(20\hat{\mathbf{j}}) + 475\mathbf{v}$

$$\therefore \qquad \mathbf{v} = \left[\left(\frac{20}{19}\,\hat{\mathbf{i}}\right) - \left(\frac{20}{19}\,\hat{\mathbf{k}}\right)\right] \text{m/s}$$ **Ans.**

27. $a = \dfrac{F}{m}$ or $a \propto F$



Hence, a-t graph is similar to F-t graph.

From *O* to t_1

$$a \propto t$$

or $\qquad a = kt \quad (k \rightarrow \text{a positive constant})$

Integrating we get,

$$v = \frac{kt^2}{2} \quad \text{or} \quad v \propto t^2$$

i.e. v-t graph is a parabola passing through origin.

From t_1 to t_2 again acceleration is positive (in the direction of velocity). So, velocity will further increase.

28. $$x_{\text{CM}} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{\pi\,(3R)^2\,(0) - (\pi R^2)\,(2R)}{\pi\,(3R)^2 - \pi R^2} = -\frac{R}{4}$$

$\therefore$ Distance of centre of mass from origin $= \dfrac{R}{4}$ **Ans.**

29. $$\mathbf{r}_{\text{CM}} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2 - A_3\mathbf{r}_3}{A_1 - A_2 - A_3}$$

$$= \frac{\pi\,(4R)^2\,(0) - (\pi R^2)\,(3R\hat{\mathbf{i}}) - (\pi R^2)\,(3R\hat{\mathbf{j}})}{\pi\,(4R)^2 - \pi R^2 - \pi R^2}$$

$$= \left(-\frac{3R}{14}\,\hat{\mathbf{i}} - \frac{3R}{14}\,\hat{\mathbf{j}}\right)$$ **Ans.**

30. Net horizontal force on system is zero, Therefore, centre of mass does not move in horizontal. Further, all surfaces are smooth. Therefore mechanical energy of the system remains constant.

31. Ler v' is the velocity of $(M + m)$ after collision. Then from conservation of linear momentum we have,

$$mv = (M + m)\,v'$$

$$\therefore \qquad v' = \left(\frac{m}{M + m}\right) v$$

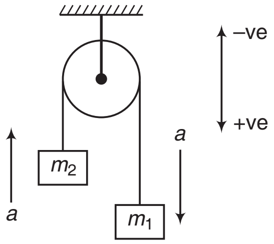
Now, $\qquad v' = \sqrt{2gh}$

$$\therefore \qquad \left(\frac{m}{M + m}\right) v = \sqrt{2gh}$$

or $\qquad v = \sqrt{2gh}\left(\dfrac{M + m}{m}\right)$

32. Net horizontal force on system is zero.

33.



$$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

$$= \frac{m_1g - m_2g}{m_1 + m_2}$$

$$= \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \qquad \text{...(i)}$$

Now, $\qquad a_{\text{CM}} = \dfrac{m_1\,a_1 + m_2\,a_2}{m_1 + m_2}$

$$a_{\text{CM}} = \frac{m_1\,(+a) + m_2\,(-a)}{m_1 + m_2}$$

$$= \left(\frac{m_1 - m_2}{m_1 + m_2}\right) a$$

Substituting the value of a from Eq. (i) we have,

$$a_{\text{CM}} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$$ **Ans.**

34. Using the equation,

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Substituting the values we get,

$$v_r = 0 - 0 + v_r \ln\left(\frac{m_0}{m}\right)$$

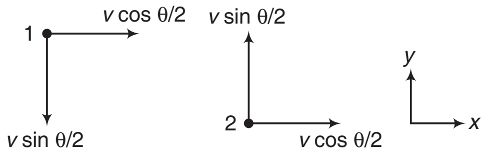
$$\therefore \qquad \frac{m_0}{m} = e = 2.178 \qquad \textbf{Ans.}$$

35. $v_i = 1\text{ m/s},\ v_f = 0$

$$|F| = \left|\frac{\Delta p}{\Delta t}\right| = \left(\frac{\Delta m}{\Delta t}\right)(v_i - v_f)$$

$$= 0.5 \times 1 = 0.5\text{ N}$$

36.



From the figure, we can see that velocity of approach is $2v \sin\frac{\theta}{2}$.

$$e = \frac{RVOS}{RVOA}$$

For completely inelastic collision, $e = 0$

$$\therefore \qquad RVOS = 0$$

$$\mathbf{p}_i = \mathbf{p}_f$$

$$\therefore \quad m\left[v\cos\frac{\theta}{2}\hat{\mathbf{i}} - v\sin\frac{\theta}{2}\hat{\mathbf{j}}\right] + m\left[v\cos\frac{\theta}{2}\hat{\mathbf{i}} + v\sin\frac{\theta}{2}\hat{\mathbf{j}}\right]$$

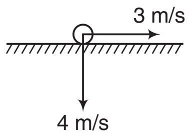
$$= 2m\mathbf{v}_c$$

$$\therefore \quad \text{Common velocity } \mathbf{v}_c = \left(v\cos\frac{\theta}{2}\right)\hat{\mathbf{i}}$$

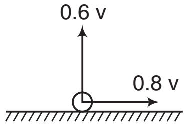
37. Component parallel to plane remains unchanged.

$$\therefore \qquad 0.8\,v = 3$$

$$\text{or} \qquad v = \frac{3}{0.8} = 3.75\text{ m/s}$$



Just before collision

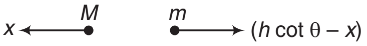


Just after collision

$|\Delta v| = (4 + 0.6\,v) = 6.25\text{ m/s}$ (in vertical direction)

$\therefore$ Impulse $= |\Delta p| = m\,|\Delta v| = 6.25$ N-s **Ans.**

38. Horizontal side of wedge is $h\cot\theta$.



$$Mx = m(h\cot\theta - x)$$

$$\therefore \qquad x = \frac{mh\cot\theta}{M + m} \qquad \textbf{Ans.}$$

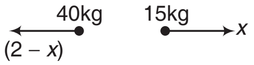
39. $$x_{CM} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4}{A_1 + A_2 + A_3 + A_4}$$

$$= \frac{(4)(1) + (4)(1) + (4)(3) + (4)(3)}{16} = 2\text{cm}$$

$$y_{CM} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4}{A_1 + A_2 + A_3 + A_4}$$

$$= \frac{(4)(1) + 4(3) + 4(3) + 4(3)}{16} = 2.5\text{cm}$$

40.



$$40(2 - x) = 15x$$

$$\therefore \qquad x = 1.46\text{ m} \qquad \textbf{Ans.}$$

41. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad m(v_0\hat{\mathbf{i}}) + m(-3\,v_0\hat{\mathbf{j}}) + m(5\,v_0\hat{\mathbf{k}})$$

$$= 3m\mathbf{v}$$

$$\therefore \qquad \mathbf{v} = \frac{v_0}{3}(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \qquad \textbf{Ans.}$$

42. $$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$\therefore \qquad R = \frac{(M/5)(0) + (4M/5)D}{M}$$

$$\therefore \qquad D = \frac{5}{4}R \qquad \textbf{Ans.}$$

Subjective Questions

1. $$x_{CM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{(ab)(0) - \left(\frac{ab}{4}\right)\left(\frac{a}{4}\right)}{ab - \frac{ab}{4}}$$

$$= -\frac{a}{12}$$

$$y_{CM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$= \frac{(ab)(0) - \left(\frac{ab}{4}\right)\left(\frac{b}{4}\right)}{ab - \frac{ab}{4}}$$

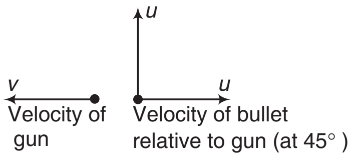
$$= -\frac{b}{12}$$

2. $$x_{CM} = \frac{V_1x_1 - V_2x_2}{V_1 - V_2} \qquad (V = \text{volume})$$

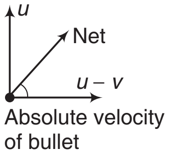
$$= \frac{\left(\frac{4}{3}\pi R^3\right)(0) - \left(\frac{4\pi a^3}{3}\right)(b)}{\left(\frac{4}{3}\pi R^3\right) - 4\left(\frac{\pi a^3}{3}\right)}$$

$$= \frac{-a^3 b}{R^3 - a^3}$$ **Ans.**

3.



At 45° means horizontal and vertical components are same (let u).



In absolute velocity, since horizontal component of velocity has been decreased ($= u - v$). Therefore, $\theta > 45°$

4. $p = \text{constant}$

$$\therefore \qquad \sqrt{2\,Km} = \text{constant}$$

$$\text{or} \qquad K \propto \frac{1}{m}$$

5. In head on elastic collision between two equal masses, velocities are interchanged.

6. At maximum elastic potential energy, velocity of both particles is same. This common velocity will be given by

$$v = \frac{\text{Total momentum}}{\text{Total mass}}$$

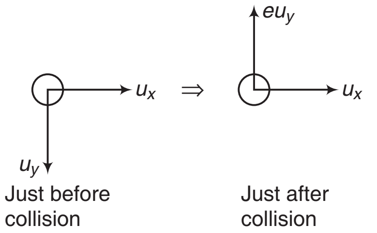
$$= \frac{\sqrt{2Km}}{2m} = \sqrt{\frac{K}{2m}}$$

$$E_i = E_f$$

$$K = \frac{1}{2} \times 2m\left(\sqrt{\frac{K}{2m}}\right)^2 + v_{\max}$$

$$\therefore \qquad v_{\max} = \frac{K}{2}$$ **Ans.**

7. Just after collision x-component of velocity remain unchanged but y-component of velocity becomes e times.



$$\text{Now,} \quad T = \frac{2u_y}{g},\ H = \frac{u_y^2}{2g} \quad \text{and} \quad R = \frac{2u_x\,u_y}{g}$$

$$\text{or} \qquad T \propto u_y\,,\ H \propto u_y^2 \text{ and } R \propto u_y$$

$$\therefore \qquad \frac{T_1}{T_2} = a = \frac{u_y}{eu_y} = \frac{1}{e}$$

$$\frac{R_1}{R_2} = b = \frac{u_y}{eu_y} = \frac{1}{e}$$

$$\text{and} \qquad \frac{H_1}{H_2} = c = \frac{(u_y)^2}{(eu_y)^2} = \frac{1}{e^2}$$

8. (a) $$\mathbf{a}_{COM} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

$$= \frac{(1)(-10\hat{\mathbf{j}}) + (2)(-10\hat{\mathbf{j}})}{1+2}$$

$$= (-10\hat{\mathbf{j}})\ \text{m/s}^2$$ **Ans.**

(b) $$\mathbf{v}_{COM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

$$= \frac{m_1(\mathbf{u}_1 + \mathbf{a}_1t) + m_2(\mathbf{u}_2 + \mathbf{a}_2t)}{m_1 + m_2}$$

$$= \frac{(1)[0 + (-10\hat{\mathbf{j}})(1)] + 2[(10\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) + (-10\hat{\mathbf{j}})(1)]}{1+2}$$

$$= \frac{10}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}})\ \text{m/s}$$ **Ans.**

(c) $$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$= \frac{m_1\left[\mathbf{r}_i + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\right]_1 + m_2\left[\mathbf{r}_i + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\right]_2}{m_1 + m_2}$$

$$= \frac{1\left[(10\hat{\mathbf{i}} + 20\hat{\mathbf{j}}) + 0 + \frac{1}{2}(-10\hat{\mathbf{j}})(1)^2\right] + 2\left[(20\hat{\mathbf{i}} + 40\hat{\mathbf{j}}) + (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})(1) + \frac{1}{2}(-10\hat{\mathbf{j}})(1)^2\right]}{1+2}$$

$$= \left(\frac{70}{3}\hat{\mathbf{i}} + 35\hat{\mathbf{j}}\right)\ \text{m}$$ **Ans.**

9. (a) $$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$\therefore \quad 24 = \frac{(0.6)\,(80) + (m_2)\,(0)}{\text{Total mass}}$$

∴ Total mass = 2 kg **Ans.**

(b) $$\mathbf{a}_{CM} = \frac{d}{dt}(\mathbf{v}_{CM}) = \frac{d}{dt}[6t^2]\,\hat{\mathbf{j}}$$

$$= (12t)\,\hat{\mathbf{j}}\ \text{m/s}^2$$ **Ans.**

(c) $$\mathbf{F}_{\text{ext}} = m_{CM}\,\mathbf{a}_{CM} = m_{CM}\frac{d}{dt}(\mathbf{v}_{CM})$$

$$= m_{CM}\,(12t)\,\hat{\mathbf{j}}$$

At $t = 3$ s , $\mathbf{F}_{\text{ext}} = (2\text{kg})\,(12)\,(3)\,\hat{\mathbf{j}}$ N

$$= (72\hat{\mathbf{j}})\ \text{N}$$ **Ans.**

10. (a) $$\mathbf{v}_c = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2} = \mathbf{0.8\,\hat{i}}\ \text{m/s}$$

(b) $\mathbf{v}_1 = -1.6\,\hat{\mathbf{i}}$ m/s

From COM, $m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$

⇒ $v_2 = 2.4\hat{\mathbf{i}}$ m/s

(c) $$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{4}{7}$$

11. Fuel is finished in 40 s

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 10 \times 40 + (2 \times 10^3)\ln\left(\frac{200}{40}\right)$$

$$= 2818\ \text{m/s} = 2.82\ \text{km/s}$$ **Ans.**

12. Let v is the horizontal velocity of platform in opposite direction. Then from momentum conservation in opposite direction we have,

$$(60 + 40)\,v = (1)\,(10\cos 45°)$$

$$\therefore \quad v = \frac{10}{100}\cos 45°\ \text{m/s}$$

$$= \frac{10}{\sqrt{2}}\ \text{cm/s}$$

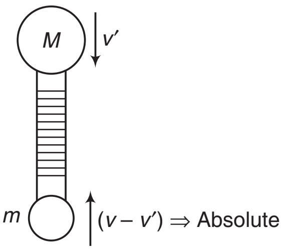
∴ Displacement of platform = vt

$$= \frac{(v)\,(2u\sin 45°)}{g}$$

$$= \left(\frac{10}{\sqrt{2}}\right) = \frac{2 \times 10 \times \frac{1}{\sqrt{2}}}{10}$$

$$= 10\ \text{cm}$$ **Ans.**

13. Since the system (or COM) was initially at rest. So it will always remain at rest. Because the net external forces are not changing.



(a) $Mv' = m\,(v - v')$

$$\therefore \quad v' = \frac{mv}{M + m}$$ **Ans.**

(b) If man stops climbing, then balloon will also stop to keep the system at rest.

14. $$\frac{F}{m} = \frac{v_r\left(-\frac{dm}{dt}\right)}{m} = a = \text{constant}$$

$$\therefore \quad \frac{u\left(-\frac{dm}{dt}\right)}{m} = a$$

$$\therefore \quad \int_{m_0}^{m} -\frac{dm}{m} = \int_0^t \frac{a}{u}\,dt$$

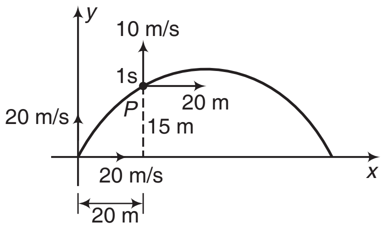
Solving this equation we get,

$$m = m_0\,e^{-at/u}$$ **Ans.**

15. At 1 seconds particle is at point P as shown in figure. Let velocity of second part is $\mathbf{v}$. Then applying momentum conservation, just before and just after explosion we have,

$$2m\,(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) = m\,(0) + m\mathbf{v}$$

$$\therefore \quad \mathbf{v} = (40\hat{\mathbf{i}} + 20\hat{\mathbf{j}})$$



Its vertical component of velocity is still 20 m/s. So total height,

$$h = h_i + \frac{u^2}{2g}$$

$$= 15 + \frac{(20)^2}{2 \times 10}$$

$$= 35\,\text{m}$$ **Ans.**

16. $h = (1+1)\text{ m} = 2\text{m}$

$\therefore \quad v = \sqrt{2gh}$

$= \sqrt{2 \times 10 \times 2}$

$= 6.32\text{ m/s}$ (downwards)

Impulse $= \Delta p = m\,(v_f - v_i)$

$= (1)\,[0 - (-6.32)] = 6.32\text{ N-s}$ (upwards)

18. In elastic collision, velocities are interchanged. So, v is minimum

$\therefore \quad \frac{1}{2} \times 0.1 \times v^2 = 0.2$

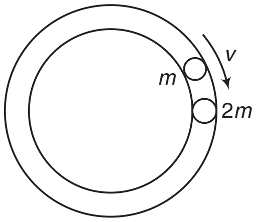
$\therefore \quad v = 2\text{ m/s} =$ minimum value **Ans.**

In perfectly inelastic collision, speed of combined mass will remain half.

$\therefore \quad \frac{1}{2} \times 0.1 \times \left(\frac{v}{2}\right)^2 = 0.2$

$\therefore \quad v = 2\sqrt{2}\text{ m/s} =$ maximum value **Ans.**

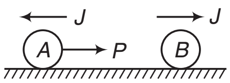
19. Since collision is elastic. It means relative velocity of separation will remain v and relative velocity means one mass is assumed at rest.



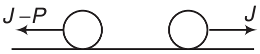
$\therefore \quad t = \frac{2\pi r}{v}$ **Ans.**

20. $J =$ impulse $=$ change in momentum

$$e = \frac{RVOS}{RVOA}$$



Initial momenta



Final momenta

$$= \frac{(J/m) + (J\text{-}P/m)}{(P/m)}$$

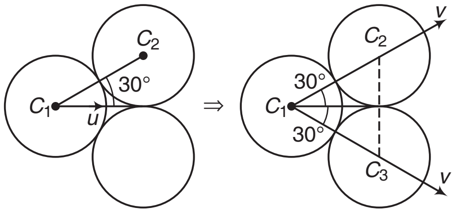
$$= \frac{2J}{P} - 1$$ **Ans.**

21. $p_i = p_f$

$\therefore \quad mu = 2\,mv\cos 30°$

or $\quad v = \frac{u}{\sqrt{3}}$

$e = \frac{RVOS}{RVOA}$ (along C_1C_2 or C_1C_3)



$$= \frac{v}{u\cos 30°} = \frac{(u/\sqrt{3})}{\sqrt{3}\,u/2} = \frac{2}{3}$$ **Ans.**

22. For the collision

$1 \times 10 = 10 \times v \quad \Rightarrow \quad v = 1\text{m/s}$

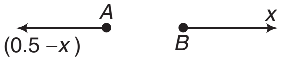
If x be the maximum compression

$$\frac{1}{2} \times 10 \times 1^2 = \mu\,(m+M)\,gx + \frac{1}{2}kx^2$$

$$5 = 10x + 120x^2$$

$\Rightarrow \quad x = \frac{1}{6}m$ **Ans.**

23.



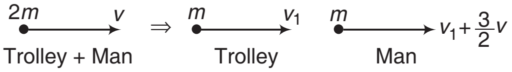
$$m_A\,x_A = m_B\,x_B$$

$\therefore \quad (5)\,(0.5 - x) = 30x$

$\therefore \quad x = 0.0714\text{ m}$

$= 71.4\text{ mm}$ **Ans.**

24. $t_1 = \frac{L}{(3/2)\,v} = \frac{2L}{3v}$



$$t_2 = \frac{L}{(3/2)\,v} = \frac{2L}{3v}$$

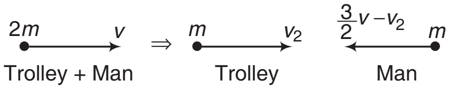
$$p_i = p_f$$

$\therefore \quad 2\,mv = mv_1 + m\left(v_1 + \frac{3}{2}v\right)$

$$v_1 = \frac{v}{4}$$

$\therefore \quad s_1 = v_1\,t_1 = \left(\frac{v}{4}\right)\left(\frac{2L}{3v}\right) = \frac{L}{6}$

In return journey



$$p_i = p_f$$

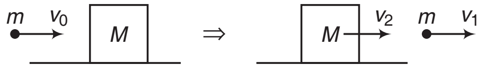
$$\therefore \quad 2\,mv = mv_2 - m\left(\frac{3}{2}v - v_2\right)$$

$$\therefore \quad v_2 = \frac{7}{4}\,v$$

or $$s_2 = v_2 t_2 = \left(\frac{7}{4}v\right)\left(\frac{2L}{3v}\right) = \frac{7L}{6v}$$

$$s_{\text{Total}} = s_1 + s_2 = \frac{4L}{3}$$ **Ans.**

25.



(a) $p_i = p_f$

$$\therefore \quad mv_0 = mv_1 + mv_2$$

$$\therefore \quad v_2 = \frac{mv_0 - mv_1}{M}$$

$$= \frac{(0.004)\,(500-100)}{1.0}$$

$$= 1.6 \text{ m/s}$$

Retardation due to friction,

$$a = \frac{\mu mg}{m} = \mu g = (10\,\mu)$$

$$s = \frac{v_2^2}{2a} = \frac{v_2^2}{20\,\mu}$$

$$\therefore \quad \mu = \frac{v_2^2}{20s}$$

$$= \frac{(1.6)^2}{20 \times 0.3} = 0.43$$ **Ans.**

(b) Decrease in kinetic energy of bullet

$$= \frac{1}{2}m\,(v_0^2 - v_1^2)$$

$$= \frac{1}{2} \times 0.004\,[\,(500)^2 - (100)^2\,]$$

$$= 480 \text{ J}$$ **Ans.**

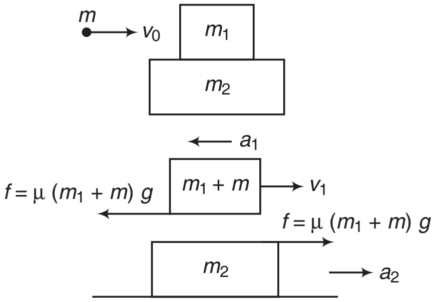
(c) Kinetic energy of block,

$$= \frac{1}{2}\,Mv_2^2$$

$$= \frac{1}{2} \times 1.0 \times (1.6)^2$$

$$= 1.28 \text{ J}$$ **Ans.**

26.



(i) Common velocity $= \dfrac{\text{Initial momentum}}{\text{Total mass}}$

$$v_c = \frac{0.25 \times 302}{0.25 + 37.5 + 1.25}$$

$$= 1.94 \text{ m/s}$$ **Ans.**

(ii) $$v_1 = \frac{\text{Initial momentum}}{m_1 + m}$$

$$= \frac{0.25 \times 302}{37.5 + 0.25} = 2 \text{ m/s}$$

$$a_1 = \frac{f}{m_1 + m} = \mu g = 5 \text{ m/s}^2$$

$$a_2 = \frac{f}{m_1 + m + m_2} = \left(\frac{m_1 + m}{m_1 + m + m_2}\right)\mu g$$

$$= \left(\frac{37.5 + 0.25}{37.5 + 0.25 + 1.25}\right)(0.5 \times 10)$$

$$= 4.84 \text{ m/s}^2$$

$$a_r = a_1 - a_r = 0.16 \text{ m/s}^2$$

Common velocity is achieved when, v_1 converts into v_c by a retardation a_1.

$$\therefore \quad v_c = v_1 - a_1 t$$

$$\therefore \quad t = \frac{v_1 - v_c}{a_1} = \frac{2 - 1.94}{5}$$

$$= 0.012 \text{ s}$$

Now, $$s_r = \frac{1}{2}\,a_r t^2$$

$$= \frac{1}{2} \times 0.16 \times (0.012)^2$$

$$= 0.011 \text{ mm}$$ **Ans.**

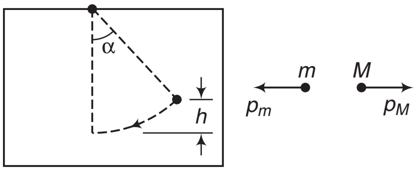
27. $h = l\,(1 - \cos\alpha)$ ...(i)

$$p_m = p_M \text{ (in opposite directions)}$$

$$\therefore \quad \sqrt{2K_m m} = \sqrt{2K_M M}$$

or $$\frac{K_m}{K_M} = \frac{M}{m}$$ ...(ii)

$$K_m + K_M = mgh = mgl\,(1 - \cos\alpha) \qquad \text{...(iii)}$$



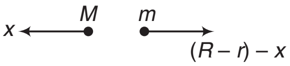
Solving these three equations we get,

$$K_M = \left(\frac{m}{m + M}\right) mgl\,(1 - \cos\alpha)$$

$$\therefore \quad \frac{1}{2} Mv_M^2 = \left(\frac{m}{M + m}\right) mgl\,(1 - \cos\alpha)$$

$$\therefore \quad v_m = 2m \sin\frac{\alpha}{2}\sqrt{\frac{gl}{M(M + m)}} \qquad \textbf{Ans.}$$

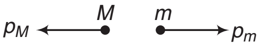
28. (i)



$$Mx = m\,(R - r - x)$$

$$\therefore \quad x = \frac{m\,(R - r)}{M + m} \qquad \textbf{Ans.}$$

(ii)



$$p_M = p_m$$

$$\therefore \quad \sqrt{2K_M M} = \sqrt{2K_m m}$$

$$\therefore \quad \frac{K_M}{K_m} = \frac{m}{M} \qquad \text{...(i)}$$

$K_m + K_m$ = decrease in potential energy of m

$$= mg\,(R - r) \qquad \text{...(ii)}$$

Solving these two equations we get,

$$K_M = \left(\frac{m}{M + m}\right) mg\,(R - r) = \frac{1}{2} Mv_M^2$$

$$\therefore \quad v_M = m\sqrt{\frac{2g\,(R - r)}{M\,(M + m)}} \qquad \textbf{Ans.}$$

29. (a) $|\Delta \mathbf{p}| = \sqrt{p_1^2 + p_2^2 - 2p_1 p_2 \cos 90°}$

$$= \sqrt{p_1^2 + p_2^2}$$

$$= \sqrt{(0.05 \times 2)^2 + (0.05 \times 2)^2}$$

$$= 0.14 \text{ kg-m/s} \qquad \textbf{Ans.}$$

(b) Initial and final velocity of wall is zero. Therefore change in momentum is zero.

30. $v = \sqrt{2gy}$

Thrust force downwards

$$F = \lambda v^2 \qquad (\lambda = \text{mass/length})$$

or $\quad F = m\,(2gy)$ or $F = 2\,mgy$

y length is lying on table. So its weight

$$W = (ym)\,g$$

Total force on table = $F + W = (3\,mgy)$

= weight of a length $3y$ of the rope

31. Relative velocity of sand is 2 m/s in backward direction. Since mass is increasing, therefore thrust force is in the direction of relative velocity (backwards).

$$\text{Thrust force} = v_r\left(+\frac{dm}{dt}\right)$$

$$= (5)\,(2) = 10 \text{ N} \qquad \text{(backwards)}$$

$\therefore$ Force needed (F_{ext}) to move the belt with constant velocity ($F_{\text{net}} = 0$) is

$$F_{\text{ext}} = F = 10 \text{ N} \qquad \text{(in forward direction)}$$

$$P = F_{\text{ext}}\, v = 20 \text{ W} \qquad \textbf{Ans.}$$

32. Impulse = $\Delta p = p_f - p_i$

$$= m\,(v_f - v_i)$$

$$= (3)\,[40 - (-50)]$$

$$= 270 \text{ N-s}$$

i.e. impulse is 270 N-s (towards right) **Ans.**

Impulse = $F_{\text{av}}\ \Delta t$

$$\therefore \quad F_{\text{av}} = \frac{\text{Impulse}}{\Delta t} = \frac{270}{0.02}$$

$$= 13500 \text{ N}$$

$$= 13.5 \text{ kN} \qquad \text{(towards right)} \qquad \textbf{Ans.}$$

33. $v_3' = \left(\frac{m_3 - m_2}{m_3 + m_2}\right) v_3 + \left(\frac{2m_2}{m_2 + m_3}\right) v_2$

$$= \left(\frac{3 - 2}{3 + 2}\right)(2) + 0 = 0.4 \text{ m/s}$$

$$v_2' = 0 + \left(\frac{2 \times 3}{2 + 3}\right)(2) = 2.4 \text{ m/s}$$

Retardation of each block,

$$a = \frac{\mu_K mg}{m} = \mu_K g = 3 \text{ m/s}^2$$

Before coming to rest

$$s_3 = \frac{v_3'^2}{2a} = \frac{(0.4)^2}{2 \times 3} = 0.03 \text{ m}$$

$$s_2 = \frac{v_2'2}{2a} = \frac{(2.4)^2}{2 \times 3} = 0.96 \text{ m}$$

$$\therefore \quad d = s_2 - s_3 = 0.93 \text{ m} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. Let velocity of m just after collision is v. Then, from conservation of momentum we have,

$$m_1v_1 = mv + m_1\frac{v_1}{3} \quad \Rightarrow \quad \therefore \quad v = \frac{2}{3}\frac{m_1v_1}{m}$$

Now, for just completing the circle,

$$v = \sqrt{5gl}$$

$$\therefore \qquad \frac{2}{3}\frac{m_1v_1}{m} = \sqrt{5gl}$$

$$\therefore \qquad v_1 = \frac{3m}{2m_1}\sqrt{5gl} \qquad \textbf{Ans.}$$

2. $v_m = \sqrt{2gl}$

From conservation of linear momentum,

$$v_{2m} = \frac{v_m}{2} = \frac{\sqrt{2gl}}{2}$$

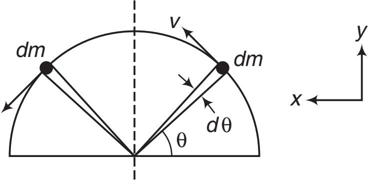
$$\text{Now,} \qquad h = \frac{v_{2m}^2}{2g} = \frac{l}{4}$$

$$\text{But} \qquad h = l(1 - \cos\theta)$$

$$\therefore \qquad \frac{l}{4} = l(1 - \cos\theta)$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{3}{4}\right)$$

3. y-components of momenta are cancelled and x-components are added.



$$P = \int_0^\pi dP_x = \int_0^\pi (dm)(v\sin\theta)$$

$$= \int_0^\pi \left(\frac{M}{\pi}\,d\theta\right)(v\sin\theta) = \frac{2Mv}{\pi} \qquad \textbf{Ans.}$$

4. Let v is the velocity of mass $2m$ in natural length of spring, then from conservation of energy we have,

$$\frac{1}{2}kx_0^2 = \frac{1}{2}(2m)v^2$$

$$\therefore \qquad v = \sqrt{\frac{k}{2m}}\,x_0$$

Velocity of centre of mass at this instant,

$$v_{CM} = \frac{\text{Total momentum}}{\text{Total mass}}$$

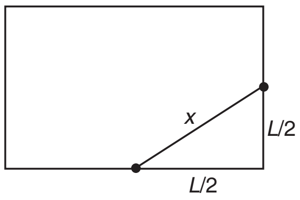
$$= \frac{2mv}{3m} = \frac{2}{3}v = \frac{1}{3}\sqrt{\frac{2k}{m}}\,x_0$$

Now, impulse on system = change in momentum of system

$$\therefore \qquad F_{av}\,\Delta t = (3m)\,v_{CM} = (\sqrt{2mk})\,x_0$$

$$\therefore \qquad F_{av} = \frac{(\sqrt{2mk})\,x_0}{\Delta t} \qquad \textbf{Ans.}$$

5. Retardation, $a = \frac{\mu\, mg}{m} = \mu g$



$$= 0.2 \times 10 = 2\ \text{m/s}^2$$

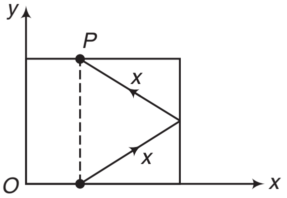
Total distance travelled before coming to rest,

$$d = \frac{u^2}{2a} = \frac{(2)^2}{2\times 2} = 1\text{m}$$

$$x = \sqrt{2}\frac{L}{2} = \frac{L}{\sqrt{2}} = \frac{1}{2}$$

$$= 0.5\ \text{m}$$

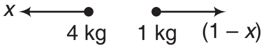
Therefore striker will stop after travelling a distance $2x$, as shown in figure.



Co-ordinates of point P are, $\left(\frac{L}{2}, L\right)$

$$\text{or} \qquad \left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

6. Displacement of M relative to m on reaching the other end is $2L\sin 30°$ or 1m.



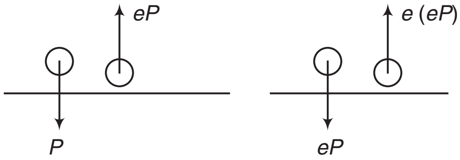
$$4x = 1\,(1 - x)$$

$$\therefore \qquad x = 0.2\ \text{m} \qquad \textbf{Ans.}$$

7. $\Delta P_1 = (1 + e)\,P$

$\Delta P_2 = (1 + e)\,eP$

and so on



$$\therefore \Delta P_{\text{Total}} = \Delta P_1 + \Delta P_2 + \ldots$$
$$= (1 + e)\, P\, (1 + e + e^2 + \ldots)$$
$$= \left(\frac{1+e}{1-e}\right) P$$

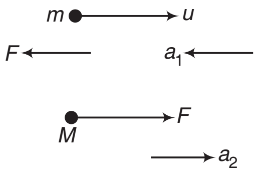
8. Assuming retardation force F to same in both cases, we have,

retardation $a = \dfrac{F}{m}$

Using, $v^2 = u^2 - 2as$, we have,

$$0 = u^2 - 2\left(\frac{F}{m}\right) h$$

$$\therefore \qquad F = \frac{mu^2}{2h} \qquad \text{...(i)}$$



In second case,

$$a_1 = \frac{F}{m},\ a_2 = \frac{F}{M}$$

Relative retardation of m,

$$a_r = a_1 + a_2 = F\left(\frac{M+m}{Mm}\right)$$

Now, $\qquad v_r^2 = u_r^2 - 2a_r s$

$$\therefore \qquad 0 = u^2 - 2F\left(\frac{M+m}{Mm}\right) s$$

or $\qquad s = \dfrac{u^2 Mm}{2F\,(M+m)}$

Substituting, $\dfrac{u^2}{F} = \dfrac{2h}{m}$ from Eq. (i) we have,

$$s = \frac{Mh}{M+m} \qquad \textbf{Ans.}$$

9. Velocity of B after collision will become,

$$\left(\frac{1+e}{2}\right) v = \left(\frac{1+e}{2}\right)(16)$$

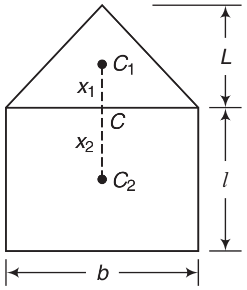
or $\qquad 8\,(1 + e)$ m/s

Now, $\quad 8\,(1 + e) = \sqrt{2gh}$

$$= \sqrt{2 \times 10 \times 5} = 10$$

$$\therefore \qquad e = 0.25 \qquad \textbf{Ans.}$$

10.

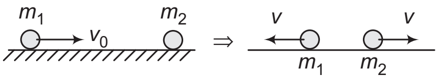


$$A_1 x_1 = A_2 x_2$$

$$\therefore \qquad \frac{1}{2}(bL)\left(\frac{L}{3}\right) = (bl)\left(\frac{l}{2}\right)$$

$$\therefore \qquad l = \frac{L}{\sqrt{3}} \qquad \textbf{Ans.}$$

11. $p_i = p_f$



$$\therefore \qquad m_1 v_0 = m_1 v + m_2 v \qquad \text{...(i)}$$

$$e = \frac{RVOS}{RVOA}$$

$e = 1$ for elastic collision

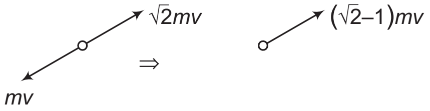
$$\therefore \qquad RVOS = RVOA$$

or $\qquad 2v = v_0 \qquad$ ...(ii)

From Eqs. (i) and (ii) we get,

$$\frac{m_1}{m_2} = \frac{1}{3}$$

12. Net momentum of three fragments



$$\mathbf{p}_i = \mathbf{p}_f = 0$$

$\therefore$ Momentum of fourth part is $(\sqrt{2} - 1)\, mv$ in opposite direction.

So, velocity of fourth part is $(\sqrt{2} - 1)\, v$

$$\therefore \qquad \text{KE} = 3\left(\frac{1}{2}mv^2\right) + \frac{1}{2}m\,[(\sqrt{2} - 1)\, v]^2$$

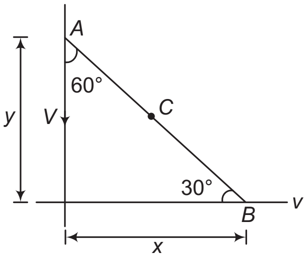
$$= (3 - \sqrt{2})\, mv^2 \qquad \textbf{Ans.}$$

13. $v_a = v_b = \dfrac{v}{2}$

$$v_c = \frac{v}{3}$$

$$\therefore \qquad v_a : v_b : v_c = 3 : 3 : 2$$

14. Distance AB = constant



$\therefore$ Component of V along AB

= component of v along AB

or $$\frac{V}{2} = \frac{\sqrt{3}\, v}{2}$$

or $$V = \sqrt{3}\, v$$

Now, $$\frac{dx}{dt} = v$$

$$\left(-\frac{dy}{dt}\right) = V = \sqrt{3}\, v$$

$$\mathbf{r}_c = \frac{x}{2}\,\hat{\mathbf{i}} + \frac{y}{2}\,\hat{\mathbf{j}}$$

$$\therefore \qquad \mathbf{v}_c = \frac{d\mathbf{r}_c}{dt} = \frac{1}{2}\left[\frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}\right]$$

$$= \frac{1}{2}[v\hat{\mathbf{i}} - \sqrt{3}\, v\hat{\mathbf{j}}]$$

$$\therefore \qquad |\mathbf{v}_c| = \left[\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,\right] v = v \qquad \textbf{Ans.}$$

15. At 5 s

$$\mathbf{r}_{\text{COM}} = \mathbf{r}_i + \mathbf{v}_{\text{CM}}\, t$$

$$= (\hat{\mathbf{i}}) + (0.2 \times 5)\,\hat{\mathbf{i}}$$

$$= 2\hat{\mathbf{i}} = \frac{m_1\mathbf{r}_i + m_2\mathbf{r}_2}{m_1 + m_2}$$

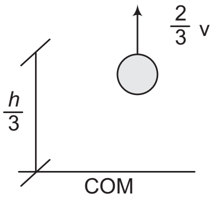
$$\therefore \quad 2\hat{\mathbf{i}} = \frac{(2/3)\,(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - -4\hat{\mathbf{k}}) + (4/3)\mathbf{r}_2}{2}$$

$$\therefore \quad \mathbf{r}_2 = (4.5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

So, the co-ordinates are (4.5m, – 1m, – 2m) **Ans.**

16. m mass falls a distance $\dfrac{h}{2}$ in time, say t.

Then,



$$\frac{h}{2} = \frac{1}{2}gt^2 \quad \Rightarrow \quad \therefore \quad t = \sqrt{\frac{h}{g}}$$

For mass $2m$,

$$\frac{h}{2} = vt - \frac{1}{2}gt^2$$

$$= v\sqrt{\frac{h}{g}} - \frac{h}{2}$$

$$\therefore \qquad v = \sqrt{gh}$$

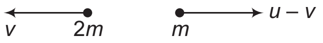
Now, let u is the velocity with which combined mass collides with ground, then

$$u = \sqrt{\left(\frac{2}{3}v\right)^2 + 2g\left(\frac{h}{3}\right)}$$

$$= \sqrt{\frac{4}{9}gh + \frac{2}{3}gh}$$

$$= \frac{\sqrt{10}}{3}\sqrt{gh} \qquad \textbf{Ans.}$$

17. $2mv = m\,(u - v)$



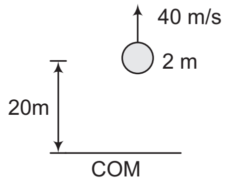
$$\therefore \qquad v = \frac{u}{3}$$

$$\therefore \qquad u - v = \frac{2u}{3}$$

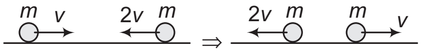
$$\text{KE} = \frac{1}{2} \times m\left(\frac{2u}{3}\right)^2 + \frac{1}{2}(2m)\left(\frac{u}{3}\right)^2$$

$$= \frac{1}{3}mu^2 \qquad \textbf{Ans.}$$

18. $h_{\max} = 20 + \dfrac{(40)^2}{2g} = 100\,\text{m}$ **Ans.**



19. Let $|\mathbf{p}| = mv = p$



In elastic head on collision with equal masses, velocities are interchanged.

Gain in kinetic energy of first

$$= \frac{1}{2} m\,[(2v)^2 - v^2]$$

$$= \frac{3}{2} mv^2 = \left(\frac{3}{2} m\right)\left(\frac{p}{m}\right)^2 = \frac{3p^2}{2m}$$ **Ans.**

20. $$\frac{(4m)\,X_1 + (4m)\left(\frac{a}{2}\right)}{8m} = \frac{(4m)\,X_1 + (m)\,(a)}{5m}$$

Solving this equation, we get

$X_1 = X$-coordinate of COM of plate

$$= \frac{a}{6}$$ **Ans.**

21. $$X_{\text{CM}} = \frac{m_1X_1 + m_2X_2 + m_3X_3 + m_4X_4}{m_1 + m_2 + m_3 + m_4}$$

Similarly Y_{CM}.

22. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore\ (m_0v_0\hat{\mathbf{i}}) = m_0\frac{v_0}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right) + 2m_0\mathbf{v}$$

$$\therefore\quad \mathbf{v} = \frac{v_0}{4}\hat{\mathbf{i}} - \frac{v_0}{4}\hat{\mathbf{j}}$$

$$\therefore\quad |\mathbf{v}| = \sqrt{2}\,\frac{v_0}{4} = \frac{v_0}{2\sqrt{2}}$$ **Ans.**

23. $$\frac{1}{2}kx_0^2 = \frac{1}{2}m_2v_2^2$$

$$\therefore\quad v_2 = \sqrt{\frac{k}{m_2}}\,x_0$$

$$v_{\text{CM}} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

$$= \left(\frac{\sqrt{k\,m_2}}{m_1 + m_2}\right)x_0 \qquad (\text{as } v_1 = 0)$$

24. Velocity of second ball

$$= \left(\frac{m/2 - m}{m/2 + m}\right)(0) + \left(\frac{2 \times m}{m + m/2}\right)v$$

$$= \frac{4}{3}v$$

Velocity of third ball will become,

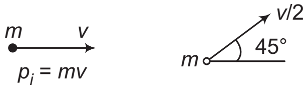
$$\frac{4}{3} \text{ times } \frac{4}{3}v \quad \text{or} \quad \left(\frac{4}{3}\right)^2 v$$

$$\therefore\quad v_n = \left(\frac{4}{3}\right)^{n-1} v = \sqrt{5gr}$$

$$\therefore\quad v = \left(\frac{3}{4}\right)^{n-1}\sqrt{5gr}$$ **Ans.**

More than One Correct Options

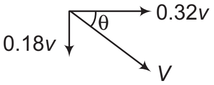
1. Momentum remains conserved in any type of equation



$$\mathbf{p}_i = \mathbf{p}_f$$

$$\therefore\quad (mv\hat{\mathbf{i}}) = \left(\frac{mv}{2\sqrt{2}}\hat{\mathbf{i}} + \frac{mv}{2\sqrt{2}}\hat{\mathbf{j}}\right) + (2m)\,\mathbf{v}$$

$\mathbf{v}$ = velocity of mass 2m



$$= 0.32\hat{\mathbf{i}} - 0.35\hat{\mathbf{j}}$$

$$= (0.32v)\,\hat{\mathbf{i}} - (0.18v)\,\hat{\mathbf{j}}$$

$$V = 0.37\,v$$

$$\tan\theta = \frac{0.18v}{0.32v} = 0.5625$$

$\therefore\ \theta = 29.35°$

Since $(\theta + 45°) < 90°$. Therefore, the angle of divergence between particles after collision is less than 90°.

Further, $K_i = \frac{1}{2}mv^2$ and

$$K_f = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}(2m)\,(0.37v)^2$$

$$K_f < K_i \quad .$$

Therefore, collision is inelastic.

2. Just after collision,

$$v_m = \left(\frac{m - 5m}{m + 5m}\right)\sqrt{2gl} = -\frac{2}{3}\sqrt{2gl}$$

$$v_{5\text{m}} = \left(\frac{2 \times m}{m + 5m}\right)\sqrt{2gl} = \frac{\sqrt{2gl}}{3}$$

$$T - mg = \frac{mv_m^2}{l} = \frac{m}{l}\left(\frac{8gl}{9}\right)$$

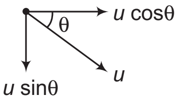
$$\therefore\quad T = \frac{17}{9}mg$$

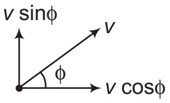
$$h_m = \frac{v_m^2}{2g} = \frac{4l}{9}$$

3. $$u\cos\theta = v\cos\phi \qquad \text{...(i)}$$

$$v\sin\phi = eu\sin\theta$$

or $$eu\sin\theta = v\sin\phi \qquad \text{...(ii)}$$





From Eqs. (i) and (ii), we can see that,

$$\tan\phi = e\tan\theta$$

Momentum or velocity changes only in vertical direction.

$$\therefore \quad |\text{Impulse}| = |\Delta p|$$

$$= m(u\sin\theta + eu\sin\theta)$$

$$= m(1+e)\,u\sin\theta$$

$$v = \sqrt{(v\cos\phi)^2 + (v\sin\phi)^2}$$

$$= \sqrt{(u\cos\theta)^2 + (eu\sin\theta)^2}$$

$$= \sqrt{u^2(\cos^2\theta + e^2\sin^2\theta)}$$

$$= u\sqrt{1-(1-e^2)\sin^2\theta}$$

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{v^2}{u^2}$$

$$= \cos^2\theta + e^2\sin^2\theta$$

4. Impulse $= \Delta\mathbf{p} = m(\mathbf{v}_f - \mathbf{v}_i)$

$\therefore$ Impulse received by m

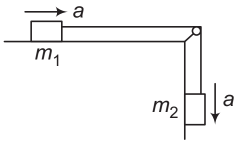
$$= m[(-2\hat{\mathbf{i}} + \hat{\mathbf{j}}) - (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})]$$

$$= m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

Impulse received by $M = -$ (impulse received by m)

$$= m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

5. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{m_2 g}{m_1 + m_2}$



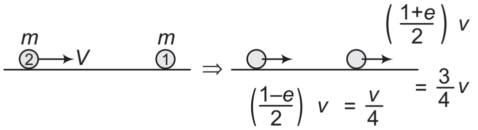
Now,

$$(a_{\text{cm}})_x = \frac{m_1 a}{m_1 + m_2} = \frac{m_1 m_2 g}{(m_1 + m_2)^2}$$

$$(a_{\text{cm}})_y = \frac{m_2 a}{m_1 + m_2} = \left(\frac{m_2}{m_1 + m_2}\right)^2 g$$

6. Out of two blocks, one block of mass m is moving in vertical direction also (downwards). Therefore COM is moving vertically downwards and momentum of the system is not conserved in vertical direction.

7. $|\text{Impulse}| = |\Delta p_1 \text{ or } \Delta p_2|$



$$= m\left(\frac{3}{4}v\right) = \frac{3mv}{4}$$

Loss of kinetic energy $= K_i - K_f$

$$= \frac{1}{2}mv^2 - \left[\frac{1}{2}m\left(\frac{3v}{4}\right)^2 + \frac{1}{2}m\left(\frac{v}{4}\right)^2\right]$$

$$= \frac{3}{16}mv^2$$

8. External force gravity acts on system. Therefore momentum of system is not conserved.

Mass keep on decreasing. Therefore acceleration will keep on increasing.

Comprehension Based Questions

1. Impulse $= mu$

$$\therefore \quad u = \frac{\text{Impulse}}{m} = \frac{4}{2} = 2 \text{ m/s}$$

Now, $\quad E_i = E_f$

$$\therefore \quad \frac{1}{2}\times 2\times(2)^2 + \frac{1}{2}\times 4000\times(0.05)^2 = \frac{1}{2}\times 2\times v^2$$

Solving we get,

$v = 3$ m/s **Ans.**

2. Again using,

$$E_i = E_f$$

$$\therefore \quad \frac{1}{2}\times 2\times(2)^2 + \frac{1}{2}\times 4000\times(0.05)^2 = \frac{1}{2}\times 4000\times x^2$$

$\therefore \quad x = 0.067$ m $= 6.7$ m $=$ compression

$\therefore \quad d = 5$ cm $+ 6.7$ cm $+ 6.7$ cm $+ 6.7$ cm

≈ 25 cm **Ans.**

3. $p_i = p_f$

$$\therefore \quad 0 = (8m)(2v) - (16m)(v) + (48m)v'$$

Here v' = absolute speed of rod

$= 0$

$\therefore$ Displacement of rod $= 0$ **Ans.**

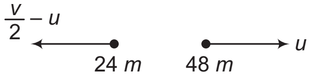
4. $x_A + x_B = 12\,L$

$$\therefore \quad 2vt + vt = 12L$$

$$\therefore \quad vt = 4L$$

$$\Rightarrow \quad X_B = vt = 4L$$

5. $(24\,m)\left(\dfrac{v}{2} - u\right) = (48\,m)\,u$



$$\therefore \quad u = \frac{v}{6} \qquad \textbf{Ans.}$$

6. $t = t_1 + t_2 = \left(\dfrac{4L}{v}\right) + \left(\dfrac{12L - 4L}{v/2}\right) = \dfrac{20L}{v}$

$$\text{But} \quad \frac{L}{v} = T = 4 \text{ s}$$

$$\therefore \quad t = 80 \text{ s} \qquad \textbf{Ans.}$$

7. Till t_1, rod is stationary. For time t_2 rod is moving with absolute speed $u\ (= v/6)$

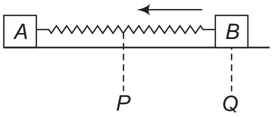
$$\therefore \quad \text{Displacement of rod} = \left(\frac{v}{6}\right) t_2$$

$$= \left(\frac{v}{6}\right)\left(\frac{16L}{v}\right) = \frac{8L}{3} \qquad \textbf{Ans.}$$

Match the Columns

1. $P \rightarrow$ compressed state of spring

$Q \rightarrow$ natural length of spring



From P to Q

$$a_B = \frac{Kx}{m_B}$$

$$a_{CM} = \frac{m_B a_B}{m_A + m_B} = \frac{(m_B)\,(Kx)}{m_A + m_B}$$

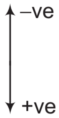
From P to Q, compression x decreases. Therefore, a_{CM} decreases. After Q, A leaves contact with wall, spring comes in its natural length. Net force on system becomes zero.

Therefore, a_{CM} becomes zero.

From P to Q velocity of B, therefore velocity of COM will increase. After that a_{CM} becomes zero. Therefore, v_{CM} becomes constant.

2. $t_1 = \dfrac{2u}{g} = \dfrac{2 \times 20}{10} = 4\,\text{s}$

$$t_2 = \sqrt{\frac{2h}{g}} = \frac{\sqrt{2 \times 180}}{10} = 6 \text{ s}$$



At $t = 0$

$$a_{CM} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} = \frac{(m)\,(10) + (m)\,(10)}{2m}$$

$$= 10 \text{ m/s}^2$$

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(m)\,(-20) + (m)\,(0)}{2m}$$

$$= -10 \text{ m/s}$$

At $t = 5$ s

$$a_{CM} = \frac{(m)\,(0) + (m)\,(10)}{2m} = 5 \text{ m/s}^2$$

$$v_{CM} = \frac{(m)\,(0) + (m)\,(50)}{2m} = 25 \text{ m/s}$$

3. (a) $p = \sqrt{2Km} = \sqrt{2 \times 0.5 \times 4} = 2$ kg-m/s

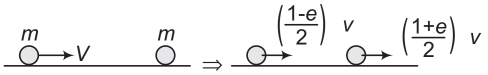
(b) $p_{CM} = p_A + p_B = 0 + 2 = 2$ kg - m/s

(c) At maximum compression, their velocities or, their momenta are same (half of p_{CM}).

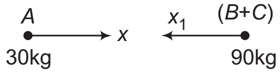
$$\therefore \quad p_A = p_B = \frac{p_{CM}}{2} = 1 \text{ kg - m / s}$$

(d) In elastic collision between two equal masses, velocities are interchanged. So, B comes to rest and A starts moving.

4.

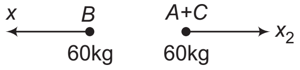

m
V
m
$\Rightarrow$
$\left(\frac{1-e}{2}\right) v$
$\left(\frac{1+e}{2}\right) v$

5. (a) If A moves towards right then B and C will move towards left.



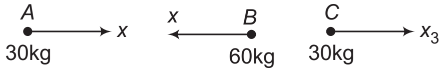
$$30x = 90x_1 \quad \Rightarrow \quad x_1 = \frac{x}{3}$$

(b) $60x = 60x_2 \quad \Rightarrow \quad x_2 = x$

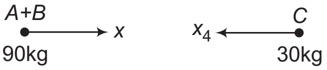


(c) $30x + 30x_3 = 60x$

$$\therefore \quad x_3 = x$$



(d) $90x = 30x_4 \quad \Rightarrow \quad x_4 = 3x$



6. (a) If T = tension on string connecting m_1 and m_2, then, $2T$ = tension in other string.

Equilibrium of m_2 gives,

$$T = m_2\, g = 200 \text{ N}$$

Equilibrium of $(M + m_1)$ gives,

$$3T = W_{\text{man}} + m_1 g$$

$$\text{or} \quad 600 = W_{\text{man}} + 100 \quad \Rightarrow \quad W_{\text{man}} = 500 \text{ N}$$

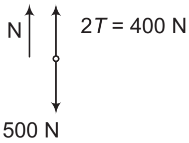
(b) Total upward force on system is $4T$ and total downward force is weight of all. To accelerate COM upwards,

$$4T > 200 + 100 + 500 \quad \text{or} \quad T > 200 \text{ N}$$

(c) To accelerate downwards,

$$T < 200 \text{ N}$$

(d) In equilibrium, forces on man



$$N + 400 = 500$$

$$\therefore \quad N = 100 \text{ N}$$

7. $$v_{\text{CM}} = \frac{\text{Total momentum}}{\text{Total mass}}$$

$$= \frac{2 \times 3}{3 + 6} = \frac{2}{3} \text{ m/s} = \text{constant}$$

At maximum deformation,

$$v_{3\text{kg}} = v_{6\text{kg}} = v_{\text{CM}} = \frac{2}{3} \text{ m/s}$$

Upto maximum elongation spring force on 6kg is towards left. So 6 kg block will accelerate and its velocity will be maximum.

8. $$V_{\text{CM}} = \frac{\text{Total momentum}}{\text{Total mass}}$$

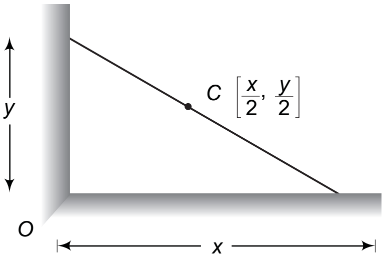
$$= \frac{(10 \times 1) - (5 \times 2)}{3} = 0 = \text{constant}$$

$$P_{\text{CM}} = (m_{\text{CM}})\,(v_{\text{CM}}) = 0 = \text{constant}$$

Finally both blocks will stop.

Subjective Questions

1. $$y = \sqrt{L^2 - x^2} \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{\sqrt{L^2 - x^2}}\,\frac{dx}{dt}$$

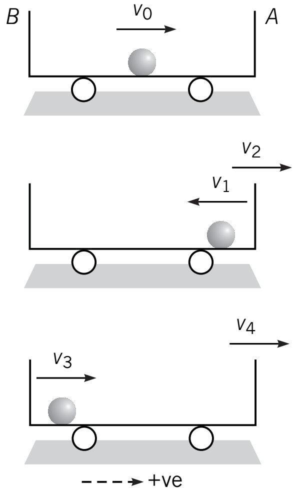


$$= -\frac{3 \times 2}{4} = -\frac{3}{2} \text{ m/s}$$

$$V_{CM} = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(1)^2 + \left(\frac{3}{4}\right)^2} = 1.25 \text{ m/s} \qquad \textbf{Ans.}$$

2. Applying conservation of linear momentum and



$$e = \frac{\text{Relative speed of separation}}{\text{Relative speed of approach}}, \quad \text{we get}$$

$$mv_0 = Mv_2 - mv_1 \qquad \ldots\text{(i)}$$

$$v_1 + v_2 = ev_0 \qquad \ldots\text{(ii)}$$

Solving these two equations, we get,

$$v_1 = \left(\frac{eM - m}{M + m}\right) v_0, \quad v_2 = m\left(\frac{e + 1}{M + m}\right) v_0 \qquad \textbf{Ans.}$$

The desired time is

$$t = \frac{d}{v_0} + \frac{2d}{ev_0} + \frac{d}{e^2 v_0}$$

$$\text{or} \qquad t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right) \qquad \textbf{Ans.}$$

3. (i) $x_1 = v_0 t - A(1 - \cos \omega t)$

$$t\, x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = v_0 t$$

$$\therefore \quad x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos \omega t)$$

(ii) $$a_1 = \frac{d^2 x_1}{dt^2} = -\omega^2 A \cos \omega t$$

The separation $x_2 - x_1$ between the two blocks will be equal to l_0 when $a_1 = 0$ or $\cos \omega t = 0$

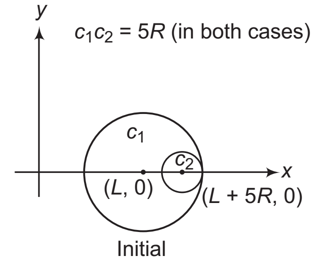
$$x_2 - x_1 = \frac{m_1}{m_2} A(1 - \cos \omega t) + A(1 - \cos \omega t)$$

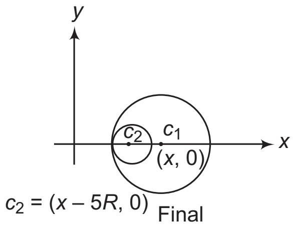
or $$l_0 = \left(\frac{m_1}{m_2} + 1\right) A \qquad (\cos \omega t = 0)$$

Thus, the relation between l_0 and A is,

$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A$$ **Ans.**

4. Since, all the surfaces are smooth, no external force is acting on the system in horizontal direction. Therefore, the centre of mass of the system in horizontal direction remains stationary.





x-coordinate of COM initially will given by

$$x_i = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_i = \frac{(4M)(L) + M(L + 5R)}{4M + M} = (L + R) \qquad \ldots\text{(i)}$$

Let $(x, 0)$ be the coordinates of the centre of large sphere in final position. Then, x-coordinate of COM finally will be

$$x_f = \frac{(4M)(x) + M(x - 5R)}{4M + M} = (x - R) \qquad \ldots\text{(ii)}$$

Equating Eqs. (i) and (ii), we have

$$x = L + 2R$$

Therefore, coordinates of large sphere, when the smaller sphere reaches the other extreme position are $(L + 2R, 0)$. **Ans.**

5. (a) Chain has a constant speed. Therefore, net force on it should be zero. Thus,

P = Weight of length y of chain + thrust force

$$= \frac{m}{l} yg + \rho v_0^2 \qquad \left(\text{here } \rho = \frac{m}{l}\right)$$

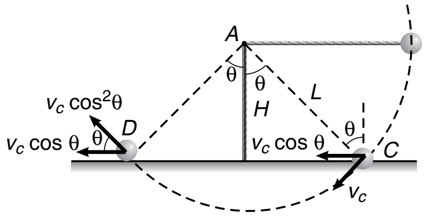
$$= \frac{m}{l}(gy + v_0^2)$$ **Ans.**

(b) Energy lost during the lifting = work done by applied force – increase in mechanical energy of chain

$$= \int_0^y P \cdot dy - \left(\frac{m}{l} y\right) g \left(\frac{y}{2}\right) - \frac{1}{2}\left(\frac{m}{l} \cdot y\right) v_0^2$$

$$= \frac{myv_0^2}{2l}$$ **Ans.**

6. In perfectly inelastic collision with the horizontal surface the component parallel to the surface will remain unchanged. Similarly when the string becomes taut again, the component perpendicular to its length will remain unchanged.



$$\cos\theta = \frac{H}{L} \quad \Rightarrow \quad v_c = \sqrt{2gH}$$

$$v_c \cos^2\theta = (\sqrt{2gH})\frac{H^2}{L^2} = v \ \text{(say)}$$

$$\therefore \qquad h = \frac{v^2}{2g} = \frac{(2gH)\frac{H^4}{L^4}}{2g} = \frac{H^5}{L^4}$$ **Ans.**

7. Applying conservation of linear momentum at the time of collision or at $t = 1$ s,

$$m\mathbf{v} + m(0) = 2m(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$$

$$\therefore \qquad \mathbf{v} = 40\hat{\mathbf{i}} + 20\hat{\mathbf{j}}$$

At 1 s, masses will be at height

$$h_1 = u_y t + \frac{1}{2} v_y t^2 = (20)(1) + \frac{1}{2}(-10)(1)^2 = 15 \text{ m}$$

After explosion other mass will further rise to a height :

$$h_2 = \frac{u_y^2}{2g} = \frac{(20)^2}{2 \times 10}$$

$$= 20 \text{ m}$$

$$\Rightarrow \qquad u_y = 20 \text{ m/s just after collision.}$$

$\therefore$ Total height $h = h_1 + h_2 = 35$ m **Ans.**

8. Let CT stands for common tangent direction and CN for common normal directions.

	Mass $m = eM$		Mass M	
	CT	**CN**	**CT**	**CN**
Befor collision	v_1 (let)	v_2 (let)	Zero (given)	Zero (given)
After collision	v_1	v_3 (suppose)	Zero	v_4 (suppose)

In the common tangent directions velocity components remain unchanged.

In common normal direction applying conservation of linear momentum and definition of *e*.

$$eMv_2 = eMv_3 + Mv_4 \qquad \text{...(i)}$$

From the definition of coefficient of restitution

$$e = \frac{v_4 - v_3}{v_2} \qquad \text{...(ii)}$$

Solving these two equations we get,

$$v_3 = 0 \text{ but } v_4 \neq 0$$

So, after collision velocity of *m* is along CT while that of *M* along CN or they are moving at right angles.

9. Muzzle velocity v_r is given to be constant.

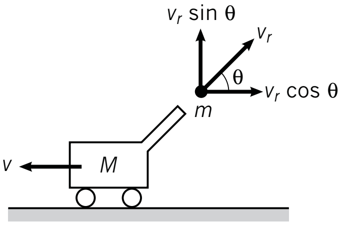
From conservation of linear momentum in horizontal direction we have,

$$Mv = m(v_r \cos\theta - v)$$

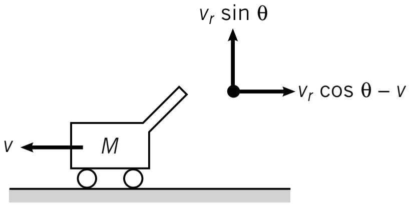
or
$$v = \frac{mv_r \cos\theta}{M + m} \qquad \text{...(i)}$$

Further, range of bullet on horizontal ground

$$R = \frac{2v_r \sin\theta}{g}(v_r \cos\theta - v)$$



Components of velocity of bullet with respect to gun



Components of velocity of bullet with respect to ground

$$= \frac{2v_r \sin\theta}{g}\left(v_r \cos\theta - \frac{mv_r \cos\theta}{M + m}\right)$$

$$= \frac{2Mv_r^2 \sin\theta \cos\theta}{(M + m)g}$$

or
$$R = \left(\frac{M}{M + m}\right)\frac{v_r^2 \sin 2\theta}{g} \qquad \text{...(ii)}$$

(a) From Eq. (ii) we see that maximum range is at

$$\theta = 45° \qquad \textbf{Ans.}$$

(b) At $\theta = 45°$,

$$R_{\max} = \left(\frac{M}{M + m}\right)\frac{v_r^2}{g} \qquad \textbf{Ans.}$$

10. (a) $u_r = 0, \quad a_r = g$

$$\therefore \quad v_r = \sqrt{2gh_1}$$

After collision relative velocity

$$v_r' = e\sqrt{2gh_1}$$

and relative retardation is still *g* (downwards). Hence,

$$h_2 = \frac{(v_r')^2}{2g} = e^2 h_1 \qquad \textbf{Ans.}$$

(b) $u_r = 0, \quad a_r = g + \dfrac{g}{4} = \dfrac{5g}{4}$

$\therefore$ Just before collision $v_r = \sqrt{2\left(\dfrac{5g}{4}\right)h_1}$

Just after collision $v_r' = ev_r$.

Relative retardation is still $\dfrac{5g}{4}$.

Hence,
$$h_2 = \frac{(v_r')^2}{2\left(\dfrac{5g}{4}\right)} = e^2 h_1 \qquad \textbf{Ans.}$$

11. Let the velocity of the block and the plank, when the block leaves the spring be *u* and *v* respectively.

By conservation of energy $\dfrac{1}{2}kx^2 = \dfrac{1}{2}mu^2 + \dfrac{1}{2}Mv^2$

[*M* = mass of the plank, *m* = mass of the block]

$$\therefore \quad 100 = u^2 + 5v^2 \qquad \text{...(i)}$$

By conservation of momentum

$$mu + Mv = 0$$

$$\Rightarrow \quad u = -5v \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii)

$$30v^2 = 100 \Rightarrow \therefore v = \sqrt{\frac{10}{3}} \text{ m/s}$$

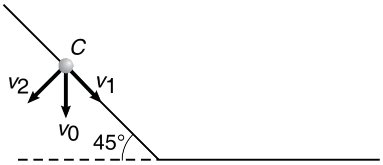
From this moment until block falls, both plank and block keep their velocity constant.

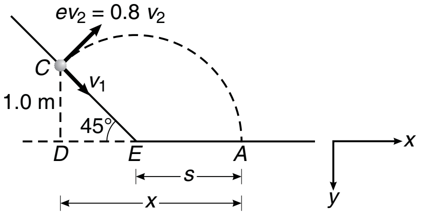
Thus, when block falls velocity of plank

$$= \sqrt{\frac{10}{3}} \text{ m/s.} \qquad \textbf{Ans.}$$

12. $v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 1.5} = 5.42 \text{ m/s}$

Component of velocity parallel and perpendicular to plane at the time of collision.





$$v_1 = v_2 = \frac{v_0}{\sqrt{2}} = 3.83 \text{ m/s}$$

Component parallel to plane (v_1) remains unchanged, while component perpendicular to plane becomes ev_2, where

$$ev_2 = 0.8 \times 3.83 = 3.0 \text{ m/s}$$

$\therefore$ Component of velocity in horizontal direction after collision

$$v_x = \frac{(v_1 + ev_2)}{\sqrt{2}} = \frac{(3.83 + 3.0)}{\sqrt{2}} = 4.83 \text{ m/s}$$

While component of velocity in vertical direction after collision.

$$v_y = \frac{v_1 - ev_2}{\sqrt{2}} = \frac{3.83 - 3.0}{\sqrt{2}} = 0.59 \text{ m/s}$$

Let t be the time, the particle takes from point C to A, then

$$1.0 = 0.59t + \frac{1}{2} \times 9.8 \times t^2$$

Solving this we get,

$$t = 0.4 \text{ s} \qquad \text{(Positive value)}$$

$\therefore$ $DA = v_x t = (4.83)(0.4) = 1.93 \text{ m}$

$\therefore$ $s = DA - DE$

$= 1.93 - 1.0$

$s = 0.93 \text{m}$ **Ans.**

$v_{yA} = v_{yc} + gt$

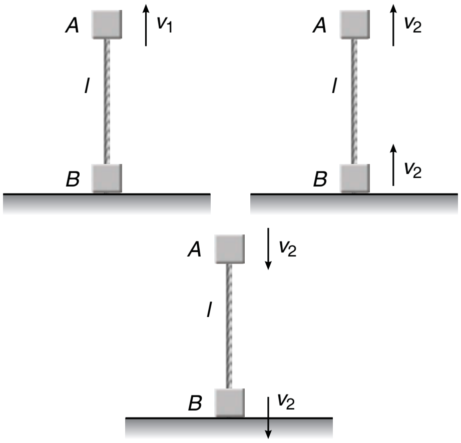
$= (0.59) + (9.8)(0.4) = 4.51 \text{ m/s}$

$v_{xA} = v_{xC} = 4.83 \text{ m/s}$

$\therefore$ $v_A = \sqrt{(v_{xA})^2 + (v_{yA})^2}$

$= 6.6 \text{ m/s}$ **Ans.**

13. String becomes tight when A moves upwards by a distance l. Let v_1 be the velocity of A at this moment, then



$$v_1^2 = (\sqrt{10gl})^2 - 2gl = 8gl \text{ or } v_1 = \sqrt{8gl}$$

Let v_2 be the common velocities of both A and B just after string becomes tight. Then from conservation of linear momentum.

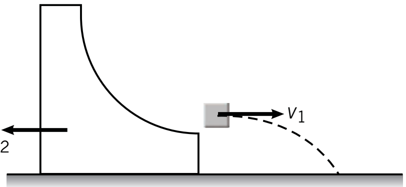
$$v_2 = \frac{v_1}{2} = \frac{\sqrt{8gl}}{2}$$

Both particles return to their original height with same speed v_2. String becomes loose after B strikes the ground and the speed v with which A strikes the ground is,

$$v^2 = v_2^2 + 2gl = \frac{8gl}{4} + 2gl$$

or $v^2 = 4gl$ or $v = 2\sqrt{gl}$ **Ans.**

14.



$$mv_1 = Mv_2 \quad \text{...(i)}$$

$$mgR = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 \quad \text{...(ii)}$$

$$t = \sqrt{\frac{2(R/2)}{g}} = \sqrt{\frac{R}{g}} \quad \text{...(iii)}$$

The desired distance is

$$S = (v_1 + v_2)t \quad \text{...(iv)}$$

Solving Eqs. (i) and (ii) for v_1 and v_2 and substituting in Eq. (iv), we get

$$S = R\sqrt{\frac{2(M+m)}{M}}$$ **Ans.**

15. Let v_r be the velocity of washer relative to centre of hoop and v the velocity of centre of hoop. Applying conservation of linear momentum and mechanical energy we have,

$$m(v_r \cos\phi - v) = Mv \qquad \text{...(i)}$$

$$mgr(1+\cos\phi) = \frac{1}{2}Mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r\cos\phi) \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have,

$$v = m\cos\phi\sqrt{\frac{2gr(1+\cos\phi)}{(M+m)(M+m\sin^2\phi)}} \qquad \textbf{Ans.}$$

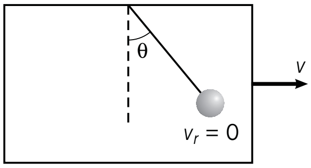
16. $$H = \frac{u^2\sin^2\theta}{2g} = \frac{(20)^2 \times \frac{3}{4}}{2\times 10} = 15\text{ m}$$

i.e. the shell strikes the ball at highest point of its trajectory. Velocity of (ball + shell) just after collision,

$$v = \frac{u\cos 60°}{2}$$

(from conservation of linear momentum)

$$= \frac{20}{2\times 2} = 5\text{ m/s}$$



At highest point combined mass is at rest relative to the trolley. Let v be the velocity of trolley at this instant. From conservation of linear momentum we have,

$$2\times 5 = \left(2+\frac{4}{3}\right)v \quad \text{or} \quad v = 3\text{ m/s}$$

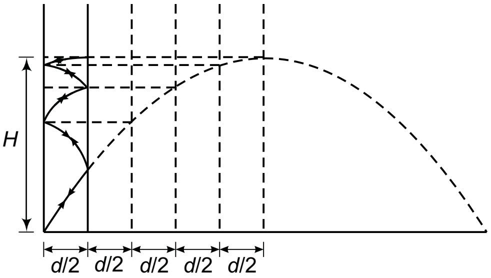
From conservation of energy, we have

$$\frac{1}{2}\times 2\times(5)^2 - \frac{1}{2}\left(2+\frac{4}{3}\right)(3)^2 = 2\times 10(1-\cos\theta)$$

Solving we get, $\cos\theta = \dfrac{1}{2}$

$$\therefore \qquad \theta = 60° \qquad \textbf{Ans.}$$

17. While colliding with the wall its vertical component (v_y) of velocity will remain unchanged (component along common tangent direction remains unchanged) while horizontal component (v_x) is reversed remaining same in magnitude. Thus, path of the particle will be as shown in figure.



(a) $H = \dfrac{u^2\sin^2\alpha}{2g}$

(b) Total number of collisions with the walls before the ball comes back to the ground are nine.

(c) Ball will return to point O (the starting point)

18. As the collisions are perfectly elastic, collision of the ball will not affect the vertical component of its velocity while the horizontal component will be simply reversed.

Hence, $$H_{\max} = \frac{v_y^2}{2g} = \frac{[20\times\sin 45°]^2}{2\times 10} = 10\text{ m}$$

Total time of flight

$$T = \frac{2v_y}{g} = \frac{2\times 20\times\left(\frac{1}{\sqrt{2}}\right)}{10} = 2\sqrt{2}\text{ s}$$

Total horizontal distance travelled before striking the ground $x = v_xT = 40$ m

$$PB + BA + AB + BA + AB = 45\text{ m}$$

Hence, total number of collision suffered by the particle with the walls before it hits ground = 4.

Ans.

19. Let v_1 = velocity of block 2 kg just before collision

v_2 = velocity of block 2 kg just after collision

and v_3 = velocity of block M just after collision.

Applying work energy theorem

(change in kinetic energy = work done by all the forces) at different stages as shown in figure.

Figure 1.

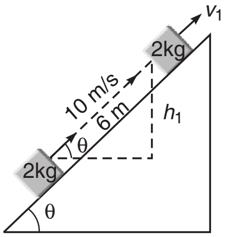


Fig. 1

$$\Delta\text{KE} = W_{\text{friction}} + W_{\text{gravity}}$$

$$\left[\frac{1}{2} m\{v_1^2 - (10)^2\}\right] = -6\mu mg\cos\theta - mgh_1$$

$(m = 2\text{ kg})$

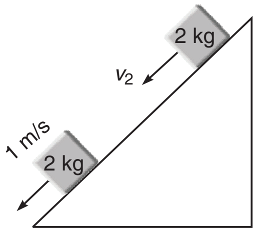
or $$v_1^2 - 100 = 2[6\mu g\cos\theta + gh_1]$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(0.05)^2} \approx 0.99$$

$\therefore$ $$v_1^2 = 100 - 2[(6)(0.25)(10)(0.99) + (10)(0.3)]$$

$\Rightarrow$ $$v_1 \approx 8\text{ m/s}$$

Figure 2.



$$\Delta\text{KE} = W_{\text{friction}} + W_{\text{gravity}}$$

$$\frac{1}{2} m[(1)^2 - (v_2^2)] = -6\mu mg\cos\theta + mgh_1$$

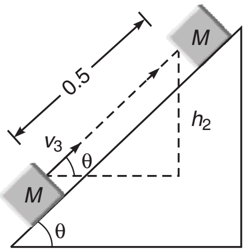
or $$1 - v_1^2 = 2[-6\mu g\cos\theta + gh_1]$$

$$= 2[(-6)(0.25)(10)(0.99) + (10)(0.3)]$$

$$= -23.7$$

$\therefore$ $$v_2^2 = 24.7 \quad\text{or}\quad v_2 \approx 5\text{ m/s}$$

Figure 3.



$$\Delta\text{KE} = W_{\text{friction}} + W_{\text{gravity}}$$

$$\frac{1}{2} M[0 - v_3^2] = -(0.5)(\mu)(M)g\cos\theta - Mgh_2$$

or $$-v_3^2 = -\mu g\cos\theta - 2gh_2$$

or $$v_3^2 = (0.25)(10)(0.99) + 2(10)(0.025)$$

or $$v_3^2 = 2.975$$

$\therefore$ $$v_3 \approx 1.72\text{ m/s}$$

Now

(i) Coefficient of restitution

$$= \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}$$

$$= \frac{v_2 + v_3}{v_1} = \frac{5 + 1.72}{8} = \frac{6.72}{8}$$

or $$e \approx 0.84$$

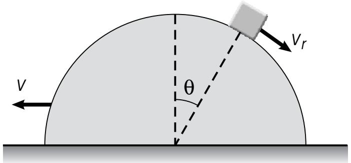
(ii) Applying conservation of linear momentum before and after collision

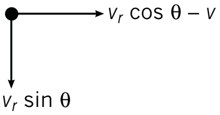
$$2v_1 = Mv_3 - 2v_2$$

$\therefore$ $$M = \frac{2(v_1 + v_2)}{v_3} = \frac{2(8+5)}{1.72} = \frac{26}{1.72}$$

$$M \approx 15.12\text{ kg}$$ **Ans.**

20. Let v_r be the relative velocity of block as it leaves contact with the sphere ($N = 0$) and v the horizontal velocity of sphere at this instant.





Absolute components of velocity of block

Applying conservation of linear momentum in horizontal direction, we get

$$mv = m(v_r\cos\theta - v)$$

or $$2v = v_r\cos\theta \qquad \ldots\text{(i)}$$

Conservation of mechanical energy gives,

$$mgr(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r\cos\theta)$$

or $$gr(1-\cos\theta) = v^2 + \frac{v_r^2}{2} - vv_r\cos\theta \qquad \ldots\text{(ii)}$$

Equation of laws of motion gives,

$$mg\cos\theta = \frac{mv_r^2}{r} \quad\text{or}\quad gr = \frac{v_r^2}{\cos\theta} \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\cos^3\theta - 6\cos\theta + 4 = 0$$ **Ans.**

Note *We have not considered pseudo force while writing the equation of motion. Think why?*

21. $$\frac{x}{u\cos\alpha} + \frac{x}{eu\cos\alpha} = T = \frac{2u\sin\alpha}{g}$$

or $$x = \frac{eu^2\sin 2\alpha}{(1+e)g} \Rightarrow x_{\max} = \frac{eu^2}{(1+e)g}$$

at $$2\alpha = 90°$$ **Ans.**

12. Rotational Mechanics

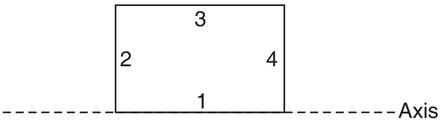
INTRODUCTORY EXERCISE 12.1

1. $K = \sqrt{\dfrac{I}{m}} = \sqrt{\dfrac{m\,(2l)^2/3}{m}} = \dfrac{2}{\sqrt{3}}\, l$ **Ans.**

2. $I = (1)\left[\sqrt{(1)^2 + (2)^2}\right]^2 + 2[\sqrt{(3^2) + (4)^2}\,]^2$

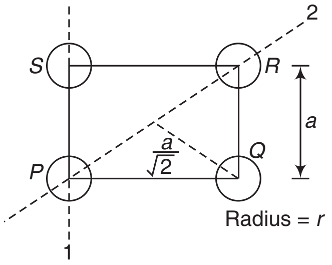
$= 55 \text{ kg-m}^2$ **Ans.**

3. $I = I_1 + I_2 + I_3 + I_4$



$= 0 + \dfrac{ml^2}{3} + ml^2 + \dfrac{ml^2}{3} = \dfrac{5}{3}\, ml^2$ **Ans.**

5. $I_1 = I_P + I_Q + I_R + I_S$



$= \left(\dfrac{2}{5}\, mr^2\right) + \left(\dfrac{2}{5}\, mr^2 + ma^2\right) + \left(\dfrac{2}{5}\, mr^2 + ma^2\right) + \dfrac{2}{5}\, mr^2$

$= \dfrac{8}{5}\, mr^2 + 2ma^2$

$I_2 = I_P + I_Q + I_R + I_S$

$= \dfrac{2}{5}\, mr^2 + \left(\dfrac{2}{5}\, mr^2 + \dfrac{ma^2}{2}\right) + \dfrac{2}{5}\, mr^2 + \left(\dfrac{2}{5}\, mr^2 + \dfrac{ma^2}{2}\right)$

$= \dfrac{8}{5}\, mr^2 + ma^2$

6. $M = \int_0^{2l} dM = \int_0^{2l} (mx) \cdot dx$

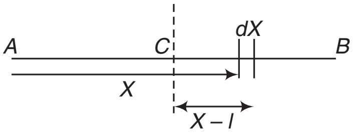
$= 2ml^2$

$\therefore \quad m = \dfrac{M}{2l^2}$

(a) $I_A = \int_0^{2l} (dM)\, X^2 = \int_0^{2l} (mXdX)\, X^2$

$= \int_0^{2l} \left(\dfrac{M}{2l^2}\, X^3\, dX\right) = 2\, Ml^2$ **Ans.**

(b) $I_C = \int_0^{2l} (dM)\,(X - l)^2$



$= \int_0^{2l} (mXdX)\,(X - l)^2$

$= \int_0^{2l} \left(\dfrac{M}{2l^2}\right)(X)\,(X - l)^2\, dX$

$= \dfrac{1}{3}\, Ml^2$ **Ans.**

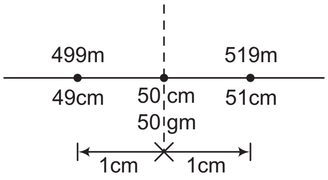
7. $I = \dfrac{5}{6}\, I_{\text{Total}} = \dfrac{5}{6}\, (0.6) \text{ kg-m}^2 = 0.5 \text{ kg-m}^2$

8. $I = \dfrac{1}{2}\, mR^2$ or $I \propto R^2$

m and thickness are same.

Therefore, radius *R* of the material having smaller density should be more, so the moment of inertia.

9. $I = 50\,(0)^2 + (49 + 51)\,(1)^2 + (48 + 52)\,(2)^2 + \ldots\ldots\ldots + (1 + 99)\,(49)^2$



$= 100\,[(1)^2 + (2)^2 + \ldots\ldots + (49)^2\,]$

$= 4.3 \times 10^6 \text{ gm-cm}^2 = 0.43 \text{ kg-m}^2$ **Ans.**

10. $l = 2\pi R \quad \Rightarrow \quad R = \dfrac{l}{2\pi}$

$\dfrac{I_1}{I_2} = \dfrac{ml^2/12}{mR^2} = \dfrac{l^2}{12R^2}$

$= \dfrac{l^2}{12\,(l/2\pi)^2}$

$= \dfrac{\pi^2}{3}$ **Ans.**

INTRODUCTORY EXERCISE 12.2

1. $\omega = \dfrac{2\pi}{T}$

$$= \frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/s} \qquad \textbf{Ans.}$$

2. $\omega = \dfrac{v_r}{PQ} = \dfrac{2v - v}{2R} = \dfrac{v}{2R}$

Rotation is clockwise. So, ω is perpendicular to paper inwards.

3. $\omega = \dfrac{\text{component of velocity perpendicular to } \mathbf{r}}{r}$

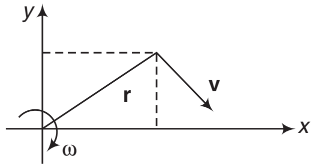
$$\mathbf{r} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$$

$$\mathbf{v} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$$

Since, $\mathbf{r} \cdot \mathbf{v} = 0$

$\therefore$ $\mathbf{v} \perp \mathbf{r}$

$\therefore$ $\omega = \dfrac{v}{r} = 1 \text{ rad/s}$

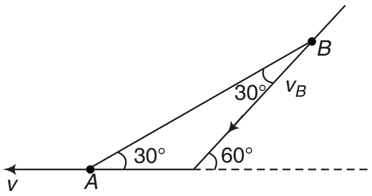


$$v = |\mathbf{v}| = 5 \text{ m/s}$$

$$r = |\mathbf{r}| = 5 \text{ m}$$

From the figure, we can see that ω is along negative *z*-axis. **Ans.**

4. Component of velocities of *A* and *B* along *AB* should be same



$\therefore$ $v_A \cos 30° = v_B \cos 30°$

$\therefore$ $v_B = v_A = v$

$$\omega = \frac{\text{Relative velocity} \perp \text{to } AB}{AB}$$

$$= \frac{v_A \sin 30° + v_B \sin 30°}{l}$$

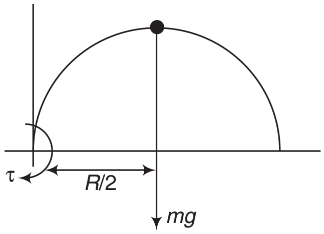
$$= \frac{v}{l} \qquad (\text{as } v_A = v_B = v)$$

INTRODUCTORY EXERCISE 12.3

1. $\mathbf{r} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

$$\tau = \mathbf{r} \times \mathbf{F} \qquad \textbf{Ans.}$$

2. $\tau = \dfrac{mgR}{2} = \dfrac{(mg)\, u^2 \sin 2\theta}{2g}$



$$= \frac{mu^2}{2} \qquad (\text{as } \theta = 45°)$$

$$= \frac{(1)\,(20\sqrt{2})^2}{2}$$

$$= 400 \text{ N-m} \qquad \textbf{Ans.}$$

3. $\tau_{10N} = 0$

$$\tau_{20N} = (20 \cos 45° \times 0.1) \qquad \text{(clockwise)}$$

$$= 1.414 \text{ N-m}$$

$$\tau_{30N} = (30 \sin 60° \times 0.05) \qquad \text{(clockwise)}$$

$$= 1.299 \text{ N-m}$$

$\therefore$ $\tau_{\text{Total}} = 2.71 \text{ N-m}$

4. $\tau_{10N} = 10 \times 0.25$

$$= 2.5 \text{ N-m} \qquad \text{(clockwise)}$$

$$\tau_{9N} = 9 \times 0.25$$

$$= 2.25 \text{ N-m} \qquad \text{(clockwise)}$$

$$\tau_{12N} = 12 \cos 60° \times 0.1$$

$$= 0.6 \text{ N} \qquad \text{(anticlockwise)}$$

$\therefore$ $\tau_{\text{net}} = 4.15 \text{ N-m}$

INTRODUCTORY EXERCISE 12.4

1. $\theta = \dfrac{1}{2} \alpha t^2$

$\therefore$ $\alpha = \dfrac{2\theta}{t^2} = \dfrac{2\,(50)\,(2\pi)}{(5)^2}$

$$= (8\pi) \text{ rad/s}^2$$

$$= 25.14 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

$$\omega = (\alpha t)$$

$$= (8\pi)\,(5)$$

$$= (40\pi) \text{ rad/s} \qquad \textbf{Ans.}$$

2. $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$\therefore \qquad \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{225 - 25}{2\times 1}$$

$$= 100 \text{ rad} \qquad \textbf{Ans.}$$

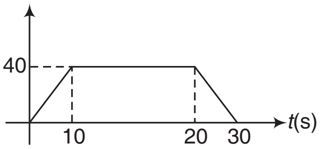
3. $0 = \omega_0 - \alpha t$

$$\Rightarrow \qquad \alpha = \frac{\omega_0}{t} = \frac{10}{10}$$

$$= 1 \text{ rad/s}^2$$

$$\tau = I\alpha = 5 \text{ N-m} \qquad \textbf{Ans.}$$

4. θ = Area under ω-t graph



$$= \frac{1}{2}(30 + 10)(40) = 800 \text{ rad} \qquad \textbf{Ans.}$$

5. $\omega^2 = \omega_0^2 - 2\alpha\theta$

$$0 = \left(\frac{100}{60}\times 2\pi\right)^2 - 2\alpha\,(10\times 2\pi)$$

$$\alpha = 0.87 N = \frac{\tau}{I}$$

$$= \frac{F\cdot R}{\frac{1}{2}mR^2}$$

$$\therefore \qquad F = (0.87)(0.5\,mR)$$

$$= (0.87)(0.5)(10)(0.2)$$

$$= 0.87 \text{ N} \qquad \textbf{Ans.}$$

6. $\omega = \dfrac{d\theta}{dt} = 6 - 6t^2$

$$\alpha = \frac{d\omega}{dt} = -12t$$

$$\omega = 0$$

at $\qquad t = 1$ s

$$\text{(a)}\ \langle\omega\rangle_{0-1} = \frac{\int_0^1 \omega dt}{1} = \int_0^1 (6 - 6t^2)\,dt$$

$$= 4 \text{ rad/s} \qquad \textbf{Ans.}$$

$$\langle\alpha\rangle_{0-1} = \frac{\int_0^1 (-12t)\,dt}{1}$$

$$= -6 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

(b) At $t = 1$ s

$$\alpha = -12 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

7. $\omega = \omega_0 - \alpha t$

$$-20 = 20 - 2t$$

$$\therefore \qquad t = 20 \text{ s} \qquad \textbf{Ans.}$$

8. (a) Angular impulse

= change in angular momentum

$$\therefore \qquad \tau_f t = I\omega$$

$$\text{or} \qquad \tau_f = \frac{I\omega}{t} = \frac{0.03\times 20}{60} = 0.01 \text{ N-m} \qquad \textbf{Ans.}$$

(b) During acceleration

$$(\tau_e - \tau_f)\,t = I\omega$$

$$\therefore \qquad \tau_e = \frac{I\omega}{t} + \tau_f = \frac{0.03\times 20}{5} + 0.01$$

$$= 0.13 \text{ N-m} \qquad \textbf{Ans.}$$

9. (a) Angular impulse

= change in angular momentum

$$\therefore \qquad \tau\, t_0 = I\omega$$

$$\therefore \qquad t_0 = \frac{I\omega}{\tau} = \frac{16\times 9}{4} = 36 \text{ s}$$

$$\text{(b)}\ \int_0^{t_0} \tau\, dt = I\omega$$

$$\therefore \qquad \int_0^{t_0} kt\,dt = I\omega$$

$$\therefore \qquad \frac{kt_0^2}{2} = I\omega$$

$$\therefore \qquad t_0 = \sqrt{I\omega}\sqrt{\frac{2}{k}}$$

$$= \sqrt{16\times 9}\sqrt{\frac{2}{k}} = 12\sqrt{\frac{2}{k}} \qquad \textbf{Ans.}$$

10. (a) $\int d\omega = \int \alpha\, dt$

$$\therefore \qquad \int_{65}^{\omega} d\omega = \int_0^3 (-10 - 5t)\,dt$$

$$\therefore \qquad \omega = 65 - [10\,t + 2.5\,t^2]_0^3$$

$$= 12.5 \text{ rad/s} \qquad \textbf{Ans.}$$

$$\text{(b)}\ \int_{65}^{\omega} d\omega = \int_0^t (-10 - 5t)\,dt$$

$$\therefore \qquad \omega = 64 - 10t - 2.5t^2$$

$$\int_0^{\theta} d\theta = \int_0^3 \omega dt = \int_0^3 (65 - 10t - 2.5t^2)\,dt$$

$$\therefore \qquad \theta = 195 - 45 - 22.5$$

$$= 127.5 \text{ rad} \qquad \textbf{Ans.}$$

11. $\theta = \displaystyle\int_0^3 \omega\, dt = \int_0^3 (12 - 3t^2)\,dt$

$$= 36 - 27 = 9 \text{ rad} \qquad \textbf{Ans.}$$

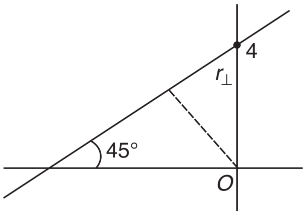
$$N = \frac{\theta}{2\pi} = \frac{9}{2\times 3.14} = 1.43 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 12.5

1. $I = \dfrac{m\,(3l)^2}{12} + m\left(\dfrac{l}{2}\right)^2 = ml^2$

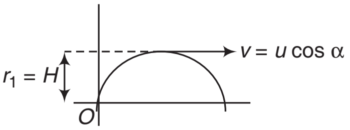
$$L = I\omega = ml^2\omega$$

2. $r_\perp = 4\cos 45^\circ = 2\sqrt{2}$ m



$$\begin{aligned} L_0 &= mvr_\perp \\ &= (1)\,(2)\,(2\sqrt{2}) \\ &= 4\sqrt{2}\ \text{kg-m}^2/\text{s} \end{aligned}$$

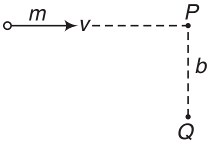
3. $L_0 = mvr_\perp$



$$= m\,(u\cos\alpha)\left(\frac{u^2\sin^2\alpha}{2g}\right)$$

$$= \frac{mu^3\cos\alpha\,\sin^2\alpha}{2g} \qquad \textbf{Ans.}$$

4.



$$L_P = 0 \qquad (\text{as } r_\perp = 0)$$

$$L_Q = mvb \qquad (\text{as } r_\perp = b)$$

5. $L = mvr_\perp + I_c\omega$

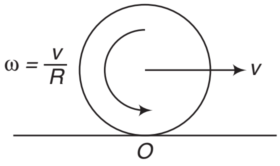
$$= mvR + \left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)$$

$$= \frac{7}{5}mvR$$

This is clockwise, or along negative z-axis.

$$\therefore \qquad \mathbf{L} = \left(-\frac{7}{5}mvR\right)\hat{\mathbf{k}} \qquad \textbf{Ans.}$$

6. $L_0 = mvR - I_c\omega$



$$= mvR - \left(\frac{1}{2}mR^2\right)(vR)$$

$$= \frac{1}{2}mvR \qquad (\text{clockwise}) \qquad \textbf{Ans.}$$

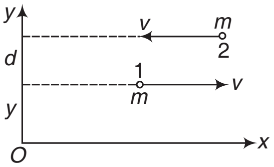
7. $L_0 = L_1 + L_2$

= (mvy in clockwise or along negative z-axis)

+ [$mv\,(y + d)$ in anticlockwise or along positive z-axis]

= mvd in anticlockwise direction or along positive z-axis.

= constant



INTRODUCTORY EXERCISE 12.6

1. $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \qquad \omega_2 = \frac{I_1}{I_2}\,\omega_1$$

$$= \left(\frac{MR^2}{MR^2 + 2\,mR^2}\right)\omega_0$$

$$= \left(\frac{M}{M + 2m}\right)\omega_0 \qquad \textbf{Ans.}$$

2. Mass will move towards equator (away from axis). So, I will increase. Therefore ω will decrease (as $I\omega$ = constant). Hence, time period will increase $\left(\text{as } T = \dfrac{2\pi}{\omega}\right)$.

3. Moment of inertia I increase. Therefore, ω decreases (as $I\omega$ = constant). Hence time period increases $\left(\text{as } T = \dfrac{2\pi}{\omega}\right)$

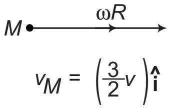
4. I will increase $\left(\text{as } I = \frac{2}{5}mR^2\right)$

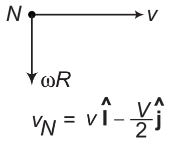
$\therefore$ ω will decrease (as $I\omega$ = constant)

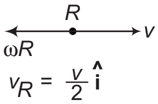
$\therefore T$ will increase (as $T = 2\pi/\omega$)

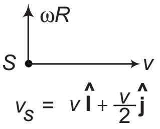
INTRODUCTORY EXERCISE 12.7

1. $\omega R = \frac{v}{2}$









2. There is only one acceleration,

$$a_r = R\omega^2 \qquad \text{(towards centre)}$$

INTRODUCTORY EXERCISE 12.8

1. $\frac{K_R}{K_T} = \frac{2}{5} \Rightarrow K_T = \frac{5}{2}K_R = \frac{5}{2} \times 10 = 25\text{ J}$

$\therefore$Total kinetic energy $= K_R + K_T = 35\text{ J}$

2. Under forward slip condition $v > R\omega$

$$K_T = \frac{1}{2}mv^2$$

$$K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}(mR^2)\omega^2$$

$$\therefore \qquad K_T > K_R$$

3. $K_T = \frac{1}{2}mv^2,\ K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2$

Since, in backward slip condition, $v < R\omega$

$\therefore$ K_R may be equal to K_T.

INTRODUCTORY EXERCISE 12.9

1. $v = r\omega$, where r is the distance from A (position of instantaneous axis of rotation)

For A, $r = 0 \Rightarrow v_A = 0$

For B and D, $r = \sqrt{2}R$

$\Rightarrow \qquad v_B = v_D = \sqrt{2}R\omega = \sqrt{2}v$

For C, $r = 2R \Rightarrow v_C = 2R\omega = 2v$

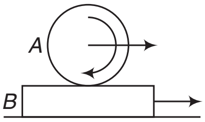
Direction of velocity of a general point P is perpendicular to AP in the sense of rotation.

2. $I_C = \frac{2}{5}mR^2,\ I_0 = \frac{7}{5}mR^2$ and $\omega = \frac{v}{R}$

Substituting these values in the given equation, we get the result.

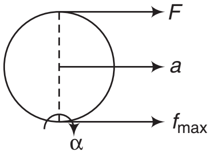
INTRODUCTORY EXERCISE 12.10

1. Net work done by friction in pure rolling is always zero.



For example, if A pure rolls over B, then work done by friction on both A and B are non-zero (one is positive and other is negative). But net work done by friction = 0.

2. Till pure rolling continues we can find α about bottommost point. About bottommost point torque of friction is zero.



$$\alpha = \frac{F\ (2R)}{(3/2)\ mR^2} = \frac{4F}{3mR}$$

$$a = R\alpha = \frac{4F}{3m}$$

$$F + f_{\text{max}} = ma = \frac{4F}{3}$$

$$F = 3\ f_{\text{max}} = 3\ \mu mg$$

$$= 3 \times 0.6 \times 4 \times 10 = 72\text{ N} \qquad \textbf{Ans.}$$

This is the maximum value of applied force, as we have taken the maximum friction.

3. (a) $\mu_{\text{min}} = \frac{\tan\theta}{1 + (mR^2/I)}$

$$= \frac{\tan 30^\circ}{1 + 5/2} = \frac{2}{7\sqrt{3}} \qquad \textbf{Ans.}$$

(b) $a = \frac{g\sin\theta}{1 + I/mR^2}$

$$= \frac{(10)\sin 30^\circ}{1 + 2/5}$$

$$= \frac{25}{7}\text{ m/s}^2 \qquad \textbf{Ans.}$$

(c) $a = g\sin\theta - \mu g\cos\theta$

$$= (10)\sin 30^\circ - \left(\frac{1}{7\sqrt{3}}\right)(10)(\cos 30^\circ)$$

$$= 5 - \frac{5}{7} = \frac{30}{7}\ \text{m/s}^2$$ **Ans.**

4. (a) Maximum friction will act

$$\therefore \quad a = \frac{mg\sin\theta - \mu\, mg\cos\theta}{m}$$

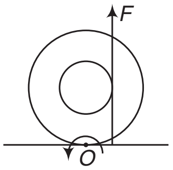
$$= g\sin\theta - \mu\, g\cos\theta$$

(b) Only friction will provide the torque.

$$\therefore \quad \alpha = \frac{\tau}{I} = \frac{(\mu\, mg\cos\theta)\,R}{\frac{2}{5}\,mR^2}$$

$$= \frac{5\mu\, g\cos\theta}{2R}$$ **Ans.**

5. τ_F about O is anticlockwise.



So, it will roll leftwards F is vertical. So, horizontal force is vertical. So, horizontal force is provided by friction (to give it linear acceleration a). So, friction acts leftwards.

INTRODUCTORY EXERCISE 12.11

1. Torque of $mg\sin\theta$ is non-zero. So, angular momentum is not constant.

2. (a) $v = \frac{J}{m}$, J and m are same for both.

(b) $\omega = \frac{Jh}{I}$

or $\qquad \omega \propto \frac{1}{I}$

J and h are same but, $I_{\text{solid}} < I_{\text{hollow}}$

$$\therefore \quad \omega_{\text{solid}} < \omega_{\text{hollow}}$$

(c) $K_R = \frac{1}{2} I\omega^2$

$$\omega \propto \frac{1}{I}$$

$$\therefore \quad K_R \propto \frac{1}{I}$$

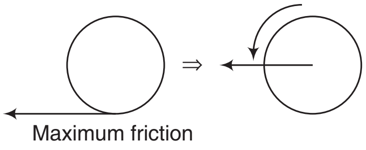
Exercise

LEVEL 1

Assertion and Reason

1. If we compare between many parallel axes, then moment of inertial is least about that axis which passes through COM.

2. Slips is towards right. So maximum friction will act towards left which will convert this slip into pure rolling.



3. Solid sphere, $\quad \frac{K_R}{K_T} = \frac{2}{5}$

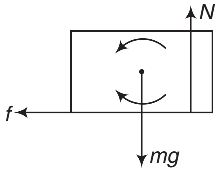
$$\therefore \quad K_R = \frac{2}{7} K_{\text{Total}} \quad \text{and} \quad K_T = \frac{5}{7} K_{\text{Total}}$$

Hollow sphere, $\quad \frac{K_R}{K_T} = \frac{2}{3}$

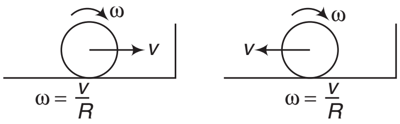
$$\therefore \quad K_R = \frac{2}{5} K_{\text{Total}} \quad \text{and} \quad K_T = \frac{3}{5} K_{\text{Total}}$$

4. In case of smooth surface whole of its potential energy (at A) will convert into its translational kinetic energy (at B).

5. f = friction force



6. In second figure, net angular momentum about bottommost axis,

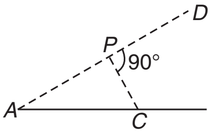


$$L = mvR - I_c\omega$$

$$= mvR - (mR^2)\left(\frac{v}{R}\right)$$

$$= 0$$

7. At P distance from C is minimum.



$$I = I_C + mr^2$$

From A to P, moment of inertia will first decrease. At P it is minimum. Then it will increase.

8. Constant linear momentum means moving with constant speed in a straight line. Therefore, in $L = mv\, r_\perp$ all three m, v and $r_\perp$ are constant.

9. At C, two terms v and $R\omega$ are at 0°. At A they are at 180° and at B they are at 90°

$$\therefore \qquad v_C = v + R\omega$$

$$v_A = v \sim R\omega$$

$$\text{and} \qquad v_B = \sqrt{v^2 + (R\omega)^2}$$

11. In first case,

$$a = R\alpha$$

$$\therefore \qquad \frac{F - \mu_1\, mg}{m} = R\left(\frac{\mu_1\, mgR}{2/5\; mR^2}\right)$$

$$= \frac{5}{2}\mu_1 g \quad \text{or} \quad \mu_1 = \frac{2}{7}\frac{F}{mg}$$

In second case,

$$a - R\alpha = a_{\text{Plank}}$$

$$\therefore \qquad \left(\frac{F - \mu_2\, mg}{m}\right) - R\left(\frac{\mu_2\, mg}{2/5\; mR^2}\right) = \frac{\mu_2\, mg}{M}$$

$$\therefore \qquad \frac{F}{m} = \mu_2\, g + \frac{5}{2}\mu_2 g + \frac{m}{M}\mu_2 g$$

$$\therefore \qquad \mu_2 = \frac{F}{\left(7/2 + \dfrac{m}{M}\right) mg}$$

$$\text{or} \qquad \mu_2 < \mu_1$$

Net work done by friction in pure rolling is zero.

Single Correct Option

2. $K = \sqrt{\dfrac{I}{M}}$

$$= \sqrt{\frac{\frac{1}{2}mR^2}{m}} = \frac{R}{\sqrt{2}}$$

$$= \frac{25}{\sqrt{2}} \approx 18 \text{ cm}$$

3. $0 = \omega_0 - \alpha t$

$$\therefore \qquad \alpha = \frac{\omega_0}{t} = \frac{(2\pi)\,(1725/60)}{20}$$

$$= 9 \text{ rad/s}^2$$

$$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 9 \times (20)^2 = 1800 \text{ rad}$$

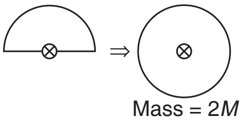
$$N = \frac{\theta}{2\pi} = 287$$ **Ans.**

4. I will decrease

$\therefore$ ω will increase (as $L = I\omega$ = constant)

KE $= \dfrac{L^2}{2I}$ will increase as I is decreasing.

5. $I_{\text{Whole disc}} = \dfrac{1}{2}(2M)\, r^2 = Mr^2$



$$\therefore \qquad I_{\text{Half disc}} = \frac{1}{2} Mr^2$$

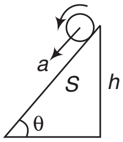
7. In case of pure rolling on ground, net work done by friction = 0

8. After melting, ice will distribute in whole pan. So, moment of inertia I will increase. Hence, angular speed ω will decrease (as $I\omega$ = constant).

9. $\dfrac{K_R}{K_T} = \dfrac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2} = \dfrac{I\omega^2}{mv^2}$

$$= \frac{(mR^2)\,(v/r)^2}{mv^2} = \frac{R^2}{r^2}$$ **Ans.**

10. $a = \dfrac{g \sin\theta}{1 + I/mR^2} = \dfrac{g\sin\theta}{1 + 2/5} = \dfrac{5}{7} g \sin\theta$



$$t = \sqrt{\frac{2S}{a}} = \sqrt{\frac{2\left(\dfrac{h}{\sin\theta}\right)}{(5/7)\, g\sin\theta}}$$

$$\text{or} \qquad t \propto \frac{1}{\sin\theta}$$

So, time will be different.

$$v = \sqrt{2aS} = \sqrt{2\left(\frac{5}{7} g \sin\theta\right)\left(\frac{h}{\sin\theta}\right)}$$

or v is independent of θ. So, v will be same.

11. Mass is distributed at maximum distance (from the axis) in case of four rods.

12. $f = \dfrac{mg \sin\theta}{1 + mR^2/I}$

$$f_{\text{disc}} = \frac{mg \sin\theta}{1+2} = \frac{1}{3}\, mg \sin\theta$$

$$f_{\text{sphere}} = \frac{mg \sin\theta}{1+5/2} = \frac{2}{7}\, mg \sin\theta$$

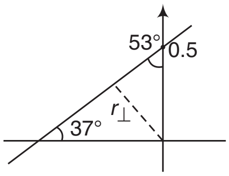
$\therefore$ The ratio is $\dfrac{7}{6}$.

13. $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore\ \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \frac{\left(\frac{1}{2}mR^2\right)}{\left(\frac{1}{2}mR^2 + mR^2\right)}\,\omega = \frac{\omega}{3}$$

14. Friction will act (due to sliding). But torque of friction about the mentioned point in the question will be zero. Hence, angular momentum will remain constant.

15. $y = \dfrac{3}{4}x + 0.5$



$$\text{Slope } = \tan\theta = \frac{3}{4}$$

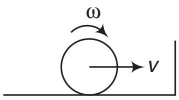
$$\therefore \qquad \theta = 37°$$

$$r_\perp = 0.5 \sin 53° = 0.4 \text{ m}$$

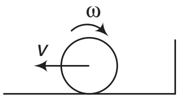
$$L = mvr_\perp = (3)\,(5)\,(0.4)$$

$$= 6 \text{ kg-m}^2/\text{s}$$

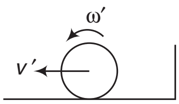
16. $v = R\omega$ and $v' = R\omega'$



(a)



(b)



(c)

From (b) to (c), forward slip will take place and backward maximum friction will convert it into pure rolling.

17. $I_\theta = \dfrac{ma^2}{12}$ (in case of square plate)

or it is independent of θ.

18. Net torque about bottommost point is clockwise. Hence, the spool will move towards right.

19. $K = \dfrac{L^2}{2I}$ or $K \propto \dfrac{1}{I}$ (as L = constant)

As I has doubled, so K will become half.

20. $\rho = \dfrac{M}{\left\{\frac{4}{3}\pi\,(2R)^3\right\} - \left\{\frac{4}{3}\pi R^3\right\}}$

$$= \frac{3M}{28\,\pi R^3}$$

$$M_{2R} = \rho\left(\frac{4}{3}\right)(\pi)\,(2R)^3$$

$$= \left(\frac{3}{28}\right)\left(\frac{4}{3}\right)(8)\,M = \frac{8}{7}M$$

$$M_R = \rho\left(\frac{4}{3}\pi R^3\right)$$

$$= \left(\frac{3}{28}\right)\left(\frac{4}{3}\right)M = \frac{1}{7}M$$

Now, $I = I_{2R} - I_R$

$$= \frac{1}{2}M_{2R}\,(2R)^2 - \frac{1}{2}M_R R^2$$

$$= \frac{1}{2}\left(\frac{8}{7}M\right)(4R^2) - \frac{1}{2}\left(\frac{M}{7}\right)R^2$$

$$= \frac{31}{14}MR^2$$ **Ans.**

21. S = area of cross-section

$$\rho_x = \rho_0 + \left(\frac{2\rho_0 - \rho_0}{L}\right)x = \rho_0 + \frac{\rho_0}{L}x$$

Now, $M = \int_0^L dM = \int_0^L (Sdx)\left(\rho_0 + \dfrac{\rho_0}{L}x\right)$

$$= \rho_0 S\left(\frac{3L}{2}\right)$$

$$\therefore \qquad \rho_0 S = \frac{2M}{3L} \qquad \text{...(i)}$$

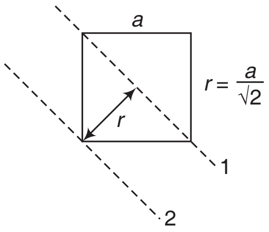
Now, $I = \int_0^L dI = \int_0^L (dM)\,x^2$

$$= \int_0^L (Sdx)\left(\rho_0 + \frac{\rho_0}{L}x\right)(x^2)$$

After substituting value of $\rho_0 S$ from Eq. (i), then we find

$$I = \frac{7}{18} ML^2$$ **Ans.**

22. $I_1 = \dfrac{ma^2}{12}$



$$I_2 = I_1 + mr^2$$

$$= \frac{ma^2}{12} + \frac{ma^2}{2}$$

$$= \frac{7}{12} ma^2$$

$$\therefore \qquad I_1 : I_2 = 1 : 7$$ **Ans.**

23. $I = \dfrac{ML^2}{12} + M\left(\dfrac{L}{4}\right)^2 = \dfrac{7}{48} ML^2$

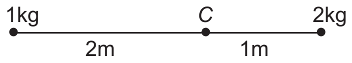
24. Decrease in gravitational kinetic energy
= increase in rotational kinetic energy

$$\therefore mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right) = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\therefore \qquad \omega = \sqrt{\frac{6g}{L}} \sin\frac{\theta}{2}$$ **Ans.**

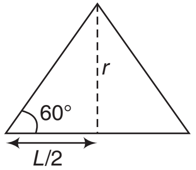
25. $I = 1\,(2)^2 + 2\,(1)^2$



$$= 6 \text{ kg-m}^2$$ **Ans.**

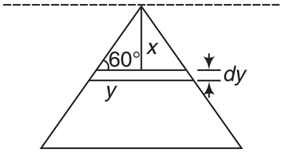
26. $r = \dfrac{L}{2} \tan 60°$

$$= \frac{\sqrt{3}}{2} L$$



$$\text{Area } = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times L \times \frac{\sqrt{3}\,L}{2} = \frac{\sqrt{3}}{4} L^2$$



$$\therefore \text{ Mass per unit area} = \frac{M}{\sqrt{3}L^2/4} = \frac{4M}{\sqrt{3}L^2}$$

or $$y = \frac{x}{\sqrt{3}}$$

$$\Rightarrow \qquad dy = \frac{dX}{\sqrt{3}}$$

$$\text{Area of strip} = (2y)\, dy = \left(\frac{2x}{\sqrt{3}}\right)\left(\frac{dx}{\sqrt{3}}\right)$$

$$= \left(\frac{2x}{3}\right) dX$$

$$\therefore \qquad \text{Mass of strip} = \left(\frac{4M}{\sqrt{3}L^2}\right)\left(\frac{2x}{3}\, dX\right) = dM$$

$$I = \int dI$$

$$= \int_0^{\frac{\sqrt{3}}{2}L} (dM)\, X^2$$

$$= \frac{3}{8\sqrt{3}} L^2$$ **Ans.**

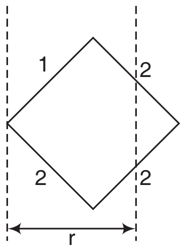
27. $r = \dfrac{l}{\sqrt{2}} + \dfrac{l}{2\sqrt{2}} = \dfrac{3l}{2\sqrt{2}}$

$$I = 2I_1 + 2I_2$$

$$= 2\,(I_1 + I_2)$$

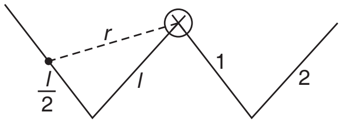
$$= 2\left[\frac{ML^2}{3}\sin^2 45° + \left(\frac{ML^2}{12}\sin^2 45° + Mr^2\right)\right]$$

$$= \frac{8}{3} ML^2$$



28. $r = \sqrt{l^2 + \frac{l^2}{4}} = \frac{\sqrt{5}}{2}\, l$

$$I = 2[I_1 + I_2] = 2\left[\frac{ml^2}{3} + \left(\frac{ml^2}{12} + mr^2\right)\right]$$



$$= \frac{10}{3}\, ml^2 \qquad \textbf{Ans.}$$

29. $\frac{x}{r} = \frac{h}{R} \quad \Rightarrow \quad r = \frac{x}{h} R$

Volume of disc = $(\pi r^2)\, dX$

Mass of disc = $(\pi r^2)(dX)\,\rho = dm$

$$dI = \frac{1}{2}(dm)\, r^2 = \frac{1}{2}(\pi r^2 dx)\,\rho r^2$$

$$= \frac{\pi\rho}{2} r^4\, dx = \left(\frac{\pi\rho}{2}\right)\left(\frac{R^4}{h^4}\right) x^4\, dx$$

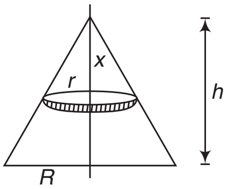
$$I = \int_0^h dI = \left(\frac{\pi\rho}{2}\right)\left(\frac{R^4}{h^4}\right)\left(\frac{h^5}{5}\right)$$

$$= \left(\frac{\pi\rho}{10}\right)(hR^4)$$

$$\rho = \frac{m}{\left(\frac{1}{3}\pi R^2 h\right)}$$

$$\therefore \qquad I = \left(\frac{\pi}{10}\right)\left(\frac{m}{\frac{1}{3}\pi R^2 h}\right)(hR^4)$$

So, $\qquad I \propto mR^2$



Since m and R are same. Therefore I is same.

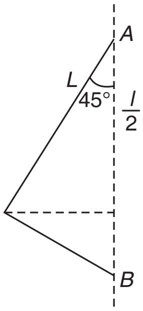
30. $L = \frac{l}{2}\sec 45° = \frac{l}{\sqrt{2}}$

$$m = Lx = \frac{lx}{\sqrt{2}}$$

$$I = 2\left[\frac{mL^2}{3}\sin^2 45°\right]$$

$$= 2\left[\left(\frac{lx}{3\sqrt{2}}\right)\left(\frac{l}{\sqrt{2}}\right)^2\left(\frac{1}{2}\right)\right]$$

$$= \frac{xl^3}{6\sqrt{2}} \qquad \textbf{Ans.}$$



Subjective Questions

1. Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5} MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5} M\left(\frac{R}{2}\right)^2 \omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5} MR^2\omega = \frac{1}{4} \times \frac{2}{5} MR^2\omega'$$

i.e. $\qquad \omega' = 4\omega$

$$T' = \frac{1}{4} T = \frac{1}{4} \times 24 = 6 \text{ h} \qquad \textbf{Ans.}$$

2. $R = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{1}{2} mR^2 + md^2}{m}}$

$$\therefore \qquad d = \frac{R}{\sqrt{2}} \qquad \textbf{Ans.}$$

3. $I_1 = I_2 \quad \Rightarrow \quad I_1 + I_2 = I_3 = \frac{Ma^2}{6}$

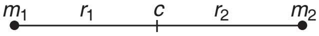
$$\therefore \qquad 2I_1 \quad \text{or} \quad 2I_2 = \frac{Ma^2}{6}$$

$$\therefore \qquad I_1 = I_2 = \frac{Ma^2}{12} \qquad \textbf{Ans.}$$

4. $\frac{7}{12}ml^2 = \frac{ml^2}{12} + mr^2$

$\therefore \quad r = \frac{l}{\sqrt{2}}$ **Ans.**

5.



$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r$$

$$r_2 = \left(\frac{m_1}{m_1 + m_2}\right) r$$

$$I_c = m_1 r_1^2 + m_2 r_2^2$$ **Ans.**

6. $I_r = I_c + mr^2$

$$mK_r^2 = mK_c^2 + mr^2$$

$\therefore \quad K_c = \sqrt{K_r^2 - r^2}$

$$= \sqrt{(10)^2 - (6)^2}$$

$= 8$ cm **Ans.**

7. Instantaneous angular velocity at time t is

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(at^2)$$

or $\quad \omega = 2at = 0.4\,t \quad$ (as $a = 0.2$ rad/s^2)

Further, instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.4\,t)$$

or $\quad \alpha = 0.4$ rad/s^2

Angular velocity at $t = 2.5$ s is

$\omega = 0.4 \times 2.5 = 1.0$ rad/s

Further, radius of the wheel

$$R = \frac{v}{\omega}$$

or $\quad R = \frac{0.65}{1.0} = 0.65$ m

Now, magnitude of total acceleration is,

$$a = \sqrt{a_n^2 + a_t^2}$$

Here, $\quad a_n = R\omega^2 = (0.65)(1.0)^2 = 0.65$ m/s^2

and $\quad a_t = R\alpha = (0.65)(0.4) = 0.26$ m/s^2

$\therefore \quad a = \sqrt{(0.65)^2 + (0.26)^2}$

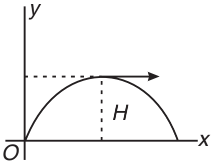
or $\quad a = 0.7$ m/s^2 **Ans.**

8. $\omega_c = \frac{v}{R} = \frac{2}{0.1} = 20$ rad/s

ω about any point on circumference.

$= \frac{\omega_c}{2} = 10$ rad/s **Ans.**

9. At the highest point it has only horizontal velocity $v_x = v\cos\theta$



Length of the perpendicular to the horizontal velocity from 'O' is the maximum height, where

$$H_{\max} = \frac{v^2 \sin^2\theta}{2g}$$

$\Rightarrow$ Angular momentum

$$L = \frac{mv^3 \sin^2\theta \cos\theta}{2g}$$

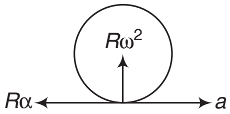
10. $I = \int_0^l dI = \int_0^l (dm)\, x^2$

$$= \int_0^l (\lambda_X\, dx)\, x^2$$

$$= \int_0^l (\alpha x + \beta)\, x^2\, dx$$

$= \frac{\alpha l^4}{4} + \frac{\beta l^3}{3}$ **Ans.**

11. $a = R\alpha$



So, net acceleration is $R\omega^2$ towards centre.

12. In case of pure rolling

$\frac{K_R}{K_T} = \frac{2}{5} \quad$ (for a solid sphere)

$\therefore \quad K_R = \frac{2}{7} \quad$ (total kinetic energy)

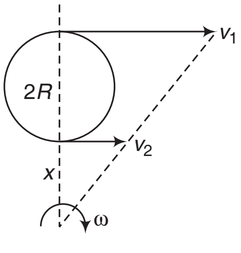
$= \frac{2}{7}(mgh)$ **Ans.**

13. $x\omega = v_2 \quad$ ...(i)

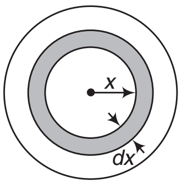
$(x + 2R)\,\omega = v_1 \quad$ ...(ii)

Solving Eqs. (i) and (ii) we get,

$\omega = \left(\frac{v_1 - v_2}{2R}\right)$ **Ans.**



14. $M = \int_0^a (dM)$



$$= \int_0^a (2\pi x)\,dX\,(kx^2) = \frac{\pi k a^4}{2}$$

$$\therefore \quad \pi k = \frac{2M}{a^4}$$

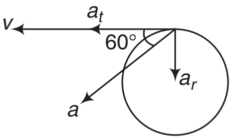
Now, $I = \int_0^a dI = \int_0^a (dM)\,X^2$

$$= \int_0^a (2\pi k x^3 dx)\,x^2 = \frac{(\pi k)\,a^6}{3}$$

Substituting the value of πk, we get

$$I = \frac{2}{3}Ma^2$$ **Ans.**

15. $\tan 60° = \frac{a_r}{a_t} = \frac{r\omega^2}{r\alpha}$



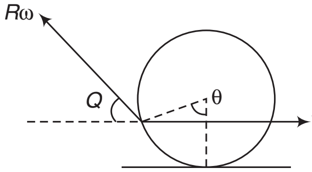
$$= \frac{\left[\int_0^t \alpha dt\right]^2}{\alpha} = \frac{(0.01\,t^2)^2}{0.02\,t}$$

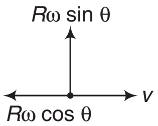
Solving we get,

$$t^3 = 346$$

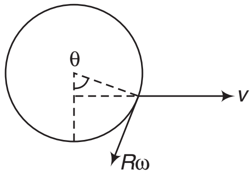
$$\therefore \quad t \approx 7 \text{ s}$$ **Ans.**

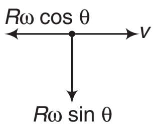
16. If $R\omega \cos\theta = v$ then velocity of point P is vertically upwards.





$$\therefore \quad \theta = \pm \cos^{-1}\left(\frac{v}{R\omega}\right)$$ **Ans.**





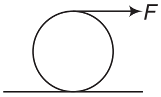
17. $a = R\alpha$ by applied forces

$$\therefore \quad \frac{F_1 + F_2}{M} = 2r\left[\frac{F_1\,(2r) - F_2\,(r)}{I}\right]$$

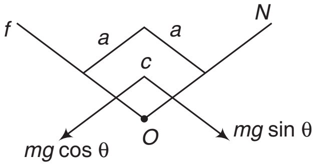
Solving these two equations we get,

$$\frac{F_1}{F_2} = \frac{I + 2\,Mr^2}{4\,Mr^2 - I}$$

18. Even a small force will produce torque about bottommost point. So, the disc starts toppling about bottommost point and here toppling means motion.



19. **For sliding** $mg\sin\theta > \mu\, mg\cos\theta$



or $\tan\theta > \mu$ ...(i)

For toppling

$$\tau_{mg\sin\theta} > \tau_{mg\cos\theta} \quad \text{(about } O\text{)}$$

$$\therefore \quad (mg\sin\theta)\left(\frac{a}{2}\right) > (mg\cos\theta)\left(\frac{a}{2}\right)$$

or $\tan\theta > 1$ ...(ii)

If $\mu > 1$, then condition (ii) is satisfied earlier. So, the cube topples before sliding.

If $\mu < 1$, then condition (i) is satisfied earlier. So, the cube slides before toppling.

20. Angular impulse = change in angular momentum

$$\therefore \quad \tau t = I\omega = \frac{1}{2}mR^2\,\omega \quad \text{...(i)}$$

$$\therefore \quad \tau = \frac{mR^2\omega}{2t}$$

$$= \frac{(20)\,(0.5)^2\left(2\pi \times \frac{240}{\pi \times 60}\right)}{2 \times 3}$$

$$= \frac{20}{3}\text{ N-m}$$ **Ans.**

During retardation using Eq. (i)

$$t = \frac{mR^2\omega}{2\tau} = \frac{mR^2\omega}{2\,(FR)}$$

$$= \frac{mR\omega}{2F}$$

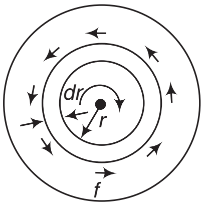
$$= \frac{(20)\,(0.5)\left(2\pi \times \frac{240}{\pi \times 60}\right)}{2 \times 10}$$

$$= 4 \text{ s}$$ **Ans.**

21. $d\tau = \mu\,(dm)\,g\cdot r$

$$= \mu\left(\frac{M}{\pi R^2}\right)(2\pi r dr)\,gr$$

$$\therefore \qquad \tau = \int_0^R d\tau = \frac{2\mu MgR}{3}$$



Angular retardation, $\alpha = \dfrac{\tau}{I}$

$$= \frac{(2\mu MgR/3)}{(1/2MR^2)} = \frac{4\mu g}{3R}$$

(a) $0 = \omega_0 - \alpha t$

$$\therefore \qquad t = \frac{\omega_0}{\alpha} = \frac{3\omega_0 R}{4\mu g} \qquad \textbf{Ans.}$$

(b) $0 = \omega_0^2 - 2\alpha\theta$

$$\therefore \qquad \theta = \frac{\omega_0^2}{2\alpha} = \frac{3\,\omega_0^2 R}{8\,\mu g} \qquad \textbf{Ans.}$$

22. $\theta = 6t - 2t^3$

$$\omega = \frac{d\theta}{dt}$$

$$= 6 - 6t^2$$

$$\alpha = \frac{d\omega}{dt} = -12t$$

$$\omega = 0$$

at $\qquad t = 1$ sec

$$\langle\omega\rangle_{0-1} = \frac{\int_0^1 \omega dt}{1} = \int_0^1 (6 - 6t^2)\,dt$$

$$= 4 \text{ rad/s} \qquad \textbf{Ans.}$$

$$\langle\alpha\rangle_{0-1} = \frac{\int_0^1 \alpha dt}{1} = \int_0^1 (-12t)$$

$$= -6 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

23. Centre of mass of both lies at the centre of ring.

$$I_c = mR^2 + \frac{m\,(2R)^2}{12}$$

$$= \frac{4}{3}\,mR^2$$

$$L = (2mvr_\perp) + I_c\omega$$

$$= (2mvR) + \frac{4}{3}\,mR^2\left(\frac{v}{R}\right)$$

$$= \frac{10}{3}\,mvR$$

This is clockwise, or along negative z-axis

$$\therefore \qquad \mathbf{L} = \left(-\frac{10}{3}\,mvR\right)\hat{\mathbf{k}} \qquad \textbf{Ans.}$$

24. $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \qquad \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1$$

$$= \frac{\left(\frac{1}{2}MR^2\right)}{\left[\frac{1}{2}MR^2 + m\left(\frac{R}{\sqrt{2}}\right)^2\right]}\,\omega_1$$

$$= \left(\frac{M}{M+m}\right)\omega_1 = \left(\frac{1}{1+0.2}\right)5$$

$$= \frac{25}{6} \text{ rad/s}$$

25. $I_1\omega_1 = I_2\,\omega_2$

$$\therefore \qquad I_1\,(100) = (I_1 + (10)\,(9)^2\,]\,(90)$$

or $\qquad I_1 + 810 = 1.11\,I_1$

$$\therefore \qquad I_1 = 7290 \text{ g-cm}^2$$

$$= 7.29 \times 10^{-4} \text{ kg-m}^2 \qquad \textbf{Ans.}$$

26. (a) $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \qquad \omega_2 = \frac{I_1}{I_2}\cdot\omega_1$$

$$= \frac{[1.6 + (2)\,(4)\,(0.9)^2\,]}{[1.6 + (2)\,(4)\,(0.15)^2\,]} \times 0.5$$

$$= \left(\frac{8.08}{1.78}\right)(0.5)$$

$$= 2.27 \text{ rev/s}$$

$$= 14.3 \text{ rad/s}$$

(b) $\qquad K_i = \dfrac{1}{2}\,I_1\,\omega_1^2$

$$= \frac{1}{2}\,[1.6 + (2)\,(4)\,(0.9)^2\,]\,(0.5 \times 2\pi)^2$$

$$= 39.9 \text{ J} \qquad \textbf{Ans.}$$

$$K_f = \frac{1}{2}\,[1.6 + (2)\,(4)\,(0.15)^2\,]\,(14.3)^2$$

$$= 181 \text{ J}$$

(c) $W = K_f - K_i = 141.1$ J

27. (a) $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \quad \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \frac{\left(\frac{1}{2}MR^2 + mR^2\right)}{\left(\frac{1}{2}mR^2\right)}\cdot\omega_0$$

$$= \left(1 + \frac{2m}{M}\right)\omega_0$$

(b) $W = K_f - K_i$

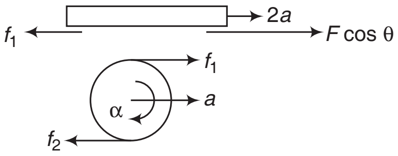
$$= \frac{1}{2}I_2\,\omega_2^2 - \frac{1}{2}I_1\,\omega_1^2$$

$$= \frac{1}{2}\left[\frac{1}{2}MR^2\right]\left[1 + \frac{2m}{M}\right]^2\omega_0^2 - \frac{1}{2}\left[\frac{1}{2}MR^2 + mR^2\right]\omega_0^2$$

$$= \frac{1}{4}MR^2\left(1 + \frac{2m}{M}\right)\omega_0^2\left[1 + \frac{2m}{M} - 1\right]$$

$$= \frac{1}{2}m\omega_0^2R^2\left(1 + \frac{2m}{M}\right) \qquad \textbf{Ans.}$$

28. Since, there is no slip at any contact. Therefore, net work done by friction = 0.



In time *t*

Work done by the applied force

= kinetic energy of plank and cylinder ($F\cos\theta$) (displacement of plank)

$$= \frac{1}{2}m\,(\text{velocity of plank})^2 + \frac{1}{2}\left(1 + \frac{1}{2}\right)M\,(\text{velocity of cylinder})^2$$

$$\therefore \quad (F\cos\theta)\left(\frac{1}{2}\times 2a\times t^2\right) = \frac{1}{2}\times m\,(2at)^2 + \frac{3}{4}\times M\times(at)^2$$

Solving, we get

$$a = \frac{4F\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

Equation of plank gives,

$$F\cos\theta - f_1 = m\,(2a)$$

$$\therefore \quad f_1 = F\cos\theta - 2ma$$

$$= F - \frac{8mF\cos\theta}{3M + 8m}$$

$$= \frac{3MF\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

Equation of cylinder gives

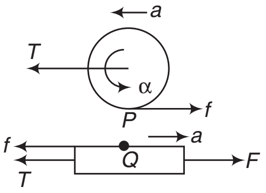
$$f_1 - f_2 = M\cdot a$$

$$\therefore \quad f_2 = f_1 - Ma$$

$$= \frac{3MF\cos\theta}{3M + 8m} - \frac{4\,MF\cos\theta}{3M + 8m}$$

or $$|f_2| = \frac{MF\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

29. For Cylinder



$$T - f = ma \qquad \text{...(i)}$$

$$a_P = a_Q$$

$$\therefore \quad R\alpha - a = a \qquad \text{...(ii)}$$

$$\alpha = \frac{\tau}{I} = \frac{f\cdot R}{\frac{1}{1}mR^2} = \frac{2f}{mR} \qquad \text{...(iii)}$$

For Plank

$$F - T - f = Ma \qquad \text{...(iv)}$$

We have four knows f, T, a and α.

Solving the equations we get,

$$a = \frac{F}{M + 3m} \qquad \textbf{Ans.}$$

30. Angular impulse = change in angular momentum

$$\therefore \quad (J\times r_\perp) = I\omega$$

where, J = linear impulse $= \dfrac{I\omega}{r_t}$

$$= \frac{(m)\left(\frac{2l}{3}\right)^2\cdot\omega}{l}$$

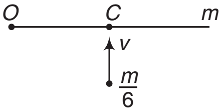
$$= \frac{4}{3}ml\omega \qquad \textbf{Ans.}$$

31. (a) Angular momentum about O remains constant just before and just after collision.

$$\therefore \quad L_i = L_f$$

or $$\left(\frac{m}{6}v\,\frac{L}{2}\right) = I\omega$$

$$= \left[\frac{mL^2}{3} + \frac{m}{6}\frac{L^2}{4}\right]\omega$$



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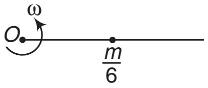


Solving, we get

$$\omega = \frac{2v}{9L} \qquad \textbf{Ans.}$$

(b) $$\frac{K_f}{K_i} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}\left(\frac{m}{6}\right)v^2}$$

$$= \frac{6\,I\omega^2}{v^2}$$

$$= \frac{6\left[\frac{mL^2}{3} + \frac{mL^2}{24}\right]\left(\frac{2v}{9L}\right)^2}{v^2}$$

$$= 6 \times \frac{9}{24} \times \frac{4}{81}$$

$$= \frac{1}{9} \qquad \textbf{Ans.}$$

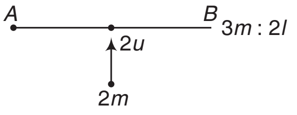
32. (a) Angular momentum of system is conserved just before and just after the impact.

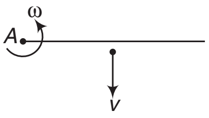
$\therefore \qquad L_i = L_f$

$\therefore \quad (2m)\,(2u)\,(l) = \dfrac{(3m)\,(2l)^2}{3}\cdot\omega - (2m)\,vl \qquad$ ...(i)

At the point of impact,

$$e = 1$$





$\therefore \qquad \dfrac{RVOS}{RVOA} = 1$

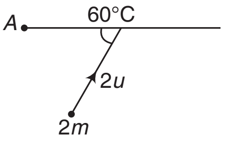
$\therefore \qquad RVOS = RVOA$

or $\qquad \omega l + v = 2u \qquad$ ...(ii)

Solving Eqs. (i) and (ii) we get,

$$v = \frac{2u}{3} \qquad \textbf{Ans.}$$

(b) Component of velocity of particle $P\,(= u)$ parallel to AB remains the same. Component perpendicular to AB produces the same impact as was done in part (a).



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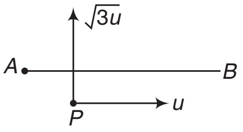
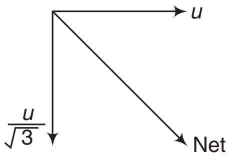


In part *(a)*, $2u$ velocity becomes

$$\frac{2u}{3}\left(\text{i.e., } \frac{1}{3}\text{ rd}\right).$$

Therefore, in this part $\sqrt{3}\,u$ will become

$$\frac{\sqrt{3}\,u}{3} \text{ or } \frac{u}{\sqrt{3}}.$$



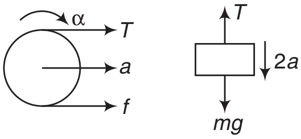
$\therefore \quad$ Net velocity $= \sqrt{u^2 + \dfrac{u^2}{3}}$

$$= \frac{2u}{\sqrt{3}} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. **For Block**



$$mg - T = m\,(2a) \qquad \text{...(i)}$$

For Ring

$$T + f = ma \qquad \text{...(ii)}$$

$$\frac{(T - f)\,R}{mR^2} = \alpha \qquad \text{...(iii)}$$

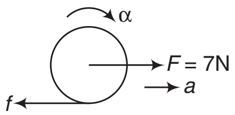
$$a = R\alpha \qquad \text{...(iv)}$$

Solving these equations we get

$$a = \frac{g}{3},\ 2a = \frac{2g}{3},\ T = \frac{mg}{3} \text{ and } f = 0$$

2. $f_{\max} = \mu mg = 0.1 \times 10 \times 10 = 10\ \text{N}$

Requirement of friction



$$a = R\alpha$$

$$\therefore \quad \left(\frac{7-f}{10}\right) = \frac{(R)\,(f\cdot R)}{2/5\ mR^2}$$

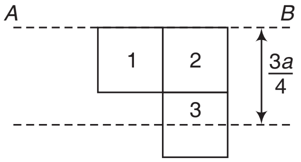
$$= \frac{2.5f}{m} = \frac{2.5f}{10}$$

Solving this equation, we get

$$f = 2$$

Since, $f < f_{\max}$, therefore $2N$ frictional force will act.

3. $I_{AB} = I_1 + I_2 + I_3$



$$= \frac{(m/4)\,(a/2)^2}{3} + \frac{(m/4)\,(a/2)^2}{3}$$

$$+ \left[\frac{(m/4)\,(a/2)^2}{12} + (m/4)\left(\frac{3a}{4}\right)^2\right]$$

$$= \frac{3}{16}ma^2$$

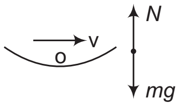
4. At bottom total kinetic energy

=decrease in potential energy

$= mgR$

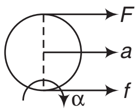
The ratio, $\dfrac{K_R}{K_T} = \dfrac{2}{5}$

$$\therefore \quad K_T = \frac{1}{2}mv^2 = \frac{5}{7}mgR$$



$$\therefore \quad \frac{mv^2}{R} = \frac{10}{7}mg = N - mg$$

$$\therefore \quad N = \frac{17}{7}mg \qquad \textbf{Ans.}$$



5. In case of pure rolling α can be obtained about bottommost point, about which torque of friction is zero.

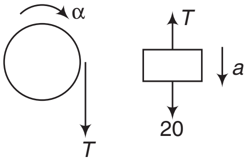
$$\alpha = \frac{F\ (2R)}{3/2\ mR^2} = \frac{4}{3}\frac{F}{mR}$$

$$\therefore \quad a = R\alpha = \frac{4F}{3m} \qquad \textbf{Ans.}$$

6. $F + f = ma = \dfrac{4F}{3}$

$$\therefore \quad f = \frac{F}{3} \quad \text{(towards right)} \qquad \textbf{Ans.}$$

7. **For Block**



$$20 - T = 2a \qquad \text{...(i)}$$

For Pulley

$$TR = I\alpha$$

$$\therefore \quad T\ (0.2) = (0.32)\,\alpha$$

$$T = 1.6\,\alpha \qquad \text{...(ii)}$$

$$a = R\alpha = 0.2\,\alpha \qquad \text{...(iii)}$$

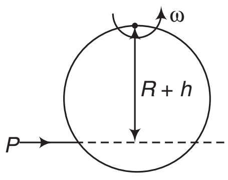
Solving these equations, we get

$$a = 2\ \text{m/s}^2 \qquad \textbf{Ans.}$$

8. $\omega = \dfrac{\text{Angular impulse}}{I}$ (about O)

$$= \frac{P\ (R+h)}{\frac{3}{2}mR^2}$$

Now, $v_{\text{COM}} = R\omega = \dfrac{PR\ (R+h)}{\frac{3}{2}mR^2}$



$$P + P_{\text{Hinge}} = mv_{\text{COM}} = \frac{mpR\ (R+h)}{\frac{3}{2}mR^2}$$

Solving this equation, we get $h = \dfrac{R}{2}$

9. $T_i = \dfrac{mg}{2}$ and $T_f = \dfrac{mg}{4}$

10. In first case $T < Mg$

In second case $T = Mg$

Torques are different, so angular accelerations are different.

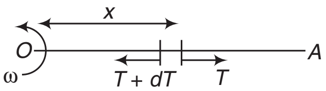
11. From energy conservation principle, maximum spring potential energy

= maximum decrease in gravitational potential energy

$$\therefore \quad \frac{1}{2}kx_m^2 = Mgh = Mg\,(x_m \sin\theta)$$

$$\therefore \quad x_m = \frac{2\,Mg\sin\theta}{k}$$

12. $-\,dT = (dm)\times\omega^2 = \left(\dfrac{m}{l}\,dx\right)x\omega^2$



$$\therefore \quad -\int_0^T dT = \int_l^x \left(\frac{m}{l}\,\omega^2\right)x\,dx$$

$$\therefore \quad T = \frac{1}{2}\,m\omega^2\,l\left(1-\frac{x^2}{l^2}\right) \qquad \textbf{Ans.}$$

13. $\omega = 10 + 5t$

$$\alpha = \frac{d\omega}{dt} = 5$$

At $\quad t = 0, \omega = 10$ rad/s

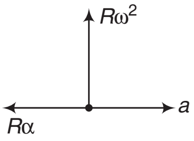
and $\quad \alpha = 5$ rad/s^2 $\Rightarrow r = OA = 1$ m

$$\mathbf{v} = (r\omega)\,\hat{\mathbf{j}} = (10\hat{\mathbf{j}}) \text{ m/s}$$

$$\mathbf{a} = (r\alpha)\,\hat{\mathbf{j}} - (r\omega^2)\,\hat{\mathbf{i}}$$

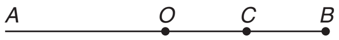
$$= (-100\,\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \text{ m/s}^2 \qquad \textbf{Ans.}$$

14. $a = R\alpha$



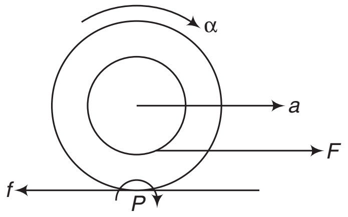
$$\therefore \quad a_{\text{net}} = R\omega^2 \quad \text{(towards centre)}$$

15. I_C is least and C lies between O and B as density is increasing from A to B.

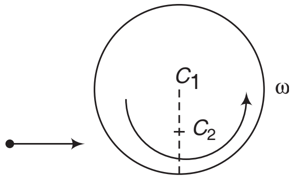


$I_A > I_B$ as more mass is concentrated towards B.

16. Torque of F about P is clockwise. So, spool will move towards right. Torque of friction will provide the clockwise torque about centre. So, f is leftwards. Further, a is towards right, so, $F > f$.



17. $C_1 C_2 = \dfrac{R}{2}$



where, C_2 = combined centre of mass of ring and particle.

From conservation of angular momentum about C_2.

$$L_i = L_f$$

$$\therefore \quad \frac{mv\cdot R}{2} = I\omega$$

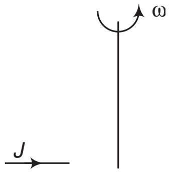
$$= \left[mR^2 + \frac{mR^2}{4} + \frac{mR^2}{4}\right]\omega$$

Solving this equation we get,

$$\omega = \frac{v}{3R} \qquad \textbf{Ans.}$$

18. Mg can't provide rotational motion. This is only N which actually provides the rotational motion.

19. Angular momentum = Angular impulse



$$\therefore \quad I\omega = Jl$$

or $\quad \omega = \dfrac{\text{Angular impulse}}{I}$

$$= \frac{Jl}{ml^2/3} = \frac{3J}{ml}$$

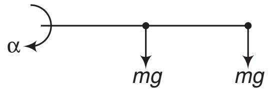
$$\text{KE} = \frac{1}{2}\,I\omega^2 = \frac{1}{2}\left(\frac{ml}{3}\right)^2\left(\frac{3J}{ml}\right)^2$$

$$= \frac{3J^2}{2m}$$

20. No rotational motion.

21. $\alpha = \dfrac{\tau_{\text{net}}}{I}$

$$= \frac{mg\left(\frac{l}{2}\right) + mgl}{(ml^2/3)}$$

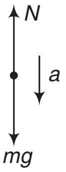


$$= \frac{4.5\,g}{l}$$

$$a_{\text{coin}} = l\alpha = 4.5\,g$$

$$mg - N = ma_{\text{coin}}$$

$\therefore \quad N = mg - ma_{\text{coin}} = -3.5\,g$



Since, N can't be negative.

So, $N = 0$ and coin can not remain in contact with rod.

22. Let F = the applied force and f = force of friction (leftwards)

Net horizontal force = 0

$\therefore \quad F\cos\theta = f$ ...(i)

Net torque about centre = 0

$\therefore \quad F\cdot r = fR$

Substituting $\dfrac{f}{R} = \dfrac{r}{R}$ in Eq. (i) we have,

$$\cos\theta = \frac{r}{R}$$

or $\quad \sin\theta = \sqrt{1 - \dfrac{r^2}{R^2}}$

$\therefore \quad \theta = \sin^{-1}\left(\sqrt{1 - \dfrac{r^2}{R^2}}\right)$

23. $a = \dfrac{mg - T}{m}$ ...(i)

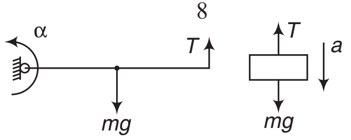
$$\alpha = \frac{\tau}{I} = \frac{T\cdot l - mg\cdot\frac{l}{2}}{(ml^2/3)}$$

or $\quad \alpha = \dfrac{1}{ml}\left[3T - \dfrac{3mg}{2}\right]$ ...(ii)

$a = l\alpha$ ...(iii)

Solving these three equations we get,

$a = \dfrac{3g}{8}$ **Ans.**

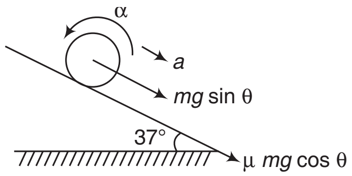


24. $a = (\mu g\cos\theta) + (g\sin\theta)$

$$= 0.5\times 10\times 0.8 + 10\times 0.6 = 10\text{ m/s}^2$$

$$\alpha = \frac{(\mu\, mg\cos\theta)\,R}{\frac{1}{2}mR^2} = \frac{2\,\mu g\cos\theta}{R}$$

$$= \frac{2\times 0.5\times 10\times 0.8}{0.4} = 20\text{ rad/s}^2$$



Pure rolling will start when,

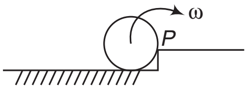
$v = R\omega$ or $at = R(\omega_0 - \alpha t)$

$\therefore \quad 10\,t = 0.4\,(54 - 20t)$

Solving this equation, we get

$t = 1.2\text{ s}$ **Ans.**

25. Angular momentum will remain conserved at the point of impact P. and just after impact it starts rotating about point P.



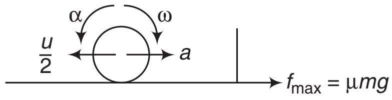
$$L_i = L_f$$

$$\therefore\ Mv_0\left(R - \frac{R}{4}\right) + \frac{1}{2}MR^2\left(\frac{v_0}{R}\right) = \frac{3}{2}MR^2\cdot\omega$$

$\therefore \quad \omega = \dfrac{5v_0}{6R}$

$\therefore \quad v_{\text{COM}} = \omega R = \dfrac{5v_0}{6}$ **Ans.**

26. $a = \dfrac{\mu mg}{m} = \mu g$



$\omega = \dfrac{u}{R} \Rightarrow \alpha = \dfrac{(\mu mg)\,R}{2/5\;mR^2} = \dfrac{5\,\mu g}{2R}$

Pure rolling will start when

$$v = R\omega$$

$$\left(\frac{u}{2} - at\right) = R\,(\alpha t - \omega)$$

or $\quad \left(\dfrac{u}{2} - \mu\,gt\right) = R\left(\dfrac{5\,\mu g}{2R}\,t - \dfrac{u}{R}\right)$

Solving we get, $\quad t = \dfrac{3u}{7\,\mu g}$ **Ans.**

27. Viscous liquid has only translational kinetic energy

$\therefore$ KE = translational + rotational kinetic energy of hollow sphere + translational kinetic energy of liquid.

$$= \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{2}{3}mR^2\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2$$

$$= \frac{4}{3}mv^2$$ **Ans.**

28. $$a = \frac{g\sin\theta}{1 + I/mR^2} = \frac{g\sin\theta}{1 + (1/2)}$$

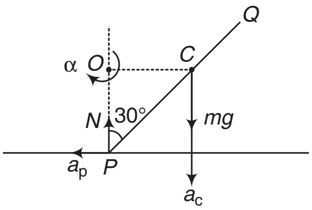
$$= \frac{2g\sin\theta}{3}$$

$$\therefore \quad g\sin\theta = \frac{3}{2}a$$

Now, $$f = \frac{mg\sin\theta}{1 + mR^2/I} = \frac{m\,(3/2\,a)}{1 + 2}$$

$$= \frac{1}{2}ma$$ **Ans.**

29. Only two forces are acting on rod, normal reaction (vertically upwards) and weight (vertically downwards). Since, both forces are vertical centre of mass falls in vertical direction (downwards).



$$OC = \frac{l}{2}\sin 30° = \frac{l}{4}$$

$$\alpha = \frac{\tau}{I} \qquad \text{(about } O\text{)}$$

$$= \frac{(mg)\left(\frac{l}{2}\sin 30°\right)}{\left(\frac{ml^2}{12} + \frac{ml^2}{16}\right)}$$

$$= \frac{12}{7}\frac{g}{l}$$

$$a_C = (OC)\,\alpha$$

$$= \left(\frac{l}{4}\right)\left(\frac{12}{7}\frac{g}{l}\right) = \frac{3}{7}g$$

Now, $$mg - N = ma_C = \frac{3\,mg}{7}$$

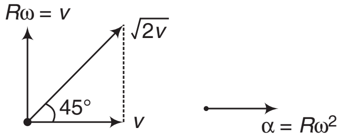
$$\therefore \quad N = \frac{4\,mg}{7}$$ **Ans.**

30. $$a_P = (OP)\,\alpha$$

$$= \left(\frac{l}{2}\cos 30°\right)\left(\frac{12}{7}\frac{g}{l}\right)$$

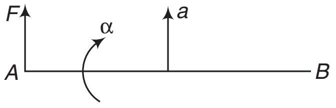
$$= \frac{3\sqrt{3}g}{7}$$ **Ans.**

31. Angle between acceleration and velocity is 45°.



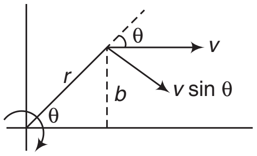
32. $$a = \frac{F}{M}$$

$$\alpha = \frac{FL/2}{ML^2/12} = \frac{6F}{ML}$$



$$a_B = \frac{L}{2}\alpha - a = \frac{2F}{M} \qquad \text{(downwards)}$$

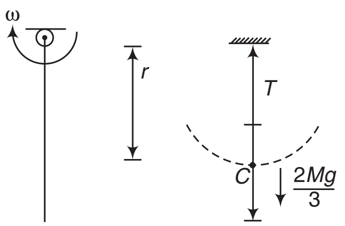
33. $$\omega = \frac{v\sin\theta}{r} = \frac{v\sin\theta}{b/\sin\theta}$$



$$= \frac{v\sin^2\theta}{b} \propto \sin^2\theta \qquad \text{(as } v \text{ and } b \text{ are constants)}$$

θ is decreasing. Therefore, ω will decrease.

34. Decrease in gravitational potential energy = increase in rotational kinetic energy



$$\therefore \quad Mg\frac{L}{2} = \frac{1}{2}\left(\frac{ML^2}{3}\right)\cdot\omega^2 \quad \therefore \quad \omega^2 = \frac{3g}{L}$$

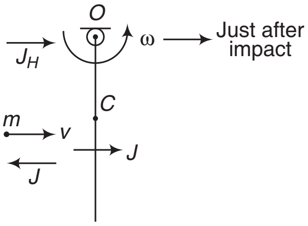
$$r = \frac{L}{3} + \frac{2L/3}{2} = \frac{2L}{3}$$

$$\frac{T-2Mg}{3}=\left(\frac{2M}{3}\right)a_C=\left(\frac{2M}{3}\right)(r\omega^2)$$

$$\therefore \qquad T=\frac{2Mg}{3}+\left(\frac{2Mg}{3}\right)\left(\frac{2L}{3}\right)\left(\frac{3g}{L}\right)$$

$$=2\,Mg$$ **Ans.**

35. From conservation of angular momentum, just before and just after impact about point O we have,



$$L_i=L_f$$

$$\therefore \qquad mv\frac{L}{2}=\left(\frac{mL^2}{3}\right)\cdot\omega$$

$$\therefore \qquad \omega=\frac{3v}{2L}$$

$$v_C=\frac{1}{2}\cdot\omega=\frac{3}{4}v$$

Impulse J has changed the momentum of particle from mv to O. Hence,

$$J=mv$$

For rod

$$J_H+J=mv_C=\frac{3}{4}mv$$

$$\therefore \qquad J_H=\frac{3}{4}mv-J=\frac{3}{4}mv-mv$$

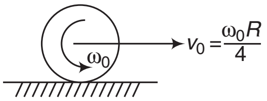
$$=-\frac{mv}{4}\text{ or }|J_H|=\frac{mv}{4}$$ **Ans.**

36. $L_i=L_f$

$$(I+mR^2)\,\omega=(mvR)+I\omega'$$

$$\therefore \qquad \omega'=\frac{(I+mR^2)\,\omega-mvR}{I}$$ **Ans.**

37. About bottommost point,



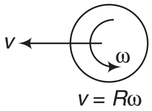
Angular momentum,

$$L=I_C\,\omega_0-mv_0R \qquad \text{(anticlockwise)}$$

$$=\frac{1}{2}mR^2\omega_0-m\left(\omega_0\frac{R}{4}\right)R$$

$$=\frac{1}{4}mR^2\omega_0$$

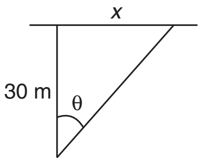
$$=+\text{ ve or anticlockwise}$$



During slip, friction acts about bottommost point. So, its torque is zero or angular momentum about bottommost point should also remain anticlockwise when pure rolling starts. So, figure should be as shown below.

So, the disc will return to its initial position for all values of μ.

38. $x=30\tan\theta$



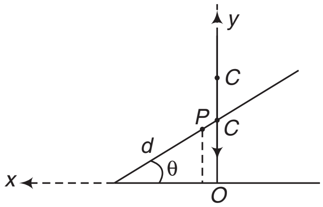
$$\therefore \qquad \left(\frac{dX}{dt}\right)=(30\sec^2\theta)\cdot\frac{d\theta}{dt}$$

or $$v_{\text{car}}=(30\sec^2\theta)\,\omega$$

$$\therefore \qquad \omega=\frac{v_{\text{car}}}{30\sec^2\theta}$$

$$=\frac{40}{(30)\sec^2 30^\circ}=1\text{ rad/s}$$

39. Two forces normal reaction and weight are the only forces acting on the rod during motion. Both forces are vertical. So centre of mass will fall downwards in a vertical line.



$$X_P=x\text{ (say)}=\frac{l}{2}\cos\theta-d\cos\theta$$

or $$\cos\theta=\frac{x}{\left(\frac{l}{2}-d\right)} \qquad \text{...(i)}$$

Similarly,

$$y_P=y=d\sin\theta$$

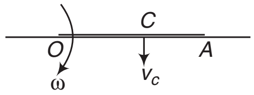
$$\therefore \qquad \sin\theta=\frac{y}{d} \qquad \text{...(ii)}$$

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{X^2}{\left(\frac{l}{2}-d\right)^2}+\frac{y^2}{d^2}=1$$

This is an equation of a circle for $d=\frac{l}{4}$. For any other value of 'd' it is equation of ellipse.

40. Instantaneous axis of rotation will pass through O.



$$\therefore \qquad v_0 = 0$$

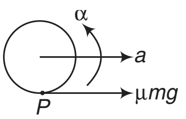
41. Decrease in gravitational potential energy
= increase in rotational kinetic energy about O

$$\therefore \qquad mg\frac{l}{2} = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\text{or} \qquad \omega = \sqrt{\frac{3g}{l}}$$

$$v_A = r\omega = l\omega = \sqrt{3gl} \qquad \textbf{Ans.}$$

42. $a = \dfrac{\mu\, mg}{m} = \mu g = \dfrac{2}{7}g$



$$\alpha = \frac{\mu\, mgR}{\frac{2}{5}mR^2} = \frac{5\mu g}{2R} = \frac{5}{7}\frac{g}{R}$$

Pure rolling will start when,

$$v_P = v + R\omega = v_0$$

$$\text{or} \qquad at + R(\alpha t) = v_0$$

$$\therefore \qquad \left(\frac{2}{7}gt\right) + \frac{5}{7}gt = v_0$$

$$\therefore \qquad t = \frac{v_0}{g}$$

$$S = \frac{1}{2}at^2$$

$$= \frac{1}{2}\left(\frac{2}{7}g\right)\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{7g} \qquad \textbf{Ans.}$$

More than One Correct Options

1. $\dfrac{K_R}{K_T} = \dfrac{I}{mr^2}$

$$\therefore \qquad K_T = \left(\frac{mr^2}{I + mr^2}\right)K_{\text{Total}}$$

$$\text{or} \qquad K_{\text{Total}} = \left(\frac{I + mr^2}{mr^2}\right)\left(\frac{1}{2}mv^2\right)$$

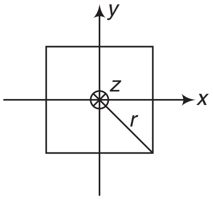
$$= mgh = mg\left(\frac{3}{4}\frac{v^2}{g}\right)$$

Solving we get $I = \dfrac{1}{2}mr^2$

So it is either solid cylinder or disc.

2. (a) $I_x = I_y = 2\left[\dfrac{ml^2}{12} + m\left(\dfrac{l}{2}\right)^2\right]$

$$= \frac{2}{3}ml^2$$



(b) $I_z = I_x + I_y = \dfrac{4}{3}ml^2$

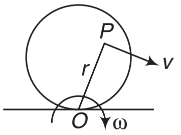
(c) $I = I_z + (4m)\,r^2$

$$= \frac{4}{3}ml^2 + (4m)\left(\frac{l}{\sqrt{2}}\right)^2$$

$$= \frac{10}{3}ml^2$$

(d) $I = 2\dfrac{ml^2}{3} + ml^2 = \dfrac{5}{3}ml^2$

3. $O \rightarrow$ instantaneous axis of rotation

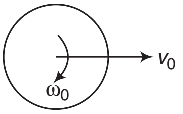


$$v = r\omega \;\text{ or }\; v \propto r$$

More the value of r from O more is the speed of point P.

4. $v_0 + R\omega_0 = v \qquad$ ...(i)

$v_0 - R\omega_0 = 3v \qquad$ ...(ii)



Solving these two equations,

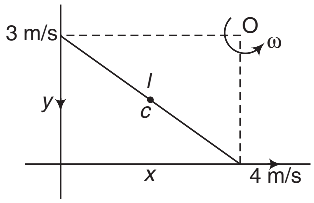
we get $\qquad v_0 = 2v$

and $\qquad \omega_0 = -\dfrac{v}{R}$

5. $x^2 + y^2 = l^2 = (0.25)\ m^2$

$$y\omega = 4 \quad \text{...(i)}$$

$$x\omega = 3 \quad \text{...(ii)}$$



Squaring and adding, we get

$$(x^2 + y^2)\,\omega^2 = 25$$

or $$(0.25)\,\omega^2 = 25$$

$$\therefore \quad \omega = 10 \text{ rad/s}$$

$$OC = \frac{l}{2} = 0.25\text{m}$$

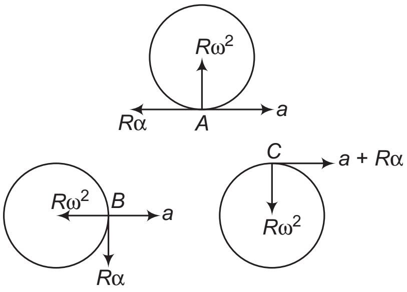
$$v_{\text{CM}} = (OC)\,\omega = 2.5 \text{ m/s}$$

$$K_{\text{Total}} = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\,\omega^2$$

$$= \frac{1}{2}\times 2\times(2.5)^2 + \frac{1}{2}\left(\frac{2\times 0.25}{12}\right)(10)^2$$

$$= \frac{25}{3}\text{ J}$$

6. Net acceleration of any point on the rim is vector sum of a, $R\omega^2$ and $R\alpha$ with $a = R\alpha$

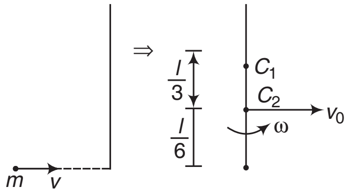


$a_A = R\omega^2 \rightarrow$ vertically upwards

If $a = R\omega^2$, a_B is vertically downwards and so on.

7. C_1 is centre of mass of rod

C_2 is centre of mass of both



$$P_i = P_f$$

$$\therefore \quad mv = 3mv_0 \Rightarrow v_0 = \frac{v}{3} \quad \text{...(i)}$$

$L_i = L_f$ about C_2 we have,

$$mv\,\frac{l}{6} = I_{C_2}\,\omega$$

$$= \left[\left(\frac{2ml^2}{12}\right) + 2m\left(\frac{l}{3}\right)^2 + m\left(\frac{l}{6}\right)^2\right]$$

$$\therefore \quad \omega = \frac{2}{5}\frac{v}{l} \Rightarrow K_i = \frac{1}{2}mv^2$$

$$K_f = \frac{1}{2}(3m)\left(\frac{v}{3}\right)^2 + \frac{1}{2}I_{C_2}\,\omega^2$$

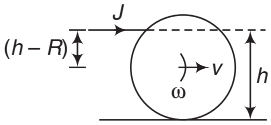
$$= \frac{1}{6}mv^2 + \frac{1}{2}\left(\frac{5}{12}ml^2\right)\left(\frac{2}{5}\frac{v}{l}\right)^2$$

$$= \frac{1}{5}mv^2$$

$\therefore$ Loss of kinetic energy $= K_i - K_f = \dfrac{3}{10}mv^2$

8. $I = mK^2$ $\left(\text{where, } K = \dfrac{R}{2} = \text{radius of gyration}\right)$

$$= m\left(\frac{R}{2}\right)^2 = \frac{1}{4}mR^2$$



Ball will roll purely if

$$v = R\omega$$

$$\therefore \quad \left(\frac{J}{m}\right) = R\left[\frac{J\,(h-R)}{(1/4)\,mR^2}\right]$$

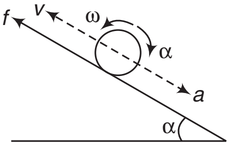
Solving this equation, we get

$$h = \frac{5R}{4}$$

Further, if ball is struck at centre of mass, there will be no rotation only translation.

9. Friction always acts upwards. If the ball moves upwards, it becomes a case of retardation with pure rolling.

$$\mu_{\min} = \frac{\tan\alpha}{1 + \dfrac{mR^2}{I}} = \frac{\tan\alpha}{1 + \dfrac{3}{2}} = \frac{2}{5}\tan\alpha$$



10. $I_1\omega_1 = I_2\omega_2$

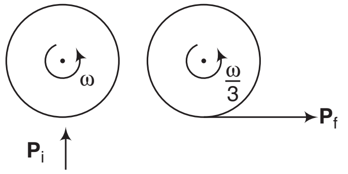
$$\therefore \quad \omega_2 = \frac{I_1}{I_2}\cdot\omega_1$$

$$= \frac{(1/2\ mR^2)}{(1/2\ mR^2 + mR^2)}\cdot\omega$$

$$= \frac{\omega}{3}$$

$$P_i = mv_i = m\,(2\omega R) = 2m\omega R$$

$$P_f = mv_f = m\left(\frac{\omega}{3}R\right) = \frac{1}{3}m\omega R$$

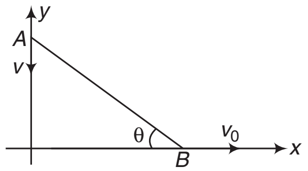


Impulse $J = |\mathbf{P}_f - \mathbf{P}_i|$

$$= \sqrt{P_i^2 + P_f^2} \qquad (\text{as}\ \theta = 90°)$$

$$= \frac{\sqrt{37}}{3}\,m\omega R$$

11. Velocity component along $AB = 0$



$$\therefore \quad v_0\cos\theta = v\sin\theta$$

or $\quad v = v_0\cot\theta = f_1(\theta)$

at $\quad \theta = 37°,\ v = \frac{4}{3}v_0$

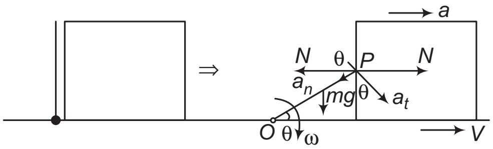
$$\omega = \frac{\text{Relative velocity} \perp \text{to } AB}{l}$$

$$= \frac{v_0\sin\theta + v\cos\theta}{l} = f_2(\theta)$$

$$= \frac{(v_0)(3/5) + (4/3\ v_0)(4/5)}{l} = \frac{5v_0}{3l}$$

Comprehension Based Questions

1. At angle θ shown in figure :



decrease in potential energy of rod
= increase in rotational kinetic energy of rod
+ translational kinetic energy of block.

But $\quad v = v_P\sin\theta = (l\omega)\sin\theta$

$$\therefore \quad mg\left(\frac{l}{2} - \frac{l}{2}\sin\theta\right) = \frac{1}{2}\left(\frac{ml^2}{3}\omega^2\right) + \frac{1}{2}M\,(\omega l\sin\theta)^2$$

From here we get,

$$l\omega^2 = a_n = \frac{mg\,(1-\sin\theta)}{m/3 + M\sin^2\theta} \qquad \text{...(i)}$$

$$a_t = l\alpha = l\left(\frac{\tau}{I}\right)_0 = l\left[\frac{mg\,\frac{l}{2}\cos\theta - Nl\sin\theta}{ml^2/3}\right]$$

$$= \frac{3}{2}g\cos\theta - \frac{3N\sin\theta}{m} \qquad \text{...(ii)}$$

For block

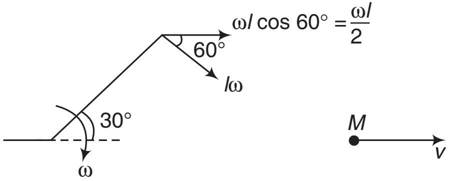
$$N = Ma = M\,[a_t\sin\theta - a_n\cos\theta] \qquad \text{...(iii)}$$

Now putting values of a_t and a_n from Eqs. (i) and (ii) in Eqs. (iii) and then putting $N = 0$ and $\theta = 30°$ in the equation we get,

$$\frac{M}{m} = \frac{4}{3} \qquad \textbf{Ans.}$$

2. $\frac{M}{m} = \frac{4}{3}$

$$M = \frac{4}{3}m$$



$$\frac{\omega l}{2} = v \quad \therefore \quad \omega = \frac{2v}{l}$$

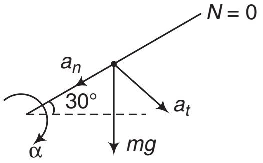
Decrease in potential energy of rod
= increase in rotational kinetic energy of rod
+ translational kinetic energy of block

$$\therefore \quad mg\left(\frac{l}{2} - \frac{l}{2}\sin 30°\right) = \frac{1}{2}\left(\frac{ml}{3}\right)^2\left(\frac{2v}{l}\right)^2 + \frac{1}{2}\left(\frac{4m}{3}\right)(v^2)$$

Solving this equation, we get

$$v = \frac{\sqrt{3gl}}{4} \qquad \textbf{Ans.}$$

3. $a_n = \frac{l}{2}\omega^2 = \frac{l}{2}\left(\frac{4v^2}{l^2}\right)$



$$= \left(\frac{2}{l}\right)\left(\frac{3gl}{16}\right) = \frac{3}{8}g$$

$$a_t = \frac{l}{2}\alpha = \frac{l}{2}\left(\frac{\tau}{I}\right) = \frac{l}{2}\left[\frac{mg\left(\frac{l}{2}\cos 30°\right)}{ml^2/3}\right]$$

$$= \frac{3\sqrt{3}\,g}{8}$$

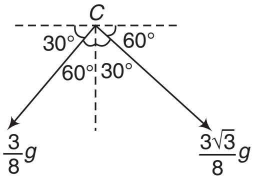
$$\therefore \qquad a = \sqrt{a_n^2 + a_t^2}$$

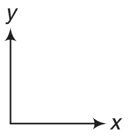
$$= \frac{3g}{4}$$

4. $\mathbf{a}_C = \left(\frac{3\sqrt{3}\,g}{8}\cos 60° - \frac{3}{8}g\cos 30°\right)\hat{\mathbf{i}}$

$$+ \left(\frac{3\sqrt{3}\,g}{8}g\cos 30° + \frac{3}{8}g\cos 60°\right)(-\hat{\mathbf{j}})$$

$$= -\left(\frac{3g}{4}\right)\hat{\mathbf{j}}$$





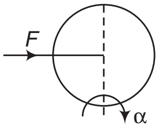
Now, $\quad mg(-\hat{\mathbf{j}}) + \mathbf{F}_{\text{Hinge}} = m\mathbf{a}_C$

Substituting the values we get

$$\mathbf{F}_{\text{Hinge}} = \left(\frac{mg}{4}\right)\hat{\mathbf{j}}$$ **Ans.**

5. Bottommost point where friction actually acts is at rest.

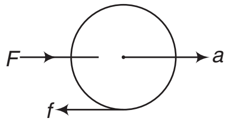
6. $\alpha = \frac{F \cdot R}{\frac{3}{2}mR^2} = \frac{2F}{3mR}$



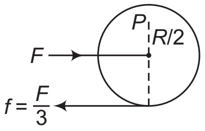
$$a = R\alpha = \frac{2F}{3m}$$

$$F - f = ma = \frac{2F}{3}$$

$$\therefore \qquad f = \frac{F}{3} = \frac{1}{2}ma$$



7. About point P net torque of F and f is zero. So, angular momentum is conserved.



8. K_{Total} after falling a height h is mgh

$$\frac{K_R}{K_T} = \frac{2}{3}$$

$$\therefore \qquad K_T = \frac{1}{2}mv^2 = \left(\frac{3}{5}\right)(mgh)$$

$$\therefore \qquad v = \sqrt{\frac{6gh}{5}}$$ **Ans.**

9. $H = \frac{v^2 \sin^2 \sin 37°}{2g}$

$$= \frac{(6gh/5)\,(3/5)^2}{2g} = \frac{27h}{125}$$ **Ans.**

10. $R = x = \frac{2u^2 \sin 37° \cos 37°}{g}$

$$= \frac{(2)\,(6gh/5)\,(3/5)\,(4/5)}{g}$$

$$= \frac{144}{125}h$$ **Ans.**

Match the Columns

1. (a) $t = \sqrt{\frac{2S}{a}}$ and $a = \frac{g\sin\theta}{1 + I/mR^2}$

I of hollow sphere is maximum.

So, a is minimum and t is maximum.

(b) $K_{\text{Total}} = mgh$ for all

(c) $\frac{K_R}{K_T} = \frac{2}{5}$ for solid sphere

$= \frac{2}{3}$ for hollow sphere

$= \frac{1}{2}$ for disc

It is maximum for hollow sphere.

But $\frac{K_T}{K_R}$ is maximum for solid sphere.

2. In all cases,

$$v = \frac{J}{m} \qquad (J = \text{linear impulse})$$

$$\omega = \frac{J \times r_\perp}{I_E}$$

Here, $r_\perp$ is from centre E. v is rightwards. So, for pure rolling ω should be clockwise. Hence, J should be applied at A.

If it is below A, angular velocity is anticlockwise and it will cause forward slip.

3. In case of pure roll (upwards or downwards) required value of friction acts in upward direction.

In case of slip (forward or backward) maximum friction will act in backward or forward direction.

4. (a) $K = \sqrt{I/m}$

$$= \sqrt{\frac{m\,(2a)^2}{3}} = \frac{2}{\sqrt{3}}\,a$$

(b) $K = \sqrt{\frac{m\,(2a)^2}{12}} = \frac{a}{\sqrt{3}}$

(c) $K = \sqrt{\frac{ma^2}{3}} = \frac{a}{\sqrt{3}}$

(d) $K = \sqrt{\frac{ma^2}{12}} = \frac{a}{\sqrt{12}}$

5. (a) $K_{\text{Total}} = mgh\,K \text{ (say)}$

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \quad K_R = \frac{2}{7}K = \frac{2}{7}mgh$$

(b) $K_{\text{Total}} = mg\frac{h}{2}$

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore\ K_T = \frac{5}{7}\left(mg\frac{h}{2}\right) = \frac{5}{14}mgh$$

(c) At 3

$$K_R = \frac{2}{7}\left(mg\frac{h}{2}\right) = \frac{1}{7}mgh$$

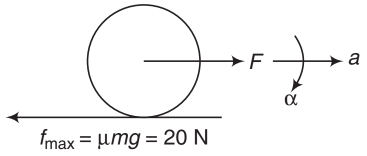
From 3 to 4, K_R will become constant.

(d) $K_T = K_{\text{Total}} - K_R$

$$= mgh - \frac{1}{7}mgh = \frac{6}{7}mgh$$

6. $a = R\alpha = R\left(\frac{\tau}{I}\right)$

$$\therefore \quad \frac{F-20}{10} = (1)\left(\frac{20 \times 1}{\frac{1}{2} \times 10 \times 1^2}\right)$$



Solving this equation, we get

$$F = 60 \text{ N}$$

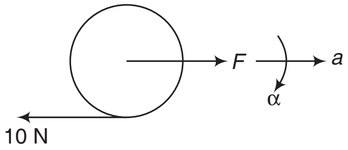
So, when $F = 60$ N, friction reaches its maximum value or slipping will start. F becomes 60 N at 6 s.

(d) $a = R\alpha = R\left(\frac{\tau}{I}\right)$

$$\text{or} \quad \frac{F-10}{10} = (1)\left[\frac{10 \times 1}{\frac{1}{2} \times 10 \times 1^2}\right]$$

$$F = 30 \text{ N}$$

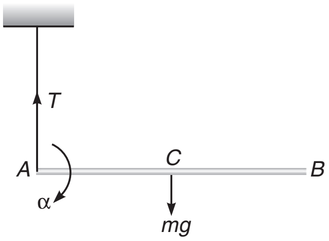
and F becomes 30 N at 3 s.



Subjective Questions

1. Torque about bottommost point in each case is clockwise.

2. $\alpha = \frac{\tau}{I} = \frac{mg\left(\frac{l}{2}\right)}{\frac{ml^2}{3}} = \frac{3}{2}\frac{g}{l}$



(a) $a_B = l\alpha = \frac{3g}{2}$ **Ans.**

(b) $a_C = \frac{l}{2}\alpha = \frac{3g}{4}$ **Ans.**

(c) $a_C = \frac{mg - T}{m}$ or $\frac{3g}{4} = \frac{mg - T}{m}$

∴ $T = \frac{mg}{4}$ **Ans.**

3. $mgh = K_R + K_T = \frac{3}{4}mv^2$

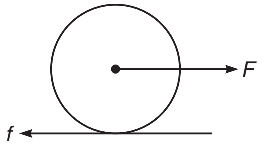
Here $h = s\sin\theta$

∴ $gs\sin\theta = \frac{3}{4}v^2$ or $s = \frac{3v^2}{4g\sin\theta}$

$$= \frac{3\times(2.0)^2}{4\times 9.8\times\frac{1}{2}} = 0.612 \text{ m}$$ **Ans.**

4. $I = MR^2$

For pure rolling to take place. $a = R\alpha$



or $\frac{F - f}{M} = R\left(\frac{f.R}{MR^2}\right) = \frac{f}{M}$

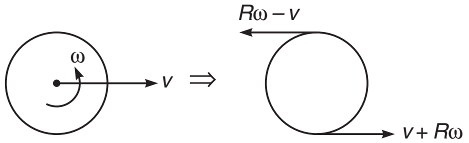
∴ $f = \frac{F}{2}$ **Ans.**

and $a = \frac{F - f}{M} = \frac{F}{2M}$ **Ans.**

5. $v + R\omega = 1.5$ ...(i)

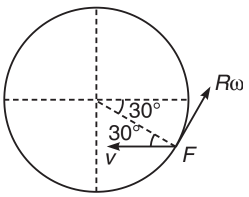
and $R\omega - v = 3.0$ ...(ii)

From Eq. (i) and (ii), we have



$R\omega = 2.25$ m/s and $v = -0.75$ m/s

Thus, velocity of point C is 0.75 m/s (towards left).



$$v_F = \sqrt{v^2 + (R\omega)^2 + 2v(R\omega)\cos(90 + 30)}$$

$$= \sqrt{0.5625 + 5.0625 + 2\times 0.75\times 2.25\times\left(-\frac{1}{2}\right)}$$

$= 1.98$ m/s **Ans.**

6. $a = \frac{Mg - T}{M}$ ...(i)

$$\alpha = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$$ ...(ii)

$$a = R\alpha$$ ...(iii)

Solving these three equations, we get

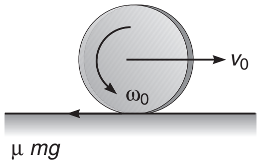
$a = \frac{2g}{3}$ and $T = \frac{Mg}{3}$ **Ans.**

7. From conservation of mechanical energy, decrease in gravitational PE = increase in rotational KE

or $mg(R) = \left[\frac{1}{2}MR^2 + mR^2\right]\left(\frac{1}{2}\omega^2\right)$

or $\omega = \sqrt{\frac{4mg}{(2m + M)R}}$ **Ans.**

8. Initially there is forward slipping. Therefore, friction is backwards and maximum.



Let velocity becomes zero in time t_1 and angular velocity becomes zero in time t_2.

Then, $0 = v_0 - at_1$

or $t_1 = \frac{v_0}{a} = \frac{v_0}{\mu g}$ ...(i)

and $0 = \omega_0 - \alpha t_2$

or $t_2 = \frac{\omega_0}{\alpha}$

Here, $\alpha = \frac{\mu mgR}{\frac{1}{2}mR^2} = \frac{2\mu g}{R}$

∴ $t_2 = \frac{\omega_0 R}{2\mu g}$ ...(ii)

Disk will return back when

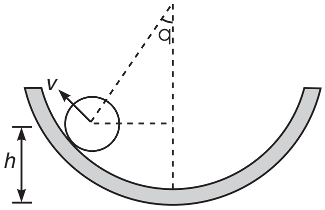
$t_2 > t_1$ or $\frac{\omega_0 R}{2\mu g} > \frac{v_0}{\mu g}$

or $\omega_0 > \frac{2v_0}{R}$ **Ans.**

9. $h = (R - r)(1 - \cos\theta)$ …(i)

Kinetic energy at angle θ is,

$$K = \frac{7}{5}\left(\frac{1}{2}mv_0^2\right) - mgh$$



∵ In case of pure rolling

$$K_T = \frac{5}{7}K$$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - \frac{5}{7}mgh$$

$$\therefore \quad v^2 = v_0^2 - \frac{10}{7}gh \qquad \text{…(ii)}$$

Equation of motion at angle θ is,

$$N - mg\cos\theta = \frac{mv^2}{(R - r)}$$

$$\therefore \quad N = mg\cos\theta + \frac{m}{(R - r)}\left(v_0^2 - \frac{10}{7}gh\right)$$

Substituting value of h from Eq. (i)

$$N = mg\cos\theta + \left(\frac{m}{R - r}\right)\left\{v_0^2 - \frac{10}{7}g(R - r)(1 - \cos\theta)\right\}$$

$$= \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R - r)} \qquad \textbf{Ans.}$$

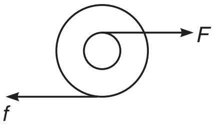
Force of friction,

$$f = \frac{mg\sin\theta}{1 + \frac{mr^2}{I}} \quad \text{(for pure rolling to take place)}$$

$$= \frac{mg\sin\theta}{1 + \frac{5}{2}} \qquad \left(I = \frac{2}{5}mr^2\right)$$

$$= \frac{2}{7}mg\sin\theta \qquad \textbf{Ans.}$$

10. (a) For pure rolling to take place,



$$a = R\alpha \quad \text{or} \quad \frac{F - f}{m} = R\left[\frac{Fr + fR}{\frac{1}{2}mR^2}\right]$$

Solving this equation, we get

$$f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F \qquad \textbf{Ans.}$$

(b) Acceleration $a = \dfrac{F - f}{m}$

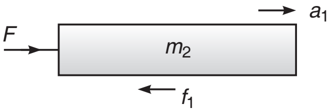
Substituting value of f from part (a), we get

$$a = \frac{2F}{3mR}(R + r) \qquad \textbf{Ans.}$$

(c) $a > \dfrac{F}{m}$ if $\dfrac{2}{3R}(R + r) > 1$ or $r > \dfrac{R}{2}$ **Ans.**

(d) In this case force of friction is in forward direction. **Ans.**

11. We can choose any arbitrary directions of frictional forces at different contacts.



In the final answer, the negative value will show the opposite directions.

Let f_1 = friction between plank and cylinder

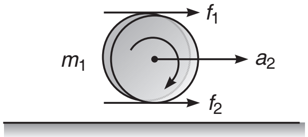
f_2 = friction between cylinder and ground

a_1 = acceleration of plank

a_2 = acceleration of centre of mass of cylinder

and α = angular acceleration of cylinder about is COM.

Directions of f_1 and f_2 are as shown here



Since, there is no slipping anywhere

$$\therefore \quad a_1 = 2a_2 \qquad \text{…(i)}$$

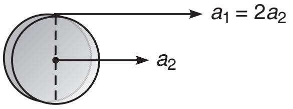
(Acceleration of plank = acceleration of top point of cylinder)

$$a_1 = \frac{F - f_1}{m_2} \qquad \text{…(ii)}$$

$$a_2 = \frac{f_1 + f_2}{m_1} \qquad \text{…(iii)}$$

$$\alpha = \frac{(f_1 - f_2)R}{I}$$

(I = moment of inertia of cylinder about COM)

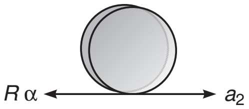


$$\therefore \qquad \alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}m_1 R^2}$$

$$\alpha = \frac{2(f_1 - f_2)}{m_1 R} \qquad \ldots\text{(iv)}$$

$$a_2 = R\alpha = \frac{2(f_1 - f_2)}{m_1} \qquad \ldots\text{(v)}$$

(Acceleration of bottommost point of cylinder = 0)



(a) Solving Eqs. (i), (ii), (iii) and (v), we get

$$a_1 = \frac{8F}{3m_1 + 8m_2}$$

and
$$a_2 = \frac{4F}{3m_1 + 8m_2}$$

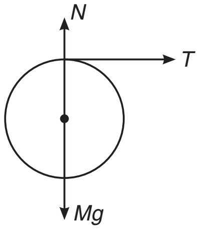
(b) $f_1 = \dfrac{3m_1 F}{3m_1 + 8m_2}$

$$f_2 = \frac{m_1 F}{3m_1 + 8m_2}$$

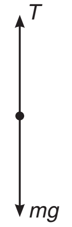
Since, all quantities are positive, they are correctly shown in figures.

Note *Above calculations have been done at $t = 0$ when $\omega = 0$.*

12. If α be the angular acceleration of the hoop and a be the acceleration of its centre, acceleration of m would be $\alpha r + a$.



FBD of hoop



FBD of block of mass m

Here, $Tr = I\alpha$ [where I = moment of inertia of the hoop about the horizontal axis passing through its centre]

Also, $\quad T = Ma \quad$ and $\quad mg - T = m[a + \alpha r]$

Solving, we get

$$a = \frac{mg}{[M + 2m]} = \frac{2}{1.4} = 1.43 \text{ m/s}^2$$

Hence, $\quad T = 1.43$ N

and $\quad \alpha = \dfrac{Tr}{I} = \dfrac{T}{Mr} = 7.15 \text{ rad/s}^2$

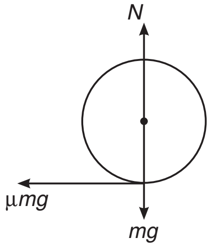
13. Initially the cylinder will slip on the plank, therefore kinetic friction will act between the cylinder and the plank.

$$a_c = -\frac{\mu mg}{m} = -\mu g$$

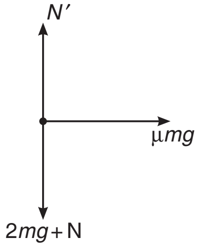
$$a_p = +\frac{\mu mg}{2m} = +\frac{\mu g}{2}$$

$$\alpha_c = +\frac{(\mu mg)(R)}{(mR^2/2)} = +\frac{2\mu g}{R}$$

For pure rolling,



FBD of cylinder



FBD of plank

$$v_p = v_c - R\omega_c$$

$$\therefore \qquad \frac{\mu g}{2}t = v_0 - \mu g t - (R)\left(\frac{2\mu g}{R}\right)(t)$$

$$\therefore \qquad t = \frac{v_0}{3.5\mu g} = \frac{7}{3.5 \times 0.1 \times 10} = 2 \text{ s}$$

$$\therefore \qquad s_c - s_p = v_0 t - \frac{1}{2} \times (\mu g)(t^2) - \frac{1}{2}\left(\frac{\mu g}{2}\right)(t^2)$$

$$= (7 \times 2) - \frac{1}{2}(0.1)(10)(4) - \frac{1}{2}\left(\frac{0.1 \times 10}{2}\right)(4) = 11 \text{ m}$$

Also, $\quad v_c - v_P = (v_0 - \mu g t) - \left(\dfrac{\mu g}{2}\right)(t)$

$$= 7 - 0.1 \times 10 \times 2 - \frac{0.1 \times 10 \times 2}{2} = 4 \text{ m/s}$$

Hence, the remaining distance $(12 - 11 = 1 \text{ m})$ is travelled in a time,

$$t' = \frac{1.0}{4} = 0.25 \text{ s}$$

$\therefore \quad$ Total time $= 2 + 0.25 = 2.25$ s

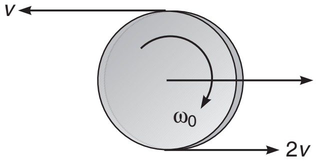
14. In rolling without sliding on a stationary ground, work done by friction is zero. Hence work done by the applied force = change in kinetic energy

$$\therefore\quad (30)(0.25)=\frac{1}{2}\times 9\times v^2 + 2\left[\frac{1}{2}\times 6\times v^2+\frac{1}{2}\times\frac{1}{2}\times 6\times r^2\times\frac{v^2}{r^2}\right]$$

or $$7.5 = 13.5v^2$$

$\therefore$ $$v = 0.745\ \text{m/s}$$ **Ans.**

15. Let v_0 be the linear velocity and ω_0 the angular velocity of the disc as shown in figure then,



$$v_0 - r\omega_0 = 2v \qquad \text{...(i)}$$

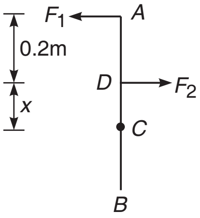
and $$v_0 + r\omega_0 = -v \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have

$$\omega_0 = -\frac{3}{2}\frac{v}{r}$$

Hence, the angular velocity of disc is $\frac{3}{2}\frac{v}{r}$ anticlockwise. **Ans.**

16. Let x be the distance of centre point C of rod from D. Then,



$$F_2 - F_1 = ma$$

or $$F_1 = 3\ \text{N}$$

Further, $$\tau_c = 0$$

$\therefore$ $$F_2x = F_1\ (0.2 + x)$$

$$5x = F_1\ (0.2 + x)$$

$\therefore$ $$5x = 3\ (0.2 + x)$$

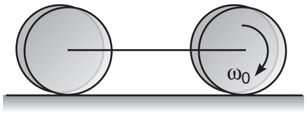
or $$x = 0.3\ \text{m}$$

$\therefore$ Length of rod = 2 (x + 0.2) = 1.0 m **Ans.**

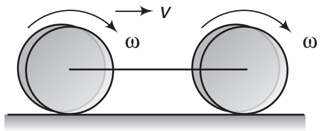
17. $L_i = L_f$ (about bottommost point)

$\therefore$ $$I\omega_0 = 2[I\omega + mRv]$$

or $$\left(\frac{1}{2}mR^2\right)\omega_0 = 2\left[\frac{1}{2}mR^2\omega + mR\ (\omega R)\right]$$



$\Rightarrow$



$\therefore$ $$\omega = \frac{\omega_0}{6}$$ **Ans.**

and $$v = \omega R = \frac{\omega_0 R}{6}$$ **Ans.**

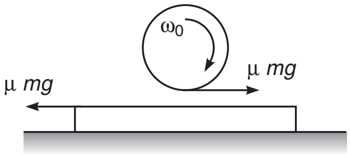
18. Let,

a_1 = linear acceleration of sphere (towards right),

a_2 = linear acceleration of plank (towards left)

and α = angular retardation of sphere

$$a_1 = a_2 = \frac{\mu mg}{m} = \mu g \quad\Rightarrow\quad \alpha = \frac{\mu mgr}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$$



Let pure rolling starts after time 't'. Then

$$\omega r - v = v$$

$\therefore$ $$\omega r = 2v$$

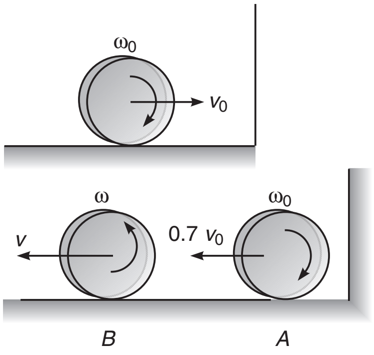
$$(\omega_0 - \alpha t)r = 2(a_1 t)$$

Substituting the values,

$$t = \frac{2}{9}\frac{\omega_0 r}{\mu g}$$

$\therefore$ $$s = \frac{1}{2}(a_2)t^2 = \frac{2\omega_0^2 r^2}{81\mu g}$$ **Ans.**

19. Between A and B, there is forward slipping. Therefore, friction will be maximum and backwards (rightwards). At point B where $v = R\omega$, ball starts rolling without slipping and force of friction becomes zero.



From conservation of angular momentum between points A and B about bottommost point (because torque of friction about this point is zero)

$$L_A = L_B$$

$\therefore \quad m(0.7v_0)R - I\omega_0 = mvR + I\omega$

Substituting $\omega_0 = \dfrac{v_0}{R}, \omega = \dfrac{v}{R}$

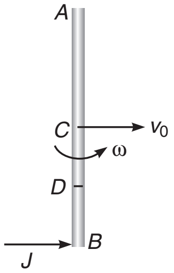
and $\quad I = \dfrac{2}{5}mR^2$, we get

$$v = \frac{3}{14}v_0 = \left(\frac{3}{14}\right)(7)\text{ m/s} = 1.5 \text{ m/s} \qquad \textbf{Ans.}$$

20. Let J be the linear impulse applied at B and ω the angular speed of rod.

$$J = mv_0 \qquad \text{...(i)}$$

$$J\left(\frac{l}{2}\right) = \frac{ml^2}{12}.\omega \qquad \text{...(ii)}$$



Solving these two equations,

$$\omega = \frac{6v_0}{l}$$

Linear speed of D (mid-point of CB) relative to C,

$$v = \omega\left(\frac{l}{4}\right) = \frac{3}{2}v_0$$

$\therefore$ Force exerted by upper half on the lower half,

$$F = \frac{\left(\frac{m}{2}\right)v^2}{\left(\frac{l}{4}\right)}$$

Substituting $\quad v = \dfrac{3}{2}v_0$, we get

$$F = \frac{9}{2}\frac{mv_0^2}{l} \qquad \textbf{Ans.}$$

21. $\dfrac{I}{mR^2} = \dfrac{2}{5} = 0.4$ for sphere

$= \dfrac{1}{2} = 0.5$ for disc $\quad$ and $\quad = 1$ for hoop

$$s = \frac{2}{\sin 30°} = 4 \text{ m}$$

For sphere

$$a = \frac{g\sin\theta}{1 + \dfrac{I}{mR^2}} = \frac{9.8 \times \dfrac{1}{2}}{1 + 0.4} = 3.5 \text{ m/s}$$

$\therefore \quad v = \sqrt{2as} = \sqrt{2 \times 3.5 \times 4} = 5.29 \text{ m/s}$

$$f = \frac{mg\sin\theta}{1 + mR^2/I} = \frac{3 \times 9.8 \times \dfrac{1}{2}}{1 + \left(\dfrac{1}{0.4}\right)} = 4.2 \text{ N}$$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 4}{3.5}} = 1.51\text{s}$$

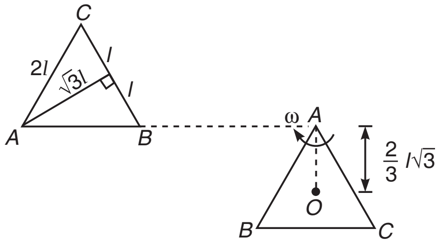
Similarly, the values for disk and hoop can be obtained.

22. $I_A = I_{AB} + I_{AC} + I_{BC}$

$$= \frac{4}{3}ml^2 + \frac{4}{3}ml^2 + \left\{\frac{1}{3}ml^2 + m\,(l\sqrt{3})^2\right\}$$

$$= 6\,ml^2$$

If ω is the angular velocity in the second position, then using conservation of mechanical energy, we have



For COM $\qquad h_i = +\dfrac{\sqrt{3}l}{3}$

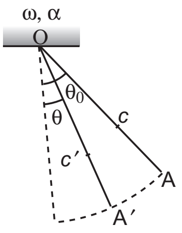
and $\qquad h_f = -\dfrac{2\sqrt{3}}{3}\,l$

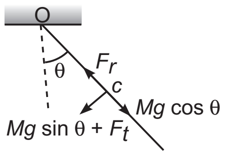
$$3mg\left(\frac{l\sqrt{3}}{3}\right) = \frac{1}{2}(6ml^2)\omega^2 + 3mg\left(\frac{-2l\sqrt{3}}{3}\right)$$

or $\qquad \omega = \sqrt{\dfrac{g\sqrt{3}}{l}}$

Now, velocity of C at this instant is $2l\omega$ or $2\sqrt{gl\sqrt{3}}$ and maximum. **Ans.**

23. (*i*) C is the centre of mass of the rod. Let ω be the angular speed of rod about point O at angle θ. From conservation of mechanical energy,





$$Mg\,\frac{L}{2}(\cos\theta - \cos\theta_0) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

$$\therefore \qquad \omega^2 = \frac{3g}{L}(\cos\theta - \cos\theta_0) \qquad \ldots\text{(i)}$$

Now, $F_r - Mg\cos\theta = M\left(\dfrac{L}{2}\right)\omega^2$ ...(ii)

From Eqs. (i) and (ii), we get

$$F_r = \frac{1}{2}Mg\,(5\cos\theta - 3\cos\theta_0)$$ **Hence proved.**

(*ii*) Angular acceleration of rod at this instant,

$$\alpha = \frac{\tau}{I} = \frac{Mg\,\frac{L}{2}\sin\theta}{\frac{ML^2}{3}} = \frac{3}{2}\,\frac{g\sin\theta}{L}$$

Tangential acceleration of COM,

$$a_t = (\alpha)\left(\frac{L}{2}\right) = \frac{3}{4}g\sin\theta \qquad \ldots\text{(iii)}$$

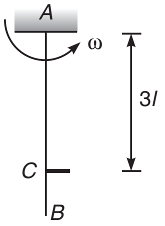
Now, $F_t + Mg\sin\theta = Ma_t$ ...(iv)

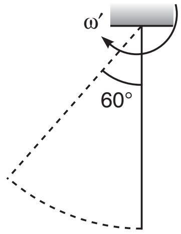
From Eqs. (iii) and (iv), we get

$$F_t = -\frac{1}{4}Mg\sin\theta$$ **Hence proved.**

Here negative sign implies that direction of F_t is opposite to the component $Mg\sin\theta$.

24. (a) From conservation of mechanical energy.





$$(3m)(g)(2l) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left[\frac{(3m)(4l)^2}{3}\right]\omega^2 = 8ml^2\omega^2$$

$$\therefore \qquad \omega = \frac{1}{2}\sqrt{\frac{3g}{l}}$$

Applying,

angular impulse = change in angular momentum

$$J(3l) = I\omega$$

or $$3Jl = (16ml^2)\left(\frac{1}{2}\sqrt{\frac{3g}{l}}\right)$$

$$\therefore \qquad J = \frac{8}{3}ml\sqrt{\frac{3g}{l}}$$

or $$J = \frac{8}{3}m\sqrt{3gl}$$ **Ans.**

(b) Let ω' be the angular speed in opposite direction. Again applying conservation of mechanical energy,

$$(3m)(g)(l) = \frac{1}{2}I(\omega')^2 = 8ml^2(\omega')^2$$

$$\therefore \qquad \omega' = \frac{1}{2\sqrt{2}}\sqrt{\frac{3g}{l}}$$

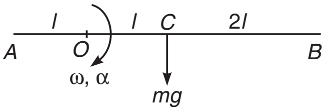
Now, applying, angular impulse = change in angular momentum

$$\therefore\ J(3l) = I(\omega + \omega') = (16ml^2)\frac{1}{2}\sqrt{\frac{3g}{l}}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$\therefore \qquad J = \frac{4}{3}m\sqrt{6gl}\,(\sqrt{2} + 1)$$ **Ans.**

25. $$\alpha = \frac{mgl}{\frac{m(4l)^2}{12} + ml^2} = \frac{3}{7}\cdot\frac{g}{l}$$

$$\therefore \qquad (a_C)_V = l\alpha = \frac{3}{7}g \qquad \text{(downwards)}$$



Let V be the vertical reaction (upwards) at axis, then

$$mg - V = ma_C = \frac{3mg}{7}$$

$$\therefore \qquad V = \frac{4}{7}mg \qquad \ldots\text{(i)}$$

If H be the horizontal reaction (towards CO) at axis, then

$$H = ml\omega^2 \qquad \ldots\text{(ii)}$$

$\therefore$ Total reaction at axis,

$$N = \sqrt{H^2 + V^2} = \frac{4}{7}mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$$ **Ans.**

(b) $$a_C = \sqrt{(a_C)_V^2 + (l\omega^2)^2} = \sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$$ **Ans.**

(c) Let ω' be the angular speed of the rod when it becomes vertical for the first time. Then from conservation of mechanical energy,

$$\frac{1}{2}I({\omega'}^2 - \omega^2) = mgl$$

$$\therefore \qquad \omega'^2 = \omega^2 + \frac{2mgl}{I}$$

$$= \omega^2 + \frac{2mgl}{\frac{7}{3}ml^2}$$

$$= \omega^2 + \frac{6}{7}\frac{g}{l}$$

Acceleration of centre of mass at this instance will be,

$$a_C = l\omega'^2 = l\omega^2 + \frac{6g}{7} \qquad \textbf{Ans.}$$

Let V be the reaction (upwards) at axis at this instant, then,

$$V - mg = ma_C = ml\omega^2 + \frac{6mg}{7}$$

$$\therefore \qquad V = \frac{13}{7}mg + ml\omega^2 \qquad \textbf{Ans.}$$

(d) From conservation of mechanical energy,

$$mgl = \frac{1}{2}I\omega_{\min}^2$$

$$\therefore \qquad \omega_{\min} = \sqrt{\frac{2mgl}{I}} = \sqrt{\frac{2mgl}{\frac{7}{3}ml^2}}$$

$$= \sqrt{\frac{6}{7}\frac{g}{l}} \qquad \textbf{Ans.}$$

26. Linear momentum, angular momentum and kinetic energy are conserved in the process.
From conservation of linear momentum,

$$Mv' = mv$$

$$\text{or} \qquad v' = \frac{m}{M}v \qquad \text{...(i)}$$

Conservation of angular momentum gives,

$$mvd = \left(\frac{Ml^2}{12}\right)\omega$$

$$\text{or} \qquad \omega = \left(\frac{12mvd}{Ml^2}\right) \qquad \text{...(ii)}$$

Collision is elastic. Hence,

$$e = 1$$

or relative speed of approach

= relative speed of separation

$$\therefore \qquad v = v' + d\omega$$

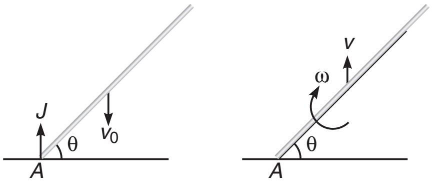
Substituting the values, we have

$$v = \frac{m}{M}v + \frac{12mvd^2}{Ml^2}$$

Solving it, we get

$$m = \frac{Ml^2}{12d^2 + l^2} \qquad \textbf{Ans.}$$

27. Let v = linear velocity of rod after impact (upwards),



At the time of impact After impact

ω = angular velocity of rod

and J = linear impulse at A during impact

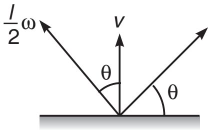
$$\text{Then,} \qquad J = \Delta P = P_f - P_i$$

$$J = mv - (-mv_0)$$

$$\therefore \qquad J = m(v + v_0) \qquad \text{...(i)}$$

Angular impulse = ΔL

$$\therefore \qquad J\left(\frac{l}{2}\cos\theta\right) = I\omega = \frac{ml^2}{12}\omega \qquad \text{...(ii)}$$



Collision is elastic ($e = 1$)

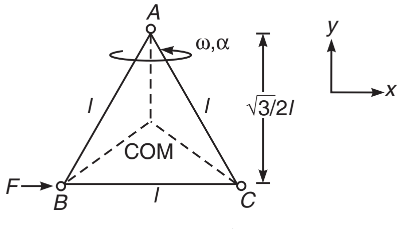
∴ Relative speed of approach = Relative speed of separation at point of impact

$$v_0 = v + \frac{l}{2}\omega\cos\theta \qquad \text{...(iii)}$$

Solving above equations, we get

$$\omega = \frac{6v_0\cos\theta}{l(1 + 3\cos^2\theta)} \qquad \textbf{Ans.}$$

28. (a) The distance of centre of mass (COM) of the system about point A will be



$$r = \frac{l}{\sqrt{3}}$$

Therefore, the magnitude of horizontal force exerted by the hinge on the body is

$$F = \text{centripetal force}$$

$$\text{or} \qquad F = (3m)r\omega^2$$

or $$F = (3m)\left(\frac{l}{\sqrt{3}}\right)\omega^2$$

or $$F = \sqrt{3}\, ml\omega^2$$

(b) Angular acceleration of system about point A is

$$\alpha = \frac{\tau_A}{I_A} = \frac{(F)\left(\frac{\sqrt{3}}{2}l\right)}{2ml^2} = \frac{\sqrt{3}\,F}{4ml}$$

Now, acceleration of COM along x-axis is

$$a_x = r\alpha = \left(\frac{l}{\sqrt{3}}\right)\left(\frac{\sqrt{3}\,F}{4ml}\right) \text{ or } a_x = \frac{F}{4m}$$

Now, let F_x be the force applied by the hinge along x-axis.

Then, $$F_x + F = (3m)a_x$$

or $$F_x + F = (3m)\left(\frac{F}{4m}\right)$$

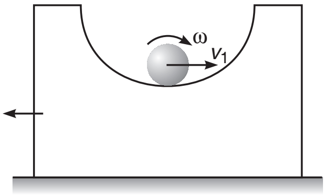
or $$F_x + F = \frac{3}{4}F \text{ or } F_x = -\frac{F}{4}$$

Further if F_y be the force applied by the hinge along y-axis. Then,

$$F_y = \text{centripetal force}$$

or $$F_y = \sqrt{3}\, ml\omega^2$$

29. From conservation of linear momentum



$$mv_1 = Mv_2 \qquad \ldots\text{(i)}$$

∴ Velocity of cylinder axis relative to block

$$v_r = v_1 + v_2 \qquad \ldots\text{(ii)}$$

Applying conservation of mechanical energy,

$$mg(R - r) = \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_2^2 \qquad \ldots\text{(iii)}$$

Here, $I = \frac{1}{2}mr^2$ and $\omega = \frac{v_r}{r}$

Solving the above equations with given data, we get

$$v_1 = 2.0 \text{ m/s}$$

and $$v_2 = 1.5 \text{ m/s}$$ **Ans.**

Further, $$N - mg = \frac{mv_r^2}{R - r}$$

$$\therefore N = mg + \frac{mv_r^2}{R - r} = (0.5)(10) + \frac{(0.5)(3.5)^2}{0.525} = 16.67 \text{ N}$$ **Ans.**

30. Given $\mu > \tan\alpha \Rightarrow \mu mg\cos\alpha > mg\sin\alpha$

$$a = (\mu g\cos\alpha - g\sin\alpha)$$

$$\alpha = \frac{(\mu mg\cos\alpha)\,r}{\frac{1}{2}mr^2} = \frac{2\mu g\cos\alpha}{r}$$

Slipping will stop when,

$$v = r\omega$$

or $$at = r(\omega_0 - \alpha t)$$

$$\therefore \quad t = \frac{r\omega_0}{a + r\alpha} = \left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)$$

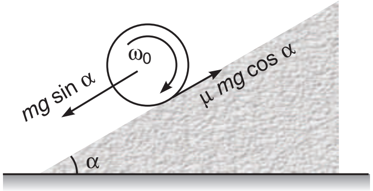
$$d_1 = \frac{1}{2}at^2$$

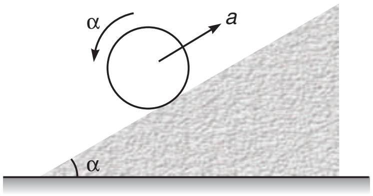
$$= \frac{1}{2}(\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)^2$$

$$= \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$$ **Ans.**

$$v = at = (\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)$$

$$= \frac{r\omega_0(\mu\cos\alpha - \sin\alpha)}{(3\mu\cos\alpha - \sin\alpha)}$$





Once slipping is stopped, retardation in cylinder,

$$a' = \frac{g\sin\alpha}{1 + \frac{I}{mr^2}} = \frac{g\sin\alpha}{1 + \frac{1}{2}} = \frac{2}{3}g\sin\alpha$$

$$d_2 = \frac{v^2}{2a'} = \frac{3r^2\omega_0^2\,(\mu\cos\alpha - \sin\alpha)^2}{(3\mu\cos\alpha - \sin\alpha)^2\,(4g\sin\alpha)}$$

$$\therefore \qquad d_{\max} = d_1 + d_2$$

$$= \frac{r^2\omega_0^2\,(\mu\cos\alpha - \sin\alpha)}{2g\,(3\mu\cos\alpha - \sin\alpha)^2}\left[1 + \frac{3\,(\mu\cos\alpha - \sin\alpha)}{2\sin\alpha}\right]$$

$$= \frac{r^2\omega_0^2\,(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha\,(3\mu\cos\alpha - \sin\alpha)}$$ **Ans.**

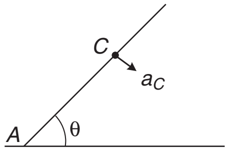
Note *Once slipping was stopped, pure rolling continues if*

$$\mu > \frac{\tan\alpha}{1 + \dfrac{mr^2}{I}}$$

or $\mu > \dfrac{\tan\alpha}{1+2}$ or $\mu > \dfrac{\tan\alpha}{3}$

and already in the question it is given that $\mu > \tan\alpha$. That's why we have taken $a' = \frac{2}{3}g\sin\alpha$.

31. Point A is momentarily at rest.



$$\alpha = \frac{mg\dfrac{l}{2}\cos\theta}{\dfrac{ml^2}{3}} = \frac{3}{2}\frac{g\cos\theta}{l}$$

$$\therefore \qquad a_C = \frac{l}{2}\alpha = \frac{3}{4}g\cos\theta$$

Now $\mu N = ma_x$

or $\mu N = ma_C\sin\theta$

or $\mu N = \dfrac{3}{4}mg\sin\theta\cos\theta$ …(i)

Further, $mg - N = ma_y$

or $N = mg - ma_C\cos\theta$

or $N = mg - \dfrac{3}{4}mg\cos^2\theta$ …(ii)

Dividing Eq. (i) by Eqs. (ii), we have

$$\mu = \frac{\dfrac{3}{4}\sin\theta\cos\theta}{1 - \dfrac{3}{4}\cos^2\theta} = \frac{3\sin\theta\cos\theta}{4 - 3\cos^2\theta}$$

$$= \frac{3\sin\theta\cos\theta}{1 + 3\sin^2\theta}$$ **Hence proved.**

32. Figure (a) and (b)

ω : Decrease in gravitational potential energy = increase in rotational kinetic energy

$$\therefore \quad mg\frac{l}{4}\sin\theta = \frac{1}{2}I_0\omega^2$$

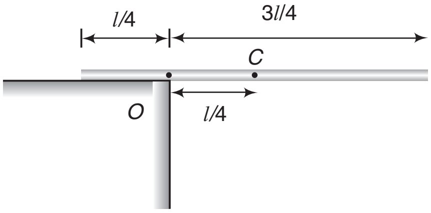
$$= \frac{1}{2}\left[\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right]\omega^2$$

$$\therefore \qquad \omega = \sqrt{\left[\frac{24g\sin\theta}{7l}\right]} \qquad \text{…(i)}$$

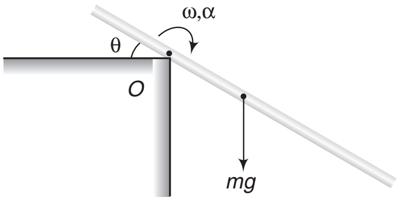
$$\alpha: \qquad \alpha = \frac{\tau}{I} = \frac{mg\dfrac{l}{4}\cos\theta}{\left[\dfrac{ml^2}{12} + m\left(\dfrac{l}{4}\right)^2\right]}$$

$$= \frac{12\,g\cos\theta}{7l} \qquad \text{…(ii)}$$

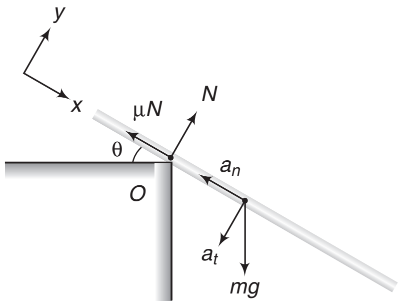
$\Sigma F_y = ma_y$ or $mg\cos\theta - N = ma_t$



(a)



(b)



(c)

or $$N = mg\cos\theta - ma_t$$

$$= mg\cos\theta - m\frac{l}{4}\alpha$$

Substituting value of α from Eq. (ii), we get

$$N = \frac{4}{7}mg\cos\theta \qquad \text{...(iii)}$$

Rod begins to slip when

$$\mu N - mg\sin\theta = ma_n$$

or $$\frac{4}{7}\mu mg\cos\theta - mg\sin\theta = m\frac{l}{4}\omega^2$$

Substitution value of ω from Eq. (i), we get

$$\tan\theta = \frac{4\mu}{13}$$

$\therefore$ $$\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$$ **Ans.**

33. Writing equations of motion, we get

$$5g - T_2 = 5a_2 \qquad \text{...(i)}$$

$$\frac{(T_2 - T_1)R_2}{\frac{1}{2}M_2R_2^2} = \alpha_2 \qquad \text{...(ii)}$$

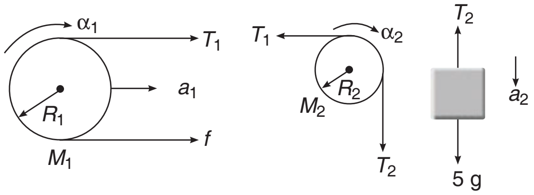
$$T_1 + f = M_1a_1 \qquad \text{...(iii)}$$

$$\frac{(T_1 - f)R_1}{\frac{1}{2}M_1R_1^2} = \alpha_1 \qquad \text{...(iv)}$$

$$a_1 = R_1\alpha_1 \qquad \text{...(v)}$$

$$a_1 + R_1\alpha_1 = R_2\alpha_2 \qquad \text{...(vi)}$$

$$R_2\alpha_2 = a_2 \qquad \text{...(vii)}$$



We have seven unknowns, $T_1, T_2, a_1, a_2, \alpha_1, \alpha_2$ and f solving above equations, we get

$$a_2 = \frac{4}{11}g = 3.6 \text{ m/s}^2$$ **Ans.**

$$v = a_2t = \frac{4gt}{11}$$ **Ans.**

34. Equations of motion are,

$$F + f_1 = Ma_1 \qquad \text{...(i)}$$

$$f_1 + f_2 = Ma_2 \qquad \text{...(ii)}$$

$$2F - f_2 = 2Ma_3 \qquad \text{...(iii)}$$

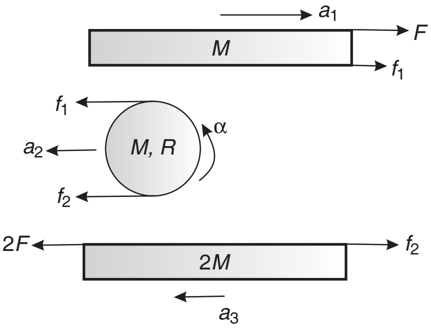
$$\alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}MR^2}$$

or $$\alpha = \frac{2(f_1 - f_2)}{MR} \qquad \text{...(iv)}$$

For no slipping condition,

$$a_2 + R\alpha = -a_1 \qquad \text{...(v)}$$

and $$a_2 - R\alpha = a_3 \qquad \text{...(vi)}$$



We have six unknowns, f_1, f_2, a_1, a_2, a_3 and α. Solving the above six equations, we get

$$a_1 = \frac{21F}{26M} \quad \text{and} \quad a_2 = \frac{F}{26M}$$ **Ans.**

35. **Angular velocity** From conservation of mechanical energy,

decrease in gravitational PE

= increase in rotational KE

or $mgr\sin 60° + mg\,(2r\sin 60°)$

$$= \frac{1}{2}\left[\frac{3mr^2}{2} + m\,(2r)^2\right]\omega^2$$

$\therefore$ $$\frac{3\sqrt{3}mgr}{2} = \frac{11}{4}mr^2\omega^2$$

$\therefore$ $$\omega = \sqrt{\frac{6g\sqrt{3}}{11r}}$$ **Ans.**

Angular acceleration

$$\alpha = \frac{\tau}{I} = \frac{mgr\cos 60° + mg\,(2r\cos 60°)}{\left[\frac{3mr^2}{2} + m\,(2r)^2\right]}$$

$$= \frac{\frac{3mgr}{2}}{\frac{11}{2}mr^2} = \frac{3g}{11r}$$ **Ans.**

36. $\mathbf{a}_B = \mathbf{a}_0 + \mathbf{a}_{B/0}$

Here, $\mathbf{a}_{B/0}$ has two components a_t (tangential acceleration) and a_n (normal acceleration)

$$a_t = r\alpha = (0.3)(5) = 1.5 \text{ m/s}^2$$

$$a_n = r\omega^2 = (0.3)(4)^2 = 4.8 \text{ m/s}^2$$

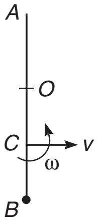
and $$a_0 = 2 \text{ m/s}^2$$

$$\therefore \quad a_B = \sqrt{(\Sigma a_x)^2 + (\Sigma a_y)^2}$$

$$= \sqrt{(2 + 4.8 \cos 45° - 1.5 \cos 45°)^2 + (4.8 \sin 45° + 1.5 \sin 45°)^2}$$

$$= 6.21 \text{ m/s}^2$$

37. C is the COM of $(M + m)$



$$BC = \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right)$$

and $$OC = \left(\frac{m}{M+m}\right)\left(\frac{l}{2}\right)$$

From conservation of linear momentum,

$$(M + m)\, v = m v_0$$

or $$v = \left(\frac{m}{M+m}\right) v_0 \qquad \ldots\text{(i)}$$

From conservation of angular momentum about point C we have,

$$m v_0 (BC) = I\omega$$

or $$\frac{m M v_0 l}{2\,(M+m)} = \left[m \left(\frac{M}{M+m}\right)^2 \left(\frac{l}{4}\right)^2 + \frac{Ml^2}{12} + M \left(\frac{m}{M+m}\right)^2 \left(\frac{l^2}{4}\right) \right] \omega$$

Putting $\dfrac{m v_0}{M+m} = v$

from Eq. (i), we have

$$\frac{v}{\omega} = \frac{l}{6}\left[\frac{4m + M}{M+m}\right]$$

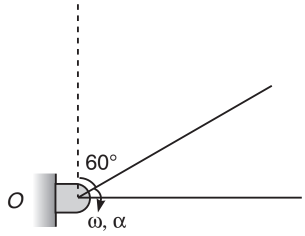
Now, a point (say P) at a distance $x = \dfrac{v}{\omega}$, from C (towards O) will be at rest. Hence, distance of point P from boy at B will be

$$BP = BC + x$$

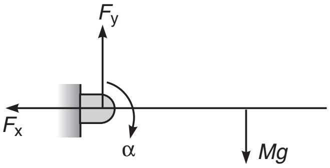
$$= \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right) + \frac{l}{6}\left[\frac{4m+M}{M+m}\right]$$

$$= \frac{2l}{3}$$ **Ans.**

38. Let ω be the angular velocity and α the angular acceleration of rod in horizontal position. Then



$$\alpha = \frac{(Mg)\dfrac{l}{2}}{\dfrac{Ml^2}{3}} = \frac{3}{2}\frac{g}{l} \qquad \ldots\text{(i)}$$



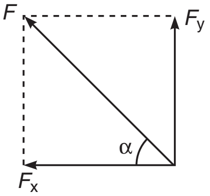
$$\frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2 = Mg\,\frac{l}{4}$$

$$\therefore \quad \omega = \frac{3}{2} \cdot \frac{g}{l} \qquad \ldots\text{(ii)}$$

$$F_x = M\left(\frac{l}{2}\right)\omega^2 = M\left(\frac{l}{2}\right)\left(\frac{3g}{2l}\right) = \frac{3}{4}Mg$$

$$Mg - F_y = M(\alpha)\left(\frac{l}{2}\right)$$

$$F_y = Mg - \frac{3}{4}Mg = \frac{Mg}{4}$$



$$\therefore \quad F = \sqrt{F_x^2 + F_y^2}$$

$$= \frac{\sqrt{10}}{4} Mg$$ **Ans.**

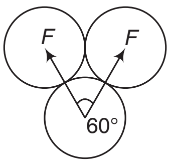
$$\tan \alpha = \frac{F_y}{F_x} = \frac{(Mg/4)}{(3Mg/4)} = \frac{1}{3}$$

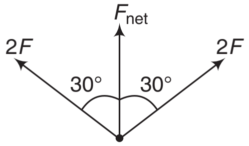
$$\therefore \quad \alpha = \tan^{-1}\left(\frac{1}{3}\right)$$ **Ans.**

13. Gravitation

INTRODUCTORY EXERCISE 13.1

1. $F_{\text{net}} = \sqrt{F^2 + F^2 + 2FF\cos 60°}$

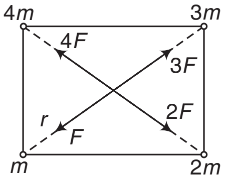


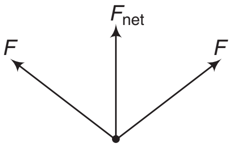


$$= \sqrt{3}\,F$$

$$= \sqrt{3}\,\frac{GMM}{(2a)^2} = \frac{\sqrt{3}\,GM^2}{4a^2}$$

2. $F_{\text{net}} = \sqrt{2}\,(2F)$





$$= 2\sqrt{2}\,\frac{G.m.m}{r^2}$$

$$= \frac{2\sqrt{2}\,Gm^2}{(a/\sqrt{2})^2} = \frac{4\sqrt{2}\,Gm^2}{a^2}$$

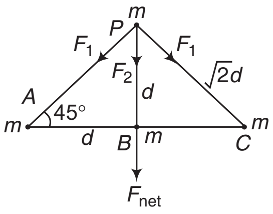
3. $a_1 = \frac{F}{m_1} = \frac{Gm_1m_2}{r^2} = \frac{Gm_2}{r^2}$

$$= \frac{(6.67\times10^{-11})\,(2)}{(0.5)^2}$$

$$= 5.3\times10^{-10}\ \text{m/s}^2$$

Similarly, $a_2 = \frac{Gm_1}{r^2}$

4. $F_{\text{net}} = 2(F_1\cos 45°) + F_2$



$$= \sqrt{2}\,F_1 + F_2 = \frac{(\sqrt{2})\,Gmm}{(\sqrt{2}d)^2} + \frac{Gmm}{d^2}$$

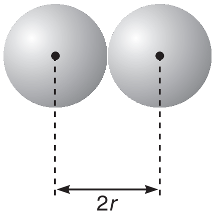
$$= \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)\frac{Gm^2}{d^2} \qquad \text{(Along } PB\text{)}$$

5. $m_1 = m_2 = (\text{volume})(\text{density})$

$$= \left(\frac{4}{3}\pi r^3\right)\rho$$

$$\therefore \quad F = \frac{Gm_1m_2}{r^2}$$

$$= \frac{G\left(\frac{4}{3}\pi r^3\right)\left(\frac{4}{3}\pi r^3\right)\rho^2}{r^2}$$



or $\quad F \propto r^4 \qquad$ **Hence proved.**

INTRODUCTORY EXERCISE 13.2

1. $g = \frac{GM}{R^2}$ or $g \propto \frac{M}{R^2}$

Mass and radius both are two times. Therefore, value of g is half.

2. (a) $g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$

At $h = R,\ g' = \frac{g}{4}$

(d) $g' = g\left(1 - \frac{d}{R}\right)$

At $d = \frac{R}{2},\ g' = \frac{g}{2}$

3. $\frac{g}{\left(1+\frac{h}{R}\right)^2} = g\left(1-\frac{h}{R}\right)$

Solving this equation, we get

$$h = \left(\frac{\sqrt{5}-1}{2}\right)R$$

4. $\Delta g = g' - g$

$$= -R\omega^2\cos^2\theta$$

$$= -(6.37\times10^6)\left(\frac{2\pi}{24\times3600}\right)^2\cos^2 45°$$

$$= -0.0168\ \text{m/s}^2$$

5. $g' = 0.64\,g = \frac{g}{\left(1+\frac{h}{R}\right)^2}$

$$\therefore \quad 1 + \frac{h}{R} = \frac{5}{4}$$

$$h = \frac{R}{4} = 1600\ \text{km}$$

6. $g' = \frac{3}{5}g = g - R\omega^2 \cos^2 0°$

$\therefore \qquad R\omega^2 = \frac{2}{5}g$

or $\qquad \omega = \sqrt{\frac{2g}{5R}} = \sqrt{\frac{2 \times 9.8}{5 \times 6400 \times 10^3}}$

$= 7.8 \times 10^{-4}$ rad/s

7. At equator, $g' = g - R\omega^2$

$$= (9.8)^2 - (6400 \times 10^3)\left(\frac{2\pi}{24 \times 3600}\right)^2$$

$$= 9.766 \text{ m/s}^2$$

Now, $mg = 1000$

$\therefore \qquad m = \frac{1000}{g}$

$w' = mg' = (1000)\left(\frac{g'}{g}\right)$

$= \frac{(1000)\,(9.766)}{(9.8)} = 997$ N

8. $g' = g - R\omega^2$

$\therefore \qquad 0 = g - R\omega^2$

$\therefore \qquad \omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{9.8}{6400 \times 1000}}$

$= 1.237 \times 10^{-3}$ rad/s and $T = \frac{2\pi}{\omega}$

By putting the value of ω, we get

$T \approx 84.6$ min

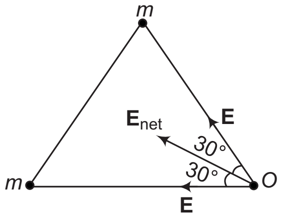
9. $\frac{g}{\left(1 + \frac{h}{R}\right)} = g - R\omega^2$

By putting the values of g, h, R and ω, we get

$h \approx 10$ km

INTRODUCTORY EXERCISE 13.3

1. $V = -\frac{Gm}{a} - \frac{Gm}{a} = -\frac{2Gm}{a}$



$E = \frac{Gm}{a^2}$

$E_{\text{net}} = \sqrt{E^2 + E^2 + 2(E)(E)\cos 60°} = \sqrt{3}E$

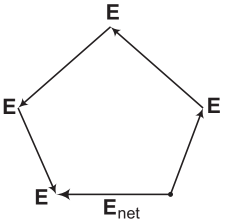
$= \frac{\sqrt{3}Gm}{a^2}$

2. $V = 5\left(\frac{-Gm}{a}\right) = -\frac{5Gm}{a}$

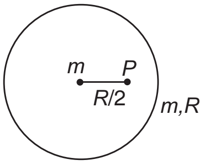
For E Five vectors of equal magnitudes, when added as per polygon law of vector addition make a closed regular pentagon. Hence, net **E** is zero.

3. $V = 4\left(-\frac{Gm}{a}\right) = -4\frac{Gm}{a}$

For **E**



$\mathbf{E}_{\text{net}} = \mathbf{E} = \frac{Gm}{a^2}$



4. V_p = due to particle + due to shell

$= -\frac{Gm}{R/2} - \frac{Gm}{R}$

$= -\frac{3Gm}{R}$

5. $\mathbf{E} = \frac{\mathbf{F}}{m} = \frac{4\hat{\mathbf{i}}}{20 \times 10^{-3}} = (200\,\hat{\mathbf{i}})$ N/kg

INTRODUCTORY EXERCISE 13.4

1. $\mathbf{E} = -\left[\frac{\partial V}{\partial X}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$

2. $|\mathbf{E}| = \sqrt{\left(-\frac{\partial V}{\partial x}\right)^2 + \left(-\frac{\partial V}{\partial y}\right)^2 + \left(\frac{-\partial V}{\partial z}\right)^2}$

$-\frac{\partial V}{\partial x} = 10$ J/kg-m is given

No information is given about

$-\frac{\partial V}{\partial y}$ and $-\frac{\partial V}{\partial z}$

So, $\qquad |\mathbf{E}| \geq 10$ N/kg

3. $\mathbf{F} = m\mathbf{E} = m\left[\left(-\frac{\partial V}{\partial x}\right)\hat{\mathbf{i}} + \left(-\frac{\partial V}{\partial y}\right)\hat{\mathbf{j}}\right]$

$= -(10\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}})$ N or $|\mathbf{F}| = 10\sqrt{2}$ N

4. $\mathbf{F} = m\mathbf{E} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ N/kg

We can check that path given (therefore displacement) is perpendicular to force.

$\therefore \qquad W = 0$

INTRODUCTORY EXERCISE 13.5

1. $K_i + U_i = K_f + U_f$

$$\therefore\ 0 - \frac{Gm_1m_2}{r_i} = \frac{1}{2} \times 10 \times (2v)^2 + \frac{1}{2} \times 20\, v^2 - \frac{Gm_1m_2}{r_f}$$

$$\therefore \qquad v = \sqrt{\frac{Gm_1m_2}{30}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 10 \times 20}{30}\left(\frac{1}{0.5} - \frac{1}{1}\right)}$$

$$= 2.1 \times 10^{-5} \text{ m/s}$$

and $\qquad 2v = 4.2 \times 10^{-5}$ m/s

2. Total pairs $= \dfrac{4(4-1)}{2} = 6$. Four pairs are at a distance of a and two pairs at distance of $\sqrt{2}a$.

3. $\Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Put $\qquad h = R$

4. *(i)* $\Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Putting $h = nR$, we get

$$\Delta U = \left(\frac{n}{n+1}\right) mgR$$

(ii) $\Delta U = \left(\dfrac{n}{n+1}\right) mgR$

$$= \frac{1}{2} mv^2$$

$$\therefore \qquad v = \sqrt{\frac{2ngR}{n+1}}$$

5. $h = \dfrac{v^2}{2g - (v^2/R)}$

$$= \frac{(10^4)^2}{2 \times 9.8 - \dfrac{(10^4)^2}{6.4 \times 10^6}}$$

$$= 2.51 \times 10^7 \text{ m}$$

$$= 2.51 \times 10^4 \text{ km}$$

6. Apply $\dfrac{1}{2} mv^2 = \dfrac{mgh}{1 + \dfrac{h}{R}}$

INTRODUCTORY EXERCISE 13.6

1. $\text{KE} = \dfrac{1}{2} mv_e^2$

$$= \frac{1}{2} m \left(\frac{2GM}{R}\right)$$

$$= \frac{m}{R}(gR^2) \qquad \text{(as } GM = gR^2\text{)}$$

$$= mgR$$

2. $v_e = \sqrt{\dfrac{2GM}{R}}$

or $\qquad v_e \propto \sqrt{\dfrac{M}{R}}$

3. (a) Total mechanical energy $= E_0 - 2E_0 = -E_0$.

Since, it is negative, it will not escape to infinity.

(b) $E_i = E_f \Rightarrow E_0 - 2E_0 = 0 + U \Rightarrow U = -E_0$

INTRODUCTORY EXERCISE 13.7

1. Delhi does not lie on equator.

2. $T \propto r^{3/2}$

$$\therefore \qquad \left(\frac{T_2}{\pi}\right) = \left(\frac{r_2}{r_1}\right)^{3/2}$$

or $\qquad T_2 = \left(\dfrac{r_2}{r_1}\right)^{3/2} T_1$

$$= \left(\frac{2 \times 10^4}{10^4}\right)^{3/2} (28) = 56\sqrt{2} \text{ h}$$

3. $r_1 = R + h_1 = R + R = 2R$

$r_2 = R + h_2 = R + 3R = 4R$

KE and PE $\propto \dfrac{1}{r}$

$$\therefore \qquad \frac{K_1}{K_2} = \frac{U_1}{U_2} = \frac{r_2}{r_1} = \frac{2}{1}$$

4. (a) $v = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{GM}{R+h}}$

(b) $\text{KE} = \dfrac{GMm}{2r} = \dfrac{GMm}{2(R+h)}$

(c) $\text{PE} = -\dfrac{GMm}{r} = -\dfrac{GMm}{(R+h)}$

(d) $T = \dfrac{2\pi}{\sqrt{GM}} r^{3/2} = \dfrac{2\pi}{\sqrt{GM}} (R+h)^{3/2}$

5. (a) W = Energy of satellite in first orbit – energy of satellite on the surface of earth

$$= -\frac{GMm}{2\,(2R)} - \left(-\frac{GMm}{R}\right)$$

$$= \frac{3}{4}\frac{GMm}{R}$$

$$= \frac{3}{4}mgR = \frac{3}{4} \times 2 \times 10^3 \times 10 \times 6.4 \times 10^6$$

$$= 9.6 \times 10^{10}\ \text{J}$$

(b) W = energy of satellite in second orbit – energy in first orbit

$$= -\frac{GMm}{2\,(3R)} - \left[\frac{-GMm}{2\,(2R)}\right]$$

$$= \frac{1}{12}\frac{GMm}{R} = \frac{1}{12}mgR$$

$$= \frac{1}{12} \times 2 \times 10^3 \times 10 \times 6.4 \times 10^6$$

$$= 1.07 \times 10^{10}\ \text{J}$$

Exercises

LEVEL 1

Assertion and Reason

1. $U = -\dfrac{G\,m_1m_2}{r}$

If r decreases, U also decreases.

2.

$$V_C = -1.5\frac{GM}{R}$$

$$V_S = -\frac{GM}{R}$$

$$E_C = 0 \quad \text{and} \quad E_S = \frac{GM}{R^2}$$

$C \to$ centre, $S \to$ surface.

3. If a mass m is displaced from centre along the line AB, force on it is away from the centre. So, it is in unstable equilibrium position at centre. So, potential energy and hence the potential at centre is maximum.

4. $\mathbf{E} = -\left[\dfrac{\partial V}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial V}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$

$\dfrac{\partial V}{\partial X} = 0$ does not mean field strength is zero. Because it also depends on $\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$.

5. $h = \dfrac{u^2}{2g}$ or $h \propto u^2$ cannot be applied in this case. Because, for higher values of v, acceleration due to gravity g does not remain constant.

6. Force on planet is towards centre of sum. Hence, torque is zero only about centre of sun.

7. Polar satellites don't have equatorial plane.

8. Earth's gravity is utilised in providing the necessary centripetal force. But weight is felt due to moon's weight is felt due to moon's gravity.

9. Geostationary satellites lies about equator and Moscow does not lies over equator.

10. $V = -\dfrac{GMm}{r}$, $K = \dfrac{GMm}{2r}$, $E = -\dfrac{GMm}{2r}$

and $V = \sqrt{\dfrac{GM}{r}}$

11. By changing the radius, moment of inertia will change. Hence, angular speed ω will change. But ω has no effect on the value of g on pole.

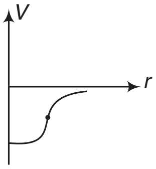
Single Correct Option

2. Angular momentum is conserved only about centre of sun.

3. $g' = g - R\omega^2\cos^2\phi$

At $\phi = 90°$, $g' = g$, independent of ω.

5. V is negative.



6. Comparing with

$$\oint \mathbf{E}\cdot d\mathbf{S} = \left(\frac{q}{\varepsilon_0}\right)$$

We have, $\dfrac{1}{4\pi\,\varepsilon_0} \equiv G$

$\Rightarrow \quad \dfrac{1}{\varepsilon_0} \equiv 4\pi G$

$\Rightarrow \quad q \equiv m$

8. $E = 0$ inside a sheet. Therefore, gravitational force on m is zero.

9. $T \propto r^{3/2}$

10. Area velocity remains constant. If area is half time taken is half.

11. $T \propto \frac{1}{\sqrt{g}}$

$$\frac{T_2}{T_1} = \sqrt{\frac{g_1}{g_2}} \quad \text{or} \quad \frac{1.05\,T}{T} = \sqrt{\frac{g}{g\left(1 - \frac{h}{R}\right)}}$$

Solving this equation we get, $h = 64$ km

12. $g = \frac{GM}{R^2} \;\Rightarrow\; \therefore\; \frac{g}{G} = \frac{M}{R^2}$

13. $E \propto \frac{1}{r^2}$

$$\frac{E_2}{E_1} = \left(\frac{r_1}{r_2}\right)^2$$

or $$\frac{1}{100} = \left(\frac{R_e}{R_e + h}\right)^2$$

or $$h = 9\,R_e$$

14. $T \propto r^{3/2}$

$$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = \left(\frac{3R + R}{R}\right)^{3/2} = 8$$

∴ $$T_2 = 8T_1$$

15. $g_{\text{equator}} = g - R\omega^2$ and $g_{\text{pole}} - R\omega^2$

∴ $$\frac{g_{\text{pole}}}{2} \quad \text{or} \quad \frac{g}{2} = g - R\omega^2$$

or $$\omega \sqrt{\frac{g}{2R}}$$

$$= \sqrt{\frac{9.8}{2 \times 6.4 \times 10^6}}$$

$$= 8.75 \times 10^{-4} \text{ rad/s}$$

16. $U = mV$

or $$U \propto V$$

U is half, it means gravitational potential is half.

$$-\frac{GM}{(R + h)} = \frac{1}{2}\left[-\frac{3}{2}\frac{GM}{R}\right]$$

Solving we get, $h = \frac{R}{3}$ **Ans.**

17. $\Delta U = \dfrac{mgh}{1 + \frac{h}{R}}$

18. $T = \sqrt{\dfrac{GM}{r}}$

T is independent of R, the radius of earth.

19. $g = \dfrac{GM}{R^2} = \dfrac{G\left(\frac{4}{3}\pi R^3 \rho\right)}{R^2}$

or $$g \propto \rho R$$

or $$g_1 = g_2$$

∴ $$\rho_1 R_1 = \rho_2 R_2$$

or $$R_2 = \frac{\rho_1}{\rho_2} \cdot R_1 = \left(\frac{1}{2}\right) R = \frac{R}{2}$$ **Ans**

20. At 84.6 min, g at equator becomes zero.

$$\frac{\omega_2}{\omega_1} = \frac{2\pi/T_2}{2\pi/T_1} = \frac{T_1}{T_2}$$

$$= \frac{24 \times 60}{84.6} = 17$$ **Ans.**

21. $\omega_1\, t + \omega_2 t = 2\pi$

$$(\omega_1)\, 6 + \left(\frac{2\pi}{23}\right)^6 = 2\pi$$

or $$6\omega_1 = \frac{3\pi}{2} \quad \text{or} \quad \omega_1 = \left(\frac{\pi}{4}\right) \text{rad/h}$$

23. Only due to M_1.

24. Time taken in one complete oscillations is 84.6 min.

25. $W = \Delta U = U_f - V_i$

$$= 3\left(-\frac{GMM}{2l}\right) - 3\left(-\frac{GMM}{l}\right)$$

$$= \frac{3}{2}\frac{GM^2}{l}$$

26. Change in kinetic energy = change in potential energy

∴ $$\frac{1}{2} mv^2 = -\frac{GMm}{R + R} - \left(-\frac{GMm}{R}\right)$$

$$v = \sqrt{\frac{GM}{R}} = \frac{\sqrt{2GM/R}}{\sqrt{2}}$$

$$= \frac{v_e}{\sqrt{2}}$$

where, v_e = escape velocity

27. $U = \dfrac{-GMm}{r}$

∴ $$(-GM) = \frac{Ur}{m}$$

$$E = -\frac{GM}{(2r)^2} = \frac{(Ur/m)}{4r^2} = \frac{U}{4mr}$$

$$F = mE = \frac{U}{4r}$$

Subjective Questions

1. Potential at the surface of sphere,

$$V = -\frac{GM}{R}$$

$$= -\frac{(6.67\times10^{-11})(20)}{1}\ \text{J/kg}$$

$$= -1.334\times10^{-9}\ \text{J/kg}$$

i.e. 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus,

$$W = 1.334\times10^{-9}\ \text{J}$$

2. $g = \dfrac{GM}{R^2}$

$$\Rightarrow \quad \frac{dg}{dR} = \frac{-2GM}{R^3}$$

$$\Rightarrow \quad \frac{dg}{h} = \frac{-2GM}{R^2}\cdot\frac{1}{R}$$

$$\Rightarrow \quad \frac{dg}{g} = -2\left(\frac{h}{R}\right)$$

3. Let v_r be their velocity of approach. From conservation of energy,

Increase in kinetic energy

= decrease in gravitational potential energy

$$\text{or} \quad \frac{1}{2}\mu v_r^2 = \frac{Gm_1m_2}{r} \qquad \text{...(i)}$$

Here, μ = reduced mass

$$= \frac{m_1m_2}{m_1+m_2}$$

Substituting in Eq. (i), we get

$$v_r = \sqrt{\frac{2G(m_1+m_2)}{r}}$$

4. Close to earth,

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R}{GM/R^2}}$$

$$\text{or} \quad T^2 = \frac{4\pi^2R^3}{GM}$$

$$= \frac{4\pi^2R^3}{G\left(\frac{4}{3}\pi R^3\rho\right)}$$

$$\text{or} \quad \rho T^2 = \frac{3\pi}{G} = a \text{ universal constant}$$

5. (a) $v_e = \sqrt{2}\,v_o$. Given $v = \sqrt{1.5}\,v_o$

If $v_o < v < v_e$, satellite will move in elliptical orbit with initial position as the perigee position.

(b) $v = 2v_o > \sqrt{2}\,v_o$ or v_e

Hence, the satellite will escape to infinity.

6. Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2\omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}MR^2\omega = \frac{1}{4}\times\frac{2}{5}MR^2\omega'$$

$$\text{i.e.} \quad \omega' = 4\omega$$

$$T' = \frac{1}{4}T = \frac{1}{4}\times24 = 6\ \text{h}$$

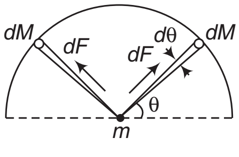
7. Particle lies between two shells. Therefore, net force in exerted only by the inner shell.

$$\therefore \quad F = \frac{Gm_1m}{\left(\frac{R_1+R_2}{2}\right)^2} = \frac{4\,Gm_1m}{(R_1+R_2)^2}$$

8. $L = \pi R$

$$\therefore \quad R = \frac{L}{\pi} \quad \Rightarrow F = \int_0^{\pi} dF\sin\theta$$

$$= \int_0^{\pi}\frac{GdM.m}{R^2}\sin\theta$$



$$= \int_0^{\pi}\frac{G\left(\frac{M}{\pi}d\theta\right)m}{L^2}(\pi^2)\sin\theta$$

$$= \frac{2\pi G\,mM}{L^2}$$

9. $K_i + U_i = K_f + U_f$

$$\therefore \quad \frac{1}{2}m\,(2\sqrt{gR})^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 + 0$$

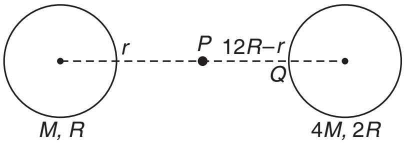
$$\therefore \qquad v^2 = 4gR - \frac{2GM}{R}$$

Putting $GM = gR^2$, we get

$$v^2 = 4gR - 2gR$$

$$\therefore \qquad v = \sqrt{2gR}$$

10. (a) $\dfrac{GM}{r^2} = \dfrac{G\,(4M)}{(12\,R - r)^2}$



Solving this equations, we get $r = 4R$

$$v_P = \frac{v_Q}{2}$$

$$\therefore \qquad \left(-\frac{GM}{r}\right) = \left[\frac{-G\,(4m)}{12\,R - r}\right]$$

$$= \frac{-G\,(4m)}{2R} - \frac{GM}{10\,R}$$

Solving this equation, we get

$$r = 7.65\,R \quad \text{and} \quad 1.49\,R$$

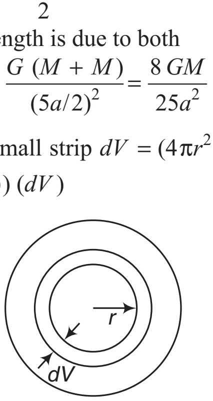
11. (a) At distance $\dfrac{3a}{2}$ which lies between solid sphere and shell field is only due to solid sphere.

$$E = \frac{GM}{(3a/2)^2} = \frac{4GM}{9a^2} \quad \text{(towards centre)}$$

(b) At distance $\dfrac{5a}{2}$ which lies outside the shell. So, field strength is due to both

$$\therefore \quad E = \frac{G\,(M + M)}{(5a/2)^2} = \frac{8\,GM}{25a^2} \quad \text{(towards centre)}$$

12. Volume of small strip $dV = (4\pi r^2)\,dr$ mass of this strip $dm = (\rho)\,(dV)$



$$= \left(\frac{\rho_0 a}{r}\right)(4\pi r^2) dr = (4\pi\; a\rho_0)\,(rdr)$$

$$\therefore \text{ Mass of whole sphere } = \int_0^a (4\pi a \rho_0)\, rdr$$

$$m = (2\pi\; a^3 \rho_0)$$

$$E = \frac{Gm}{(2a)^2} = \frac{G\,(2\pi a^3 \rho_0)}{4a^2}$$

$$= \frac{\pi G \rho_0 a}{2}$$

13. By conservation of momentum, their speeds are same. Using the energy equations we have

$$K_i + U_i = K_f + U_f$$

$$0 - \frac{Gmm}{r_i} = 2\left(\frac{1}{2}\,mv^2\right) - \frac{Gmm}{r_f}$$

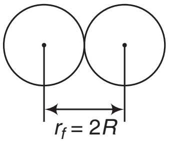
$$\therefore \qquad v = \sqrt{Gm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)} \qquad \ldots\text{(i)}$$

$$= \sqrt{\frac{Gm}{r_i}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 10^{30}}{10^{10}}}$$

$$= 8.16 \times 10^4 \text{ m/s}$$

$$= 81.6 \text{ km/s}$$ **Ans.**



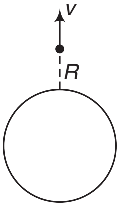
Using Eq. (i), we have

$$v = \sqrt{Gm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$$

$$= \sqrt{6.67 \times 10^{-11} \times 10^{30}\left(\frac{1}{2 \times 10^5} - \frac{1}{10^{10}}\right)}$$

$$= 1.8 \times 10^7 \text{ m/s} = 1.8 \times 10^4 \text{ km/s}$$ **Ans.**

14. Using the equation



$$K_i + U_i = K_f + U_f$$

$$\therefore \quad \frac{1}{2}\,mv^2 - \frac{GmM}{2R} = 0 + 0$$

$$\therefore \qquad v = \sqrt{\frac{GM}{R}}$$

15. (a) $W = U_f - U_i = U_B - U_A$

$$= mV_B - mV_A \qquad (V = \text{gravitational potential})$$

$$= m\,(V_B - V_A)$$

$$= (1)\left[-\frac{100}{2} - \frac{400}{8} + \frac{100}{8} + \frac{400}{2}\right](6.67 \times 10^{-11})$$

$$= 7.5 \times 10^{-9} \text{ J}$$

(b) First let us find the point (say) c between A and B where field strength due to 400 kg and 100 kg masses is zero. Let its distance from 400 kg is r.

Then, $$\frac{G(400)}{r^2} = \frac{G\,(100)}{(10-r)^2}$$

$$r = \frac{20}{3}\text{ m} \quad\text{and}\quad (10-r) = \frac{10}{3}\text{ m}$$

So, we have to move the body only from A to C. After that 100 kg mass will pull 1 kg mass by its own.

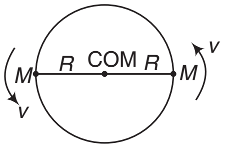
$\therefore$ Minimum kinetic energy $= \Delta U = U_C - U_A$

$$= m\,(V_C - V_A)$$

$$= (1)\left[-\frac{400}{20/3} - \frac{100}{10/3} + \frac{100}{8} + \frac{400}{2}\right](6.67\times10^{-11})$$

$$= 8.17\times10^{-9}\text{ J}$$

16. (a) $$F = \frac{GM.M}{(2R)^2} = \frac{GM^2}{4R^2}$$



(b) $$\frac{GM^2}{4R^2} = \frac{MV^2}{R}$$

$$\therefore\quad v = \sqrt{\frac{GM}{4R}}$$

$$T = \frac{2\pi(R)}{v} = \frac{2\pi R}{\sqrt{GM/4R}} = \frac{4\pi R^{3/2}}{\sqrt{GM}}$$

(c) $W = E_f - E_i = [U_f - U_i] + [K_f - K_i]$

$$= \left[0 - \left(\frac{-GMM}{2R}\right)\right] + \left[0 - 2\times\frac{1}{2}\times M\left(\sqrt{\frac{GM}{4R}}\right)^2\right]$$

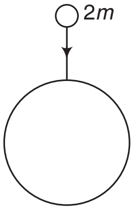
$$= \frac{GM^2}{4R}$$

17. (a) $E = 2$ (energy of one satellite)

$$= 2\left(-\frac{GMm}{2r}\right) = \frac{-GMm}{r}$$

(b) Immediately after collision velocity of combined mass = 0

Path is straight line as shown only potential energy is there.



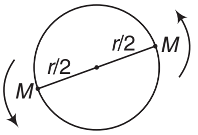
$$v = -\frac{G\,(2m)\,M}{r}$$

$$= -\frac{2GMm}{r}$$

18. $$\frac{2\pi}{\sqrt{GM}}r^{3/2} = 1\text{yr} \qquad \text{...(i)}$$

$$\frac{G\,(M)\,(M)}{r^2} = \frac{Mv^2}{(r/2)}$$

$$\therefore\quad v = \sqrt{\frac{GM}{2r}}$$



$$T = \frac{2\pi\,(r/2)}{v} = \frac{\pi r}{\sqrt{\dfrac{GM}{2r}}}$$

$$= \sqrt{2}\left(\frac{\pi}{\sqrt{GM}}\right)r^{3/2}$$

$$= \sqrt{2}\left(\frac{1}{2}\right)\text{yr} = 0.71\text{ yr} \qquad \left[\text{as } \frac{\pi r^{3/2}}{\sqrt{GM}} = \frac{1}{2}\text{ yr}\right]$$

19. $$W_1 = \Delta U = -\frac{GMm}{(R+h)} + \frac{GMm}{R} = \frac{GMmh}{R\,(R+h)}$$

$$W_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2$$

$$= \frac{1}{2}m\left(\frac{GM}{R+h}\right) = \frac{GMm}{2\,(R+h)}$$

$$W_1 > W_2$$

If $\quad \dfrac{h}{R} > \dfrac{1}{2} \quad$ or $\quad h > \dfrac{R}{2}$

or $\quad h > \dfrac{6370}{2}$ km

or $\quad h > 3185$ km

LEVEL 2

Single Correction Option

1. Just before collision,

$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+R}} = \sqrt{\frac{GM}{2R}}$$

From conservation of linear momentum,

$$mv_0 = \frac{m}{2}\times0 + \frac{m}{2}v$$

$$\therefore\quad v = 2v_0 = \sqrt{\frac{2GM}{R}}$$

Increase in mechanical energy

$$= K_f - K_i$$

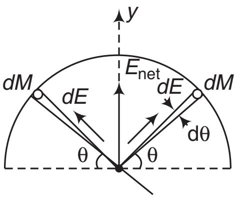
$$= \frac{1}{2}\frac{m}{2}v^2 - \frac{1}{2}mv_0^2$$

$$= \frac{1}{4}m\,(2v_0)^2 - \frac{1}{2}mv_0^2$$

$$= \frac{1}{2} m v_0^2 = \frac{1}{2} m \left(\frac{GM}{2R}\right) = \frac{1}{4} \frac{GMm}{R}$$

$$= \frac{1}{4} mgR \qquad \left(\text{as } g = \frac{GM}{R^2}\right)$$

2. $E_{\text{net}} = \int_0^{\pi} dE \sin\theta$



$$= \int_0^{\pi} \frac{G \cdot dM}{R^2} \sin\theta$$

$$= \int_0^{\pi} \frac{G\left(\frac{M}{\pi} d\theta\right)}{R^2} \sin\theta$$

$$= \frac{2GM}{\pi R^2} = \frac{2\,G\pi M}{l^2}$$

$$l = \pi R$$

$$\therefore \qquad R = \frac{l}{\pi}$$

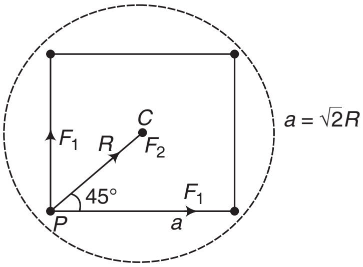
3. Net force on P towards centre

$$F = \sqrt{2}\, F_1 + F_2$$

$$\therefore \qquad \frac{Mv^2}{R} = \sqrt{2}\left[\frac{G.MM}{a^2}\right] + \frac{GMM}{(2R)^2}$$

$$\text{or} \qquad \frac{v^2}{R} = \frac{\sqrt{2}\, GM}{(\sqrt{2}R)^2} + \frac{GM}{4R^2}$$

$$\therefore \qquad v = \sqrt{\frac{GM}{R}\,\frac{2\sqrt{2}+1}{4}}$$



4. $K_i + U_i = K_f + U_f$

$$\frac{1}{2} m\,(kv_e)^2 - \frac{GMm}{R} = 0 - \frac{GMm}{r}$$

$$\therefore \qquad \frac{1}{2} k^2\,(2gR) - \frac{GM}{R} = -\frac{GM}{r}$$

$$\text{or} \qquad \frac{GM}{r}\,(1-k^2) = \frac{GM}{r} \qquad \left(\text{as } gR = \frac{GM}{R}\right)$$

$$\text{So,} \qquad r = \frac{R}{1-k^2}$$

5. $a = E = \left(\dfrac{GM}{R^3}\right) y \qquad \left(\text{as } a = E = \dfrac{F}{m}\right)$

$$= \frac{G\left(\frac{4}{3}\pi R^3 \rho\right)}{R^3}\, y = \frac{4}{3}\,\pi G \rho y$$

6. Earth rotates from west to east. Net velocity of train

$$= (\omega\, R - \theta)^2$$

$$\text{Now, } mg - N = \frac{m\,(\omega R - \theta)^2}{R}$$

$$\therefore \qquad N = mg - \frac{m\,(\omega R - v)^2}{R}$$

$$= mg\left[1 - \frac{\omega^2 R^2 + v^2 - 2\,vR\omega}{gR}\right]$$

$$= mg\left[1 - \frac{\omega\,(\omega R - 2v)}{g} - \frac{v^2}{Rg}\right] \quad \textbf{Ans.}$$

7. Gravitational field at any point inside the cavity is uniform (both in magnitude as well as direction). So, let us find its value at centre of cavity.

$E_R = E_T - E_C$ (at centre of cavity)

$R \to$ Remaining, $T \to$ total, $C \to$ cavity

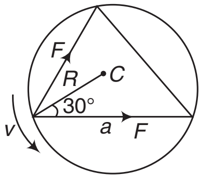
So, $E_R = E_T$ (as $E_C = 0$)

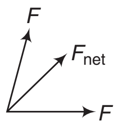
$$= \left(\frac{GM}{R^3}\right) a \quad \text{or} \quad E_R \propto a$$

8. Potential (and hence potential energy) at centre is $\frac{3}{2}$ times the value on surface. So, required kinetic energy is $\frac{3}{2}$ times or required speed is $\sqrt{\frac{3}{2}}$ times.

9. $\int dV = -\int E dr$

10. $a = 2R \cos 30° = \sqrt{3}\, R$





$F_{\text{net}} = \sqrt{3}\, F$ is towards centre C

$$\therefore \qquad \sqrt{3}\, F = \frac{MV^2}{R}$$

or $$\sqrt{3}\,\frac{GMM}{a^2} = \frac{MV^2}{R}$$

or $$\frac{\sqrt{3}GM^2}{(\sqrt{3}R)^2} = \frac{MV^2}{R}$$

or $$v = \sqrt{\frac{GM}{\sqrt{3}R}}$$

11. $T^2 \propto R^3 \;\Rightarrow\; \therefore\; T^{2/3} \propto R$ or $T^2 \propto \dfrac{1}{(1/R^3)}$

12. $W = \Delta E = E_A - E_\alpha$

$$= (K_A + U_A) - (K_\alpha + U_\alpha)$$
$$= (K_A + mV_A) - (0 + 0)$$

$\therefore\; -5.5 = \frac{1}{2} \times 1 \times (3)^2 + (1)\,V_A$

or $V_A = -10$ J/kg

13. $K_i + U_i = K_f + U_f$

$\therefore\; \frac{1}{2} mv^2 - 2\frac{GMm}{\sqrt{2}R} = 0 + 0$

$\therefore\; v = \sqrt{\frac{2\sqrt{2}GM}{R}}$

14. $W = -\Delta U = U_i - U_f$

$$= -\frac{Gmm}{a}\left[\left\{\frac{(1)\,(2)}{1} + \frac{(1)\,(3)}{1} + \frac{(2)\,(3)}{\sqrt{2}}\right\} - \left\{\frac{(1\times 2)}{1} + \frac{(1\times 3)}{1} + \frac{(2\times 3)}{1}\right\}\right]$$

$$= \frac{6\,Gm^2}{a}\left(1 - \frac{1}{\sqrt{2}}\right)$$

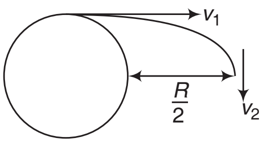
15. $K_i + U_i = K_f + U_f$

$\therefore\; \frac{1}{2} mv^2 - \frac{3}{2}\frac{GMm}{R} = 0 - \frac{GMm}{R}$

Solving, we get

$$v = \sqrt{\frac{GM}{R}}$$

16. $v_1 = nv_e = n\sqrt{2gR} = n\sqrt{\frac{2GM}{R}}$



From mechanical energy conservative, we have

$$\frac{1}{2}m\left[n\sqrt{\frac{2GM}{R}}\right]^2 - \frac{GMm}{R} = \frac{1}{2}mv_2^2 - \frac{GMm}{R + \frac{R}{2}} \quad \text{...(i)}$$

From conservation of angular momentum about centre of earth

$$m\left[n\sqrt{\frac{2GM}{R}}\right]R = mv_2\left(R + \frac{R}{2}\right) \quad \text{...(ii)}$$

Solving these two equations we get,

$$n = \sqrt{0.6}$$

17. Let v is the velocity at centre then.

$$K_i + U_i = K_f + U_f$$

or $$0 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{3}{2}\frac{GMm}{R}$$

$\therefore$ $$v = \sqrt{\frac{GM}{R}}$$

From conservation of linear momentum.

$$2mv - mv = (2m + m)\,v$$

$\therefore$ $$v' = \frac{v}{3} = \frac{1}{3}\sqrt{\frac{GM}{R}}$$

Again applying energy conservation,

$$K_i + U_i = K_f + U_f$$

$\therefore$ $$\frac{1}{2}(3m)\left[\frac{1}{3}\sqrt{\frac{GM}{R}}\right]^2 - \frac{3}{2}\frac{GM\,(3m)}{R}$$
$$= 0 - 3m\left[\frac{GM}{R^3}(1.5\,R^2 - 0.5\,r^2)\right]$$

Solving this equations, we get

$$r = \frac{R}{3}$$

So, $\frac{R}{3}$ is the amplitude of oscillations.

18. $v_e = \sqrt{2gR}$

or $v_e \propto \sqrt{gR}$

19. $F = -\frac{dU}{dr} = -6r^2$

Now $\frac{mv^2}{r} = 6r^2$

$\therefore\; \frac{1}{2}mv^2 = 3r^3$

$$E = K + U = 3r^3 + 2r^3 = 5r^3$$

At $r = 5\,\text{m},\; E = 625$ J **Ans.**

20. E on the surface of earth,

$$E_1 = \frac{F}{m} = \frac{10}{1}$$
$$= 10\ \text{N/kg}$$
$$E \propto \frac{1}{r^2}$$

$$\therefore \qquad \frac{E_2}{E_1} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{R}{3R/2}\right)^2 = \frac{4}{9}$$

$$\therefore \qquad E_2 = \frac{4}{9} E_f = \frac{40}{9} \text{ N/kg}$$

$$F_2 = m\,E_2 = (200)\left(\frac{40}{9}\right) = 889 \text{ N}$$

21. $g = \dfrac{GM}{R^2} \quad \Rightarrow \quad \therefore \quad GM = gR^2$

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+x}} = \sqrt{\frac{gR^2}{R+x}}$$

22. $C \to$ cavity, $T \to$ Total, $R \to$ remaining

$$F_1 = \frac{GMm}{(2R)^2} \qquad \ldots\text{(i)}$$

$$F_R = F_2 = F_T - F_C = F_1 - F_C$$

or
$$F_2 = \frac{GMm}{(2R)^2} - \frac{G\left(\dfrac{M}{8}\right)m}{(3R/2)^2}$$

or
$$F_2 = \frac{14\,GMm}{72R^2} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii) we get,

$$\frac{F_2}{F_1} = \frac{7}{9}$$

More than One Correct Options

1. $g = \dfrac{GM}{R^2} = \dfrac{G\left(\dfrac{4}{3}\pi R^3\rho\right)}{R^2}$

or
$$g \propto R \qquad \text{(as } \rho \text{ is same)}$$

$$v_e = \sqrt{2gR} = \sqrt{\frac{2GM}{R}}$$

$$= \sqrt{\frac{2G\left(\dfrac{4}{3}\pi R^3\rho\right)}{R}}$$

So, v_e or $v \propto R$ (as ρ is same).

3. $K_i + U_i = K_f + U_f$

$$\therefore \qquad 0 - \frac{GMm}{2R} = \frac{1}{2}mv^2 - \frac{3}{2}\frac{GMm}{R}$$

or
$$v = \sqrt{\frac{2GM}{R}}$$

4. E due to point mass is $E = \dfrac{Gm}{r^2}$

As $r \to 0, E \to \propto$

So, just over the point masses, $E = \infty$. Hence, in moving from one point mass to other point mass, E first decreases and then increases.

V due to a point mass is

$$V = -\frac{Gm}{r}$$

As $\qquad r \to 0, V \to -\infty$

So, just over the point mass, V is $-\infty$. Hence, in moving from one point mass to other point mass, V first increases and then decreases.

5. Inside a shell, $V =$ constant and $E = 0$.

Between A and B, $E_{\text{net}} = 0$, $V_{\text{net}} =$ constant because these points μ inside both shells.

Between *B* and *C*

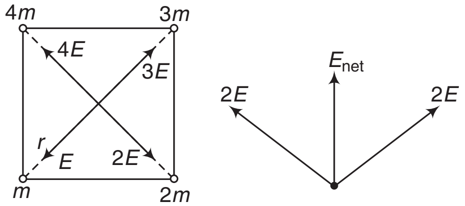
E of $m \neq 0$, E of $2m = 0$

V of $m \neq$ constant, V of $2m =$ constant.

Beyond *C*

E and V due to both shells are neither zero nor constant.

6. $E = \dfrac{Gm}{r^2}$

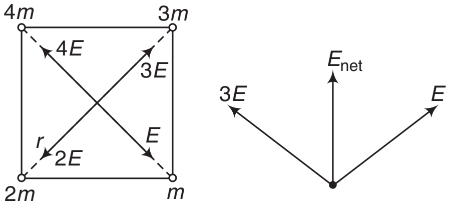


$$E'_{\text{net}} = 2\sqrt{2}\,E = \frac{2\sqrt{2}\,Gm}{r^2}$$

$$E'_{\text{net}} = \sqrt{(9E)^2 + E^2} = \sqrt{10}\,E$$

$$= \frac{\sqrt{10}\,Gm}{r^2}$$

$$E'_{\text{net}} > E_{\text{net}}$$



Potential in both cases is

$$V_{\text{net}} = -\frac{G}{r}(m + 2m + 3m + 4m)$$

$$= -\frac{10\,Gm}{r}$$

7. At highest point velocity of particle-1will becomes zero. But velocity of particle-2 is non zero.

8. At maximum distance (at A) kinetic energy is minimum. But angular momentum about centre of sun always remains constant.

9. $E_1 = -\dfrac{GMm}{R}$

$$U_2 = -\frac{GMm}{2R} \qquad (r = R + R = 2R)$$

$$K_2 = \frac{GMm}{4R}$$

and $$E_2 = -\frac{GMm}{4R}$$

10. $v = \sqrt{\dfrac{GM}{r}}$

or $$v \propto \sqrt{GM}$$

$$T = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

or $$T \propto \frac{1}{\sqrt{GM}}$$

Match the Columns

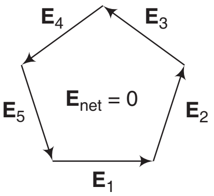
1. Inside the shell, V, U, K, and v remains constant. F, E and a are zero.

From A to B, kinetic energy increase and potential energy decreases.

2. Resultant of five field strength vectors of equal magnitude acting at same angles is zero.

(b) $V = -\dfrac{5\,GMm}{r}$

(c) If $\mathbf{E}_5$ is removed then resultant of rest four vectors is equal and opposite of $\mathbf{E}_5$.



$\therefore$ $$|\mathbf{E}_{\text{net}}| = |\mathbf{E}_5| = \frac{Gm}{r^2}$$

(d) $V = -\dfrac{4Gm}{r}$

3. $\dfrac{GM}{y} = x$ or $GM = xy$

(a) $E = \dfrac{GM}{(2y)^2} = \dfrac{xy}{4y^2} = \dfrac{x}{4y}$

(b) $|V| = \dfrac{GM}{y^3}\left(1.5\, y^2 - 0.5\,\dfrac{y^3}{4}\right)$

$$= \left(\frac{xy}{y^3}\right)\left(\frac{11}{8}\, y^2\right) = \frac{11}{8}\, x$$

(c) $E = \dfrac{GM}{y^3}\left(\dfrac{y}{2}\right) = \dfrac{xy}{y^3}\,\dfrac{y}{2} = \dfrac{x}{2y}$

(d) $|V| = \dfrac{GM}{2y} = \dfrac{xy}{2y} = \dfrac{x}{2}$

4. (a) $W = \Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Put $h = R$, $W = \dfrac{1}{2}\, mgR$

(b) $\text{KE} = \dfrac{GMm}{2r} = \dfrac{GMm}{4R} = \dfrac{1}{4}\, mgR \quad (\text{as}\, GM = gR^2)$

(c) $E = -\dfrac{GMm}{2r}$

$$E_1 = -\frac{GMm}{2\,(2R)} = -\frac{GMm}{4R} = -\frac{1}{4}\, mgR$$

$$E_2 = -\frac{GMm}{2\,(3R)} = -\frac{1}{6}\, mgR$$

$$E_1 \sim E = \frac{1}{12}\, mgR$$

(d) $\text{KE} = \Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Putting $h = R$, $\text{KE} = \dfrac{1}{2}\, mgR$

Subjective Questions

1. $W = \Delta U = U_f - U_i$

$$= 3\left[\frac{-Gmm}{2a}\right] - 3\left[-\frac{Gmm}{a}\right]$$

$$= \frac{3\,Gmm}{2a} = \frac{3Gm^2}{2a} \qquad \textbf{Ans.}$$

2. $h = \dfrac{u^2}{2g_e}$

$\therefore$ $$u = \sqrt{2g_e h} \qquad \ldots\text{(i)}$$

For the asked planet this u should be equal to the escape velocity from its surface.

$\therefore$ $$\sqrt{2g_e h} = \sqrt{2g_p R_p}$$

or $$g_e h = g_p R_p$$

$$\frac{GM_e}{R_e^2}.h = \frac{GM_p}{R_p^2}.\, R_p$$

or $$\frac{\left(\frac{4}{3}\pi R_e^3\right)\rho h}{R_e^2} = \frac{\left(\frac{4}{3}\pi R_p^3\right)\rho R_p}{R_p^2}$$

or $$R_P = \sqrt{R_e h}$$

$$= \sqrt{(6.41 \times 10^6)(1.5)}$$

$$= 3.1 \times 10^3 \text{ m}$$ **Ans.**

3. (a) $v_o = \frac{v_e}{2}$

$$\therefore \quad \sqrt{\frac{GM}{r}} = \frac{\sqrt{\frac{2GM}{R}}}{2}$$

$$\therefore \quad r = 2R$$

or $h = r - R = R$ or height = radius of earth.

(b) Increase in kinetic energy = decrease in potential energy

$$\therefore \quad \frac{1}{2}mv^2 = \frac{mgh}{1 + \frac{h}{R}}$$

$$\therefore \quad v = \sqrt{\frac{2hg}{1 + \frac{h}{R}}}$$

Substituting the values we have,

$$v = \sqrt{\frac{2 \times 9.81 \times 6400 \times 10^3}{1 + \frac{R}{R}}}$$

$$= 7924 \text{ m/s} \approx 7.92 \text{ km/s}$$ **Ans.**

4. (i) At point A, field strength due to shell will be zero.

Net field is only due to metal sphere. Distance between centre of metal sphere and point A is $4R$.

$$\therefore \quad E_A = \frac{G(m)}{(4R)^2} = \frac{Gm}{16R^2}$$

(ii) At point B, net field is due to both, due to shell and due to metal sphere.

$$\therefore \quad E_B = \frac{Gm}{(5R)^2} + \frac{Gm}{(6R)^2}$$

$$= \frac{61\,Gm}{900\,R^2}$$ **Ans.**

5. Radius of hollow sphere is $\frac{R}{2}$, so mass in this hollow portion would had been, $\frac{M}{8}$.

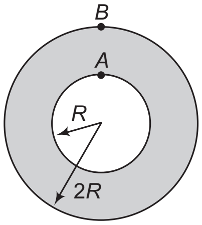
Now, net force on m due to whole sphere = force due to remaining mass + force due to cavity mass.

$\therefore$ Force due to remaining mass = force due to whole sphere − force due to cavity mass

$$= \frac{G\,Mm}{d^2} - \frac{G\,Mm}{8(d - R/2)^2}$$

$$= \frac{G\,Mm}{d^2}\left[1 - \frac{1}{8\left(1 - \frac{R}{2d}\right)^2}\right]$$

6. Let m_1 be the mass of the core and m_2 the mass of outer shell.



$$g_A = g_B \qquad \text{(given)}$$

Then $$\frac{Gm_1}{R^2} = \frac{G(m_1 + m_2)}{(2R)^2}$$

$$\therefore \quad 4m_1 = (m_1 + m_2)$$

or $$4\left\{\frac{4}{3}\pi R^3 \rho_1\right\} = \frac{4}{3}\pi R^3 \cdot \rho_1 + \left\{\frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3\right\}\rho_2$$

$$\therefore \quad 4\rho_1 = \rho_1 + 7\rho_2$$

$$\therefore \quad \frac{\rho_1}{\rho_2} = \frac{7}{3}$$ **Ans.**

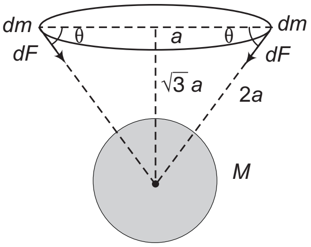
7. Total mechanical energy of a satellite in an elliptical orbit of semi major axis 'a' is $-\frac{GMm}{2a}$.

$$E = K + U$$

$$\therefore \quad -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

or $$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$ **Hence Proved.**

8. dF = force on a small mass 'dm' of the ring by the sphere.



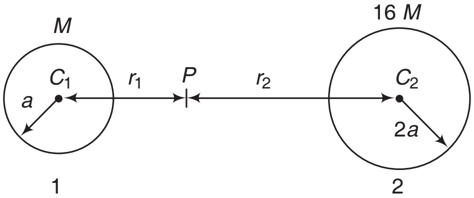
Net force on ring $= \Sigma(dF \sin\theta)$ or $\int dF \sin\theta$

$$= \Sigma \frac{GM(dm)}{(2a)^2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}GM}{8a^2}\Sigma(dm)$$

But $\Sigma(dm) = m$, the mass of whole ring.

$$\therefore \qquad \text{Net force} = \frac{\sqrt{3}GMm}{8a^2} \qquad \textbf{Ans.}$$

9. Let there are two stars 1 and 2 as shown below.



Let P is a point between C_1 and C_2, where gravitational field strength is zero. Or at P field strength due to star 1 is equal and opposite to the field strength due to star 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G(16M)}{r_2^2} \quad \text{or} \quad \frac{r_2}{r_1} = 4$$

also $\qquad r_1 + r_2 = 10a$

$$\therefore \qquad r_2 = \left(\frac{4}{4+1}\right)(10a) = 8a$$

and $\qquad r_1 = 2a$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.

From conservation of mechanical energy $\frac{1}{2}mv_{\min}^2$

= Potential energy of the body at P

– Potential energy at the surface of the larger star.

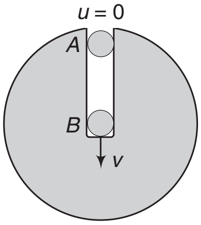
$$\therefore \quad \frac{1}{2}mv_{\min}^2 = \left[-\frac{GMm}{r_1} - \frac{16GMm}{r_2}\right] - \left[-\frac{GMm}{10a-2a} - \frac{16GMm}{2a}\right]$$

$$= \left[-\frac{GMm}{2a} - \frac{16GMm}{8a}\right] - \left[-\frac{GMm}{8a} - \frac{8GMm}{a}\right]$$

$$\text{or} \qquad \frac{1}{2}mv_{\min}^2 = \left(\frac{45}{8}\right)\frac{GMm}{a}$$

$$\therefore \qquad v_{\min} = \frac{3\sqrt{5}}{2}\left(\sqrt{\frac{GM}{a}}\right) \qquad \textbf{Ans.}$$

10. Let mass of the ball be m.



$$\frac{1}{2}mv^2 = m(V_A - V_B)$$

$$= m\left[-\frac{GM}{R} - \left(-1.5\frac{GM}{R}\right)\right]$$

$$= \frac{GMm}{2R}$$

$$\therefore \qquad v = \sqrt{\frac{GM}{R}}$$

Velocity of ball just after collision,

$$v' = ev = \frac{1}{2}\sqrt{\frac{GM}{R}}$$

Let r be the distance from the centre upto where the ball reaches after collision. Then,

$$\frac{1}{2}mv'^2 = m[V(r) - V(\text{centre})]$$

$$\text{or} \quad \frac{1}{8}\frac{GMm}{R} = m\left[\frac{3GM}{2R} - \frac{GM}{R^3}\left(\frac{3R^2}{2} - \frac{r^2}{2}\right)\right]$$

$$\text{or} \qquad \frac{1}{8} = \frac{3}{2} - \frac{3}{2} + \frac{r^2}{2R^2}$$

$$\therefore \qquad \frac{r^2}{R^2} = \frac{1}{4} \quad \text{or} \quad r = \frac{R}{2}$$

∴ The desired distance,

$$s = R + \frac{R}{2} + \frac{R}{2} = 2R \qquad \textbf{Ans.}$$

11. Applying conservation of mechanical energy,

Increase in kinetic energy

= decrease in gravitational potential energy

$$\text{or} \qquad \frac{1}{2}m_0v^2 = U_B - U_A = m_0(V_B - V_A)$$

$$\therefore \qquad v = \sqrt{2(V_B - V_A)} \qquad \ldots\text{(i)}$$

Potential at A

V_A = potential due to complete sphere

– potential due to cavity

$$= -\frac{1.5\,GM}{R} - \left[-\frac{Gm}{R/2}\right]$$

$$= \frac{2Gm}{R} - \frac{1.5\,GM}{R}$$

Here, $m = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 \rho = \frac{\pi\rho R^3}{6}$

and $M = \frac{4}{3}\pi R^3 \rho$

Substituting the values, we get

$$V_A = \frac{G}{R}\left[\frac{\pi\rho R^3}{3} - 2\pi\rho R^3\right] = -\frac{5}{3}\pi G\rho R^2$$

Potential at *B*

$$V_B = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(\frac{R}{2}\right)^2\right] + \frac{1.5Gm}{R/2}$$

$$= -\frac{11}{8}\frac{GM}{R} + \frac{3Gm}{R}$$

$$= \frac{G}{R}\left[\frac{\pi\rho R^3}{2} - \frac{11}{6}\cdot\pi\rho R^3\right] = -\frac{4}{3}\pi G\rho R^2$$

$$\therefore \quad V_B - V_A = \frac{1}{3}\pi G\rho R^2$$

So, from Eq. (i)

$$v = \sqrt{\frac{2}{3}\pi G\rho R^2} \qquad \textbf{Ans.}$$

12. (a) Let x be the displacement of ring. Then displacement of the particle is $x_0 - x$, or $(3.0 - x)$ m. Centre of mass will not move. Hence,

$$(5.4 \times 10^9)x = (6 \times 10^8)(3 - x)$$

Solving, we get

$$x = 0.3 \text{ m} \qquad \textbf{Ans.}$$

(b) Apply conservation of linear momentum and conservation of mechanical energy.

13. (a) Mean radius of planet,

$$m_2 = \frac{r_1 + r_2}{2} = 1.4 \times 10^8 \text{ km}$$

Now, $T \propto r^{3/2}$

$\therefore$ Time period of m_2 :

$$T_2 = T_1\left(\frac{1.4 \times 10^8}{10^8}\right)^{3/2}$$

or $T_2 = 2\,(1.4)^{3/2}$

$$= 3.31 \text{ yr} \qquad \textbf{Ans.}$$

(b) For m_2, point P is perigee position. So, speed at this point is greater than orbital speed for circular orbit.

$$\therefore \quad U_{m_2} = U_{m_1}$$

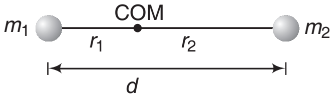
$$\therefore \quad E_{m_2} > E_{m_1}$$

$$K_{m_2} > K_{m_1}$$

(c) vr = constant

14. (a) $r_1 + r_2 = d$ ...(i)

$m_1 r_1 = m_2 r_2$ ...(ii)



Solving these two equations we get,

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right)d \quad \text{or} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right)d$$

The centripetal force is provided by gravitational force,

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{d^2}$$

Solving these equations, we get

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{d^3}}$$

$$\therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d^3}{G(m_1 + m_2)}} \qquad \textbf{Ans.}$$

(b) $$\frac{K_1}{K_2} = \frac{\frac{1}{2}I_1\omega^2}{\frac{1}{2}I_2\omega^2} = \frac{I_1}{I_2} = \frac{m_1 r_1^2}{m_2 r_2^2}$$

$$= \left(\frac{m_1}{m_2}\right)\left(\frac{r_1}{r_2}\right)^2 = \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_1}\right)^2 = \frac{m_2}{m_1} \qquad \textbf{Ans.}$$

(c) $$\frac{L_1}{L_2} = \frac{I_1\omega}{I_2\omega} = \frac{I_1}{I_2} = \frac{m_2}{m_1} \qquad \textbf{Ans.}$$

(d) $$L = L_1 + L_2 = (I_1 + I_2)\,\omega$$

$$= (m_1 r_1^2 + m_2 r_2^2)\,\omega$$

$$= \left[\frac{m_1 m_2^2 d^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 d^2}{(m_1 + m_2)^2}\right]\omega$$

$$= \mu\omega d^2 \qquad \textbf{Ans.}$$

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass

(e) $$K = \frac{1}{2}(I_1 + I_2)\,\omega^2 = \frac{1}{2}\mu\omega^2 d^2 \qquad \textbf{Ans.}$$

14. Simple Harmonic Motion

INTRODUCTORY EXERCISE 14.1

1. $a = -4x$

Comparing with $a = -\omega^2 x$, we get

$$\omega = 2 \text{ rad/s}$$

Now, $\quad T = \frac{2\pi}{\omega} = (\pi)\text{sec}$

2. $E = \frac{1}{2}kA^2$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{4}\right)^2$$

$$= \frac{1}{16}\left(\frac{1}{2}kA^2\right)$$

$$= \frac{1}{16}E$$

Hence, $\frac{1}{16}$ fraction is potential energy and $\frac{15}{16}$ fraction is kinetic energy.

3. (a) $A =$ initial displacement from mean position

$= 15$ cm

(b) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.0}{150}} = 0.726$ s

(c) $F = \frac{1}{T} = 1.38$ Hz **Ans.**

(d) $E = \frac{1}{2}kA^2$

$= \frac{1}{2} \times 150 \times (0.15)^2 = 1.69$ J **Ans.**

(e) $v_{\max} = \omega A$

$= \left(\frac{2\pi}{T}\right)A = \left(\frac{2\pi}{0.726}\right)(0.15)$

$= 1.30$ m/s **Ans.**

4. $v_{\max} = \omega A = 2\pi fA$

$= (2\pi)(2)(8 \times 10^{-3})$

$= 0.101$ m/s **Ans.**

$a_{\max} = \omega^2 A = (2\pi f)^2 A$

$= 4 \times \pi^2 \times 4 \times 8 \times 10^{-3}$

$= 1.264 \text{ m/s}^2$ **Ans.**

$F_{\max} = ma_{\max}$

$= (0.5)(1.264)$

$= 0.632$ N **Ans.**

5. No, as acceleration in SHM $a = -\omega^2 x$ is variable.

INTRODUCTORY EXERCISE 14.2

1. (a) $E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}(m)\omega^2 A^2$

$= \frac{1}{2} \times 2.0 \times \left(\frac{\pi}{4}\right)^2 \times (1.5)^2$

$= 1.39$ J **Ans.**

(b) $0.5 = 1.5 \sin\left(\frac{\pi t_1}{4} + \frac{\pi}{6}\right)$

From here find t.

Then, $\quad -0.75 = 1.5 \sin\left(\frac{\pi t_2}{4} + \frac{\pi}{6}\right)$

From here find t_2.

Now, $t_1 \sim t_2$ is the required time.

2. $\omega = 3$ rad/s

$$A = 0.2 \text{ m}$$

At $x = 5$ cm

$v = \pm\,\omega\sqrt{A^2 - x^2}$

$= \pm\, 3.0\sqrt{(0.2)^2 - (0.05)^2}$

$= \pm\, 0.58$ m/s **Ans.**

$a = -\omega^2 x = -(3)^2(0.05)$

$= -0.45 \text{ m/s}^2$ **Ans.**

At $x = 0$

$v = \pm\,\omega A$

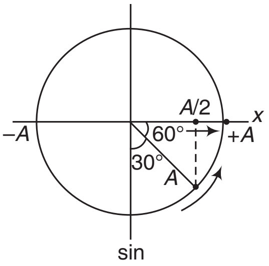
$= \pm\,(3.0)(0.2)$

$= \pm\, 0.6$ m/s **Ans.**

$a = -\omega^2 x = -(3)^2(0)$

$= 0$ **Ans.**

3.



$$x = A\sin(\omega t + 30°)$$

$\therefore \quad \delta = 30°$ **Ans.**

4. (b) $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = 1.57$ rad/s **Ans.**

(c) $\omega = \sqrt{\dfrac{k}{m}}$

$\therefore\ k = m\omega^2 = (0.8)(1.57)^2 = 1.97\ \text{N/m}$ **Ans.**

(d) At $t = 1$ s,

$v = \text{slope of } x\text{-}t \text{ graph} = 0$

(e) At $t = 1$ s, x is maximum $(= +A)$. Therefore. magnitude of acceleration is also maximum.

$a = \text{maximum} = \omega^2 A = (1.57)^2\,(0.08)$

$= 0.197\ \text{m/s}^2$ **Ans.**

5. $X = 5\sin\left(20t + \dfrac{\pi}{3}\right)$

$$v = \frac{dx}{dt} = 100\cos\left(20\,t + \frac{\pi}{3}\right)$$

$$a = \frac{dv}{dt} = -\,2000\sin\left(20t + \frac{\pi}{3}\right)$$

(a) $v = 0$ for the first time when,

$$20t + \frac{\pi}{3} = \frac{\pi}{2}$$

$\therefore$ $t = \dfrac{\pi}{120}$ s **Ans.**

(b) $a = 0$ for the first time, when

$$20t + \frac{\pi}{3} = \pi$$

$\therefore$ $t = \dfrac{\pi}{30}$ s **Ans.**

(c) When $a = 0$ for the first time, its speed will be maximum.

$t = \dfrac{\pi}{30}$ s **Ans.**

6. Simple harmonic with mean at $x = 10$, amplitude 4 and extreme positions at $x = 6$ and $x = 14$. At $t = 0$, it starts from $x = 6$. Here, x is coordinate, not displacement from mean position.

INTRODUCTORY EXERCISE 14.3

1. $a = -16x$

Comparing with $a = -\omega^2 x \Rightarrow \omega = 4$ rad/s

Now, $T = \dfrac{2\pi}{\omega} = \left(\dfrac{\pi}{2}\right)$ sec

2. $2 = 2\pi\sqrt{\dfrac{M}{k}}$ …(i)

$3 = 2\pi\sqrt{\dfrac{M+4}{k}}$ …(ii)

Solving two equations, we get

$M = 3.2$ kg

3. $2 = 2\pi\sqrt{\dfrac{(0.3 + 0.1)}{k}}$ …(i)

$T = 2\pi\sqrt{\dfrac{0.1}{k}}$ …(ii)

Solving these two equations we get,

$T = 1$ s

4. Comparing with $a = -\,\omega^2 x$

$$\omega^2 = p$$

$\therefore$ $\omega = \sqrt{p}$

5. $T = 2\pi\sqrt{\dfrac{M}{k}}$

$\dfrac{5T}{3} = 2\pi\sqrt{\dfrac{M+m}{k}}$

Solving these two equations, we get

$$\frac{m}{M} = \frac{16}{9}$$

6. $T \propto \sqrt{l}$

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{1.21\,l}{l}} = 1.1$$

$\therefore$ $T' = 1.1\,T$

Hence, percentage increase in T is 11%.

7. $\omega^2 x = \omega\sqrt{A^2 - x^2} = \dfrac{\sqrt{A^2 - x^2}}{x} = 2\pi f$

$\therefore$ $f = \dfrac{\sqrt{A^2 - x^2}}{2\pi x}$

Substitute, $A = 2$ cm and $x = 1$ cm

INTRODUCTORY EXERCISE 14.4

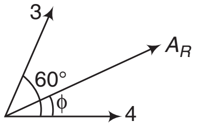
1. In such situation, amplitudes are added by vector method.

$$A_R = \sqrt{(4)^2 + (3)^2 + 2\,(4)\,(3)\cos\phi}$$

$= \sqrt{25 + 24\cos\phi}$ …(i)

Now, we can substitute different values of ϕ given in different parts in the question and can find the value of A_R.

2. $A_R = \sqrt{(4)^2 + (3)^2 + 2\,(4)\,(3)\cos 60^\circ}$



$= 6.1$ units

$$\tan\phi = \frac{3\sin 60^\circ}{4 + 3\cos 60^\circ} = 0.472$$

$\therefore \quad \phi = 25.3°$

$\therefore \quad x = 6.1 \sin (100\pi t + 25.3°)$

(a) At $t = 0$, $x = 6.1 \sin 25.3°$

$= 2.6$ unit

(b) $v_{\max} = \omega A$

$= (100\pi)(6.1)$

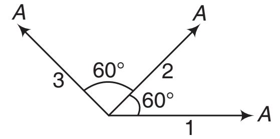
$= 1917$ unit

(c) $a_{\max} = \omega^2 A$

$= (100\pi)^2 (6.1)$

$= 6.0 \times 10^5$ units

3. Resultant of 1 and 3 is also A in the direction of 2.



$\therefore \quad A_R = 2A$

4. $A = \sqrt{A^2 + A^2 + 2A.A \cos\phi}$

Solving we get,

$$\cos\phi = -\frac{1}{2} \quad \text{or} \quad \phi = 120° \quad \text{or} \quad \frac{2\pi}{3}$$

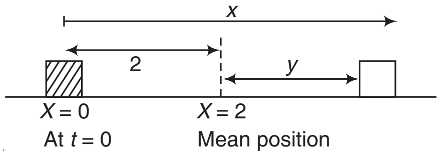
Exercises

LEVEL 1

Assertion and Reason

1. x is measured from the mean position.

2. $y = (-A \cos\omega t)$ at the particle starts from $-A$. So, displacement from the mean position will be $-A \cos\omega t$.

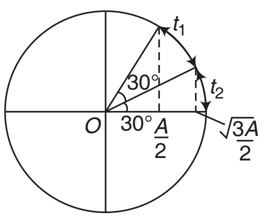


$\therefore \quad x = y + 2$

$= -2\cos\omega t + 2$

3. By applying a constant force on spring-block system mean position is changed but time period remains unchanged.

4. $t_1 = t_2 = \dfrac{\theta}{\omega} = \dfrac{\pi/6}{2\pi/T} = \dfrac{T}{12}$



5. In angular SHM, path is not straight line.

6. $k \propto \dfrac{1}{l} \quad \Rightarrow \quad \omega = \sqrt{\dfrac{k}{m}}$

$\therefore \quad \omega \propto \dfrac{1}{\sqrt{ml}}$

It m and l both are halved then ω will become 2 times.

7. $F = -kx$. For all displacements. From the mean position, whether they are small or large.

8. $\dfrac{\pi}{2\omega} = \dfrac{\pi}{2(2\pi/T)} = \dfrac{T}{4}$. In the given time particle moves from $x = A$ to $x = 0$.

9. $\left|\dfrac{x}{a}\right| = \left(\dfrac{T}{2\pi}\right)^2 = \dfrac{1}{\omega^2} =$ constant for a given SHM.

10. If a particle P rotates in a circle with constant angle speed ω and we draw a perpendicular on and diameter. This perpendicular cuts the diameter at point Q then motion of Q is simple harmonic. But motion of P is circular.

Single Correct Option

1. In equation $x = A\cos\omega t$, putting $x = \dfrac{A}{2}$ we get

$$\omega t = \frac{\pi}{3}$$

$\therefore \quad \left(\dfrac{2\pi}{T}\right) t = \dfrac{\pi}{3}$

or $\quad t = \dfrac{T}{6}$

2. Kinetic energy and potential energy in SHM oscillate with double frequency.

3. $v = \omega\sqrt{A^2 - x^2}$

$\therefore \quad \dfrac{v^2}{\omega^2} + \dfrac{x^2}{(1)^2} = A^2$

Hence, $v - x$ graph is an ellipse.

4. l in the equation $T = 2\pi\sqrt{\dfrac{l}{g}}$ is measured from centre of mass and centre of mass in both cases remains at same location.

5. **At $t = 1$ s**

$$\phi_1 = \frac{\pi}{2} + \phi$$

and $$\phi_2 = \frac{2\pi}{3} + \phi$$

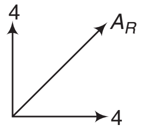
∴ $$\Delta\phi = \phi_2 - \phi_1 = \frac{\pi}{6}$$

6. $v = \omega\sqrt{A^2 - x^2}$

∴ $\dfrac{\omega A}{2} = \omega A\sqrt{A^2 - x^2}$ or $x = \dfrac{\sqrt{3}}{2}A$

7. All other equation can be converted into the from $y = a\sin(\omega t \pm \phi)$ or $a\cos(\omega t \pm \theta)$.

8. Phase difference between cos πt and sin πt is 90°.



$$A_R = 4\sqrt{2} \text{ units}$$

9. $\omega_1 t = \omega_2 t + 2\pi \qquad (\theta_1 = \theta_2 + 2\pi)$

∴ $$t = \frac{2\pi}{\omega_1 - \omega_2}$$

$$= \frac{2\pi}{(2\pi/T_1) - (2\pi/T_2)}$$

$$= \frac{T_1 T_2}{T_2 - T_1}$$

$$= \frac{4 \times 4.2}{4.2 - 4} = 84$$

Number of vibration of X in this time are,

$$N_1 = \frac{t}{T_1} = \frac{84}{4} = 21$$

10. $mg = kx \quad \Rightarrow \quad \therefore \quad k = \dfrac{mg}{x}$

$$T = 2\pi\sqrt{\frac{M+m}{R}} = 2\pi\sqrt{\frac{(M+m)\,x}{mg}}$$

11. $\omega_1 A_1 = \omega_2 A_2$

∴ $$\frac{A_1}{A_2} = \frac{\omega_2}{\omega_1} = \frac{\sqrt{k_2/m}}{\sqrt{k_1/m}} = \sqrt{\frac{k_2}{k_1}}$$

12. $I = \dfrac{1}{2}mR^2 + mR^2 = \dfrac{3}{2}mR^2$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Here $\quad l = R$

13. $U = \dfrac{1}{2}kx^2$

i.e. U *versus* x^2 graph is a straight line passing through origin.

14. $T = 2\pi\sqrt{\dfrac{1}{g\left(\dfrac{1}{l} + \dfrac{1}{R}\right)}}$

Put $\quad l = R$

15. $x = A\cos\omega t$

$$F \propto -x$$

Hence, F is $-\cos\omega t$ graph.

16. In equilibrium, let x_0 is extension in spring then

$$kx_0 = mg\sin\theta$$

∴ $$x_0 = \frac{mg\sin\theta}{k}$$

= amplitude of oscillations

17. $\dfrac{d^2x}{dt^2} = -\pi^2 x$

Comparing with $\dfrac{d^2x}{dt^2} = -\omega^2 x$

We have, $\quad \omega = \pi$

∴ $\quad 2\pi f = \pi$

$$f = \frac{1}{2}\text{ Hz}$$

18. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{\mu}}$

where, $\quad \mu$ = reduced mass

$$= \frac{M\,m}{M+m}$$

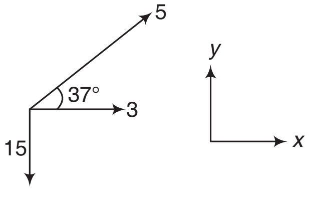
19. $T \propto \dfrac{1}{\sqrt{g}} \propto \dfrac{1}{\sqrt{M/R^2}}$

or $\quad T \propto \dfrac{R}{\sqrt{M}}$

R and M both are doubled. So, T will become $\sqrt{2}$ times.

∴ $\quad T^{-1} = \sqrt{2}\,T = 2\sqrt{2}$ s $\qquad$ (as $T = 2$ s)

20. x_3 can be written as



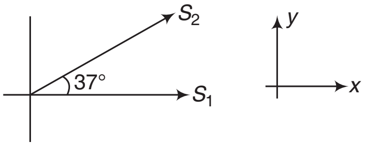
$$x_3 = 15\sin(\omega t - \pi/2)$$

$$\mathbf{A} = 3\,\hat{\mathbf{i}} + 5\cos 37°\,\hat{\mathbf{j}} + 5\sin 37°\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}}$$

$$= 7\hat{\mathbf{i}} + 12\hat{\mathbf{j}}$$

$$\therefore \quad |\mathbf{A}| = \sqrt{(7)^2 + (12)^2} = 13.89$$

21. $X = S_1 + S_2 \cos 37°$



$$= (a \sin \omega t + 0.8\, b \sin \omega t) \qquad \ldots\text{(i)}$$

$$y = S_2 \sin 37°$$

$$= 0.6\, b \sin \omega t$$

$$\sin \omega t = \frac{y}{0.6\, b}$$

Substituting this value in Eq. (i), we have

$$x = \left(\frac{a + 0.8\, b}{0.6\, b}\right) y$$

This is equation of a straight line passing through origin.

22. (a) Kinetic energy is 0.64 times, it means speed is 0.8 times

$$0.8\,\omega A = \omega\sqrt{A^2 - x^2}$$

$$\therefore \quad x = 0.6\, A = 6\,\text{cm}$$

(b) $$\frac{\omega A}{2} = \omega\sqrt{A^2 - x^2}$$

$$\therefore \quad x = \frac{\sqrt{3}}{2} A$$

23. $$T = 2\pi \sqrt{\left|\frac{x}{a}\right|}$$

$$\therefore \quad \omega = \frac{2\pi}{T} = \sqrt{\left|\frac{x}{a}\right|} = \sqrt{\frac{0.5}{4 \times 10^{-2}}}$$

$$= 3.53\ \text{rad/s}$$

24. Let x_0 is the compression in equilibrium. Then,

$$kx_0 = ma$$

$$x_0 = \frac{ma}{k} = \frac{1 \times 2}{100}$$

$$= 0.02\ \text{m}$$

Amplitude $= x_0 = 0.02$ m **Ans.**

Subjective Question

1. Resultant of k and k is $2k$. Then, resultant of $2k$ and $2k$ is k.

Now $$T = 2\pi \sqrt{\frac{m}{k}}$$

2. $$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{0.2}{80}} = \frac{\pi}{10}\ \text{s}$$

$$= 0.314\ \text{s}$$ **Ans.**

3. $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{9}{0.05} = 180\ \text{N/m}$$

$$m = \frac{W}{g} = \frac{27}{10} = 2.7\ \text{kg}$$

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{2.7}{180}}$$

$$= 0.78\ \text{s}$$ **Ans.**

4. $\Delta T = \frac{1}{2} T\alpha\, \Delta\theta$

$$\Rightarrow \quad \Delta t = \frac{\Delta T}{T^1} t$$

But $T' \approx T$

$$\therefore \quad \Delta t = \frac{1}{2} \alpha\, \Delta\theta\, (t)$$

$$= \frac{1}{2} \times 0.000012 \times 20 \times 24 \times 3600$$

$$= 10.37\ \text{s}$$ **Ans.**

At higher temperature, length of pendulum clock will be more. So, time period will be more and it will lose the time.

5. (a) $kx = mg$

$$\therefore \quad k = \frac{mg}{x} = \frac{(20 \times 10^{-3})(10)}{7 \times 10^{-2}}$$

$$= \frac{20}{7}\ \text{N/m}$$ **Ans.**

(b) $$T = 2\pi \sqrt{\frac{m}{k}}$$

$$= 2\pi \sqrt{\frac{50 \times 10^{-3}}{(20/7)}}$$

$$= 0.84\ \text{s}$$ **Ans.**

6. $k \propto \frac{1}{l}$

Length is halved, so value of force constant of each part will become $2k$. Now net force constant of $2k$ and $2k$ as shown in figure will becomes $4k$.

$$T = 2\pi \sqrt{\frac{m}{k}}$$

or $$T \propto \frac{1}{\sqrt{k}}$$

k has becomes 4 times. Therefore, T will remain half.

8. $$g_e = g - \frac{\text{Upthrust}}{\text{mass}}$$

$$= g - \frac{V\,(\delta/10)\,g}{V\delta} = \frac{9}{10}g$$

$$T' = 2\pi\sqrt{\frac{l}{g_e}} = 2\pi\sqrt{\frac{10}{9}\left(\frac{l}{g}\right)}$$

$$= \left(\sqrt{\frac{10}{9}}\right)T$$

9. For small values of x, the term x^2 can be neglected.

$\therefore$ $$F = -6\,x$$

Comparing with $F = -kx$, we get

$$k = 6\text{ N/m}$$ **Ans.**

10. $$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.25} = (8\pi)\text{ rad/s}$$

$$x = A\sin(\omega t + \phi)$$

$$U = \frac{dx}{dt} = \omega A\cos(\omega t + \phi)$$

At $t = 0$

$$x = A\sin\phi$$

$\therefore$ $$5 = A\sin\phi \qquad \ldots\text{(i)}$$

$$U = \omega A\cos\phi = (8\pi)\,A\cos\phi$$

$\therefore$ $$218 = (8\pi\,A)\cos\phi \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii) we get,

$$A = 10\text{ cm} \quad \text{and} \quad \phi = \frac{\pi}{6}\text{ or }30°$$

11. KE at mean position

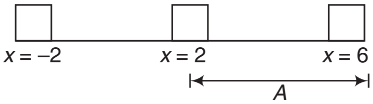
$$K = \frac{1}{2}m\,v_{\max}^2 = \frac{1}{2}m\,\omega^2A^2$$

$\therefore$ $$\omega = \left(\sqrt{\frac{2K}{m}}\right)\frac{1}{A}$$

$$= \left(\sqrt{\frac{2\times10^{-3}\times8}{0.1}}\right)\frac{1}{0.1} = 4\text{ rad/s}$$

Now, $y = A\sin(\omega t + \pi/4)$ is the required equation.

12. U_0 = minimum potential energy at mean position



$$= 10\text{ J}$$

At extreme position

U = Total mechanical energy

$$= 26\text{ J} = 10 + (x-2)^2$$

$\therefore$ $$(x-2) = \pm 4$$

Hence, $x = 6$ m and $x = -2$ m are the extreme positions.

(a) $$K_{\max} = E - U_0 = 16\text{J}$$

$\therefore$ $$\frac{1}{2}m\omega^2A^2 = 16$$

or $$\frac{1}{2}\times2\times\omega^2\times(4)^2 = 16$$

or $$\omega = 1\text{ rad/s}$$

(b) At mean position

$$E = 26\text{J}$$

$$U = U_0 = 10\text{J}$$

$\therefore$ $$K = 16\text{J}$$

At extreme position

$$K = 0$$

$\therefore$ $$U = E = 26\text{ J}$$

13. $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

Putting $h = R$

$$g' = \frac{g}{4}$$

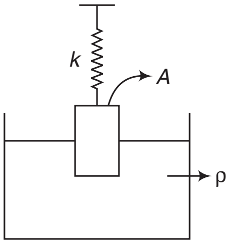
$$T = 2\pi\sqrt{\frac{l}{(g/4)}}$$

$$= 4\pi\sqrt{\frac{l}{g}}$$

$$= 4\sqrt{l} = 4\sqrt{1.0} \qquad (\text{as } g = \pi^2)$$

$$= 4\text{ s}$$ **Ans.**

14. In such situation, upthrust also behaves like a spring force of force constant $= \rho Ag$



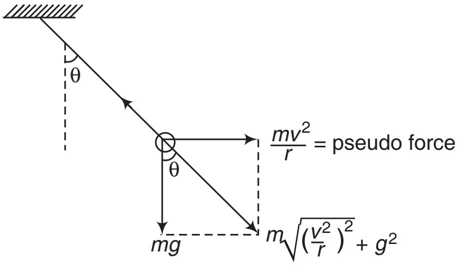
k_{net} in the given situation is $(k + \rho Ag)$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k+\rho Ag}}$$

$$= 2\pi\sqrt{\frac{10}{100 + 1000\times20\times10^{-4}\times10}}$$

$$= 1.8\text{ s}$$ **Ans.**

15. $g_e = \sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$


θ
θ
$\frac{mv^2}{r}$ = pseudo force
mg
$m\sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$

$$T = 2\pi\sqrt{\frac{g}{g_e}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

16. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^3}{0.1}} = 200$ rad/s

$$X = 10 \sin (200\, t)$$

$$6 = 10 \sin 200\, t_1$$

$$\therefore \quad 200\, t_1 = \sin^{-1}\left(\frac{3}{5}\right) = 0.646 \text{ rad}$$

$$\therefore \quad t_1 = 3.23 \text{ ms}$$

$$8 = 10 \sin 200\, t_2$$

$$\text{or} \quad 200\, t_2 = \sin^{-1}\left(\frac{4}{5}\right) = 0.925 \text{ rad}$$

$$\therefore \quad t_2 = 4.62 \text{ ms}$$

$$\therefore \quad \Delta t = 1.4 \text{ ms} = 1.4\times10^{-3}\text{s}$$ **Ans.**

17. (a) For small displacement x, the term x^2 can be neglected.

$$\therefore \quad F = -100x$$

Comparing with $F = -kx$ we have,

$$k = 100 \text{ N/m}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{0.2}{100}} = 0.28 \text{ s}$$ **Ans.**

(b) $|\Delta F| = 10x^2 = 10\,(0.04)^2 = 0.016$ N

$$|F| = 100\,x = 100\,(0.04) = 4\text{N}$$

$$\therefore \quad \%\text{ error} = \frac{|\Delta F|}{|F|}\times100 = 0.4\%$$

18. It is a physical pendulum, the time period of which is,

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Here, I = moment of inertia of the ring about point of suspension

$$= mr^2 + mr^2 = 2mr^2$$

and l = distance of point of suspension from centre of gravity $= r$

$$\therefore \quad T = 2\pi\sqrt{\frac{2mr^2}{mgr}}$$

$$= 2\pi\sqrt{\frac{2r}{g}}$$

∴ Angular frequency

$$\omega = \frac{2\pi}{T}$$

$$\text{or} \quad \omega = \sqrt{\frac{g}{2r}}$$ **Ans.**

19. (a) Frequency $= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

(Frequency is independent of g in spring)

(b) Extension in spring in equilibrium

$$\text{initial} = \frac{mg}{k}$$

Extension in spring in equilibrium in accelerating lift $= \frac{m\,(g+a)}{k}$

$$\therefore \quad \text{Amplitude} = \frac{m\,(g+a)}{k} - \frac{mg}{k} = \frac{ma}{k}$$ **Ans.**

20. $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.05} = (40\pi)$ rad/s

Since, the body starts from the extreme position, we can write

$$\theta = \theta_0 \cos\omega t$$

$$\text{or} \quad \theta = \left(\frac{\pi}{10}\right)\cos 40\pi t$$ **Ans.**

21. $\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$ rad/s

If at $t = 0$, particle passes through its mean position ($x = A \sin \omega t$) with maximum speed its v - t equation can be written as

$$v = \omega A \cos \omega t$$

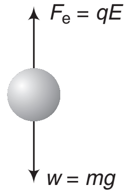
Substituting the given values, we have

$$2 = \left(\frac{\pi}{8}\right) A \cos\left(\frac{\pi}{8}\right)(2)$$

$$\therefore \quad A = \frac{16\sqrt{2}}{\pi}\text{ m}$$

$$= 7.2 \text{ m}$$ **Ans.**

22. The two forces acting on the bob are shown in figure



g_{eff} in this case will be $\dfrac{w - F_e}{m}$

or $$g_{\text{eff}} = \frac{mg - qE}{m}$$

$$= g - \frac{qE}{m}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g_{\text{eff}}}}$$

$$= 2\pi\sqrt{\frac{l}{g - \dfrac{qE}{m}}} \qquad \textbf{Ans.}$$

23. (a) $g_e = g + a$

(b) $g_e = g - a$

(c) $g_e = 0$

(d) $g_e = \sqrt{g^2 + a^2}$

24. (a) Before the lump of putty is dropped the total mechanical energy of the block and spring is

$$E_1 = \frac{1}{2}kA_1^2.$$

Since, the block is at the equilibrium position, $U = 0$, and the energy is purely kinetic. Let v_1 be the speed of the block at the equilibrium position, we have

$$E_1 = \frac{1}{2}Mv_1^2 = \frac{1}{2}kA_1^2$$

$$\therefore \quad v_1 = \sqrt{\frac{k}{M}}\,A_1$$

During the process momentum of the system in horizontal direction is conserved. Let v_2 be the speed of the combined mass, then

$$(M + m)v_2 = Mv_1$$

$$\therefore \quad v_2 = \frac{M}{M + m}v_1$$

Now, let A_2 be the amplitude afterwards. Then,

$$E_2 = \frac{1}{2}kA_2^2 = \frac{1}{2}(M + m)v_2^2$$

Substituting the proper values, we have

$$A_2 = A_1\sqrt{\frac{M}{M + m}} \qquad \textbf{Ans.}$$

Note *$E_2 < E_1$, as some energy is lost into heating up the block and putty.*

Further, $$T_2 = 2\pi\sqrt{\frac{M + m}{k}} \qquad \textbf{Ans.}$$

(b) When the putty drops on the block, the block is instantaneously at rest. All the mechanical energy is stored in the spring as potential energy. Again the momentum in horizontal direction is conserved during the process. But now it is zero just before and after putty is dropped. So, in this case, adding the extra mass of the putty has no effect on the mechanical energy, i.e.

$$E_2 = E_1 = \frac{1}{2}kA_1^2$$

and the amplitude is still A_1. Thus,

$$A_2 = A_1$$

and $$T_2 = 2\pi\sqrt{\frac{M + m}{k}} \qquad \textbf{Ans.}$$

25. Let v is speed of combined mass just after collision. Then, from conservation of linear momentum, we have

$$(M + m)v = mv_0$$

$$\therefore \quad v = \frac{mv_0}{M + m}$$

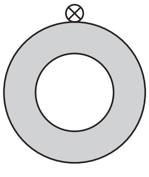
This is maximum speed of combined mass at mean position

$$\therefore \quad v = \omega A$$

or $$\left(\frac{m\,v_0}{M + m}\right) = \sqrt{\left(\frac{k}{M + m}\right)}\,A$$

$$\therefore \quad A = \frac{mv_0}{\sqrt{k\,(M + m)}}$$

26. Mass per unit area $= \dfrac{m}{\pi\,(R^2 - r^2)} = \sigma$ (say)



Whole mass $m_1 = (\pi R^2)\sigma$

$$= \left(\frac{R^2}{R^2 - r^2}\right)m$$

Mass of cavity $m_2 = (\pi r^2)\,\sigma$

$$= \left(\frac{r^2}{R^2 - r^2}\right) m$$

$$I = \frac{3}{2}\, m_1 R^2 - \left[\frac{1}{2}\, m_2 r^2 + m_2 R^2\right]$$

$$= \frac{3}{2}\left[\frac{R^2}{R^2 - r^2}\right] mR^2 - \left[\frac{1}{2}\left(\frac{r^2}{R^2 - r^2}\right) mr^2 + \left(\frac{r^2}{R^2 - r^2}\right) mR^2\right]$$

$$= \frac{m}{2\,(R^2 - r^2)}\,[3R^4 - r^4 - 2r^2R^2]$$

$$= \frac{m\,(3R^2 + r^2)}{2}$$

Now, $\quad T = 2\pi \sqrt{\dfrac{I}{mgl}} \quad$ …(i)

Here, $\quad l = R$

$\therefore \quad \dfrac{I}{mR} = \dfrac{3R}{2} + \dfrac{r^2}{2R}$

Substituting in Eq. (i) we have,

$$T = 2\pi \sqrt{\frac{\dfrac{3R}{2} + \dfrac{r^2}{2R}}{g}}$$

Comparing with

$$T = 2\pi \sqrt{\frac{l}{g}}$$

l of pendulum $= \dfrac{3R}{2} + \dfrac{r^2}{2R}$

$l = \dfrac{3R}{2} \quad$ as $r \to 0$

and $\quad l = 2R \quad$ as $r \to R$

27. $2\pi \sqrt{\dfrac{l}{g}} = 2\pi \sqrt{\dfrac{I}{mgl'}}$

$\therefore \quad l = \dfrac{I}{ml'}$

or $\quad I = (m)\,(l)\,(l')$

$= (200)\,(20)\,(35)$

$= 1.4 \times 10^5$ g-cm^2

28. At earth's surface, the value of time period is given by

$$T = 2\pi \sqrt{\frac{L}{g}} \quad \text{or} \quad T \propto \frac{1}{\sqrt{g}}$$

At a depth h below the surface,

$$g' = g\left(1 - \frac{h}{R}\right)$$

$\therefore \quad \dfrac{T'}{T} = \sqrt{\dfrac{g}{g'}}$

$$= \sqrt{\frac{1}{\left(1 - \dfrac{h}{R}\right)}} = \sqrt{\frac{R}{R - h}}$$

$\therefore \quad T' = T\sqrt{\dfrac{R}{R - h}}$

or $\quad T' \propto \dfrac{1}{\sqrt{R - h}} \quad$ **Hence proved.**

Further, $\quad T_{R/2} = 2\sqrt{\dfrac{R}{R - R/2}}$

$= 2\sqrt{2}$ s $\quad$ **Ans.**

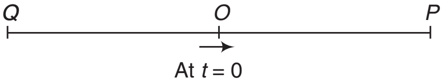
29. (a) Distance travelled in first 4 s

$= OP + PO = A + A = 2A$

Distance travelled in next 4 s

$= OQ + QO = A + A = 2A$

Two distances are equal.



(b) Distance travelled in first 2 s $= OP = A$ and distance travelled in next 2 s $= PO = A$

Again the two distances are same.

30. (a) $u = \omega\sqrt{A^2 - x_1^2} \quad$ … (i)

$v = \omega\sqrt{A^2 - x_2^2} \quad$ …(ii)

Solving Eqs. (i) and (ii), we get the result,

$$A \text{ or } a = \sqrt{\frac{v^2x_1^2 - u^2x_2^2}{v^2 - u^2}}$$

(b) $u_1 = \omega\sqrt{A^2 - x_1^2} \quad$ …(iii)

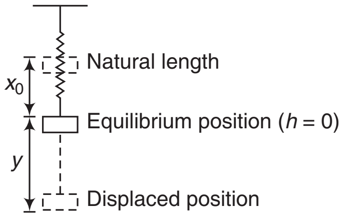
$u_2 = \omega\sqrt{A^2 - x_2^2} \quad$ …(iv)

Solving Eqs. (iii) and (iv), we find

$$\omega = \sqrt{\frac{u_1^2 - u_2^3}{x_2^2 - x_1^2}} = \frac{2\pi}{T}$$

$\therefore \quad T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{u_1^2 - u_2^2}}$

31. At equilibrium position, $kx_0 = mg$



In displaced position,

$$U = \frac{1}{2}k(x_0 + y)^2 - mg\ y$$

$$= \frac{1}{2}kx_0{}^2 + \frac{1}{2}ky^2 + kx_0y - mgy$$

Substituting $kx_0 = mg$ we get,

$$U = \frac{1}{2}kx_0^2 + \frac{1}{2}ky^2$$

$$= \frac{1}{2}kx_0^2 + \frac{1}{2}ky^2 + kx_0y - mgy$$

But $\quad \frac{1}{2}kx_0^2 =$ constant say U_0

$$\therefore \quad U = U_0 + \frac{1}{2}ky^2$$

32. While returning to equilibrium position,

$$\frac{1}{2}kd^2 = \frac{1}{2}(m_1 + m_2)v^2$$

$$\therefore \quad v = \left(\sqrt{\frac{k_1}{m_1 + m_2}}\right)d$$

Now, after mean position m_2 is detached from m_1 and keeps on moving with this constant velocity v towards right. Block $m_1$starts SHM with spring and this v becomes its maximum velocity at mean position.

$$\therefore \quad v = \omega A$$

$$\therefore \quad \left(\sqrt{\frac{k}{m_1 + m_2}}\right)d = \left(\sqrt{\frac{k}{m_1}}\right)A$$

$$\therefore \quad A = \left(\sqrt{\frac{m_1}{m_1 + m_2}}\right)d$$ **Ans.**

33. (a) In equilibrium, let x_0 is the elongation then,

$$F = kx_0$$

$$\therefore \quad x_0 = \frac{F}{k}$$

This x_0 is the amplitude

$$\therefore \quad A = x_0 = \frac{F}{k}$$

$$T = 2\pi\sqrt{\frac{M}{k}}$$

(b) $E = \frac{1}{2}kx_0^2 = \frac{1}{2}k\left(\frac{F}{k}\right)^2 = \frac{F^2}{2k}$

(c) Kinetic energy at mean position

$$= E = \frac{F^2}{2k}$$

34. (a) $F = kx_0$

$$\therefore \quad x_0 = \frac{F}{k} = \frac{10}{100} = 0.1 \text{ m}$$ **Ans.**

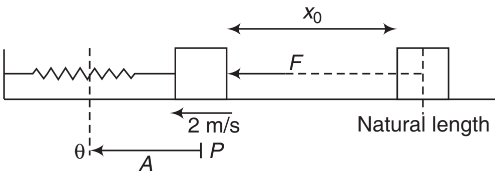
(b) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx_0^2$

$$= \frac{1}{2} \times 1 \times (2)^2 + \frac{1}{2} \times 100 \times (0.1)^2 = 2.5 \text{ J}$$ **Ans.**

(c) $T = 2\pi\sqrt{\frac{M}{k}}$

$$= 2\pi\sqrt{\frac{1}{100}} = \frac{\pi}{5} \text{ sec}$$ **Ans.**

(d) From P to Q



Work done by applied force = change in mechanical energy.

$$\therefore \quad FA = E_Q - E_P$$

$$\therefore \quad (10)\ A = \frac{1}{2}k(A + x_0)^2 - 2.5$$

$$= \frac{1}{2} \times 100\,(A + 0.1)^2 - 2.5$$

Solving this equation, we get

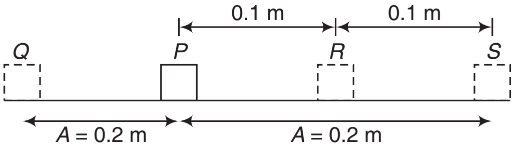
$$A = 0.2 \text{ m}$$ **Ans.**

(e) $U_Q = \frac{1}{2}k(A + x_0)^2$

$$= \frac{1}{2} \times 100\,(0.2 + 0.1)^2$$

$$= 4.5 \text{ J}$$ **Ans.**

(f) P = mean position



$Q, S =$ extreme position

$R =$ natural length

$$U_S = \frac{1}{2}k\,(0.1)^2$$

$$= \frac{1}{2} \times 100 \times (0.1)^2 = 0.5\ \text{J}$$

Due to work done by the applied fore, F, answer are different.

35. $$T_A = 2\pi\sqrt{\frac{I}{mgl}}$$

$$= 2\pi\sqrt{\frac{(md^2/3)}{mg\,(d/2)}} = \left(\frac{\sqrt{2}}{3}\right) 2\pi\sqrt{\frac{d}{g}}$$

and $$T_B = 2\pi\sqrt{\frac{d}{g}}$$

$\therefore$ $$\frac{T_A}{T_B} = \sqrt{\frac{2}{3}} = 0.816$$ **Ans.**

36. In displaced position,

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\,\omega^2$$

Putting $$I = \frac{1}{2}mR^2$$

and $$\omega = \frac{v}{R}$$

we get $$E = \frac{1}{2}kx^2 + \frac{3}{4}mv^2$$

Since, $E =$ constant

$\therefore$ $$\frac{dE}{dt} = 0$$

or $$0 = \frac{1}{2}k\left(\frac{dx}{dt}\right)(2x) + \frac{3}{4}m\,(2v)\frac{dv}{dt}$$

Putting, $$\frac{dx}{dt} = v \text{ and } \frac{dv}{dt} = a$$

we get, $$F = (ma) = -\left(\frac{2k}{3}\right)x$$

Since, $F \propto -x$ motion is simple harmonic

$$k_e = -\frac{2k}{3}$$

$$T = 2\pi\sqrt{\frac{m}{k_e}} = 2\pi\sqrt{\frac{3m}{2k}}$$ **Hence Proved.**

37. In equilibrium

$$k\,x_0 = mg$$

$\therefore$ $$x_0 = \frac{mg}{k} = \frac{0.5 \times 10}{20}$$

$$= 0.25\ \text{m}$$

When displaced downwards by x from the mean position, total mechanical energy,

$$E = -\,mgx + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}(x + x_0)^2$$

Substituting $I = 0.6\,MR^2$ and $\omega = \dfrac{v}{R}$

We have,

$$E = -\,mgx + \frac{1}{2}mv^2 + \frac{1}{2}(0.6\,MR^2)\left(\frac{v}{R}\right)^2$$

$$+\frac{1}{2}k\,(x + x_0)^2$$

Since $E =$ constant

$\therefore$ $$\frac{dE}{dt} = 0$$

or $$0 = -\,mg\left(\frac{dx}{dt}\right) + \frac{1}{2}m\left(2v\cdot\frac{dv}{dt}\right)$$

$$+ (0.6\,M) + (v)\frac{dv}{dt} + \frac{1}{2}k\,[2\,(x + x_0)]\frac{dx}{dt}$$

Putting $\dfrac{dx}{dt} = v$, $kx_0 = mg$ and $\dfrac{dv}{dt} = a$

We have

$$0 = (ma + 0.6\,Ma) + kx$$

or $$a = -\left(\frac{k}{m + 0.6\,M}\right)x$$

Since, $a \propto -x$ motion is SHM.

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m + 0.6\,M}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{20}{0.5 \times 0.6 \times 5}}$$

$$= 0.38\ \text{Hz}$$ **Ans.**

38. $x = A\sin\omega t$ ...(i)

$$y = A\sin(2\omega t + \pi/2)$$

$$= A\cos 2\omega t$$

$$= A\,(1 - 2\sin^2\omega t)$$

From Eq. (i),

$$\sin\omega t = \frac{x}{A}$$

$\therefore$ $$y = A\left(1 - \frac{2x^2}{A}\right)$$

39. (a) $x = x_1 + x_2$

At $t = 0.0125$ s

$$x_1 = -\sqrt{2}\text{ cm}$$

and $$x_2 = -1\text{ cm}$$

$\therefore$ $$x = x_1 + x_2 = -2.41\text{ cm}$$

(b) At $t = 0.025$ s.

$$x_1 = 2\text{cm}$$

and $$x_2 = -1.73\text{cm}$$

$\therefore$ $$x = x_1 + x_2$$
$$= 0.27\text{cm}$$

LEVEL 2

Single Correct Option

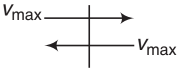
1. $\frac{1}{2}kA^2 = 1.0$

$$\frac{1}{2}k\,(0.4)^2 = 1.0$$

or $$k = 12.5 \quad \text{or} \quad \frac{25}{2}\text{ N/m}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{\frac{25}{2}}} = \frac{4\pi}{5}\text{ s}$$

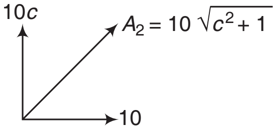
2. $v_{\max} = \omega A = \left(\sqrt{\frac{K}{m}}\right)a$



$$\Delta P = 2m\,v_{\max} = 2m\left(\sqrt{\frac{K}{m}}\right)a$$

$$F = \frac{\Delta P}{\Delta t} = \frac{\Delta P}{T/2} = \frac{2ma\sqrt{k/m}}{\pi\sqrt{m/k}} = \frac{2aK}{\pi}$$

3. $A_1 = 40$ units



Putting $A_1 = A_2$ we get,

$$c = \sqrt{15}$$

4. $T = \frac{1}{F} = \frac{1}{2.5} = 0.4$ s

$$t = 0.3\text{ s} = \frac{3T}{4}$$

At the given time, particle will be in its extreme position if at $t = 0$ it crosses the mean position.

5. $\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$ rad/s

At $t = 0$, particle crosses the means position. Hence its velocity is maximum. So, velocity as a function of time can be written as

$$v = v_{\max}\cos\omega t$$

or $$v = \omega A\cos\omega t$$

$\therefore$ $$1 = \left(\frac{\pi}{8}\right)A\cos\left(\frac{\pi}{8}\right)(2)$$
$$= \left(\frac{\pi}{8\sqrt{2}}\right)A$$

$\therefore$ $$A = \frac{8\sqrt{2}}{\pi}\text{m}$$

6. $g_e = \sqrt{g^2 + a^2} = \sqrt{(10)^2 + (4)^2}$

$$= 10.77\text{ m/s}^2$$

$$T = 2\pi\sqrt{\frac{l}{g_e}} = 2\pi\sqrt{\frac{1}{10.77}}$$
$$= 1.90\text{ s}$$

7. If $v = v_0\sin\omega t$, then

$$a = \frac{dv}{dt} = v_0\omega\cos\omega t$$

$$\sin\omega t = \frac{v}{v_0} \quad \ldots\text{(i)}$$

$$\cos\omega t = \frac{a}{v_0\omega} \quad \ldots\text{(ii)}$$

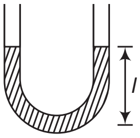
Squaring and adding these equations, we get

$$1 = \frac{v_2}{v_0^2} + \frac{a^2}{v_0\omega}$$

$\therefore$ $$v^2 = \left(-\frac{v_0}{\omega}\right)a^2 + v_0^2$$

Hence, v^2 *versus* a^2 equation is a straight line with positive intercept and negative slope.

8. $T = 2\pi\sqrt{\frac{l}{g}}$



$$2l = \frac{1000}{20}\text{ g}$$
$$= 50\text{ cm} = 0.5\text{ cm}$$

$\Rightarrow$ $$l = 0.25\text{ m}$$

Now $$T = 2\pi\sqrt{\frac{0.25}{9.8}} = 1\text{ s}$$

9. $T = 2\pi\sqrt{\dfrac{m}{k}}$

Here $\qquad m = (\rho_0)(a^3)$

and $\qquad K = \rho_l\, A\, g = \rho a^2 g$

$\therefore \qquad T = 2\pi\sqrt{\dfrac{\rho_0 a}{\rho g}}$

10. $T = 2\pi\sqrt{\dfrac{I}{mgd}}$

$$= 2\pi\sqrt{\frac{\dfrac{ml^2}{12} + md^2}{mgd}}$$

$$= 2\pi\sqrt{\frac{l^2 + 12d^2}{12\,d}}$$

$$= 2\pi\sqrt{\left(\frac{l^2}{12d} + d\right)}$$

T is minimum when first derivation of the quantity inside the root with respect to d is zero.

$$-\frac{l^2}{12d^2} + 1 = 0$$

or $\qquad \dfrac{d}{l} = \dfrac{1}{\sqrt{12}}$

11. In solved example, we have shown that

$$T_1 = 2\pi\sqrt{\frac{m}{k}}$$

$$T_2 = 2\pi\sqrt{\frac{4m}{k}}$$

and $\qquad T_3 = 2\pi\sqrt{\dfrac{m}{4k}}$

12. $T = 2\pi\sqrt{\dfrac{m + m_s/3}{k}}$ (in general)

Here $\qquad m_s =$ mass of spring

and $\qquad m =$ mass of block

13. Half the oscillation is completed with one spring and the other half with other spring. Hence,

$$T = \frac{T_1}{2} + \frac{T_2}{2} = \frac{2\pi\sqrt{\dfrac{M}{k}}}{2} + \frac{2\pi\sqrt{\dfrac{M}{4k}}}{2}$$

$$= \pi\sqrt{\frac{M}{k}}\left[1 + \frac{1}{2}\right] = \frac{3\pi}{2}\sqrt{\frac{M}{k}}$$

14. $F = -\dfrac{dU}{dx} = 20 - 10x$

$$F = 0$$

at $\qquad x = 0$

So, from $x = -3$ to $x = 2$, amplitude in 5.

Hence, other extreme position will be

$$x = 2 + 5 = 7$$

15. $k \propto \dfrac{1}{\text{Length of spring}}$

Length	Force constant
L	k
nL	k/n
$(1-n)L$	$k/(1-n)$

$$m_1 = nm$$

and $\qquad m_2 = (1-n)\,m$

$$\frac{f_1}{f_2} = \frac{\dfrac{1}{2\pi}\sqrt{\dfrac{k/n}{(1-n)\,m}}}{\dfrac{1}{2\pi}\sqrt{\dfrac{k/(1-n)}{nm}}} = 1$$

16. $\dfrac{1}{2}k\,(A^2 - x^2) = \dfrac{1}{4}\left(\dfrac{1}{2}kA^2\right)$

$\therefore \qquad x = \dfrac{\sqrt{3}}{2}A$

$\therefore \qquad CD = CB = \dfrac{\sqrt{3}}{2}R$

or $\qquad BD = 2\,(CD)$

or $\qquad 2\,(CB) = \sqrt{3}R$

17. $K_Q = \dfrac{YA}{L}$

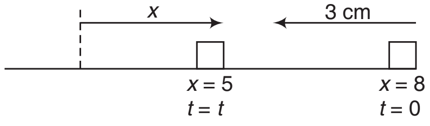
$$K_p = \frac{(2Y)\,(A/2)}{L} = \frac{YA}{L}$$

$\therefore \qquad K_P = K_Q = K = \dfrac{YA}{L}$

Half oscillation is completed with P and half with Q. But their value of K is same. Hence, we can say that in one oscillation one time period is completed with spring.

$$T = \left(\frac{2L}{v}\right) + 2\pi\sqrt{\frac{m}{K}} = \frac{2L}{v} + 2\pi\sqrt{\frac{mL}{AY}}$$

18. $X = A\cos\omega t$



$$5 = 8\cos\omega t$$

$$\omega t = \cos^{-1}(5/8) = 0.9$$

$$\therefore \quad t = \frac{0.9}{\omega} = \frac{0.9}{(2\pi/T)}$$

$$= \frac{0.9T}{2\pi} = \frac{0.9 \times 1.2}{2\pi}$$

$$= 0.17 \text{ s}$$

Ans.

19. $OP = a \sin 60° = \dfrac{\sqrt{3}\,a}{2}$

$$OC = l = \frac{2}{3}(OP) = \frac{a}{\sqrt{3}}$$

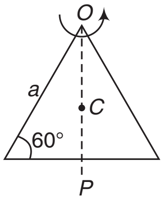
$$I = I_0 = 2\left(\frac{ma^2}{3}\right) + m\,(OP)^2 + \frac{ma^2}{12}$$

$$= \frac{2}{3}ma^2 + \frac{3}{4}ma^2 + \frac{1}{12}ma^2 = \frac{3}{2}ma^2$$

$$T = 2\pi\sqrt{\frac{I}{(3m)\,gl}}$$

$$= 2\pi\sqrt{\frac{(3/2m\,a^2)}{(3m)\,g\left(\dfrac{a}{\sqrt{3}}\right)}}$$

$$= 2\pi\sqrt{\frac{\sqrt{3}}{2}\,\frac{a}{g}}$$



Putting $a = \dfrac{1}{\sqrt{3}}$ m and $g = 10 \text{ m/s}^2$

We get, $T = \dfrac{\pi}{\sqrt{5}}$ s **Ans.**

20. $t = \dfrac{2.5\,\pi}{\omega} = \dfrac{2.5\,\pi}{(2\pi/T)}$

$$= (2.5)\left(\frac{T}{2}\right)$$

$$= 5\left(\frac{T}{4}\right)$$

Particle starts from extreme position and in every $\dfrac{T}{4}$ time it travels a distance A. So, in time $t = 5\,(T/4)$ it will travel a distance $5\,A$.

21. Half of the oscillation is completed with length l and rest half with $l/4$.

$$\therefore \text{ Time period} = \frac{T_1}{2} + \frac{T_2}{2}$$

$$= \frac{1}{2}\left[2\pi\sqrt{\frac{l}{g}} + 2\pi\sqrt{\frac{l/4}{g}}\right]$$

$$= \frac{3}{4}\left[2\pi\sqrt{\frac{l}{g}}\right] = \frac{3}{4}T$$

22. $A = \text{radius} = \dfrac{\text{Diameter}}{2} = 0.4$ m

$$f = \frac{30}{60} = \frac{1}{2} \text{ rps or } \frac{1}{2}\text{Hz}$$

$$\therefore \quad T = \frac{1}{f} = 2 \text{ s.}$$

23. $T = 0.6\pi$

$$\therefore \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{0.6\,\pi}$$

$$= \frac{10}{3} \text{ rad/s}$$

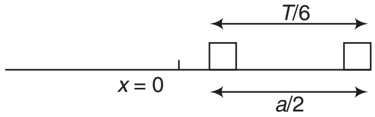
Particle starts from $y = -A = -2$ cm

$$\therefore \quad y = -A\cos\omega t$$

$$= A\sin(\omega t - \pi/2)$$

$$= 2\sin\left(\frac{10}{3}t - \pi/2\right)$$

24. Mean velocity $= \dfrac{\text{displacement}}{\text{time}}$



$$= \frac{(a/2)}{(T/6)} = \frac{3a}{T}$$

25. Maximum acceleration $= g$

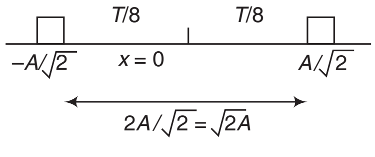
$$\therefore \quad \omega^2 A = g$$

$$\therefore \quad (2\pi f)^2\,(0.5) = g$$

$$f^2 = \frac{2g}{(2\pi)^2}$$

$$\therefore \quad f = \frac{\sqrt{2g}}{2\pi}$$

26. $\dfrac{1}{2}kX^2 = \dfrac{1}{2}k\,(A^2 - X^2)$

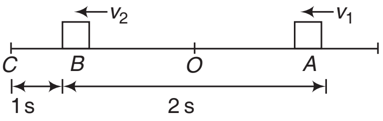


$$\therefore \quad X = \pm\frac{A}{\sqrt{2}}$$

$$\text{Average speed} = \frac{d}{t} = \frac{\sqrt{2}A}{T/8 + T/8}$$

$$= \frac{4\sqrt{2}\,A}{T}$$

27. $V_1 = V_2$



$$t_{AB} = 2\text{ s}$$

$$\therefore \quad t_{OB} = 1\text{ s}$$

$$t_{BCB} = 2\text{ s}$$

$$\therefore \quad _{BC} = 1\text{ s}$$

$$\therefore \quad t_{OB} = t_{BC}$$

$$\therefore \quad OB = \frac{A}{\sqrt{2}}$$

or

$$\frac{OB}{A} = \frac{1}{\sqrt{2}}$$

28. $t = \dfrac{\pi}{6\omega}$

$$\therefore \quad \omega t = (\pi/6)$$

According to the equation,

$$X = A\cos\omega t$$

$$X = A \text{ at } t = 0$$

and

$$X = \frac{\sqrt{3}A}{2} \text{ at } \omega t = \pi/6$$

$\therefore$ Distance travelled

$$= A - \frac{\sqrt{3}}{2}A$$

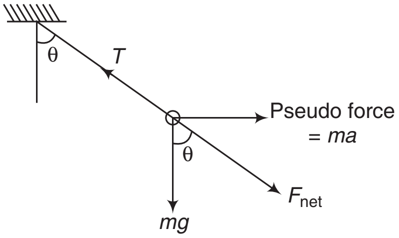
$$= (2-\sqrt{3})\frac{A}{2}$$

$$\text{Average speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{(2-\sqrt{3})\,A/2}{(\pi/6\omega)} = \frac{3\omega A}{\pi}(2-\sqrt{3})$$

More than One Correct Options

1. $\tan\theta = \dfrac{ma}{mg} = \dfrac{a}{g}$

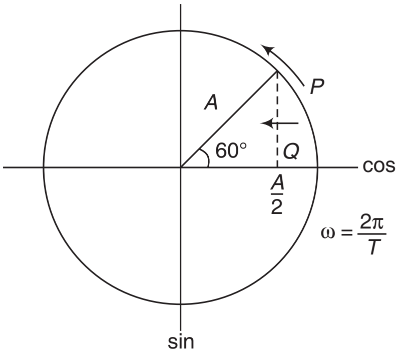


$$\therefore \quad \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

$$T = F_{\text{net}}$$

$$= m\sqrt{a^2 + g^2}$$

2. $x = A\sin(\omega t + 150°)$



$$= A\sin\left(\frac{2\pi}{T}t + \frac{5\pi}{6}\right)$$

or

$$x = A\cos(\omega t + 60°)$$

$$= A\cos\left(\frac{2\pi}{T}t + \frac{\pi}{3}\right)$$

3. $kx_0 = mg$

$$\therefore \quad x_0 = \frac{mg}{k} = \frac{1\times 10}{500}$$

$$= 0.02\text{ m} = 2\text{ cm}$$

So, equilibrium is obtained after an extension of 2 cm of at a length of 42 cm. But it is released from a length of 45 cm.

$$\therefore \quad A = 3\text{cm} = 0.03\text{ m}$$

(b) $v_{\max} = \omega A = \sqrt{\dfrac{k}{m}}A$

$$= \left(\sqrt{\frac{500}{1}}\right)(0.03) = 0.3\sqrt{5}\text{ m/s}$$

$$= 30\sqrt{5}\text{ cm/s}$$

(c) $a_{\max} = \omega^2 A$

$$= \left(\frac{k}{m}\right)(A) = \left(\frac{500}{1}\right)(0.03) = 15\text{ m/s}^2$$

(d) Mean position is at 42 cm length and amplitude is 3 cm. Hence block oscillates between 45 cm length and 39 cm. Natural length 40 cm lies in between these two, where elastic potential energy = 0.

4. v or KE = 0 at $y = \pm A$

v or KE = maximum at $y = 0$

F or a is maximum at $y = \pm A$

F or a is zero at $y = 0$

5. $v = \omega\sqrt{A^2 - x^2}$

$$\therefore \qquad \frac{v^2}{\omega^2} + \frac{x^2}{(1)^2} = A^2$$

i.e. v - x graph is an ellipse.

$$a = -\omega^2 x$$

i.e. a - x graph is a straight line passing through origin with negative slope.

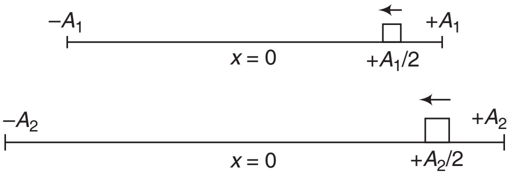
6. $a = 0$ at $x = 0.5$ m

and particle is released from $x = 2$m

Hence,

$$A = 2 - 0.5$$
$$= 1.5 \text{ m}$$
$$\omega^2 = 100$$
$$\therefore \qquad \omega = 10 \text{ rad/s}$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10}$$
$$= 0.63 \text{ s}$$
$$v_{\max} = \omega A = (10)(1.5)$$
$$= 15 \text{m/s}$$

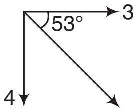
7. Two particles shown in figure are in same phase, although they have distances from the mean position at $t = 0$



8. The given equation can be written as

$$y = 3 \sin 100\,\pi t + (4 - 4 \cos 100\,\pi t) - 6$$
$$= 3 \sin 100\,\pi t + 4 \sin(100\pi t + \pi/2) - 2$$

or $\quad y = 5 \sin(100\pi - 53°) - 2$



$$y_{\max} = 5 - 2 = 3$$
$$y_{\min} = -5 - 2 = -7$$

$$\text{Mean position} = \frac{y_{\max} + y_{\min}}{2} = -2 \text{cm}$$

Comprehension Based Questions

1. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{2}} = 10\sqrt{2}$ rad/s

2. $Kx_0 = ma$

$$x_0 = \frac{ma}{K} = \frac{2 \times 5}{400} \text{ m}$$
$$= 0.025 \text{ m}$$
$$= 2.5 \text{cm}$$
$$\therefore \qquad A = x_0 = 2.5 \text{ cm}$$

Match the Columns

1. $x_{\min} = 2 - 2 = 0 \rightarrow$ extreme position

$x_{\min} = 2 + 2 = 4 \rightarrow$ extreme position

$$\text{Mean position} = \frac{x_{\min} + x_{\min}}{2} = 2$$

Maximum potential energy is at extreme positions.

2. $U_0 + \frac{1}{2}KA^2 = U_{\max}$

$$4 + \frac{1}{2}kA^2 = -20 \quad \text{or} \quad \frac{1}{2}kA^2 = 16\text{J}$$

(a) $\qquad U = U_0 + \frac{1}{2}kX^2$

$$= U_0 + \frac{1}{2}k\left(\frac{A}{2}\right)^2$$
$$= 4 + \frac{16}{4} = 8 \text{ J}$$

(b) $\text{KE} = \frac{1}{2}k\left(A^2 - \frac{A^2}{116}\right)$

$$= \frac{15}{16}\left(\frac{1}{2}kA^2\right) = \frac{15}{16} \times 16 = 15\text{J}$$

(c) $\text{KE} = \frac{1}{2}k\,(A^2 - 0)$

$$= \frac{1}{2}kA^2 = 16\text{J}$$

(d) $\text{KE} = \frac{1}{2}k\left(A^2 - \frac{A^2}{4}\right)$

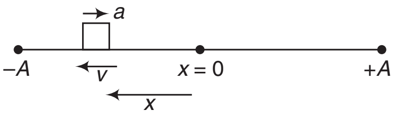
$$= \frac{3}{4}\left(\frac{1}{2}kA^2\right) = \frac{3}{4}(16\text{J}) = 12\text{J}$$

3. (a) $x \propto -\alpha$

At time t_1 acceleration is positive. Hence, X will be negative.

(b) At time $t_2 \quad a = 0 \quad \therefore \quad x = 0$

(c) At time t_1, x is negative and a is increasing. So, the particle is moving toward extreme position.



$\therefore \quad$ Velocity is negative.

(d) At mean position, velocity is maximum. Just after few seconds acceleration becomes positive. So displacement will becomes negative. Hence, the velocity should be negative.

4. $v_{\max} = \omega A = (4\pi)(2)$

$$= 25.12 \text{ m/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi}$$

$$= 0.5 \text{ s}$$

Hence the given time

$$t = 1 \text{ s} = 2T$$

$$\text{Maximum KE} = \frac{1}{2} m v_{\max}^2$$

$$= \frac{1}{2} \times 2 \times (25.12)^2$$

$$= 631 \text{ J}$$

Given kinetic energy of 400 J is less than this maximum kinetic energy. So in one time period kinetic energy becomes 400 J four times. In time $2T$ it becomes 400 J eight times.

(d) $|a_{\max}| = \omega^2 A$

$$= (4\pi)^2(2)$$

$$= 315.5 \text{ m/s}^2$$

Given acceleration is less than this. So, it becomes two times in one time period and four times in time $2T$.

5. $x_{\max} = 4 + 6 = 10 \text{m}$

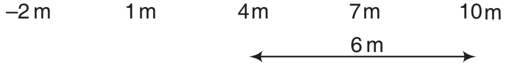
$x_{\min} = 4 - 6 = -2 \text{m}$

$$\text{Mean position} = \frac{x_{\max} + x_{\min}}{2} = 4 \text{ m}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \text{ s}$$

(a) $x = 10\text{m}$ to $x = 4 \text{ m}$ $t = \frac{T}{4} = \frac{1}{2}$ s

(b) $x = 10\text{m}$ to $x = 7\text{m}$, $t = \frac{T}{6} = \frac{1}{3}$ s



(c) $x = 7\text{m}$ to x , $t = 2\left(\frac{T}{12}\right) = \frac{1}{3}$ s

(d) $x = 10\text{m}$ to $x = -2\text{m}$, $t = \frac{T}{2} = 1$ s

Subjective Questions

1. $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{100}{4}} = \frac{2.5}{\pi} = 0.8 \text{ Hz}$ **Ans.**

with 1 kg mass, $f_0 = \frac{1}{2\pi}\sqrt{\frac{100}{1}} = \frac{5}{\pi}$ Hz

Further, from conservation of linear momentum (at mean position)

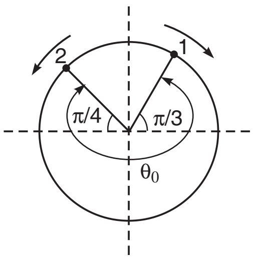
$$\omega A = \frac{1}{4}\omega_0 A_0$$

or $$fA = \frac{1}{4} f_0 A_0$$

or $$A = \frac{f_0 A_0}{4 f} - \frac{(5/\pi)(0.1)}{(4)(2.5/\pi)}$$

$$= 0.05 \text{ m}$$ **Ans.**

2. (a) $\theta_0 = \omega_1 t + \omega_2 t$



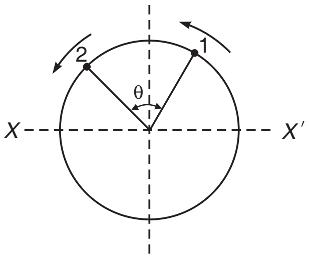
but $$\omega_1 = \omega_2 = \frac{2\pi}{T}$$

and $$\theta_0 = \pi + \frac{\pi}{3} + \frac{\pi}{4} = \frac{19}{12}\pi$$

$\therefore$ $$\frac{19\pi}{12} = \frac{2\pi}{T}t + \frac{2\pi}{T}t$$

or $$t = \frac{19}{48}T$$

(b) $\theta = 2\pi - \theta_0 = 2\pi - \frac{19}{12}\pi = \frac{5\pi}{12}$



Two particles will collide when line XX' becomes the line of bisector of angle θ.

$\therefore$ Any one of the particles (say-2) has rotated an angle

$$\omega t = \pi/4 + \theta/2$$

or $$\frac{2\pi}{T} t = \frac{\pi}{4} + \frac{5\pi}{24} = \frac{11\pi}{24}$$

$\therefore$ $$t = \frac{11T}{48}$$

3. $1.5 = \omega A = 2A \quad \Rightarrow \quad \therefore \quad A = 0.75$ m

Further, the particle will start its journey from its mean position in downward direction (−ve direction).

4. (a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{600}{2+1}} = 10\sqrt{2}$ rad/s

From conservation of linear momentum (at mean position) velocity of combined mass will be 2 m/s.

This is the maximum velocity of combined mass.

$\therefore$ $v = \omega A$

or $$A = \frac{v}{\omega} = \frac{2}{10\sqrt{2}} \text{ m}$$

$$= 0.141 \text{ m} = 14.1 \text{ cm}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{2}} = 0.44 \text{ s}$$

(b) If the collision is elastic,

$$\omega' = \sqrt{\frac{600}{2}} = 10\sqrt{3} \text{ rad/s}$$

In elastic collision,

$$v_2' = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1$$

$$= \left(\frac{2 \times 1}{1+2}\right)(6) = 4 \text{ m/s} \qquad (\text{As } v_2 = 0)$$

This is maximum velocity.

$\therefore$ $v_2' = \omega' A'$

or $$A' = \frac{v_2'}{\omega'} = \frac{4}{10\sqrt{3}}$$

$$= 0.23 \text{ m} = 23 \text{ cm}$$

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{10\sqrt{3}} = 0.36 \text{ s}$$

(c) In both cases journey is started from mean position

$\therefore$ $x = \pm A \sin \omega t$

will be the displacement-time equation. For impulse we can apply the equation. Impulse = change in linear momemtum.

5. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{4}} = 10$ rad/s

Let t_1 be the time from $x = 0$ to $x = 12$ cm and t_2 the time from $x = 0$ to $x = 9$ cm. Then,

$$12 = 15 \sin (10t_1)$$

or $$t_1 = 0.093 \text{ s}$$

$$9 = 15 \sin (10t_2)$$

or $$t_2 = 0.064 \text{ s}$$

$\therefore$ Total time $= t_1 + t_2 = 0.157$ s

6. (a) $y = a(1 - \cos \omega t)$

$$\frac{d^2y}{dt^2} = a\omega^2 \cos \omega t$$

$$N - mg = m \cdot \frac{d^2y}{dt^2}$$

or $$N = mg + ma\omega^2 \cos \omega t$$

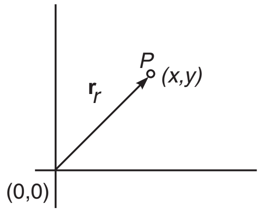
or $$N = m(g + a\omega^2 \cos \omega t)$$ **Ans.**

(b) $\left(\frac{d^2y}{dt^2}\right)_{\max} = a\omega^2$ or $a\omega^2 = g$

$\therefore$ $$a = \frac{g}{\omega^2} = \frac{980}{(11)^2}$$

$$= 8.1 \text{ cm}$$ **Ans.**

7. $\mathbf{F} = -kx\hat{\mathbf{i}} - ky\hat{\mathbf{j}}$



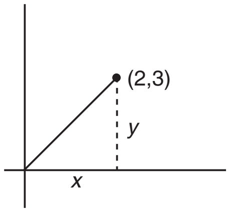
$\mathbf{F} = 0$ at $(0, 0)$

When it is displaced to a point P whose position vector is

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Force on it is $\mathbf{F} = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = -k\mathbf{r}$

Since, $\mathbf{F} \propto -\mathbf{r}$, motion is simple harmonic. At $t = 0$ particle is at (2, 3)



$$\frac{y}{x}=\frac{3}{2} \quad \text{or} \quad 2y-3x=0$$

i.e. the particle will oscillate simple harmonically along this line.

8. In equilibrium, $mg\sin\theta = kx_0$ ...(i)

When displaced by x,

$$E=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2+\frac{1}{2}k(x+x_0)^2-mgx\sin\theta$$

Since, $E = \text{constant}$

$$\frac{dE}{dt}=0$$

$$0=mv\left(\frac{dv}{dt}\right)+I\omega\left(\frac{d\omega}{dt}\right)+k(x+x_0)\frac{dx}{dt}-mg\sin\theta\frac{dx}{dt}$$

Substituting,

$$\frac{dv}{dt}=a,\quad \omega=\frac{v}{R},\quad I=\frac{1}{2}mR^2$$

$$\frac{d\omega}{dt}=\alpha=\frac{a}{R},\quad \frac{dx}{dt}=v$$

and $kx_0 = mg\sin\theta$

We get, $3ma = -2kx$

$$\therefore \quad f=\frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}=\frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$$

Substituting the values,

$$f=\frac{1}{2\pi}\sqrt{\frac{2\times 200}{\frac{3\times 100}{9.8}}}$$

$$= 0.56 \text{ Hz}$$ **Ans.**

9. In the displaced position,

$$E=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2+\frac{1}{2}k(2x)^2$$

$$I=\frac{1}{2}mR^2 \quad \text{and} \quad \omega=\frac{v}{R}$$

$$\therefore \quad E=\frac{3}{4}mv^2+2kx^2$$

$$E = \text{constant}$$

$$\therefore \quad \frac{dE}{dt}=0$$

$$\text{or} \quad \frac{3}{2}mv\frac{dv}{dt}+4kx\frac{dx}{dt}=0$$

Substituting, $\frac{dx}{dt}=v$ and $\frac{dv}{dt}=a$

$$a=-\frac{8k}{3m}.x$$

Comparing with, $a=-\omega^2x$

We have

$$\omega=\sqrt{\frac{8k}{3m}}=\sqrt{\frac{8\times 1000}{\frac{3\times 100}{9.8}}}=16.16\text{ rad/s}$$

$$\therefore \quad \theta=\theta_0\cos\omega t$$

$$\text{or} \quad \theta=0.4\cos(16.16t)$$ **Ans.**

10. Similar to Q.9

11. Let F be the restoring force (extra tension) on block m when displaced by x from its equilibrium position.

$$x=2x_1+2x_2$$

$$=2\left[\frac{2F}{k_1}+\frac{2F}{k_2}\right]$$

$$=4F\left(\frac{k_1+k_2}{k_1k_2}\right)$$

$$\text{or} \quad F=-\frac{k_1k_2}{4(k_1+k_2)}x$$

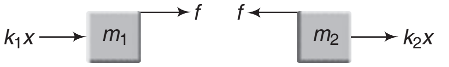
$$\therefore \quad a=-\frac{k_1k_2}{4m(k_1+k_2)}x$$

$$\omega=\sqrt{\frac{k_1k_2}{4m(k_1+k_2)}}$$ **Ans.**

12. (a) $f=\frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{\text{total mass}}}=\frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m_1+m_2}}$

(b) Suppose the system is displaced towards left by a distance x.

Restoring force on m_1 :



$$F=m_1\omega^2x \quad \text{(towards right)}$$

$$=m_1\left(\frac{k_1+k_2}{m_1+m_2}\right)x$$

Friction f on it will be towards right if,

$$k_1x<F$$

$$\text{or} \quad k_1x<m_1\left(\frac{k_1+k_2}{m_1+m_2}\right)x$$

$$\text{or} \quad \frac{k_1}{k_2}<\frac{m_1}{m_2}$$ **Ans.**

(c) $k_1A_m+\mu m_2g=m_1\left(\frac{k_1+k_2}{m_1+m_2}\right)A_m$

$$A_m\left(\frac{m_1k_1+m_1k_2}{m_1+m_2}-k_1\right)=\mu m_2g$$

$$\text{or} \quad A_m=\frac{\mu(m_1+m_2)m_2g}{m_1k_2-m_2k_1}$$ **Ans.**

13. (a) $\frac{1}{2}kx_0^2 = \frac{1}{2}\mu v_r^2$

Here, μ = reduced mass $= \frac{6 \times 3}{6+3}$

$$= 2 \text{ kg}$$

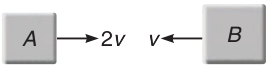
$$\therefore \quad v_r = \sqrt{\frac{k}{\mu}}\, x_0 = \left(\sqrt{\frac{200}{2}}\right)(3 \times 10^{-2})$$

$$= 0.3 \text{ m/s} = 2v + v$$

$$\therefore \quad v = 0.1 \text{ m/s}$$

$$\therefore \quad v_1 = 0.2 \text{ m/s}$$

and $\quad v_2 = 0.1 \text{ m/s}$ **Ans.**



Angular frequency

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{200}{2}}$$

$$= 10 \text{ rad/s}$$

(b) $$v_{\text{cm}} = \frac{m_1v_1 + m_2v_2 + m_3v_3}{m_1 + m_2 + m_3}$$

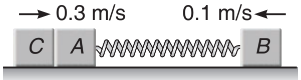
$$= \frac{(3)(0.2) - (6)(0.1) + 3(0.4)}{3 + 6 + 3}$$

$$= 0.1 \text{ m/s} \quad \text{(towards right)}$$ **Ans.**

(c) After collision velocity of combined blocks $(A + C)$

$$v_0 = \frac{(3 \times 0.2) + (3)(0.4)}{3 + 3} = 0.3 \text{ m/s}$$

and velocity of block B is $v_2 = 0.1$ m/s



The spring will compress till velocity of all the blocks become equal to the centre of mass. Applying conservation of mechanical energy,

$$\frac{1}{2}(3+3)(0.3)^2 + \frac{1}{2}(6)(0.1)^2 = \frac{1}{2}(3+3+6)(0.1)^2 + \frac{1}{2}kA^2$$

Solving this we get, $A = 0.048$ m

or $\quad A = 4.8$ cm **Ans.**

(d) $$\Delta E = \frac{1}{2}(3)(0.4)^2 + \frac{1}{2}(3)(0.2)^2 - \frac{1}{2}(3+3)(0.3)^2$$

$$= 0.24 + 0.06 - 0.27 = 0.03 \text{ J}$$ **Ans.**

14. Restoring torque

$$\tau = -(kl\theta)l - \left(k\frac{l}{2}\theta\right)\frac{l}{2} = -\frac{5}{4}kl^2\theta$$

Now, $$\left(\frac{ml^2}{3}\right)\alpha = -\left(\frac{5}{4}kl^2\right)\theta$$

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|} = \frac{1}{2\pi}\sqrt{\frac{15k}{4m}}$$

15. When the mass m is displaced from its mean position by a distance x, let F be the restoring (extra tension) force produced in the string. By this extra tension further elongation in the springs are $\frac{2F}{k_1}, \frac{2F}{k_2}, \frac{2F}{k_3}$ and $\frac{2F}{k_4}$ respectively.

Then,

$$x = 2\left(\frac{2F}{k_2}\right) + 2\left(\frac{2F}{k_2}\right) + 2\left(\frac{2F}{k_3}\right) + 2\left(\frac{2F}{k_4}\right)$$

or $$F\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right) = -x$$

Here negative sign shows the restoring nature of force.

$$a = -\frac{x}{m\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right)}$$

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

$$= 4\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}\right)}$$ **Ans.**

16. Let F be the extra tension in the string, when the block is displaced x from its mean position.

Extension in spring-2 is

$$x_2 = \frac{F}{k_2}$$

Extension is spring-1 is

$$x_1 = \frac{2F}{k_1}$$

$$x = 2x_1 + x_2 = \frac{4F}{k_1} + \frac{F}{k_2}$$

Extra tension F will become restoring force for the block. Therefore, above equation can be written as,

$$F = -\left(\frac{1}{\frac{4}{k_1} + \frac{1}{k_2}}\right)x = \left(\frac{k_1k_2}{4k_2 + k_1}\right)x$$

or $$k_e = \frac{k_1 k_2}{4k_2 + k_1}$$

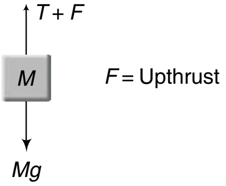
$$T = 2\pi\sqrt{\frac{m}{k_e}}$$

$$= 2\pi\sqrt{\frac{m(4k_2 + k_1)}{k_1 k_2}} \qquad \textbf{Ans.}$$

17. In equilibrium,

$$T + F = Mg \qquad \ldots\text{(i)}$$

When the block is further depressed by x, weight Mg remains unchanged, upthrust F increases by ρAxg and let ΔT be the increase in tension.



If a is the acceleration of block then,

$$\Delta T + \rho Axg = Ma \qquad \ldots\text{(ii)}$$

Restoring torque on the cylinder,

$$\tau = \left[\frac{kx}{2}\frac{R}{2} - \Delta T\, R\right] = \left[\frac{kxR}{4} - (Ma - \rho Axg)\, R\right]$$

$$\frac{1}{2}MR^2\alpha = \left[\frac{kR^2\theta}{4} - (MR\alpha - \rho AgR\theta)\, R\right]$$

or $$\frac{3}{2}MR^2\alpha = \left[\frac{kR^2}{4} + \rho AgR^2\right]\theta$$

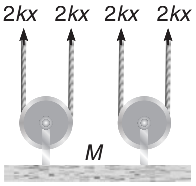
or $$\alpha = \frac{-\left[\frac{k}{4} + \rho Ag\right]}{\frac{3}{2}M}\theta$$

Here negative sign has been used for restoring nature of torque.

$$\therefore \qquad f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k + 4\rho Ag}{6M}} \qquad \textbf{Ans.}$$

18. If the mass M is displaced by x from its mean position each spring further stretches by $2x$.



Net restoring force

$$F = -8kx$$

$$\therefore \quad M \cdot a = -8kx$$

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\frac{8k}{M}}$$

$$= \frac{1}{\pi}\sqrt{\frac{2k}{M}} \qquad \textbf{Ans.}$$

15 Elasticity

INTRODUCTORY EXERCISE 15.1

1. $\Delta l = \dfrac{Fl}{AY}$ or $\Delta l \propto \dfrac{1}{Y}$

2. $\Delta l = \dfrac{Fl}{AY} = \dfrac{1000\times100}{4\times2\times10^6}$

$= 0.0125\,\text{cm}$

3. $\sigma = \dfrac{F}{A} = \dfrac{F}{\pi(R^2 - r^2)}$

$$R = \sqrt{\frac{F}{\sigma\pi} + r^2}$$

$$= \sqrt{\frac{1.6\times10^6}{90\times10^6\times3.14} + (0.1)^2}$$

$= 0.1251\ \text{m} = 125.1\ \text{mm}$

Diameter $= 2R = 250.2$ mm

4. Stress $= \dfrac{\text{Force}}{\text{Area}}$

Strain $= \dfrac{\Delta l}{l}$

Modulus of elasticity $= \dfrac{\text{Stress}}{\text{Strain}}$

INTRODUCTORY EXERCISE 15.2

1. Energy density = energy per unit volume

2. (a) Energy stored $U = \dfrac{1}{2}$ (stress) (strain) (volume)

or $U = \dfrac{1}{2}\left(\dfrac{F}{A}\right)\left(\dfrac{\Delta l}{l}\right)(Al)$

$= \dfrac{1}{2} F\cdot\Delta l$

$= \dfrac{1}{2}(100)(0.3\times10^{-3})$

$= 0.015$ J **Ans.**

(b) Work done = Potential energy stored

$= \dfrac{1}{2}k(\Delta l)^2$

$= \dfrac{1}{2}\left(\dfrac{YA}{l}\right)(\Delta l)^2$ $\left(\text{as } k = \dfrac{YA}{l}\right)$

Substituting the values, we have

$$W = \frac{1}{2}\frac{(2.0\times10^{11})(10^{-6})}{(2)}(0.1\times10^{-3})^2$$

$= 5.0\times10^{-4}$ J **Ans.**

Exercises

LEVEL 1

Assertion and Reason

2. At higher pressure, it is difficult to press the gas more. So, bulk modulus is high.

3. If length is doubled, Δl will also becomes two times and Y will remain same.

4. Bulk modulus is related to volume change and volume change is possible in all three states.

Young's modulus is related to length change, which is possible only in solids.

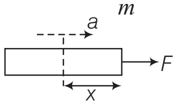
6. Stress ∝ strain only in proportionality limit.

7. $B_T = P$

$\Rightarrow \qquad B_S = \gamma p$

$\therefore \qquad \dfrac{B_T}{B_S} = \dfrac{1}{\gamma}$

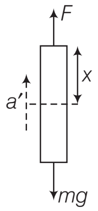
8. ***Case* I** $F = ma \Rightarrow a = \dfrac{F}{m}$



$F - T = m'a = \dfrac{m}{L}xa$

$T = F - \dfrac{m}{L}x\cdot\dfrac{F}{m} = F\left(1 - \dfrac{x}{L}\right)$ …(i)

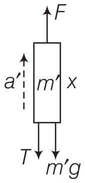
***Case* II** $F - mg = ma'$



$$a' = \frac{F}{m} - g$$

$$F - T - m'g = m'a'$$

$$F - T - m'g = m'\left(\frac{F}{m} - g\right)$$



$$F - T = m' \cdot \frac{F}{m}$$

$$F - T = \frac{m}{L}x \cdot \frac{F}{m} = \frac{x}{L}F$$

$$\Rightarrow \qquad T = F\left(1 - \frac{x}{L}\right) \qquad \text{...(ii)}$$

Tension at a point on the rod (of length L) at a distance x from point of application of force is same in both cases. [from Eqs. (i) and (ii)]

Hence, weight has no effect on tension in case (II).

Note *You can appreciate the extension of rod in first case, by comparing it with a case of many identical blocks connected by ideal springs.*

Extension in rod occurs due to force acting at any point on the rod. In certain cases, when net force acts at the centre of rod like weight, extension may not occur like the given case (II).

9. $Y_{\text{steel}} > Y_{\text{copper}}$. So, for the same strain, stress to be produced in steel is more.

Also, Work done $= \frac{1}{2}\text{Stress} \times \text{Strain}$

Volume $= \frac{1}{2} Y \times \varepsilon^2$

Single Correct Option

1. $$B = -\frac{\Delta p}{\Delta V / V}$$

$\Delta V = 0$ for an incompressible liquid.

$\therefore \quad B = \infty$

2. Young's modulus of elasticity is a materials property.

3. $$\sigma_{\max} = \frac{F_{\max}}{A}$$

$\therefore \quad F_{\max} = (\sigma_{\max})A,$

which is independent of length of wire.

4. $T = 0$ is free fall.

5. $$B = -\frac{\Delta p}{\Delta V / V}$$

$$= \frac{(1.2 \times 10^5)(1 \times 10^{-3})}{0.3 \times 10^{-3}}$$

$$= 4 \times 10^5 \text{ N/m}^2$$

Ans.

6. Length of the wire is $l = 20$ m

Radius of the wire is $r = 2 \times 10^{-3}$ m

Increase in length is

$$\Delta l = 0.031 \times 10^{-3} \text{ m}$$

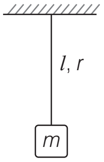
$$g = 3.1 \times \pi \text{ ms}^{-2}$$

Young's modulus is

$$Y = \frac{Fl}{A\,\Delta l} = \frac{(mg)l}{\pi r^2 (\Delta l)}$$

$$\therefore \qquad Y = \frac{4 \times 3.1 \times \pi \times 20}{\pi\,(2 \times 10^{-3})^2 \times 0.031 \times 10^{-3}}$$

$$= 2 \times 10^{12} \text{ Nm}^{-2}$$



7. Volume of wire is $V = L \times \pi r^2$

$$= 1 \times \pi (10^{-3})^2$$

$$= \pi \times 10^{-6} \text{m}^3$$

Area of square cross-section

$$= (2 \times 10^{-3})^2$$

$$= 4 \times 10^{-6} \text{m}^2$$

Length of new wire $= \frac{\text{Volume}}{\text{Area}} = \frac{\pi \times 10^{-6}}{4 \times 10^{-6}} = \frac{\pi}{4}$ m

Initially extension is $x = \frac{FL}{AY} = \frac{F}{Y}\left(\frac{1}{\pi \times 10^{-6}}\right)$

$$\therefore \qquad \frac{F}{Y} = (\pi \times 10^{-6})x$$

Finally extension is $x' = \frac{FL'}{A'Y} = \frac{F}{Y} \cdot \frac{L'}{A'}$

$$x' = (\pi \times 10^{-6})\,x \cdot \frac{\pi/4}{4 \times 10^{-6}} = \frac{\pi^2}{16}x$$

8. In the figure, the reciprocal of slope of stress-strain (x and y-axes) curve, upto proportionality limit, gives Young's modulus. The measure of ductility is obtained as the length of the stress-strain curve between yield point and ultimate load.

9. $\rho' = \dfrac{\rho}{1-\dfrac{dp}{B}} \approx \rho\left(1+\dfrac{dp}{B}\right)$

$$\Delta\rho = \rho' - \rho = \frac{\rho\,(dp)}{B} \qquad \text{...(i)}$$

$$\frac{\Delta\rho}{\rho} \times 100 = 0.1$$

$$\therefore \qquad \frac{\Delta\rho}{\rho} = 0.001$$

From Eq. (i),

$$dp \quad \text{or} \quad \Delta p = B\frac{\Delta\rho}{\rho}$$

$$= (2\times 10^9)\,(0.001) = 2\times 10^6 \text{ N/m}^2$$

10. $W = U = \dfrac{1}{2}F\,(\Delta l) = \dfrac{1}{2}\cdot\dfrac{YA\,(\Delta l)^2}{l}$

$\therefore \qquad U \propto \dfrac{A}{l}$ ($\because$ In both the cases Δl is same)

$$\frac{U_2}{U_1} = \frac{A_2}{A_1}\cdot\frac{l_1}{l_2} = \left(\frac{r_2}{r_1}\right)^2\cdot\frac{l_1}{l_2} = (4)(2)$$

$\therefore \qquad U_2 = 8U_1 = 8(2) = 16$ J

11. $U = \dfrac{1}{2}\sigma\varepsilon = \dfrac{1}{2}(Y\varepsilon)\varepsilon$

$\Rightarrow \quad U = \dfrac{1}{2}Y\varepsilon^2$ is similar to $x = ky^2$

Which is a parabola passing through origin and symmetric about x-axis.

12. Change in volume, $\Delta V = \dfrac{-\Delta p\cdot V_i}{B}$

Hence, density at depth of about 11 km is

$$= \frac{\text{Mass}}{\text{Volume}} = \frac{\rho_0\times V_i}{V_i - \dfrac{\Delta p v_i}{B}} = \frac{\rho_0 B}{(B-\Delta p)}$$

$$= \frac{\rho_0}{1-\dfrac{\Delta p}{B}} = \frac{\rho_0}{1-\dfrac{1\times 10^8}{2\times 10^9}}$$

$$= \frac{\rho_0}{1-\dfrac{1}{20}} = \frac{\rho_0}{0.95} \quad \Rightarrow \quad \rho = \frac{\rho_0}{0.95}$$

$\Rightarrow \qquad \rho_0 = 0.95\rho$

$\therefore$ % change in density $= \dfrac{\rho-\rho_0}{\rho_0}\times 100$

$$= \left[\frac{\dfrac{1}{0.95}-1}{1}\right]\times 100$$

$$\approx 5\ \%$$

13. $\Delta l = \dfrac{FL}{AY} = \dfrac{FL}{(\pi r^2)Y}$

$\therefore \qquad \Delta l \propto \dfrac{L}{r^2}$

$\therefore \qquad \dfrac{\Delta l_1}{\Delta l_2} = \dfrac{L/R^2}{2L/(2R)^2} = 2$

14. The net pressure at a depth of 1 km in the ocean is

$$p = \rho gh + p_{\text{atm}}$$
$$= (10^3)\,(9.8)\,(10^3) + 10^5$$
$$= 99\times 10^5 \text{ Pa} = 9.9\times 10^6 \text{ Pa}$$

This pressure acts uniformly on all sides of the balloon (which is in equilibrium) and the restoring forces with in the balloon are equal to external forces. So, normal stress will be same as external pressure.

15. Let T be the tension in the rope. Then,

$$2T = 10 \text{ kN}$$

$\Rightarrow \qquad T = 5000$ N

Longitudinal stress in the rope is

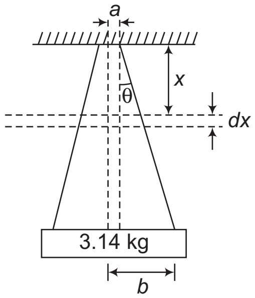
$$\sigma = \frac{T}{A} = \frac{5000 \text{ N}}{10^3 \text{ mm}^2} = 5 \text{N mm}^{-2}$$

Extension in the rope $= \dfrac{\text{Stress}}{Y}\times L$

$$= \frac{5}{10^3}\times(600+900) = 7.5 \text{ mm}$$

$\therefore$ Downward deflection of the load $= \dfrac{7.5}{2} = 3.75$ mm

16. Change in length of element is



$$d\,(\Delta L) = \frac{F\,dx}{AY}$$

Cross-sectional area of the element is

$$A = \pi\left[a + \frac{b-a}{L}x\right]^2$$

where, $a = \dfrac{b-a}{L}x$ is radius of the wire at the location of element.

$$d(\Delta L)=\frac{F}{\pi\left[a+\frac{b-a}{L}x\right]^2 Y}\,dx$$

Total change in length of wire is

$$\Delta L=\frac{F}{\pi Y}\int_0^L \frac{1}{\left(a+\frac{b-a}{L}x\right)^2}\,dx$$

Let $\quad a+\frac{b-a}{L}x=t$

$$\Rightarrow\quad \frac{b-a}{L}dx=dt \quad\Rightarrow\quad dx=\frac{L}{b-a}dt$$

When $x=0$, $t=a$ and when $x=L$, $t=b$

$$\therefore\quad \Delta L=\frac{F}{\pi Y}\int_a^b t^{-2}\cdot\frac{L}{b-a}\,dt$$

$$=\frac{FL}{\pi Y(b-a)}(-1)\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$=\frac{FL}{\pi Y(b-a)}\cdot\frac{(b-a)}{ab}=\frac{FL}{\pi Y ab}$$

$$=\frac{(3.14)\,(9.8)\,(10)}{(3.14)\,(2\times10^{11})\,(5\times10^{-4})\,(9.8\times10^{-4})}$$

$$=10^{-3}\,\text{m}\quad\Rightarrow\quad\therefore\quad \Delta L=10^{-3}\,\text{m}$$

Subjective Questions

1. The changed density, $\rho'=\dfrac{\rho}{1-\dfrac{dp}{B}}$

Substituting the value, we have

$$\rho'=\frac{11.4}{1-\dfrac{2.0\times10^8}{8.0\times10^9}}\quad\text{or}\quad \rho'=11.69\ \text{g/cm}^3$$ **Ans.**

2. $\Delta l=\dfrac{Fl}{AY}=\dfrac{Fl}{(\pi d^2/4)\,Y}$

$$\therefore\quad d=\sqrt{\frac{4\,Fl}{\pi\,(\Delta l)\,Y}}$$

$$=\sqrt{\frac{4\times400\times3}{3.14\times0.2\times10^{-2}\times2.1\times10^{11}}}$$

$$=1.91\times10^{-3}\,\text{m}=1.91\ \text{mm}$$ **Ans.**

3. $\dfrac{m\,(g+a)}{A}=\dfrac{1}{3}\sigma_{\max}$

$$\therefore\quad a=\frac{\sigma_{\max}A}{3\text{m}}-g$$

$$=\frac{(3\times10^8)\,(4\times10^{-4})}{3\times900}-9.8$$

$$=34.64\ \text{m/s}^2$$ **Ans.**

4. Refer solved example 1

$$\Delta l=\frac{mgl}{2AY}=\frac{\rho\,(\pi r^2 l)\,gl}{2\,(\pi r^2)\,Y}$$

$$=\frac{gl^2\rho}{2Y}=\frac{(9.8)\,(5)^2\,(8000)}{2\times2\times10^{11}}$$

$$=4.9\times10^{-6}\text{m}$$

$$U=\frac{1}{2}k\,(\Delta l)^2=\frac{1}{2}\frac{YA}{l}(\Delta l)^2$$

$$=\frac{1}{2}\times\frac{Y\times\pi r^2}{l}(\Delta l)^2$$

$$=\frac{(2\times10^{11})\,(3.14)\,(6\times10^{-3})^2\,(4.9\times10^{-6})^2}{2\times5}$$

$$=5.43\times10^{-5}\text{J}$$ **Ans.**

5. $\sigma_{\max}$ on upper string $=\dfrac{(m_1+m_2+m)\,g}{0.006\times10^{-4}}$

$$\therefore\qquad 8\times10^8=\frac{(10+20+m)\times10}{0.006\times10^{-4}}$$

or $\qquad m=18$ kg

$\sigma_{\max}$ on lower string $=\dfrac{(m_1+m')\,g}{0.003\times10^{-4}}$

or $\qquad 8\times10^8=\dfrac{(10+m')\times10}{0.003\times10^{-4}}$

or $\qquad m'=14$ kg

So, answer is 14 kg and lower string will break earlier.

6. (a) $\sigma=\dfrac{F}{A}$

F and A both are same. Hence, the ratio is 1.

(b) Strain $=\dfrac{\text{Stress}}{Y}\propto\dfrac{1}{Y}$

$$\therefore\quad \frac{(\text{Strain})_{\text{steel}}}{(\text{Strain})_{\text{copper}}}=\frac{Y_{\text{copper}}}{Y_{\text{steel}}}=\frac{13}{20}$$ **Ans.**

7. $\rho'=\dfrac{\rho}{1-\dfrac{dp}{B}}\approx\rho\left(1+\dfrac{dp}{B}\right)$

$$\therefore\qquad \Delta\rho=\rho'-\rho=\frac{\rho\,(dp)}{B}$$

$$=\frac{\rho\,(\rho gh)}{B}$$

$$=\frac{(1030)^2\times9.8\times400}{2\times10^9}$$

$$=2.0\ \text{kg/m}^3$$ **Ans.**

8. $B = \dfrac{\Delta p}{\Delta V/V} = \dfrac{\rho g h}{(\Delta V/V)}$

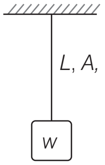
$$= \frac{(10^3)\,(9.8)\,(180)}{(0.1/100)}$$

$$= 1.76 \times 10^9 \text{ N/m}^2$$ **Ans.**

LEVEL 2

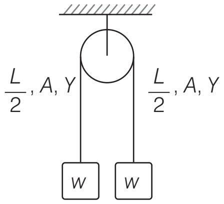
Single Correct Option

1. Extension in the first case is



$$l = \frac{FL}{AY} = \frac{wL}{AY}$$

Extension in the second case is



$$l' = \frac{w(L/2)}{AY} + \frac{w(L/2)}{AY} = \frac{wL}{AY}$$

It is clear that $l' = l$

Note *In the above problem, even if lengths of wire are unequal on two sides of the pulley, the elongation will still be l.*

2. Force on wire = Load carried

$$= m\,(g + a) = m\left(g + \frac{g}{2}\right) = \frac{3mg}{2}$$

Stress $\sigma = \dfrac{F}{A} \Rightarrow A = \dfrac{F}{\sigma}$

Hence, $A_{\min} = \dfrac{F}{\sigma_{\max}}$

$$\frac{\pi\, d_{\min}^2}{4} = \frac{\frac{3}{2} m \cdot g}{\sigma}$$

$$\Rightarrow \quad d_{\min}^2 = \frac{\frac{3}{2} mg \times 4}{\sigma \pi}$$

$$d_{\min} = \sqrt{\frac{6}{\pi} \cdot \frac{mg}{\sigma}}$$

3. $T = ml\,\omega^2$

$$\frac{T}{A} = \sigma_{\max} = \frac{ml\omega^2}{A}$$

$$\therefore \quad \omega = \sqrt{\frac{\sigma_{\max}\, A}{ml}}$$

$$= \sqrt{\frac{4.8 \times 10^7 \times 10^{-6}}{10 \times 0.3}}$$

$= 4$ rad/s **Ans.**

4. $T = \dfrac{M}{2} g = \left(\dfrac{L}{2} s\rho\right) g$

$\therefore \quad \sigma = \dfrac{T}{S} = \dfrac{1}{2}\rho g L$ **Ans.**

5. $F = -\dfrac{dU}{dr} = -$ slope of U - r graph

At B, slope $= 0$

$\therefore \quad F = 0$

Hence, atoms are in equilibrium at B. From A to B or even before A slope is negative. Hence, force is positive and positive force mean repulsion.

6.
$$l_1 - l = \frac{T_1 l}{AY} \qquad \ldots\text{(i)}$$
$$l_2 - l = \frac{T_2 l}{AY} \qquad \ldots\text{(ii)}$$

Dividing Eq. (i) by Eq. (ii) and then simplifying, we get

$$l = \frac{l_2 T_1 - l_1 T_2}{T_1 - T_2}$$ **Ans.**

7. $\Delta l = \dfrac{Fl}{AY} = \dfrac{Fl}{(V/l)\,Y}$ (V = Volume)

$$= \frac{Fl^2}{VY}$$

$\therefore \quad \Delta l \propto l^2$ (as F, V and Y = constants)

8. $\Delta l = l\alpha\, \Delta\theta$

$\therefore \quad$ Strain $= \dfrac{\Delta l}{l} = (\alpha \Delta\theta)$

Stress = $Y \times$ Strain $= Y\alpha\Delta\theta$

Energy stored per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times Y \times (\alpha\, \Delta\theta)^2$$

$$= \frac{1}{2} \times 10^{11} \times (12 \times 10^{-6} \times 20)^2$$

$= 2880$ J/m^3 **Ans.**

9. Δl due to temperature rise

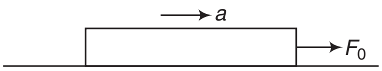
$$\begin{aligned}&= l\alpha\,\Delta\theta\\&= (1000)\,(10^{-4})\,(20)\\&= 2\,\text{mm}\end{aligned}$$

But 1 mm is allowed (1000 mm to 1001 mm)

$$\therefore \quad \Delta l_e = 1\text{ mm}$$

$$\begin{aligned}\text{Stress} &= Y \times \text{Strain}\\&= (10^{11})\left(\frac{1\text{ mm}}{1000\text{ mm}}\right)\\&= 10^8\,\text{N/m}^2\end{aligned}$$ **Ans.**

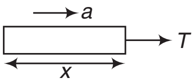
10.



$$a = \frac{F_0}{m}$$

$$T = m_x\,a = \left(\frac{m}{l}\right)x\left(\frac{F_0}{m}\right) = \left(\frac{F_0}{l}\right)x$$

$$\Delta l = \int_0^L \frac{Tdx}{SY}$$



$$= \left(\frac{F_0}{lsY}\right) \times \frac{l^2}{2}$$

$$\text{strain} = \frac{\Delta l}{l} = \frac{F_0}{2SY}$$ **Ans.**

More than One Correct Options

1. $U = \frac{1}{2}kx^2$. It is a parabola symmetric about U-axis.

At $x = 0.2$ mm, $U = 0.2$ J (from the figure)

$$\therefore \quad 0.2 = \frac{1}{2}k(2\times 10^{-4})^2$$

$$\Rightarrow \quad k = 10^7\ \text{Nm}^{-1}$$

$$k = \frac{YA}{L}$$

$$\Rightarrow \quad \frac{A}{L} = \frac{k}{Y} = \frac{10^7}{2\times 10^{11}} = 5\times 10^{-5} \qquad \ldots\text{(i)}$$

$$AL = \text{Volume} = 200\times 10^{-6}\ \text{m}^3 \qquad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$A = 10^{-4}\,\text{m}^2 \text{ and } L = 2\,\text{m}$$

2. W_F = elastic potential energy stored in the wire

$$\begin{aligned}&= \frac{1}{2}k\,(\Delta l)^2\\&= \frac{1}{2}\left(\frac{YA}{L}\right)(l)^2 = \frac{YAl^2}{2L}\end{aligned}$$

3.

$$\Delta x = \frac{Fl}{AY}$$

$$\therefore \quad F = \left(\frac{AY}{L}\right)\Delta x$$

i.e. F *versus* Δx graph is a straight line of slope $\frac{YA}{L}$.

$$(\text{Slope})_B > (\text{Slope})_A$$

$$\therefore \quad \left(\frac{YA}{L}\right)_B > \left(\frac{YA}{L}\right)_A$$

or $\quad (A)_B > (A)_A$

They are of same material. Hence,

$$Y_B = Y_A$$

4. Half of energy is lost in heat and rest half is stored as elastic potential energy.

$$l = \frac{Mgl}{AY} \qquad \ldots\text{(i)}$$

$$U = \frac{1}{2}Kl^2 = \frac{1}{2}\left(\frac{YA}{L}\right)l^2 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we can prove that

$$U = \frac{1}{2}Mgl$$

5. When, $T_Q = \frac{mg}{3}$

and $\quad T_P = mg + \frac{mg}{3} = \frac{4mg}{3}$

Stress, $\quad \sigma = \frac{T}{\pi r^2}$

$$\frac{\sigma_P}{\sigma_Q} = \frac{T_P}{T_Q}\cdot\left(\frac{r_Q}{r_P}\right)^2$$

If $r_P = r_Q$, $\sigma_P = 4\sigma_Q$, then P breaks.

If $r_P < 2r_Q$, $\sigma_P > \sigma_Q$, then P breaks.

If $r_P = 2r_Q$, $\sigma_A = \sigma_B$, then either P or Q may break.

6. Area of cross-section is $A = \frac{\pi d^2}{4}$

$$\Delta L = \frac{F}{\pi d^2/4}\cdot\frac{L}{Y}$$

$$\Rightarrow \quad \Delta L \propto \frac{1}{d^2}$$

$$\therefore \quad \frac{\Delta L_B}{\Delta L_A} = \frac{(2d)^2}{(d)^2}$$

$$\Rightarrow \quad \Delta L_B = 4\,\Delta L_A$$

$$\text{Strain} = \frac{\Delta L}{L} = \frac{4F}{\pi d^2 Y}$$

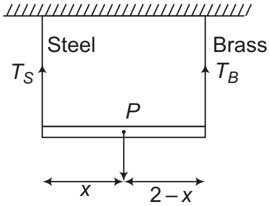
$$\Rightarrow \quad \text{Strain} \propto \frac{1}{d^2}$$

$$\therefore \quad (\text{Strain})_B = 4\,(\text{Strain})_A$$

7. If stress in steel = stress in brass,

then $$\frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\frac{T_S}{T_B} = \frac{A_S}{A_B} = \frac{10^{-3}}{2 \times 10^{-3}} = \frac{1}{2} \qquad \ldots(\text{i})$$



System is in equilibrium. So, taking moments about P

$$T_S \cdot x = T_B (2 - x)$$

$$\Rightarrow \quad \frac{T_S}{T_B} = \frac{2-x}{x} \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get $x = 1.33$ m

$$\text{Strain} = \frac{\text{Stress}}{Y}$$

If strain in steel = strain in brass,

then $$\frac{T_S/A_S}{Y_S} = \frac{T_B/A_B}{Y_B}$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{A_S Y_S}{A_B Y_B} = \frac{(10^{-3})(2 \times 10^{11})}{(2 \times 10^{-3})(10^{11})} = 1 \qquad \ldots(\text{iii})$$

From Eqs. (ii) and (iii), we get $x = 1\,\text{m}$

8. Area of steel rod, $A_S = 16\ \text{cm}^2$

Area of two brass rods,

$$A_B = 2 \times 10 = 20\ \text{cm}^2$$

$$F = 5000\ \text{kg}$$

$$\sigma_S = \text{Stress in steel}$$

and $$\sigma_B = \text{Stress in brass}$$

Decrease in length of steel rod = Decrease in length of brass rod

$$\frac{\sigma_S}{Y_S} \cdot L_S = \frac{\sigma_B}{Y_B} \cdot L_B$$

$$\Rightarrow \quad \sigma_S = \frac{Y_S}{Y_B} \cdot \frac{L_B}{L_S} \cdot \sigma_B$$

or $$\sigma_S = \left(\frac{2 \times 10^6}{10^6}\right)\left(\frac{20}{30}\right)\sigma_B$$

$$\Rightarrow \quad \sigma_S = \frac{4}{3}\sigma_B \qquad \ldots(\text{i})$$

Now, $$F = \sigma_S A_S + \sigma_B A_B$$

or $$5000 = \sigma_S \times 16 + \sigma_B \times 20 \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$\sigma_B = 120.9\ \text{kg cm}^{-2}$$

and $$\sigma_S = 161.2\ \text{kg cm}^{-2}$$

Comprehension Based Questions

1. Initially, $$\Delta L = \frac{FL}{AY}$$

$$= \frac{10 \times 1}{10^{-3} \times 2 \times 10^5}$$

$$= 0.05\ \text{m} = 5\ \text{cm}$$

2. Force constant of string is

$$k = \frac{\text{Force}}{\text{Elongation}}$$

$$= \frac{F}{FL/AY}$$

or $$k = \frac{YA}{L} = \frac{1 \times 10^5 \times 10^{-3}}{1}$$

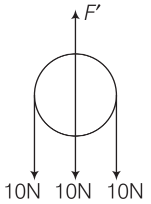
$$= 200\ \text{Nm}^{-1}$$

Initial elastic potential energy of string

$$= \frac{1}{2}k\,(0.05)^2$$

$$= \frac{1}{2}\,(200)\,(25 \times 10^{-4})$$

$$= 25 \times 10^{-2}\ \text{J}$$



Let after force $F = 10$ N is applied extra elongation is x, then

$$0.25 + 30x = \frac{1}{2}(200)\,(x^2 + 25 \times 10^{-4} + 0.1x)$$

$$0.25 + 30x = 100x^2 + 0.25 + 10x$$

$$\therefore \quad 100x^2 = 20x$$

$$\Rightarrow \quad x = \frac{20}{100} = \frac{1}{5}\,\text{m} = 0.2\ \text{m}$$

$$= 20\ \text{cm}$$

$$\therefore \quad x_{\text{max}} = 20 + 5 = 25\text{cm}$$

3. When the displacement of the pulley is 25 cm, the string gets loosened on both the sides. So, point A moves down by 50 cm.
4. Young's modulus is a material constant.
5. On crossing the yield region, the material will experience the breaking stress and further elongation causes reduction in stress and breaking of the wire.
6. Stress-strain curve flattens on crossing elastic region.

Match the Columns

1. Stress, modulus of elasticity, energy density and pressure have SI units J/m^3 or N/m^2.

   Strain, coefficient of friction and relative density are dimensionless. Force constant have SI unit N/m.
2. $\Delta l = \frac{Fl}{AY}$ and stress $= \frac{F}{A}$
3. For A and B, the rod is in equilibrium and hence internal restoring force developed per unit area across any cross-section is same and thus stress is uniform. But in C and D, the case is opposite.

Subjective Questions

1. Increase in pressure is $\Delta p = \frac{Mg}{A}$

   Bulk modulus is $B = \frac{\Delta p}{(\Delta V/V)}$

$$\therefore \qquad \frac{\Delta V}{V} = \frac{\Delta p}{B} = \frac{Mg}{AB} \qquad \ldots\text{(i)}$$

   Also, the volume of the sphere is

$$V = \frac{4}{3}\pi R^3 \Rightarrow \frac{\Delta V}{V} = \frac{3\Delta R}{R}$$

$$\text{or} \qquad \frac{\Delta R}{R} = \frac{1}{3}\cdot\frac{\Delta V}{V},$$

   Using Eq. (i), we get

$$\frac{\Delta R}{R} = \frac{Mg}{3AB} \Rightarrow \therefore \alpha = 3$$

2. Force constant of the wire is

$$k = \frac{F}{\Delta L} = \frac{F}{FL/AY} = \frac{YA}{L}$$

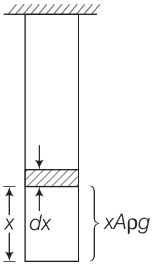
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{YA}{Lm}} \Rightarrow \therefore \sqrt{\frac{YA}{Lm}} = 140$$

$$\Rightarrow \qquad \frac{Y \times 4.9 \times 10^{-7}}{1 \times 0.1} = 140 \times 140$$

$$Y = \frac{140 \times 140}{49 \times 10^{-7}} = 4 \times 10^9$$

$$\Rightarrow \qquad p = 4$$

3. Consider an element as shown in the figure.



$$\text{Stress in the element } = \frac{\text{Force}}{\text{Area}} = \frac{xA\rho g}{A} = x\rho g$$

   Now, elastic potential energy stored in the wire is

$$dU = \frac{1}{2}\text{(Stress) (Strain) (Volume)}$$

$$= \frac{1}{2}\cdot\frac{(\text{Stress})^2}{Y}\text{(Volume)}$$

$$dU = \frac{1}{2}\cdot\frac{(x\rho g)^2}{Y} A\,dx = \frac{1}{2}\cdot\frac{\rho^2 g^2 A}{Y} x^2 dx$$

$$\text{Total elastic potential energy} = \frac{1}{2}\cdot\frac{\rho^2 g^2 A}{Y}\int_0^L x^2 dx$$

$$= \frac{\rho^2 g^2 A L^3}{6Y}$$

4. $\Delta l = \frac{Fl}{\Delta Y}$

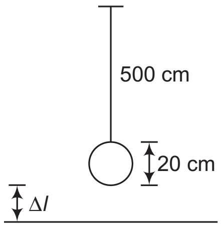
   Here, $F = \text{Upthrust} = V_i \rho_l g \Rightarrow A = \frac{\pi d^2}{4}$

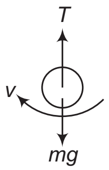
$$\therefore \qquad \Delta l = \frac{4V_i \rho_l g l}{\pi d^2 Y}$$

$$= \frac{(4)(10^{-3})(800)(9.8)(3)}{(3.14)(0.4 \times 10^{-3})^2 (8 \times 10^{10})}$$

$$= 2.34 \times 10^{-3}\text{m}$$ **Ans.**

5. $\Delta l = 521 - 500 - 20 = 1 \text{ cm} = 0.01 \text{ m}$





$$T - mg = \frac{mv^2}{R}$$

$$\therefore \qquad T = m\left(g + \frac{v^2}{R}\right) = m\left(g + \frac{v^2}{l}\right)$$

$$\Delta l = \frac{Tl}{AY} = \frac{m\left(g + \frac{v^2}{l}\right)l}{(\pi d^2/4)Y}$$

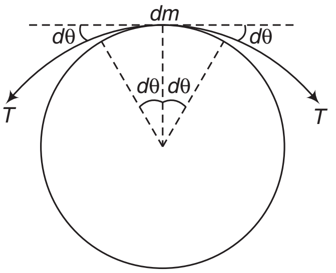
$$= \frac{mgl + mv^2}{(\pi d^2/4)\,Y}$$

$$\therefore \quad v = \sqrt{\frac{\pi d^2\,\Delta l Y}{4m} - gl}$$

$$= \sqrt{\frac{(3.14)(4\times10^{-3})^2(0.01)(2\times10^{11})}{4\times25} - 9.8\times5}$$

≈ 31 m/s **Ans.**

6. $2T\sin d\theta = (dm)\,R\omega^2$



For small angles, $\sin d\theta \approx d\theta$

$$\therefore \quad 2T(d\theta) = (2R\,d\theta)(\pi r^2)(\rho)(R)(2\pi f)$$

$$T = (4\pi^3 f^2R^2r^2\rho)$$

Now $\quad \Delta l = \dfrac{Tl}{AY}$

$$\frac{\Delta l}{l} = \frac{T}{AY} = \frac{T}{\pi r^2 Y}$$

$$l = 2\pi R$$

$$\Delta l = 2\pi(\Delta R)$$

$$\therefore \quad \frac{\Delta l}{l} = \frac{\Delta R}{R} = \frac{T}{\pi r^2 Y}$$

$$= \frac{4\pi^3 f^2R^2r^2\rho}{\pi r^2 Y}$$

7. $T - mg = ml\omega^2 = ml\,(2\pi f)^2$

$$\therefore \quad T = mg + 4\pi^2 mlf^2$$

$$= 6\times9.8 + 4\pi^2(6)(0.6)\,(2)^2$$

$$= 628 \text{ N}$$

Now, $\quad \Delta l = \dfrac{Tl}{AY}$

$$= \frac{(628)\,(0.6)}{(0.05\times10^{-4})\,(2\times10^{11})}$$

$= 3.8\times10^{-4}$ m **Ans.**

8. $2T_C + T_S = mg$ …(i)

$$\Delta l_C = \Delta l_s$$

$$\therefore \quad \frac{T_C\,l_C}{A_C Y_C} = \frac{T_S\,l_S}{A_S Y_S}$$

$$\therefore \quad T_S = 2T_C$$

(as $Y_S = 2Y_C$, $l_S = l_C$, $A_S = A_C$)

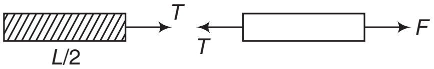
or $\quad T_S = 2T_C$ …(ii)

Solving these two equations we get,

$$T_C = \frac{mg}{4}$$ and

$$T_S = 2T_C$$

9.



(a) $T = \left(\dfrac{M}{2}\right)a_0 = \left(\dfrac{L}{2}S\rho\right)a_0$

$\therefore \quad$ Stress $= \dfrac{T}{S} = \dfrac{L\rho a_0}{2}$ **Ans.**

(b) $\quad T_x = M_x\,a_0 = x\rho Sa_0$

$$\Delta l = \int_0^L \frac{T_x dX}{SY} = \int_0^L \frac{x\rho Sa_0}{SY}\,dX$$

$= \dfrac{\rho a_0 L^2}{2Y}$ **Ans.**

10. $\Delta l = \dfrac{Fl}{AY}$

$$U = \frac{1}{2}K\,(\Delta l^2)$$

$$= \frac{1}{2}\left(\frac{YA}{l}\right)(\Delta l^2)$$

$$= \frac{1}{2}\left(\frac{YA}{l}\right)\left(\frac{Fl}{AY}\right)^2$$

$$= \frac{F^2 l}{2AY} = mS\Delta\theta$$

$$\therefore \quad \Delta\theta = \frac{F^2 l}{2AYms}$$

$$= \frac{(Mg)^2 l}{2(\pi r^2)\,Y\,(l\pi r^2)\rho s}$$

$$= \frac{M^2g^2}{2\pi^2 r^4 Y\rho s}$$

$$= \frac{(100\times9.8)^2}{2\pi^2(2\times10^{-3})^4\,(2.1\times10^{11})\,(7860)\,(420)}$$

$\approx 4.568\times10^{-3}$ °C **Ans.**

16. Fluid Mechanics

INTRODUCTORY EXERCISE 16.1

1. $p_0 + \rho_1 g h_1 = p_0 + \rho_2 g h_2$

$$\therefore \quad h_1 = \frac{\rho_2 h_2}{\rho_1} = \frac{20 \times 0.9}{1} = 18 \text{ cm}$$ **Ans.**

2. Suppose the atmospheric pressure = p_0.

Pressure at $A = p_0 + h(1000 \text{ kgm}^{-3})g$

Pressure at $B = p_0 + (0.02\text{m})(13600 \text{kgm}^{-3})g$

These pressures are equal as A and B which are at the same horizontal level. Thus,

$$h = (0.02\text{m}) \times 13.6$$
$$= 0.27\text{m} = 27\text{cm}$$

3. (a) $p = p_0 + \rho_{Hg} g \Delta h$

(b) $p = p_0 + \rho_{Hg} \; g h_{RHS}$

Here, $\Delta h = (8 - 2) = 6 \text{ cm} = 6 \times 10^{-2}$ m

and $h_{RHS} = 8 \text{ cm} = 8 \times 10^{-2}$ m

4. Let the pressure of the liquid just below the piston be p. The forces acting on the piston are:

(a) its weight, mg (downward)

(b) force due to the air above it, $p_0 A$ (downward)

(c) force due to the liquid below it, pA (upward)

It the piston is in equilibrium,

$$pA = p_0 A + mg$$

or

$$p = p_0 + \frac{mg}{A}$$

5. In equilibrium, the pressure at the two surfaces should be equal as they lie in the same horizontal level. If the atmospheric pressure is p and a force F is applied to maintain the equilibrium, the pressure are

$$p_0 + \frac{5\text{N}}{1 \text{ cm}^2} \text{ and } p_0 + \frac{F}{10 \text{ cm}^2} \text{ respectively.}$$

This gives $F = 50$ N

6. $\Delta p_1 = \Delta p_2$

$$\frac{F}{A} = (\Delta h) \rho g$$

$$\therefore \quad \Delta h = \frac{F}{A\rho g} = \frac{45g}{(900 \times 10^{-4})(10^3)g}$$

$$= 0.5 \text{ m} = 50 \text{ cm}$$

INTRODUCTORY EXERCISE 16.2

1. In vertical direction pressure increases with depth. In horizontal direction pressure decreases in the direction of acceleration.

2. $p_N + \rho_g h - \rho a \left(\frac{h}{2}\right) = p_M$

But $\quad p_N = p_M$

$\therefore \quad a = 2g$

3. $p_B = p_A + \dfrac{\rho \omega^2 x^2}{2} = p_0 + \dfrac{\rho \omega^2 x^2}{2}$

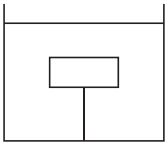
INTRODUCTORY EXERCISE 16.3

1. Weight = upthrust

$$V \rho_1 g = \left(\frac{3}{4} V\right) \rho_2 g$$

$$\therefore \quad \rho_2 = \frac{4}{3} \rho_1$$

2.



T + weight = upthrust

$$\therefore \quad T = \text{upthrust} - \text{weight}$$
$$= (71.2)\left(\frac{1}{0.75}\right) - 71.2$$
$$= 23.7 \text{ N}$$ **Ans.**

3. Reading = weight of water + magnitude of upthrust on piece of metal (acting downwards)

$$= (0.02 \times 9.8) + (10^{-6})(10^3)(9.8)$$
$$= 0.206 \text{ N}$$ **Ans.**

4. (a) $a = \dfrac{\text{upthrust} - \text{weight}}{\text{mass}}$

$$= \frac{(V)(1000)(9.8) - (V)(0.4 \times 10^{-3})(9.8)}{(V)(0.4 \times 10^3)}$$

$$= 14.7 \text{ m/s}^2$$ **Ans.**

(b) $t = \sqrt{\dfrac{2s}{a}}$

$$= \sqrt{\frac{2 \times 2.9}{14.7}}$$

$$= 0.63 \text{ s}$$ **Ans.**

INTRODUCTORY EXERCISE 16.4

1. (a) Apply continuity equation

(b) Apply Bernoulli's equation

2. (a) $\frac{dV}{dt} = Av$

(b) $\frac{dV}{dt} = Av$

(c) Apply Bernoulli's equation

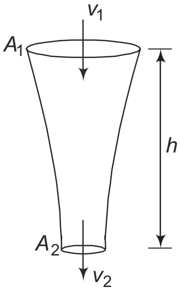
3. From conservation of energy

$$v_2^2 = v_1^2 + 2gh \qquad \text{...(i)}$$

[can also be found by applying Bernoulli's theorem]

From continuity equation

$$A_1 v_1 = A_2 v_2$$



$$v_2 = \left(\frac{A_1}{A_2}\right) v_1 \qquad \text{...(ii)}$$

Substituting value of v_2 from Eq. (ii) in Eq. (i),

$$\frac{A_1^2}{A_2^2} \cdot v_1^2 = v_1^2 + 2gh$$

or $$A_2^2 = \frac{A_1^2 v_1^2}{v_1^2 + 2gh}$$

$\therefore$ $$A_2 = \frac{A_1 v_1}{\sqrt{v_1^2 + 2gh}}$$

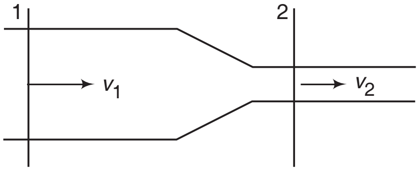
Substituting the given values, we get

$$A_2 = \frac{(10^{-4})(1.0)}{\sqrt{(1.0)^2 + 2\,(10)\,(0.15)}}$$

$$A_2 = 5.0 \times 10^{-5}\ \text{m}^2$$

4. From continuity equation,

$$A_1 v_1 = A_2 v_2$$



$\therefore$ $$v_2 = \left(\frac{A_1}{A_2}\right) v_1 = \left(\frac{10}{5}\right)(1) = 2\ \text{m/s}$$

Applying Bernoulli's theorem at 1 and 2

$$p_2 + \frac{1}{2}\rho v_2^2 = p_1 + \frac{1}{2}\rho v_1^2$$

$\therefore$ $$p_2 = p_1 + \frac{1}{2}\rho\,(v_1^2 - v_2^2)$$

$$= \left(2000 + \frac{1}{2} \times 10^3\ (1 - 4)\right)$$

$\therefore$ $$p_2 = 500\,\text{Pa}$$

INTRODUCTORY EXERCISE 16.5

1. $\Delta p = \frac{1}{2}\rho v^2$

$\therefore$ $$v = \sqrt{\frac{2\Delta p}{\rho}}$$

$$= \sqrt{\frac{2\,(3\ \text{atm} - 1\ \text{atm})}{\rho}}$$

$$= \sqrt{\frac{2 \times 2 \times 10^5}{10^3}} = 20\ \text{m/s}$$

2. $t = \frac{A}{a}\sqrt{\frac{2H}{g}}$ (to empty the complete tank)

Now, $t_{H \to 0} = t_{H \to \frac{H}{3}} + t_{\frac{H}{3} \to 0}$

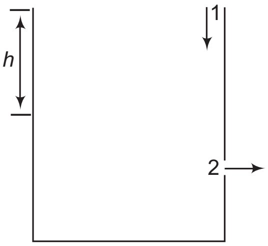
Given, $\frac{A}{a}\sqrt{\frac{2H}{g}} = t_0 + \frac{A}{a}\sqrt{\frac{2\,(H/3)}{g}}$

From here find,

$$\frac{A}{a}\sqrt{\frac{2\,(H/3)}{g}}$$

3. Applying continuity equation at 1 and 2, we have

$$A_1 v_1 = A_2 v_2 \qquad \text{...(i)}$$



Further applying Bernoulli's equation at these two points, we have

$$p_0 + \rho gh + \frac{1}{2}\rho v_1^2 = p_0 + 0 + \frac{1}{2}\rho v_2^2 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have $v_2^2 = \dfrac{2gh}{1 - \dfrac{A_2^2}{A_1^2}}$

Substituting the values, we have

$$v_2^2 = \frac{2 \times 10 \times 2.475}{1 - (0.1)^2} = 50\ \text{m}^2/\text{s}^2$$

INTRODUCTORY EXERCISE 16.6

1. Using, $V = \dfrac{2}{9}\dfrac{r^2(\rho - \sigma)g}{\eta}$

$$\eta = 1.0\ \text{mPi} = 1.0 \times 10^{-3}\ \text{Pi}$$

$$= \frac{2}{9}\frac{(20 \times 10^{-6})^2 \times (2 \times 10^3 - 10^3) \times 9.8}{1.0 \times 10^{-3}}$$

$$= 0.871\ \text{mm/s}$$

2. ∵ Volume of bigger drop

$$= n \times \text{volume of smaller drop}$$

$$\frac{4}{3}\pi R^3 = 2 \times \frac{4}{3}\pi r^3$$

$$R = 2^{1/3} r$$

Terminal velocity $\propto r^2$

∴ $$v' = 2^{2/3} v$$

3. $F = \eta A \dfrac{\Delta V}{\Delta x}$

$$= 10^{-3} \times 10 \times 2 \times 10^{-1}$$

$$= 2 \times 10^{-2}$$

$$= 0.02\ \text{N}$$

4. ∵ $F = \eta A \dfrac{\Delta V}{\Delta x}$

$$\frac{F}{A} = \text{shearing stress} = \eta \frac{\Delta V}{\Delta x}$$

$$= 10^{-3} \times \frac{5}{5} = 10^{-3}\ \text{N/m}^2$$

INTRODUCTORY EXERCISE 16.7

1. $\dfrac{4}{3}\pi R^3 = 10^6\left(\dfrac{4}{3}\pi r^3\right)$

⇒ $$r = (10^{-2}) R = 10^{-2}\ \text{cm}$$

$$A_i = 4\pi R^2$$

$$A_f = (10^6)(4\pi r^2)$$

$$\Delta A = A_f - A_i$$

∴ $$\Delta U = (T)(\Delta A)$$

2. $W = T(\Delta A)$

$$= T(2)(l)(\Delta d)$$

$$= 7.2 \times 10^{-2} \times 2 \times 0.10 \times 10^{-3}$$

$$= 1.44 \times 10^{-5}\ \text{J}$$

3. $W = T(\Delta A)$

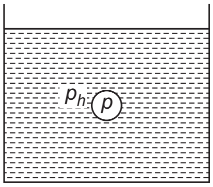
Soap bubble has two free surfaces.

∴ $$W = T(8\pi R^2)$$

4. $p - p_h = \dfrac{2T}{r}$

∴ $$p = p_h + \frac{2T}{r}$$

$$= p_0 + h\rho g + \frac{2T}{r}$$



INTRODUCTORY EXERCISE 16.8

1. $h = \dfrac{2T\cos\theta}{r\rho g}$

∴ $$hr = \frac{2T\cos\theta}{\rho g} = \text{constant}$$

∴ $$h_1 r_1 = h_2 r_2 \quad \text{or} \quad h_2 = \frac{h_1 r_1}{r_2}$$

Substituting the values, we get

$$h_2 = (2.0)(3) \qquad \left(\frac{r_2}{r_1} = \frac{1}{3}\right)$$

$$= 6.0\ \text{cm}$$ **Ans.**

2. ∵ $h = \dfrac{2T\cos\theta}{r\rho g}$

$$T_1 = \frac{h_1 \rho_1 r g}{2\cos\theta_1}$$

$$T_2 = \frac{h_2 \rho_2 r g}{2\cos\theta_2}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} \cdot \frac{\rho_2}{\rho_1} \cdot \frac{\cos\theta_1}{\cos\theta_2}$$

$$= 7.23$$

3. $h = \dfrac{2T}{R\rho g}$ (R = radius of meniscus at the top)

∴ $$R = \frac{2T}{h\rho g} = \frac{2 \times 0.07}{(10^{-2})(10^3)(10)}$$

$$= 1.4 \times 10^{-3}\ \text{m}$$

$$= 1.4\ \text{mm}$$

4. $p_A = p_C = (p_B + \rho g h)$

Exercise

LEVEL 1

Assertion and Reason

1. Direction of force in $p = \frac{F}{A}$ is always perpendicular to the surface or it is always specific. Hence, it is not a vector.

3.
$$p_1 = p_0 + \rho g h$$
$$p_2 = p_0 + \rho g\,(2h) \neq 2p_1$$

5. Area of cross-section is different. So, heights are different. In pressure height is more important.

6. Speed will also depend on h.

9. On moon, atmospheric pressure is zero. Hence barometer height is zero.

10. Pressure at P is less. As liquid is flowing at P while liquid is at rest at Q.

11. Force of buoyancy,
$$U = V\rho_{\text{air}}\,g$$
Since, ρ_{air} is negligible. Hence, U is negligible.

Single Correct Option

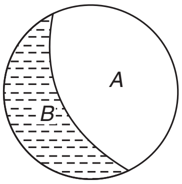
1. $v = \text{constant}$
$$\therefore \quad a = 0$$
$$\text{or} \quad F_{\text{net}} = 0$$

2. $$\left[\frac{\eta}{\rho}\right] = \left[\frac{\text{ML}^{-1}\,\text{T}^{-1}}{\text{ML}^{-3}}\right] = [\text{M}^0\text{L}^2\text{T}^{-1}]$$

9. $n = n_1 + n_2$
$$\frac{pV}{RT} = \frac{p_1V_1}{RT} + \frac{p_2V_2}{RT}$$
$$\text{or} \quad pV = p_1V_1 + p_2V_2$$
$$\therefore \quad \left(\frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(\frac{4T}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(\frac{4T}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right)$$
$$\therefore \quad r = \sqrt{r_1^2 + r_2^2}$$
$$= 5 \text{ cm}$$

10. Drag force + upthrust = weight
$$\therefore \quad ku + F = mg$$
$$\text{or} \quad u = \frac{mg - F}{k}$$

11. B will have a tendency to keep its area as low as possible.



12. $m \propto r^3$ and terminal velocity $\propto r^2$ mass is eight times. So, radius is two times and terminal velocity will be four times.

14. $N\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$
$$\therefore \quad r = R\left(\frac{1}{N}\right)^{1/3}$$
$$W = \sigma\,(\Delta A)$$
$$= \sigma\,(A_f - A_i)$$
$$= \sigma\,[N\,(4\pi r^2) - 4\pi R^2]$$
$$= 4\pi\sigma\left[NR^3\left(\frac{1}{N}\right)^{2/3} - R^2\right]$$
$$= 4\sigma\pi R^2\,(N^{1/3} - 1) \qquad \textbf{Ans.}$$

15. $(p_2 - p_1) = \frac{4T}{R}$

where R = radius of common surface
$$\therefore \quad \frac{4T}{R_2} - \frac{4T}{R_1} = \frac{4T}{R}$$
$$\text{or} \quad \frac{1}{R} = \frac{1}{R_2} - \frac{1}{R_1}$$

16. Pressure (and hence the level of liquid) will keep on decreasing in the direction of motion of liquid.

17. $v_t = \frac{2}{9}\,\frac{r^2\,(\rho - \sigma)\,g}{\eta}$

Ignoring the density of air σ we have,
$$v_T = \frac{2}{9}\,\frac{r^2\rho g}{\eta}$$
$$\therefore \quad r = \sqrt{\frac{9\eta\,v_T}{2\rho g}}$$
$$= \sqrt{\frac{9 \times 1.8 \times 10^{-5} \times 0.3}{2 \times 10^3 \times 9.8}}$$
$$= 0.49 \times 10^{-4} \text{ m}$$
$$\approx 0.05 \text{ mm} \qquad \textbf{Ans.}$$

18. If angle of contact in 90°, then liquid will neither rise nor fall.

19. $v_T \propto r^2$

Radius or diameter is half. So, uniform or terminal speed is $\frac{1}{4}$ th.

20.
$$h\rho g = 10^5 \text{ Pa (given)}$$
$$\therefore \quad p_x = \left(h - \frac{h}{5}\right)\rho g = 0.8\, h\rho g$$
$$= 0.8 \times 10^5 \text{ Pa}$$

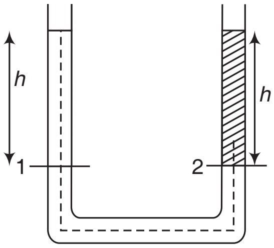
21. Fraction of volume immersed
$$f = \frac{\rho_s}{\rho_l} \quad \text{or} \quad r \propto \frac{1}{\rho_l}$$
$$\frac{f_1}{f_2} = \frac{\rho_{l_2}}{\rho_{l_1}} \quad \text{or} \quad \rho_{l_2} = \left(\frac{f_1}{f_2}\right)\rho_{l1}$$
$$= \frac{(V - V/3)}{(V/3)}\rho_{l_1}$$
$$= 2\rho_{l1}$$

22. Let ρ = density of sugar solution
$$> \rho_w$$
$$m = \text{mass of ice.}$$
In floating condition weight = upthrust
$$\therefore \quad mg = V_D \rho g$$
$$\text{or} \quad V_D = \frac{m}{\rho} \qquad \text{...(i)}$$
When ice melts, m mass of ice becomes m mass of water. Volume of this water formed.
$$V_F = \frac{m}{\rho_w} \qquad \text{...(ii)}$$
Since, $\rho_w < \rho$, $V_F > V_D$. Hence, level will increase.

23. Upthrust = $V_i \rho_l g_e$

Value of g_e will decrease. So upthrust will decrease.

24.



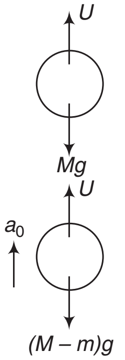
$$p_1 = p_2$$
$$\therefore \quad p_0 + \rho_w gh = p_0 + \rho gh$$
$$\Rightarrow \quad \rho = \rho_w$$

25. F_1 will decrease and F_2 will increase. So f_1 may or may not be greater than f_2.

Total weight to system in both conditions will remain same. Hence,
$$f_1 + f_2 = F_1 + F_2$$

26. $\Delta p = \frac{1}{2}\rho v^2$
$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2 \times 0.5 \times 10^5}{10^3}}$$
$$= 10 \text{ m/s}$$

27.



$$Mg - U = Ma_0 \qquad \text{...(i)}$$
$$U = (M - m)g = (M - m)a_0 \qquad \text{...(ii)}$$
Solving these two equations we get,
$$m = \frac{2Ma_0}{g + a_0}$$

28. Diameter of left hand side is $\frac{1}{5}$ times. So area will be $\frac{1}{25}$ times.
$$\Delta p_{\text{LHS}} = \Delta p_{\text{RHS}} \quad \Rightarrow \quad \frac{F}{(A/25)} = \frac{(Mg)}{A}$$
$$\therefore \quad F = \frac{Mg}{25}$$
Equating the volumes

Displacement on LHS = 5 times

Displacement on RHS
$$\therefore \quad S = (5)(0.5) = 2.5 \text{ cm}$$
$$W = FS$$
$$\therefore \quad (500) = \left(\frac{Mg}{25}\right)(2.5 \times 10^{-2})$$
$$\therefore \quad M = 5 \times 10^4 \text{ kg} \qquad (\text{with } g = 10 \text{ m/s}^2)$$

29. $4 = \frac{Vd_1 + Vd_2}{2V}$
$$\text{or} \quad d_1 + d_2 = 8 \qquad \text{...(i)}$$

$$3 = \frac{m+m}{(m/d_1)+(m/d_2)} \quad \text{or} \quad \frac{2d_1d_2}{d_1+d_2} = 3 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$d_1 = 6 \quad \text{and} \quad d_2 = 2$$

30. Total weight = total upthrust

$(A \times 0.5 \times 900)\, g + (100)\, g = A \times 0.5 \times 1000 \times g$

$\therefore \qquad A = 2 \text{ m}^2$ **Ans.**

31. Change in weight = upthrust.

$\therefore (38.2 - 36.2)\, g = (V_{\text{gold}} + V_{\text{cavity}})\, \rho_w g$

$$\therefore \qquad 2 = \left(\frac{38.2}{19.3}\right) + V_{\text{cavity}} \left(\text{as } \rho_w = \frac{1 \text{ g}}{\text{cm}^3}\right)$$

$\therefore \qquad V_{\text{cavity}} = 0.02 \text{ cm}^3$ **Ans.**

32. See the hint of Q-No. 23 (a) of subjective questions of Level 1.

33. $v = \sqrt{2gH} \quad \Rightarrow \quad \dfrac{dV}{dt} = av = a\sqrt{2gH}$

This is independent of ρ_{liquid}.

34. $(\Sigma Q)_{\text{inflow}} = (\Sigma Q)_{\text{outflow}}$

$\therefore (4 \times 10^{-6}) + Q + (4 \times 10^{-6})$

$= (8 \times 10^{-6}) + (2 \times 10^{-6}) + (5 \times 10^{-6}) + (6 \times 10^{-4})$

$\therefore \qquad Q = 13 \times 10^{-6} \text{ m}^3/\text{s}$

35. Weight = Upthrust

$$\therefore \qquad Mg = \left(\frac{3}{4}a^3\right)\rho g$$

$$\therefore \qquad a = \left(\frac{4M}{3\rho}\right)^{1/3}$$

36. $h = \dfrac{2T\cos\theta}{r\rho g}$

$$10 = \frac{2T_1 \cos 0°}{r\rho_1 g} \qquad \ldots\text{(i)}$$

$$-3.42 = \frac{2T_2 \cos 135°}{r\rho_2 g} \qquad \ldots\text{(ii)}$$

From these two equations, we get

$$\frac{T_1}{T_2} = \frac{10.\,(\cos 135°)}{(-3.42)\,(\cos 0°)\,\rho_2}$$

$$= \frac{(10)\,(-1/\sqrt{2})\,(1)}{(-3.42)\,(1)\,(13.6)}$$

$$= 1 : 6.5$$

37. Area is halved, means radius of tube is made $\dfrac{1}{\sqrt{2}}$ times.

$$h = \frac{2T\cos\theta}{r\rho g} \quad \text{or} \quad h \propto \frac{1}{r}$$

Hence, h will become $\sqrt{2}$ times.

38. In steady state,

Volume flow rate entering the vessel

= volume flow rate leaving the vessel

$$\therefore \quad Q = av = a\sqrt{2gh} \quad \text{or} \quad h = \frac{Q^2}{2ga^2}$$

$$= \frac{(10^2)^2}{(2 \times 1000)\,(1)^2} = 5 \text{ cm}$$ **Ans.**

39. Area at other point is half. So, speed will be double. Now,

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore \qquad p_2 = p_1 + \frac{1}{2}\rho\,(v_1^2 - v_2^2)$$

$$= (8000) + \frac{1}{2} \times 1000\,(4 - 16)$$

$$= 2000 \text{ Pa}$$ **Ans.**

40. Equating the volume, we have

$$8\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3 \quad \text{or} \quad R = 2r$$

$$v_T \propto (\text{radius})^2$$

Radius has become two times.Therefore, terminal velocity will become 4 times.

41. Under normal conditions, upward surface tension force supports the weight of liquid of height h in the capillary. In artificial satellite effective weight will be zero, but upward surface tension force will be there. Hence, liquid rises upto the top.

42. Pressure inside a soap bubble is given by,

$$P = P_0 + \frac{4T}{r}$$

So, pressure inside a smaller soap bubble will be more and air will flow from smaller drop to bigger drop.

43. $h = \dfrac{2T\cos\theta}{R\rho g} \quad \text{or} \quad h \propto \dfrac{1}{R}$

Now, $\qquad M = (\pi R^2 h)\,\rho$

or $\qquad M \propto R^2 h$

or $\qquad M \propto R$

Radius is doubled, so mass in the capillary tube will also become two times.

44. $\rho g h = \dfrac{4T}{r} \quad \Rightarrow \quad T = \dfrac{rh\rho g}{4}$

45. $h\rho g = \dfrac{2T}{r} \quad \Rightarrow \quad h = \dfrac{2T}{r\rho g}$

$$= \frac{2 \times 70}{0.005 \times 1 \times 1000} = 28 \text{ cm}$$

46. $N\left(\dfrac{4}{3}\pi a^3\right) = \dfrac{4}{3}\pi b^3 \quad \Rightarrow \quad N = \dfrac{b^3}{a^3}$

$$A_i = N\ (4\pi a^2) = \frac{4\pi b^3}{a}$$

$$A_f = 4\ \pi b^2$$

$$\Delta A = A_i - A_f = 4\ \pi b^2\left(\frac{b}{a} - 1\right)$$

$$W = T\ (\Delta A) = 4\ \pi b^2\left(\frac{b}{a} - 1\right)T$$

$$= \frac{1}{2}\ mv^2 = \frac{1}{2}\left(\frac{4}{3}\ \pi b^3 \rho\right) v^2$$

Solving we get

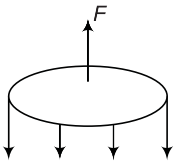
$$v = \sqrt{\frac{6T}{\rho}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

47. $p = \dfrac{2T}{r} + p_0$

$$= \frac{2 \times 0.7}{0.14 \times 10^{-3}} + 10^5$$

$$= 1.01 \times 10^5 \text{ N/m}^2$$

48. From two surfaces,



S.T force

$$F = 2\ (Tl) = 2\ (T)\ (2\ \pi r)$$

$$= 2\ (T)\ (2\ \pi)\ (1.5)$$

$$= (6\ \pi T) \text{ dyne}$$

49. $\Delta p = h\rho g + \dfrac{2T}{r}$

$$= 2 \times 1 \times 980 + \frac{2 \times 70}{0.05}$$

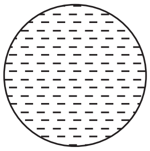
$$= 4760 \text{ dyne/cm}^2$$

Subjective Questions

1. In floating condition,
weight of liquid displaced = weight of solid

2. Weight = $V\rho_w g$

Upthrust = $V\rho_e g = V\rho_w g$



Since, weight = upthrust

Apparent weight = 0

3. $F = pA\ (\rho gh)\ B$

$\therefore \qquad F \propto h$

4. If volume is doubled, then radius becomes $(2)^{\frac{1}{3}}$ times.

$$W = T\ (8\ \pi R^2)$$

or $\qquad W \propto R^2$

Therefore $\quad W' = (2)^{\frac{2}{3}}\ W.$

5. Concept is same as in Q.No-2 of same exercise.

6. Relative density of metal in this case is given by

$$\text{RD} = \frac{\text{weight in air}}{\text{change in weight in water}}$$

$$= \frac{0.096}{0.096 - 0.071} = 3.84$$

$\therefore$ Density of metal = $3.84\ \rho_w$

$= 3840 \text{ kg/m}^3$ **Ans.**

7. Weight = upthrust

$\therefore \qquad (25 + 5)\ g = (V + 2)\ \rho_w g$

Putting $\qquad \rho_w = 1 \text{ g/cm}^3$

We get, $\qquad V = 28 \text{ cm}^3$ **Ans.**

8. Weight of both (block + woman)

= upthrust on 100% volume of block

$\therefore \quad 50\ g + V\ (850)\ g = V\ (1000)\ g$

$\therefore \qquad V = 0.33 \text{ m}^3$ **Ans.**

9. Weight of ice block + weight of metal piece
= upthrust on 100% volume of ice cube.

Let a = side of ice cube. Then,

$$[(a^3) \times 900 \times g + 0.5 \text{ g}] = (a^3)\ (1000)\ (g)$$

$\therefore \qquad a^3 = (5 \times 10^{-3})\text{m}^3$

or $\qquad a \approx 0.17 \text{ m} \quad \text{or} \quad 17 \text{cm}$ **Ans.**

10. Fraction of volume immersed is given by

$$f = \frac{\rho_s}{\rho_l}$$

In first case $\quad 0.6 = \dfrac{\rho_s}{\rho_w} = \dfrac{\rho_s}{1}$

or $\rho_s = 0.6\ \text{g/cm}^3 =$ density of wood

In second case

$$0.8 = \frac{\rho_s}{\rho_l} = \frac{0.6}{\rho_e}$$

or $$\rho_l = \frac{0.6}{0.85}$$

$$= 0.705\ \text{g/cm}^3$$ **Ans.**

11. Extra weight = extra upthrust on extra immersed volume.

$\therefore \quad (\Delta m)\, g = (\pi r^2 \Delta h)\, \rho_w g$

$\therefore \quad \Delta m = \pi r^2 \Delta h\, \rho_w$

$= (\pi)\,(0.8)^2\,(3)\,(1)$

$= 6.03$ g **Ans.**

12. $$p = \frac{F}{A} = \frac{3000 \times 9.8}{425 \times 10^{-4}}$$

$= 6.92 \times 10^5\ \text{N/m}^2$ **Ans.**

13. $p_0 + \rho_w h_w g = p_0 + \rho_s h_s g$

$$\therefore \quad \frac{\rho_s}{\rho_w} = \frac{h_w}{h_s}$$

$$= \frac{10}{12.5} = 0.8$$

= relative density of mercury

14. $(p_0 + \rho_w h_w g) - (p_0 + \rho_s h_s g) = \rho_{\text{Hg}} h_{\text{Hg}}\, g$

$$\therefore \quad h_{\text{Hg}} = \frac{\rho_w \rho_w - \rho_s h_s}{\rho_{\text{Hg}}}$$

$$= \frac{(1)\,(25) - (0.8)\,(27.5)}{13.6}$$

$= 0.221$ cm **Ans.**

15. (i) Absolute pressure in part (*a*)

= 76 cm of Hg + 20 cm of Hg

= 96 cm of Hg

Gauge pressure in part (*a*) = 20 cm of Hg

Absolute pressure in part (*b*)

= 76 cm of Hg – 18 cm of Hg = 58 cm of Hg

Gauge pressure in part (*b*) = – 18 cm of Hg.

(ii) 13.6 cm of water is equivalent to 1 cm of Hg. So the new level difference will become 19 cm.

16. $$v = \frac{p_1 - p_2}{4\eta L}\,(R^2 - r^2)$$

or $\quad v \propto R^2 - r^2$

(a) $$\frac{v_1}{v_2} = \frac{R^2 - r_1^2}{R^2 - r_2^2}$$

or $$\frac{3}{v_2} = \frac{400 - 0}{400 - 100}$$

$\therefore \quad v_2 = 2.25$ m/s **Ans.**

(b) $$v_2 = \left(\frac{R^2 - r_2^2}{R^2 - r_1^2}\right) v_1$$

$v_2 = 0 \ $ at $r_2 = R$

17. (a) $$Q = \frac{\pi}{8}\left(\frac{R^4}{\eta}\right)\left(\frac{p_1 - p_2}{L}\right)$$

$$= \left(\frac{\pi}{8}\right)\frac{(0.04)^4}{1.005 \times 10^{-1}} \times \frac{1400}{0.2}$$

$= 7 \times 10^{-2}\ \text{m}^3/\text{s}$

(b) $Q_1 = Q_2$

$\therefore \quad [(p_1 - p_2)\, R^4]_i = [(p_1 - p_2)\, R^4]_f$

Since, diameter or radius has decreased to half. Therefore gauge pressure should become 16 times.

or $\quad (p_1 - p_2)_f = 16 \times 1400$

$= 2.24 \times 10^4$ Pa **Ans.**

(c) $$Q \propto \frac{1}{\eta}$$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{\eta_2}{\eta_1}$$

or $$Q_2 = \left(\frac{\eta_1}{\eta_2}\right) Q_1$$

$$= \left(\frac{1.005}{0.469}\right)(7 \times 10^{-2})$$

$= 0.15\ \text{m}^3/\text{s}$ **Ans.**

18. (a) Relative density of metal

$$= \frac{\text{weight in air}}{\text{change in weight in water}} = \frac{10}{2} = 5$$

$\therefore$ Density of metal $= 5\rho_w = 5000\ \text{kg/m}^3$

Now, $$\text{volume} = \frac{\text{mass}}{\text{density}}$$

$$= \frac{10 \times 10^{-3}}{5000}$$

$= 2 \times 10^{-6}\ \text{m}^3$

(b) Change in weight

= upthrust on 100% volume of solid

or $\quad \Delta w = V_s \rho_l\, g$

$\therefore \quad \Delta w \propto \rho_l$

∴ $$\frac{\Delta w_l}{\Delta w_w} = \frac{\rho_l}{\rho_w}$$

or $$\rho_l = \left(\frac{\Delta w_l}{\Delta w_w}\right)\rho_w = \left(\frac{1.5}{2}\right)(1000)$$

$$= 750 \text{ kg/m}^3$$

19. Let (x) mm is in water and ($60 - x$) mm is in mercury.

Total upthrust = weight

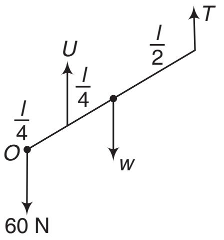
$$= (0.06)^2\,(x \times 10^{-3})\,(10^3)\,g + (0.06)^2\,(60 - x)\,(10^{-3})\,(13.6 \times 10^3)\,g$$

$$= (0.06)^3\,(7.7 \times 10^{-3})\,(g)$$

Solving this equation, we get

$$x = 28 \text{ mm} \quad \text{and} \quad 60 - x = 32 \text{ mm}$$

20. $T + U = 60 + w$ …(i)



Σ (Moments) about = O

∴ $$= U\left(\frac{l}{4}\right) + T\,(l) = W\left(\frac{l}{2}\right)$$

or $U + 4T = 2W$ …(ii)

$w = 120$ N …(iii)

Solving these three equations we get $T = 20$ N and $U = 160$ N

Now, $160\left(\frac{V}{2}\right)\rho_w g = \left(\frac{V}{2}\right)(1000)\,(10)$

∴ $V = 32 \times 10^{-3}$ m^3 **Ans.**

21. (a) Upthrust − Weight − $T = ma$

∴ T = Upthrust − Weight − ma

$$= \left(\frac{2}{500}\right)(1000)\,(10 + 2) - 20 - 4$$

$$= 48 - 20 - 4 = 24\text{N}$$ **Ans.**

(b) Downward force T suddenly becomes zero. Therefore

$$a = \frac{\text{upthrust} - \text{weight}}{m} = \frac{48 - 20}{2}$$

$$= 14 \text{ m/s}^2$$

∴ Acceleration w.r.t. tank.

$$= 14 - 2$$

$$= 12 \text{ m/s}^2$$ **Ans.**

22. (a) Force from left hand side

= force from right hand side

∴ $kx = \Delta pA$

or $$x = \frac{(\Delta p)\,A}{k}$$

$$= \frac{(30 + 101) \times 10^3 \times 0.5 \times 10^{-4}}{60}$$

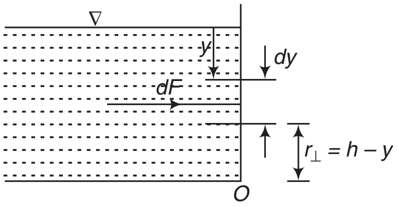
$= 0.109$ m or 10.9 cm **Ans.**

(b) $$x = \frac{(\Delta p) A}{k}$$

$$= \frac{30 \times 10^3 \times 0.5 \times 10^{-4}}{60}$$

$= 0.025$ m $= 2.5$ cm **Ans.**

23. (a) $dF = (p_y)\,(dA) = (\rho g y)\,(l dy)$



∴ $$F = \int_0^h dF = \frac{\rho g l h^2}{2}$$

(b) Perpendicular distance of about force from point O is

$$r_\perp = h - y$$

∴ $d\tau = (dF)\,r_\perp = (\rho g l y)\,(h - y)\,dy$

$$= \text{total torque } = \int_0^h d\tau$$

After substituting the values we get,

$$\tau = \frac{\rho g l h^3}{6}$$ **Ans.**

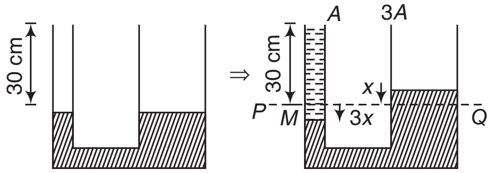
(c) $F \times r_\perp = \tau$

∴ $$r_\perp = \frac{\tau}{F}$$

Substituting the values we get,

$$r_\perp = \frac{h}{3}$$ **Ans.**

24. PQ is original level of mercury. Equating the pressures at the level of M from left hand side and right hand side.



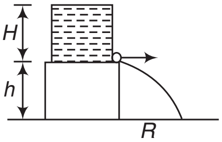
$$p_0 + \rho_w g\, h_w = p_0 + \rho_{Hg}\, g h_{Hg}$$

or $$\rho_w\, h_w = \rho_{Hg} h_{Hg}$$

∴ $$(1)\,(30 + 3x) = (13.6)\,(4x)$$

Solving we get, $x = 0.58$ cm **Ans.**

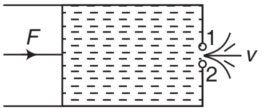
25.



$$R = 2\sqrt{Hh}$$

∴ $$H = \frac{R^2}{4h}$$ **Ans.**

26.



$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$\left(p_0 + \frac{F}{A}\right) + 0 = p_0 + \frac{1}{2}\rho v^2$$

∴ $$v = \sqrt{\frac{2F}{\rho A}}$$ **Ans.**

27. $\frac{1}{2}\rho_{air}\, v^2 = \rho_w g h_w$

$$v = \sqrt{\frac{2\rho_w g\, h_w}{\rho_{air}}}$$

$$= \sqrt{\frac{2 \times 10^3 \times 10 \times 10^{-2}}{1.3}}$$

$$= 12.4 \text{ m/s}$$ **Ans.**

28. $\left(\frac{Mg}{A}\right) + \rho g h = \frac{1}{2}\rho v^2$

∴ $$v = \sqrt{\frac{2Mg}{\rho A} + 2gh}$$

$$= \sqrt{\frac{2 \times 20 \times 10}{0.5 \times 10^3} + 2 \times 10 \times 0.5}$$

$$= 3.28 \text{ m/s}$$ **Ans.**

29. $$v_{max} = \frac{(p_1 - p_2)\, R^2}{4\eta L}$$

∴ $$(p_1 - p_2) = \frac{4\eta L\, v_{max}}{R^2}$$

$$= \frac{(4 \times 4 \times 10^{-3})\,(10^{-3})\,(0.66 \times 10^{-3})}{(2 \times 10^{-6})^2}$$

$$= 2.64 \times 10^3 \text{ N/m}^2 = h\rho g$$

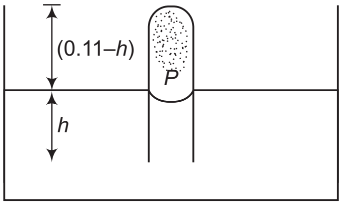
$$h = \frac{2.64 \times 10^3}{\rho g}$$

$$= \frac{2.64 \times 10^3}{13.6 \times 10^3 \times 9.81} \text{ m of Hg}$$

$$= 0.0195 \text{ m of Hg}$$

$$\approx 19.5 \text{ mm of Hg}$$

30.



$$p - \frac{2T}{R} = p_0$$

∴ $$P = \frac{2T}{R} + P_0$$

$$R = \frac{r}{\cos\theta}$$

For water $\theta = 0°$

∴ $R = r$ or $P = \frac{2T}{r} + P_0$

As temperature is constant.

$$p_1 V_1 = p_2 V_2$$

∴ $$p_0\,(0.11) = p\,(0.11 - h)$$

or $$p_0\,(0.11) = \left(p_0 + \frac{2T}{r}\right)(0.11 - h)$$

Substituting the values, we get

$$(1.01 \times 10^5\,(0.11) = \left[1.01 \times 10^5 + \frac{2 \times 5.06 \times 10^{-2}}{10^{-5}}\right](0.11 = h)$$

Solving this equation we get,

$$h = 0.01 \text{ m or } 1 \text{ cm}$$

If seal is broken, then water will rise in the capillary.

31. $W = T\,(\Delta A)$

$$= T\,(2ld)$$

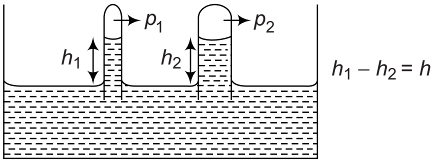
$$= (72 \times 10^{-3})\,(2)\,(0.1)\,(10^{-3})$$

$$= 14.4 \times 10^{-6} \text{ J}$$ **Ans.**

32. Let the pressures in the wide and narrow capillaries of radii r_1 and r_2 respectively be p_1 and p_2. Then pressure just below the meniscus in the wide and narrow tubes respectively are

$$\left(p_1 - \frac{2T}{r_1}\right) \text{ and } \left(p_2 - \frac{2T}{r_2}\right)$$

$$\left[\text{excess pressure} = \frac{2T}{r}\right]$$



Difference in these pressures

$$\left(p_1 - \frac{2T}{r_1}\right) - \left(p_2 - \frac{2T}{r_2}\right) = h\rho g$$

$\therefore$ True pressure difference $= p_1 - p_2$

$$= h\rho g + 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 0.2 \times 10^3 \times 9.8 + 2 \times 72 \times 10^{-3}$$

$$\left[\frac{1}{1.44 \times 10^{-3}} - \frac{1}{7.2 \times 10^{-4}}\right]$$

$$= 1.86 \times 10^3 = 1860\,\text{N/m}^2$$

33. The surface tension of the liquid is

$$T = \frac{rh\rho g}{2}$$

$$= \frac{(0.025\text{cm})(3.0\text{cm})(1.5\text{gm / cm}^3)(980\text{cm / s}^2)}{2}$$

= 55 dyne/cm

Hence, excess pressure inside a spherical bubble

$$p = \frac{4T}{R} = \frac{4 \times 55\text{dyne/cm}}{(0.5\text{cm})}$$

$$= 440 \text{ dyne / cm}^2$$

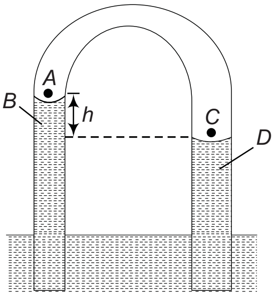
34. Suppose pressure at the points, A, B, C and D be p_A, p_B, p_C and p_D respectively.

The pressure on the concave side of the liquid surface is greater than that on the other side by $\frac{2T}{R}$.

An angle of contact θ is given to be 0°, hence

$$R\cos 0° = r \quad \text{or} \quad R = r$$

$$\therefore \quad p_A = p_B + \frac{2T}{r_1} \quad \text{and} \quad p_C = p_D + \frac{2T}{r_2}$$



where, r_1 and r_2 are the radii of the two limbs

But $\quad p_A = p_C$

$$\therefore \quad p_B + \frac{2T}{r_1} = p_D + \frac{2T}{r_2}$$

$$\text{or} \quad p_D - p_B - 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where, h is the difference in water levels in the two limbs

$$\text{Now,} \quad h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Given that $\quad T = 0.07\ \text{Nm}^{-1}, \rho = 1000\ \text{kgm}^{-3}$

$$r_1 = \frac{3}{2}\text{mm} = \frac{3}{20}\text{cm} = \frac{3}{20 \times 100}\text{m}$$

$$= 1.5 \times 10^{-3}\text{m},\ r_2 = 3 \times 10^{-3}\ \text{m}$$

$$\therefore \quad h = \frac{2 \times 0.07}{1000 \times 9.8}\left(\frac{1}{1.5 \times 10^{-3}} - \frac{1}{3 \times 10^{-3}}\right)\text{m}$$

$$= 4.76 \times 10^{-3}\text{m} = 4.76\text{mm}$$

35. The total pressure inside the bubble at depth h_1 is (p is atmospheric pressure)

$$= (p + h_1\rho g) + \frac{2T}{r_1} = p_1$$

and the total pressure inside the bubble at depth h_2 is

$$= (p + h_2\rho g) + \frac{2T}{r_2} = p_2$$

Now, according to Boyle's law

$$p_1V_1 = p_2V_2$$

$$\text{where,} \quad V_1 = \frac{4}{3}\pi r_1^3 \quad \text{and} \quad V_2 = \frac{4}{3}\pi r_2^3$$

Hence, we get

$$\left[(p + h_1\rho g) + \frac{2T}{r_1}\right]\frac{4}{3}\pi r_1^3$$

$$= \left[(p + h_2\rho g) + \frac{2T}{r_2}\right]\frac{4}{3}\pi r_2^3$$

or $\left[(p+h_1\rho g)+\dfrac{2T}{r_1}\right]r_1^3=\left[(p+h_2\rho g)+\dfrac{2T}{r_2}\right]r_2^3$

Given that $h_1 = 100$ cm, $r_1 = 0.1$ mm $= 0.01$ cm, $r_2 = 0.126$ mm $= 0.0126$ cm, $T = 567$ dyne/cm, $p = 76$ cm of mercury. Substituting all the values, we get

$$h_2 = 9.48\,\text{cm}.$$

36. Let n be the number of little droplets.

Since, volume will remain constant, hence volume of n little droplets = volume of single drop

$$\therefore \qquad n\times\frac{4}{3}\pi r^3=\frac{4}{3}\pi R^3$$

$$\text{or} \qquad nr^3 = R^3$$

Decrease in surface area $= n\times 4\pi r^2 - 4\pi R^2$

$$\text{or } \Delta A = 4\pi[nr^2 - R^2]$$

$$= 4\pi\left[\frac{nr^3}{r}-R^2\right]=4\pi\left[\frac{R^3}{r}-R^2\right]$$

$$= 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$$

Energy evolved $W = T\times$ decrease in surface area

$$= T\times 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$$

Heat produced, $Q=\dfrac{W}{J}=\dfrac{4\pi TR^3}{J}\left[\dfrac{1}{r}-\dfrac{1}{R}\right]$

But $\qquad Q = msd\theta$

where, m is the mass of big drop, s is the specific heat of water and $d\theta$ is the rise in temperature.

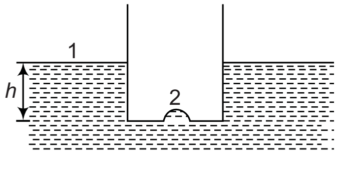
$$\therefore \quad \frac{4\pi TR^3}{J}\left[\frac{1}{r}-\frac{1}{R}\right]=\text{Volume of big drop}$$

$\times$ density of water $\times$ sp. heat of water $\times\, d\theta$

$$\text{or} \quad \frac{4}{3}\pi R^3\times1\times1\times d\theta=\frac{4\pi TR^3}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$\text{or} \qquad d\theta=\frac{3T}{J}\left[\frac{1}{r}-\frac{1}{R}\right]$$

37.



Applying pressure equations between 1 and 2,

$$p_0+\rho gh-\frac{2T}{r}=p_0$$

$$\Rightarrow \qquad h=\frac{2T}{r\rho g}$$

LEVEL 2

Single Correct Option

1. When ice melts into water its volume decreases. Hence, over all level should decrease.

Now suppose m is the mass of ice, V_1 is volume immersed in water and V_2 the volume immersed in oil. Then in floating condition,

weight = upthrust

$$\therefore \qquad mg = V_1\rho_w g + V_2\rho_0 g$$

$$\therefore \qquad V_1=\frac{m-V_2\rho_0}{\rho_w}$$

$$=\frac{m}{\rho_w}-\frac{V_2\rho_0}{\rho_w} \qquad \text{...(i)}$$

When ice melts, m mass of ice converts into m mass of water. Volume of water so formed is

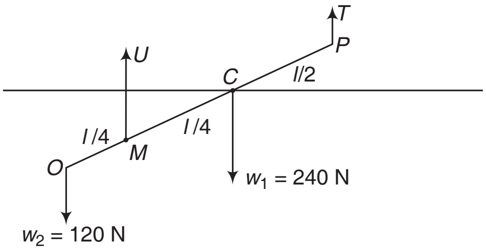
$$V_3=\frac{m}{\rho_w} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $V_3 > V_1$

$\therefore$ Interface level will rise.

2. $g_e = 0$ so, pressure inside and pressure outside the hole will be same.

3.



$$T+U=w_1+w_2=360\text{ N} \qquad \text{...(i)}$$

$$U=\frac{V}{2}\rho_w g=(V/2)\,(10^3)\,(10)$$

$$\text{or} \qquad U = 0.5\times10^4\,V \qquad \text{...(ii)}$$

(Σ Moments) about $M = 0$

$$\therefore \qquad 120\left(\frac{l}{4}\right)+T\left(\frac{3l}{4}\right)=240\left(\frac{l}{4}\right)$$

$$\text{or} \qquad T=\frac{240-120}{3}$$

$$= 40\text{ N} = 4\text{g}$$

Now from Eqs. (i) and (ii), we get

$$V = 6.4\times10^{-2}\text{ m}^3$$ **Ans.**

4. Volume flow rate $= av = \dfrac{V}{t}$

$$\therefore \qquad v_1=\frac{V}{A_1t}=\frac{120\times10^{-3}}{5\times10^{-4}\times2\times60}$$

Now, $A_1 v_1 = A_2 v_2$

A_2 is $\frac{1}{5}$ th of A_1.

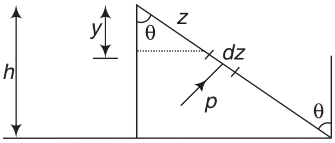
Hence, v_2 is five times of v_1 or 10 m/s.

$$t_0 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1}{10}}$$

$$= 0.447 \text{ s}$$

$$R = v_2 t_0 = 4.47 \text{ m}$$

5.



$$z = y \sec\theta$$

$$dz = dy \sec\theta$$

$$p = (\rho g y)$$

$$dA = (b)\, dz = (b)\, dy \sec\theta$$

$$dF = p\,dA = (\rho g b \sec\theta)\; y\, dy$$

$$\therefore \quad F = \int_0^h dF = \frac{1}{2} \rho b h^2 g \sec\theta$$

6. Velocity of body just before touching the lake surface is,

$$v = \sqrt{2gh}$$

Retardation in the lake,

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$$

$$= \frac{V\rho g - V\rho g}{V\rho} = \left(\frac{\sigma - \rho}{\rho}\right) g$$

Maximum depth $d_{\max} = \frac{v^2}{2a} = \frac{h\rho}{\sigma - \rho}$

7. $\rho g h = \rho a L$

$$\therefore \quad h = \frac{aL}{g}$$

8. Radius of meniscus $= \frac{R}{\cos\theta} = r$

Δp due to spherical surface $= \frac{2\sigma}{r}$

$$= \frac{2\sigma \cos\theta}{R}$$

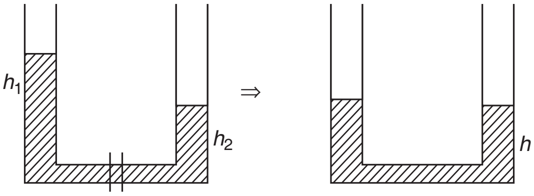
$$p_B = p_{\text{atm}}$$

9. $a_1 v_1 = a_2 v_2$

$$\therefore \quad (L^2)\sqrt{2gh} = (\pi R^2)\sqrt{2g\,(4h)}$$

$$R = \frac{L}{\sqrt{2\pi}}$$

10. Let h = final height of liquid.



Equating the volumes we have,

$$Ah_1 + Ah_2 = A\,(2h)$$

$$\therefore \quad h = \frac{h_1 + h_2}{2}$$

$$W_{mg} = -\Delta U = U_i - U_f$$

$$= \left[\left(Ah_1 \rho g \frac{h_1}{2}\right) + \left(Ah_2 \rho g \frac{h_2}{2}\right)\right] - \left[2A\left(\frac{h_1 + h_2}{2}\right)\rho\left(\frac{h_1 + h_2}{4}\right)\right]$$

Simplifying this expression we get the result.

11. $\Delta p = \rho_{0\text{il}}\; g\; h_{0\text{il}} + \rho_w\, g h_w$

$$= (600)(10)(10 \times 10^{-2}) + (1000 \times 10 \times 2 \times 10^{-2})$$

$$= 800 \text{ N/m}^2$$

12. $\Delta p = \frac{1}{2} \rho v^2$

$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2 \times 5 \times 10^5}{1000}}$$

$$= 31.5 \text{ m/s}$$

13. From continuity equation,

$$v_A = v_B = v_0$$

$$\therefore \quad p_A + \rho g h = p_B + 0$$

$$\therefore \quad p_B - p_A = \rho g h \qquad \text{...(i)}$$

Now, let us make pressure equation from manometer.

$$p_A + \rho g\,(h + H) - \rho_{\text{Hg}}\; g h = p_3$$

Putting $p_B - p_A = \rho g h$ we get $h = 0$

14. $F = \frac{\Delta p}{\Delta t} = \left(\frac{\Delta m}{\Delta t}\right)(\Delta v)$

$$= \rho\left(\frac{\Delta V}{\Delta t}\right)(2v)$$

$$= \rho\,(Av)\,(2v) = 2\rho A v^2$$

15. $\Delta p_A = \rho g h$ or $\rho a l$

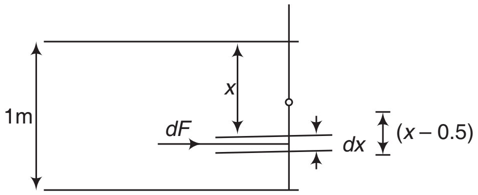
16. At every instant half of the length remains up the surface and half below the surface of liquid.

Since, $D >> d$, over all level will remain unchanged.

Let at $t = 0$, length of candle is 10 cm.

Time (s)	Length of candle (cm)	Length about the surface (cm)	Length below the surface (cm)
0	10	5	5
1	8	4	4

17.



Net torque about hinge = 0

$$dF = (\rho g x)(1 \cdot dx)$$

$$d\tau = (\rho g x)(x - 0.5)\,dx$$

∴ Net anticlockwise torque

$$= \int_0^1 d\tau = \left(\frac{\rho g}{12}\right)$$

Net clockwise torque of applied force $= F\,(0.5) = \dfrac{F}{2}$

Equating the two torques we get,

$$F = \frac{\rho g}{6}$$

18. See the typed example 12.

$$\tan\theta = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \tan 30° = \frac{1}{\sqrt{3}}$$

Solving this equation we get,

$$\frac{\rho_1}{\rho_2} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

19. $F = \dfrac{\Delta p}{\Delta t} = \left(\dfrac{\Delta m}{\Delta t}\right)(\Delta v)$

$$= \rho\left(\frac{\Delta V}{\Delta t}\right)(v_1 + v_2) = \rho V\,(v_1 + v_2)$$

20.

$$v_T = \frac{2}{9}\,\frac{r^2(\rho - \sigma)\,g}{\eta}$$

$$= \frac{2 \times (0.003)^2\,(1260 \times 2 - 1260) \times 10}{9 \times 1.26}$$

$$= 0.02 \text{ m/s}$$

$$t = \frac{d}{v_T} = \frac{10 \times 10^{-2}}{0.02} = 5 \text{ s.}$$

21. Torque of hydrostatic force about centre of sphere is already zero as hydrostatic force passes through the centre.

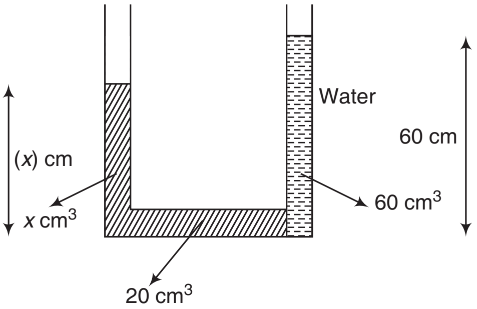
$$\therefore \quad F_{\text{LHS}} = F_{\text{RHS}}$$

For finding force refer Q-No. 23 (a) subjective questions of section Level 1.

$$\therefore \quad \frac{1}{2}(2\rho)(g)\,h^2 = \frac{1}{2}(3\rho)(g)\,R^2$$

$$\therefore \quad h = \sqrt{\frac{3}{2}}\,R$$

22.



$$\rho_1 g h_1 = \rho_2 g h_2$$

$$\therefore \quad (4)(g)(x) = (1)(g)(60)$$

or $\quad x = 15$ cm

Total volume of liquid $= (20 + 15)\text{ cm}^3$

$$= 35 \text{ cm}^3$$

23. Viscous force $= mg \sin\theta$

$$\therefore \quad \eta\,(a^2)\,\frac{v}{t} = mg \sin 37° = \frac{3}{5}\,mg$$

$$= \frac{3}{5}(a^3\rho)\,g \Rightarrow \therefore\ \eta = \frac{3\rho a g t}{5v}$$

24. Reading = weight of bucket of water + magnitude of upthrust on block

$$= (10g) + \frac{1}{2}\left(\frac{7.2}{7.2\,\rho_w}\right)\rho_w g$$

$$= 10.5 \text{ g} = 10.5 \text{ kg}$$

25. Sum of all three terms are different at three points A, B and C.

26.

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}} \text{ (upwards)}$$

$$= \frac{V\sigma g - V\rho g}{V\rho} = \left(\frac{\sigma}{\rho} - 1\right) g \qquad \text{(upwards)}$$

27. $h_{\text{Top}} = \dfrac{2H + H}{2} = 1.5H$

Since, this point lies in the tank. So hole should be made at this point.

28. Viscous force $F \propto$ (area)

So let $\quad F = kA$

$$F_0 = k\,(A_1 + A_2) \qquad \ldots\text{(i)}$$

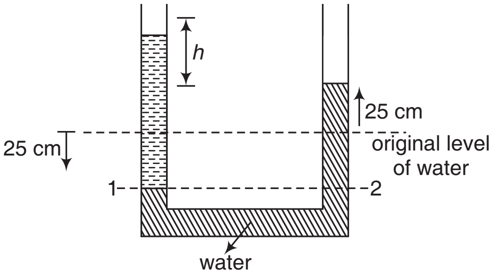
and $\quad T = k\,A_1 \qquad \ldots$(ii)

Dividing Eq. (ii) by Eq.(i) we get,

$$\frac{T}{F_0} = \frac{A_1}{A_1 + A_2}$$

$$\therefore \qquad T = \frac{F_0 A_1}{A_1 + A_2}$$

29.



$$p_1 = p_2$$

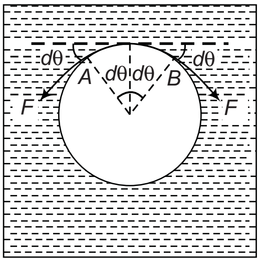
$$\therefore \qquad p_0 + \rho_0 g h_0 = p_0 + \rho_w g\, h_w$$

or $$\rho_0 h_0 = \rho_w h_w$$

or $$(0.8)\,(h + 50) = (1)\,(50)$$

or $$h = 12.5 \text{ cm}$$

30.



$$F = \text{Tension}$$

Surface tension force is radially outwards.

On AB :

$2F \sin d\theta$ = Surface tension force from two films from two sides

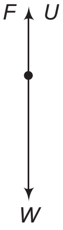
For small angle,

$$\sin d\theta \approx d\theta$$

$$\therefore \qquad 2F\, d\theta = 2\,(Tl) = 2\,(T)\,(2Rd\theta)$$

$$\therefore \qquad F = 2\,TR$$

31. Surface tension force on liquid is downwards. But on the disc it is upwards.



In the figure, F = Surface tension force

$$= Tl \cos\theta = T\;(2\,\pi r)\cos\theta$$

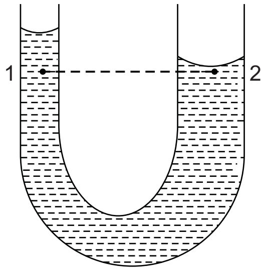
$$U = \text{upthrust} = \omega$$

$$W = \text{weight of disc}$$

For equilibrium of disc,

$$W = U + F = \omega + 2\,\pi T r \cos\theta$$

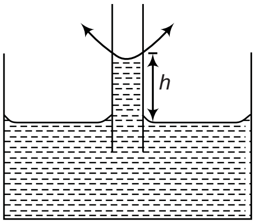
32. $p_1 = p_2$

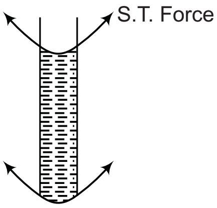


$$p_0 - \frac{2T}{r_1} + \rho g h = p_0 - \frac{2T}{r_2}$$

$$\therefore \qquad T = \frac{\rho g h r_1 r_2}{2\,(r_2 - r_1)}$$

33. When the capillary is inside the liquid, the surface tension force supports the weight of liquid of height 'h'.





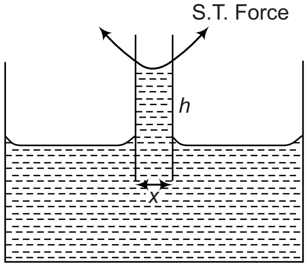
When the capillary is taken out from the liquid similar type of surface tension force acts at the bottom also, as shown in second figure. Hence, now it can support weight of a liquid of height $2h$.

34. Let L be the width of plates (perpendicular to paper inwards).

Surface tension force in upward direction

= weight of liquid of height h

$$\therefore \qquad Tl\cos\theta = Vdg \qquad (V = \text{volume})$$



or $$T\;(2L)\cos\theta = (Lxh)\;dg$$

$$\therefore \qquad h = \frac{2T\cos\theta}{xdg}$$

35. Let R_1 and R_2 are the radii of soap bubbles before and after collapsing. Then given that,

$$V = 2\left(\frac{4}{3}\pi R_1^3\right) - \frac{4}{3}\pi R_2^3 \qquad \text{...(i)}$$

$$S = 2(8\pi R_1^2) - 8\pi R_2^2 \qquad \text{...(ii)}$$

$$P_1V_1 = P_2V_2$$

$$\therefore \left(P_0 + \frac{4T}{R_1}\right)\left(2\times\frac{4}{3}\pi R_1^3\right) = \left(P_0 + \frac{4T}{R_2}\right)\left(\frac{4}{3}\pi R_2^3\right) \qquad \text{...(iii)}$$

Solving these three equation we get the desired result.

36. $p_iV_i = p_fV_f$

$$\therefore \left(p_1 + \frac{4\sigma}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(p_2 + \frac{4\sigma}{r/2}\right)\left(\frac{4}{3}\pi\right)\left(\frac{r}{2}\right)^3$$

Solving this equation we get,

$$p_2 = 8p_1 + \frac{24\sigma}{r}$$

37. $$W = 2\pi(r_1 + r_2)T$$

$$\therefore \quad T = \frac{W}{2\pi(r_1 + r_2)} = \frac{7.48\times 10\times 10^{-3}}{2\pi\times 17\times 10^{-2}}$$

$$= 70\times 10^{-3}\ \text{N/m}$$

More than One Correct Options

1. Velocity gradient $= \dfrac{\Delta v}{\Delta h} = \dfrac{2\text{m/s}}{1\text{ m}} = 2\text{s}^{-1}$

$$F = \eta A\frac{\Delta v}{\Delta h} = (10^{-3})(10)(2)$$

$$= 0.02\ \text{N}$$

2. For contact angle $\theta = 90°$, liquid neither rises nor falls.

3. Restoring force $= -(\rho A g)x$ or $F \propto -x$

This is just like a spring-block system of force constant

$$K = \rho A g$$

4. From continuity equation $Av =$ constant

$$v_2 > v_2 \quad \text{as} \quad A_2 < A_1$$

From Bernoulli's equation,

$$p + \frac{1}{2}\rho v^2 = \text{constant} \quad (\text{as } h = \text{constant})$$

$$p_2 < p_1 \quad \text{as} \quad v_2 > v_2$$

5. Fraction of volume immersed,

$$f = \frac{\rho_s}{\rho_l}$$

This fraction is independent of atmospheric pressure. With increase in temperature ρ_s and ρ_l both will decrease.

6. $U = \sqrt{2gh_T} : t = \sqrt{\dfrac{2h_B}{g}},\ R = 2\sqrt{h_Th_B}$

Here, $h_T =$ distance of hole from top surface of liquid

and $h_B =$ distance of hole from bottom surface

7. Pressure increase with depth in vertical direction and in horizontal direction it increases in opposite direction of acceleration based on this concept pressure is maximum at point D and minimum at B.

8. In air, $a_1 = g$ (downwards)

$$\text{In liquid, } a_2 = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$$

$$= \frac{(V)(2\rho)(g) - (V)(\rho)(g)}{(V\rho)}$$

$$= g \text{ (upwards)}$$

$$\therefore \quad a_1 \neq a_2$$

9. Initially

$$\frac{dV_1}{dt} = v_1a_1 = (\sqrt{2gh})(\pi)(2R)^2 \qquad \text{...(i)}$$

$$\frac{dV_2}{dt} = v_2a_2 = (\sqrt{2g(16h)})(\pi)(R)^2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

After some time v_1 and v_2 both will decrease, but decrease in the value of v_1 is more dominating. So,

$$a_1v_1 \quad \text{or} \quad \frac{dV_1}{dt} < a_2v_2 \quad \text{or} \quad \frac{dV_2}{dt}$$

10. Fraction of volume immersed is given by

$$f = \frac{\rho_s}{\rho_l}$$

ρ_s and ρ_l are same. Hence :

$$f_1 = f_2 = f_3$$

Base area in third case is uniform. Hence h_3 is minimum.

Comprehension Based Questions

2. $$x_3 = 2\sqrt{h_{\text{Top}}\times h_{\text{bottom}}}$$

$$= 2\sqrt{3a\times a} = 2\sqrt{3}a$$

3. $$x_1 = 2\sqrt{a\times 3a} = 2\sqrt{3}a$$

$$x_2 = 2\sqrt{2a\times 2a} = 2\sqrt{4}a$$

$$x_3 = 2\sqrt{3}a$$

$$\therefore \quad x_1 = x_3 < x_2$$

4. Weight = upthrust

$$\therefore \left(L\frac{A}{5}Dg\right)=\frac{L}{4}\times\frac{A}{5}\times 2d\times g+\frac{3L}{4}\times\frac{A}{5}\times d\times g$$

$$\therefore \qquad D=\frac{5d}{4}$$

5. $pA = p_0A +$ weight of two liquids + weight of cylinder

$$= p_0A+\left(\frac{H}{2}\right)(A)\,(d)\,(g)+\left(\frac{H}{2}\right)(A)\,(2d)\,(g)$$

$$+\,L\left(\frac{A}{5}\right)\left(\frac{5d}{4}\right)g$$

$$\therefore \qquad p=p_0+\frac{(L+6H)}{4}\,dg$$

6. Applying Bernoulli's equation just inside and just outside the hole,

$$p_0+\left(\frac{H}{2}\right)(d)g+\left(\frac{H}{2}-h\right)(2d)(g)=\frac{1}{2}\,(2d)\,v^2+p_0$$

$$\therefore \qquad v=\sqrt{\frac{g}{2}\,(3H-4h)}$$

7. $t=\sqrt{\dfrac{2h}{g}} \quad \Rightarrow \quad \therefore \quad x=vt=\sqrt{(3H-4h)\,h}$

Match the Columns

1. $v=\sqrt{2gh_T}$ and $R=2\sqrt{h_Th_B}$

Here, h_T = height of hole from top surface

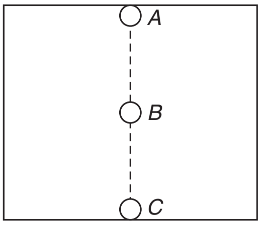
and h_B = height of hole from bottom

2. (a) and (b) In floating condition upthrust = weight

By increasing temperature of density liquid, weight remains unchanged. Hence, upthrust is unchanged.

(c) When density of solid is increased (with constant volume) mass and hence weight of solid will increase. So, upthrust will increase.

3. Ball will oscillate simple harmonically. The mean position in at depth



$$\rho=\alpha h \quad \text{or} \quad h=\frac{\rho}{\alpha}=AB$$

$$h_{\max}=\frac{2\rho}{\alpha}=AC$$

Amplitude $=\dfrac{\rho}{\alpha}=AB$ or BC

From A to B : $\rho>\rho_l$, weight > upthrust

At B, $\rho=\rho_l$, weight = upthrust

From B to C, $\rho_e>\rho$, upthrust > weight.

From A to C → upthrust will increase and gravitational potential energy will decrease.

From C to A, upthrust will decrease and gravitational potential energy will increase.

From A to B, speed will increase.

From B to C, speed will decrease.

4. Surface tension $=\dfrac{F}{e}$

Viscous force $f=\eta A\dfrac{\Delta v}{\Delta y}$

Energy density $=\dfrac{\text{Energy}}{\text{Volume}}$

Volume flow rate $=\dfrac{\text{Volume}}{\text{Time}}$

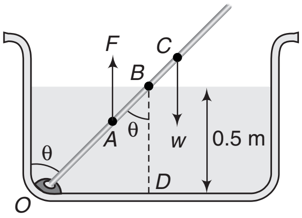
5. F_2-F_1 = upthrust

= weight of cylinder in equilibrium or floating condition.

From liquid-1, horizontal pair of forces cancel out. So, net force = 0

Subjective Questions

1. w = Weight, F = Upthrust



$$OA=\frac{OB}{2}=\frac{0.5\sec\theta}{2}=0.25\sec\theta$$

About point O, clockwise moment of w = anticlockwise moment of F.

$$\therefore \quad w\left(\frac{L}{2}\sin\theta\right)=F(OA\sin\theta)=F(0.25\sin\theta\cdot\sec\theta)$$

Given, $\qquad L=1\text{ m}$

$$\therefore \qquad \cos\theta=\frac{F}{2w}=\frac{(0.5\sec\theta)(A)(1.0)g}{2(1)(A)((0.5)g}$$

or $\qquad \cos^2\theta=\dfrac{1}{2},\ \cos\theta=\dfrac{1}{\sqrt{2}}$

$\therefore \qquad \theta=45°$ **Ans.**

2. Fraction of volume immersed before putting the new weight $= \dfrac{\rho_{\text{block}}}{\rho_{\text{water}}} = \dfrac{800}{1000} = 0.8$

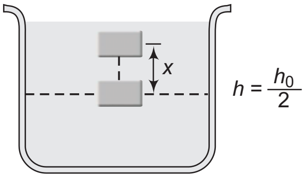
i.e. 20% of 3 cm or 0.6 cm is above water. Let w is the new weight, then spring will be compressed by 0.6 cm.

$\therefore$ w + weight of block = Upthrust on whole volume of block + spring force

or
$$w = (3\times10^{-2})^3 \times 1000\times 10 + 50\times (0.6\times10^{-2}) - (3\times10^{-2})^3\times 800\times 10$$

$\therefore$ $w = 0.354$ N **Ans.**

3. Net force on the block at a height h from the bottom is



$$F_{\text{net}} = \text{upthrust} - \text{weight} \qquad \text{(upwards)}$$
$$= \left(\frac{m}{\frac{5}{2}\rho_0}\right)\rho_0\left(4 - \frac{3h}{h_0}\right)g - mg$$
$$F_{\text{net}} = 0 \text{ at } h = \frac{h_0}{2}$$

So, $h = \dfrac{h_0}{2}$ is the equilibrium position of the block.

For $h > \dfrac{h_0}{2}$, weight > upthrust

i.e. net force is downwards and for $h < \dfrac{h_0}{2}$

weight < upthrust

i.e. net force is upwards.

For upward displacement x from mean position, net downward force is
$$F = -\left[\left(\frac{m}{\frac{5}{2}\rho_0}\right)\rho_0\left\{4 - \frac{3(h+x)}{h_0}\right\}g - mg\right]\left(h = \frac{h_0}{2}\right)$$
$$\therefore \quad F = -\frac{6mg}{5h_0}x \qquad \ldots\text{(i)}$$

(because at $h = \dfrac{h_0}{2}$ upthrust and weight are equal)

Since $\quad F \propto -x$

Oscillations are simple harmonic in nature. Rewriting Eq. (i)
$$ma = -\frac{6mgx}{5h_0}$$
or
$$a = -\frac{6g}{5h_0}x$$
$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{6g}{5h_0}}$$ **Ans.**

4. (a) $v = \sqrt{2gh} = \sqrt{2\times10\times5}$
$= 10$ m/s

(b) From conservation of energy,
$$v'^2 = v^2 + 2gH$$
$$= 100 + 2\times10\times5 = 200$$
$$v' = 14.1 \text{ m/s}$$

(c)
$$t = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$
$$= \frac{2\times\pi\times(1)^2}{10^{-4}\sqrt{2\times10}}[\sqrt{5} - \sqrt{2.5}]$$
$$= 9200 \text{ s}$$

5. Writing equation of motion for the block
$$T - mg\sin 30° = ma \qquad \ldots\text{(i)}$$
For the sphere

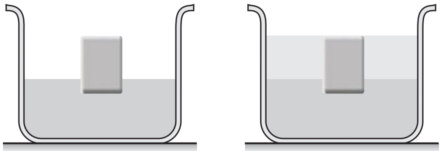
Weight − Buoyant force − $T = ma$ $\qquad\ldots$(ii)

or
$$mg - \frac{mg}{2} - T = ma$$

Solving, we get $\quad a = 0$

6. $w = (0.25)V\rho_{\text{Hg}}\, g \qquad \ldots$(i)

Let x fraction of volume is immersed in mercury in the second case. Then,



$$w = xV\rho_{\text{Hg}}g + (1-x)V\rho_w g \qquad \ldots\text{(ii)}$$

Equating Eqs. (i) and (ii), we have
$$\frac{\rho_{\text{Hg}}}{4} = x\rho_{\text{Hg}} + (1-x)\rho_w$$
$$\therefore \quad 12.6x = 2.4$$
or $\quad x = 0.19$ **Ans.**

7. (a) $p_A - \frac{1}{2}\rho v^2 + \rho g h = p_D$

But $\quad p_A = p_D = p_0$

$$\therefore \quad \frac{1}{2}\rho v^2 = \rho g h$$

$$\therefore \quad v = \sqrt{2gh}$$

Here, $\quad h = (4 + 1) = 5$ m

$$\therefore \quad v = \sqrt{2 \times 9.8 \times 5} = 9.9 \text{ m/s}$$

(b) Applying Bernoulli's equation at A and B,

$$p_A + 0 + 0 = p_B + \frac{1}{2}\rho v^2 + \rho g(1.5)$$

or $\quad p_0 = p_B + \frac{1}{2}\rho v^2 + 1.5\,\rho g$

$$\therefore \quad p_B = 1.01 \times 10^5 - \frac{1}{2} \times 900 \times (9.9)^2 - 1.5 \times 900 \times 9.8$$

$$= 4.36 \times 10^4 \text{ Pa}$$ **Ans.**

(c) Applying Bernoulli's equation at A and C,

$$p_0 = p_C + \frac{1}{2}\rho v^2 - \rho g(1.0)$$

$$\therefore \quad p_C = p_0 + \rho g - \frac{1}{2}\rho v^2$$

$$= 1.01 \times 10^5 + 900 \times 9.8 - \frac{1}{2} \times 900 \times (9.9)^2$$

$$= 6.6 \times 10^4 \text{ Pa}$$ **Ans.**

8. $\frac{1}{2}\rho v^2 = \rho g h + \frac{mg}{A}$

Here, $\quad h = 1.0 - 0.5 = 0.5$ m,

A = Area of piston = 0.5 m^2

$$\therefore \quad v = \sqrt{2gh + \frac{2mg}{\rho A}}$$

$$= \sqrt{2 \times 9.8 \times 0.5 + \frac{2 \times 20 \times 9.8}{10^3 \times 0.5}}$$

$$= 3.25 \text{ m/s}$$

∴ Speed with which it hits the surface is

$$v' = \sqrt{v^2 + 2gh'}$$

$$= \sqrt{(3.25)^2 + (2 \times 9.8 \times 0.5)}$$

$$= 4.51 \text{ m/s}$$ **Ans.**

9. $a\sqrt{2gy} = \pi x^2 \left(-\frac{dy}{dt}\right)$ ...(i)

Here, $\quad -\frac{dy}{dt} = \frac{4 \times 10^{-2}}{3600} = 1.11 \times 10^{-5}$ m/s

$$a = \pi r^2 = \pi\,(2 \times 10^{-3})^2$$

$$= 1.26 \times 10^{-5} \text{ m}^2$$

Substituting these values in Eq. (i), we have

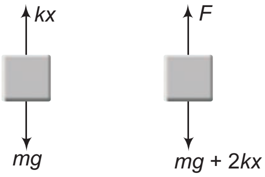
$$(1.26 \times 10^{-5})\sqrt{2 \times 9.8 \times y} = \pi(1.11 \times 10^{-5})x^2$$

or $\quad y = 0.4x^4$

This is the desired x-y relation.

10. Initially, $kx = mg$ ...(i)

In the second case,



$$F = mg + 2kx$$

or $\quad kx = \frac{F - mg}{2}$ ...(ii)

From Eqs. (i) and (ii), we have

$$mg = \frac{F - mg}{2}$$

or $\quad F = 3\,mg$

Let V be the total volume then,

$$V\rho_w g = 3mg$$

$$\therefore \quad V = \frac{3mg}{\rho_w g} = \frac{(3)(8)}{10^3} \text{ m}^3$$

$$= 0.024 \text{ m}^3$$

$$\text{Volume of wood} = \frac{\text{mass}}{\text{density}}$$

$$= \frac{8}{840} = 0.0095$$

∴ Volume of cavity = 0.024 – 0.0095

= 0.0145

∴ Percentage volume of cavity

$$= \frac{0.0145}{0.024} \times 100$$

$$= 60.41\%$$

11. $v = \sqrt{2g(10 - h)}$ ...(i)

Component of its velocity parallel to the plane is $v \cos 30°$.

Let the stream strikes the plane after time t. Then

$$0 = v\cos 30° - g\sin 30°\, t$$

$$\therefore \quad t = \frac{v\cot 30°}{g}$$

Further $$x = vt = \frac{v^2 \cot 30°}{g}$$

$$= \sqrt{3}\, y$$

or $$\frac{v^2 \cot 30°}{g} = \sqrt{3}\left(h - \frac{1}{2} g t^2\right)$$

∴ $$\frac{\sqrt{3} v^2}{g} = \sqrt{3}\left(h - \frac{g}{2}\, \frac{v^2 \cot^2 30°}{g^2}\right)$$

or $$\frac{v^2}{g} = h - \frac{3}{2}\, \frac{v^2}{g}$$

∴ $$\frac{5}{2}\, \frac{v^2}{g} = h \quad \text{or} \quad 5(10 - h) = h$$

∴ $h = 8.33$ m **Ans.**

12. In elastic collision with the surface, direction of velocity is reversed but its magnitude remains the same.

Therefore, time of fall = time of rise.

or time of fall $= \dfrac{t_1}{2}$

Hence, velocity of the ball just before it collides with liquid is

$$v = g\frac{t_1}{2} \qquad \text{...(i)}$$

Retardation inside the liquid

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}} = \frac{Vd_L g - Vdg}{Vd}$$

$$= \left(\frac{d_L - d}{d}\right) g \qquad \text{...(ii)}$$

Time taken to come to rest under this retardation will be

$$t = \frac{v}{a} = \frac{gt_1}{2a} = \frac{gt_1}{2\left(\frac{d_L - d}{d}\right) g}$$

$$= \frac{dt_1}{2(d_L - d)}$$

Same will be the time to come back on the liquid surface.

Therefore,

(a) t_2 = time the ball takes to came back to the position from where it was released

$$= t_1 + 2t = t_1 + \frac{dt_1}{d_L - d}$$

$$= t_1\left[1 + \frac{d}{d_L - d}\right] \quad \text{or} \quad t_2 = \frac{t_1 d_L}{d_L - d}$$

(b) The motion of the ball is periodic but not simple harmonic because the acceleration of the ball is g in air and $\left(\dfrac{d_L - d}{d}\right) g$ inside the liquid which is not proportional to the displacement, which is necessary and sufficient condition for SHM.

(c) When $d_L = d$, retardation or acceleration inside the liquid becomes zero (upthrust = weight). Therefore, the ball will continue to move with constant velocity $v = \dfrac{gt_1}{2}$ inside the liquid.

13. Pressure inside the bubble $= p_0 + \rho gh + \dfrac{2T}{r}$

∴ Amount of pressure inside the bubble greater than the atmospheric pressure $= \rho gh + \dfrac{2T}{r}$

Substituting the values we get,

$$\Delta p = 10^3 \times 9.8 \times 0.1 + \frac{2 \times 0.075}{10^{-3}}$$

$$= 980 + 150 = 1130 \text{ N/m}^2$$

14. (a) $F_r = 6\pi\eta r v$

$$= 6\pi(0.8)(10^{-3})(10^{-2})$$

$$= 1.5 \times 10^{-4} \text{ N}$$ **Ans.**

(b) Hydrostatic force = Upthrust

$$= \left(\frac{4}{3}\pi r^3\right)\rho g = \frac{4}{3}\pi\, (10^{-3})^3 \times 1260 \times 9.8$$

$$= 5.2 \times 10^{-5} \text{ N}$$ **Ans.**

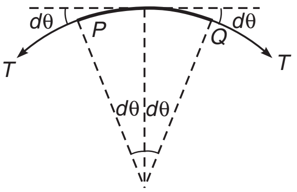
(c) At terminal velocity

w = Upthrust + viscous force

or $$(50 \times 10^{-3} \times 9.8) = (5.2 \times 10^{-5}) + 6\pi(0.8)(10^{-3})\, v_T$$

Solving we get $v_T = 32.5$ m/s **Ans.**

15. The loop will take circular shape after pricking. Radius of which is given by the relation.



$$l = 2\pi R \quad \text{or} \quad R = \frac{l}{2\pi} = \frac{6.28}{2 \times 3.14} = 1 \text{ cm}$$

$$= 10^{-2} \text{ m}$$

$2T\sin(d\theta)$ force in inward direction is balanced by surface tension force in outward direction.

$\therefore$ $2T\sin(d\theta) = (\text{Surface tension}) \times (\text{length of arc})$

For small angles, $\sin d\theta \approx d\theta$

$$\therefore \quad 2T\,d\theta = S(2Rd\theta) \quad (S = \text{Surface tension})$$

$$\therefore \quad T = SR$$

$$= (0.030)(10^{-2})$$

$$= 3.0 \times 10^{-4}\ \text{N}$$ **Ans.**

16. (a) Time taken to empty the tank (has been derived in theory) is

$$t = \frac{2A}{a\sqrt{2g}}\sqrt{H}$$

Given, $\frac{A}{a} = 400$

Substituting the values we have,

$$t = \frac{2 \times 400}{\sqrt{2 \times 9.8}}\sqrt{1}$$

$$= 180\ \text{s} = 3\ \text{min}$$ **Ans.**

(b) Rate of flow of water $Q = a\sqrt{2gH}$ = constant

Total volume of water $V = AH$

$\therefore$ Time take to empty the tank with constant rate

$$t = \frac{V}{Q} = \frac{AH}{a\sqrt{2gH}}$$

$$= \frac{400 \times 1}{\sqrt{2 \times 9.8 \times 1}}$$

$$= 90\ \text{s} = 1.5\ \text{min}$$ **Ans.**

17. (a) $\Delta p = h(\rho_w - \rho_0)g = (10)(1000 - 500)\,9.8$

$$= 49000\ \text{N/m}^2$$

Now, $\Delta p = \frac{1}{2}\rho_w v^2$

$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho_w}} = \sqrt{\frac{2 \times 49000}{1000}}$$

$$= 9.8\ \text{m/s}$$ **Ans.**

The flow will stop when,

(b) $(10 + 5)\rho_0 g = 5\rho_0 g + h\rho_w g$

$$10\rho_0 = h\rho_w$$

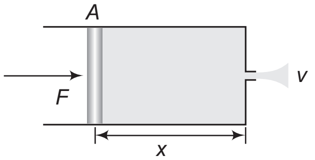
$$\therefore \quad h = \frac{10 \times 500}{1000} = 5\ \text{m}$$

i.e. flow will stop when the water-oil interface is at a height of 5.0 m. **Ans.**

18. $\frac{F}{A} = \frac{1}{2}\rho v^2$

or $F = \frac{1}{2}\rho A v^2$ ...(i)

Here, v is the velocity of liquid, with which it comes out of the hole.



Further $V = Ax$ ...(ii)

$t = \frac{V}{sv}$ ...(iii)

and $w = F \cdot x$ ...(iv)

From the above four equations,

$$w = \left(\frac{1}{2}A\rho v^2\right)\left(\frac{V}{A}\right)$$

$$= \frac{1}{2}\rho\frac{V^2}{s^2t^2}\cdot V = \frac{1}{2}\frac{\rho V^3}{s^2t^2}$$ **Ans.**

19. (a) $p_0 A = kx$

$$\therefore \quad x = \frac{p_0 A}{k} = \frac{(p_0)(\pi r^2)}{k}$$

$$= \frac{(1.01 \times 10^5)(\pi)(0.025)^2}{3600}$$

$$= 0.055\ \text{m} = 5.5\ \text{cm}$$ **Ans.**

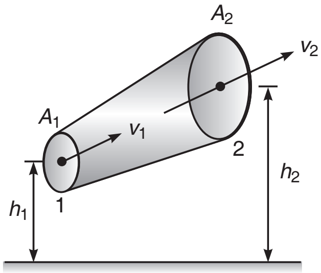
(b) Work done by atmospheric pressure

$$W = \frac{1}{2}kx^2 = \frac{1}{2}(3600)(0.055)^2$$

$$= 5.445\ \text{J}$$ **Ans.**

20. Given, $A_1 = 4 \times 10^{-3}\ \text{m}^2, A_2 = 8 \times 10^{-3}\ \text{m}^2$,

$h_1 = 2\ \text{m}, h_2 = 5\ \text{m}$, $v_1 = 1\ \text{m/s}$ and $\rho = 10^3\ \text{kg/m}^3$



From continuity equation, we have

$$A_1 v_1 = A_2 v_2$$

or $v_2 = \left(\frac{A_1}{A_2}\right) v_1$ or $v_2 = \left(\frac{4 \times 10^{-3}}{8 \times 10^{-3}}\right)(1 \text{ m/s})$

$$v_2 = \frac{1}{2} \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

or $p_1 - p_2 = \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2)$ ...(i)

(i) Work done per unit volume by the pressure as the fluid flows from *P* to *Q*.

$$w_1 = p_1 - p_2$$

$$= \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) \quad \text{[From Eq. (i)]}$$

$$= \left\{(10^3)(9.8)(5 - 2) + \frac{1}{2}(10)^3\left(\frac{1}{4} - 1\right)\right\} \text{ J/m}^3$$

$$= [29400 - 375] \text{ J/m}^3 = 29025 \text{ J/m}^3$$

(ii) Work done per unit volume by the gravity as the fluid flows from *P* to *Q*.

$$W_2 = \rho g(h_2 - h_1) = \{(10^3)(9.8)(5 - 2)\} \text{ J/m}^3$$

or $W_2 = 29400 \text{ J/m}^3$

21. Volume of the portion of the plate immersed in water is

$$10 \times \frac{1}{2}(1.54) \times 0.2 = 1.54 \text{ cm}^3$$

Therefore, if the density of water is taken as 1, then upthrust

= weight of the water displaced

$= 1.54 \times 1 \times 980 = 1509.2$ dynes

Now, the total length of the plate in contact with the water surface is $2(10 + 0.2) = 20.4$ cm

∴ Downward pull upon the plate due to surface tension

$= 20.4 \times 73 = 1489.2$ dynes

∴ Resultant upthrust

$= 1509.2 - 1489.2$

$= 20.0 \text{ dynes} = \frac{20}{980}$

$= 0.0204$ gm-wt

∴ Apparent weight of the plate in water

= weight of the plate in air − resultant upthrust

$= 8.2 - 0.0204 = 8.1796$ gm **Ans.**

22. Given that $r_1 = \frac{3.0}{2} = 1.5 \text{ mm} = 1.5 \times 10^{-3}$ m,

$r_2 = \frac{6.0}{2} = 3.0 \text{ mm} = 3.0 \times 10^{-3}$ m,

$T = 7.3 \times 10^{-2} \text{ Nm}^{-1}$, $\theta = 0°$ $\rho = 1.0 \times 10^3 \text{ kg/m}^3$,

$g = 9.8 \text{ m/s}^2$

When angle of contact is zero degree, the radius of the meniscus equals radius of bore.

Excess pressure in the first bore, $p_1 = \frac{2T}{r_1}$

$$= \frac{2 \times 7.3 \times 10^{-2}}{1.5 \times 10^{-3}} = 97.3 \text{ Pa}$$

Excess pressure in the second bore, $p_2 = \frac{2T}{r_2}$

$$= \frac{2 \times 7.3 \times 10^{-2}}{3 \times 10^{-3}} = 48.7 \text{ Pa}$$

Hence, pressure difference in the two limbs of the tube

$$\Delta p = p_1 - p_2 = h\rho g$$

or $h = \frac{p_1 - p_2}{\rho g}$

$$= \frac{97.3 - 48.7}{1.0 \times 10^3 \times 9.8}$$

$$\approx 5 \times 10^{-3} \text{ m}$$

$$= 5.0 \text{ mm}$$

23. $p_1 V_1 = p_2 V_2$

$$\therefore \left(p_0 + \frac{4T}{r}\right)\left(2 \times \frac{4}{3}\pi r^3\right) = \left(p_0 + \frac{4T}{R}\right)\left(\frac{4}{3}\pi R^3\right)$$

From here we can find expression of *T*.

24. $p_1 V_1 = p_2 V_2$

$$\therefore \left(p_0 + \frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(p_0 + \frac{4T}{R} - \frac{\sigma^2}{2\varepsilon_0}\right)\left(\frac{4}{3}\pi R^3\right)$$

Here, $V = \frac{\sigma R}{\varepsilon_0}$

or $\sigma = \frac{V\varepsilon_0}{R}$

25. $p_1 V_1 = p_2 V_2$

$$\left(p_0 + \frac{4T}{R}\right)\left(\frac{4}{3}\pi R^3\right) = \left(p_0 + \frac{4T}{2R} - \frac{\sigma^2}{2\varepsilon_0}\right)\left(\frac{4}{3}\pi\right)(2R)^3$$

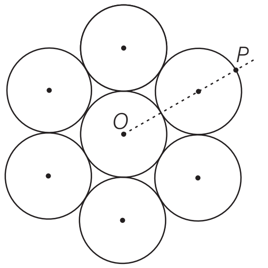
Here, $\sigma = \frac{q}{A} = \frac{q}{4\pi\,(2R)^2}$

JEE Main and Advanced

Previous Years' Questions (2018-13)

JEE Main

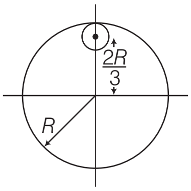
1. Seven identical circular planar discs, each of mass M and radius R are welded symmetrically as shown in the figure. The moment of inertia of the arrangement about the axis normal to the plane and passing through the point P is **(2018)**



(a) $\frac{19}{2}MR^2$ (b) $\frac{55}{2}MR^2$

(c) $\frac{73}{2}MR^2$ (d) $\frac{181}{2}MR^2$

2. From a uniform circular disc of radius R and mass $9M$, a small disc of radius $\frac{R}{3}$ is removed as shown in the figure. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through centre of disc is **(2018)**



(a) $4MR^2$ (b) $\frac{40}{9}MR^2$ (c) $10MR^2$ (d) $\frac{37}{9}MR^2$

3. A solid sphere of radius r made of a soft material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area a floats on the surface of the liquid, covering entire cross-section of cylindrical container. When a mass m is placed on the surface of the piston to compress the liquid, the fractional decrement in the radius of the sphere, $\left(\frac{dr}{r}\right)$ is **(2018)**

(a) $\frac{Ka}{mg}$ (b) $\frac{Ka}{3mg}$ (c) $\frac{mg}{3Ka}$ (d) $\frac{mg}{Ka}$

4. The mass of a hydrogen molecule is 3.32×10^{-27} kg. If 10^{23} hydrogen molecules strike per second, a fixed wall of area 2 cm^2 at an angle of 45° to the normal and rebound elastically with a speed of 10^3 m/s, then the pressure on the wall is nearly **(2018)**

(a) $2.35 \times 10^3\text{ N/m}^2$ (b) $4.70 \times 10^3\text{ N/m}^2$

(c) $2.35 \times 10^2\text{ N/m}^2$ (d) $4.70 \times 10^2\text{ N/m}^2$

5. The moment of inertia of a uniform cylinder of length l and radius R about its perpendicular bisector is I. What is the ratio l/R such that the moment of inertia is minimum? **(2017)**

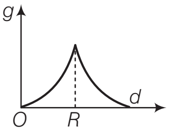
(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\frac{3}{\sqrt{2}}$ (d) $\sqrt{\frac{3}{2}}$

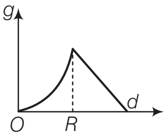
6. A slender uniform rod of mass M and length l is pivoted at one end so that it can rotate in a vertical plane (see the figure). There is negligible friction at the pivot. The free end is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical, is **(2017)**

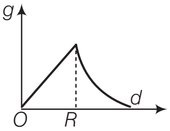
(a) $\frac{2g}{3l}\sin\theta$ (b) $\frac{3g}{2l}\cos\theta$

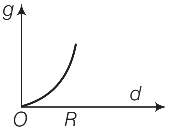
(c) $\frac{2g}{3l}\cos\theta$ (d) $\frac{3g}{2l}\sin\theta$

7. The variation of acceleration due to gravity g with distance d from centre of the Earth is best represented by (R = Earth's radius) **(2017)**

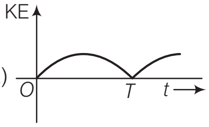
(a)


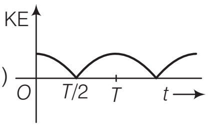
(b)


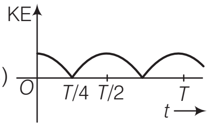
(c)


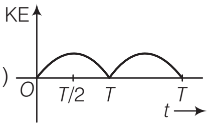
(d)


8. A particle is executing simple harmonic motion with a time period T. At time $t = 0$, it is at its position of equilibrium. The kinetic energy-time graph of the particle will look, like **(2017)**

(a)


(b)


(c)


(d)


9. A magnetic needle of magnetic moment 6.7×10^{-2} Am^2 and moment of inertia 7.5×10^{-6} kg m^2 is performing simple harmonic oscillations in a magnetic field of 0.01 T. Time taken for 10 complete oscillations is **(2017)**

(a) 8.89 s (b) 6.98 s (c) 8.76 s (d) 6.65s

10. The following observations were taken for determining surface tension T of water by capillary method. Diameter of capillary, $d = 1.25 \times 10^{-2}$ m rise of water, $h = 1.45 \times 10^{-2}$m. Using $g = 9.80$ m/s^2 and the simplified relation $T = \frac{rhg}{2} \times 10^3$ N/m, the possible error in surface tension is closest to **(2017)**

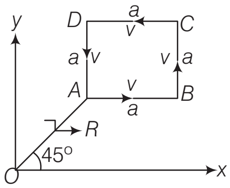
(a) 1.5% (b) 2.4% (c) 10% (d) 0.15%

11. A man grows into a giant such that his linear dimensions increase by a factor of 9. Assuming that his density remains same, the stress in the leg will change by a factor of **(2017)**

(a) $\frac{1}{9}$ (b) 81

(c) $\frac{1}{81}$ (d) 9

12. A particle of mass m is moving along the side of a square of side a, with a uniform speed v in the xy-plane as shown in the figure.



Which of the following statements is **false** for the angular momentum **L** about the origin? **(2016)**

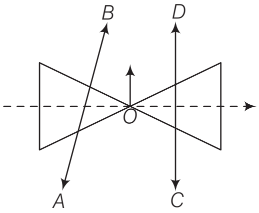
(a) $\mathbf{L} = \frac{-mv}{\sqrt{2}} R\hat{\mathbf{k}}$ when the particle is moving from A to B

(b) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} - a\right)\hat{\mathbf{k}}$ when the particle is moving from C to D

(c) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} + a\right)\hat{\mathbf{k}}$ when the particle is moving from B to C

(d) $\mathbf{L} = \frac{mv}{\sqrt{2}} R\hat{\mathbf{k}}$ when the particle is moving from D to A

13. A roller is made by joining together two corners at their vertices O. It is kept on two rails AB and CD which are placed asymmetrically (see the figure), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see the figure). It is given a light push, so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to **(2016)**



(a) turn left

(b) turn right

(c) go straight

(d) turn left and right alternately

14. A satellite is revolving in a circular orbit at a height h from the Earth's surface (radius of earth R, $h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the Earth's gravitational field, is close to (Neglect the effect of atmosphere) **(2016)**

(a) $\sqrt{2gR}$ (b) $\sqrt{gR}$

(c) $\sqrt{gR/2}$ (d) $\sqrt{gR}\,(\sqrt{2}-1)$

15. A student measures the time period of 100 oscillations of a simple pendulum four times. The data set is 90 s, 91s, 92s and 95s. If the minimum division in the measuring clock is 1s, then the reported mean time should be **(2016)**

(a) (92 ± 2 s) (b) (92 ± 5 s)

(c) (92 ± 1.8s) (d) (92 ± 3 s)

16. A particle performs simple harmonic motion with amplitude A. Its speed is trebled at the instant that it is at a distance $\frac{2}{3}A$ from equilibrium position. The new amplitude of the motion is **(2016)**

(a) $\frac{A}{3}\sqrt{41}$ (b) $3A$ (c) $A\sqrt{3}$ (d) $\frac{7}{3}A$

17. Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h, then z_0 is equal to **(2015)**

(a) $\frac{3h}{4}$ (b) $\frac{h^2}{4R}$ (c) $\frac{5h}{8}$ (d) $\frac{3h^2}{8R}$

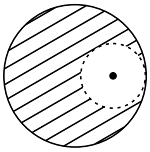
18. A particle of mass m moving in the x-direction with speed $2v$ is hit by another particle of mass $2m$ moving in the y-direction with speed v. If the collision is perfectly inelastic, the percentage loss in the energy during the collision is close to **(2015)**

(a) 50 % (b) 56 % (c) 62 % (d) 44 %

19. From a solid sphere of mass M and radius R, a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its centre and perpendicular to one of its faces is **(2015)**

(a) $\frac{MR^2}{32\sqrt{2}\pi}$ (b) $\frac{4MR^2}{9\sqrt{3}\pi}$ (c) $\frac{MR^2}{16\sqrt{2}\pi}$ (d) $\frac{4MR^2}{3\sqrt{3}\pi}$

20. From a solid sphere of mass M and radius R, a spherical portion of radius $\left(\frac{R}{2}\right)$ is removed as shown in the figure. Taking gravitational potential $V = 0$ at $r = \infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant) **(2015)**

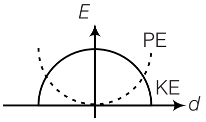


(a) $\frac{-GM}{R}$ (b) $\frac{-GM}{2R}$

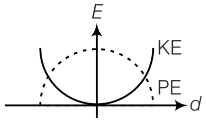
(c) $\frac{-2\,GM}{3R}$ (d) $\frac{-2\,GM}{R}$

21. For a simple pendulum, a graph is plotted between its Kinetic Energy (KE) and Potential Energy (PE) against its displacement d. Which one of the following represents these correctly? (graphs are schematic and not drawn to scale) **(2015)**

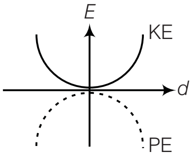
(a)



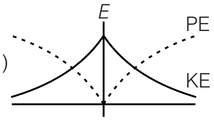
(b)



(c)



(d)



22. A pendulum made of a uniform wire of cross-sectional area A has time period T. When an additional mass M is added to its bob, the time period changes to T_M. If the Young's modulus of the material of the wire is Y, then $1/Y$ is equal to (g = gravitational acceleration) **(2015)**

(a) $\left[1-\left(\frac{T}{T_M}\right)^2\right]\frac{A}{Mg}$

(b) $\left[\left(\frac{T_M}{T}\right)^2-1\right]\frac{Mg}{A}$

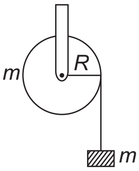
(c) $\left[1-\left(\frac{T_M}{T}\right)^2\right]\frac{A}{Mg}$

(d) $\left[\left(\frac{T_M}{T}\right)^2-1\right]\frac{A}{Mg}$

23. A bob of mass m attached to an inextensible string of length l is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical support. About the point of suspension **(2014)**

(a) angular momentum is conserved
(b) angular momentum changes in magnitude but not in direction
(c) angular momentum changes in direction but not in magnitude
(d) angular momentum changes both in direction and magnitude

24. A mass m supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release? **(2014)**



(a) $2g/3$ (b) $g/2$
(c) $5g/6$ (d) g

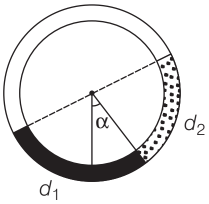
25. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction, the speed of each particle is **(2014)**

(a) $\sqrt{\frac{GM}{R}}$ (b) $\sqrt{2\sqrt{2}\frac{GM}{R}}$
(c) $\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$ (d) $\frac{1}{2}\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$

26. A particle moves with simple harmonic motion in a straight line. In first τ sec, after starting from rest it travels a distance a and in next τ sec, it travels $2a$, in same direction, then **(2014)**

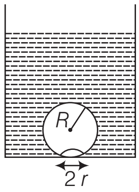
(a) amplitude of motion is $3a$
(b) time period of oscillations is 8π
(c) amplitude of motion is $4a$
(d) time period of oscillations is 6π

27. There is a circular tube in a vertical plane. Two liquids which do not mix and of densities d_1 and d_2 are filled in the tube. Each liquid subtends 90° angle at centre. Radius joining their interface makes an angle α with vertical. Ratio d_1 / d_2 is **(2014)**



(a) $\frac{1+\sin\alpha}{1-\sin\alpha}$ (b) $\frac{1+\cos\alpha}{1-\cos\alpha}$
(c) $\frac{1+\tan\alpha}{1-\tan\alpha}$ (d) $\frac{1+\sin\alpha}{1-\cos\alpha}$

28. On heating water, bubbles beings formed at the bottom of the vessel detach and rise. Take the bubbles to be spheres of radius R and making a circular contact of radius r with the bottom of the vessel. If $r << R$ and the surface tension of water is T, value of r just before bubbles detach is (density of water is ρ) **(2014)**



(a) $R^2\sqrt{\frac{2\rho_w g}{3T}}$ (b) $R^2\sqrt{\frac{\rho_w g}{6T}}$
(c) $R^2\sqrt{\frac{\rho_w g}{T}}$ (d) $R^2\sqrt{\frac{3\rho_w g}{T}}$

29. The pressure that has to be applied to the ends of a steel wire of length 10 cm to keep its length constant when its temperature is raised by 100°C is (For steel, Young's modulus is $2\times10^{11}\,\text{Nm}^{-2}$ and coefficient of thermal expansion is $1.1\times10^{-5}\,\text{K}^{-1}$) **(2014)**

(a) 2.2×10^{8} Pa
(b) 2.2×10^{9} Pa
(c) 2.2×10^{7} Pa
(d) 2.2×10^{6} Pa

30. An open glass tube is immersed in mercury in such a way that a length of 8 cm extends above the mercury level. The open end of the tube is then closed and sealed and the tube is raised vertically up by additional 46 cm. What will be length of the air column above mercury in the tube now? (Atmospheric pressure = 76 cm of Hg) **(2014)**

(a) 16 cm (b) 22 cm
(c) 38 cm (d) 6 cm

31. This question has statement I and statement II. Of the four choices given after the statements, choose the one that best describes the two statements. **(2013)**

Statement I A point particle of mass m moving with speed v collides with stationary point particle of mass M. If the maximum energy loss possible is given as $f\left(\frac{1}{2}mv^2\right)$, then $f = \left(\frac{m}{M+m}\right)$.

Statement II Maximum energy loss occurs when the particles get stuck together as a result of the collision.

(a) Statement I is true, Statement II is true, and Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is true, but Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

32. A hoop of radius r and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of the hoop when it ceases to slip? **(2013)**

(a) $r\omega_0/4$ (b) $r\omega_0/3$
(c) $r\omega_0/2$ (d) $r\omega_0$

33. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of $2R$? **(2013)**

(a) $\frac{5GmM}{6R}$ (b) $\frac{2GmM}{3R}$ (c) $\frac{GmM}{2R}$ (d) $\frac{GmM}{3R}$

34. Assume that a drop of liquid evaporates by decrease in its surface energy, so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, denstiy of liquid is ρ and L is its latent heat of vaporisation **(2013)**

(a) $\frac{\rho L}{T}$ (b) $\sqrt{\frac{T}{\rho L}}$ (c) $\frac{T}{\rho L}$ (d) $\frac{2T}{\rho L}$

35. A uniform cylinder of length L and mass M having cross-sectional area A is suspended, with its length vertical from a fixed point by a massless spring such that it is half submerged in a liquid of density σ at equilibrium position. The extension x_0 of the spring when it is in equilibrium is **(2013)**

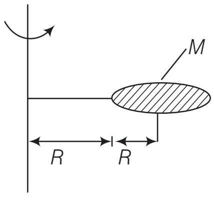
(a) $\frac{Mg}{k}$ (b) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{M}\right)$
(c) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{2M}\right)$ (d) $\frac{Mg}{k}\left(1+\frac{LA\sigma}{M}\right)$

Answer *with* Explanations

1. (d) **Key Idea** *First we found moment of inertia (MI) of system using parallel axis theorem about centre of mass, then we use it to find moment of inertia about given axis*.

Moment of inertia of an outer disc about the axis through centre is

$$= \frac{MR^2}{2} + M(2R)^2 = MR^2\left(4 + \frac{1}{2}\right) = \frac{9}{2}MR^2$$



For 6 such discs,

$$\text{moment of inertia} = 6 \times \frac{9}{2}MR^2 = 27MR^2$$

So, moment of inertia of system

$$= \frac{MR^2}{2} + 27MR^2 = \frac{55}{2}MR^2$$

Hence, $I_P = \frac{55}{2}MR^2 + (7M \times 9R^2)$

$$\Rightarrow \quad I_P = \frac{181}{2}MR^2 \quad \text{and} \quad I_{\text{system}} = \frac{181}{2}MR^2$$

2. (a) Moment of inertia of remaining solid
= Moment of intertia of complete solid
– Moment of inertia of removed portion

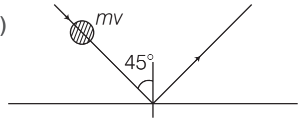
$$\therefore \quad I = \frac{9MR^2}{2} - \left[\frac{M(R/3)^2}{2} + M\left(\frac{2R}{3}\right)^2\right] \Rightarrow I = 4MR^2$$

3. **(c)** $\therefore$ Bulk modulus, $K = \dfrac{\text{Volumetric stress}}{\text{Volumetric strain}} = \dfrac{\Delta p}{\dfrac{\Delta V}{V}}$

$$\Rightarrow \quad K = \frac{mg}{a\left(\dfrac{3\Delta r}{r}\right)} \qquad \left[\because V = \frac{4}{3}\pi r^3, \text{ so } \frac{\Delta V}{V} = \frac{3\Delta r}{r}\right]$$

$$\Rightarrow \quad \frac{\Delta r}{r} = \frac{mg}{3Ka}$$

4. **(a)**



Momentum imparted due to first collision

$$= 2mv\sin 45°$$

$$= \sqrt{2}mv \qquad \left[\because \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$\therefore \text{Pressure on surface} = \frac{n\sqrt{2}mv}{\text{Area}}$$

$$= \frac{10^{23} \times \sqrt{2} \times 3.32 \times 10^{-27} \times 10^{3}}{(2 \times 10^{-2})^2}$$

$$p = 2.35 \times 10^3 \text{ N/m}^2$$

5. **(d)** MI of a solid cylinder about its perpendicular bisector of length is

$$I = M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$$

$$\Rightarrow \quad I = \frac{mR^2}{4} + \frac{ml^2}{12} = \frac{m^2}{4\pi\rho l} + \frac{ml^2}{12} \qquad [\because \rho\pi r^2 l = m]$$

For I to be maximum,

$$\frac{dI}{dl} = -\frac{m^2}{4\pi\rho}\left(\frac{1}{l^2}\right) + \frac{ml}{6} = 0 \Rightarrow \frac{m^2}{4\mu\pi\rho} = \frac{ml^3}{6}$$

$$\Rightarrow \quad l^3 = \frac{3m}{2\,\pi\rho} \Rightarrow l = \left(\frac{3}{2}\right)^{1/3}\left(\frac{m}{\pi\rho}\right)^{1/3}$$

$$\rho = \frac{m}{\pi R^2 l} \Rightarrow R^2 = \frac{m}{\pi\rho l}$$

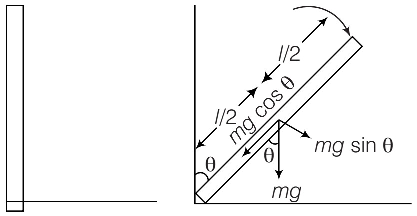
$$\Rightarrow \quad R^2 = \frac{m}{\pi\rho}\left(\frac{2}{3}\right)^{1/3}\left(\frac{\pi\rho}{m}\right)^{1/3} = \left(\frac{m}{\pi\rho}\right)^{2/3}\left(\frac{2}{3}\right)^{1/3}$$

$$\Rightarrow \quad R = \left(\frac{m}{\pi\rho}\right)^{1/3}\left(\frac{2}{3}\right)^{1/6}$$

$$\frac{l}{R} = \frac{\left(\dfrac{3}{2}\right)^{1/3}\left(\dfrac{m}{\pi\rho}\right)^{1/3}}{\left(\dfrac{m}{\pi\rho}\right)^{1/3}\left(\dfrac{2}{3}\right)^{1/6}}$$

$$= \left(\frac{3}{2}\right)^{1/3} + \left(\frac{3}{2}\right)^{1/6} \Rightarrow \frac{l}{R} = \sqrt{\frac{3}{2}}$$

6. **(d)** As the rod rotates in vertical plane so a torque is acting on it, which is due to the vertical component of weight of rod.



Initial condition At any time t

Now, Torque τ = force × perpendicular distance of line of action of force from axis of rotation

$$= mg\sin\theta \times \frac{l}{2}$$

Again, Torque, $\tau = I\alpha$

where, I = moment of inertia $= \dfrac{ml^2}{3}$

[Force and Torque frequency along axis of rotation passing through in end]

α = angular acceleration

$$\therefore \quad mg\sin\theta \times \frac{l}{2} = \frac{ml^2}{3}\alpha \Rightarrow \alpha = \frac{3g\sin\theta}{2l}$$

7. **(c)** Inside the earth surface

$$g = \frac{GM}{R^3}r \quad \text{i.e.} \quad g \propto r$$

Out the earth surface $g = \dfrac{Gm}{r^2}$ i.e. $g \propto \dfrac{1}{r^2}$

So, till earth surface g increases linearly with distance r, shown only in graph (c).

8. **(c)** KE is maximum at mean position and minimum at extreme position $\left(\text{at } t = \dfrac{T}{4}\right)$.

9. **(d)** Time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{MB}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{7.5 \times 10^{-6}}{6.7 \times 10^{-2} \times 0.01}} = 0.665 \text{ s}$$

Hence, time for 10 oscillations is $t = 6.65$ s.

10. **(a)** By ascent formula, we have surface tension,

$$T = \frac{rhg}{2} \times 10^3 \frac{\text{N}}{\text{m}} = \frac{dhg}{4} \times 10^3 \frac{\text{N}}{\text{m}}$$

$$\Rightarrow \quad \frac{\Delta T}{T} = \frac{\Delta d}{d} + \frac{\Delta h}{h} \qquad \text{[given, } g \text{ is constant]}$$

$$\text{So, percentage} = \frac{\Delta T}{T} \times 100 = \left(\frac{\Delta d}{d} + \frac{\Delta h}{h}\right) \times 100$$

$$= \left(\frac{0.01\times 10^{-2}}{1.25\times 10^{-2}} + \frac{0.01\times 10^{-2}}{1.45\times 10^{-2}}\right)\times 100 = 1.5\%$$

$\therefore \quad \frac{\Delta T}{T}\times 100 = 1.5\%$

11. **(d)** Stress $= \frac{\text{Weight}}{\text{Area}} = \frac{9^3 \times W_0}{9^2 \times A_0} = 9\left(\frac{W_0}{A_0}\right)$

Hence, the stress increases by a factor of 9.

12. **(b, d)** We can apply $\mathbf{L} = m(\mathbf{r}\times\mathbf{v})$ for different parts.

For example :

In part (a), coordinates of A are $\left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right)$

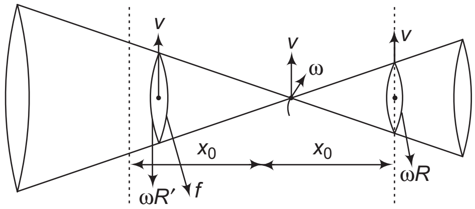
Therefore, $\quad \mathbf{r} = \frac{R}{\sqrt{2}}\hat{\mathbf{i}} + \frac{R}{\sqrt{2}}\hat{\mathbf{j}}$ and $\mathbf{v} = v\hat{\mathbf{i}}$

So, substituting in $\quad \mathbf{L} = m(\mathbf{r}\times\mathbf{v})$ we get,

$$\mathbf{L} = -\frac{mvR}{\sqrt{2}}\hat{\mathbf{k}}$$

Hence, option (a) is correct. Similarly, we can check other options also.

13. **(a)**



At distance x_0 from O, $v = \omega R$

Distance less than x_0, $v > \omega R$

Initially, there is pure rolling at both the contacts. As the cone moves forward, slipping at AB will start in forward direction, as radius at left contact decreases.

Thus, the cone will start turning towards left. As it moves, further slipping at CD will start in backward direction which will also turn the cone towards left.

14. **(d)** $v_{\text{orbital}} = \sqrt{\frac{GM}{R}} = \sqrt{gR} \quad\Rightarrow\quad v_{\text{escape}} = \sqrt{2gR}$

$\therefore$ Extra velocity required

$$= v_{\text{escape}} - v_{\text{orbital}} = \sqrt{gR}\,(\sqrt{2}-1)$$

15. **(a)** True value $= \frac{90+91+95+92}{4} = 92$

Mean absolute error

$$= \frac{|92-90|+|92-91|+|92-95|+|92-92|}{4}$$

$$= \frac{2+1+3+0}{4} = 1.5$$

Value = (92 ± 1.5)

Since, least count is 1 sec

$\therefore \qquad$ Value = (92 ± 2 s)

16. **(d)** $v = \omega\sqrt{A^2 - x^2}$ At, $x = \frac{2A}{3}$

$$v = \omega\sqrt{A^2 - \left(\frac{2A}{3}\right)^2} = \frac{\sqrt{5}}{3}\omega A$$

As, velocity is trebled, hence $v' = \sqrt{5}A\omega$

This leads to new amplitude A'

$$\therefore \qquad \omega\sqrt{A'^2 - \left(\frac{2A}{3}\right)^2} = \sqrt{5}A\omega$$

$$\Rightarrow \qquad \omega^2\left[A'^2 - \frac{4A^2}{9}\right] = 5A^2\omega^2$$

$$\Rightarrow A'^2 = 5A^2 + \frac{4}{9}A^2 = \frac{49}{9}A^2 \;\Rightarrow\; A' = \frac{7}{3}A$$

17. **(a)** Centre of mass of uniform solid cone of height h is at a height of $\frac{h}{4}$ from base. Therefore from vertex it's $\frac{3h}{4}$.

18. **(b)** In all type of collisions, momentum of the system always remains constant. In perfectly inelastic collision, particles stick together and move with a common velocity.

Let this velocity is $\mathbf{v}_c$. Then,

initial momentum of system = final momentum of system

or $\qquad m(2v)\hat{\mathbf{i}} + 2m(v)\hat{\mathbf{j}} = (m+2m)\mathbf{v}_c$

$$\therefore \qquad \mathbf{v}_c = \frac{2}{3}(v\hat{\mathbf{i}} + v\hat{\mathbf{j}})$$

$$|\mathbf{v}_c| \text{ or } v_c \text{ or speed} = \sqrt{\left(\frac{2}{3}v\right)^2 + \left(\frac{2}{3}v\right)^2} = \frac{2\sqrt{2}}{3}v$$

Initial kinetic energy

$$K_i = \frac{1}{2}(m)(2v)^2 + \frac{1}{2}(2m)(v)^2 = 3mv^2$$

Final kinetic energy

$$K_f = \frac{1}{2}(3m)\left(\frac{2\sqrt{2}}{3}v\right)^2 = \frac{4}{3}mv^2$$

$$\text{Fractional loss} = \left(\frac{K_i - K_f}{K_i}\right)\times 100$$

$$= \left[\frac{(3mv^2) - [(4/3)mv^2]}{(3mv^2)}\right]\times 100 = 56\%$$

19. **(b)** Maximum possible volume of cube will occur when

$$\sqrt{3}a = 2R \qquad (a = \text{side of cube})$$

$$\therefore \qquad a = \frac{2}{\sqrt{3}}R$$

Now, density of sphere, $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$

Mass of cube, $m =$ (volume of cube)$(\rho) = (a^3)(\rho)$

$$= \left[\frac{2}{\sqrt{3}}R\right]^3\left[\frac{m}{(4/3)\pi R^3}\right] = \left(\frac{2}{\sqrt{3}\pi}\right)M$$

Now, moment of inertia of the cube about the said axis is

$$I = \frac{ma^2}{6} = \frac{\left(\frac{2}{\sqrt{3}\pi}\right)M\left(\frac{2}{\sqrt{3}}R\right)^2}{\sigma} = \frac{4MR^2}{9\sqrt{3}\pi}$$

20. **(a)** $V_R = V_T - V_C$

$V_R =$ Potential due to remaining portion

$V_T =$ Potential due to total sphere

$V_C =$ Potential due to cavity

Radius of cavity is $\frac{R}{2}$. Hence, volume and mass is $\frac{M}{8}$.

$$\therefore\ V_R = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(\frac{R}{2}\right)^2\right] + \frac{G(M/8)}{(R/2)}\left(\frac{3}{2}\right)$$

$$= -\frac{GM}{R}$$

21. **(a)** Taking minimum potential energy at mean position to be zero, the expression of KE and PE are

$$\text{KE} = \frac{1}{2}m\omega^2(A^2 - d^2) \text{ and PE} = \frac{1}{2}m\omega^2 d^2$$

Both graphs are parabola. At $d = 0$, the mean position,

$\text{PE} = 0$ and $\text{KE} = \frac{1}{2}m\omega^2A^2 =$ maximum

At $d = \pm A$, the extreme positions,

$\text{KE} = 0$ and $\text{PE} = \frac{1}{2}m\omega^2A^2 =$ maximum

Therefore, the correct graph is (a).

22. **(d)**

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \ldots\text{(i)}$$

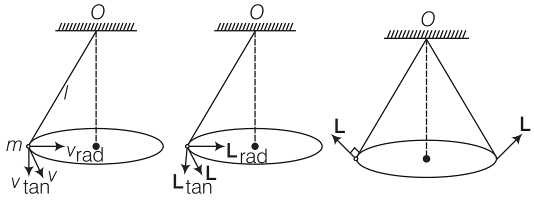
$$T_M = 2\pi\sqrt{\frac{L+\Delta L}{g}}$$

Here, $\Delta L = \frac{FL}{AY} = \frac{MgL}{AY} \Rightarrow T_M = 2\pi\sqrt{\frac{L + \frac{MgL}{AY}}{g}} \qquad \ldots\text{(ii)}$

Solving Eqs. (i) and (ii), we get

$$\frac{1}{Y} = \frac{A}{Mg}\left[\left(\frac{T_M}{T}\right)^2 - 1\right]$$

23. **(c)** Angular momentum of the pendulum about the suspension point O is

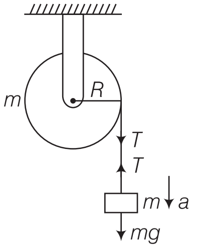


Then, v can be resolved into two components, radial component r_{rad} and tangential component $r_{\tan}$. Due to v_{rad}, L will be tangential and due to $v_{\tan}$, L will be radially outwards as shown. So, net angular momentum will be as shown in figure whose magnitude will be constant ($|L| = mvl$). But its direction will change as shown in the figure.

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$

where, $r =$ radius of circle.

24. **(b)** For the mass m, $mg - T = ma$



As we know, $a = R\alpha$

So, $mg - T = mR\alpha \qquad \ldots\text{(i)}$

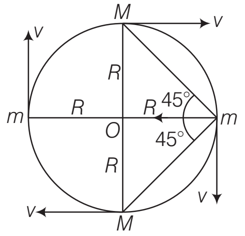
Torque about centre of pully

$$T \times R = mR^2\alpha \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get, $a = g/2$

Hence, the acceleration with the mass of a body fall is $g/2$.

25. **(d)** Net force acting on any one particle M,



$$= \frac{GM^2}{(2R)^2} + \frac{GM^2}{(R\sqrt{2})^2}\cos 45° + \frac{GM^2}{(R\sqrt{2})^2}\cos 45°$$

$$= \frac{GM^2}{R^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$$

This force will equal to centripetal force.

So, $\frac{Mv^2}{R} = \frac{GM^2}{R^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$

$$v = \sqrt{\frac{GM}{4R}(1 + 2\sqrt{2})}$$

$$= \frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}$$

Hence, speed of each particle in a circular motion is

$$\frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}.$$

26. **(d)** In SHM, a particle starts from rest, we have

i.e. $x = A\cos\omega t$, at $t = 0$, $x = A$

When $t = \tau$, then $x = A - a$ ...(i)

When $t = 2\tau$, then $x = A - 3a$ ...(ii)

On comparing Eqs. (i) and (ii), we get

$$A - a = A\cos\omega\tau \Rightarrow A - 3a = A\cos 2\omega\tau$$

As $\cos 2\omega\tau = 2\cos^2\omega\tau - 1$

$$\Rightarrow \frac{A-3a}{A} = 2\left(\frac{A-a}{A}\right)^2 - 1$$

$$\Rightarrow \frac{A-3a}{A} = \frac{2A^2 + 2a^2 - 4Aa - A^2}{A^2}$$

$$A^2 - 3aA = A^2 + 2a^2 - 4Aa$$

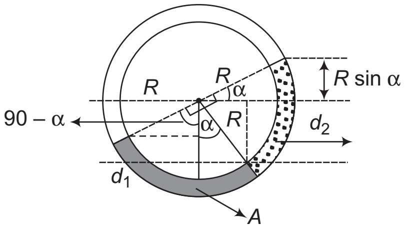
$$a^2 = 2aA \Rightarrow A = 2a$$

Now, $A - a = A\cos\omega\tau$

$$\Rightarrow \cos\omega\tau = 1/2 \Rightarrow \frac{2\pi}{T}\tau = \frac{\pi}{3} \Rightarrow T = 6\pi$$

27. **(c)** Equating pressure at A, we get

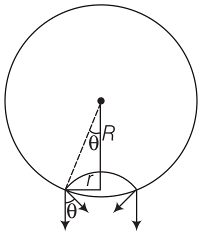
$$R\sin\alpha\, d_2 + R\cos\alpha\, d_2 + R(1-\cos\alpha)\, d_1 = R(1-\sin\alpha)\, d_1$$



$$(\sin\alpha + \cos\alpha)\, d_2 = d_1(\cos\alpha - \sin\alpha)$$

$$\Rightarrow \frac{d_1}{d_2} = \frac{1+\tan\alpha}{1-\tan\alpha}$$

28. **(a)** The bubble will detach if,



$$\int \sin\theta T \times dl = T(2\pi r)\sin\theta$$

Buoyant force ≥ Surface tension force

$$\frac{4}{3}\pi R^3 \rho_w g \geq \int T \times dl \sin\theta$$

$$(\rho_w)\left(\frac{4}{3}\pi R^3\right)g \geq (T)(2\pi r)\sin\theta \Rightarrow \sin\theta = \frac{r}{R}$$

Solving, $r = \sqrt{\dfrac{2\rho_w R^4 g}{3T}} = R^2\sqrt{\dfrac{2\rho_w g}{3T}}$

29. **(a)** If the deformation is small, then the stress in a body is directly proportional to the corresponding strain.

According to Hooke's law i.e.

$$\text{Young's modulus } (Y) = \frac{\text{Tensile stress}}{\text{Tensile strain}}$$

So, $Y = \dfrac{F/A}{\Delta L/L} = \dfrac{FL}{A\Delta L}$

If the rod is compressed, then compressive stress and strain appear. Their ratio Y is same as that for tensile case.

Given, length of a steel wire $(L) = 10$ cm

Temperature $(\theta) = 100°\text{C}$

As length is constant.

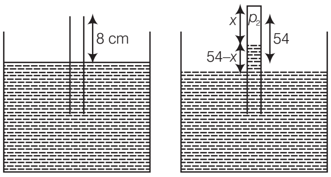
$\therefore$ $\text{Strain} = \dfrac{\Delta L}{L} = \alpha\,\Delta\theta$

Now, pressure = stress = $Y \times$ strain

[Given, $Y = 2\times10^{11}\ \text{N/m}^2$ and $\alpha = 1.1\times10^{-5}\ \text{K}^{-1}$]

$$= 2\times10^{11}\times1.1\times10^{-5}\times100 = 2.2\times10^{8}\ \text{Pa}$$

30. **(a)** **Thinking Process** In this question, the system is accelerating horizontally i.e. no component of acceleration in vertical direction. Hence, the pressure in the vertical direction will remain unaffected.



i.e. $p_1 = p_0 + \rho g h$

Again, we have to use the concept that the pressure in the same level will be same.

For air trapped in tube, $p_1V_1 = p_2V_2$

$p_1 = p_{\text{atm}} = \rho g 76 \Rightarrow V_1 = A\cdot 8$

[A = area of cross-section]

$p_2 = p_{\text{atm}} - \rho g(54 - x) = \rho g(22 + x)$

$V_2 = A\cdot x \qquad \rho g 76\times 8A = \rho g(22 + x)A\,x$

$$x^2 + 22x - 78\times 8 = 0 \Rightarrow x = 16\ \text{cm}$$

31. **(d)** Maximum energy loss

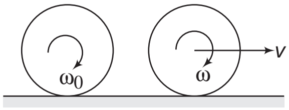
$$= \frac{p^2}{2m} - \frac{p^2}{2(m+M)} \qquad \left(\because \text{KE} = \frac{p^2}{2m}\right)$$

Before collision the mass m and after collision the mass is $m + M$

$$= \frac{p^2}{2m}\left[\frac{M}{(m+M)}\right]$$

$$= \frac{1}{2}mv^2\left\{\frac{M}{m+M}\right\} \qquad \left(f = \frac{M}{m+M}\right)$$

32. (c)



$$\omega = v/r$$

From conservation of angular momentum about bottommost point

$$mr^2\omega_0 = mvr + mr^2 \times v/r \;\Rightarrow\; v = \frac{\omega_0 r}{2}$$

33. (a) E = Energy of satellite – energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $\quad r = R + 2R = 3R$

Substituting in about equation we get, $E = \dfrac{5GMm}{6R}$

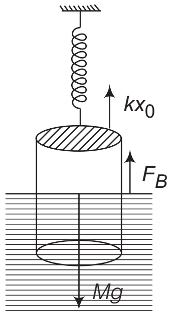
34. (d) Decrease in surface energy = heat required in vaporisation.

$\therefore \quad T(dS) = L(dm) \Rightarrow T(2)(4\pi r)dr = L(4\pi r^2 dr)\rho$

$\therefore \qquad r = \dfrac{2T}{\rho L}$

35. (c) In equilibrium, Upward force = Downward force

$$kx_0 + F_B = mg$$



Here, kx_0 is restoring force of spring and F_B is buoyancy force.

$$kx_0 + \sigma\frac{L}{2}Ag = Mg$$

$$x_0 = \frac{Mg - \dfrac{\sigma LAg}{2}}{k}$$

JEE Advanced

1. The potential energy of mass m at a distance r from a fixed point O is given by $V(r) = kr^2/2$, where k is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius R about the point O. If v is the speed of the particle and L is the magnitude of its angular momentum about O, which of the following statements is (are) true ?

(More than One Correct Option, 2018)

(a) $v = \sqrt{\dfrac{k}{2m}}R$ (b) $v = \sqrt{\dfrac{k}{m}}R$

(c) $L = \sqrt{mk}R^2$ (d) $L = \sqrt{\dfrac{mk}{2}}R^2$

2. Consider a body of mass 1.0 kg at rest at the origin at time $t = 0$. A force $\mathbf{F} = (\alpha t\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}})$ is applied on the body, where $\alpha = 1.0\,\text{Ns}^{-1}$ and $\beta = 1.0\,\text{N}$. The torque acting on the body about the origin at time $t = 1.0\,\text{s}$ is τ. Which of the following statements is (are) true ?

(More than One Correct Option, 2018)

(a) $|\tau| = \dfrac{1}{3}\,\text{N-m}$

(b) The torque τ is in the direction of the unit vector $+\hat{\mathbf{k}}$

(c) The velocity of the body at $t = 1\,\text{s}$ is $\mathbf{v} = \dfrac{1}{2}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\,\text{ms}^{-1}$

(d) The magnitude of displacement of the body at $t = 1\,\text{s}$ is $\dfrac{1}{6}\,\text{m}$

3. A uniform capillary tube of inner radius r is dipped vertically into a beaker filled with water. The water rises to a height h in the capillary tube above the water surface in the beaker. The surface tension of water is σ. The angle of contact between water and the wall of the capillary tube is θ. Ignore the mass of water in the meniscus. Which of the following statements is (are) true ?

(More than one Correct Option, 2018)

(a) For a given material of the capillary tube, h decreases with increase in r

(b) For a given material of the capillary tube, h is independent of σ

(c) If this experiment is performed in a lift going up with a constant acceleration, then h decreases

(d) h is proportional to contact angle θ

4. A ring and a disc are initially at rest, side by side, at the top of an inclined plane which makes an angle 60° with the horizontal. They start to roll without slipping at the same instant of time along the shortest path. If the time difference between their reaching the ground is $(2-\sqrt{3})/\sqrt{10}$ s, then the height of the top of the inclined plane, in metres, is .... (Take, $g = 10\text{ms}^{-2}$) **(Numerical Value, 2018)**

5. Consider a thin square plate floating on a viscous liquid in a large tank. The height h of the liquid in the tank is much less than the width of the tank. The floating plate is pulled horizontally with a constant velocity u_0. Which of the following statements is (are) true? **(More than One Correct Option, 2018)**

(a) The resistive force of liquid on the plate is inversely proportional to h

(b) The resistive force of liquid on the plate is independent of the area of the plate

(c) The tangential (shear) stress on the floor of the tank increases with u_0

(d) The tangential (shear) stress on the plate varies linearly with the viscosity η of the liquid

6. A steel wire of diameter 0.5 mm and Young's modulus 2×10^{11} N m^{-2} carries a load of mass m. The length of the wire with the load is 1.0 m. A vernier scale with 10 divisions is attached to the end of this wire. Next to the steel wire is a reference wire to which a main scale, of least count 1.0 mm, is attached. The 10 divisions of the vernier scale correspond to 9 divisions of the main scale. Initially, the zero of vernier scale coincides with the zero of main scale. If the load on the steel wire is increased by 1.2 kg, the vernier scale division which coincides with a main scale division is .... (Take, $g = 10$ ms^{-2} and $\pi = 3.2$). **(Numerical Value, 2018)**

7. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2, respectively. Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1; and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given, $m_1/m_2 = 2$ and $R_1/R_2 = 1/4$, match the ratios in List-I to the numbers in List-II. **(Matching Type, 2018)**

List-I	List-II
P. v_1/v_2	1. 1/8
Q. L_1/L_2	2. 1
R. K_1/K_2	3. 2
S. T_1/T_2	4. 8

(a) P → 4; Q → 2; R → 1; S → 3

(b) P → 3; Q → 2; R → 4; S → 1

(c) P → 2; Q → 3; R → 1; S → 4

(d) P → 2; Q → 3; R → 4; S → 1

8. In the List-I below, four different paths of a particle are given as functions of time. In these functions, α and β are positive constants of appropriate dimensions and $\alpha \neq \beta$. In each case, the force acting on the particle is either zero or conservative. In List-II, five physical quantities of the particle are mentioned: $\mathbf{p}$ is the linear momentum, $\mathbf{L}$ is the angular momentum about the origin, K is the kinetic energy, U is the potential energy and E is the total energy. Match each path in List-I with those quantities in List-II, which are conserved for that path **(Matching Type, 2018)**

	List-I	List-II
P.	$\mathbf{r}(t) = \alpha t\ \mathbf{i} + \beta t\ \mathbf{j}$	1. $\mathbf{p}$
Q.	$\mathbf{r}(t) = \alpha \cos \omega t\ \mathbf{i} + \beta \sin \omega t\ \mathbf{j}$	2. $\mathbf{L}$
R.	$\mathbf{r}(t) = \alpha(\cos \omega t\ \mathbf{i} + \sin \omega t\ \mathbf{j})$	3. K
S.	$\mathbf{r}(t) = at\ \mathbf{i} + \frac{\beta}{2}t^2\ \mathbf{j}$	4. U
		5. E

(a) P → 1, 2, 3, 4, 5; Q → 2, 5; R → 2, 3, 4, 5; S → 5

(b) P → 1, 2, 3, 4, 5; Q → 3, 5; R → 2, 3, 4, 5; S → 2, 5

(c) P → 2, 3, 4; Q → 5; R → 1, 2, 4; S → 2, 5

(d) P → 1, 2, 3, 5; Q → 2, 5; R → 2, 3, 4, 5; S → 2, 5

Passage (Q. Nos. 9-10)

One twirl is a circular ring (of mass M and radius R) near the tip of one's finger as shown in Fig. 1. In the process the finger never loses contact with the inner rim of the ring. The finger traces out the surface of a cone, shown by the dotted line. The radius of the path traced out by the point where the ring and the finger is in contact is r. The finger rotates with an angular velocity ω_0.

The rotating ring rolls without slipping on the outside of a smaller circle described by the point where the ring and the finger is in contact (Fig. 2). The coefficient of friction between the ring and the finger is μ and the acceleration due to gravity is g.

(Passage Type, 2017)

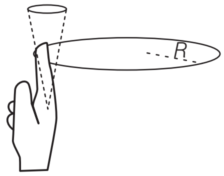


Fig. 1

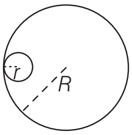


Fig. 2

9. The total kinetic energy of the ring is

(a) $M\omega_0^2(R-r)^2$ (b) $\frac{1}{2}M\omega_0^2(R-r)^2$

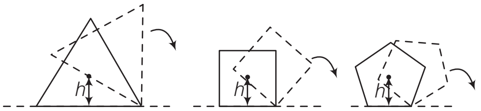
(c) $M\omega_0^2R^2$ (d) $\frac{3}{2}M\omega_0^2(R-r)^2$

10. The minimum value of ω_0 below which the ring will drop down is

(a) $\sqrt{\frac{g}{2\mu(R-r)}}$ (b) $\sqrt{\frac{3g}{2\mu(R-r)}}$

(c) $\sqrt{\frac{g}{\mu(R-r)}}$ (d) $\sqrt{\frac{2g}{\mu(R-r)}}$

11. Consider regular polygons with number of sides $n = 3, 4, 5$ …… as shown in the figure. The centre of mass of all the polygons is at height h from the ground. They roll on a horizontal surface about the leading vertex without slipping and sliding as depicted. The maximum increase in height of the locus of the centre of mass for each each polygon is Δ. Then, Δ depends on n and h as **(Single Correct Option, 2017)**

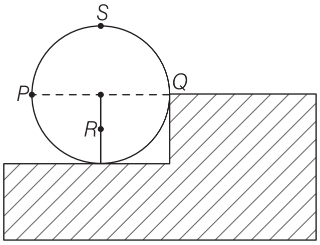


(a) $\Delta = h\sin^2\left(\frac{\pi}{n}\right)$ (b) $\Delta = h\sin\left(\frac{2\pi}{n}\right)$

(c) $\Delta = h\tan^2\left(\frac{\pi}{2n}\right)$ (d) $\Delta = h\left(\frac{1}{\cos\left(\frac{\pi}{n}\right)} - 1\right)$

12. A wheel of radius R and mass M is placed at the bottom of a fixed step of height R as shown in the figure. A constant force is continuously applied on the surface of the wheel so that it just climbs the step without slipping. Consider the torque τ about an axis normal to the plane of the paper passing through the point Q. Which of the following option(s) is/are correct?

(More than One Correct Option, 2017)



(a) If the force is applied normal to the circumference at point P, then τ is zero

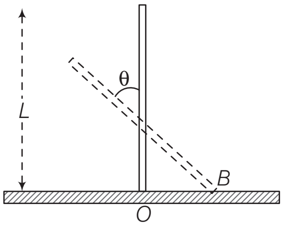
(b) If the force is applied tangentially at point S, then $\tau \neq 0$ but the wheel never climbs the step

(c) If the force is applied at point P tangentially, then τ decreases continuously as the wheel climbs

(d) If the force is applied normal to the circumference at point X, then τ is constant

13. A rigid uniform bar AB of length L is slipping from its vertical position on a frictionless floor (as shown in the figure). At some instant of time, the angle made by the bar with the vertical is θ. Which of the following statements about its motion is/are correct?

(More than One Correct Option, 2017)



(a) Instantaneous torque about the point in contact with the floor is proportional to $\sin\theta$

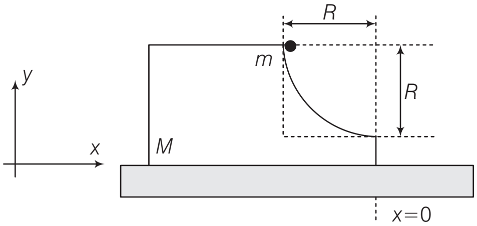
(b) The trajectory of the point A is parabola

(c) The mid-point of the bar will fall vertically downward

(d) When the bar makes an angle θ with the vertical, the displacement of its mid-point from the initial position is proportional to $(1-\cos\theta)$

14. A block of mass M has a circular cut with a frictionless surface as shown. The block rests on the horizontal frictionless surface of a fixed table. Initially, the right edge of the block is at $x = 0$, in a coordinate system fixed to the table. A point mass m is released from rest at the topmost point of the path as shown and it slides down. When the mass loses contact with the block, its position is x and the velocity is v. At that instant, which of the following option(s) is/are correct?

(More than One Correct Option, 2017)



(a) The velocity of the point mass m is $v = \sqrt{\dfrac{2gR}{1+\dfrac{m}{M}}}$

(b) The x component of displacement of the centre of mass of the block M is $-\dfrac{mR}{M+m}$

(c) The position of the point mass is $x = -\sqrt{2}\dfrac{mR}{M+m}$

(d) The velocity of the block M is $v = -\dfrac{m}{M}\sqrt{2gR}$

15. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2 \text{ km s}^{-1}$. The minimum initial velocity (v_s) required for the rocket to be able to leave the Sun-Earth system is closest to (Ignore the rotation and revolution of the Earth and the presence of any other planet)

(Single Correct Option, 2017)

(a) $v_s = 72 \text{ km s}^{-1}$

(b) $v_s = 22 \text{ km s}^{-1}$

(c) $v_s = 42 \text{ km s}^{-1}$

(d) $v_s = 62 \text{ km s}^{-1}$

16. Consider an expanding sphere of instantaneous radius R whose total mass remains constant. The expansion is such that the instantaneous density ρ remains uniform throughout the volume. The rate of fractional change in density $\left(\dfrac{1}{p}\dfrac{d\rho}{dt}\right)$ is constant. The velocity v of any point of the surface of the expanding sphere is proportional to **(Single Correct Option, 2017)**

(a) R (b) $\dfrac{1}{R}$

(c) R^3 (d) $R^{\frac{2}{3}}$

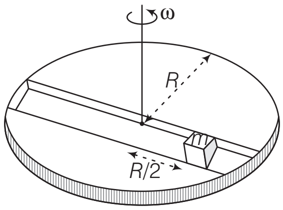
17. A drop of liquid of radius $R = 10^{-2}$ m having surface tension $S = \dfrac{0.1}{4\pi} \text{ Nm}^{-1}$ divides itself into K identical drops. In this process the total change in the surface energy $\Delta U = 10^{-3}$ J. If $K = 10^{\alpha}$, then the value of α is

(Single Integer Type, 2017)

Passage (Q. Nos. 18-19)

A frame of the reference that is accelerated with respect to an inertial frame of reference is called a non-inertial frame of reference. A coordinate system fixed on a circular disc rotating about a fixed axis with a constant angular velocity ω is an example of a non-inertial frame of reference.

The relationship between the force $\mathbf{F}_{\text{rot}}$ experienced by a particle of mass m moving on the rotating disc and the force $\mathbf{F}_{\text{in}}$ experienced by the particle in an inertial frame of reference is, $\mathbf{F}_{\text{rot}} = \mathbf{F}_{\text{in}} + 2m\,(\mathbf{v}_{\text{rot}} \times \vec{\omega}) + m\,(\vec{\omega} \times \mathbf{r}) \times \vec{\omega}$, where, $\mathbf{v}_{\text{rot}}$ is the velocity of the particle in the rotating frame of reference and $\mathbf{r}$ is the position vector of the particle with respect to the centre of the disc



Now, consider a smooth slot along a diameter of a disc of radius R rotating counter-clockwise with a constant angular speed ω about its vertical axis through its centre.

We assign a coordinate system with the origin at the centre of the disc, the X-axis along the slot, the Y-axis perpendicular to the slot and the Z-axis along th rotation axis $(\omega = \omega\mathbf{k})$. A small block of mass m is gently placed in the slot at $\mathbf{r} = (R/2)\,\hat{\mathbf{i}}$ at $t = 0$ and is constrained to move only along the slot. **(Passage Type, 2016)**

18. The distance r of the block at time t is

(a) $\frac{R}{2}\cos 2\omega t$ (b) $\frac{R}{2}\cos\omega t$

(c) $\frac{R}{4}(e^{\omega t} + e^{-\omega t})$ (d) $\frac{R}{4}(e^{2\omega t} + e^{-2\omega t})$

19. The net reaction of the disc on the block, is

(a) $m\omega^2 R\sin\omega t\,\hat{\mathbf{j}} - mg\hat{\mathbf{k}}$

(b) $\frac{1}{2}m\omega^2 R(e^{\omega t} - e^{-\omega t})\hat{\mathbf{j}} + mg\hat{\mathbf{k}}$

(c) $\frac{1}{2}m\omega^2 R(e^{2\omega t} - e^{-\omega t})\hat{\mathbf{j}} + mg\hat{\mathbf{k}}$

(d) $-m\omega^2 R\cos\omega t\,\hat{\mathbf{j}} - mg\hat{\mathbf{k}}$

20. The position vector $\mathbf{r}$ of particle of mass m is given by the following equation $\mathbf{r}(t) = \alpha t^3\hat{\mathbf{i}} + \beta t^2\hat{\mathbf{j}}$ where, $\alpha = \frac{10}{3}\,\text{ms}^{-3}$, $\beta = 5\,\text{ms}^{-2}$ and $m = 0.1$ kg. At $t = 1$ s, which of the following statement(s) is (are) true about the particle? **(More than One Correct Option, 2016)**

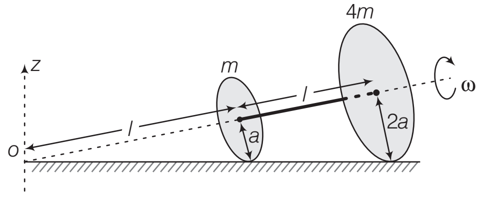
(a) The velocity $\mathbf{v}$ is given by $\mathbf{v} = (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\text{ms}^{-1}$

(b) The angular momentum $\mathbf{L}$ with respect to the origin is given by $\mathbf{L} = (5/3)\hat{\mathbf{k}}$Nm-s

(c) The force $\mathbf{F}$ is given by $\mathbf{F} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ N

(d) The torque τ with respect to the origin is given by $\tau = -\frac{20}{3}\hat{\mathbf{k}}$N-s

21. Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, rigid rod of length $l = \sqrt{24}\,a$ through their centres. This assembly is laid on a firm and flat surface and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\mathbf{L}$ (see the figure). Which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**



(a) The magnitude of the z-component of $\mathbf{L}$ is $55\,ma^2\omega$

(b) The magnitude of angular momentum of centre of mass of the assembly about the point O is $81\,ma^2\omega$

(c) The centre of mass of the assembly rotates about the Z-axis with an angular speed of $\omega/5$

(d) The magnitude of angular momentum of the assembly about its centre of mass is $17\,ma^2\omega/2$

22. A block with mass M is connected by a massless spring with stiffness constant k to a rigid wall and moves without friction on a horizontal surface. The block oscillates with small amplitude A about an equilibrium position x_0. Consider two cases : (i) when the block is at x_0 and (ii) when the block is at $x = x_0 + A$. In both the cases, a particle with mass m ($< M$) is softly placed on the block after which they stick to each other. Which of the following statement(s) is (are) true about the motion after the mass m is placed on the mass M?

(More than One Correct Option, 2016)

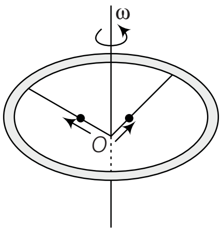
(a) The amplitude of oscillation in the first case changes by a factor of $\sqrt{\frac{M}{m+M}}$, whereas in the second case it remains unchanged

(b) The final time period of oscillation in both the cases is same

(c) The total energy decreases in both the cases

(d) The instantaneous speed at x_0 of the combined masses decreases in both the cases

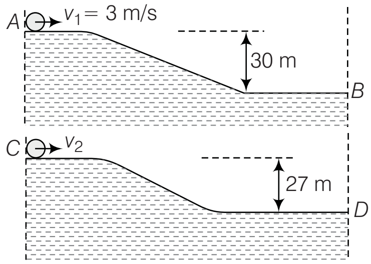
23. A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $M/8$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant, the angular speed of the system is $(8/9)\omega$ and one of the masses is at a distance of $\frac{3}{5}R$ from O. At this instant, the distance of the other mass from O is



(More than One Correct Option, 2015)

(a) $\frac{2}{3}R$ (b) $\frac{1}{3}R$ (c) $\frac{3}{5}R$ (d) $\frac{4}{5}R$

24. Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces.



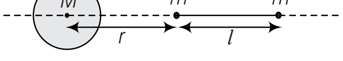
If they reach B and D with the same linear speed and $v_1 = 3$ m/s, then v_2 in m/s is ($g = 10\,\text{m/s}^2$) **(Single Integer Type, 2015)**

25. The densities of two solids spheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\frac{r}{R}\right)$ and $\rho_B(r) = k\left(\frac{r}{R}\right)^5$, respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively. If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is

(Single Integer Type, 2015)

26. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is $1/4$ th of its value at the surface of the planet. If the escape velocity from the planet is $v_{\text{sec}} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere) **(Single Integer Type, 2015)**

27. A large spherical mass M is fixed at one position and two identical masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them.



All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is **(Single Integer Type, 2015)**

28. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If $P(r)$ is the pressure at $r(r < R)$, then the correct options is/are

(More than One Correct Option, 2015)

(a) $p(r = 0) = 0$ (b) $\frac{p\left(r = \frac{3R}{4}\right)}{p\left(r = \frac{2R}{3}\right)} = \frac{63}{80}$

(c) $\frac{p\left(r = \frac{3R}{5}\right)}{p\left(r = \frac{2R}{5}\right)} = \frac{16}{21}$ (d) $\frac{p\left(r = \frac{R}{2}\right)}{p\left(r = \frac{R}{3}\right)} = \frac{20}{27}$

29. A person in a lift is holding a water jar, which has a small hole at the lower end of its side. When the lift is at rest, the water jet coming out of the hole hits the floor of the lift at a distance d of 1.2 m from the person.

In the following, state of the lift's motion is given in Column I and the distance where the water jet hits the floor of the lift is given in Column II. Match the statements from Column I with those in Column II and select the correct answer using the code given below the columns.

	Column I		Column II
P.	Lift is accelerating vertically up.	1.	$d = 1.2m$
Q.	Lift is accelerating with an acceleration loss than the gravitational acceleration.	2.	$d > 1.2m$
R.	Lift is moving vertically up with constant speed.	3.	$d < 1.2m$
S.	Lift is falling freely.	4.	No water leaks out of the jar

(More than One Correct Option, 2015)

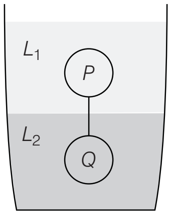
Codes

	P	Q	R	S		P	Q	R	S
(a)	2,	3,	2,	4	(b)	2,	3,	1,	4
(c)	1,	1,	1,	4	(d)	2,	3,	1,	1

30. Two spheres P and Q for equal radii have densities ρ_1 and ρ_2, respectively. The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities σ_1 and σ_2 and viscosities η_1 and η_2, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut (see figure).

If sphere P alone in L_2 has terminal velocity $\mathbf{v}_P$ and Q alone in L_1 has terminal velocity $\mathbf{v}_Q$, then

(Single Correct Option, 2015)

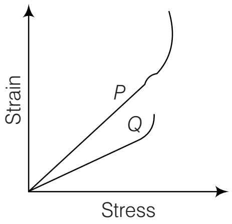


(a) $\frac{|\mathbf{v}_P|}{|\mathbf{v}_Q|} = \frac{\eta_1}{\eta_2}$ (b) $\frac{|\mathbf{v}_P|}{|\mathbf{v}_Q|} = \frac{\eta_2}{\eta_1}$

(c) $\mathbf{v}_P \cdot \mathbf{v}_Q > 0$ (d) $\mathbf{v}_P \cdot \mathbf{v}_Q < 0$

31. In plotting stress *versus* strain curves for two materials P and Q, a student by mistake puts strain on the y-axis and stress on the x-axis as shown in the figure. Then, the correct statements is/are

(More than One Correct Option, 2015)



(a) P has more tensile strength than Q

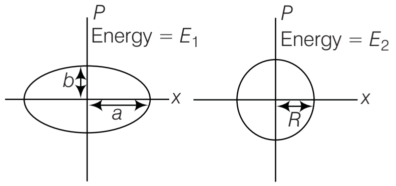
(b) P is more ductile than Q

(c) P is more brittle than Q

(d) The Young's modulus of P is more than that of Q

32. Two independent harmonic oscillators of equal masses are oscillating about the origin with angular frequencies ω_1 and ω_2 and have total energies E_1 and E_2, respectively. The variations of their momenta p with positions x are shown in the figures. If $\frac{a}{b} = n^2$ and $\frac{a}{R} = n$, then the correct equations is/are

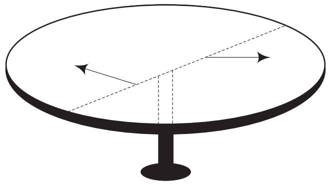
(More than One Correct Option, 2015)



(a) $E_1\omega_1 = E_2\omega_2$ (b) $\frac{\omega_2}{\omega_1} = n^2$

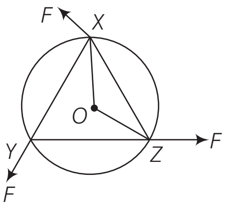
(c) $\omega_1\omega_2 = n^2$ (d) $\frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$

33. A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy-guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance 0.25 m from the centre on its either sides along its diameter (see figure). Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform, the balls have horizontal speed of $9\,\text{ms}^{-1}$ with respect to the ground. The rotational speed of the platform in rads^{-1} after the balls leave the platform is **(Single Integer Type, 2014)**



34. A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude $F = 0.5\,\text{N}$ are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad s^{-1} is

(Single Integer Type, 2014)



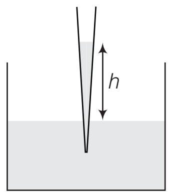
35. A planet of radius $R = 1/10 \times$ (radius of earth) has the same mass density as earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density $10^{-3}\ \text{kgm}^{-1}$ into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of earth $= 6 \times 10^6$ m and the acceleration due to gravity of earth is $10\ \text{ms}^{-2}$) **(Single Correct Option, 2014)**

(a) 96 N (b) 108 N

(c) 120 N (d) 150 N

36. A glass capillary tube is the shape of truncated cone with an apex angle α so that its two ends have cross-sections of different radii. When dipped in water vertically, water rises in it to a height h, where the radius of its cross-section is b. If the surface tension of water is S, its density is ρ, and its contact angle with glass is θ, the value of h will be (g is the acceleration due to gravity)

(Single Correct Option, 2014)



(a) $\frac{2S}{b\rho g}\cos(\theta - \alpha)$ b) $\frac{2S}{b\rho g}\cos(\theta + \alpha)$

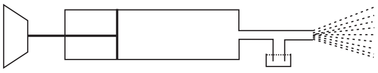
(c) $\frac{2S}{b\rho g}\cos(\theta - \alpha/2)$ (d) $\frac{2S}{b\rho g}\cos(\theta + \alpha/2)$

37. During Searle's experiment, zero of the vernier scale lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale. The 20th division of the vernier scale exactly coincides with one of the main scale divisions.

When an additional load of 2 kg is applied to the wire, the zero of the vernier scale still lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale but now the 45th division of vernier scale coincides with one of the main scale divisions. The length of the thin metallic wire is 2 m and its cross-sectional area is 8×10^{-7} m^2. The least count of the vernier scale is 1.0×10^{-5} m. The maximum percentage error in the Young's modulus of the wire is **(Single Integer Type, 2014)**

Passage (Q. Nos. 38-39)

A spray gun is shown in the figure where a piston pushes air out of nozzle. A thin tube of uniform cross-section is connected to the nozzle. The other end of the tube is in a small liquid container. As the piston pushes air through the nozzle, the liquid from the container rises into the nozzle and is sprayed out. For the spray gun shown, the radii of the piston and the nozzle are 20 mm and 1 mm respectively. The upper end of the container is open to the atmosphere. **(Passage Type, 2014)**



38. If the piston is pushed at a speed of $5\,\text{ms}^{-1}$, the air comes out of the nozzle with a speed of

(a) $0.1\,\text{ms}^{-1}$ (b) $1\,\text{ms}^{-1}$ (c) $2\,\text{ms}^{-1}$ (d) $8\,\text{ms}^{-1}$

39. If the density of air is ρ_a and that of the liquid ρ_l, then for a given piston speed the rate (volume per unit time) at which the liquid is sprayed will be proportional to

(a) $\sqrt{\frac{\rho_a}{\rho_l}}$ (b) $\sqrt{\rho_a \rho_l}$ (c) $\sqrt{\frac{\rho_l}{\rho_a}}$ (d) ρ_l

40. A particle of mass m is projected from the ground with an initial speed u_0 at an angle α with the horizontal. At the highest point of its trajectory, it makes a completely inelastic collision with another identical particle, which was thrown vertically upward from the ground with the same initial speed u_0. The angle that the composite system makes with the horizontal immediately after the collision is **(Single Correct Option, 2013)**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4} + \alpha$ (c) $\frac{\pi}{4} - \alpha$ (d) $\frac{\pi}{2}$

41. A bob of mass m, suspended by a string of length l_1, is given a minimum velocity required to complete a full circle in the vertical plane, At the highest point, it collides elastically with another bob of mass m suspended by a string of length l_2, which is initially at rest. Both the strings are massless and inextensible. If the second bob, after collision acquires the minimum speed required to complete a full circle in the vertical plane, the ratio l_1/l_2 is **(Single Integer Type, 2013)**

42. A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad/s about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is **(Single Integer Type, 2013)**

43. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the mid-point of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are) **(Single Correct Option, 2013)**

(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$

(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$

(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$

(d) The energy of the mass m remains constant

44. One end of a horizontal thick copper wire of length $2L$ and radius $2R$ is welded to an end of another horizontal thin copper wire of length L and radius R. When the arrangement is stretched by applying forces at two ends, the ratio of the elongation in the thin wire to that in the thick wire is **(Single Correct Option, 2013)**

(a) 0.25 (b) 0.50 (c) 2.00 (d) 4.00

45. A solid sphere of radius R and density ρ is attached to one end of a massless spring of force constant k. The other end of the spring is connected to another solid sphere of radius R and density 3ρ. The complete arrangement is placed in a liquid of density 2ρ and is allowed to reach equilibrium. The correct statement(s) is (are) **(Single Correct Option, 2013)**

(a) the net elongation of the spring is $\frac{4\pi R^3 \rho g}{3k}$

(b) the net elongation of the spring is $\frac{8\pi R^3 \rho g}{3k}$

(c) the light sphere is partially submerged

(d) the light sphere is completely submerged displayed by the sensor when the temperature of the metal surface is raised to 2767°C?

46. A particle of mass m is attached to one end of a mass less spring of force constant k, lying on a frictionless horizontal plane. The other end of the spring is fixed. The particle starts moving horizontally from its equilibrium position at time $t = 0$ with an initial velocity u_0. When the speed of the particle is $0.5\,u_0$, it collides elastically with a rigid wall. After this collision **(More than One Correct Option, 2013)**

(a) the speed of the particle when it returns to its equilibrium position is u_0

(b) the time at which the particle passes through the equilibrium position for the first time is $t = \pi\sqrt{\frac{m}{k}}$

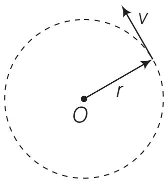
(c) the time at which the maximum compression of the spring occurs is $t = \frac{4\pi}{3}\sqrt{\frac{m}{k}}$

(d) the time at which the particle passes through the equilibrium position for the second time is $t = \frac{5\pi}{3}\sqrt{\frac{m}{k}}$

Answer *with* Explanations

1. **(b, c)** $V = \frac{kr^2}{2}$

$$F = -\frac{dV}{dr} = -kr \text{ (towards centre) } \left[F = -\frac{dV}{dr}\right]$$



At $r = R$,

$$kR = \frac{mv^2}{R} \text{ (Centripetal force)}$$

$$v = \sqrt{\frac{kR^2}{m}} = \sqrt{\frac{k}{m}}R \Rightarrow L = mvR = \sqrt{\frac{k}{m}}R^2$$

2. **(a, c)** $\mathbf{F} = (\alpha t)\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$ [at $t = 0, v = 0, \mathbf{r} = 0$]

$$\alpha = 1, \beta = 1$$

$$\mathbf{F} = t\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$$m\frac{d\mathbf{v}}{dt} = t\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

On integrating, $m\mathbf{v} = \frac{t^2}{2}\hat{\mathbf{i}} + t\hat{\mathbf{j}}$ [$m = 1$ kg]

$$\frac{d\mathbf{r}}{dt} = \frac{t^2}{2}\hat{\mathbf{i}} + t\hat{\mathbf{j}} \quad [\mathbf{r} = 0 \text{ at } t = 0]$$

Again, on integrating,

$$\mathbf{r} = \frac{t^3}{6}\hat{\mathbf{i}} + \frac{t^2}{2}\hat{\mathbf{j}}$$

At $t = 1\,\text{s}$, $\tau = (\mathbf{r}\times\mathbf{F}) = \left(\frac{1}{6}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}\right)\times(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

$$\tau = \frac{1}{3}\hat{\mathbf{k}}$$

$$v = \frac{t^3}{2}\hat{\mathbf{i}}+t\hat{\mathbf{j}}$$

At $t = 1\,\text{s}$, $\mathbf{v} = \left(\frac{1}{2}\hat{\mathbf{i}}+\hat{\mathbf{j}}\right) = \frac{1}{2}(\hat{\mathbf{i}}+2\hat{\mathbf{j}})\,\text{m/s}$

At $t = 1\,\text{s}$, $\mathbf{r}_1 - \mathbf{r}_0 = \left[\frac{1}{6}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}\right] - [0]$

$$\mathbf{s} = \frac{1}{6}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}$$

$$|\mathbf{s}| = \sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{2}\right)^2} \Rightarrow \frac{\sqrt{10}}{6}\,\text{m}$$

3. **(a, c)** $h = \dfrac{2\sigma\cos\theta}{r\rho g}$

(a) $\rightarrow h \propto \dfrac{1}{r}$

(b) h depends upon σ.

(c) If lift is going up with constant acceleration.

$$g_{\text{eff}} = (g+a) \quad \Rightarrow h = \frac{2\sigma\cos\theta}{r\rho(g+a)}$$

It means h decreases.

(d) h is proportional to $\cos\theta$.

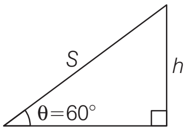
4. **(0.75 m)** $a = \dfrac{g\sin\theta}{1+\dfrac{I}{MR^2}}$

$$a_{\text{ring}} = \frac{g\sin\theta}{2} \qquad (I = MR^2)$$

$$a_{\text{disc}} = \frac{2g\sin\theta}{3} \qquad \left(I = \frac{MR^2}{2}\right)$$

$$s = \frac{h}{\sin\theta} = \frac{1}{2}at^2$$

$$= \frac{1}{2}\left(\frac{g\sin\theta}{2}\right)t_1^2$$



$$\Rightarrow \qquad t_1 = \sqrt{\frac{4h}{g\sin^2\theta}} = \sqrt{\frac{16h}{3g}}$$

$$s = \frac{h}{\sin\theta} = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{2g\sin\theta}{3}\right)t_2^2$$

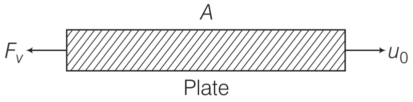
$$\Rightarrow \qquad t_2 = \sqrt{\frac{3h}{g\sin^2\theta}} = \sqrt{\frac{4h}{g}}$$

$$t_2 - t_1 = \sqrt{\frac{16h}{3g}} - \sqrt{\frac{4h}{g}} = \frac{2-\sqrt{3}}{\sqrt{10}}$$

$$\sqrt{h}\left[\frac{4}{\sqrt{3}} - 2\right] = 2 - \sqrt{3}$$

Soving this equation we get, $h = 0.75\,\text{m}$.

5. **(a, c, d)**



$$F_v = -\eta A\left(\frac{dv}{dy}\right)$$

Since, height h of the liquid in tank is very small.

$$\Rightarrow \qquad \frac{dv}{dy} = \frac{\Delta v}{\Delta y} = \left(\frac{u_0}{h}\right) \quad \Rightarrow \quad F_v = -(\eta)A\left(\frac{u_0}{h}\right)$$

$$F_v \propto \left(\frac{1}{h}\right), F_v \propto u_0, F \propto A, F_v \propto \eta$$

6. **(3)** Given, $d = 0.5\,\text{mm}$,

$Y = 2\times10^{11}\,\text{Nm}^{-2}$,

$l = 1\,\text{m}$

$$\Delta l = \frac{Fl}{AY} = \frac{mgl}{\frac{\pi d^2}{4}Y}$$

$$= \frac{1.2\times10\times1}{\frac{\pi}{4}\times(5\times10^{-4})^2\times2\times10^{11}}$$

$$= 0.3\,\text{mm}$$

LC of vernier $= \left(1-\dfrac{9}{10}\right)\text{mm} = 0.1\,\text{mm}$

So, 3rd division of vernier scale will coincide with main scale.

7. **(b)** $v = \sqrt{\dfrac{GM}{R}}$

Let $R_1 = R$, then $R_2 = 4R$

If $m_2 = m$, then $m_1 = 2m$

List-I

(P) $\dfrac{v_1}{v_2} = \sqrt{\dfrac{R_2}{R_1}} = \sqrt{\dfrac{4R}{R}} = 2:1$

(Q) $L = mvR$

$$\frac{L_1}{L_2} = \frac{R(2m)v_1}{4R(m)v_2} = \frac{1}{2}(2\ ) = 1:1$$

(R) $\dfrac{K_1}{K_2} = \dfrac{\frac{1}{2}(2m)v_1^2}{\frac{1}{2}(m)v_2^2} = 2\,(4) = 8:1$

(S) $\dfrac{T_1}{T_2} = \left(\dfrac{R_1}{R_2}\right)^{3/2} = \left(\dfrac{1}{4}\right)^{3/2} = 1:8$

8. **(a)** When force $F = 0 \Rightarrow$ potential energy $U =$ constant

$F \neq 0 \Rightarrow$ force is conservative $\Rightarrow$ Total energy $E =$ constant

List-I

(P) $\mathbf{r}(t) = \alpha t\hat{\mathbf{i}} + \beta t\hat{\mathbf{j}}$

$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} = \text{constant} \Rightarrow \mathbf{p} = \text{constant}$

$|\mathbf{v}| = \sqrt{\alpha^2 + \beta^2} = \text{constant}$

$\Rightarrow K = \text{constant}$

$\frac{d\mathbf{v}}{dt} = \mathbf{a} = 0 \Rightarrow F = 0 \Rightarrow U = \text{constant}$

$E = U + K = \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = 0$

$\mathbf{L} = \text{constant}$

$P \rightarrow 1, 2, 3, 4, 5$

(Q) $\mathbf{r}(t) = \alpha\cos\omega t\hat{\mathbf{i}} + \beta\sin\omega t\hat{\mathbf{j}}$

$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\omega\sin\omega t(-\hat{\mathbf{i}}) + \beta\omega\cos\omega t\hat{\mathbf{j}} \neq \text{constant}$

$\Rightarrow \mathbf{p} \neq \text{constant}$

$|\mathbf{v}| = \omega\sqrt{(\alpha\sin\omega t)^2 + (\beta\cos\omega t)^2} \neq \text{constant}$

$\Rightarrow K \neq \text{constant}$

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2\mathbf{r} \neq 0$

$\Rightarrow E = \text{constant} = K + U$

But $K \neq \text{constant} \Rightarrow U \neq \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\omega\alpha\beta(\hat{\mathbf{k}}) = \text{constant}$

$Q \rightarrow 2, 5$

(R) $\mathbf{r}(t) = \alpha(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}})$

$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\omega[\sin\omega t(-\hat{\mathbf{i}}) + \cos\omega t\hat{\mathbf{j}}] \neq \text{constant}$

$\Rightarrow \mathbf{p} \neq \text{constant}$

$|\mathbf{v}| = \alpha\omega = \text{constant} \Rightarrow K = \text{constant}$

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2\mathbf{r} \neq 0 \Rightarrow E = \text{constant}, U = \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\omega\alpha^2\hat{\mathbf{k}} = \text{constant}$

$R \rightarrow 2, 3, 4, 5$

(S) $\mathbf{r}(t) = \alpha t\hat{\mathbf{i}} + \frac{\beta}{2}t^2\hat{\mathbf{j}}$

$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\hat{\mathbf{i}} + \beta t\hat{\mathbf{j}} \neq \text{constant} \Rightarrow \mathbf{p} \neq \text{constant}$

$|\mathbf{v}| = \sqrt{\alpha^2 + (\beta t)^2} \neq \text{constant} \Rightarrow K \neq \text{constant}$

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \beta\hat{\mathbf{j}} \neq 0 \Rightarrow E = \text{constant} = K + U$

But $K \neq \text{constant}$

$\therefore U \neq \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = \frac{1}{2}\alpha\beta t^2\hat{\mathbf{k}} \neq \text{constant}$

$S \rightarrow 5$

9. Question is not very clear.

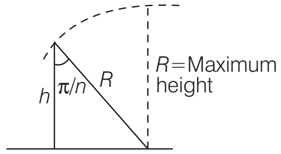
10. **(c)** If height of the cone $h >> r$

Then, $\mu N = mg$

$\mu m(R - r)\omega_0^2 = mg$

$\omega_0 = \sqrt{\frac{g}{\mu(R - r)}}$

11. **(d)**

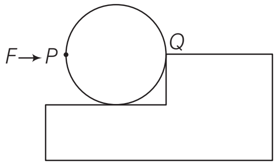


$\cos\left(\frac{\pi}{n}\right) = \frac{h}{R}$

$\Delta = R - h = \frac{h}{\cos(\pi/n)} - h$

$= h\left[\frac{1}{\cos(\pi/n)} - 1\right]$

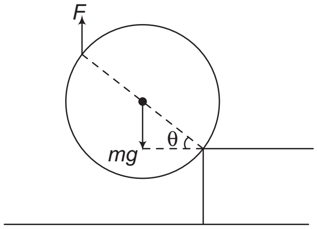
12. **(a, c)**



(a) If force is applied normal to surface at P, then line of action of force will pass from Q and thus, $\tau = 0$.

(b) Wheel can climb.

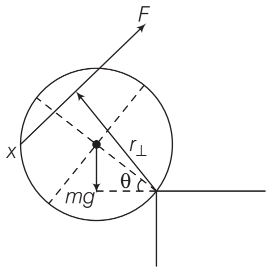
(c) $\tau = F(2R\cos\theta) - mgR\cos\theta$, $\tau \propto \cos\theta$



Hence, as θ increases

τ decreases. So its correct.

(d)



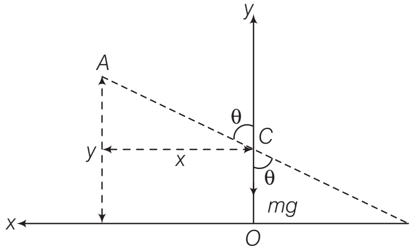
$\tau = Fr_\perp - mg\cos\theta$; τ increases with θ.

13. **(a, c, d)** When the bar makes an angle θ, the height of its COM (mid-point) is $\frac{L}{2}\cos\theta$.

$\therefore$ Displacement $= L - \frac{L}{2}\cos\theta = \frac{L}{2}(1-\cos\theta)$

Since, force on COM is only along the vertical direction, hence COM is falling vertically downward. Instantaneous torque about point of contact is

$$\tau = mg \times \frac{L}{2}\sin\theta \quad \text{or} \quad \tau \propto \sin\theta$$



Now,
$$x = \frac{L}{2}\sin\theta$$
$$y = L\cos\theta$$
$$\frac{x^2}{(L/2)^2} + \frac{y^2}{L^2} = 1$$

Path of A is an ellipse.

14. **(a, b)** Δx_{cm} of the block and point mass system $= 0$

$\therefore$ $$m(x+R) + Mx = 0$$

where, x is displacement of the block.

Solving this equation, we get

$$x = -\frac{mR}{M+m}$$

From conservation of momentum and mechanical energy of the combined system

$$0 = mv - MV$$
$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

Solving these two equations, we get

$\therefore$ $$v = \sqrt{\frac{2gR}{1+\frac{m}{M}}}$$

15. **(c)** Given, $v_e = 11.2\ \text{km/s} = \sqrt{\frac{2GM_e}{R_e}}$

From energy conservation,

$$K_i + U_i = K_f + U_f$$
$$\frac{1}{2}mv_s^2 - \frac{GM_s m}{r} - \frac{GM_e m}{R_e} = 0 + 0$$

Here, r = distane of rocket from sun

$$\Rightarrow \quad v_s = \sqrt{\frac{2GM_e}{R_e} + \frac{2GM_s}{r}}$$

Given, $M_s = 3\times10^5\ M_e$

and $r = 2.5\times10^4\ R_e$

$$\Rightarrow v_s = \sqrt{\frac{2GM_e}{R_e} + (2G)\left(\frac{3\times10^5 M_e}{2.5\times10^4 R_e}\right)}$$
$$= \sqrt{\frac{2GM_e}{R_e}\left(1+\frac{3\times10^5}{2.5\times10^4}\right)}$$
$$= \sqrt{\frac{2GM_e}{R_e}\times 13}$$
$$\Rightarrow v_s \simeq 42\ \text{km/s}$$

16. **(a)** $m = \frac{4\pi R^3}{3}\times\rho$

On taking log both sides, we have

$$\ln(m) = \text{in}\left(\frac{4\pi}{3}\right) + \ln(\rho) + 3\ln(R)$$

On differentiating with respect to time,

$$0 = 0 + \frac{1}{\rho}\frac{d\rho}{dt} + \frac{3}{R}\frac{dR}{dt}$$
$$v = \frac{dR}{dt} = KR \quad \Rightarrow \quad v \propto R$$

17. **(6)** From mass conservation,

$$\rho\cdot\frac{4}{3}\pi R^3 = \rho\cdot K\cdot\frac{4}{3}\pi r^3 \quad \Rightarrow \quad R = K^{1/3}r$$

$\therefore$ $$\Delta U = T\Delta A = T(K\cdot 4\pi r^2 - 4\pi R^2)$$
$$= T(K\cdot 4\pi R^2 K^{-2/3} - 4\pi R^2)$$
$$\Delta U = 4\pi R^2 T[K^{1/3}-1]$$

Putting the values, we get

$$10^{-3} = \frac{10^{-1}}{4\pi}\times 4\pi\times10^{-4}[K^{1/3}-1]$$
$$100 = K^{1/3} - 1 \quad \Rightarrow \quad K^{1/3} \cong 100 = 10^2$$

Given that $K = 10^\alpha$

$\therefore$ $$10^{\alpha/3} = 10^2 \Rightarrow \frac{\alpha}{3} = 2 \Rightarrow \alpha = 6$$

18. **(c)** Force on block along slot

$$= m\omega^2 r = ma = m\left(\frac{vdv}{dr}\right)$$
$$\int_0^v vdv = \int_{R/2}^r \omega^2 rdr$$
$$\Rightarrow \frac{v^2}{2} = \frac{\omega^2}{2}\left(r^2 - \frac{R^2}{4}\right)$$
$$\Rightarrow \quad v = \omega\sqrt{r^2 - \frac{R^2}{4}} = \frac{dr}{dt}$$
$$\Rightarrow \int_{R/4}^r \frac{dr}{\sqrt{r^2-\frac{R^2}{4}}} = \int_0^t \omega\, dt$$

$$\ln\left(\frac{r+\sqrt{r^2-\frac{R^2}{4}}}{\frac{R}{2}}\right)-\ln\left(\frac{R/2+\sqrt{\frac{R^2}{4}-\frac{R^2}{4}}}{\frac{R}{2}}\right)=\omega t$$

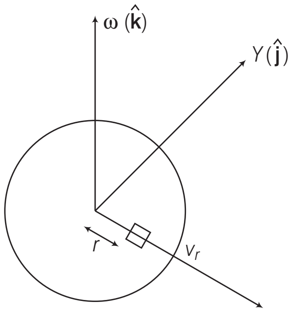
$$\Rightarrow \quad r+\sqrt{r^2-\frac{R^2}{4}}=\frac{R}{2}e^{\omega t}$$

$$\Rightarrow \quad r^2-\frac{R^2}{4}=\frac{R^2}{4}e^{2\omega t}+r^2-2r\frac{R}{2}e^{\omega t}$$

$$\Rightarrow \quad r=\frac{\frac{R^2}{4}e^{2\omega t}+\frac{R^2}{4}}{Re^{\omega t}}$$

$$=\frac{R}{4}(e^{\omega t}+e^{-\omega t})$$

19. (b)



$$\mathbf{F}_{\text{rot}}=\mathbf{F}_{\text{in}}+2m(v_{\text{rot}}\hat{\mathbf{i}})\times\omega\hat{\mathbf{k}}+m(\omega\hat{\mathbf{k}}\times r\hat{\mathbf{i}})\times\omega\hat{\mathbf{k}}$$

$$mr\omega^2\hat{\mathbf{i}}=\mathbf{F}_{\text{in}}+2mv_{\text{rot}}\omega(-\hat{\mathbf{j}})+m\omega^2r\hat{\mathbf{i}}$$

$$\mathbf{F}_{\text{in}}=2mv_r\omega\hat{\mathbf{j}}$$

$$r=\frac{R}{4}[e^{\omega t}+e^{-\omega t}]$$

$$\frac{dr}{dt}=v_r=\frac{R}{4}[\omega e^{\omega t}-\omega e^{-\omega t}]$$

$$\mathbf{F}_{\text{in}}=2m\frac{R\omega}{4}[e^{\omega t}-e^{-\omega t}]\omega\hat{\mathbf{j}}$$

$$\mathbf{F}_{\text{in}}=\frac{mR\omega^2}{2}[e^{\omega t}-e^{-\omega t}]\hat{\mathbf{j}}$$

Also, reaction is due to disc surface then

$$\mathbf{F}_{\text{reaction}}=\frac{mR\omega^2}{2}[e^{\omega t}-e^{-\omega t}]\hat{\mathbf{j}}+mg\hat{\mathbf{k}}$$

20. (a,b,d) $\mathbf{r}=\alpha t^3\hat{\mathbf{i}}+\beta t^2\hat{\mathbf{j}}$

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=3\alpha t^2\hat{\mathbf{i}}+2\beta t\hat{\mathbf{j}}$$

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=6\alpha t\hat{\mathbf{i}}+2\beta\hat{\mathbf{j}}$$

At $t=1$ s,

(a) $\mathbf{v}=3\times\frac{10}{3}\times1\hat{\mathbf{i}}+2\times5\times1\hat{\mathbf{j}}$

$=(10\hat{\mathbf{i}}+10\hat{\mathbf{j}})$m / s

(b) $\mathbf{L}=\mathbf{r}\times\mathbf{p}=\left(\frac{10}{3}\times1\hat{\mathbf{i}}+5\times1\hat{\mathbf{j}}\right)\times0.1(10\hat{\mathbf{i}}+10\hat{\mathbf{j}})$

$=\left(-\frac{5}{3}\hat{\mathbf{k}}\right)$N - ms

(c) $\mathbf{F}=m\mathbf{a}=m\times\left(6\times\frac{10}{3}\times1\hat{\mathbf{i}}+2\times5\hat{\mathbf{j}}\right)$

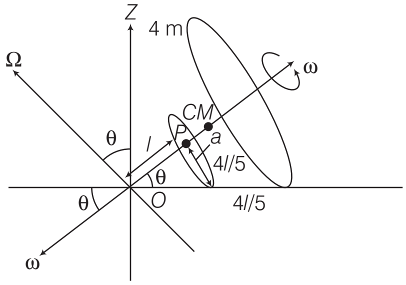
$=(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$ N

(d) $\tau=r\times\mathbf{F}=\left(\frac{10}{3}\hat{\mathbf{i}}+5\hat{\mathbf{j}}\right)\times(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$

$=+\frac{10}{3}\hat{\mathbf{k}}+10(-\hat{\mathbf{k}})$

$=\left(-\frac{20}{3}\hat{\mathbf{k}}\right)$ N - m

21. (c,d)



$$\cos\theta=\frac{l}{\sqrt{l^2+a^2}}=\frac{\sqrt{24}}{5}$$

(a) $L_z=L_{\text{CM}-O}\cos\theta-L_{D-\text{CM}}\sin\theta$

$$=\frac{81\sqrt{24}}{5}a^2m\omega\times\frac{\sqrt{24}}{5}-\frac{17\,ma^2\omega}{2}\times\frac{1}{\sqrt{24}}$$

$$=\frac{81\times24ma^2\omega}{25}-\frac{17ma^2\omega}{2\sqrt{24}}$$

(b) $L_{\text{CM}-O}=(5m)\left[\frac{9l}{5}\Omega\right]\frac{9l}{5}=\frac{81ml^2\omega}{5}=\frac{81ml^2}{5}\times\frac{a\omega}{l}$

$$L_{\text{CM}-O}=\frac{81mla\omega}{5}=\frac{81\sqrt{24}\,a^2m\omega}{5}$$

(c) Velocity of point P : $a\omega=1\,\Omega$ then

$\Omega=\frac{a\omega}{1}$ = Angular velocity of C.M. w.r.t point O.

Angular velocity fo CM w.r.t Z-axis

$=\Omega\cos\theta$

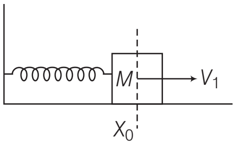
$$\omega_{\text{CM}-z}=\frac{a\omega}{1}\frac{\sqrt{24}}{5}$$

$$\omega_{\text{CM}-z}=\frac{a\omega}{5}=\frac{a\omega}{\sqrt{24}a}\frac{\sqrt{24}}{5}$$

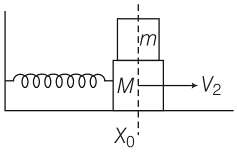
(d) $L_{D-\text{CM}}=\frac{ma^2}{2}\omega+\frac{4m(2a)^2}{2}\omega=\frac{17\,ma^2\omega}{2}$

22. **(a,b,d)** **Case 1**

Just before m is placed



Just after m is placed



In case 1, $Mv_1 = (M+m)v_2$

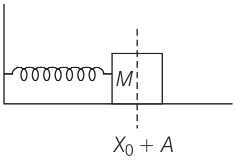
$$v_2 = \left(\frac{M}{M+m}\right)v_1$$

$$\sqrt{\frac{k}{M+m}}\,A_2 = \left(\frac{M}{M+m}\right)\sqrt{\frac{k}{M}}\,A_1$$

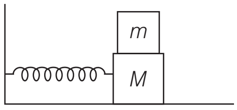
$$A_2 = \sqrt{\frac{k}{M+m}}\,A_1$$

Case 2

Just before m is placed



Just after m is placed



In case 2, $A_2 = A_1$

$$T = 2\pi\sqrt{\frac{M+m}{k}} \text{ in both cases.}$$

Total energy decreases in first case whereas remain same in 2nd case. Instantaneous speed at x_0 decreases in both cases.

23. **(d)** Let the other mass at this instant is at a distance of x from the centre O.

Applying law of conservation of angular momentum, we have $I_1\omega_1 = I_2\omega_2$

$$\therefore (MR^2)(\omega) = \left[MR^2 + \frac{M}{8}\left(\frac{3}{5}R\right)^2 + \frac{M}{8}x^2\right]\left(\frac{8}{9}\omega\right)$$

Solving this equation, we get $x = \frac{4}{5}R$.

24. **(7)** In case of pure rolling, mechanical energy remains constant (as work-done by friction is zero). Further in case of a disc,

$$\frac{\text{translational kinetic energy}}{\text{rotational kinetic energy}} = \frac{K_T}{K_R} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}I\omega^2}$$

$$= \frac{mv^2}{\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2} = \frac{2}{1}$$

or, $K_T = \frac{2}{3}$ (Total kinetic energy)

or, Total kinetic energy

$$K = \frac{3}{2}K_T = \frac{3}{2}\left(\frac{1}{2}mv^2\right) = \frac{3}{4}mv^2$$

Decrease in potential energy = increase in kinetic energy

or, $mgh = \frac{3}{4}m(v_f^2 - v_i^2)$ or $v_f = \sqrt{\frac{4}{3}gh + v_i^2}$

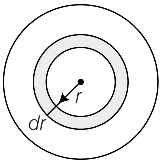
As final velocity in both cases is same.

So, value of $\sqrt{\frac{4}{3}gh + v_i^2}$ should be same in both cases.

$$\therefore \quad \sqrt{\frac{4}{3}\times 10\times 30 + (3)^2} = \sqrt{\frac{4}{3}\times 10\times 27 + (v_2)^2}$$

Solving this equation we get, $v_2 = 7$ m/s

25. **(6)** Consider a shell of radius r and thickness dr



$$dI = (dm)\,r^2 \Rightarrow dI = \frac{2}{3}(\rho 4\pi r^2 dr)r^2 \Rightarrow I = \int dI$$

$$\frac{I_B}{I_A} = \frac{\int_0^R \frac{2}{3}k\frac{r^5}{R^5}\cdot 4\pi r^2 dr\, r^2}{\int_0^R \frac{2}{3}k\frac{r}{R}\,4\pi r^2 dr\, r^2} = \frac{6}{10}$$

So, $n = 6$

26. **(2)** At height h

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \qquad \ldots\text{(i)}$$

Given, $g' = \frac{g}{4}$

Substituting in Eq. (i) we get, $h = R$

Now, from A to B,

decrease in kinetic energy = increase in potential energy

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{gh}{1+\frac{h}{R}} = \frac{1}{2}gR \qquad (h = R)$$

$$\Rightarrow \quad v^2 = gR$$

or $\quad v = \sqrt{gR}$

Now, $\quad v_{\text{esc}} = \sqrt{2gR} = v\sqrt{2}$

$$\Rightarrow \quad N = 2$$

27. **(7)** For point mass at distance $r = 3l$

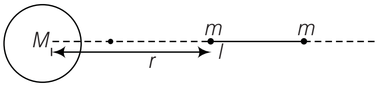
$$\frac{GMm}{(3l)^2} - \frac{Gm^2}{l^2} = ma \qquad \text{...(i)}$$

For point mass at distance $r = 4l$

$$\frac{GMm}{(4l)^2} + \frac{Gm^2}{l^2} = ma \qquad \text{...(ii)}$$

Equating the two equations we have,

$$\frac{GMm}{9l^2} - \frac{Gm^2}{l^2} = \frac{GMm}{16l^2} + \frac{Gm^2}{l^2}$$



$$\frac{7GMm}{144} = \frac{2Gm^2}{l^2}$$

$$m = \frac{7M}{288}$$

$$\therefore \quad k = 7$$

28. **(c)** Gravitational field at a distance r due to mass 'm'

$$E = \frac{G\rho\frac{4}{3}\pi r^3}{r^2} = \frac{4G\rho\pi r}{3}$$

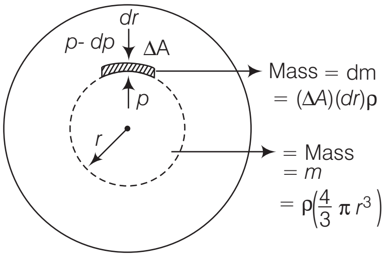
Consider a small element of width dr and area ΔA at a distance r.

Pressure force on this element outwards

= gravitational force on 'dm' from 'm' inwards

$$\Rightarrow \quad (dp)\Delta A = E(dm)$$

$$\Rightarrow \quad -dp \cdot \Delta A = \left(\frac{4}{3}G\pi\rho r\right)(\Delta A\, dr \cdot \rho)$$



$$-\int_O^P dp = \int_R^r \left(\frac{4G\rho^2\pi}{3}\right) r\,dr$$

$$-p = \frac{4G\rho^2\pi}{3\times 2}[r^2 - R^2]$$

$$\Rightarrow \quad p = c(R^2 - r^2)$$

$$r = \frac{3R}{4}, \; p_1 = c\left(R^2 - \frac{9R^2}{16}\right)$$

$$= c\left(\frac{7R^2}{16}\right)$$

$$r = \frac{2R}{3}, \quad p_2 = c\left(R^2 - \frac{4R^2}{9}\right) = c\left(\frac{5R^2}{9}\right)$$

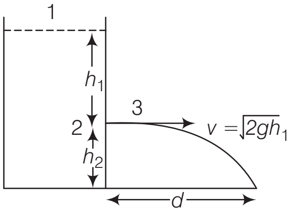
$$\frac{p_1}{p_2} = \frac{63}{80}\; r = \frac{3R}{5}, \; p_3 = c\left(R^2 - \frac{9}{25}R^2\right)$$

$$= c\left(\frac{16R^2}{25}\right)$$

$$r = \frac{2R}{5}, \; p_4 = c\left(R^2 - \frac{4R^2}{25}\right) = c\left(\frac{21R^2}{25}\right)$$

$$\Rightarrow \quad \frac{p_3}{p_4} = \frac{16}{21}$$

29. **(c)** $d = 2\sqrt{h_1h_2} = \sqrt{4h_1h_2}$



This is independent of the value of g.

(P) $g_{\text{eff}} > g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(Q) $g_{\text{eff}} < g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(R) $g_{\text{eff}} = g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(S) $g_{\text{eff}} = 0$

No water leaks out of jar. As there will be no pressure difference between top of the container and any other point. $P_1 = P_2 = P_3 = P_0$

30. **(a)** For floating, net weight of system = net upthrust

$$\Rightarrow \quad (\rho_1 + \rho_2)Vg = (\sigma_1 + \sigma_2)Vg$$

Since string is taut, $\rho_1 < \sigma_1$ and $\rho_2 > \sigma_2$

$$v_P = \frac{2r^2g}{2\eta_2}(\sigma_2 - \rho_1) \text{ (upward terminal velocity)}$$

$$v_Q = \frac{2r^2g}{9\eta_1}(\rho_2 - \sigma_1)$$

(downward terminal velocity)

$$\left|\frac{v_P}{v_Q}\right| = \frac{\eta_1}{\eta_2}$$

Further, $\bar{v}_P \cdot \bar{v}_Q$ will be negative as they are opposite to each other.

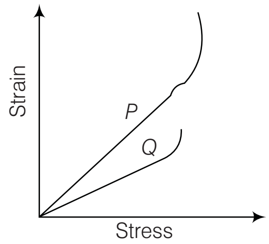
31. **(a, b)** $Y = \frac{\text{stress}}{\text{strain}}$ or $Y \propto \frac{1}{\text{strain}}$

(for same stress say σ)

$$(\text{strain})_Q < (\text{strain})_P$$

$$\Rightarrow \quad Y_Q > Y_P$$

So, P is more ductile than Q. Further, from the given figure we can also see that breaking stress of P is more than Q. So, it has more tensile strength.



32. **(b,d)** **Ist Particle**

$$P = 0 \text{ at } x = a$$

$\Rightarrow$ 'a' is the amplitude of oscillation 'A_1'.

At $x = 0, P = b$ (at mean position)

$$\Rightarrow mv_{\max} = b \Rightarrow v_{\max} = \frac{b}{m}$$

$$E_1 = \frac{1}{2}mv_{\max}^2 = \frac{m}{2}\left[\frac{b}{m}\right]^2 = \frac{b^2}{2m}$$

$$A_1\omega_1 = v_{\max} = \frac{b}{m}$$

$$\Rightarrow \quad \omega_1 = \frac{b}{ma} = \frac{1}{mn^2} \qquad \left(A_1 = a, \frac{a}{b} = n^2\right)$$

IInd Particle

$$P = 0 \text{ at } x = R \Rightarrow A_2 = R$$

At $\qquad x = 0, P = R$

$$\Rightarrow \quad v_{\max} = \frac{R}{m}$$

$$E_2 = \frac{1}{2}mv_{\max}^2 = \frac{m}{2}\left[\frac{R}{m}\right]^2 = \frac{R^2}{2m}$$

$$A_2\omega_2 = \frac{R}{m}$$

$$\Rightarrow \quad \omega_2 = \frac{R}{mR} = \frac{1}{m}$$

(b) $\frac{\omega_2}{\omega_1} = \frac{1/m}{1/mn^2} = n^2$

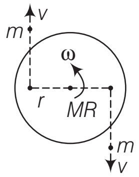
(c) $\omega_1\omega_2 = \frac{1}{mn^2} \times \frac{1}{m} = \frac{1}{m^2n^2}$

(d) $\frac{E_1}{\omega_1} = \frac{b^2/2m}{1/mn^2} = \frac{b^2n^2}{2} = \frac{a^2}{2n^2} = \frac{R^2}{2}$

$$\frac{E_2}{\omega_2} = \frac{R^2/2m}{1/m} = \frac{R^2}{2} \Rightarrow \frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$$

33. **(4)** Applying conservation of angular momentum

$$2mvr - \frac{MR^2}{2}\omega = 0, \omega = \frac{4mvr}{MR^2}$$



Substituting the values, we get

$$\omega = \frac{(4)(5 \times 10^{-2})(9)\left(\frac{1}{4}\right)}{45 \times 10^{-2} \times \frac{1}{4}}$$

$$\omega = 4 \text{ rad/s}$$

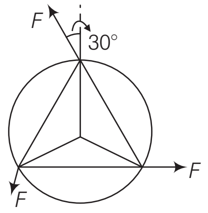
34. **(2)** Angular impulse = change in angular momentum

$$\therefore \int \tau\, dt = I\omega$$

$$\Rightarrow \qquad \omega = \frac{\int \tau\, dt}{I} = \frac{\int_0^t 3F \sin 30^\circ R\, dt}{I}$$

Substituting the values, we have

$$\omega = \frac{3\,(0.5)\,(0.5)\,(0.5)\,(1)}{\frac{1.5\,(0.5)^2}{2}} = 2 \text{ rad/s}$$



35. **(b)** Given, $R_{\text{planet}} = \frac{R_{\text{earth}}}{10}$ and density,

$$\rho = \frac{M_{\text{earth}}}{\frac{4}{3}\pi R^3_{\text{earth}}} = \frac{M_{\text{planet}}}{\frac{4}{3}\pi R^3_{\text{planet}}}$$

$$\Rightarrow \quad M_{\text{planet}} = \frac{M_{\text{earth}}}{10^3}$$

$$g_{\text{surface of planet}} = \frac{GM_{\text{planet}}}{R^2_{\text{planet}}} = \frac{GM_e \cdot 10^2}{10^3 \cdot R_e^2}$$

$$= \frac{GM_e}{10R_e^2} = \frac{g_{\text{surface of earth}}}{10}$$

$$g_{\text{depth of planet}} = g_{\text{surface of planet}} \left(\frac{x}{R}\right)$$

where, x = distance from centre of planet.

$\therefore$ Total force on wire

$$F = \int_{4R/5}^{R} \lambda\, dx\, g \left(\frac{x}{R}\right) = \frac{\lambda g}{R}\left[\frac{x^2}{2}\right]_{4R/5}^{R}$$

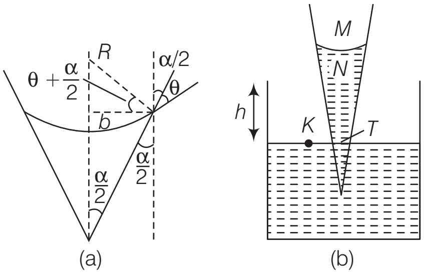
Here, $g = g_{\text{surface of planet}}$,

$R = R_{\text{planet}}$

Substituting the given values, we get

$$F = 108\text{ N}$$

36. **(d)** Using geometry



(a) (b)

$$\frac{b}{R} = \cos\left(\theta + \frac{\alpha}{2}\right)$$

$$\Rightarrow \qquad R = \frac{b}{\cos\left(\theta + \frac{\alpha}{2}\right)}$$

Using pressure equation along the path *MNTK*

$$p_0 - \frac{2S}{R} + h\rho g = p_0$$

Substituting the value of R, we get

$$h = \frac{2S}{R\rho g} = \frac{2S}{b\rho g}\cos\left(\theta + \frac{\alpha}{2}\right)$$

37. **(4)** $Y = \dfrac{F/A}{\dfrac{\Delta l}{l}}$, $\Delta l = 25 \times 10^{-50}$ m

$$\frac{\Delta Y}{Y} \times 100 = \frac{10^{-5}}{25 \times 10^{-5}} \times 100 = 4\%$$

38. **(c)** From continuity equation,

$$A_1 v_1 = A_2 v_2$$

Here, $A_1 = 400 A_2$

because $r_1 = 20 r_2$

and $A = \pi r^2$

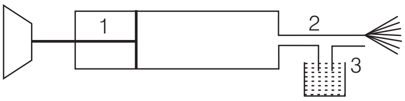
$$\therefore \qquad v_2 = \frac{A_1}{A_2}(v_1) = 400 v_1$$

$$= 400(5)\text{ mm/s}$$

$$= 2000\text{ mm/s}$$

$$= 2\text{ m/s}$$

39. **(a)** $p_1 - p_2 = \frac{1}{2}\rho_a v_a^2$



$$p_3 - p_2 = \frac{1}{2}\rho_l v_l^2$$

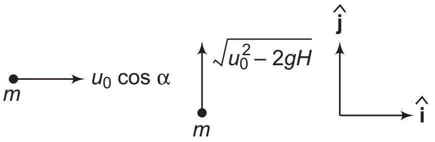
$$p_3 = p_1$$

$$\therefore \qquad \frac{1}{2}\rho_l v_l^2 = \frac{1}{2}\rho_a v_a^2$$

$$\Rightarrow \qquad v_l = \sqrt{\frac{\rho_a}{\rho_l}}\, v_a$$

$$\therefore \qquad \text{Volume flow rate} \propto \sqrt{\frac{\rho_a}{\rho_l}}$$

40. **(a)** From momentum conservation equation, we have,



$$\mathbf{p}_i = \mathbf{p}_f$$

$$\therefore\ m(u_0\cos\alpha)\hat{\mathbf{i}} + m(\sqrt{u_0^2 - 2gH})\hat{\mathbf{j}}$$

$$= (2m)\mathbf{v} \qquad \text{...(I)}$$

$$H = \frac{u_0^2 \sin^2\alpha}{2g} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$\mathbf{v} = \frac{u_0\cos\alpha}{2}\hat{\mathbf{i}} + \frac{u_0\cos\alpha}{2}\hat{\mathbf{j}}$$

Since both components of **v** are equal. Therefore, it is making 45° with horizontal.

41. **(5)** Velocity of first bob at highest point

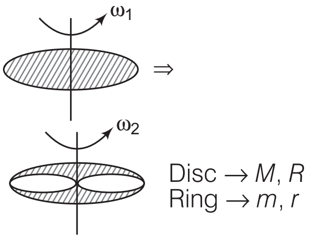
$.v_1 = \sqrt{gR} = \sqrt{gl_1}$ (to just complete the vertical circle)

= velocity of second bob just after elastic collision.

= velocity of second bob at the bottommost point

$= \sqrt{5gl_2} \quad \Rightarrow \quad \dfrac{l_1}{l_2} = 5$

42. **(8)** $I_1\omega_1 = I_2\omega_2$

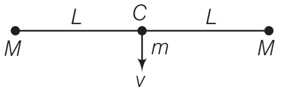


$$\therefore\ \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \left[\frac{\frac{1}{2}MR^2}{\frac{1}{2}MR^2 + 2(mr^2)}\right]\omega_1$$

$$= \left[\frac{50(0.4)^2}{50(0.4)^2 + 8\times(6.25)\times(0.2)^2}\right](10)$$

$$= 8 \text{ rad/s}$$

43. **(b)**



Let v is the minimum velocity. From energy conservation,

$$U_c + K_c = U_\infty + K_\infty$$

$$\therefore\qquad mV_c + \frac{1}{2}mv^2 = 0 + 0$$

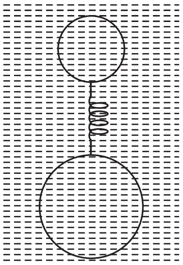
$$\therefore\qquad v = \sqrt{-2V_c} = \sqrt{(-2)\left(\frac{-2GM}{L}\right)} = 2\sqrt{\frac{GM}{L}}$$

44. **(c)** $\Delta l = \dfrac{FL}{AY} = \dfrac{FL}{(\pi r^2)Y}$

$$\therefore\qquad \Delta l \propto \frac{L}{r^2}$$

$$\therefore\qquad \frac{\Delta l_1}{\Delta l_2} = \frac{L/R^2}{2L/(2R)^2} = 2$$

45. **(a)**



On small sphere

$$\frac{4}{3}\pi R^3(\rho)g + kx = \frac{4}{3}\pi R^3(2\rho)g \qquad \text{...(i)}$$

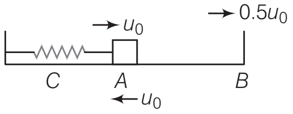
On second sphere (large)

$$\frac{4}{3}\pi R^3(3\rho)g = \frac{4}{3}\pi R^3(2\rho)g + kx \qquad \text{...(ii)}$$

By Eqs. (i) and (ii), we get

$$x = \frac{4\pi R^3\rho g}{3k}$$

46. **(a,d)** (a) At equilibrium ($t = 0$) particle has maximum velocity u_0. Therefore velocity at time t can be written as



$$u = u_{\max}\cos\omega t = u_0\cos\omega t$$

when, $\quad u = 0.5\,u_0 = u_0\cos\omega t$

$$\therefore\qquad \omega t = \frac{\pi}{3}$$

$$\therefore\qquad \frac{2\pi}{T}t = \frac{\pi}{3}$$

$$\therefore\qquad t = \frac{T}{6}$$

(b) $t = t_{AB} + t_{BA} = \dfrac{T}{6} + \dfrac{T}{6}$

$$= \frac{T}{3} = \frac{2\pi}{3}\sqrt{\frac{m}{k}}$$

(c) $t = t_{AB} + t_{BA} + t_{AC}$

$$= \frac{T}{6} + \frac{T}{6} + \frac{T}{4}$$

$$= \frac{7}{12}T = \frac{7\pi}{6}\sqrt{\frac{m}{k}}$$

(d) $t = t_{AB} + t_{BA} + t_{AC} + t_{CA}$

$$= \frac{T}{6} + \frac{T}{6} + \frac{T}{4} + \frac{T}{4} = \frac{5}{6}T$$

$$= \frac{5\pi}{3}\sqrt{\frac{m}{k}}$$

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced, the NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. The exercises in this book have been divided into two sections viz., JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions. I am extremely thankful to (Dr.)Mrs. Sarita Pandey, Mr. Anoop Dhyani and Nisar Ahmad for their endless efforts during the project.

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

HEAT AND THERMODYNAMICS

Heat, temperature, thermal expansion; specific heat capacity, calorimetry; change of state, latent heat. Heat transfer-conduction, convection and radiation, Newton's law of cooling.

Thermal equilibrium, Zeroth law of thermodynamics, concept of temperature. Heat, work and internal energy. First law of thermodynamics. Second law of thermodynamics: reversible and irreversible processes. Carnot engine and its efficiency.

Equation of state of a perfect gas, work done on compressing a gas. Kinetic theory of gases – assumptions, concept of pressure. Kinetic energy and temperature: rms speed of gas molecules; Degrees of freedom, Law of equipartition of energy, applications to specific heat capacities of gases; Mean free path, Avogadro's number.

WAVES

Wave motion. Longitudinal and transverse waves, speed of a wave. Displacement relation for a progressive wave. Principle of superposition of waves, reflection of waves, Standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler's effect in sound.

JEE Advanced

GENERAL

Specific heat of a liquid using calorimeter. Speed of sound using resonance column.

WAVES

Wave motion (plane waves only), longitudinal and transverse waves, superposition of waves, Progressive and stationary waves, Vibration of strings and air columns, Resonance, Beats, Speed of sound in gases, Doppler's effect (in sound).

HEAT and THERMODYNAMICS

Thermal expansion of solids, liquids and gases, calorimetry, latent heat, Heat conduction in one dimension, Elementary concepts of convection and radiation, Newton's law of cooling, Ideal gas laws, Specific heats (and for monoatomic and diatomic gases), Isothermal and adiabatic processes, bulk modulus of gases, Equivalence of heat and work, First law of thermodynamics and its applications (only for ideal gases), Black body radiation: absorptive and emissive powers, Kirchhoff's law, Wien's displacement law, Stefan's law.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

17

Wave Motion

Chapter Contents

17.1 Introduction
17.2 Classification of a Wave
17.3 Equation of a Travelling Wave
17.4 Sine Wave
17.5 Two Graphs in Sine Wave
17.6 Wave Speed
17.7 Energy in Wave Motion

17.1 Introduction

In every wave, a physical quantity y is made to oscillate at one point and gradually these oscillations of y propagate to other points also without transport of matter. In general, a wave also transports energy and momentum (in addition to oscillations of y).

Thus, in a wave, the following three physical quantities transfer from one point to another point.

(i) oscillations of y
(ii) energy and
(iii) momentum

In different situations, the physical quantity y may be different. For example:

y is displacement of medium particles (between $+A$ and $-A$) from the mean position in case of string wave, where A is the amplitude of oscillations. Similarly, y is electric and magnetic fields for electromagnetic waves and it is displacement, pressure and density which oscillate in longitudinal (or sound) wave. A wave is said to be **travelling** or **progressive** if it travels from one point to another. A **plane wave** is a wave of constant frequency and amplitude with wavefronts that are infinitely long straight lines. Plane waves travel in the perpendicular direction to the wavefronts. Many physical waves are approximate plane waves far from their sources.

17.2 Classification of a Wave

A wave can be classified in the following three manners :

1-D, 2-D and 3-D Wave

In 1-D wave, oscillations of y (or energy and momentum also) transfer in a straight line or we can say that wave travels in a straight line. String wave is the best example of a 1-D wave. In 2-D wave, it travels in a plane. Wave travelling on the surface of water is an example of a 2-D wave. In 3-D wave, it travels in whole space. Sound or light wave produced by a point source is an example of a 3-D wave.

Transverse and Longitudinal Wave

In transverse waves, oscillations of y are perpendicular to wave velocity and wave velocity is the direction in which oscillations (or energy and momentum) transfer from one point to another.

Electromagnetic waves and string waves are transverse in nature.In longitudinal waves, oscillations are along the wave velocity. Sound wave is a longitudinal wave.

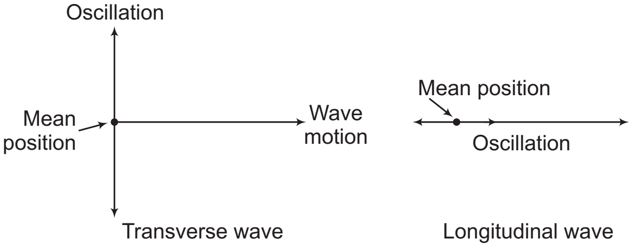


Fig. 17.1

Mechanical and Non-mechanical Wave

The waves which require medium for their propagation from one point to another point are called mechanical waves. Although this medium may be a gas also. Sound wave is a mechanical wave. Over the moon, sound waves cannot travel, as there is no gas on its surface.

Non-mechanical waves can travel with or without medium. Electromagnetic waves are non-mechanical waves.

✓ Extra Points to Remember

- Apart from mechanical and non-mechanical waves, there is also another kind of waves called "**matter waves**".
- Sound wave is basically a part of longitudinal waves with frequency varying from 20 Hz to 20,000 Hz. Similarly, light wave is a part of electromagnetic waves with wavelength varying from approximately 4000 Å to 7000 Å.
- In the propagation of mechanical waves, elasticity and inertia of the medium play an important role.
- Waves traveling through a solid medium can be either transverse or longitudinal waves. But waves travelling through a fluid (such as a liquid or a gas) are always longitudinal waves. Transverse waves require a relatively rigid medium in order to transmit their energy. As one particle begins to move it must be able to exert a pull on its nearest neighbor. If the medium is not rigid as is the case with fluids, the particles will slide past each other. This sliding action that is characteristic of liquids and gases prevents one particle from displacing its neighbour in a direction perpendicular to wave direction. It is for this reason that only longitudinal waves are observed moving through the bulk of liquids such as our ocean.
- While waves that travel within the depths of the ocean are longitudinal waves, the waves that travel along the surface of the oceans are referred to as surface waves. A **surface wave** is a wave in which particles of the medium undergo a circular motion. Surface waves are neither longitudinal nor transverse. The radius of the circles decreases as the depth into the water increases.
- Earthquakes are capable of producing both types of waves transverse as well as longitudinal. The **P-waves (Primary waves)** in an earthquake are examples of longitudinal waves. The P-waves travel with the fastest velocity and are the first to arrive. The **S-waves (Secondary waves)** in an earthquake are examples of transverse waves. S-waves propagate with a velocity slower than P-waves, arriving several seconds later.

17.3 Equation of a Travelling Wave

As we have already read, in a wave motion a physical quantity y is made to oscillate at one point and these oscillations of y propagate to other places also.

Therefore, in a wave, several particles oscillate (unlike SHM in which normally a single oscillates).

So, to determine the value of y (from its mean position between $+A$ and $-A$) we will have to tell position of the particle and time. Thus,

$$\boxed{y = f\ (\text{position of particle, time})}$$

In three dimensional space, position of the particle can be represented by three variable co-ordinates. Thus, in general, y is a function of four variables, three in co-ordinates and the fourth one is time.

But in physics, we normally keep least number of variables. If the wave is one-dimensional, then position of the particle can be represented by a single variable co-ordinate (say x).

Thus, in a one-dimensional wave y is a function of two variables x and t. Here, x is used for position of the particle and t for time or

$$\boxed{y = f\,(x, t)}$$

Note *Since, there are two variables in y, therefore whenever required, y is always differentiated partially. For example,*

$$\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x^2} \text{ etc.}$$

Now, the question is, there may be infinite number of functions of x and t, like

$$y = x + t$$

$$y = xt + t^3 x^2 \text{ etc}$$

Obviously, all functions will not represent a wave function. Only those functions of x and t will represent a wave function which satisfies the following three conditions.

Condition 1 The given function must satisfy the following differential equation :

$$\boxed{\left(\frac{\partial^2 y}{\partial t^2}\right) = k\left(\frac{\partial^2 y}{\partial x^2}\right)} \qquad \text{...(i)}$$

Here, k is a constant which is equal to square of the wave velocity or

$$\boxed{k = v^2}$$

Condition 2 The wave function must be single valued. For given values of x and t there should be only one value of y.

Condition 3 The wave function and its first derivative must be continuous. Therefore, there should not be a sudden change in the value of y and its first derivative (in some cases it will be called slope).

Note *Last two conditions are slightly difficult to explain at this stage. So, students need not to go in detail of those.*

The general solution of Eq. (i) discussed above is of the form :

$$\boxed{y(x, t) = f\,(ax \pm bt)} \qquad \text{...(ii)}$$

Thus, any function of x and t which satisfies Eq. (i) and which can be written as Eq. (ii) represents a wave, provided conditions (2) and (3) are also satisfied.

Further, if these conditions are satisfied, then speed of wave (v) is given by

$$\boxed{v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}}$$

If ax and bt are of same sign (both positive or both negative), then wave will be travelling along negative x-direction. If one is positive and other is negative, then wave travels in negative x-direction.

Alternate Method of Understanding the Wave Equation

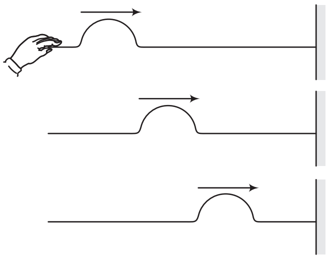


Fig. 17.2

Let us consider a long string with one end fixed to a wall and the other held by a person. The person pulls on the string keeping it tight. Suppose the person snaps his hand a little up and down producing a bump in the string near his hand.

Experiments shows that, as time passes the bump travels on the string towards right.

Suppose the man starts snapping his hand at time $t=0$ and finishes his job at $t=\Delta t$. The vertical displacement y of the left end of the string is a function of time.

$$y=0 \text{ for } t<0 \text{ and } t>\Delta t$$

$$y\neq 0 \text{ for } 0<t<\Delta t$$

Let us represent this function by $f(t)$.

Thus,

$$y(x=0,t)=f(t)$$

Here, $x=0$ is the extreme left end of the string.

The disturbance travels on the string towards right with a constant speed v. Thus, the displacement produced at the left end at time t, reaches the point x at time $t+\frac{x}{v}$.

But the displacement of the particle at point x at time t was originated at $x=0$ at time $t-\frac{x}{v}$.

$$\therefore \qquad y(x,t)=y\left(x=0,\,t-\frac{x}{v}\right)=f\left(t-\frac{x}{v}\right)$$

Thus, $$\boxed{y(x,t)=f\left(t-\frac{x}{v}\right)} \qquad \text{...(iii)}$$

Eq. (iii) represents a wave travelling in the positive x-direction with a constant speed v. The time t and the position x must appear in the wave equation in the combination $t-\frac{x}{v}$. If the wave travels in the negative x-direction, its general equation may be written as

$$y(x,t)=f\left(t+\frac{x}{v}\right)$$

Note *In the wave equation,*

$$y=(x,t)=f\left(t-\frac{x}{v}\right)$$

Coefficient of t is 1 and coefficient of x is $\frac{1}{v}$.

$$\therefore \qquad \text{wave speed}=\frac{\text{coefficient of } t}{\text{coefficient of } x}=\frac{1}{(1/v)}=v$$

Example 17.1 *In a wave motion* $y = a\sin(kx - \omega t)$, *y can represent:* **(JEE 1999)**

(a) *electric field* (b) *magnetic field*

(c) *displacement* (d) *pressure*

Solution (a, b, c, d)

In case of sound wave, y can represent pressure and displacement, while in the case of electromagnetic wave, it represents electric and magnetic fields.

Example 17.2 *Show that the equation,* $y = a\sin(\omega t - kx)$ *satisfies the wave equation* $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. *Find speed of wave and the direction in which it is travelling.*

Solution $\frac{\partial^2 y}{\partial t^2} = -\omega^2 a\sin(\omega t - kx)$ and $\frac{\partial^2 y}{\partial x^2} = -k^2 a\sin(\omega t - kx)$

We can write these two equations as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\omega^2}{k^2}\cdot\frac{\partial^2 y}{\partial x^2}$$

Comparing this with,
$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We get wave speed $v = \frac{\omega}{k}$ **Ans.**

The negative sign between ωt and kx implies that wave is travelling along positive x-direction.

INTRODUCTORY EXERCISE 17.1

1. Prove that the equation $y = a\sin\omega t$ does not satisfy the wave equation and hence it does not represent a wave.
2. A wave pulse is described by $y(x, t) = ae^{-(bx - ct)^2}$, where a, b and c are positive constants. What is the speed of this wave?
3. You have learnt that a travelling wave in one dimension is represented by a function $y = f(x, t)$ where x and t must appear in the combination $ax \pm bt$ or $x - vt$ or $x + vt$, i.e. $y = f(x \pm vt)$. Is the converse true? Examine if the following functions for y can possibly represent a travelling wave
   (a) $(x - vt)^2$ (b) $\log[(x + vt)/x_0]$ (c) $1/(x + vt)$
4. The equation of a wave travelling on a string stretched along the X-axis is given by
$$y = Ae^{-\left(\frac{x}{a} + \frac{t}{T}\right)^2}$$
   (a) Write the dimensions of A, a and T.
   (b) Find the wave speed.
   (c) In which direction is the wave travelling?
   (d) Where is the maximum of the pulse located at $t = T$ and at $t = 2T$?

17.4 Sine Wave

If oscillations of y are simple harmonic in nature, then $y(x, t)$ function is sine or cosine function. Such type of wave is called **sine wave** or **sinusoidal wave**. For better understanding, we can visualize a sine wave travelling on a string.

The general expression of a sine wave is

$$\boxed{y = A\sin(\omega t \pm kx \pm \phi)} \quad \text{and} \quad \boxed{y = A\cos(\omega t \pm kx \pm \phi)}$$

In the above equations,

(i) A is the amplitude of oscillations of y.

(ii) ω is called the angular frequency. Here,

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T} \quad \text{and} \quad f = \frac{1}{T}}$$

where, f is normal frequency of oscillations and T is the time period of oscillations. SI unit of ω is radian/sec while that of f is Hz or s^{-1}.

(iii) k is called the angular wave number, where

$$\boxed{k = \frac{2\pi}{\lambda}}$$

Here, λ is the wavelength of wave.

Wavelength (λ)

In a **transverse wave motion,** the particles of the medium oscillate about their mean or equilibrium position at right angles to the direction of propagation of wave motion itself.

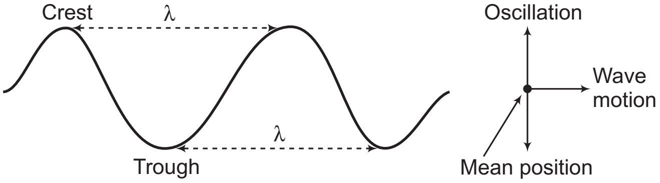


Fig. 17.3

This form of wave motion travels in the form of **crests and troughs**, for example, waves travelling along a stretched string. The distance between two successive crests or troughs is known as **wavelength (λ)** of the wave. In a **longitudinal wave motion,** particles of the medium oscillate about their mean or equilibrium position along the direction of propagation of the wave motion itself. This type of wave motion travels in the form of **compressions** and **rarefactions**.

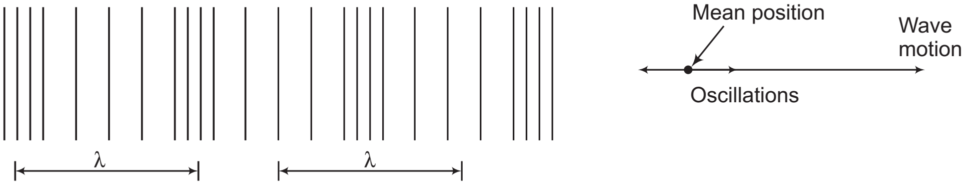


Fig. 17.4

The distance between two successive compressions or rarefactions constitute one wavelength.

Note *Numerically, $k=\frac{2\pi}{\lambda}$ or angular wave number is defined as the number of waves in 2π length. Similarly, there is one another word called wave number. This is equal to $\frac{1}{\lambda}$ and numerically this is equal to number of waves in unit length.*

(iv) Wave speed v,

$$=\frac{\text{coefficient of } t}{\text{coefficient of } x}$$

$$\therefore \qquad \boxed{v=\frac{\omega}{k}}$$

But $$\omega=2\pi f$$

and $$k=\frac{2\pi}{\lambda}$$

Substituting in the above equation, we also get

$$\boxed{v=f\lambda}$$

(v) $\frac{\partial y}{\partial t}$ is SHM velocity of the particle executing oscillations (lying between $+\omega A$ and $-\omega A$).

Similarly, $\frac{\partial^2 y}{\partial t^2}$ is SHM acceleration (lying between $+\omega^2 A$ and $-\omega^2 A$).

Let us take an example: suppose a sine wave travelling on a string is

$$y=A\sin(\omega t-kx+\phi) \qquad \ldots\text{(i)}$$

Then, $$\frac{\partial y}{\partial t}=\omega A\cos(\omega t-kx+\phi) \qquad \ldots\text{(ii)}$$

and $$\frac{\partial^2 y}{\partial \tau^2}=-\omega^2 A\sin(\omega t-kx+\phi) \qquad \ldots\text{(iii)}$$

From Eq. (i), we can determine y-displacement of SHM of any particle at position x and at time t. Value of y lies between $+A$ and $-A$.

From Eq. (ii), we can determine $\frac{\partial y}{\partial t}$ or SHM velocity and from Eq. (iii), we can find $\frac{\partial^2 y}{\partial t^2}$ or SHM acceleration.

Angle $(\omega t-kx+\phi)$ is same in all three equations which is also called phase angle of three functions. With the help of this phase angle, we can determine values of y, $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ at any position x at time t. If we put $x=0$ and $t=0$, then this angle is only ϕ. This is called initial phase angle of the particle at co-ordinate $x=0$. With the help of this angle, we can find the initial values of y, $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ of this string particle (which is executing SHM at position $x=0$).

Extra Points to Remember

- Alternate expressions of a sine wave travelling along positive x-direction are

$$y = A \sin k(x - vt) = A \sin (kx - \omega t) = A \sin \frac{2\pi}{\lambda}(x - vt) = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Similarly, the expression, $y = A \sin k(x + vt) = A \sin (kx + \omega t) = A \sin 2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right)$ etc.

represent a sine wave travelling along negative x-direction.

- **Difference between two equations**

$$y = A\sin(kx - \omega t) \text{ and } y = A\sin(\omega t - kx)$$

It hardly matters whether we write the first or the second equation. Both the equations represent a wave travelling in positive x-direction with speed $v = \frac{\omega}{k}$. The difference between them is that they are out of phase, i.e. phase difference between them is π. It means, if a particle in position $x = 0$ at time $t = 0$ is in its mean position and moving upwards (represented by first wave) then the same particle will be in its mean position but moving downwards (represented by the second wave). Similarly, the waves

$$y = A \sin (kx - \omega t) \text{ and } y = -A \sin (kx - \omega t) \text{ are also out of phase.}$$

Example 17.3 *A wave travelling along a string is described by*

$$y(x, t) = 0.005 \sin (80.0\, x - 3.0\, t).$$

in which the numerical constants are in SI units (0.005 *m*, 80.0 *rad* m^{-1} *and* 3.0 *rad* s^{-1}). *Calculate (a) the amplitude. (b) the wavelength (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance x* = 30.0 *cm and time t* = 20 *s?* **(NCERT Solved Example)**

Solution On comparing the given equation with

$$y(x, t) = A \sin (kx - \omega t), \text{ we find}$$

(a) the amplitude $A = 0.005 \text{ m} = 5 \text{ mm}$.

(b) the angular wave number k and angular frequency ω are

$$k = 80.0 \text{ m}^{-1} \quad \text{and} \quad \omega = 3.0 \text{ s}^{-1}$$

$$\therefore \quad \lambda = 2\pi/k = \frac{2\pi}{80.0} = 0.0785 \text{ m}$$

$$= 7.85 \text{ cm}$$ **Ans.**

(c) $T = 2\pi/\omega = \frac{2\pi}{3.0} = 2.09 \text{ s}$ **Ans.**

and frequency, $f = 1/T = 0.48 \text{ Hz}$ **Ans.**

The displacement y at $x = 30.0$ cm and time $t = 20$ s is given by

$$y = (0.005 \text{ m}) \sin (80.0 \times 0.3 - 3.0 \times 20)$$
$$= (0.005 \text{ m}) \sin (-36)$$
$$= (0.005 \text{ m}) \sin (-36 + 12\pi)$$
$$= (0.005 \text{ m}) \sin (1.714)$$
$$= (0.005 \text{ m}) \sin (98.27^\circ) = 4.94 \text{ mm}$$ **Ans.**

Example 17.4 *The equation of a wave is*

$$y(x,t)=0.05\sin\left[\frac{\pi}{2}(10x-40t)-\frac{\pi}{4}\right]m$$

Find : *(a) the wavelength, the frequency and the wave velocity*

(b) the particle velocity and acceleration at $x=0.5$ *m and* $t=0.05$ *s.*

Solution (a) The equation may be rewritten as

$$y(x,t)=0.05\sin\left(5\pi x-20\pi t-\frac{\pi}{4}\right)\text{ m}$$

Comparing this with equation of plane progressive harmonic wave,

$$y(x,t)=A\sin(kx-\omega t+\phi)\text{ we have,}$$

Wave number, $k=\dfrac{2\pi}{\lambda}=5\pi$ rad/m

$\therefore$ $\lambda=0.4$ m **Ans.**

The angular frequency is

$$\omega=2\pi f=20\pi\text{ rad/s}$$

$\therefore$ $f=10$ Hz **Ans.**

The wave velocity is,

$$v=f\lambda=\frac{\omega}{k}=4\text{ m/s in }+x\text{-direction}$$ **Ans.**

(b) The particle velocity and particle acceleration at the given values of x and t are

$$\frac{\partial y}{\partial t}=-(20\pi)(0.05)\cos\left(\frac{5\pi}{2}-\pi-\frac{\pi}{4}\right)$$

$$=2.22\text{ m/s}$$ **Ans.**

$$\frac{\partial^2 y}{\partial t^2}=-(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2}-\pi-\frac{\pi}{4}\right)$$

$$=140\text{ m/s}^2$$ **Ans.**

INTRODUCTORY EXERCISE 17.2

1. Consider the wave $y=(5\text{ mm})\sin[1\text{cm}^{-1}x-(60\text{s}^{-1})t]$. Find (a) the amplitude, (b) the angular wave number, (c) the wavelength, (d) the frequency, (e) the time period and (f) the wave velocity.

2. A wave is described by the equation $y=(1.0\text{ mm})\sin\pi\left(\dfrac{x}{2.0\text{ cm}}-\dfrac{t}{0.01\text{ s}}\right)$.

   (a) Find time period and wavelength.
   (b) Find the speed of particle at $x=1.0$ cm and time $t=0.01$ s.
   (c) What are the speeds of the particles at $x=3.0$ cm, 5.0 cm and 7.0 cm at $t=0.01$ s?
   (d) What are the speeds of the particles at $x=1.0$ cm at $t=0.011, 0.012$ and 0.013 s?

17.5 Two Graphs in Sine Wave

As we have learned in the above article that $y(x, t)$ equation of a sine wave is either sine or cosine equation. Now, corresponding to this equation we can have two graphs and two simple equations.

First Graph

In $y(x, t)$ equation, if value of t is fixed (or substituted), then the equation left is $y(x)$ equation.So, we can plot y-x graph corresponding to this equation. And obviously the graph (or the equation) will be a sine or cosine graph.

For example Suppose a sine wave is travelling along positive x-direction on a string. At a given time (say at 9 AM), the y-x graph may be as shown in the figure.

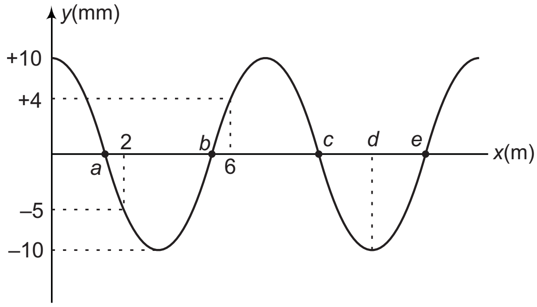


Fig. 17.5

The important points in the above graph are

(i) amplitude of oscillation is 10 mm.

(ii) at 9 AM, y-displacement of the string particle at $x = 2$ m is -5 mm and of the particle at $x = 6$ m is $+4$ mm. Or, we can say that this graph represents SHM y-displacements of different string particles (at different x-coordinates) at 9 AM. Hence, this is a photograph (or snapshot) of the string at 9 AM.

(iii) Slope of this graph at any point is $\frac{\partial y}{\partial x}\left(\text{not } \frac{dy}{dx}\right)$, as y has two variables x and t.

(iv) Two string particles at different locations are in different phases. The phase difference between them is given by

$$\boxed{\Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x)}$$

Here, Δx is the path difference between them.

For example

Particles 'a' and 'b' have a path difference of $\frac{\lambda}{2}$. So, phase difference between them is π.

Particle 'a' and 'c' have a path difference of λ. So, phase difference is 2π.

Similarly, particles 'c' and 'd' have a path difference of $\frac{\lambda}{4}$. Therefore, the phase difference is $\frac{\pi}{2}$.

(v) From the above graph, we cannot determine the direction of wave velocity. For that, wave equation will be required.

Second Graph

In $y(x, t)$ equation, if the value of x is fixed (or substituted) then the equation left is $y(t)$ equation. So, we can plot y-t graph corresponding to this equation. Again, the graph (or the equation) will be a sine or cosine graph.

For example Suppose a sine wave is travelling along positive x-direction on a string. At a given position (say $x = 4$ m), the y-t graph may be as shown in the figure.

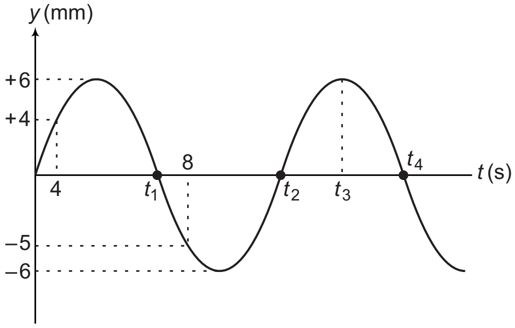


Fig. 17.6

The important points in this graph are

(i) amplitude of oscillations is 6 mm.

(ii) y-displacement of the particle at $x = 4$ m is 4 mm at 4 s. Similarly, y-displacement of the particle at $x = 4$ m is -5 mm at 8 s or we can say that this graph is basically SHM y-t graph of the string particle at $x = 4$ m, which shows different SHM y-displacements of this particle at different times or it is videography of SHM oscillations of this particle.

(iii) Slope of this particle at any time is $\frac{\partial y}{\partial t}$ and this is SHM velocity $(= \pm\omega\sqrt{A^2 - y^2})$ of the particle at $x = 4$ m at that time.

(iv) The same particle at two different times will have different phase angles.

The phase difference in a time interval of Δt is given by

$$\boxed{\Delta\phi = \left(\frac{2\pi}{T}\right)\Delta t}$$

For example,

t_1 and t_2 have a time interval of $\frac{T}{2}\cdot$ So, phase difference is π.

t_2 and t_3 have a time interval of $\frac{T}{4}\cdot$ So, the phase difference is $\frac{\pi}{2}$.

Similarly, t_1 and t_4 have a time interval of T.

Therefore, phase difference is 2π.

Wave velocity (v), particle velocity (v_p) and particle acceleration (a_p) in a Sinusoidal wave

Wave velocity (v) is the velocity by which oscillations of y (or energy) transfer from one point to another point. Later, we will see that this velocity depends on characteristics of medium.

Particle velocity (v_P) and particle acceleration (a_P) are different.

In a sinusoidal wave, particles of the medium oscillate simple harmonically about their mean position. Therefore, all the formulae we have read in SHM apply to the particles here also. For example, maximum particle velocity is $\pm A\omega$ at mean position and it is zero at extreme positions etc. Similarly, maximum particle acceleration is $\pm \omega^2 A$ at extreme positions and zero at mean position.

However, the wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between $+A\omega$ and $-A\omega$) the wave velocity is constant for given characteristics of the medium.

Suppose, a sine wave travelling along positive x-axis is

$$y(x, t) = A \sin(kx - \omega t) \quad \text{...(i)}$$

Let us differentiate this function partially with respect to t and x.

$$\frac{\partial y(x, t)}{\partial t} = -A\omega \cos(kx - \omega t) \quad \text{...(ii)}$$

$$\frac{\partial y(x, t)}{\partial x} = Ak \cos(kx - \omega t) \quad \text{...(iii)}$$

Now, these can be written as

$$\frac{\partial y(x, t)}{\partial t} = -\left(\frac{\omega}{k}\right)\frac{\partial y(x, t)}{\partial x}$$

Here, $$\frac{\partial y(x, t)}{\partial t} = \text{particle velocity } v_P$$

$$\frac{\omega}{k} = \text{wave velocity } v$$

and $$\frac{\partial y(x, t)}{\partial x} = \text{slope of the wave}$$

Thus, $$\boxed{v_P = -v\,(\text{slope})} \quad \text{...(iv)}$$

i.e. particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.

Acceleration of the particle is the second partial derivative of $y(x, t)$ with respect to t,

$$\therefore \quad a_P = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y(x, t)$$

i.e. acceleration of the particle equals $-\omega^2$ times its displacement, which is the same result we obtained in SHM. Thus,

$$\boxed{a_P = -\omega^2\,(\text{displacement})} \quad \text{...(v)}$$

We can also show that,

$$\frac{\partial^2 y(x,t)}{\partial t^2}=\left(\frac{\omega^2}{k^2}\right)\cdot\frac{\partial^2 y(x,t)}{\partial x^2}$$

or $$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2}=v^2\frac{\partial^2 y(x,t)}{\partial x^2}} \quad \ldots\text{(vi)}$$

which is also the wave equation.

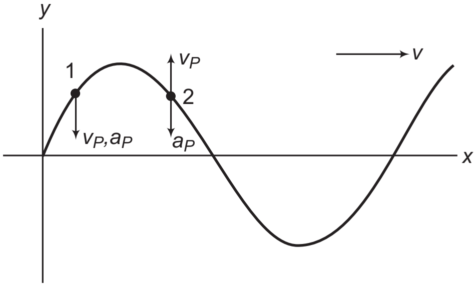


Fig. 17.7

Fig. 17.7 shows velocity (v_P) and acceleration (a_P) given by Eqs. (iv) and (v) for two points 1 and 2 on a string. The sinusoidal wave is travelling along positive x-direction.

At 1 Slope of the curve is positive. Hence, from Eq. (iv) particle velocity (v_P) is negative or downwards. Similarly, displacement of the particle is positive, so from Eq. (v) acceleration will also be negative or downwards.

At 2 Slope is negative while displacement is positive. Hence, v_P will be positive (upwards) and a_P is negative (downwards).

Note *Direction of v_P will change if the wave travels along negative x-direction.*

Example 17.5 *Under what condition, maximum particle velocity is four times the wave velocity corresponding to the equation,*

$$y=A\sin(\omega t-kx)$$

Solution Maximum particle velocity is ωA and wave velocity is $\frac{\omega}{k}$.

Now, given that

maximum particle velocity = 4 (wave velocity)

or $$\omega A=4\left(\frac{\omega}{k}\right)$$

∴ $$A=\frac{4}{k}$$

So, this is the required condition.

Example 17.6 *A transverse sinusoidal wave moves along a string in the positive x-direction at a speed of 10 cm/s. The wavelength of the wave is 0.5 m and its amplitude is 10 cm. At a particular time t, the snapshot of the wave is shown in figure. The velocity of point P when its displacement is 5 cm is*

(JEE 2008)

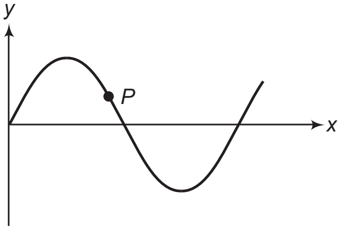


Fig. 17.8

(a) $\frac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}\ m/s$ (b) $-\frac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}\ m/s$

(c) $\frac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}\ m/s$ (d) $-\frac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}\ m/s$

Solution Particle velocity $v_P = -v$ (slope of y-x graph)

Here, $v = +$ ve, as the wave is travelling in positive x-direction.Slope at P is negative.

$\therefore$ Velocity of particle is in positive y (or $\hat{\mathbf{j}}$) direction.

$\therefore$ Correct option is (a).

Note *We can also find the magnitude of particle velocity by the relation.*

$$v_P = \omega\sqrt{A^2 - y^2}$$

Here, $A = 10$ cm $= 0.1$m, $y = 5$ cm $= 0.05$ m *and* $\omega = 2\pi f = 2\pi\left(\frac{v}{\lambda}\right) = 2\pi\left(\frac{0.1}{0.5}\right) = \frac{2\pi}{5}$ *rad* s^{-1}

Example 17.7 *Equation of a transverse wave travelling in a rope is given by*

$$y = 5\sin(4.0t - 0.02x)$$

where y and x are expressed in cm and time in seconds. Calculate

(a) the amplitude, frequency, velocity and wavelength of the wave.

(b) the maximum transverse speed and acceleration of a particle in the rope.

Solution (a) Comparing this with the standard equation of wave motion,

$$y = A\sin(\omega t - kx) = A\sin\left(2\pi ft - \frac{2\pi}{\lambda}x\right)$$

where A, f and λ are amplitude, frequency and wavelength respectively.

Thus, amplitude $A = 5$ cm

$\Rightarrow$ $2\pi f = 4 \Rightarrow$ Frequency, $f = \frac{4}{2\pi} = 0.637$ Hz

Again $\frac{2\pi}{\lambda} = 0.02$ or Wavelength, $\lambda = \frac{2\pi}{0.02} = (100\pi)$ cm

Velocity of the wave, $v = f\lambda = \frac{4}{2\pi}\ \frac{2\pi}{0.02} = 200$ cm/s **Ans.**

(b) Transverse velocity of the particle,

$$v_P = \frac{\partial y}{\partial t} = 5\times 4\cos(4.0t - 0.02x) = 20\cos(4.0t - 0.02x)$$

Maximum velocity of the particle = 20 cm/s

Particle acceleration, $a_P = \dfrac{\partial^2 y}{\partial t^2} = -20 \times 4 \sin(4.0t - 0.02x)$

Maximum particle acceleration = $80\ \text{cm/s}^2$

Example 17.8 *In the above example, find phase difference* $\Delta\phi$.

(a) of same particle at two different times with a time interval of 1 *s*

(b) of two different particles located at a distance of 10 *cm at same time*

Solution (a) $\Delta\phi = \left(\dfrac{2\pi}{T}\right)\Delta t$, but $\dfrac{2\pi}{T} = \omega = 4.0$ rad/s

$$\therefore \qquad \Delta\phi = \omega\,\Delta t = (4.0)(1.0)$$
$$= 4 \text{ rad}$$ **Ans.**

(b) $\Delta\phi = \left(\dfrac{2\pi}{\lambda}\right)\Delta x$, but $\dfrac{2\pi}{\lambda} = k = 0.02\ \text{cm}^{-1}$

$$\therefore \qquad \Delta\phi = (0.02)(10)$$
$$= 0.2 \text{ rad}$$ **Ans.**

INTRODUCTORY EXERCISE 17.3

1. The equation of a wave travelling on a string is
$$y = (0.10\ \text{mm}) \sin[(31.4\ \text{m}^{-1})x + (314\ \text{s}^{-1})t]$$
   (a) In which direction does the wave travel?
   (b) Find the wave speed, the wavelength and the frequency of the wave.
   (c) What is the maximum displacement and the maximum speed of a portion of the string?
2. The equation for a wave travelling in x-direction on a string is
$$y = (3.0\ \text{cm}) \sin[(3.14\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t].$$
   (a) Find the maximum velocity of a particle of the string.
   (b) Find the acceleration of a particle at $x = 6.0$ cm at time $t = 0.11$ s
3. The equation of a travelling wave is
$$y(x, t) = 0.02 \sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right) \text{m}$$
   Find
   (a) the wave velocity and
   (b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.
   Given $\cos\theta = -0.85$, where $\theta = 34$ rad
4. A wave of frequency 500 Hz has a wave velocity of 350 m/s.
   (a) Find the distance between two points which are 60° out of phase.
   (b) Find the phase difference between two displacements at a certain point at time 10^{-3} s apart.

17.6 Wave Speed

Wave speed (v) depends on the medium or characteristics of the medium. Waves of all frequencies or all wavelengths travel in a given medium with same speed. For examples, all electromagnetic waves travel in vacuum with same speed ($\approx 3 \times 10^8$ m/s).

In water, this speed will be different. Similarly, all longitudinal waves (including the sound wave) travel in air with the same speed (≈ 330 m/s).

Now, frequency of a wave depends on source and wavelength is self adjusted in a value,

$$\boxed{\lambda = \frac{v}{f}}$$

In the above expression, v is same (obviously in a given medium) for all sources (or frequencies). So, with increase in value of f, wavelength λ automatically decreases.

For example, usually frequency of female voice is more than frequency of male voice.

So, wavelength of female voice will be less.

Example 17.9 *Speed of sound in air is 330 m/s. Find maximum and minimum wavelength of audible sound in air.*

Solution Frequency of audible sound varies from 20 Hz to 20000 Hz.

Speed of all frequencies in air will be same (= 330 m/s).

$$\therefore \qquad \lambda_{\min} = \frac{v}{f_{\max}} = \frac{330}{20000}$$

$$= 0.0165 \text{ m}$$ **Ans.**

Similarly,

$$\lambda_{\max} = \frac{v}{f_{\min}} = \frac{330}{20}$$

$$= 16.5 \text{ m}$$ **Ans.**

INTRODUCTORY EXERCISE 17.4

1. Speed of light in vacuum is 3×10^8 m/s. Range of wavelength of visible light is 4000 Å - 7000 Å. Find the range of frequency of visible light.
2. Speed of sound in air is 330 m/s. Frequency of Anoop's voice is 1000 Hz and of Shubham's voice is 2000 Hz. Find the wavelength corresponding to their voice.

Speed of Different Waves

Normally, two wave speeds are required at this stage.

(i) Transverse wave speed on a string.

(ii) Longitudinal wave speed in all three states: solid, liquid and gas.

Transverse Wave Speed on a String

Speed of transverse wave on a string is given by

$$v = \sqrt{\frac{T}{\mu}}$$

Here, μ = mass per unit length of the string

$$= \frac{m}{l} = \frac{mA}{lA} \qquad (A = \text{area of cross-section of the string})$$

$$= \left(\frac{m}{V}\right) A \qquad (V = \text{volume of string})$$

$$= \rho A \qquad (\rho = \text{density of string})$$

Hence, the above expression can also be written as

$$\boxed{v = \sqrt{\frac{T}{\rho A}}}$$

Proof

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse is stationary and the string moves with a speed v to the left.

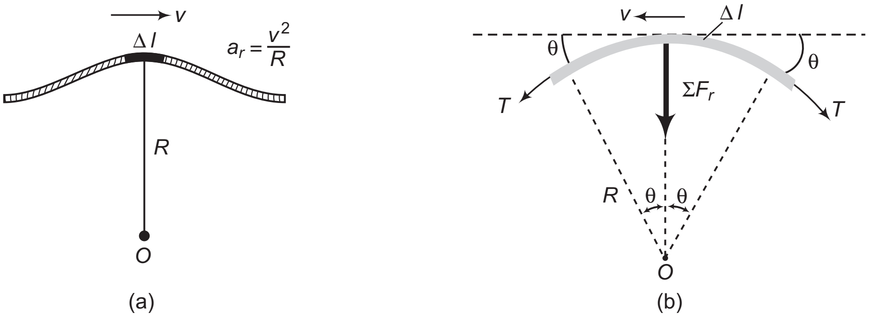


Fig. 17.9 (a) To obtain the speed v of a wave on a stretched string, it is convenient to describe the motion of a small segment of the string in a moving frame of reference. (b) In the moving frame of reference, the small segment of length Δl moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension forces are cancelled.

Figure shows a small segment of the string of length Δl. This segment forms a part of a circular arc of radius R. Instantaneously, the segment is moving with speed v in a circular path, so it has a centripetal acceleration $\frac{v^2}{R}$. The forces acting on the segment are the tension T at each end. The horizontal components of these forces are equal and opposite and thus cancel. The vertical components of these forces point radially inward toward the centre of the circular arc. These radial forces provide the centripetal acceleration. Let the angle subtended by the segment at centre be 2θ. The net radial force acting on the segment is

$$\Sigma F_r = 2T \sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta \approx \theta$ for small θ.

If μ is the mass per unit length of the string, the mass of the segment of length Δl is

$$m = \mu\Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$$

From Newton's second law, $\Sigma F_r = ma = \dfrac{mv^2}{R}$ or $2T\theta = (2\mu R\theta)\left(\dfrac{v^2}{R}\right)$

$$\therefore \qquad v = \sqrt{\frac{T}{\mu}}$$

Longitudinal Wave Speed in Three States

Speed of longitudinal wave through a gas (or a liquid) is given by

$$\boxed{v = \sqrt{\frac{B}{\rho}}}$$

Here, B = Bulk modulus of the gas (or liquid)

and ρ = density of the gas (or liquid)

Now Newton, who first deduced this relation for v, assumed that during the passage of a sound wave through a gas (or air), the temperature of the gas remains constant, i.e. sound wave travels under isothermal conditions and hence took B to be the isothermal elasticity of the gas and which is equal to its pressure p. So, **Newton's** formula for the velocity of a sound wave (or a longitudinal wave) in a gaseous medium becomes

$$\boxed{v = \sqrt{\frac{p}{\rho}}}$$

If, however, we calculate the velocity of sound in air at NTP with the help of this formula by substituting.

$$p = 1.01 \times 10^5 \text{ N/m}^2 \quad \text{and} \quad \rho = 1.29 \times 10^{-3} \text{ kg/m}^3$$

then v comes out to be nearly 280 m/s. Actually, the velocity of sound in air at NTP as measured by Newton himself, is found to be 332 m/s. Newton could not explain this large discrepancy between his theoretical and experimental results.

Laplace after 140 years correctly argued that a sound wave passes through a gas (or air) very rapidly. So, adiabatic conditions are developed. So, he took B to be the adiabatic elasticity of the gas, which is equal to γp where γ is the ratio of C_p (molar heat capacity at constant pressure) and C_V (molar heat capacity at constant volume). Thus, Newton's formula as corrected by Laplace becomes

$$\boxed{v = \sqrt{\frac{\gamma p}{\rho}}}$$

For air, $\gamma = 1.41$, so that in air, $v = \sqrt{\dfrac{1.41\,p}{\rho}}$

which gives 331.6 m/s as the velocity of sound (in air) at NTP which is in agreement with Newton's experimental result.

Speed of longitudinal wave in a thin rod or wire is given by

$$\boxed{v = \sqrt{\frac{Y}{\rho}}}$$

Here, Y is the Young's modulus of elasticity.

Note *In the chapter of sound wave (Chapter-19), we will discuss longitudinal wave speed in detail (with proof and examples). In the present chapter, we are taking examples of only transverse wave speed on a string.*

Example 17.10 *One end of 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to a fixed support. A cord attached to the other end passes over a pulley and supports an object with a mass of 5.0 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end.* $(g = 9.8\ m/s^2)$

Solution Tension in the rubber tube AB, $T = mg$

or $$T = (5.0)(9.8) = 49\text{ N}$$

Mass per unit length of rubber tube, $\mu = \dfrac{0.9}{12} = 0.075\text{ kg/m}$

$\therefore$ Speed of wave on the tube, $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{49}{0.075}} = 25.56\text{ m/s}$

$\therefore$ The required time is, $t = \dfrac{AB}{v} = \dfrac{12}{25.56} = 0.47\text{ s}$ **Ans.**

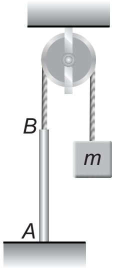


Fig. 17.10

Example 17.11 *A wire of uniform cross-section is stretched between two points 100 cm apart. The wire is fixed at one end and a weight is hung over a pulley at the other end. A weight of 9 kg produces a fundamental frequency of 750 Hz.*

(a) What is the velocity of the wave in wire?

(b) If the weight is reduced to 4 kg, what is the velocity of wave?

Note *Fundamental frequency is given by*

$$f = \frac{v}{2L}$$

Solution (a) $L = 100\text{ cm}$, $f_1 = 750\text{ Hz}$

$$v_1 = 2Lf_1 = 2 \times 100 \times 750$$

$$= 150000\text{ cms}^{-1} = 1500\text{ ms}^{-1}$$ **Ans.**

(b) $v_1 = \sqrt{\dfrac{T_1}{\mu}}$ and $v_2 = \sqrt{\dfrac{T_2}{\mu}}$

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

$\therefore$ $$\frac{v_2}{1500} = \sqrt{\frac{4}{9}}$$

$\therefore$ $$v_2 = 1000\text{ ms}^{-1}$$ **Ans.**

INTRODUCTORY EXERCISE 17.5

1. Figure shows a string of linear mass density $1.0\ \text{g cm}^{-1}$ on which a wave pulse is travelling. Find the time taken by the pulse in travelling through a distance of 50 cm on the string. Take $g = 10\ \text{ms}^{-2}$.

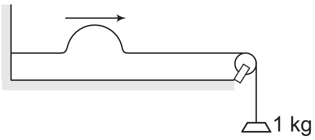


Fig. 17.11

2. A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8 N, what would be the speed of a transverse wave passing on it?

3. Two blocks each having a mass of 3.2 kg are connected by a wire *CD* and the system is suspended from the ceiling by another wire *AB*. The linear mass density of the wire *AB* is $10\ \text{gm}^{-1}$ and that of *CD* is $8\ \text{g m}^{-1}$. Find the speed of a transverse wave pulse produced in *AB* and in *CD*.

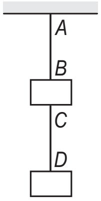


Fig. 17.12

4. In the arrangement shown in figure, the string has a mass of 4.5 g. How much time will it take for a transverse disturbance produced at the floor to reach the pulley? Take $g = 10\ \text{ms}^{-2}$.

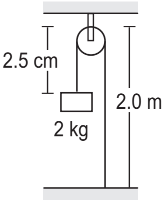


Fig. 17.13

5. A copper wire 2.4 mm in diameter is 3 m long and is used to suspend a 2 kg mass from a beam. If a transverse disturbance is sent along the wire by striking it lightly with a pencil, how fast will the disturbance travel? The density of copper is $8920\ \text{kg/m}^3$.

6. One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates at 120 Hz. The other end passes over a pulley and supports a 1.50 kg mass. The linear mass density of the rope is 0.0550 kg/m.
   (a) What is the speed of a transverse wave on the rope?
   (b) What is the wavelength?
   (c) How would your answers to parts (a) and (b) change if the mass were increased to 3.00 kg?

17.7 Energy in Wave Motion

In a wave, many particles oscillate. In a sinusoidal wave, these oscillations are simple harmonic in nature. Each particle has some energy of oscillation. At the same time, energy transfer also takes place. Related to energy of oscillation and energy transfer, there are three terms, namely, energy density (u), power (P) and intensity (I).

Energy density (u)

Energy of oscillation per unit volume is called energy density (u). Its SI unit is $\mathrm{J/m^3}$. In case of SHM, energy of oscillation (of a single particle) is $E = \frac{1}{2} m\omega^2 A^2$. In a sinusoidal wave, each particle of the string oscillates simple harmonically.

Therefore,

$$\text{Energy density} = \frac{\text{energy of oscillation}}{\text{volume}} \quad \text{or} \quad u = \frac{E}{V} = \frac{\frac{1}{2} m\omega^2 A^2}{V}$$

But, $$\frac{m}{V} = \text{density or } \rho$$

∴ $$\boxed{u = \frac{1}{2} \rho\omega^2 A^2}$$

Power (P)

Energy transferred per unit time is called power. The expression of power is

$$\boxed{P = \frac{1}{2} \rho\omega^2 A^2 S v}$$

Here, S is area of cross-section of the medium in which wave is travelling and v is the wave speed. SI unit of power is J/s or Watt. Now, let us derive the above expression for a sinusoidal travelling wave on a string. Area of cross-section of string is S and the wave speed is v.

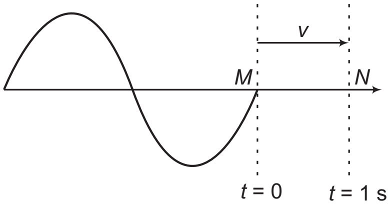


Fig. 17.14

Suppose at time $t = 0$, wave is at point M. In 1 s, it will travel a distance v and it will reach at point N. Or, we can say that v new length of string will start oscillating. Area of cross-section of string is S. Therefore, in 1 s, Sv new volume of string will start oscillating. Energy of oscillation per unit volume is called energy density (u). So, in 1 s, (uSv) new energy will be added or (uSv) energy will be transferred from the source. Energy transferred in one second is called power.

$$\therefore \quad P = uSv$$
$$= \frac{1}{2} \rho\omega^2 A^2 S v \qquad \left(\text{as } u = \frac{1}{2} \rho\omega^2 A^2\right)$$

or $$P = \frac{1}{2} \rho\omega^2 A^2 S v$$

This is the desired expression of power.

Intensity (*I*)

Energy transferred per unit cross-sectional area per unit time is called intensity.

Thus,

$$I = \frac{\text{Energy transferred}}{(\text{time})(\text{cross - sectional area})} = \frac{\text{Power}}{\text{cross - sectional area}} = \frac{P}{S}$$

$$= \frac{\frac{1}{2}\rho\omega^2 A^2 S v}{S} \quad \text{or} \quad \boxed{I = \frac{1}{2}\rho\omega^2 A^2 v}$$

The SI unit of intensity is J/s-m^2 or Watt/m^2.

Extra Points to Remember

- Although the above relations for power and intensity have been discussed for a transverse wave on a string, they hold good for other waves also.
- In SHM, potential energy is maximum (and kinetic energy is zero) at the extreme positions.
- For a string segment, the potential energy depends on the slope of the string and is maximum when the slope is maximum, which is at the equilibrium position of the segment, the same position for which the kinetic energy is maximum.

  At *A* : Kinetic energy and potential energy both are zero.

  At *B* : Kinetic energy and potential energy both are maximum.

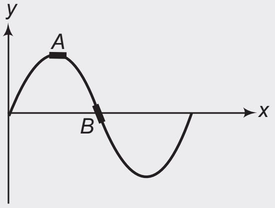


Fig. 17.15

- **Intensity due to a point source** If a point source emits wave uniformly in all directions, the energy at a distance *r* from the source is distributed uniformly on a spherical surface of radius *r* and area $S = 4\pi r^2$. If *P* is the power emitted by the source, the power per unit area at a distance *r* from the source is $\frac{P}{4\pi r^2}$. The average power per unit area that is incident perpendicular to the direction of propagation is called the intensity. Therefore,

$$I = \frac{P}{4\pi r^2} \quad \text{or} \quad I \propto \frac{1}{r^2}$$

Example 17.12 *A stretched string is forced to transmit transverse waves by means of an oscillator coupled to one end. The string has a diameter of 4 mm. The amplitude of the oscillation is 10^{-4} m and the frequency is 10 Hz. Tension in the string is 100 N and mass density of wire is $4.2 \times 10^3 \text{ kg/m}^3$. Find*

(a) the equation of the waves along the string
(b) the energy per unit volume of the wave
(c) the average energy flow per unit time across any section of the string and
(d) power required to drive the oscillator.

Solution (a) Speed of transverse wave on the string is

$$v = \sqrt{\frac{T}{\rho S}} \qquad (\text{as } \mu = \rho S)$$

Substituting the values, we have

$$v = \sqrt{\frac{100}{(4.2 \times 10^3)\left(\frac{\pi}{4}\right)(4.0 \times 10^{-3})^2}} = 43.53 \text{ m/s}$$

$$\omega = 2\pi f = 20\pi = 62.83 \text{ rad/s}$$

$$k = \frac{\omega}{v} = 1.44 \text{ m}^{-1}$$

$\therefore$ Equation of the waves along the string, $y(x, t) = A \sin(kx - \omega t)$

$$= (10^{-4} \text{ m}) \sin [(1.44 \text{ m}^{-1})x - (62.83 \text{ rad/s})t]$$ **Ans.**

(b) Energy per unit volume of the string, u = energy density $= \frac{1}{2}\rho\omega^2 A^2$

Substituting the values, we have

$$u = \left(\frac{1}{2}\right)(4.2 \times 10^3)(62.83)^2 (10^{-4})^2$$

$$= 8.29 \times 10^{-2} \text{ J/m}^3$$ **Ans.**

(c) Average energy flow per unit time, $P = \text{power} = \left(\frac{1}{2}\rho\omega^2 A^2\right)(Sv) = (u)(Sv)$

Substituting the values, we have $P = (8.29 \times 10^{-2})\left(\frac{\pi}{4}\right)(4.0 \times 10^{-3})^2 (43.53)$

$$= 4.53 \times 10^{-5} \text{ J/s}$$ **Ans.**

(d) Therefore, power required to drive the oscillator is 4.53×10^{-5} W. **Ans.**

INTRODUCTORY EXERCISE 17.6

1. Spherical waves are emitted from a 1.0 W source in an isotropic non-absorbing medium. What is the wave intensity 1.0 m from the source?
2. A line source emits a cylindrical expanding wave. Assuming the medium absorbs no energy, find how the amplitude and intensity of the wave depend on the distance from the source?
3. A certain 120 Hz wave on a string has an amplitude of 0.160 mm. How much energy exists in an 80 g length of the string?
4. A taut string for which $\mu = 5.00 \times 10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?
5. A 200 Hz wave with amplitude 1 mm travels on a long string of linear mass density 6 g/m kept under a tension of 60 N.
   (a) Find the average power transmitted across a given point on the string.
   (b) Find the total energy associated with the wave in a 2.0 m long portion of the string.
6. A transverse wave of amplitude 0.50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m/s. What average power is the source transmitting to the wire?

List of Formulae

1. **Wave Equation** Only those functions of x and t or $y(x, t)$ represent a wave function which satisfies the following three conditions

   Condition 1
   $$\left(\frac{\partial^2 y}{\partial t^2}\right) = k\left(\frac{\partial^2 y}{\partial x^2}\right)$$

   Here, $k = v^2$ (v = wave speed)

   Condition 2 $y(x, t)$ should be a single valued function for all values of x and t.

   Condition 3 The wave function and its first derivative must be continuous.

2. If $y(x, t)$ is of type $f(ax \pm bt)$, then

   $$\text{wave speed } v = \frac{\text{coefficient of } t}{\text{coefficient of } x}$$

   Further, wave travels in positive x-direction if ax and bt are of opposite sign and it travels along negative x-direction if ax and bt are of same sign.

3. **Sine wave** General equation of this wave is

   $$y = A\sin(\omega t \pm kx \pm \phi)$$

   or
   $$y = A\cos(\omega t \pm kx \pm \phi)$$

   In these equations,

   (i) A is amplitude of oscillation,

   (ii) ω is angular frequency,

   $$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad \omega = 2\pi f$$

   and
   $$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

   (iii) k is angular wave number,

   $$k = \frac{2\pi}{\lambda}$$ ($\lambda \rightarrow$ wavelength)

   (iv) Wave speed, $v = \frac{\omega}{k} = f\lambda$

   (v) ϕ is initial phase angle at $x = 0$ and

   (vi) $(\omega t \pm kx \pm \phi)$ is phase angle at time t at coordinate x.

4. **Particle velocity (v_P) and wave velocity (v) in sine wave**

   (i) $y = f(x, t)$

   Then,
   $$v_P = \frac{\partial y}{\partial x}$$

   (ii) In sine wave, particles are executing SHM. Therefore, all equations of SHM can be applied for particles also.

   (iii) Relation between v_P and v

   $$v_P = -v \cdot \frac{\partial y}{\partial x}$$

   Here, $\partial y / \partial x$ is the slope of y - x graph when t is kept constant.

5. Phase Difference ($\Delta\phi$)

Case I $\Delta\phi = \omega(t_1 - t_2)$

or $\Delta\phi = \frac{2\pi}{T} \cdot \Delta t$

$=$ phase difference of one particle at a time interval of Δt.

Case II $\Delta\phi = k(x_1 \sim x_2)$

$= \frac{2\pi}{\lambda} \cdot \Delta x$

$=$ phase difference at one time between two particles at a path difference of Δx.

6. Wave Speed

(i) Speed of transverse wave on a stretched wire

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho S}}$$

(ii) Speed of longitudinal wave

$$v = \sqrt{\frac{E}{\rho}}$$

(a) In solids, $E = Y =$ Young's modulus of elasticity

$\therefore$ $$v = \sqrt{\frac{Y}{\rho}}$$

(b) In liquids, $E = B =$ Bulk modulus of elasticity

$\therefore$ $$v = \sqrt{\frac{B}{\rho}}$$

(c) In gases, $E =$ adiabatic bulk modulus $= \gamma p$

$\therefore$ $$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

7. Energy Density (u), Power (P) and Intensity (I) in sine Wave

(i) Energy density, $u = \frac{1}{2}\rho\omega^2 A^2 =$ energy of oscillation per unit volume

(ii) Power, $P = \frac{1}{2}\rho\omega^2 A^2 S v =$ energy transferred per unit time

(iii) Intensity, $I = \frac{1}{2}\rho\omega^2 A^2 v =$ energy transferred per unit time per unit area

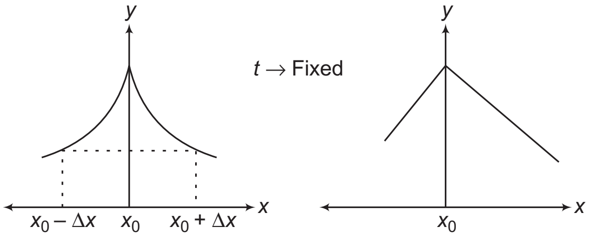
Solved Examples

TYPED PROBLEMS

Type 1. *Based on symmetry of a wave pulse*

Concept

To check symmetry of a wave pulse or wave velocity, always concentrate on maximum or minimum value of y. For example,



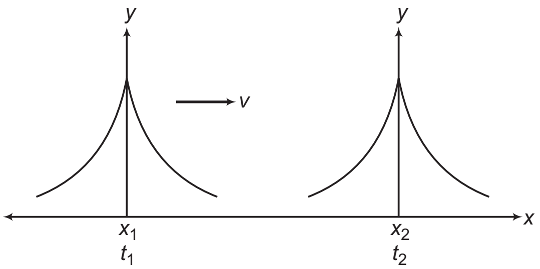
(i) (ii)

Wave pulse shown in Fig. (i) is symmetric but the wave pulse shown in Fig. (ii) is asymmetric. Now, the question is how will you check whether the pulse is symmetric or not.

The answer is

In both cases maximum value of y at the given time (say t_0) is at $x = x_0$.

Now, if $y(t = t_0, x = x_0 + \Delta x) = y(t = t_0, x = x_0 - \Delta x)$, then the pulse is symmetric otherwise not. So, at a given time (or any other time of your choice) first of all you have to find that x coordinate where you are getting the maximum value of y.



Further, suppose that maximum value of y is at x_1 at time t_1 and at x_2 at time $t_2 (> t_1)$. Then, from the figure we can see that peak of the wave pulse is travelling towards positive x-direction. In time $(t_2 - t_1)$, it has travelled a distance $(x_2 - x_1)$ along positive x-direction. Hence, the wave velocity is

$$v = + \frac{(x_2 - x_1)}{(t_2 - t_1)}$$

Example 1 *$y(x,t) = \dfrac{0.8}{[(4x+5t)^2+5]}$ represents a moving pulse where x and y are in metre and t in second. Then, choose the correct alternative(s):* **(JEE 1999)**

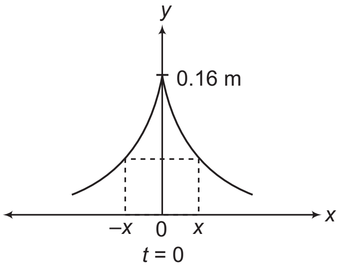
(a) pulse is moving in positive x-direction

(b) in 2 s it will travel a distance of 2.5 m

(c) its maximum displacement is 0.16 m

(d) it is a symmetric pulse

Solution (b), (c) and (d) are correct options.



The shape of pulse at $x=0$ and $t=0$ would be as shown in figure

$$y\,(0,0) = \frac{0.8}{5} = 0.16 \text{ m}$$

From the equation it is clear that $y_{\max} = 0.16$ m

Pulse will be symmetric (symmetry is checked about $y_{\max}$) if

At $t=0$; $\quad y\,(x) = y\,(-x)$

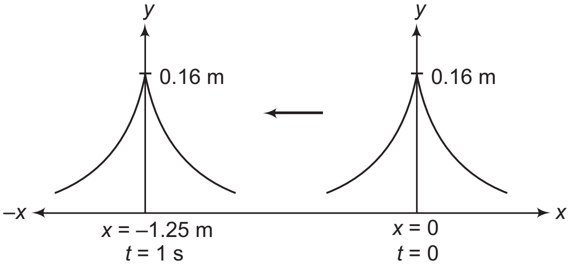
From the given equation

$$\left.\begin{aligned} y\,(x) &= \frac{0.8}{16x^2+5} \\ \text{and} \qquad y\,(-x) &= \frac{0.8}{16x^2+5} \end{aligned}\right\} \text{at } t=0$$

or $$y\,(x) = y\,(-x)$$

Therefore, pulse is symmetric.

Speed of pulse



At $t = 1$ s and $x = -1.25$ m, value of y is again 0.16 m, i.e. pulse has travelled a distance of 1.25 m in 1 second in negative x-direction or we can say that the speed of pulse is 1.25 m/s and it is travelling in negative x-direction. Therefore, it will travel a distance of 2.5 m in 2 seconds.

Type 2. *To find wave velocity from two $y(x)$ equations given at two different times.*

Concept

In $y(x,t)$ equation, if value of t is substituted then the equation left is $y(x)$ equation.

In this type of problem, $y(x)$ equation will be given at two different times and we have to find the wave velocity.

How to Solve?

- At the given time find the x-coordinates where you are getting the maximum (or minimum) value of y. From these two x-coordinates and two times we can calculate the wave velocity by the method discussed in Type 1.

Example 2 *At time $t=0$, $y(x)$ equation of a wave pulse is*

$$y=\frac{10}{2+(x-2)^2}$$

and at $t=2\,s$, $y(x)$ equation of the same wave pulse is

$$y=\frac{10}{2+(x+4)^2}$$

Here, y is in mm and x in metres. Find the wave velocity.

Solution From the given $y(x)$ equations at two different times we can see that value of y is maximum $\left(=\frac{10}{2}\text{ or }5\text{ mm}\right)$ at $x=2\,\text{m}$ at time $t=0$ and at $x=-4\,\text{m}$ at time $t=2\,\text{s}$.

So, peak of the wave pulse has travelled a distance of 6 m (from $x=2$ m to $x=-4$ m) in 2 s along negative x-direction.

Hence, the wave velocity is

$$v=-\frac{6}{2}=-3\text{ m/s}$$ **Ans.**

Type 3. *To make complete $y(x,t)$ function if $y(x)$ function at some given time and wave velocity are given*

Concept

(i) Wave speed, $v=\dfrac{\text{coefficient of } t}{\text{coefficient of } x}$

$\therefore$ Coefficient of $t=(v)(\text{coefficient of } x)$

(ii) Sign of coefficient of t will be opposite to the sign of coefficient of x if wave travels along positive x-direction and they are of same sign if the wave travels along negative x-direction.

Example 3 *A wave is travelling along positive x-direction with velocity 2 m/s. Further, $y(x)$ equation of the wave pulse at $t = 0$ is*

$$y = \frac{10}{2 + (2x + 4)^2}$$

(a) From the given information make complete $y(x, t)$ equation.

(b) Find $y(x)$ equation at $t = 1$ s

Solution (a) Here, coefficient to x is 2.

Wave speed is 2 m/s. Therefore, coefficient of $t = v$ (coefficient of x) $= 2 \times 2 = 4$ units.

Further, coefficient of x is positive and the wave is travelling along positive x-direction. Hence, coefficient of t must be negative.

Now, suppose the $y(x, t)$ function is

$$y = \frac{10}{2 + (2x - 4t + \alpha)^2} \qquad \text{...(i)}$$

Here, α is a constant.

At time $t = 0$, Eq. (i) becomes

$$y = \frac{10}{2 + (2x + \alpha)^2}$$

and the given function is

$$y = \frac{10}{2 + (2x + 4)^2}$$

Therefore, the value of α is 4.

Substituting in Eq. (i), we have

$$y = \frac{10}{2 + (2x - 4t + 4)^2}$$

(b) At $t = 1$ s

$$y = \frac{10}{2 + (2x - 4 \times 1 + 4)^2}$$

or

$$y = \frac{10}{2 + 4x^2}$$

Ans.

Type 4. *Based on transverse wave speed on a string*

Concept

(i) We know that transverse wave speed is given by

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad \sqrt{\frac{T}{\rho S}}$$

(ii) If tension is uniform, then v is also uniform and we can calculate the time taken by the wave pulse in travelling from one point to another point by the direct relation

$$t = \frac{\text{distance}}{\text{speed}}$$

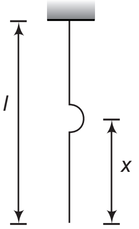
If tension is non-uniform, then v will be non-uniform and in that case time can be obtained by integration.

Example 4 *A uniform rope of mass* 0.1 *kg and length* 2.45 *m hangs from a ceiling.*

(a) Find the speed of transverse wave in the rope at a point 0.5 *m distant from the lower end.*

(b) Calculate the time taken by a transverse wave to travel the full length of the rope.

Solution (a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So, if m is the mass of string of length l, the mass of length x of the string will be $\left(\frac{m}{l}\right)x$.



$$\therefore \quad T = \left(\frac{m}{l}\right) xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$$

$$\therefore \quad \frac{T}{\mu} = xg$$

$$\text{or} \quad v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \qquad \text{...(i)}$$

$$\text{At } x = 0.5 \text{ m}, \quad v = \sqrt{0.5 \times 9.8}$$

$$= 2.21 \text{ m/s}$$

(b) From Eq. (i), we can see that velocity of the wave is different at different points. So, if at point x the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$$\therefore \quad \int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

$$\text{or} \quad t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}}$$

$$= 1.0 \text{ s}$$

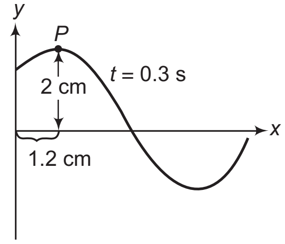
Ans.

Type 5. *To write the equation corresponding to given y - x graph (or a snapshot) at a given time*

Concept

From the given y - x graph, we can easily determine A, ω and k. Secondly, we have to check whether the wave is travelling along positive x -direction or negative x -direction. Because this factor will decide whether, ωt and kx should be of same sign or opposite sign.

Example 5 *Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at $x = 0$ at $t = 0$, write the equation of travelling wave.*



Solution Given, $A = 2$ cm, $\lambda = 7.5$ cm

$$\therefore \qquad k = \frac{2\pi}{\lambda} = 0.84 \text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave,

$$v = \frac{1.2}{0.3} = 4 \text{ cm/s}$$

$\therefore$ Angular frequency $\omega = (v)(k) = 3.36$ rad/s

Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as

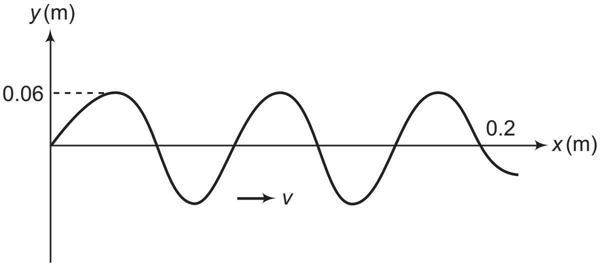
$$y(x, t) = A \cos (kx - \omega t)$$

or $$y(x, t) = A \cos (\omega t - kx) \quad \text{as} \quad \cos(-\theta) = \cos\theta$$

Therefore, the desired equation is

$$y(x, t) = (2 \text{ cm}) \cos [(0.84 \text{ cm}^{-1})\, x - (3.36 \text{ rad/s})\, t] \text{ cm}$$ **Ans.**

Example 6 *For the wave shown in figure, write the equation of this wave if its position is shown at $t = 0$. Speed of wave is $v = 300$ m/s.*



Solution The amplitude $A = 0.06$ m

$$\frac{5}{2}\lambda = 0.2 \text{ m}$$

$$\therefore \qquad \lambda = 0.08 \text{ m}$$

$$f = \frac{v}{\lambda} = \frac{300}{0.08} = 3750 \text{ Hz}$$

$$k = \frac{2\pi}{\lambda} = 78.5 \text{ m}^{-1}$$

and $\omega = 2\pi f = 23562$ rad/s

At $t = 0$, $x = 0$, $\frac{\partial y}{\partial x} = \text{positive}$

and the given curve is a sine curve.

Hence, equation of wave travelling in positive x-direction should have the form,

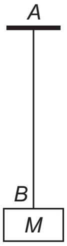
$$y(x, t) = A \sin (kx - \omega t)$$

Substituting the values, we have

$$y(x, t) = (0.06 \text{ m}) \sin [(78.5 \text{ m}^{-1})\, x - (23562 \text{ s}^{-1})\, t] \text{ m}$$ **Ans.**

Miscellaneous Examples

Example 7 *A block of mass $M = 2$ kg is suspended from a string AB of mass 6 kg as shown in figure. A transverse wave pulse of wavelength λ_0 is produced at point B. Find its wavelength while reaching at point A.*



Concept

While moving from B to A, tension will increase. So, wave speed will also increase (as $v = \sqrt{T/\mu}$). Frequency will remain unchanged because it depends on source. Therefore, wavelength will also increase (as $\lambda = \frac{v}{f}$ or $\lambda \propto v$).

Solution $\lambda = \frac{v}{f} = \frac{\sqrt{T/\mu}}{f}$

or $\lambda \propto \sqrt{T}$ (as μ and f are constants)

$\therefore$ $$\frac{\lambda_B}{\lambda_A} = \sqrt{\frac{T_B}{T_A}}$$

$\therefore$ $$\lambda_B = \left(\sqrt{\frac{T_B}{T_A}}\right) \lambda_A$$

$$= \left[\sqrt{\frac{(2+6)g}{2g}}\right]$$

$$= 2\lambda$$ **Ans.**

Example 8 *A wave moves with speed* 300 *m/s on a wire which is under a tension of* 500 *N. Find how much tension must be changed to increase the speed to* 312 *m/s?*

Solution Speed of a transverse wave on a wire is,

$$v = \sqrt{\frac{T}{\mu}} \qquad \text{...(i)}$$

Differentiating with respect to tension, we have

$$\frac{dv}{dT} = \frac{1}{2\sqrt{\mu T}} \qquad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we have

$$\frac{dv}{v} = \frac{1}{2}\frac{dT}{T} \quad \text{or} \quad dT = (2T)\frac{dv}{v}$$

Substituting the proper values, we have

$$dT = \frac{(2)(500)(312 - 300)}{300}$$

$$= 40 \text{ N}$$ **Ans.**

i.e. tension should be increased by 40 N.

Example 9 *For a wave described by* $y = A \sin(\omega t - kx)$*, consider the following points (a)* $x = 0$ *(b)* $x = \frac{\pi}{4k}$ *(c)* $x = \frac{\pi}{2k}$ *and (d)* $x = \frac{3\pi}{4k}$.

For a particle at each of these points at $t = 0$*, describe whether the particle is moving or not and in what direction and describe whether the particle is speeding up, slowing down or instantaneously not accelerating?*

Solution $y = A \sin(\omega t - kx)$

Particle velocity $v_P(x, t) = \frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx)$ and particle acceleration

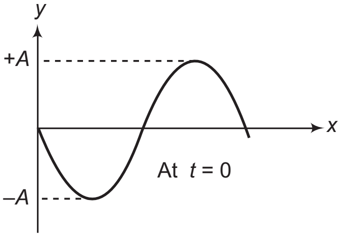
$$a_P(x, t) = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(\omega t - kx)$$

(a) $t = 0,\ x = 0:$ $v_P = +\omega A$ and $a_P = 0$

i.e. particle is moving **upwards** but its **acceleration is zero**.

Note *Direction of velocity can also be obtained in a different manner as under,*

At $t = 0$, $y = A \sin(-kx) = -A \sin kx$



i.e. y-x graph is as shown in figure. At $x = 0$*, slope is negative. Therefore, particle velocity is positive* ($v_P = -v \times$ *slope), as the wave is travelling along positive x-direction.*

(b) $t = 0, x = \frac{\pi}{4k}$ $\quad x = \frac{\pi}{4k}$

$\therefore$ $$kx = \frac{\pi}{4}$$

$$v_P = \omega A \cos\left(-\frac{\pi}{4}\right) = +\frac{\omega A}{\sqrt{2}}$$

and $$a_P = -\omega^2 A \sin\left(-\frac{\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is positive, i.e. the particle is moving **upwards** (along positive y-direction). Further v_P and a_P are in the same direction (both are positive). Hence, the particle is **speeding up**.

(c) $t = 0, x = \frac{\pi}{2k}$ $\quad x = \frac{\pi}{2k} \quad \therefore \quad kx = \frac{\pi}{2}$

$$v_P = \omega A \cos(-\pi/2) = 0$$

$$a_P = -\omega^2 A \sin(-\pi/2) = \omega^2 A$$

i.e. particle is **stationary** or at its extreme position ($y = -A$). So, it is **speeding up** at this instant.

(d) $t = 0, \ x = \frac{3\pi}{4k}$ $\quad x = \frac{3\pi}{4k} \quad \therefore \quad kx = \frac{3\pi}{4}$

$\therefore$ $$v_P = \omega A \cos\left(-\frac{3\pi}{4}\right) = -\frac{\omega A}{\sqrt{2}}$$

$$a_P = -\omega^2 A \sin\left(-\frac{3\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is negative, i.e. the particle is moving **downwards**. Further v_P and a_P are in opposite directions, i.e. the particle is **slowing down**.

Example 10 *A thin string is held at one end and oscillates vertically so that,*

$$y\,(x = 0, t) = 8 \sin 4t \text{ (cm)}$$

Neglect the gravitational force. The string's linear mass density is 0.2 kg/m and its tension is 1 N. The string passes through a bath filled with 1 kg water. Due to friction heat is transferred to the bath. The heat transfer efficiency is 50%. Calculate how much time passes before the temperature of the bath rises one degree kelvin?

Solution Comparing the given equation with equation of a travelling wave,

$$y = A \sin(kx \pm \omega t) \quad \text{at} \quad x = 0 \quad \text{we find,}$$

$$A = 8 \text{ cm} = 8 \times 10^{-2} \text{ m}$$

$$\omega = 4 \text{ rad/s}$$

Speed of travelling wave, $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1}{0.2}} = 2.236 \text{ m/s}$$

Further, $\rho S = \mu = 0.2$ kg/m

The average power over a period is

$$P = \frac{1}{2}(\rho S)\,\omega^2 A^2 v$$

Substituting the values, we have

$$P = \frac{1}{2}(0.2)(4)^2(8\times10^{-2})^2(2.236)$$

$$= 2.29\times10^{-2}\text{ J/s}$$

The power transferred to the bath is,

$$P' = 0.5\,P = 1.145\times10^{-2}\text{ J/s}$$

Now let, it takes t second to raise the temperature of 1 kg water by 1 degree kelvin. Then

$$P'\,t = ms\Delta t$$

Here, s = specific heat of water $= 4.2\times10^3$ J/kg-° K

$$\therefore \qquad t = \frac{ms\Delta t}{P'} = \frac{(1)(4.2\times10^3)(1)}{1.145\times10^{-2}}$$

$$= 3.6\times10^5\text{ s} \approx 4.2\text{ day}$$ **Ans.**

Example 11 *Consider a wave propagating in the negative x-direction whose frequency is 100 Hz. At t = 5 s, the displacement associated with the wave is given by*

$$y = 0.5\cos(0.1\,x)$$

where x and y are measured in centimetres and t in seconds. Obtain the displacement (as a function of x) at t = 10 s. What is the wavelength and velocity associated with the wave?

Solution A wave travelling in negative x-direction can be represented as

$$y(x, t) = A\cos(kx + \omega t + \phi)$$

At $t = 5$ s,

$$y(x, t = 5) = A\cos(kx + 5\omega + \phi)$$

Comparing this with the given equation,

We have,

$$A = 0.5\text{ cm}, k = 0.1\text{ cm}^{-1}$$

and $$5\omega + \phi = 0 \qquad \text{...(i)}$$

Now, $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.1} = (20\pi)\text{ cm}$$ **Ans.**

$$\omega = 2\pi f = (200\pi)\text{ rad/s}$$

$$\therefore \qquad v = \frac{\omega}{k} = \frac{200\pi}{0.1} = (2000\,\pi)\text{ cm/s}$$ **Ans.**

From Eq. (i), $$\phi = -5\omega$$

At $t = 10$ s,

$$y(x, t = 10) = 0.5\cos(0.1\,x + 10\,\omega - 5\,\omega)$$

$$= 0.5\cos(0.1\,x + 5\,\omega)$$

Substituting $\omega = 200\,\pi$,

$$y(x, t = 10) = 0.5\cos(0.1\,x + 1000\pi)$$

$$= 0.5\cos(0.1\,x)$$ **Ans.**

Example 12 *A simple harmonic wave of amplitude 8 units travels along positive x-axis. At any given instant of time, for a particle at a distance of 10 cm from the origin, the displacement is + 6 units, and for a particle at a distance of 25 cm from the origin, the displacement is + 4 units. Calculate the wavelength.*

Solution $y = A \sin \frac{2\pi}{\lambda}(vt - x)$

or $$\frac{y}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

In the first case,

$$\frac{y_1}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

Here, $y_1 = +6, \quad A = 8, \quad x_1 = 10$ cm

$\therefore$ $$\frac{6}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) \qquad \text{...(i)}$$

Similarly in the second case,

$$\frac{4}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \text{...(ii)}$$

From Eq. (i),

$$2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) = \sin^{-1}\left(\frac{6}{8}\right) = 0.85 \text{ rad}$$

or $$\frac{t}{T} - \frac{10}{\lambda} = 0.14 \qquad \text{...(iii)}$$

Similarly from Eq. (ii),

$$2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) = \sin^{-1}\left(\frac{4}{8}\right) = \frac{\pi}{6} \text{ rad}$$

or $$\frac{t}{T} - \frac{25}{\lambda} = 0.08 \qquad \text{...(iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$\frac{15}{\lambda} = 0.06$$

$\therefore$ $$\lambda = 250 \text{ cm}$$ **Ans.**

Example 13 *A wave pulse on a horizontal string is represented by the function*

$$y(x, t) = \frac{5.0}{1.0 + (x - 2t)^2} \qquad (\textit{CGS units})$$

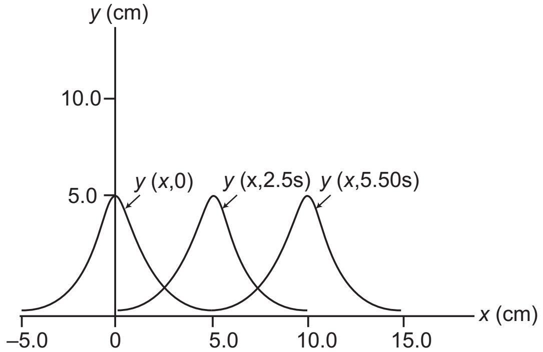
Plot this function at $t = 0$, 2.5 and 5.0 s.

Solution At the given times the function representing the wave pulse is

$$y\,(x, 0) = \frac{5.0}{1.0 + x^2}$$

$$y\,(x, 2.5 \text{ s}) = \frac{5.0}{1.0 + (x - 5.0)^2}$$

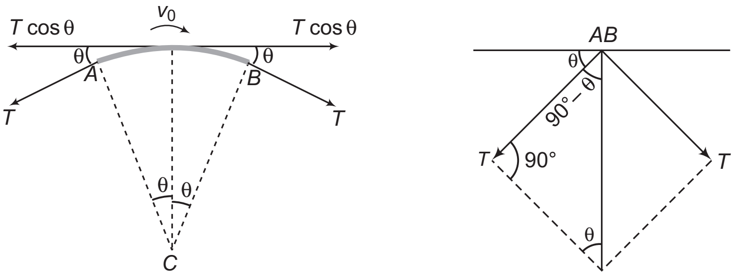
$$y\,(x, 5.0\text{ s}) = \frac{5.0}{1.0 + (x - 10.0)^2}$$



The maximum of $y(x, 0)$ is 5.0 cm; at $t = 0$, it is located at $x = 0$. At $t = 2.5$ and 5.0 s, the maximum of the pulse has moved to $x = 5.0$ and 10.0 cm, respectively. So, in each 2.5 s time interval, the pulse moves 5.0 cm in the positive x-direction. Its velocity is therefore +2.0 cm/s.

Example 14 *A uniform circular hoop of string is rotating clockwise in the absence of gravity. The tangential speed is* v_0. *Find the speed of the wave travelling on this string.*

Solution



(a) (b)

Let T be the tension in the string. Consider a small circular element AB of the string of length,

$$\Delta l = R\,(2\theta) \qquad (R = \text{radius of hoop})$$

The components of tension $T\cos\theta$ are equal and opposite and thus cancel out. The components towards centre C (i.e. $T\sin\theta$) provides the necessary centripetal force to element AB.

$$\therefore \qquad 2T\sin\theta = \frac{mv_0^2}{R} \qquad \text{...(i)}$$

Here, $$m = \mu\Delta l = 2\mu R\theta \qquad \left(\mu = \frac{\text{mass}}{\text{length}}\right)$$

As θ is small, $$\sin\theta \approx \theta$$

Substituting in Eq. (i), we get

$$2T\theta = \frac{2\mu R\theta\, v_0^2}{R}$$

or $$\frac{T}{\mu} = v_0^2$$

or $$\sqrt{\frac{T}{\mu}} = v_0 \qquad \text{...(ii)}$$

Speed of wave travelling on this string,

$$v = \sqrt{\frac{T}{\mu}} = v_0 \qquad \text{[from Eq. (ii)]}$$

i.e. the velocity of the transverse wave along the hoop of string is the same as the velocity of rotation of the hoop, *viz.* v_0. **Ans.**

Example 15 *A sinusoidal wave travelling in the positive direction on a stretched string has amplitude* 2.0 *cm, wavelength* 1.0 *m and wave velocity* 5.0 *m/s. At* $x = 0$ *and* $t = 0$, *it is given that* $y = 0$ *and* $\frac{\partial y}{\partial t} < 0$. *Find the wave function* $y(x, t)$.

Solution We start with a general equation for a rightward moving wave,

$$y(x, t) = A \sin (kx - \omega t + \phi)$$

The amplitude given is

$$A = 2.0 \text{ cm} = 0.02 \text{ m}$$

The wavelength is given as

$$\lambda = 1.0 \text{ m}$$

$\therefore$ Angular wave number,

$$k = \frac{2\pi}{\lambda} = (2\pi) \text{ m}^{-1}$$

Angular frequency,

$$\omega = vk = (10\pi) \text{ rad/s}$$

$$\therefore \qquad y(x, t) = (0.02) \sin [2\pi (x - 5.0t) + \phi]$$

We are given that for $x = 0$, $t = 0$,

$$y = 0$$

and $$\frac{\partial y}{\partial t} < 0$$

i.e. $$0.02 \sin \phi = 0 \qquad (\text{as } y = 0)$$

and $$-0.2\pi \cos \phi < 0$$

From these conditions, we may conclude that

$$\phi = 2n\pi \qquad \text{where } n = 0, 2, 4, 6 \ldots$$

Therefore,

$$y(x, t) = (0.02 \text{ m}) \sin [(2\pi \text{ m}^{-1}) x - (10\pi \text{ rad s}^{-1}) t] \text{ m}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Mechanical transverse waves can't travel in gaseous medium.
Reason : They do not possess modulus of rigidity.

2. **Assertion :** Surface waves are neither transverse nor longitudinal.
Reason : In surface wave particles undergo circular motion.

3. **Assertion :** Two equations of wave are $y_1 = A \sin(\omega t - kx)$ and $y_2 = A \sin(kx - \omega t)$. These two waves have a phase difference of π.
Reason : They are travelling in opposite directions.

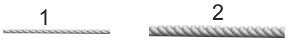
4. **Assertion :** Wave speed is given by $v = f\lambda$. If frequency f is doubled, v will become two times.
Reason : For given conditions of medium wave speed remains constant.

5. **Assertion :** On moon you cannot hear your friend standing at some distance from you.
Reason : There is a vacuum on moon.

6. **Assertion :** Wave number is the number of waves per unit length.
Reason : Wave number $= \dfrac{1}{\lambda}$.

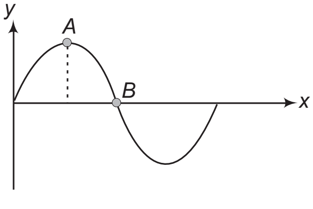
7. **Assertion :** Electromagnetic waves do not require medium for their propagation.
Reason : They can't travel in a medium.

8. **Assertion :** Two strings shown in figure have the same tension. Speed of transverse waves in string-1 will be more.

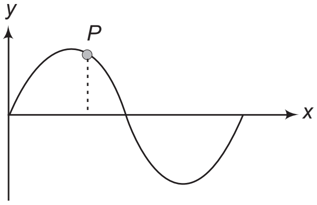


Reason : $v \propto \dfrac{1}{\sqrt{\mu}}$, Here μ is mass per unit length of string.

9. **Assertion :** y-x graph of a transverse wave on a string is as shown in figure. At point A potential energy and kinetic energy both are minimum.
Reason : At point B kinetic energy and potential energy both are maximum.



10. **Assertion :** y-x graph of a transverse wave on a string is as shown in figure. At the given instant point P is moving downwards. Hence, we can say that wave is moving towards positive y-direction.



Reason : Particle velocity is given by

$$v_P = -v\frac{\partial y}{\partial x}$$

Objective Questions

1. Equation of progressive wave is given by $y = 4\sin\left[\pi\left(\frac{t}{5}-\frac{x}{9}\right)+\frac{\pi}{6}\right]$, where x and y are in metre.

Then,

(a) $v = 5$ m/s (b) $\lambda = 18$ m
(c) $A = 0.04$ m (d) $f = 50$ Hz

2. The equation of a wave is given by $Y = 5\sin 10\pi(t - 0.01x)$ along the x-axis. (All the quantities are expressed in SI units). The phase difference between the points separated by a distance of 10 m along x-axis is

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) $\frac{\pi}{4}$

3. The displacement function of a wave travelling along positive x-direction is $y = \frac{1}{2+3x^2}$ at $t = 0$ and by $y = \frac{1}{2+3(x-2)^2}$ at $t = 2$ s, where y and x are in metre. The velocity of the wave is

(a) 2 m/s (b) 0.5 m/s
(c) 1 m/s (d) 3 m/s

4. The angle between wave velocity and particle velocity in a travelling wave may be

(a) zero (b) $\frac{\pi}{2}$ (c) π (d) All of these

5. A source oscillates with a frequency 25 Hz and the wave propagates with 300 m/s. Two points A and B are located at distances 10 m and 16 m away from the source. The phase difference between A and B is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

6. The equation of a transverse wave propagating in a string is given by

$$y = 0.02\sin(x + 30t)$$

where, x and y are in metre and t is in second.

If linear density of the string is 1.3×10^{-4} kg/m, then the tension in the string is

(a) 0.12 N (b) 1.2 N (c) 12 N (d) 120 N

7. A harmonic oscillator vibrates with amplitude of 4 cm and performs 150 oscillations in one minute. If the initial phase is 45° and it starts moving away from the origin, then the equation of motion is

(a) $0.04 \sin\left(5\pi t + \frac{\pi}{4}\right)$ (b) $0.04 \sin\left(5\pi t - \frac{\pi}{4}\right)$

(c) $0.04 \sin\left(4\pi t + \frac{\pi}{4}\right)$ (d) $0.04 \sin\left(4\pi t - \frac{\pi}{4}\right)$

Subjective Questions

1. A certain transverse wave is described by

$$y(x, t) = (6.50 \text{ mm}) \cos 2\pi \left(\frac{x}{28.0 \text{ cm}} - \frac{t}{0.0360 \text{ s}}\right).$$

Determine the wave's

(a) amplitude (b) wavelength

(c) frequency (d) speed of propagation and

(e) direction of propagation.

2. For the wave $y = 5 \sin 30\pi\,[t - (x/240)]$, where x and y are in cm and t is in seconds, find the

(a) displacement when $t = 0$ and $x = 2$ cm

(b) wavelength

(c) velocity of the wave and

(d) frequency of the wave

3. The displacement of a wave disturbance propagating in the positive x-direction is given by

$$y = \frac{1}{1 + x^2} \text{ at } t = 0 \quad \text{and} \quad y = \frac{1}{1 + (x - 1)^2} \quad \text{at} \quad t = 2 \text{ s}$$

where, x and y are in metre. The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave?

4. A travelling wave pulse is given by $y = \dfrac{10}{5 + (x + 2t)^2}$

Here, x and y are in metre and t in second. In which direction and with what velocity is the pulse propagating? What is the amplitude of pulse?

5. Is there any relationship between wave speed and the maximum particle speed for a wave travelling on a string? If so, what is it?

6. Calculate the velocity of a transverse wave along a string of length 2 m and mass 0.06 kg under a tension of 500 N.

7. Calculate the speed of a transverse wave in a wire of 1.0 mm^2 cross-section under a tension of 0.98 N. Density of the material of wire is $9.8 \times 10^3 \text{ kg/m}^3$

8. If at $t = 0$, a travelling wave pulse on a string is described by the function,

$$y = \frac{10}{(x^2 + 2)}$$

Here, x and y are in metre and t in second. What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 2 m/s?

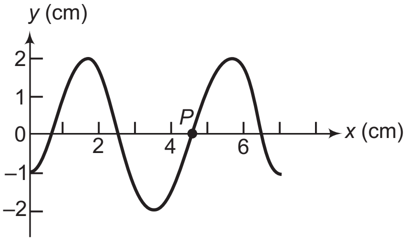
9. Consider a sinusoidal travelling wave shown in figure. The wave velocity is + 40 cm/s.
Find

(a) the frequency

(b) the phase difference between points 2.5 cm apart

(c) how long it takes for the phase at a given position to change by 60°

(d) the velocity of a particle at point P at the instant shown.



10. The equation of a travelling wave is

$$y\,(x,t) = 0.02 \sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right) \text{ m}$$

Find

(a) The wave velocity and

(b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.

Given, $\cos\theta = -\,0.85$, where, $\theta = 34$ rad

11. Transverse waves on a string have wave speed 12.0 m/s, amplitude 0.05 m and wavelength 0.4 m. The waves travel in the + x- direction and at $t = 0$ the $x = 0$ end of the string has zero displacement and is moving upwards.

(a) Write a wave function describing the wave.

(b) Find the transverse displacement of a point at $x = 0.25$ m at time $t = 0.15$ s.

(c) How much time must elapse from the instant in part (b) until the point at $x = 0.25$ m has zero displacement?

12. A wave is described by the equation

$$y = (1.0 \text{ mm}) \sin\pi\left(\frac{x}{2.0 \text{ cm}} - \frac{t}{0.01 \text{ s}}\right)$$

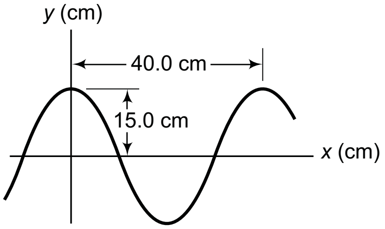
(a) Find the time period and the wavelength.

(b) Write the equation for the velocity of the particles. Find the speed of the particle at $x = 1.0$ cm at time $t = 0.01$ s.

(c) What are the speeds of the particles at $x = 3.0$ cm, 5.0 cm and 7.0 cm at $t = 0.01$ s ?

(d) What are the speeds of the particles at $x = 1.0$ cm at $t = 0.011$, 0.012 and 0.013 s ?

13. A sinusoidal wave travelling in the positive x-direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm and a frequency of 8.00 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15.0 cm as shown in figure.

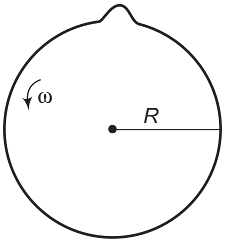


(a) Find the angular wave number k, period T, angular frequency ω and speed v of the wave.

(b) Write a general expression for the wave function.

14. A flexible steel cable of total length L and mass per unit length μ hangs vertically from a support at one end. (a) Show that the speed of a transverse wave down the cable is $v = \sqrt{g(L - x)}$, where x is measured from the support. (b) How long will it take for a wave to travel down the cable?

15. A loop of rope is whirled at a high angular velocity ω, so that it becomes a taut circle of radius R. A kink develops in the whirling rope.



(a) Show that the speed of the kink in the rope is $v = \omega R$.

(b) Under what conditions does the kink remain stationary relative to an observer on the ground?

16. A non-uniform wire of length L and mass M has a variable linear mass density given by $\mu = kx$, where x is distance from one end of wire and k is a constant. Find the time taken by a pulse starting at one end to reach the other end when the tension in the wire is T.

LEVEL 2

Single Correct Option

1. The speed of propagation of a wave in a medium is 300 m/s. The equation of motion of point at $x = 0$ is given by $y = 0.04 \sin 600\,\pi t$ (metre). The displacement of a point $x = 75$ cm at $t = 0.01$ s is

(a) 0.02 m (b) 0.04 m (c) zero (d) 0.028 m

2. A 100 Hz sinusoidal wave is travelling in the positive x-direction along a string with a linear mass density of 3.5×10^{-3} kg/m and a tension of 35 N. At time $t = 0$, the point $x = 0$, has maximum displacement in the positive y-direction. Next when this point has zero displacement, the slope of the string is $\pi/20$. Which of the following expression represent (s) the displacement of string as a function of x (in metre) and t (in second)

(a) $y = 0.025 \cos (200\pi t - 2\pi x)$ (b) $y = 0.5 \cos (200\,\pi t - 2\pi x)$

(c) $y = 0.025 \cos (100\,\pi t - 10\,\pi x)$ (d) $y = 0.5 \cos (100\,\pi t - 10\,\pi x)$

3. Vibrations of period 0.25 s propagate along a straight line at a velocity of 48 cm/s. One second after the emergence of vibrations at the initial point, displacement of the point, 47 cm from it is found to be 3 cm. [Assume that at initial point particle is in its mean position at $t = 0$ and moving upwards]. Then,

(a) amplitude of vibrations is 6 cm (b) amplitude of vibrations is $3\sqrt{2}$ cm

(c) amplitude of vibrations is 3 cm (d) None of these

4. Transverse waves are generated in two uniform steel wires A and B by attaching their free ends to a fork of frequency 500 Hz. The diameter of wire A is half that of B and tension in wire A is half the tension in wire B. What is the ratio of velocities of waves in A and B?

(a) 1 : 2 (b) $\sqrt{2} : 1$ (c) 2 : 1 (d) $1 : \sqrt{2}$

5. The frequency of A note is 4 times that of B note. The energies of two notes are equal. The amplitude of B note as compared to that of A note will be

(a) double (b) equal

(c) four times (d) eight times

6. If at $t = 0$, a travelling wave pulse on a string is described by the function.

$$y = \frac{6}{x^2 + 3}$$

What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 4 m/s?

(a) $y = \frac{6}{(x + 4t)^2 + 3}$ (b) $y = \frac{6}{(x - 4t)^2 + 3}$

(c) $y = \frac{6}{(x - t)^2}$ (d) $y = \frac{6}{(x - t)^2 + 12}$

More than One Correct Options

1. A transverse wave travelling on a stretched string is represented by the equation $y = \frac{2}{(2x - 6.2\,t)^2 + 20}$. Then,

(a) velocity of the wave is 3.1 m/s

(b) amplitude of the wave is 0.1 m

(c) frequency of the wave is 20 Hz

(d) wavelength of the wave is 1 m

2. For energy density, power and intensity of any wave choose the correct options.

(a) u = energy density $= \frac{1}{2}\rho\omega^2 A^2$ (b) P = power $= \frac{1}{2}\rho\,\omega^2 A^2 v$

(c) I = intensity $= \frac{1}{2}\rho\,\omega^2 A^2 S v$ (d) $I = \frac{P}{S}$

3. For the transverse wave equation $y = A \sin(\pi x + \pi t)$, choose the correct options at $t = 0$

(a) points at $x = 0$ and $x = 1$ are at mean positions

(b) points at $x = 0.5$ and $x = 1.5$ have maximum accelerations

(c) points at $x = 0.5$ and $x = 1.5$ are at rest

(d) the given wave is travelling in negative x- direction

4. In the wave equation,

$$y = A \sin \frac{2\pi}{a}(x - bt)$$

(a) speed of wave is a (b) speed of wave is b

(c) wavelength of wave is a/b (d) wavelength of wave is a

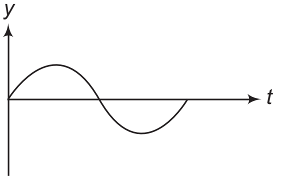
5. In the wave equation,

$$y = A \sin 2\pi\left(\frac{x}{a} - \frac{t}{b}\right)$$

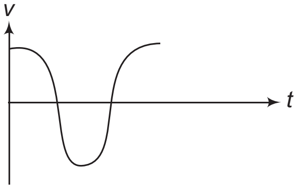
(a) speed of wave is a/b (b) speed of wave is b/a

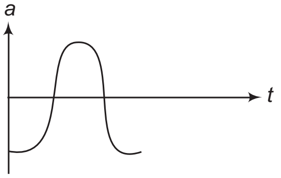
(c) wavelength of wave is a (d) time period of wave is b

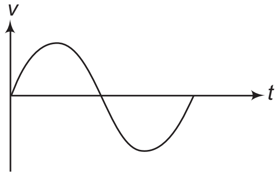
6. Corresponding to y-t graph of a transverse harmonic wave shown in figure,

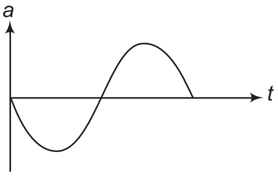


choose the correct options at same position.

(a)


(b)


(c)


(d)


Match the Columns

1. For the wave equation, $y = a \sin(bt - cx)$

Match the following two columns.

Column I	Column II
(a) wave speed	(p) $\frac{b}{2\pi}$
(b) maximum particle speed	(q) $\frac{c}{2\pi}$
(c) wave frequency	(r) $\frac{b}{c}$
(d) wavelength	(s) None of these

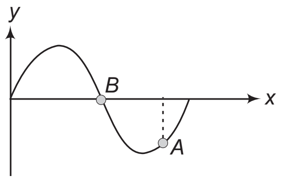
2. For the wave equation,

$$y = (4\text{ cm}) \sin[\pi t + 2\pi x]$$

Here, t is in second and x in metres.

Column I	Column II
(a) at $x = 0$, particle velocity is maximum at $t =$	(p) 0.5 s
(b) at $x = 0$, particle acceleration is maximum at $t =$	(q) 1.0 s
(c) at $x = 0.5$ m, particle velocity is maximum at $t =$	(r) zero
(d) at $x = 0.5$ m, particle acceleration is maximum at $t =$	(s) 1.5 s

3. y-x graph of a transverse wave at a given instant is shown in figure. Match the following two columns.



Column I	Column II
(a) velocity of particle A	(p) positive
(b) acceleration of particle A	(q) negative
(c) velocity of particle B	(r) zero
(d) acceleration of particle B	(s) can't tell

4. For a travelling wave, match the following two columns.

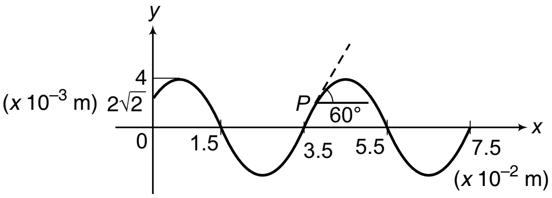
Column I	Column II
(a) energy density	(p) $[\mathrm{ML}^2\mathrm{T}^{-3}]$
(b) power	(q) $\frac{1}{2}\rho\omega^2A^2Sv$
(c) intensity	(r) $[\mathrm{M}^0\mathrm{L}^{-1}\mathrm{T}]$
(d) wave number	(s) None

5. Match the following two columns.

Column I	Column II
(a) $y = A\sin(\omega t - kx)$	(p) travelling in positive x-direction
(b) $y = A\sin(kx - \omega t)$	(q) travelling in negative x-direction
(c) $y = -A\cos(\omega t + kx)$	(r) at $t = 0$, velocity of particle is positive at $x = 0$
(d) $y = -A\cos(kx - \omega t)$	(s) at $t = 0$ acceleration of particle is positive at $x = 0$

Subjective Questions

1. The figure shows a snap photograph of a vibrating string at $t = 0$. The particle P is observed moving up with velocity $20\sqrt{3}$ cm / s. The tangent at P makes an angle 60° with x-axis.



(a) Find the direction in which the wave is moving.

(b) Write the equation of the wave.

(c) The total energy carried by the wave per cycle of the string. Assuming that the mass per unit length of the string is 50 g/m.

2. A long string having a cross-sectional area 0.80 mm^2 and density 12.5 g/cm^3 is subjected to a tension of 64 N along the positive x-axis. One end of this string is attached to a vibrator at $x = 0$ moving in transverse direction at a frequency of 20 Hz. At $t = 0$, the source is at a maximum displacement $y = 1.0$ cm.

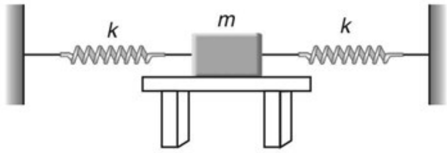
(a) Find the speed of the wave travelling on the string.

(b) Write the equation for the wave.

(c) What is the displacement of the particle of the string at $x = 50$ cm at time $t = 0.05$ s?

(d) What is the velocity of this particle at this instant?

3. One end of each of two identical springs, each of force constant 0.5 N/m are attached on the opposite sides of a wooden block of mass 0.01 kg. The other ends of the springs are connected to separate rigid supports such that the springs are unstretched and are collinear in a horizontal plane. To the wooden piece is fixed a pointer which touches a vertically moving plane paper. The wooden piece kept on a smooth horizontal table is now displaced by 0.02 m along the line of springs and released. If the speed of paper is 0.1 m/s, find the equation of the path traced by the pointer on the paper and the distance between two consecutive maximas on this path.

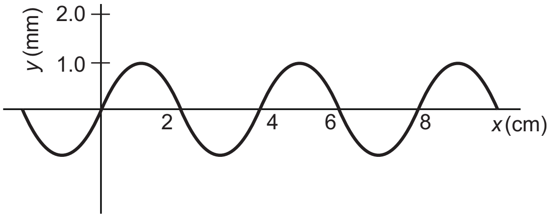


4. A wave pulse is travelling on a string with a speed v towards the positive x-axis. The shape of the string at $t = 0$ is given by $y(x) = A \sin(x/a)$, where A and a are constants.

(a) What are the dimensions of A and a?

(b) Write the equation of the wave for a general time t, if the wave speed is v.

5. Figure shows a plot of the transverse displacement of the particle of a string at $t = 0$ through which a travelling wave is passing in the positive x-direction. The wave speed is 20 cm/s. Find (a) the amplitude (b) the wavelength (c) the wave number and (d) the frequency of the wave.



6. Two wires of different densities but same area of cross-section are soldered together at one end and are stretched to a tension T. The velocity of a transverse wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.

7. Two long strings A and B, each having linear mass density 1.2×10^{-2} kg/m are stretched by different tensions 4.8 N and 7.5 N respectively and are kept parallel to each other with their left ends at $x = 0$. Wave pulses are produced on the strings at the left ends at $t = 0$ on string A and at $t = 20$ ms on string B. When and where will the pulse on B overtake that on A?

8. A sinusoidal transverse wave travels on a string. The string has length 8.00 m and mass 6.00 g. The wave speed is 30.0 m/s and the wavelength is 0.200 m. (a) If the wave is to have an average power of 50.0 W, what must be the amplitude of the wave? (b) For this same string, if the amplitude and wavelength are the same as in part (a) what is the average power for the wave if the tension is increased such that the wave speed is doubled?

9. A uniform rope with length L and mass m is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. Find the time required for a transverse wave to travel from one end of the rope to the other.

Hint : $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

Answers

Introductory Exercise 17.1

2. c/b

3. The converse is not true. Only function (c) satisfies this condition.

4. (a) L, L, T (b) a/T (c) negative x - direction (d) $x = -a$ and $x = -2a$

Introductory Exercise 17.2

1. (a) 5 mm (b) 1 cm^{-1} (c) 2π cm (d) $\frac{30}{\pi}$ Hz (e) $\left(\frac{\pi}{30}\right)$ sec, (f) 60 cm/s

2. (a) 20 ms 4.0 cm (b) zero (c) zero (d) 9.7 cm s^{-1}, 18 cm s^{-1}, 25 cm s^{-1}

Introductory Exercise 17.3

1. (a) negative x-direction (b) 10 ms^{-1}, 20 cm, 50 Hz (c) 0.10 mm, 3.14 cm s^{-1} **2.** (a) 9.4 m/s (b) zero

3. (a) –5 m/s (b) –1.7 m/s **4.** (a) 0.116 m (b) π

Introductory Exercise 17.4

1. 7.5×10^{14} Hz – 4.3×10^{14} Hz **2.** 0.33 m, 0.165 m

Introductory Exercise 17.5

1. 0.05 s **2.** 32 ms^{-1} **3.** 79 ms^{-1} and 63 ms^{-1} **4.** 0.02 s

5. 22 m/s **6.** (a) 16.3 m/s (b) 0.136 m (c) both increase by $\sqrt{2}$ times

Introductory Exercise 17.6

1. $\frac{1}{4\pi}$ W/m^2 **2.** $A \propto \frac{1}{\sqrt{r}}, I \propto \frac{1}{r}$ **3.** 0.58 mJ **4.** 512 W

5. (a) 0.47 W (b) 9.4 mJ **6.** 49 mW

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (a) **7.** (c) **8.** (d) **9.** (b) **10.** (d)

Objective Questions

1. (b) **2.** (b) **3.** (c) **4.** (d) **5.** (c) **6.** (a) **7.** (a)

Subjective Questions

1. (a) 6.50 mm (b) 28.0 cm (c) 27.8 Hz (d) 7.8 m/s (e) positive x

2. (a) –3.535 cm (b) 16 cm (c) 240 cm/s (d) 15 Hz **3.** 0.5 m/s

4. The pulse is travelling along negative x -axis with velocity 2 m/s. The amplitude of the pulse is 2 m.

5. Yes, $(v_P)_{\text{max}} = (kA)\,v$ **6.** 129.1 m/s **7.** 10 m/s

8. $y = \dfrac{10}{(x - 2t)^2 + 2}$ **9.** (a) 10 Hz (b) $\dfrac{5\pi}{4}$ rad (c) $\dfrac{1}{60}$ s (d) –1.26 m/s

10. (a) –5 m/s (b) –1.7 m/s

11. (a) $y(x, t) = (0.05\text{ m}) \sin[(60\pi\text{ s}^{-1})t - (5\pi\text{ m}^{-1})x]$ (b) – 3.54 cm (c) 4.2 ms

12. (a) 20 ms, 4.0 cm (b) $v = -(\pi/10\text{ m/s}) \cos \pi\left[\dfrac{x}{2.0\text{ cm}} - \dfrac{t}{0.01\text{ s}}\right]$, zero

(c) 0 m/s, 0 m/s, 0 m/s (d) 9.7 cm/s, 18 cm/s, 25 cm/s

13. (a) 0.157 rad/cm, 0.125 s, 50.3 rad/s, 320 cm/s (b) $y = (15.0\text{ cm}) \cos(0.157x - 50.3t)$

14. (b) $t = 2\sqrt{\dfrac{L}{g}}$

15. (b) The kink will be stationary with respect to the ground if it moves clockwise with respect to the rope.

16. $\dfrac{2}{3}\sqrt{\dfrac{2ML}{T}}$

LEVEL 2

Single Correct Option

1.(b) **2.**(a) **3.**(a) **4.**(b) **5.**(c) **6.**(b)

More than One Correct Options

1.(a, b) **2.**(a, d) **3.**(all) **4.**(b, d) **5.**(a, c, d) **6.**(a,d)

Match the Columns

1. (a) → r (b) → s (c) → p (d) → s

2. (a) → q,r (b) → p,s (c) → q,r (d) → p,s

3. (a) → s (b) → p (c) → s (d) → r

4. (a) → s (b) → p,q (c) → s (d) → s

5. (a) → p,r (b) → p (c) → q,s (d) → p,s

Subjective Questions

1. (a) Negative x (b) $y = (0.4\text{ cm}) \sin\left(10\pi t + \dfrac{\pi}{2}x + \dfrac{\pi}{4}\right)$ s (c) 1.6×10^{-5} J

2. (a) 80 m/s (b) $y = (1.0\text{ cm}) \cos\left[(40\pi\text{ s}^{-1})t - \left(\dfrac{\pi}{2}\text{ m}^{-1}\right)x\right]$ (c) $\dfrac{1}{\sqrt{2}}$ cm (d) 89 cm/s

3. $y = 0.02 \cos(10t - 100x)$ m, 0.0628 m **4.** (a) L, L (b) $y(x, t) = A \sin\left(\dfrac{x - vt}{a}\right)$

5. (a) 1.0 mm (b) 4 cm (c) 1.6 cm^{-1} (d) 5 Hz **6.** 0.25 **7.** at $t = 0.1$ s, at $x = 2.0$ m

8. (a) 7.07 cm (b) 400.0 W **9.** $\dfrac{\pi}{\sqrt{2}\omega}$

18

Superposition of Waves

Chapter Contents

18.1 Principle of Superposition

18.2 Resultant Amplitude and Intensity due to Coherent Sources

18.3 Interference

18.4 Standing Wave

18.5 Normal Modes of a String

18.6 Reflection and Transmission of a Wave

18.1 Principle of Superposition

Suppose there are two sources of waves S_1 and S_2.

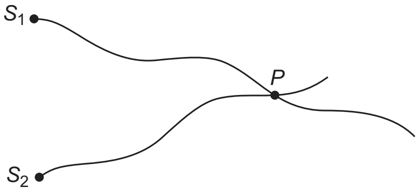


Fig. 18.1

Now, the two waves from S_1 and S_2 meet at some point (say P). Then, according to principle of superposition net displacement at P (from its mean position) at any time is given by

$$\boxed{y = y_1 + y_2}$$

Here, y_1 and y_2 are the displacements of P due to two waves individually.

For example, suppose at 9AM, displacement of P above its mean position should be 6 mm accordingly to wave-1 and the same time its displacement should be 2 mm below its mean position accordingly to wave-2, then at 9AM net displacement of P will be 4 mm above its mean position.

Now, based upon the principle of superposition we have two phenomena in physics, interference and beats. Stationary waves (or standing waves) and Young's double slit experiment (or YDSE) are two examples of interference.

Based on principle of superposition means two or more than two waves meet at one point or several points and at every point net displacement is $y = y_1 + y_2$ or $y = y_1 + y_2 + y_3$ etc.

18.2 Resultant Amplitude and Intensity due to Coherent Sources

In article 18.1, we have seen that the two waves from two sources S_1 and S_2 were meeting at point P. Suppose they meet at P in a phase difference $\Delta\phi$ (or ϕ). If this phase difference remains constant with time, then sources are called coherent, otherwise incoherent.

For sources to be coherent the frequencies (f, ω or T) of the two sources must be same. This can be explained by the following example.

Suppose the phase difference is 0°. It means they are in same phase. Both reach their extremes ($+A$ or $-A$), simultaneously. They cross their mean positions (in the same direction) simultaneously. Now, if we want their phase difference to remain constant or we want that the above situation is maintained all the time, then obviously their time periods (or frequencies) must be same.

Resultant Amplitude

1. Consider the superposition of two sinusoidal waves of same frequency (means sources are coherent) at a point. Let us assume that the two waves are travelling in the same direction with same velocity. The equation of the two waves reaching at a point can be written as

$$y_1 = A_1 \sin (kx - \omega t)$$

and $$y_2 = A_2 \sin (kx - \omega t + \phi)$$

The resultant displacement of the point where the waves meet is

$$\begin{aligned} y &= y_1 + y_2 \\ &= A_1 \sin (kx - \omega t) + A_2 \sin (kx - \omega t + \phi) \\ &= A_1 \sin (kx - \omega t) + A_2 \sin (kx - \omega t) \cos \phi + A_2 \cos (kx - \omega t) \sin \phi \\ &= (A_1 + A_2 \cos \phi) \sin (kx - \omega t) + A_2 \sin \phi \cos (kx - \omega t) \\ &= A \cos \theta \sin (kx - \omega t) + A \sin \theta \cos (kx - \omega t) \end{aligned}$$

or $$y = A \sin (kx - \omega t + \theta)$$

Here, $$A_1 + A_2 \cos \phi = A \cos \theta$$

and $$A_2 \sin \phi = A \sin \theta$$

or $$A^2 = (A_1 + A_2 \cos \phi)^2 + (A_2 \sin \phi)^2$$

or $$\boxed{A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos \phi}} \qquad \text{...(i)}$$

and $$\tan \theta = \frac{A \sin \theta}{A \cos \theta} = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}$$

2. The above result can be obtained by graphical method as well. Assume a vector $\mathbf{A}_1$ of length A_1 to represent the amplitude of firstwave.

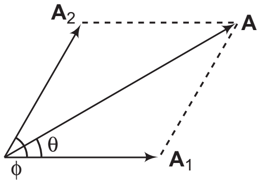


Fig. 18.2

Another vector $\mathbf{A}_2$ of length A_2, making an angle ϕ with $\mathbf{A}_1$ represent the amplitude of second wave. The resultant of $\mathbf{A}_1$ and $\mathbf{A}_2$ represent the amplitude of resulting function y. The angle θ represents the phase difference between the resulting function and the first wave.

Resultant Intensity

In the previous chapter, we have read that intensity of a wave is given by

$$I = \frac{1}{2} \rho \omega^2 A^2 v \quad \text{or} \quad I \propto A^2$$

So, if ρ, ω and v are same for the both interfering waves, then Eq. (i) can also be written as

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \phi} \qquad \text{...(ii)}$$

Here, proportionality constant ($I \propto A^2$) cancels out on right hand side and left hand side.

Note *(i) Eqs (i) and (ii) are two equations for finding resultant amplitude and resultant intensity at some point due to two coherent sources.*

(ii) In the above equations ϕ is the constant phase difference at that point. As the sources are coherent value of this constant phase difference will be different at different points.

(iii) *The special case of above two equations is, when the individual amplitudes (or intensities) are equal.*

or $A_1 = A_2 = A_0$ *(say)* $\Rightarrow$ $I_1 = I_2 = I_0$ *(say)*

In this case, Eqs. (i) and (ii) become

$$\boxed{A = 2A_0 \cos\frac{\phi}{2}} \qquad \text{...(iii)}$$

and

$$\boxed{I = 4I_0 \cos^2\frac{\phi}{2}} \qquad \text{...(iv)}$$

(iv) *From Eqs. (i) to (iv) we can see that, for given values of A_1, A_2, I_1 and I_2 the resultant amplitude and the resultant intensity are the functions of only ϕ.*

(v) *If three or more than three waves (due to coherent sources) meet at some point then there is no direct formula for finding resultant amplitude or resultant intensity. In this case, first of all we will find resultant amplitude by vector method (either by using polygon law of vector addition or component method) and then by the relation $I \propto A^2$, we can also determine the resultant intensity.*

For example, if resultant amplitude comes out to be $\sqrt{2}$ times then resultant intensity will become two times.

18.3 Interference

For interference phenomena to take place, sources must be coherent. So, phase difference at some point should remain constant. The value of this constant phase difference will be different at different points. And since the sources are coherent, therefore the following four equations can be applied for finding resultant amplitude and intensity (in case of two sources)

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \qquad \text{...(i)}$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \qquad \text{...(ii)}$$

$$A = 2A_0\cos\frac{\phi}{2} \qquad (\text{if } A_1 = A_2 = A_0) \qquad \text{...(iii)}$$

$$I = 4I_0\cos^2\frac{\phi}{2} \qquad (\text{if } I_1 = I_2 = I_0) \qquad \text{...(iv)}$$

For given values of A_1, A_2, I_1 and I_2 the resultant amplitude and resultant intensity are the functions of only ϕ.

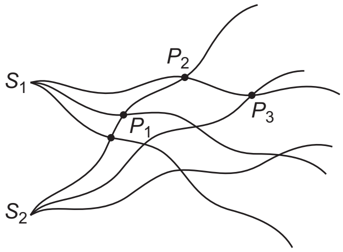


Fig. 18.3

Now, suppose S_1 and S_2 are two coherent sources, then we can see that the two waves are meeting at several points (P_1, P_2, P_3 … etc). At different points path difference Δx will be different and therefore phase difference $\Delta\phi$ or ϕ will also be different. Because the phase difference depends on the path difference ($\Delta\phi$ or $\phi = \frac{2\pi}{\lambda}\cdot\Delta x$).

And since phase difference at different points is different, therefore from the above four equations we can see that resultants amplitude and intensity will also be different. But whatever is the intensity at some point, it will remain constant at that point because the sources are coherent and the phase difference is constant at that point.

Constructive Interference

These are the points where resultant amplitude or intensity is maximum or

$$A_{\max} = A_1 + A_2 \qquad \text{[from Eq. (i)]}$$

or $$A_{\max} = \pm 2A_0 \qquad \text{[from Eq. (iii)]}$$

and $$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2 \qquad \text{[from Eq. (ii)]}$$

or $$I_{\max} = 4I_0 \qquad \text{[from Eq. (iv)]}$$

at those points where

$$\cos\phi = +1 \qquad \text{[from Eqs. (i) or (ii)]}$$

or $$\phi = 0, 2\pi, 4\pi, \ldots, 2n\pi \qquad \text{(where, } n = 0, 1, 2\text{)}$$

$\therefore$ $$\Delta x = 0, \lambda, 2\lambda, \ldots, n\lambda \qquad \left[\text{as } \Delta x = \phi\left(\frac{\lambda}{2\pi}\right)\right]$$

Destructive Interference

These are the points where resultant amplitude or intensity is minimum or

$$A_{\min} = A_1 \sim A_2 \qquad \text{[from Eq. (i)]}$$

or $$A_{\min} = 0 \qquad \text{[from Eq. (iii)]}$$

and $$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \text{[from Eq. (ii)]}$$

or $$I_{\min} = 0 \qquad \text{[from Eq. (iv)]}$$

at those points where,

$$\cos\phi = -1 \qquad \text{[from Eqs. (i) or (ii)]}$$

or $$\phi = \pi, 3\pi \ldots (2n-1)\pi \qquad \text{(where, } n = 1, 2\ldots\text{)}$$

$\therefore$ $$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \ldots (2n-1)\frac{\lambda}{2} \qquad \left[\text{as } \Delta x = \phi\left(\frac{\lambda}{2\pi}\right)\right]$$

✓ Extra Points to Remember

- In amplitude it hardly matters whether its $+2A_0$ or $-2A_0$. This is the reason we have taken, $A_{\max} = \pm 2A_0$
- In interference, two or more than two waves from coherent sources meet at several points. At different points Δx, $\Delta\phi$ or ϕ, resultant amplitude and therefore resultant intensity will be different (varying from $I_{\max}$ to $I_{\min}$). But whatever is the resultant intensity at some point, it remains constant at that point.
- In interference,

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{I_1/I_2} + 1}{\sqrt{I_1/I_2} - 1}\right)^2$$

$$= \left(\frac{A_1/A_2 + 1}{A_1/A_2 - 1}\right)^2 = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = \left(\frac{A_{\max}}{A_{\min}}\right)^2$$

Example 18.1 *In interference, two individual amplitudes are A_0 each and the intensity is I_0 each. Find resultant amplitude and intensity at a point, where*

(a) phase difference between two waves is 60°

(b) path difference between two waves is $\frac{\lambda}{3}$.

Solution (a) Substituting $\phi = 60°$ in the equations,

$$A = 2A_0 \cos\frac{\phi}{2} \qquad \ldots\text{(i)}$$

and

$$I = 4I_0 \cos^2\frac{\phi}{2} \qquad \ldots\text{(ii)}$$

We get,

$$A = \sqrt{3}A_0 \text{ and } I = 3I_0 \qquad \textbf{Ans.}$$

(b) Given,

$$\Delta x = \frac{\lambda}{3}$$

$$\therefore \qquad \phi \text{ or } \Delta\phi = \left(\frac{2\pi}{\lambda}\right)\cdot\Delta x$$

$$= \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{3}\right) = \frac{2\pi}{3} \text{ or } 120°$$

Now, substituting $\phi = 120°$ in the above two equations we get

$$A = A_0 \text{ and } I = I_0 \qquad \textbf{Ans.}$$

Example 18.2 *Three waves from three coherent sources meet at some point. Resultant amplitude of each is A_0. Intensity corresponding to A_0 is I_0. Phase difference between first wave and second wave is* 60°. *Path difference between first wave and third wave is* $\frac{\lambda}{3}$. *The first wave lags behind in phase angle from second and third wave. Find resultant intensity at this point.*

Solution Here, the sources are three. So, we don't have any direct formula for finding the resultant intensity. First we will find the resultant amplitude by vector method and then by the relation $I \propto A^2$, we can also find the resulting intensity.

Further, a path difference of $\frac{\lambda}{3}$ is equivalent to a phase difference of 120° ($\Delta\phi$ or $\phi = \frac{2\pi}{\lambda}\cdot\Delta x$).

Hence, the phase difference first and second is 60° and between first and third is 120°. So, vector diagram for amplitude is as shown below.

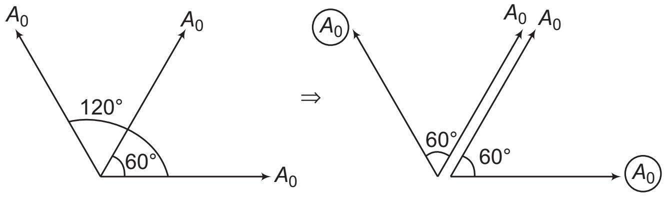


Fig. 18.4

Now, resultant of first and third acting at 120° is also A_0 (as $A = 2A_0 \cos\frac{\phi}{2}$ and $\phi = 120°$) and since the first and third are equal, so this resultant A_0 passes through the bisector line of these two or in the direction of second amplitude vector. Therefore, the resultant amplitude is

$$A = A_0 + A_0 = 2A_0$$

and the resultant intensity is

$$I = 4I_0 \qquad (\text{as } I \propto A^2) \qquad \textbf{Ans.}$$

Example 18.3 *Two waves of equal frequencies have their amplitudes in the ratio of* $3:5$. *They are superimposed on each other. Calculate the ratio of maximum and minimum intensities of the resultant wave.*

Solution Given, $\frac{A_1}{A_2} = \frac{3}{5}$

$$\therefore \qquad \sqrt{\frac{I_1}{I_2}} = \frac{3}{5} \qquad (\text{as } I \propto A^2)$$

Maximum intensity is obtained, where

$$\cos\phi = 1$$

and

$$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

Minimum intensity is found, where

$$\cos\phi = -1$$

and

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Hence,

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}} + 1}{\sqrt{\frac{I_1}{I_2}} - 1}\right)^2$$

$$= \left(\frac{3/5 + 1}{3/5 - 1}\right)^2 = \frac{64}{4} = \frac{16}{1} \qquad \textbf{Ans.}$$

Example 18.4 *In interference,* $\frac{I_{\max}}{I_{\min}} = \alpha$, *find*

(a $\frac{A_{\max}}{A_{\min}}$ *(b)* $\frac{A_1}{A_2}$ *(c)* $\frac{I_1}{I_2}$

Solution (a) $$\frac{A_{\max}}{A_{\min}} = \sqrt{\frac{I_{\max}}{I_{\min}}} = \sqrt{\alpha} \qquad \textbf{Ans.}$$

(b) $$\frac{A_{\max}}{A_{\min}} = \sqrt{\alpha} = \frac{A_1 + A_2}{A_1 - A_2} = \frac{A_1/A_2 + 1}{A_1/A_2 - 1}$$

Solving this equation, we get

$$\frac{A_1}{A_2} = \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}$$ **Ans.**

(c) $$\frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 \left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^2$$

INTRODUCTORY EXERCISE 18.1

1. The ratio of intensities of two waves is 9:16. If these two waves interfere, then determine the ratio of the maximum and minimum possible intensities.
2. In Interference two individual amplitudes are 5 units and 3 units. Find

   (a) $\frac{A_{\max}}{A_{\min}}$ (b) $\frac{I_{\max}}{I_{\min}}$
3. Three waves due to three coherent sources meet at one point. Their amplitudes are $\sqrt{2}A_0$, $3A_0$ and $\sqrt{2}A_0$. Intensity corresponding to A_0 is I_0. Phase difference between first and second is 45°. Path difference between first and third $\frac{\lambda}{4}$. In phase angle, first wave lags behind from the other two waves. Find resultant intensity at this point.

18.4 Standing Wave

Standing wave is an example of interference. When two identical waves travel in opposite directions, then they superimpose almost at every point. Now, since the waves are identical (or their frequencies are same), sources are coherent. So, interference will take place. By their superposition (or interference) standing waves (also called stationary waves) are formed. By identical wave we mean that all properties (like f, ω, T, λ and k) are same. Only amplitudes may be different but still we prefer equal amplitudes ($A_1 = A_2 = A_0$). The following are listed some of the important points in standing wave.

(i) Here, the constructive interference points are called antinodes (denoted by A) and destructive interference points are called nodes (denoted by N).

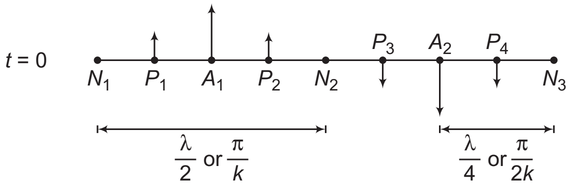
(ii) Distance between two successive nodes (or two successive antinodes) is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$ $\left(\text{as } k = \frac{2\pi}{\lambda}\right)$.

Similarly, distance between a node and its adjacent antinode is $\frac{\lambda}{4}$ or $\frac{\pi}{2k}$. Here, λ and k are the values corresponding to constituent waves by which stationary waves are formed.

(iii) Amplitude of oscillation in stationary wave varies from a maximum value ($A_1 + A_2$ or $2A_0$ at antinode) to a minimum value ($A_1 \sim A_2$ or zero at node). But now onwards (unless mentioned in the question) we will talk about the equal amplitudes. So, amplitude at antinode will be $2A_0$ and at node it is zero.

Thus, now we can say that, in stationary wave all particles (except nodes for the case when $A_1 = A_2$) oscillate with same frequency but different amplitudes.

(iv) All points lying between two successive nodes are in same phase. They cross their mean positions (in the same direction) simultaneously and reach their extreme positions also simultaneously. But they are out of phase with the particles lying between adjacent two nodes. This can be shown by the following figure.



t = 0
N_1
P_1
A_1
P_2
N_2
P_3
A_2
P_4
N_3
$\frac{\lambda}{2}$ or $\frac{\pi}{k}$
$\frac{\lambda}{4}$ or $\frac{\pi}{2k}$

Fig. 18.5

In the above figure N_1, N_2 and N_3 are nodes and A_1, A_2 are antinodes. Amplitude at nodes is zero and amplitude at antinodes is $2A_0$. In between them we have taken four more points P_1, P_2, P_3 and P_4. Suppose amplitude at these points is A_0 (between $2A_0$ and 0). Now, at $t = 0$ all particles are at their mean positions. Particles P_1, A_1 and P_2 are moving upwards (as they are in same phase) and the particles P_3, A_2 and P_4 are moving downwards. At this time, they have maximum speed. For example, in case of sine wave,

$$v_{P_1} = \omega A_0 \qquad (v_{\max} = \omega A \text{ in SHM})$$

$$v_{A_1} = \omega(2A_0)$$

Now, in the chapter of SHM we have read that at time $t = \frac{T}{12}$ a particle reaches from its mean position to half its extreme position and in time $t = \frac{T}{4}$ it reaches from mean position to extreme position. So, at different times positions of different particles are as shown below.

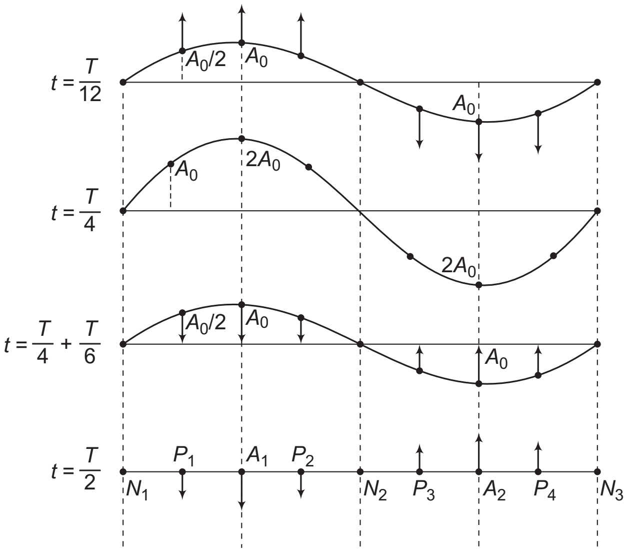


Fig. 18.6

(v) Equation of a stationary wave depends on the equations of constituent waves by which stationary wave is formed. Let us take an example

Suppose the two identical waves travelling in opposite directions are

$$y_1 = A\sin(kx - \omega t)$$

and $$y_2 = A\sin(kx + \omega t)$$

By the principle of superposition, their sum is

$$y = y_1 + y_2$$

or $$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

By using the identity,

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \quad \text{we obtain}$$

$$\boxed{y = 2A\sin kx\cos\omega t} \qquad \text{...(i)}$$

This expression is different from wave representations that we have encountered upto now. It doesn't have the form $f(x \pm vt)$ or $f(ax \pm bt)$ and therefore, does not describe a travelling wave. Instead Eq. (i) represents what is known as a **standing wave.**

Eq. (i) can also be written as

$$y = A(x)\cos\omega t \qquad \text{...(ii)}$$

where, $$A_x = 2A\sin kx \qquad \text{...(iii)}$$

This equation of a standing wave [Eq. (ii)] is really an equation of simple harmonic motion, whose amplitude [Eq. (iii)] is a function of x.

$$A_x = 0, \text{ where } \sin kx = 0$$

or $$kx = 0, \pi, 2\pi, \dots, n\pi \qquad (n = 0, 1, 2, \dots)$$

Substituting $k = \frac{2\pi}{\lambda}$, we have

$$A_x = 0 \qquad \text{where,} \quad \boxed{x = 0, \frac{\lambda}{2}, \lambda, \dots, \frac{n\lambda}{2}}$$

These are the points which never displace from their mean position. These are known as the **nodes** of the standing wave. The distance between two adjacent nodes is $\frac{\lambda}{2}$.

Further, from Eq. (iii), we can see that maximum value of $|A_x|$ is $2A$, where

$$\sin kx = \pm 1$$

or $$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, (2n-1)\frac{\pi}{2} \qquad (n = 0, 1, 2, \dots)$$

or $$\boxed{x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \dots, (2n-1)\frac{\lambda}{4}} \qquad \left(k = \frac{2\pi}{\lambda}\right)$$

These are the points of maximum displacement called **antinodes**. The distance between two adjacent antinodes is also $\frac{\lambda}{2}$, while that between a node and an antinode is $\frac{\lambda}{4}$.

Extra Points to Remember

Table 18.1 comparison between travelling and stationary waves

S.No.	Travelling waves	Stationary waves
1.	In these waves, all particles of the medium oscillate with same frequency and amplitude	In these waves, all particles except nodes oscillate with same frequency but different amplitudes. Amplitudes is zero at nodes and maximum at antinodes
2.	At any instant phase difference between any two particles can have any value between 0 and 2π.	At any instant phase difference between any two particles can be either zero or π.
3.	In these waves, at no instant all the particles of the medium pass through their mean positions simultaneously.	In these waves, all particles of the medium pass through their mean positions simultaneously twice in each time period.
4.	These waves transmit energy in the medium.	These waves do not transmit energy in the medium provided $A_1 = A_2$.

Example 18.5 *The displacement of a standing wave on a string is given by*

$$y(x, t) = 0.4 \sin(0.5x) \cos(30t)$$

where x and y are in centimetres.

(a) Find the frequency, amplitude and wave speed of the component waves.

(b) What is the particle velocity at x = 2.4 cm at t = 0.8 s ?

Solution (a) The given equation can be written as the sum of two component waves as

$$y(x, t) = 0.2 \sin(0.5x - 30t) + 0.2 \sin(0.5x + 30t)$$

Hence, the two component waves are

$$y_1(x, t) = 0.2 \sin(0.5x - 30t)$$

travelling in positive x-direction and

$$y_2(x, t) = 0.2 \sin(0.5x + 30t)$$

travelling in negative x-direction.

Now, $\omega = 30 \text{ rad/s}$

and $k = 0.5 \text{cm}^{-1}$

$$\text{Frequency, } f = \frac{\omega}{2\pi} = \frac{15}{\pi} \text{ Hz}$$ **Ans.**

$$\text{Amplitude, } A = 0.2 \text{ cm}$$ **Ans.**

and $$\text{wave speed, } v = \frac{\omega}{k} = \frac{30}{0.5}$$

$$= 60 \text{ cm/s}$$ **Ans.**

(b) Particle velocity,

$$v_P(x,t) = \frac{\partial y}{\partial t} = -12\sin(0.5x)\sin(30t)$$

$$\therefore \quad v_P(x = 2.4\text{ cm},\ t = 0.8\text{ s})$$

$$= -12\sin(1.2)\sin(24)$$

$$= 10.12\text{ cm/s}$$

Ans.

Example 18.6 *The vibrations of a string of length* 60 *cm fixed at both ends are represented by the equation* $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$, *where x and y are in cm and t in seconds.*

(a) What is the maximum displacement of a point at x = 5 *cm?*

(b) Where are the nodes located along the string?

(c) What is the velocity of the particle at x = 7.5 *cm and t =* 0.25 *s?*

(d) Write down the equations of the component waves whose superposition gives the above wave.

Solution (a) At $x = 5$ cm, the standing wave equation gives

$$y = 4\sin\left(\frac{5\pi}{15}\right)\cos(96\pi t)$$

$$= 4\sin\frac{\pi}{3}\cos(96\pi t)$$

$$= 4\times\frac{\sqrt{3}}{2}\cos(96\pi t)$$

$\therefore$ Maximum displacement $= 2\sqrt{3}$ cm

(b) The nodes are the points of permanent rest. Thus, they are those points for which

$$\sin\left(\frac{\pi x}{15}\right) = 0$$

i.e. $$\frac{\pi x}{15} = n\pi, \text{ where } n = 0, 1, 2, 3, 4 \ldots$$

$x = 15n$, i.e. at $x = 0$, 15, 30, 45 and 60 cm.

(c) The particle velocity is equal to

$$\left(\frac{\partial y}{\partial t}\right) = 4\sin\left(\frac{\pi x}{15}\right)(96\pi)(-\sin 96\pi t)$$

$$= -384\,\pi\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

At $x = 7.5$ cm and $t = 0.25$ s, we get

$$\left(\frac{\partial y}{\partial t}\right) = -384\,\pi\sin\left(\frac{\pi}{2}\right)\sin(24\pi) = 0$$

(d) The equations of the component waves are

$$y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right)$$

and $$y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

as we can see that $$y = y_1 + y_2$$

INTRODUCTORY EXERCISE 18.2

1. In stationary wave, phase difference between two particles can't be $\frac{\pi}{3}$. Is this statement true or false?
2. A string vibrates according to the equation $y = 5\sin\frac{\pi x}{3}\cos 40\pi t$

   where, x and y are in centimetres and t is in seconds
   (a) What is the speed of the component wave?
   (b) What is the distance between the adjacent nodes?
   (c) What is the velocity of the particle of the string at the position $x = 1.5$ cm when $t = \frac{9}{8}$ s?
3. If two waves differ only in amplitude and are propagated in opposite directions through a medium, will they produce standing waves. Is energy transported?
4. Two sinusoidal waves travelling in opposite directions interfere to produce a standing wave described by the equation

   $$y = (1.5\text{ m})\sin(0.400\,x)\cos(200\,t)$$

   where, x is in metres and t is in seconds. Determine the wavelength, frequency and speed of the interfering waves.

18.5 Normal Modes of a String

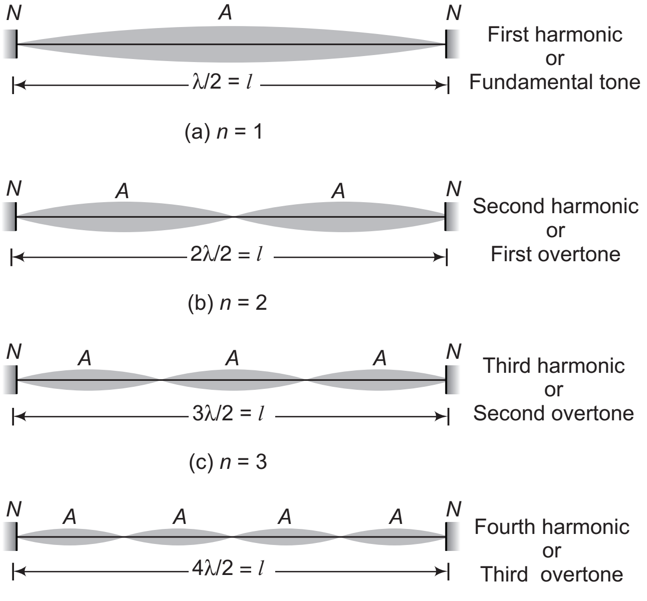
In an unbounded continuous medium, there is no restriction on the frequencies or wavelengths of the standing waves. However, if the waves are confined in space, for example, when a string is tied at both ends,standing waves can be set-up for a discrete set of frequencies or wavelengths. Consider a string of definite length *l,* rigidly held at both ends. When we setup a sinusoidal wave on such a string, it gets reflected from the fixed ends. By the superposition of two identical waves travelling in opposite directions standing waves are established on the string. The only requirement we have to satisfy is that the end points be nodes as these points cannot oscillate. They are permanently at rest. There may be any number of nodes in between or none at all, so that the wavelength associated with the standing waves can take many different values. The distance between adjacent nodes is $\lambda/2$, so that in a string of length l there must be exactly an integral number n of half wavelengths $\lambda/2$. That is,

$$\frac{n\lambda}{2} = l$$

or $$\lambda = \frac{2l}{n} \qquad (n = 1, 2, 3 \ldots)$$

But $\lambda = \dfrac{v}{f}$ and $v = \sqrt{\dfrac{T}{\mu}}$, so that the natural frequencies of oscillation of the system are

$$\boxed{f = n\left(\frac{v}{2l}\right) = \frac{n}{2l}\sqrt{\frac{T}{\mu}}} \qquad (n = 1, 2, 3 \ldots)$$



(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

(d) $n = 4$

Fig. 18.7

The smallest frequency f_1 corresponds to the largest wavelength $(n = 1)$, $\lambda_1 = 2l$.

$$f_1 = \frac{v}{2l}$$

This is called the **fundamental frequency**. The other standing wave frequencies are

$$f_2 = \frac{2v}{2l} = 2f_1$$

$$f_3 = \frac{3v}{2l} = 3f_1 \quad \text{and so on}$$

These frequencies are called **harmonics**. Musicians sometimes call them **overtones**. Students are advised to remember these frequencies by name.

For example,

f_1 = fundamental tone or first harmonic

$f_2 = 2f_1$ = first overtone or second harmonic

$f_3 = 3f_1$ = second overtone or third harmonic and so on.

Extra Points to Remember

- We have seen that the natural frequencies of oscillation of a stretched wire are given by

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$$

Now, tension in the wire can be produced by the following two methods.

Method 1

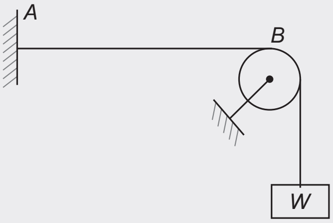


Fig. 18.8

By suspending a weight W from the wire AB as shown. In this case, transverse stationary waves are produced in wire AB.

Method 2 Wire AB is stretched at some higher temperature (than the normal room temperature). When temperature decreases, thermal stresses are produced and the wire comes under tension. Detailed discussion of thermal expansion will be covered in chapter 20.

- If by somehow, tension is decreased (for example: when weight is partially or fully immersed in a liquid), then from the above expression we can see that the set of frequencies decrease.
- As n increases, T, μ, l and therefore $v\left(=\sqrt{\frac{T}{\mu}}\right)$ remain unchanged. Frequency increases, therefore wavelength $\left(\lambda=\frac{v}{f}\right)$ and the loop size $(=\lambda/2)$ decrease.
- Even and odd both harmonics $(n=1, 2, 3, \ldots)$ are obtained with stretched wire fixed at both ends.

Example 18.7 *In terms of T, μ and l, find frequency of*

(a) fourth overtone mode

(b) third harmonic

Solution (a) Fourth overtone mode means

$$n = 5 \qquad (n=1, 2, 3, 4, 5)$$

Therefore, using the equation

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}} = \frac{5}{2l}\sqrt{\frac{T}{\mu}}$$ **Ans.**

(b) Third harmonic means

$$n = 3$$

$$\therefore \qquad f = \frac{3}{2l}\sqrt{\frac{T}{\mu}}$$ **Ans.**

Example 18.8 *A massless rod BD is suspended by two identical massless strings AB and CD of equal lengths. A block of mass m is suspended from point P such that BP is equal to x. If the fundamental frequency of the left wire is twice the fundamental frequency of right wire, then the value of x is* **(JEE 2006)**

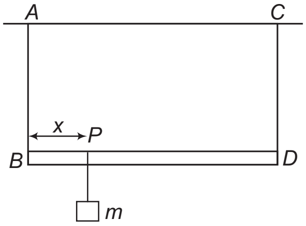


Fig. 18.9

(a) $l/5$ (b) $l/4$ (c) $4l/5$ (d) $3l/4$

Solution $f \propto v \propto \sqrt{T} \quad \Rightarrow \quad f_{AB} = 2f_{CD}$

$\therefore$ $$T_{AB} = 4T_{CD} \qquad \text{...(i)}$$

Further $$\Sigma\tau_P = 0$$

$\therefore$ $$T_{AB}(x) = T_{CD}(l - x)$$

or $$4x = l - x \qquad (\text{as } T_{AB} = 4T_{CD})$$

or $$x = l/5$$

$\therefore$ The correct option is (a).

Example 18.9 *An object of specific gravity* ρ *is hung from a thin steel wire. The fundamental frequency for transverse standing waves in the wire is* 300 *Hz. The object is immersed in water, so that one half of its volume is submerged. The new fundamental frequency (in Hz) is* **(JEE 1995)**

(a) $300\left(\dfrac{2\rho - 1}{2\rho}\right)^{1/2}$ (b) $300\left(\dfrac{2\rho}{2\rho - 1}\right)^{1/2}$ (c) $300\left(\dfrac{2\rho}{2\rho - 1}\right)$ (d) $300\left(\dfrac{2\rho - 1}{2\rho}\right)$

Solution The diagrammatic representation of the given problem is shown in figure. The expression of fundamental frequency is

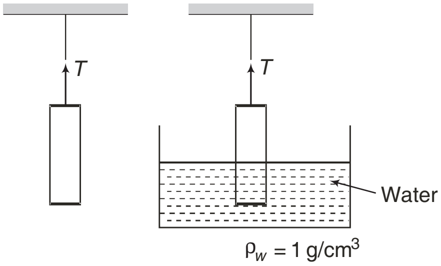


Fig. 18.10

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

In air, $T = mg = (V\rho)g$

$\therefore$ $$\nu = \frac{1}{2l}\sqrt{\frac{V\rho g}{\mu}} \qquad \text{...(i)}$$

When the object is half immersed in water, then

$$T' = mg - \text{upthrust} = V\rho g - \left(\frac{V}{2}\right)\rho_w g = \left(\frac{V}{2}\right) g\,(2\rho - \rho_w)$$

The new fundamental frequency is

$$\nu' = \frac{1}{2l} \times \sqrt{\frac{T'}{\mu}} = \frac{1}{2l}\sqrt{\frac{(Vg/2)(2\rho - \rho_w)}{\mu}} \qquad \text{...(ii)}$$

$\therefore$ $$\frac{\nu'}{\nu} = \left(\sqrt{\frac{2\rho - \rho_w}{2\rho}}\right) \quad \text{or} \quad \nu' = \nu\left(\frac{2\rho - \rho_w}{2\rho}\right)^{1/2}$$

$$= 300\left(\frac{2\rho - 1}{2\rho}\right)^{1/2} \text{Hz}$$ **Ans.**

$\therefore$ The correct option is (a).

INTRODUCTORY EXERCISE 18.3

1. Sound waves of frequency 660 Hz fall normally on a perfectly reflecting wall. The shortest distance from the wall at which the air particles have maximum amplitude of vibration is …… m. Speed of sound = 330 m/s. **(JEE 1984)**
2. If the frequencies of the second and fifth harmonics of a string differ by 54 Hz. What is the fundamental frequency of the string?
3. A wire is attached to a pan of mass 200 g that contains a 2.0 kg mass as shown in the figure. When plucked, the wire vibrates at a fundamental frequency of 220 Hz. An additional unknown mass *M* is then added to the pan and a fundamental frequency of 260 Hz is detected. What is the value of *M*?

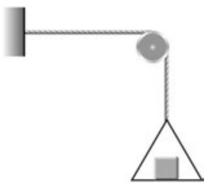


Fig. 18.11

4. A wire fixed at both ends is 1.0 m long and has a mass of 36 g. One of its oscillation frequencies is 250 Hz and the next higher one is 300 Hz.
   (a) Which harmonics do these frequencies represent?
   (b) What is the tension in the wire?
5. Two different stretched wires have same tension and mass per unit length. Fifth overtone frequency of the first wire is equal to second harmonic frequency of the second wire. Find the ratio of their lengths.

18.6 Reflection and Transmission of a Wave

Following points are important in reflection and transmission of a wave :

(i) Speed From wave point of view that medium is called a denser medium in which speed of wave is less.

For example, for a sound wave (or longitudinal wave) air is denser medium compared to water because speed of sound in air is less. On the other hand, air is rarer medium for an electromagnetic wave as speed of electromagnetic wave in air is more. When a wave strikes the boundary separating two different media, part of it is reflected and part is transmitted.

Since, wave speed depends upon the medium, in reflection wave speed does not change because medium does not change. In transmission, medium changes, therefore wave speed also changes. In denser medium it decreases.

(ii) Frequency Frequency (ω and T also) depends on source. In reflection as well as transmission source does not change. Therefore, frequency does not change.

(iii) Wavelength Wavelength (and therefore wave number $k = \frac{2\pi}{\lambda}$ also) is self adjusted in a value, $\lambda = \frac{v}{f}$. In reflection v and f do not change, therefore λ and k do not change. In transmission, f remains unchanged but v changes. Therefore, λ and k both change. In a denser medium, v decreases, therefore λ also decreases but k increases.

(iv) Phase change A phase change of π or 180° occurs only during the reflection from a denser medium. In any other case (in transmission or reflection from a rarer medium) phase change is zero.

(v) Amplitude In reflection as well as transmission, amplitude changes.

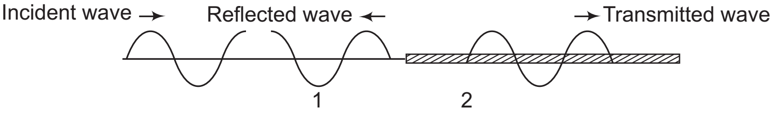


Fig. 18.12

If amplitude of incident wave in medium-1 is A_i, it is partly reflected and partly transmitted at the boundary of two media-1 and 2. Wave speeds in two media are v_1 and v_2. If amplitudes of reflected and transmitted waves are A_r and A_t, then

$$\boxed{A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i} \quad \text{and} \quad \boxed{A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i}$$

From the above two expressions, we can make the following conclusions :

Conclusion 1 If $v_1 = v_2$, then $A_r = 0$ and $A_t = A_i$

Basically $v_1 = v_2$ means both media are same from wave point of view. So, in this case there is no reflection ($A_r = 0$), only transmission ($A_t = A_i$) is there.

Conclusion 2 If $v_2 < v_1$, then A_r comes out to be negative. Now, $v_2 < v_1$ means the second medium is denser. A_r in this case is negative means, there is a phase change of π.

Conclusion 3 If $v_2 > v_1$, then $A_t > A_i$. This implies that amplitude always increases as the wave travels from a denser medium to rarer medium (as $v_2 > v_1$, so second medium is rarer).

(vi) Power At the boundary of two media,
energy incident per second = energy reflected per second + energy transmitted per second.

or

Power incident = power reflected + power transmitted

or

$$\boxed{P_i = P_r + P_t}$$

Let us summarise the above discussion in tabular form as below.

Table 18.2

Wave property	Reflection	Transmission (Refraction)
v	does not change	changes
f, T, ω	do not change	do not change
λ, k	do not change	change
A, I	change	change
ϕ	$\Delta\phi = 0$, from a rarer medium $\Delta\phi = \pi$, from a denser medium	$\Delta\phi = 0$

Example 18.10 *Two strings* 1 *and* 2 *are taut between two fixed supports (as shown in figure) such that the tension in both strings is same. Mass per unit length of* 2 *is more than that of* 1. *Explain which string is denser for a transverse travelling wave.*

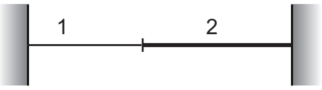


Fig. 18.13

Solution Speed of a transverse wave on a string,

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad v \propto \frac{1}{\sqrt{\mu}}$$

Now, $\mu_2 > \mu_1$ (given)

$\therefore$ $v_2 < v_1$

i.e. medium 2 is denser and medium 1 is rarer.

Example 18.11 *A sound wave and a light wave are reflected and refracted (or transmitted) from water surface. Show the changes in different physical quantities associated with a wave.*

Solution **Sound wave** We have seen in article 18.6, that water is a rarer medium for sound. When a wave travels from a denser to rarer medium, it bends away from normal. Its speed, wavelength and amplitude increase but angular wave number decreases. Further, in reflection from a rarer medium and in transmission there is no change in phase angle.

In the figure shown, 1 is incident wave 2 is reflected wave and 3 is transmitted wave.

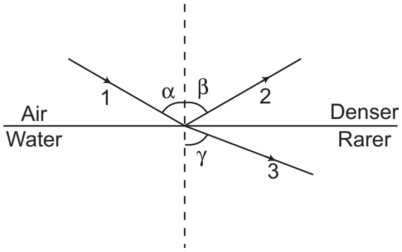


Fig. 18.14

$\alpha = \beta, \gamma > \alpha, f_1 = f_2 = f_3, v_1 = v_2, \lambda_1 = \lambda_2, k_1 = k_2, v_1 < v_3, \lambda_3 > \lambda_1 = \lambda_2, k_3 < k_1$

$\Delta\phi_{12} = \Delta\phi_{13} = 0°$.

Electromagnetic Wave For electromagnetic wave, water is denser medium. So, in water, ray of light will bend towards normal.

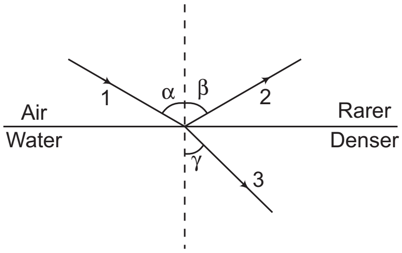


Fig. 18.15

Further, speed and wavelength in this medium will decrease. But value of k will increase.

$\alpha = \beta, \gamma < \alpha, f_1 = f_2 = f_3, v_1 = v_2, \lambda_1 = \lambda_2, k_1 = k_2, v_3 < v_1, \lambda_3 < \lambda_1, k_3 > k_1$

$\Delta\phi_{12} = \pi$ and $\Delta\phi_{13} = 0°$

Example 18.12 *A harmonic wave is travelling on string 1. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and that the boundary between the two strings is at $x = 0$. If the expression for the incident wave is*

$$y_i = A_i \cos(k_1 x - \omega_1 t)$$

(a) What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?

(b) Show that the average power carried by the incident wave is equal to the sum of the average power carried by the transmitted and reflected waves.

Solution (a) Since, $v = \sqrt{T/\mu}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

We have,
$$v_2 = \frac{v_1}{2} \qquad \text{...(i)}$$

From Table 18.2, we can see that the frequency does not change, that is

$$\omega_1 = \omega_2 \qquad \text{...(ii)}$$

Also, because $k = \dfrac{\omega}{v}$, the wave numbers of the harmonic waves in the two strings are related by

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\frac{\omega_1}{v_1} = 2k_1 \qquad \ldots\text{(iii)}$$

The amplitudes are $$A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i = \left[\frac{2(v_1/2)}{v_1 + (v_1/2)}\right] = \frac{2}{3}A_i \qquad \ldots\text{(iv)}$$

and $$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i = \left[\frac{(v_1/2) - v_1}{v_1 + (v_1/2)}\right]A_i = -\frac{A_i}{3} \qquad \ldots\text{(v)}$$

Now with Eqs. (ii), (iii) and (iv), the transmitted wave can be written as

$$y_t = \frac{2}{3}A_i \cos(2k_1 x - \omega_1 t)$$ **Ans.**

Similarly the reflected wave can be expressed as

$$y_r = -\frac{A_i}{3}\cos(k_1 x + \omega_1 t)$$

$$= \frac{A_i}{3}\cos(k_1 x + \omega_1 t + \pi)$$ **Ans.**

(b) The average power of a harmonic wave on a string is given by

$$P = \frac{1}{2}\rho A^2 \omega^2 S v = \frac{1}{2}A^2\omega^2 \mu v \qquad (\text{as } \rho S = \mu)$$

Now, $$P_i = \frac{1}{2}\omega_1^2 A_i^2 \mu_1 v_1 \qquad \ldots\text{(vi)}$$

$$P_t = \frac{1}{2}\omega_1^2\left(\frac{2}{3}A_i\right)^2 (4\mu_1)\left(\frac{v_1}{2}\right) = \frac{4}{9}\omega_1^2 A_i^2 \mu_1 v_1 \qquad \ldots\text{(vii)}$$

and $$P_r = \frac{1}{2}\omega_2^2\left(-\frac{A_i}{3}\right)^2 (\mu_1)(v_1) = \frac{1}{18}\omega_1^2 A_i^2 \mu_1 v_1 \qquad \ldots\text{(viii)}$$

From Eqs. (vi), (vii) and (viii), we can show that

$$P_i = P_t + P_r$$ **Hence proved.**

Example 18.13 *From energy conservation principle prove the relations,*

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i \quad \text{and} \quad A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i$$

Here, symbols have their usual meanings.

Solution By conservation of energy, the average incident power equals the average reflected power plus the average transmitted power

or $$P_i = P_r + P_t$$

$\therefore$ $$\frac{1}{2}\mu_1\omega^2 A_i^2 v_1 = \frac{1}{2}\mu_1\omega^2 A_r^2 v_1 + \frac{1}{2}\mu_2\omega^2 A_t^2 v_2$$

or $$\left(\frac{T}{v_1^2}\right)\omega^2 A_i^2 v_1 = \left(\frac{T}{v_1^2}\right)\omega^2 A_r^2 v_1 + \left(\frac{T}{v_2^2}\right)\omega^2 A_t^2 v_2$$

or $$\frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2} \qquad \ldots\text{(i)}$$

Further $$A_i + A_r = A_t \qquad \ldots\text{(ii)}$$

Solving these two equations for A_r and A_t, we get

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i$$

and $$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$ **Hence proved.**

INTRODUCTORY EXERCISE 18.4

1. Two pulses of identical shape overlap such that the displacement of the rope is momentarily zero at all points, what happens to the energy at this time?
2. The pulse shown in figure has a speed of 10 cm/s.

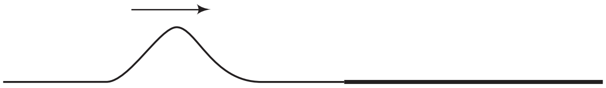


Fig. 18.16

   (a) If the linear mass density of the right string is 0.25 that of the left string, at what speed does the transmitted pulse travel?
   (b) Compare the heights of the transmitted pulse and the reflected pulse to that of the incident pulse.
3. The harmonic wave $y_i = (2.0 \times 10^{-3})\cos \pi(2.0x - 50t)$ travels along a string toward a boundary at $x = 0$ with a second string. The wave speed on the second string is 50 m/s. Write expressions for reflected and transmitted waves. Assume SI units.

Final Touch Points

1. Until now, we have come across the following three sets of equations

$$\left.\begin{array}{l} y = A\sin(\omega t \pm \phi) \\ y = A\cos(\omega t \pm \phi) \end{array}\right\} \text{SHM}$$

$$\left.\begin{array}{l} y = A\sin(kx \pm \omega t \pm \phi) \\ y = A\cos(kx \pm \omega t \pm \phi) \end{array}\right\} \text{Travelling wave}$$

$$\left.\begin{array}{lll} y = A\sin kx\cos\omega t & \text{or} & 2A\sin kx\cos\omega t \\ y = 2A\sin\omega t\cos kx & \text{or} & A\sin\omega t\cos kx \\ y = A\sin kx\sin\omega t & \text{or} & 2A\sin kx\sin\omega t \\ y = 2A\cos kx\cos\omega t & \text{or} & A\cos kx\cos\omega t \end{array}\right\} \text{Standing wave}$$

Note *In standing waves, we have given four sets of equations. The equation of standing wave basically depends on the component waves. Further, if the maximum amplitude is 2A, it means amplitude of travelling waves is A and if it is A, then amplitude of travelling waves is* $\frac{A}{2}$.

2. **Resonance** If the string (discussed in Art 18.5) is set vibrating in anyone of the normal modes and left to itself, the oscillations gradually die out. The motion is damped by dissipation of energy through the elastic supports at the ends and by the resistance of the air to the motion. We can pump energy into the system by applying a driving force. If the driving frequency is equal to any natural frequency of the string, the string will vibrate at that frequency with a larger amplitude. This phenomenon is called **resonance**. Because the string has a large number of natural frequencies, resonance can occur at many different frequencies.

3. **Reason of phase change of π in reflection from a denser medium** When a transverse wave is reflected at the rigid support, a phase change of π takes place in the displacement. Suppose a transverse pulse is produced along a string PQ whose end Q is attached to a rigid support. If the pulse travels in the form of a crest, then on reaching the rigid support, the pulse would exert an upward force. Since the support is rigid, it remains unaffected but a reaction force act in the downward direction on the string which reverses the sign of the displacement of the particle at Q. As a result, an inverted pulse in the form of a trough travels back as shown.

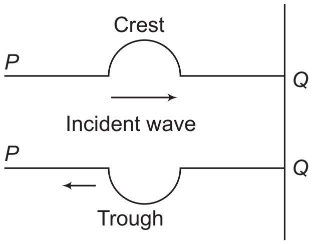


Fig. 18.17

This concept can also be understood by a second thought process. Incident wave and reflected wave are two identical waves travelling in opposite directions. So, they should produce a stationary wave. Point Q is a fixed point, which cannot oscillate at all. So, Q should become a node. This is possible only when the two displacements (one due to incident wave y_1 and the other due to reflected wave y_2) are always equal and opposite ($y = y_1 + y_2 = 0$ or $y_1 = -y_2$). Or, they have a phase difference of π.

Solved Examples

TYPED PROBLEMS

Type 1. *To find length of oscillating wire from the equation of stationary wave*

How to Solve?

- From the given equation of stationary wave find the value of wave number k and then λ. Now, from the given mode of oscillations we can also find number of loops. Suppose number of loops are n and one loop size is $\frac{\lambda}{2}$. Then, total length of wire will be

$$l = n\left(\frac{\lambda}{2}\right)$$

Example 1 *A stretched wire is oscillating in third overtone mode. Equation of transverse stationary wave produced in this wire is*

$$y = A\sin(6\pi x)\sin(20\pi t)$$

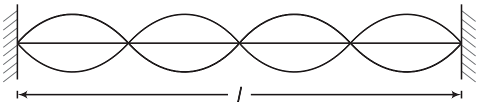
Here, x is in metres. Find the length of the wire.

Solution From the given equation we can see that

$$k = 6\pi \text{ m}^{-1}$$

$$\therefore \quad \lambda = \frac{2\pi}{k} = \frac{2\pi}{6\pi} = \frac{1}{3}\text{ m}$$

Now, third overtone mode means four loops as shown below



One loop size is $\frac{\lambda}{2}$, therefore total length of wire,

$$l = 4\left(\frac{\lambda}{2}\right) = 2\lambda$$

$$= 2\left(\frac{1}{3}\right)\text{ m}$$

$$= \frac{2}{3}\text{ m}$$

Ans.

Type 2. *To find amplitude at some given point in stationary wave*

Concept

In a stationary wave amplitude at different points varies from 0 at node (when $A_1 = A_2 = A_0$) to maximum (= $2A_0$) at antinode. If by some how, $A_{\max}$, k and position of node or position of antinode is known then we can find amplitude at any general point. For example, if position of node is known, then take this point as $x = 0$. Now, amplitude at any general point x is

$$A_x = A_{\max} \sin kx$$

In this expression we can see that $A_x = 0$ at $x = 0$.

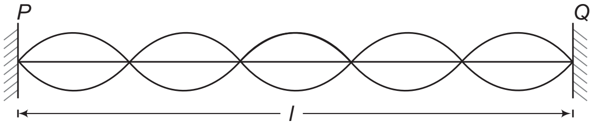
Similarly, if position of antinode is known then amplitude at general point x is

$$A_x = A_{\max} \cos kx$$

Again we can see that, $A_x = A_{\max}$ at $x = 0$.

Example 2 *Length of a stretched wire is 2m. It is oscillating in its fourth overtone mode. Maximum amplitude of oscillations is 2mm. Find amplitude of oscillation at a distance of 0.2 m from one fixed end.*

Solution Fourth overtone mode means five loops.



$$\therefore \qquad l = 5\left(\frac{\lambda}{2}\right)$$

$$\text{or} \qquad \lambda = \frac{2l}{5}$$

$$= \frac{2 \times 2}{5} = 0.8 \text{ m}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8}$$

$$= (2.5\,\pi)\ \text{m}^{-1}$$

Now, P is a node. So, take $x = 0$ at P.

Then, at a distance x from P,

$$A_x = A_{\max} \sin kx$$

$$= (2 \text{ mm}) \sin (2.5\,\pi)(0.2)$$

$$= (2 \text{ mm}) \sin \left(\frac{\pi}{2}\right)$$

$$= 2 \text{ mm}$$

Ans.

Note *At $x = 0.2$ m, we are getting $A_x = A_{\max}$.*

Hence, $x = 0.2$ m is an antinode.

Type 3. *To find energy of oscillation in a given portion in stationary wave*

Concept

In SHM, energy of oscillation of a single particle is given by

$$E = \frac{1}{2} m\omega^2 A^2$$

In a travelling sine wave, all particles oscillate with same amplitude and same frequency. Energy density (or energy per unit volume) is given by

$$u = \frac{1}{2}\rho\omega^2 A^2$$

Since ρ, ω and A is same at all points. Therefore, energy density will be same at all points. So, in each unit volume equal amount of energy will be stored. Therefore, to find total energy of oscillation in a given volume, we just multiply the given volume by this constant energy density.

In stationary wave, amplitude is different at different positions. So, energy density will be non-uniform. Hence, total energy of oscillation in a given volume can be obtained by integration.

There is an alternate method also. By adding two energy densities of the travelling waves (from which standing wave is formed) first of all we find the total energy density. Then, we multiply the given volume with this energy density.

Example 3 *A standing wave is formed by two harmonic waves, $y_1 = A\sin(kx - \omega t)$ and $y_2 = A\sin(kx + \omega t)$ travelling on a string in opposite directions. Mass density of the string is ρ and area of cross-section is s. Find the total mechanical energy between two adjacent nodes on the string.*

Solution The distance between two adjacent nodes is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$.

∴ Volume of string between two nodes will be

$$V = (\text{area of cross-section}) \times (\text{distance between two nodes})$$

$$= (S)\left(\frac{\pi}{k}\right)$$

Energy density (energy per unit volume) of a travelling wave is given by

$$u = \frac{1}{2}\rho A^2\omega^2$$

A standing wave is formed by two identical waves travelling in opposite directions. Therefore, the energy stored between two nodes in a standing wave.

$$E = 2\left[\text{energy stored in a distance of } \frac{\pi}{k} \text{ of a travelling wave}\right]$$

$$= 2\,(\text{energy density})\,(\text{volume})$$

$$= 2\left(\frac{1}{2}\rho A^2\omega^2\right)\left(\frac{\pi S}{k}\right)$$

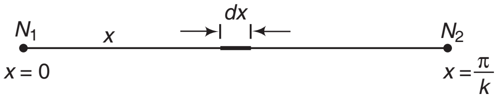
or
$$E = \frac{\rho A^2\omega^2\pi S}{k}$$
Ans.

Alternate Method The equation of the standing wave

$$y = y_1 + y_2 = 2A \sin kx \cos \omega t = A_x \cos \omega t$$

Here, $A_x = 2A \sin kx$

i.e. first node is at $x = 0$ and the next node is at $x = \frac{\pi}{k}$. Let us take an element of length dx at a distance x from N_1. Mass of this element is dm $(= \rho S dx)$. This can be treated as point mass. This element oscillates simple harmonically with angular frequency ω and amplitude $2A \sin kx$.



Hence, energy of this element

$$dE = \frac{1}{2}(dm)\,(2A \sin kx)^2(\omega^2)$$

$$dE = \frac{1}{2}(\rho S dx)\,(2A \sin kx)^2 \omega^2$$

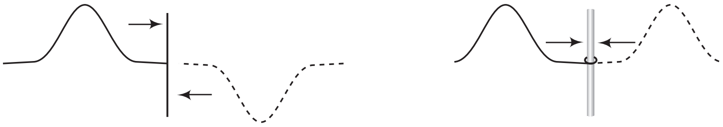
Integrating this with the limits from $x = 0$ to $x = \frac{\pi}{k}$, we get the same result.

Type 4. *Based on the formation of the reflected pulse, either from a rigid boundary or from a free boundary*

Concept

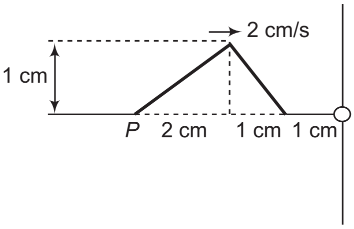
The formation of the reflected pulse can be obtained by overlapping two pulses travelling in opposite directions.

The net displacement at any point is given by the principle of superposition.



(a) Rigid boundary (b) Free boundary

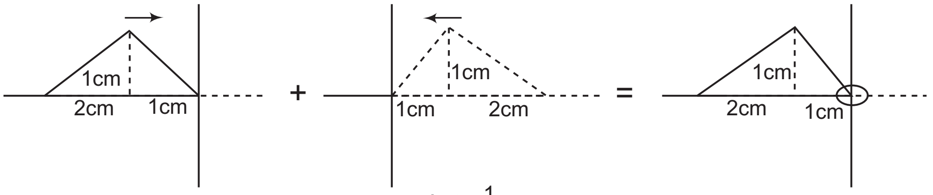
Example 4 *A triangular wave pulse moving at 2 cm/s on a rope approached an end at which it is free to slide on a vertical pole.*



(a) *Draw the pulse at $\frac{1}{2}$ s interval until it is completely reflected.*

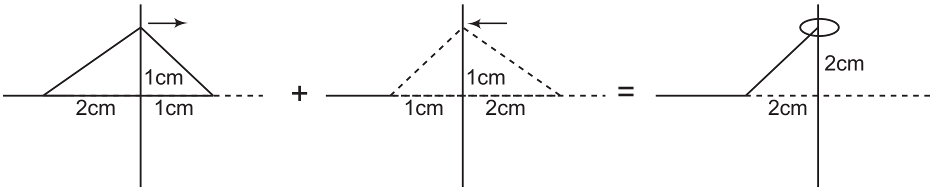
(b) *What is the particle speed on the trailing edge P at the instant depicted?*

Solution (a) Reflection of pulse from a free boundary is really the superposition of two identical waves travelling in opposite directions. This can be shown as under.



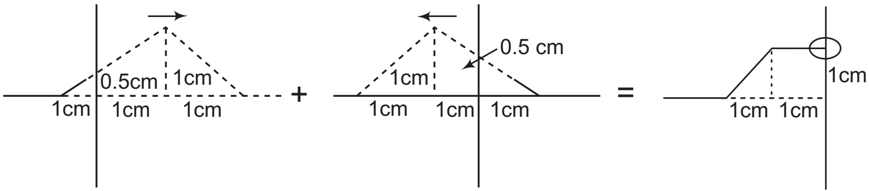
At $t = \frac{1}{2}$ s

(a) (b) (c)



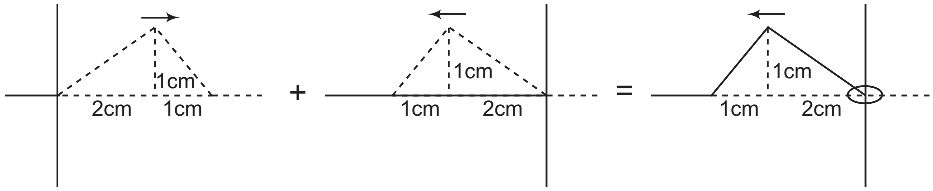
At $t = 1$ s

(d) (e) (f)



At $t = 1\frac{1}{2}$ s

(g) (h) (i)



At $t = 2$ s

(j) (k) (l)

In every $\frac{1}{2}$ s, each pulse (one real moving towards right and one imaginary moving towards left) travels a distance of 1 cm as the wave speed is 2 cm/s.

(b) Particle speed, $v_P = |-v(\text{slope})|$

Here, $v = \text{wave speed} = 2$ cm/s

and $\text{slope} = \frac{1}{2}$

$\therefore$ Particle speed $= 1$ cm/s

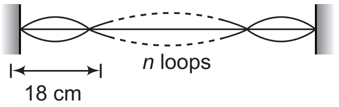
Miscellaneous Examples

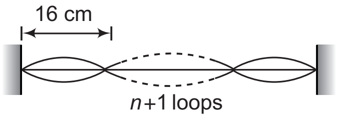
Example 5 *A string fixed at both ends has consecutive standing wave modes for which the distances between adjacent nodes are* 18 *cm and* 16 *cm, respectively.*

(a) What is the minimum possible length of the string?

(b) If the tension is 10 *N and the linear mass density is* 4 *g/m, what is the fundamental frequency?*

Solution (a)





Let l be the length of the string. Then,

$$18n = l \qquad \text{...(i)}$$

$$16(n + 1) = l \qquad \text{...(ii)}$$

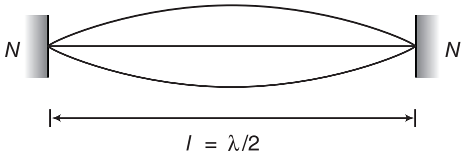
From Eqs. (i) and (ii), wet get

$$n = 8$$

and $$l = 144 \text{ cm}$$

Therefore, the minimum possible length of the string can be 144 cm. **Ans.**

(b) For fundamental frequency, $l = \dfrac{\lambda}{2}$



or $$\lambda = 2l = 288 \text{ cm}$$
$$= 2.88 \text{ m}$$

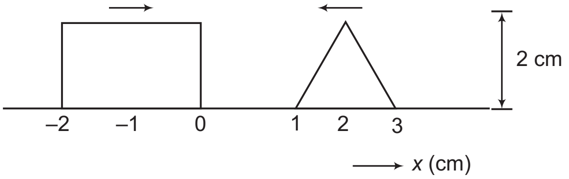
Speed of wave on the string,

$$v = \sqrt{\frac{T}{\mu}}$$
$$= \sqrt{\frac{10}{4 \times 10^{-3}}} = 50 \text{ m/s}$$

$\therefore$ Fundamental frequency,

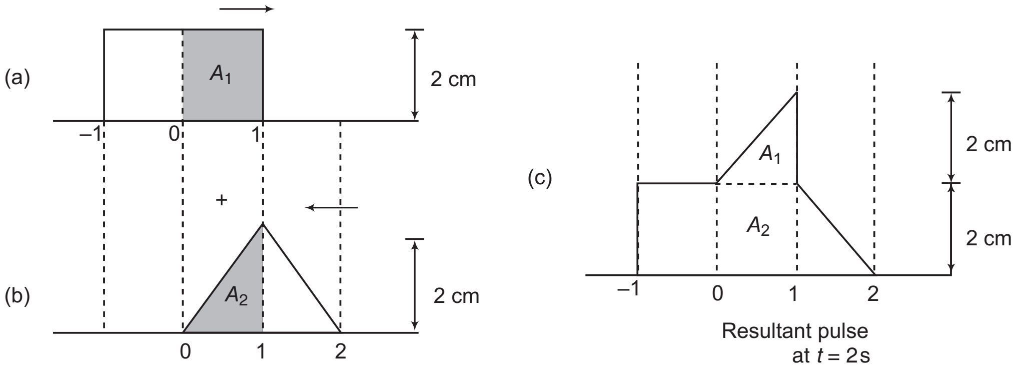
$$f = \frac{v}{\lambda} = \frac{50}{2.88}$$
$$= 17.36 \text{ Hz}$$ **Ans.**

Example 6 *Figure shows a rectangular pulse and triangular pulse approaching towards each other. The pulse speed is* 0.5 *cm/s. Sketch the resultant pulse at* $t = 2$ *s.*



Solution In 2 s each pulse will travel a distance of 1 cm.

The two pulses overlap between 0 and 1 cm as shown in figure. So, A_1 and A_2 can be added as shown in Fig. (c).



Example 7 *Two wires are fixed on a sonometer. Their tensions are in the ratio* $8:1$, *their lengths are in the ratio* $36:35$, *the diameters are in the ratio* $4:1$ *and densities are in the ratio* $1:2$. *Find the value of lower frequency if higher frequency is* 360 *Hz.*

Solution Given,

$$\frac{T_1}{T_2} = \frac{8}{1}, \frac{L_1}{L_2} = \frac{36}{35}$$

$$\frac{D_1}{D_2} = \frac{4}{1}, \frac{\rho_1}{\rho_2} = \frac{1}{2}$$

Let μ_1 and μ_2 be the linear mass densities, then

$$\therefore \quad \mu_1 = \pi \times \frac{D_1^2}{4} \times \rho_1 \text{ and } \mu_2 = \pi \times \frac{D_2^2}{4} \times \rho_2 \qquad (\mu = \rho S)$$

$$\therefore \quad \frac{\mu_1}{\mu_2} = \left(\frac{D_1}{D_2}\right)^2 \times \frac{\rho_1}{\rho_2} = \left(\frac{4}{1}\right)^2 \times \frac{1}{2} = \frac{8}{1}$$

$$\therefore \quad \frac{f_1}{f_2} = \frac{L_2}{L_1} \times \sqrt{\frac{T_1}{T_2} \times \frac{\mu_2}{\mu_1}} = \frac{35}{36}\sqrt{\frac{8}{1} \times \frac{1}{8}} = \frac{35}{36} \qquad \left(f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}\right)$$

$$f_2 > f_1$$

We have $f_2 = 360$

$\therefore$ $f_1 = 350$ **Ans.**

Example 8 *In a stationary wave pattern that forms as a result of reflection of waves from an obstacle the ratio of the amplitude at an antinode and a node is* $\beta = 1.5$. *What percentage of the energy passes across the obstacle?*

Solution
$$\frac{A_{\max}}{A_{\min}} = \frac{A_i + A_r}{A_i - A_r}$$

This ratio is given as 1.5 or $\frac{3}{2}$.

$$\therefore \qquad \frac{A_i + A_r}{A_i - A_r} = \frac{3}{2} \quad \text{or} \quad \frac{1 + \frac{A_r}{A_i}}{1 - \frac{A_r}{A_i}} = \frac{3}{2}$$

Solving this equation, we get
$$\frac{A_r}{A_i} = \frac{1}{5}$$

$$\therefore \qquad \frac{I_r}{I_i} = \left(\frac{A_r}{A_i}\right)^2 = \frac{1}{25}$$

or
$$I_r = 0.04\, I_i$$

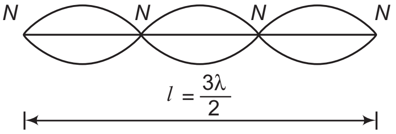
i.e. 4% of the incident energy is reflected or 96% energy passes across the obstacle. **Ans.**

Example 9 *A string of linear mass density* 5.0×10^{-3} *kg/m is stretched under a tension of* 65 N *between two rigid supports* 60 cm *apart.*

(a) If the string is vibrating in its second overtone so that the amplitude at one of its antinodes is 0.25 cm*, what are the maximum transverse speed and acceleration of the string at antinodes?*

(b) What are these quantities at a distance 5.0 cm *from an node?*

Solution (a) In second overtone,



$$l = \frac{3\lambda}{2}$$

or
$$\lambda = \frac{2l}{3} = \frac{2 \times 60}{3}$$
$$= 40 \text{ cm} = 0.4 \text{ m}$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi \text{ m}^{-1}$$
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{65}{5.0 \times 10^{-3}}}$$
$$= 114 \text{ m/s}$$
$$\therefore \qquad \omega = kv = 570\pi \text{ rad/s}$$

Maximum transverse speed at antinode = $A_0\omega$

Here, A_0 = amplitude of antinode = 0.25 cm = 2.5×10^{-3} m

$\therefore$ $\qquad$ Maximum speed $= (2.5 \times 10^{-3})(570\,\pi)$ m/s

$= 4.48$ m/s **Ans.**

Maximum acceleration $= \omega^2 A_0$

$= (570\pi)^2\,(2.5 \times 10^{-3})$ m/s^2

$= 8.0 \times 10^3$ m/s^2 **Ans.**

(b) At a distance *x* from the node, the amplitude can be written as

$$A = A_0 \sin kx = (2.5 \times 10^{-3}) \sin(5\pi x) \text{ metre}$$

Here, *x* is in metres.

Therefore, at $\qquad x = 5.0 \text{ cm} = 5.0 \times 10^{-2}$ m

$$A = (2.5 \times 10^{-3}) \sin(5\pi \times 5.0 \times 10^{-2})$$

$$= 1.8 \times 10^{-3} \text{ m}$$

Maximum speed $= A\omega$

$= (1.8 \times 10^{-3})\,(570\pi)$ m/s

$= 3.22$ m/s **Ans.**

and maximum acceleration $= \omega^2 A$

$= (570\pi)^2\,(1.8 \times 10^{-3})$ m/s^2

$= 5.8 \times 10^3$ m/s^2 **Ans.**

Example 10 *An aluminium wire of cross-sectional area* 10^{-6} m^2 *is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer pulled by a weight of* 10 *kg. The total length of the compound wire between the bridges is* 1.5 *m of which the aluminium wire is* 0.6 *m and the rest is steel wire. Transverse vibrations are setup in the wire by using an external source of variable frequency. Find the lowest frequency of excitation for which the standing waves are formed such that the joint in the wire is a node. What is the total number of nodes at this frequency? The density of aluminium is* 2.6×10^3 kg/m^3 *and that of steel is* 1.04×10^4 $kg/m^3 (g = 10\ m/s^2)$.

Solution Let n_a loops are formed in aluminium wire and n_s in steel. Then,

$$f_a = f_s$$

or
$$n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

or
$$\frac{n_a}{n_s} = \left(\frac{v_s}{v_a}\right)\left(\frac{l_a}{l_s}\right)$$

But,
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho s}} \propto \frac{1}{\sqrt{\rho}}$$

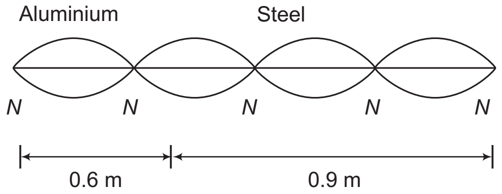
Therefore,
$$\frac{v_s}{v_a} = \sqrt{\frac{\rho_a}{\rho_s}}$$

$\therefore$
$$\frac{n_a}{n_s} = \sqrt{\frac{\rho_a}{\rho_s}} \cdot \frac{l_a}{l_s}$$

Substituting the values, we have

$$\frac{n_a}{n_s} = \sqrt{\frac{2.6 \times 10^3}{1.04 \times 10^4}} \cdot \frac{0.6}{0.9} = \frac{1}{3}$$

i.e. at lowest frequency, one loop is formed in aluminium wire and three loops are formed in steel wire as shown in figure.



$$\therefore \qquad f_{\min} = n_a\left(\frac{v_a}{2l_a}\right) = \frac{1}{2l_a}\sqrt{\frac{T}{\rho_a S}} \qquad (T = mg = 100\text{ N})$$

$$= \frac{1}{2 \times 0.6}\sqrt{\frac{10 \times 10}{2.6 \times 10^3 \times 10^{-6}}}$$

or $f_{\min} = 163.4$ Hz **Ans.**

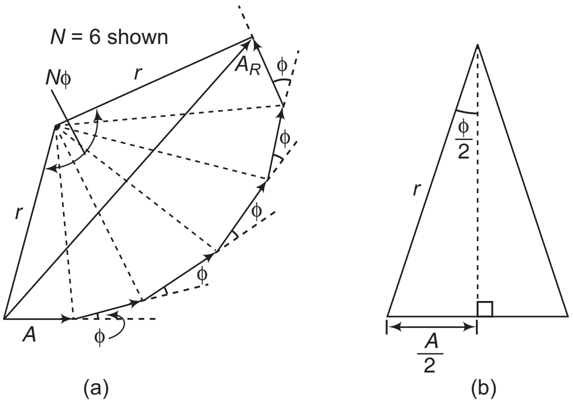
Total number of nodes are five as shown is figure. **Ans.**

Note *In such type of problems nature of junction will be known to us. Then, we have to equate frequencies on the two sides. By equating the frequencies, we find* $\frac{n_1}{n_2}$. *Suppose this comes out to be* 0.4 . *Write it,* $\frac{n_1}{n_2} = \frac{2}{5}$.

At lowest oscillation frequency 2 *loops are formed on side* 1 *and* 5 *on side* 2. *At next higher frequency* 4 *loops will be formed on side* 1 *and* 10 *on side* 2 *and so on.*

Example 11 *Find the resultant amplitude and phase of a point at which N sinusoidal waves interfere. All the waves have same amplitude A and their phases increase in arithmetic progression of common difference* ϕ.

Solution The diagram for their sum is shown in figure for $N = 6$. The resultant amplitude is A_R. The apex angle of every isosceles triangle is ϕ. So, the angle subtended by the resultant is $N\phi$. Since, the heads of the vectors are all at the same distance r from the apex of the diagram.



(a) (b)

From Fig. (a),

$$\frac{A_R}{2} = r \sin \frac{N\phi}{2} \qquad \text{...(i)}$$

and from Fig. (b),

$$\frac{A}{2} = r \sin \frac{\phi}{2} \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

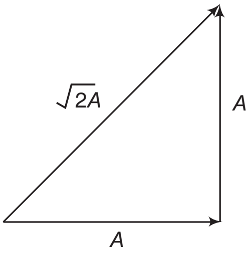
$$\frac{A_R}{A} = \frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}$$

or

$$\boxed{A_R = A\left(\frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}\right)} \qquad \textbf{Ans.}$$

Note *Suppose $N = 2$ and $\phi = 90°$, then*

$$A_R = \sqrt{2}A$$



Similarly, we can also check the above result for other special cases.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Two waves $y_1 = A\sin(\omega t + kx)$ and $y_2 = A\cos(\omega t - kx)$ are superimposed, then $x = 0$ becomes a node.

Reason : At node net displacement due to two waves should be zero.

2. Assertion : Stationary waves are so called because particles are at rest in stationary waves.

Reason : They are formed by the superposition of two identical waves travelling in opposite directions.

3. Assertion : When a wave travels from a denser medium to rarer medium, then amplitude of oscillation increases.

Reason : In denser medium, speed of wave is less compared to a rarer medium.

4. Assertion : A wire is stretched and then fixed at two ends. It oscillates in its second overtone mode. There are total four nodes and three antinodes.

Reason : In second overtone mode, length of wire should be $l = \frac{3\lambda}{2}$, where λ is wavelength.

5. Assertion : If we see the oscillations of a stretched wire at higher overtone mode, frequency of oscillation increases but wavelength decreases.

Reason : From $v = f\lambda$, $\lambda \propto \frac{1}{f}$ as v = constant.

6. Assertion : Standing waves are formed when amplitudes of two constituent waves are equal.

Reason : At any point net displacement at a given time is resultant of displacement of constituent waves.

7. Assertion : In a standing wave $x = 0$ is a node. Then, total mechanical energy lying between $x = 0$ and $x = \frac{\lambda}{8}$ is not equal to the energy lying between $x = \frac{\lambda}{8}$ and $x = \frac{\lambda}{4}$.

Reason : In standing waves different particles oscillate with different amplitudes.

8. Assertion : Ratio of maximum intensity and minimum intensity in interference is 25 : 1.

The amplitude ratio of two waves should be 3 : 2.

Reason : $\frac{I_{\max}}{I_{\min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2$

9. **Assertion :** Three waves of equal amplitudes interfere at a point. Phase difference between two successive waves is $\frac{\pi}{2}$. Then, resultant intensity is same as the intensity due to individual wave.

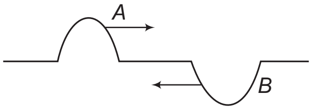
   Reason : For interference to take place sources must be coherent.

10. **Assertion :** For two sources to be coherent phase difference between two waves at all points should be same.

    Reason : Two different light sources are never coherent.

Objective Questions

1. Two identical harmonic pulses travelling in opposite directions in a taut string approach each other. At the instant when they completely overlap, the total energy of the string will be



   (a) zero (b) partly kinetic and partly potential
   (c) purely kinetic (d) purely potential

2. Three coherent waves having amplitudes 12 mm, 6 mm and 4 mm arrive at a given point with successive phase difference of $\pi/2$. Then, the amplitude of the resultant wave is
   (a) 7 mm (b) 10 mm
   (c) 5 mm (d) 4.8 mm

3. Two transverse waves A and B superimpose to produce a node at $x = 0$. If the equation of wave A is $y = a\cos(kx + \omega t)$, then the equation of wave B is
   (a) $+a\cos(kx - \omega t)$ (b) $-a\cos(kx + \omega t)$
   (c) $-a\cos(kx - \omega t)$ (d) $+a\cos(\omega t - kx)$

4. In a stationary wave system, all the particles of the medium
   (a) have zero displacement simultaneously at some instant
   (b) have maximum displacement simultaneously at some instant
   (c) are at rest simultaneously at some instant
   (d) all of the above

5. In a standing wave on a string
   (a) In one time period all the particles are simultaneously at rest twice
   (b) All the particles must be at their positive extremes simultaneously once in a time period
   (c) All the particles may be at their positive extremes simultaneously twice in a time period
   (d) All the particles are never at rest simultaneously

6. If λ_1, λ_2 and λ_3 are the wavelength of the waves giving resonance to the fundamental, first and second overtone modes respectively in a string fixed at both ends. The ratio of the wavelengths $\lambda_1 : \lambda_2 : \lambda_3$ is
   (a) 1 : 2 : 3 (b) 1 : 3 : 5
   (c) $1 : \frac{1}{2} : \frac{1}{3}$ (d) $1 : \frac{1}{3} : \frac{1}{5}$

7. In a standing wave, node is a point of
   (a) maximum strain (b) maximum pressure
   (c) maximum density (d) All of these

8. For a certain stretched string, three consecutive resonance frequencies are observed as 105, 175 and 245 Hz respectively. Then, the fundamental frequency is

(a) 30 Hz
(b) 45 Hz
(c) 35 Hz
(d) None of these

9. A string 1 m long is drawn by a 300 Hz vibrator attached to its end. The string vibrates in three segments. The speed of transverse waves in the string is equal to

(a) 100 m/s
(b) 200 m/s
(c) 300 m/s
(d) 400 m/s

10. Three resonant frequencies of string with both rigid ends are 90, 150 and 210 Hz. If the length of the string is 80 cm, what is the speed of the transverse wave in the string?

(a) 45 m/s
(b) 75 m/s
(c) 48 m/s
(d) 80 m/s

11. The period of oscillations of a point is 0.04 s and the velocity of propagation of oscillation is 300 m/s. The difference of phases between the oscillations of two points at distance 10 m and 16 m respectively from the source of oscillations is

(a) 2π
(b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$
(d) π

12. A transverse wave described by an equation $y = 0.02 \sin (x + 30t)$, where x and t are in metre and second, is travelling along a wire of area of cross-section 1 mm^2 and density 8000 kgm^{-3}. What is the tension in the string?

(a) 20 N
(b) 7.2 N
(c) 30 N
(d) 14.4 N

Subjective Questions

1. Two waves are travelling in the same direction along a stretched string. The waves are 90° out of phase. Each wave has an amplitude of 4.0 cm. Find the amplitude of the resultant wave.

2. Two wires of different densities are soldered together end to end then stretched under tension T. The wave speed in the first wire is twice that in the second wire.

(a) If the amplitude of incident wave is A, what are amplitudes of reflected and transmitted waves?
(b) Assuming no energy loss in the wire, find the fraction of the incident power that is reflected at the junction and fraction of the same that is transmitted.

3. A wave is represented by

$$y_1 = 10 \cos (5x + 25\,t)$$

where, x is measured in metres and t in seconds. A second wave for which

$$y_2 = 20 \cos \left(5x + 25t + \frac{\pi}{3}\right)$$

interferes with the first wave. Deduce the amplitude and phase of the resultant wave.

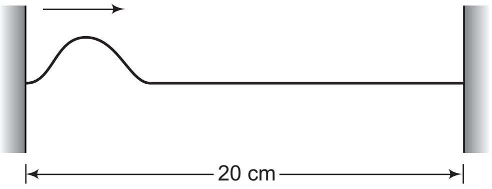
4. Two waves passing through a region are represented by

$$y_1 = (1.0\text{ cm}) \sin [(3.14\text{ cm}^{-1})\ x - (157\text{ s}^{-1})\ t]$$

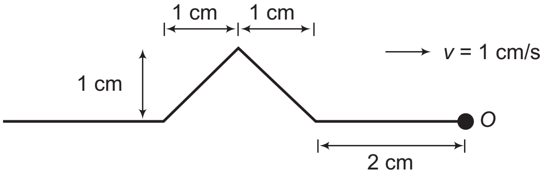
and
$$y_2 = (1.5\text{ cm}) \sin [(1.57\text{ cm}^{-1})\ x - (314\text{ s}^{-1})\ t]$$

Find the displacement of the particle at $x = 4.5$ cm at time $t = 5.0$ ms.

5. A string of length 20 cm and linear mass density 0.4 g/cm is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in figure which travels towards the other end. (a) When will the string have the shape shown in the figure again? (b) Sketch the shape of the string at a time half of that found in part (a).



6. A wave pulse on a string has the dimensions shown in figure. The wave speed is $v = 1$ cm/s.



(a) If point O is a fixed end, draw the resultant wave on the string at $t = 3$ s and $t = 4$ s.

(b) Repeat part (a) for the case in which O is a free end.

7. Two sinusoidal waves combining in a medium are described by the equations

$$y_1 = (3.0\text{ cm})\sin\pi(x + 0.60\,t)$$

and
$$y_2 = (3.0\text{ cm})\sin\pi(x - 0.60\,t)$$

where, x is in centimetres and t is in seconds. Determine the maximum displacement of the medium at

(a) $x = 0.250$ cm,

(b) $x = 0.500$ cm and

(c) $x = 1.50$ cm.

(d) Find the three smallest values of x corresponding to antinodes.

8. A standing wave is formed by the interference of two travelling waves, each of which has an amplitude $A = \pi$ cm, angular wave number $k = (\pi/2)$ per centimetre.

(a) Calculate the distance between two successive antinodes.

(b) What is the amplitude of the standing wave at $x = 0.50$ cm from a node?

9. Find the fundamental frequency and the next three frequencies that could cause a standing wave pattern on a string that is 30.0 m long, has a mass per unit length of 9.00×10^{-3} kg/m and is stretched to a tension of 20.0 N.

10. A string vibrates in its first normal mode with a frequency of 220 vibrations/s. The vibrating segment is 70.0 cm long and has a mass of 1.20 g.

(a) Find the tension in the string.

(b) Determine the frequency of vibration when the string vibrates in three segments.

11. A 60.0 cm guitar string under a tension of 50.0 N has a mass per unit length of 0.100 g/cm. What is the highest resonance frequency of the string that can be heard by a person able to hear frequencies upto 20000 Hz?

12. A wire having a linear density of 0.05 g/cm is stretched between two rigid supports with a tension of 450 N. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire.

13. The vibrations from an 800 Hz tuning fork set up standing waves in a string clamped at both ends. The wave speed in the string is known to be 400 m/s for the tension used. The standing wave is observed to have four antinodes. How long is the string?

14. A string vibrates in 4 segments to a frequency of 400 Hz.

(a) What is its fundamental frequency?

(b) What frequency will cause it to vibrate into 7 segments?

15. A sonometer wire has a total length of 1 m between the fixed ends. Where should the two bridges be placed below the wire so that the three segments of the wire have their fundamental frequencies in the ratio 1 : 2 : 3 ?

16. A guitar string is 90 cm long and has a fundamental frequency of 124 Hz. Where should it be pressed to produce a fundamental frequency of 186 Hz?

17. Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the $+x$-axis and is fixed at $x = 0$.

(a) Find the displacement of a point on the string as a function of position and time.

(b) Find the speed of propagation of a transverse wave in the string.

(c) Find the amplitude at a point 3.0 cm to the right of an antinode.

18. A 1.50 m long rope is stretched between two supports with a tension that makes the speed of transverse waves 48.0 m/s. What are the wavelength and frequency of

(a) the fundamental?

(b) the second overtone?

(c) the fourth harmonic?

19. A thin taut string tied at both ends and oscillating in its third harmonic has its shape described by the equation $y(x,t) = (5.60 \text{ cm}) \sin[(0.0340 \text{ rad/cm})x] \sin[(50.0 \text{ rad/s})t]$, where the origin is at the left end of the string, the x-axis is along the string and the y-axis is perpendicular to the string.

(a) Draw a sketch that shows the standing wave pattern.

(b) Find the amplitude of the two travelling waves that make up this standing wave.

(c) What is the length of the string?

(d) Find the wavelength, frequency, period and speed of the travelling wave.

(e) Find the maximum transverse speed of a point on the string.

(f) What would be the equation $y(x, t)$ for this string if it were vibrating in its eighth harmonic?

20. A wire with mass 40.0 g is stretched so that its ends are tied down at points 80.0 cm apart. The wire vibrates in its fundamental mode with frequency 60.0 Hz and with an amplitude at the antinodes of 0.300 cm.

(a) What is the speed of propagation of transverse wave in the wire?

(b) Compute the tension in the wire.

(c) Find the maximum transverse velocity and acceleration of particles in the wire.

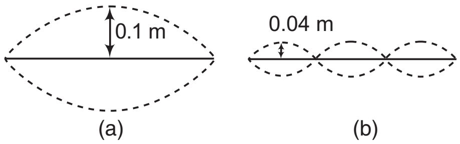
21. Two harmonic waves are represented in SI units by

$$y_1(x, t) = 0.2 \sin(x - 3.0\,t) \quad \text{and} \quad y_2(x, t) = 0.2 \sin(x - 3.0\,t + \phi)$$

(a) Write the expression for the sum $y = y_1 + y_2$ for $\phi = \dfrac{\pi}{2}$ rad.

(b) Suppose the phase difference ϕ between the waves is unknown and the amplitude of their sum is 0.32 m, what is ϕ?

22. Figure shows different standing wave patterns on a string of linear mass density 4.0×10^{-2} kg/m under a tension of 100 N. The amplitude of antinodes is indicated in each figure. The length of the string is 2.0 m.



(i) Obtain the frequencies of the modes shown in figures (a) and (b).

(ii) Write down the transverse displacement y as a function of x and t for each mode. (Take the initial configuration of the wire in each mode to be as shown by the dark lines in the figure).

23. A 160 g rope 4 m long is fixed at one end and tied to a light string of the same length at the other end. Its tension is 400 N.

(a) What are the wavelengths of the fundamental and the first two overtones?

(b) What are the frequencies of these standing waves?

[**Hint :** In this case, fixed end is a node and the end tied with the light string is antinode.]

24. A string fastened at both ends has successive resonances with wavelengths of 0.54 m for the nth harmonic and 0.48 m for the $(n+1)$ th harmonic.

(a) Which harmonics are these?

(b) What is the length of the string?

(c) What is the wavelength of the fundamental frequency?

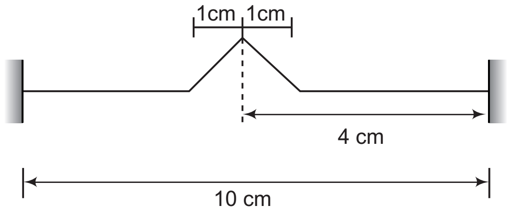
25. A wave $y_i = 0.3\cos(2.0x - 40t)$ is travelling along a string toward a boundary at $x = 0$. Write expressions for the reflected waves if

(a) the string has a fixed end at $x = 0$ and

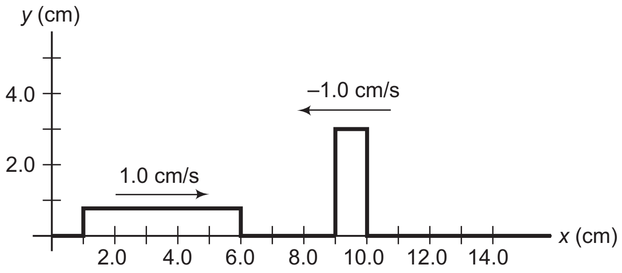
(b) the string has a free end at $x = 0$.

Assume SI units.

26. A string that is 10 cm long is fixed at both ends. At $t = 0$, a pulse travelling from left to right at 1 cm/s is 4.0 cm from the right end as shown in figure. Determine the next two times when the pulse will be at that point again. State in each case whether the pulse is upright or inverted.



27. Two pulses travelling in opposite directions along a string are shown for $t = 0$ in the figure. Plot the shape of the string at $t = 1.0$, 2.0, 3.0, 4.0 and 5.0 s respectively.



LEVEL 2

Single Correct Option

1. When tension of a string is increased by 2.5 N, the initial frequency is altered in the ratio of 3 : 2. The initial tension in the string is

(a) 6 N (b) 5 N (c) 4 N (d) 2 N

2. The lengths of two wires of same material are in the ratio 1 : 2, their tensions are in the ratio 1 : 2 and their diameters are in the ratio 1 : 3. The ratio of the notes they emit when sounded together by the same source is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$

3. If f_1, f_2 and f_3 are the fundamental frequencies of three segments into which a string is divided, then the original fundamental frequency f_0 of the whole string is

(a) $f_0 = f_1 + f_2 + f_3$ (b) $\frac{1}{f_0} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$

(c) $\frac{1}{\sqrt{f_0}} = \frac{1}{\sqrt{f_1}} + \frac{1}{\sqrt{f_2}} + \frac{1}{\sqrt{f_3}}$ (d) None of these

4. Three one-dimensional mechanical waves in an elastic medium is given as

$$y_1 = 3A\sin(\omega t - kx), \quad y_2 = A\sin(\omega t - kx + \pi) \text{ and } \quad y_3 = 2A\sin(\omega t + kx)$$

are superimposed with each other. The maximum displacement amplitude of the medium particle would be

(a) $4A$ (b) $3A$
(c) $2A$ (d) A

5. A string is stretched so that its length is increased by $\frac{1}{\eta}$ of its original length. The ratio of fundamental frequency of transverse vibration to that of fundamental frequency of longitudinal vibration will be

(a) $\eta : 1$ (b) $1 : \eta$
(c) $\sqrt{\eta} : 1$ (d) $1 : \sqrt{\eta}$

6. A string of length 1 m and linear mass density 0.01 kg/m is stretched to a tension of 100 N. When both ends of the string are fixed, the three lowest frequencies for standing wave are f_1, f_2 and f_3. When only one end of the string is fixed, the three lowest frequencies for standing wave are n_1, n_2 and n_3. Then,

(a) $n_3 = 5n_1 = f_3 = 125$ Hz (b) $f_3 = 5f_1 = n_2 = 125$ Hz

(c) $f_3 = n_2 = 3f_1 = 150$ Hz (d) $n_2 = \frac{f_1 + f_2}{2} = 75$ Hz

7. A sonometer wire has a length 114 cm between two fixed ends. Where should two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio 1 : 3 : 4

(a) $l_1 = 72$ cm, $l_2 = 24$ cm, $l_3 = 18$ cm (b) $l_1 = 60$ cm, $l_2 = 40$ cm, $l_3 = 14$ cm
(c) $l_1 = 52$ cm, $l_2 = 30$ cm, $l_3 = 32$ cm (d) $l_1 = 65$ cm, $l_2 = 30$ cm, $l_3 = 19$ cm

8. The frequency of a sonometer wire is f. The frequency becomes $f/2$ when the mass producing the tension is completely immersed in water and on immersing the mass in a certain liquid, frequency becomes $f/3$. The relative density of the liquid is

(a) 1.32 (b) 1.67 (c) 1.41 (d) 1.18

9. A string of length 1.5 m with its two ends clamped is vibrating in fundamental mode. Amplitude at the centre of the string is 4 mm. Distance between the two points having amplitude 2 mm is

(a) 1 m
(b) 75 cm
(c) 60 cm
(d) 50 cm

10. A man generates a symmetrical pulse in a string by moving his hand up and down. At $t = 0$, the point in his hand moves downwards from mean position. The pulse travels with speed 3 m/s on the string and his hand passes 6 times in each second from the mean position. Then, the point on the string at a distance 3 m will reach its upper extreme first time at $t =$

(a) 1.25 s
(b) 1 s
(c) 11/12 s
(d) $\frac{23}{24}$ s

11. Among two interfering sources, let S_1 be ahead of the phase by 90° relative to S_2. If an observation point P is such that $PS_1 - PS_2 = 1.5\,\lambda$, the phase difference between the waves from S_1 and S_2 reaching P is

(a) 3π
(b) $\frac{5\pi}{2}$
(c) $\frac{7\pi}{2}$
(d) 4π

12. A wire having a linear density 0.1 kg/m is kept under a tension of 490 N. It is observed that it resonates at a frequency of 400 Hz. The next higher frequency is 450 Hz. Find the length of the wire.

(a) 0.4 m
(b) 0.7 m
(c) 0.6 m
(d) 0.49 m

13. A string vibrates in 5 segments to a frequency of 480 Hz. The frequency that will cause it to vibrate in 2 segments will be

(a) 96 Hz
(b) 192 Hz
(c) 1200 Hz
(d) 2400 Hz

14. A wave travels on a light string. The equation of the waves is $y = A \sin(kx - \omega t + 30°)$. It is reflected from a heavy string tied to an end of the light string at $x = 0$. If 64% of the incident energy is reflected, then the equation of the reflected wave is

(a) $y = 0.8\,A \sin(kx - \omega t + 30° + 180°)$
(b) $y = 0.8\,A \sin(kx + \omega t + 30° + 180°)$
(c) $y = 0.8\,A \sin(kx - \omega t - 30°)$
(d) $y = 0.8\,A \sin(kx - \omega t + 30°)$

15. The tension, length, diameter and density of a string B are double than that of another string A. Which of the following overtones of B is same as the fundamental frequency of A?

(a) 1st
(b) 2nd
(c) 3rd
(d) 4th

More than One Correct Options

1. If the tension in a stretched string fixed at both ends is increased by 21%, the fundamental frequency is found to be changed by 15 Hz. Then, the

(a) original frequency is 150 Hz
(b) velocity of propagation of the transverse wave along the string increases by 5%
(c) velocity of propagation of the transverse wave along the string increases by 10%
(d) fundamental wavelength on the string does not change

2. For interference to take place

(a) sources must be coherent
(b) sources must have same amplitude
(c) waves should travel in opposite directions
(d) sources must have same frequency

3. Regarding stationary waves, choose the correct options.

(a) This is an example of interference
(b) Amplitudes of waves may be different
(c) Particles at nodes are always at rest
(d) Energy is conserved

4. When a wave travels from a denser to rarer medium, then

(a) speed of wave increases
(b) wavelength of wave decreases
(c) amplitude of wave increases
(d) there is no change in phase angle

5. A wire is stretched and fixed at two ends. Transverse stationary waves are formed in it. It oscillates in its third overtone mode. The equation of stationary wave is

$$y = A \sin kx \cos \omega t$$

Choose the correct options.

(a) Amplitude of constituent waves is $\frac{A}{2}$
(b) The wire oscillates in three loops
(c) The length of the wire is $\frac{4\pi}{k}$
(d) Speed of stationary wave is $\frac{\omega}{k}$

6. Which of the following equations can form stationary waves?

(i) $y = A \sin(\omega t - kx)$
(ii) $y = A \cos(\omega t - kx)$
(iii) $y = A \sin(\omega t + kx)$
(iv) $y = A \cos(\omega t + kx)$

(a) (i) and (ii)
(b) (i) and (iii)
(c) (iii) and (iv)
(d) (ii) and (iv)

7. Two waves

$$y_1 = A \sin(\omega t - kx)$$

and

$$y_2 = A \sin(\omega t + kx)$$

superimpose to produce a stationary wave, then

(a) $x = 0$ is a node
(b) $x = 0$ is an antinode
(c) $x = \frac{\pi}{k}$ is a node
(d) $\pi = \frac{2\pi}{k}$ is an antinode

Comprehension Based Questions

Passage

Incident wave $y = A \sin\left(ax + bt + \frac{\pi}{2}\right)$ *is reflected by an obstacle at* $x = 0$ *which reduces intensity of reflected wave by 36%. Due to superposition, the resulting wave consists of a standing wave and a travelling wave given by*

$$y = -1.6 \sin ax \sin bt + cA \cos(bt + ax)$$

where A, a, b *and* c *are positive constants.*

1. Amplitude of reflected wave is

(a) $0.6\,A$
(b) $0.8\,A$
(c) $0.4\,A$
(d) $0.2\,A$

2. Value of c is

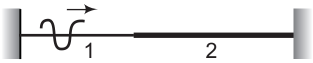
(a) 0.2
(b) 0.4
(c) 0.6
(d) 0.3

3. Position of second antinode is

(a) $x = \frac{\pi}{3a}$
(b) $x = \frac{3\pi}{a}$
(c) $x = \frac{3\pi}{2a}$
(d) $x = \frac{2\pi}{3a}$

Match the Columns

1. In the figure shown, mass per unit length of string-2 is nine times that of string-1. Tension in both the strings is same. A transverse wave is incident at the boundary as shown. Part of wave is reflected in medium-1 and part is transmitted to medium-2. Match the following two columns.



Column I	Column II
(a) $\lvert A_1/A_2 \rvert$	(p) 9
(b) v_1/v_2	(q) 1
(c) I_1/I_2	(r) 3
(d) P_2/P_1	(s) data insufficient

Here A is amplitude, v the speed of wave, I the intensity and P the power. Abbreviation 1 is used for reflected wave and 2 for transmitted wave.

2. Transverse waves are produced in a stretched wire. Both ends of the string are fixed. Let us compare between second overtone mode (in numerator) and fifth harmonic mode (in denominator). Match the following two columns.

Column I	Column II
(a) Frequency ratio	(p) 2/3
(b) Number of nodes ratio	(q) 4/5
(c) Number of antinodes ratio	(r) 3/5
(d) Wavelength ratio	(s) 5/3

3. A wave travels from a denser medium to a rarer medium, then match the following two columns.

Column I	Column II
(a) speed of wave	(p) will increase
(b) wavelength of wave	(q) will decrease
(c) amplitude of wave	(r) will remain unchanged
(d) frequency of wave	(s) may increase or decrease

4. Two waves of same amplitude and same intensity interfere at one point. Phase difference between them is θ. Match the following two columns.

Column I	Column II
(a) Resultant amplitude for $\theta = 60°$	(p) 2 times
(b) Resultant amplitude for $\theta = 120°$	(q) 3 times
(c) Resultant intensity for $\theta = 90°$	(r) 4 times
(d) Resultant intensity for $\theta = 0°$	(s) None of these

5. A wire is stretched and fixed at its two ends. Its second overtone frequency is 210 Hz. Then, match the following two columns.

Column I	Column II
(a) Fundamental frequency	(p) 210 Hz
(b) Third harmonic frequency	(q) 350 Hz
(c) Third overtone frequency	(r) 280 Hz
(d) Second harmonic frequency	(s) None of these

Subjective Questions

1. Three pieces of string, each of length L, are joined together end-to-end, to make a combined string of length $3L$. The first piece of string has mass per unit length μ_1, the second piece has mass per unit length $\mu_2 = 4\mu_1$ and the third piece has mass per unit length $\mu_3 = \mu_1/4$.

(a) If the combined string is under tension F, how much time does it take a transverse wave to travel the entire length $3L$? Give your answer in terms of L, F and μ_1.

(b) Does your answer to part (a) depend on the order in which the three piece are joined together? Explain.

2. In a stationary wave that forms as a result of reflection of waves from an obstacle, the ratio of the amplitude at an antinode to the amplitude at node is 6. What percentage of energy is transmitted?

3. A standing wave $y = a \sin kx \cos \omega t$ is maintained in a homogeneous rod with cross-sectional area S and density ρ. Find the total mechanical energy confined between the sections corresponding to the adjacent nodes.

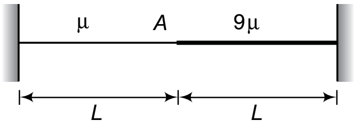
4. A string of mass per unit length μ is clamped at both ends such that one end of the string is at $x = 0$ and the other is at $x = l$. When string vibrates in fundamental mode, amplitude of the mid-point of the string is a and tension in the string is T. Find the total oscillation energy stored in the string.

5. A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has a length 4.8 m and mass 0.06 kg. QR has length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse of amplitude 3.5 cm is sent along the wire PQ from the end P. No power is dissipated during the propagation of the wave pulse. Calculate :

(a) The time taken by the wave pulse to reach the other end R.

(b) The amplitude of the reflected and transmitted wave pulse after the incident wave pulse crosses the joint Q.

6. A light string is tied at one end to fixed support and to a heavy string of equal length L at the other end as shown in figure. Mass per unit length of the strings are μ and 9μ and the tension is T. Find the possible values of frequencies such that point A is a node/antinode.

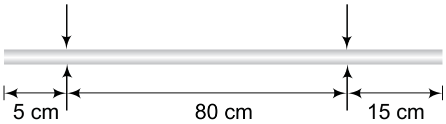


7. A string fixed at both ends is vibrating in the lowest possible mode of vibration for which a point at quarter of its length from one end is a point of maximum displacement. The frequency of vibration in this mode is 100 Hz. What will be the frequency emitted when it vibrates in the next mode such that this point is again a point of maximum displacement ?

8. Sources separated by 20 m vibrate according to the equation $y_1 = 0.06 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$. They send out waves along a rod with speed 3 m/s. What is the equation of motion of a particle 12 m from the first source and 8 m from the second, y_1, y_2 are in m?

9. Three component sinusoidal waves progressing in the same direction along the same path have the same period but their amplitudes are $A, \frac{A}{2}$ and $\frac{A}{3}$. The phases of the variation at any position x on their path at time $t = 0$ are $0, -\frac{\pi}{2}$ and $-\pi$ respectively. Find the amplitude and phase of the resultant wave.

10. A metal rod of length 1 m is clamped at two points as shown in the figure. Distance of the clamp from the two ends are 5 cm and 15 cm, respectively. Find the minimum and next higher frequency of natural longitudinal oscillation of the rod. Given that Young's modulus of elasticity and density of aluminium are $Y = 1.6 \times 10^{11}\ \text{Nm}^{-2}$ and $\rho = 2500\ \text{kgm}^{-3}$, respectively.



11. $y_1 = 8 \sin(\omega t - kx)$ and $y_2 = 6 \sin(\omega t + kx)$ are two waves travelling in a string of area of cross-section s and density ρ. These two waves are superimposed to produce a standing wave.

(a) Find the energy of the standing wave between two consecutive nodes.

(b) Find the total amount of energy crossing through a node per second.

Answers

Introductory Exercise 18.1

1. 49 : 1 **2.** (a) 4 : 1 (b) 16 : 1 **3.** 25 I_0

Introductory Exercise 18.2

1. True **2.** (a) 120 cm/s (b) 3 cm (c) zero
3. Yes, Yes **4.** 15.7 m, 31.8 Hz, 500 m/s

Introductory Exercise 18.3

1. 0.125 **2.** 18 Hz **3.** 0.873 kg
4. (a) 5th and 6th (b) 360 N **5.** $\frac{l_1}{l_2} = 3$

Introductory Exercise 18.4

1. Energy is in the form of kinetic energy **2.** (a) 20 cms^{-1} (b) $\frac{A_r}{A_i} = \frac{1}{3}, \frac{A_t}{A_i} = \frac{4}{3}$

3. $y_r = \frac{2}{3} \times 10^{-3} \cos \pi(2.0x + 50t)$, $y_t = \frac{8}{3} \times 10^{-3} \cos \pi(x - 50t)$

Exercises

LEVEL 1

Assertion and Reason

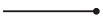
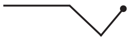
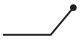
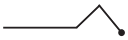
1. (d) **2.** (d) **3.** (b) **4.** (b) **5.** (a) **6.** (d) **7.** (a) **8.** (a) **9.** (b) **10.** (d)

Objective Questions

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (c) **7.** (d) **8.** (c) **9.** (b) **10.** (c)
11. (d) **12.** (b)

Subjective Questions

1. 5.66 cm **2.** (a) $-\frac{A}{3}, \frac{2}{3}A$ (b) $\frac{1}{9}, \frac{8}{9}$ **3.** 26.46 cm, $(5x + 25t + 0.714)$ rad **4.** $\frac{-1}{2\sqrt{2}}$ cm

5. (a) 0.02 s **6.** (a) (i)  (ii)  (b) (i)  (ii) 

7. (a) 4.24 cm (b) 6.00 cm (c) 6.00 cm (d) 0.500 cm, 1.50 cm, 2.50 cm
8. (a) 2 cm (b) $\sqrt{2}\pi$ cm **9.** 0.786 Hz, 1.57 Hz, 2.36 Hz, 3.14 Hz
10. (a) 163 N (b) 660 Hz **11.** 19977 Hz **12.** 2.142 m **13.** 1.0 m
14. (a) 100 Hz (b) 700 Hz
15. One bridge at 6/11 m from one end and the other at 2/11 m from the other end.
16. The string should be pressed at 60 cm from one end.
17. (a) $y(x, t) = (0.85 \text{ cm}) \sin\left[\frac{2\pi x}{0.3 \text{ cm}}\right] \sin\left[\frac{2\pi t}{0.075 \text{ s}}\right]$ (b) 4.00 m/s (c) 0.688 cm
18. (a) 3.0 m, 16.0 Hz (b) 1.0 m, 48.0 Hz (c) 0.75 m, 64.0 Hz

19. (b) 2.80 cm (c) 277 cm (d) 185 cm, 7.9 Hz, 0.126 s, 1470 cm/s (e) 280 cm/s
(f) $y(x, t) = (5.6 \text{ cm}) \sin[(0.0907 \text{ rad/cm})x] \sin[(133 \text{ rad/s})t]$

20. (a) 96.0 m/s (b) 461 N (c) 1.13 m/s, 426.4 m/s^2

21. (a) $y = 0.28 \sin\left(x - 3.0t + \dfrac{\pi}{4}\right)$ (b) $\phi = \pm 1.29$ rad

22. (i) Fundamental 12.5 Hz, third harmonic 37.5 Hz
(ii) (a) $y = 0.1 \sin \dfrac{\pi x}{2} \sin 25\pi t$ (b) $y = 0.04 \sin \dfrac{3\pi x}{2} \sin 75\pi t$

23. (a) 16 m, 5.33 m, 3.2 m (b) 6.25 Hz, 18.75 Hz, 31.25 Hz

24. (a) 8th and 9th (b) 2.16 m (c) 4.32 m

25. (a) $y(x, t) = 0.3 \cos(2.0x + 40t + \pi)$
(b) $y(x, t) = 0.3 \cos(2.0x + 40t)$ SI units

26. 8 s, inverted, 20 s upright

27. See the hints

LEVEL 2

Single Correct Option

1.(d) **2.**(d) **3.**(b) **4.**(a) **5.**(d) **6.**(d) **7.**(a) **8.**(d) **9.**(a) **10.**(c)
11.(b) **12.**(b) **13.**(b) **14.**(b) **15.**(c)

More than One Correct Options

1.(a,c,d) **2.**(a,d) **3.**(a,b,d) **4.**(a,c,d) **5.**(a,c) **6.**(b,d) **7.**(b,d)

Comprehension Based Questions

1.(b) **2.**(a) **3.**(c)

Match the Columns

1. (a) → q (b) → r (c) → s (d) → r **2.** (a) → r (b) → p (c) → r (d) → s
3. (a) → p (b) → p (c) → p (d) → r **4.** (a) → s (b) → s (c) → p (d) → r
5. (a) → s (b) → p (c) → r (d) → s

Subjective Questions

1. (a) $\dfrac{7L}{2}\sqrt{\dfrac{\mu_1}{F}}$ (b) No **2.** 49% **3.** $\dfrac{\pi S \rho \omega^2 a^2}{4k}$ **4.** $\dfrac{\pi^2 a^2 T}{4l}$

5. (a) 0.14 s (b) –1.5 cm, 2.0 cm

6. $\dfrac{f}{2}, f, \dfrac{3f}{2}$... etc. when A is a node, $\dfrac{3}{4}f, \dfrac{5}{4}f, \dfrac{7}{4}f$... etc. when A is an antinode. Here, $f = \dfrac{1}{L}\sqrt{\dfrac{T}{\mu}}$

7. 300 Hz **8.** $0.05 \sin \pi t - 0.0173 \cos \pi t$ **9.** $\dfrac{5}{6}A, -\tan^{-1}\left(\dfrac{3}{4}\right)$

10. 40 kHz, 120 kHz **11.** (a) $\dfrac{50\pi}{k}\rho\omega^2 S$ (b) $\dfrac{2\rho\omega^3 S}{k}$

19

Sound Waves

Chapter Contents

19.1 Introduction

19.2 Displacement Wave, Pressure Wave and Density Wave

19.3 Speed of a Longitudinal Wave

19.4 Sound Waves in Gases

19.5 Sound Intensity

19.6 Interference in Sound Wave and Stationary Wave

19.7 Standing Longitudinal Waves in Organ Pipes

19.8 Beats

19.9 The Doppler's Effect

19.1 Introduction

Of all the mechanical waves that occur in nature, the most important in our everyday life are longitudinal waves in a medium, usually air, called **sound waves.** The reason is that the human ear is tremendously sensitive and can detect sound waves even of very low intensity. The human ear is sensitive to waves in the frequency range from about 20 to 20000 Hz called the **audible range**, but we also use the term sound for similar waves with frequencies above **(ultrasonic)** and below **(infrasonic)** the range of human hearing. Our main concern in this chapter is with sound waves in air, but sound can travel through any gas, liquid or solid.

In this chapter, we will discuss several important properties of sound waves, including **pressure wave** and **density wave.** We will find that superposition of two sound waves differing slightly in frequency causes a phenomenon called **beats.** When a source of sound or a listener moves through the air, the listener may hear a different frequency than the one emitted by the source. This is the **Doppler's effect.**

Note *Sound wave (rather we can call audible sound wave) is a part of longitudinal wave with frequency varying from 20 Hz to 20,000 Hz.*

19.2 Displacement Wave, Pressure Wave and Density Wave

Upto this point we have described mechanical waves primarily in terms of displacement. As we have discussed in chapter-17, a sinusoidal wave equation $y(x, t)$ travelling in positive x-direction can be written as

$$\boxed{y(x, t) = A \sin(kx - \omega t)}$$

which gives the instantaneous displacement y of a particle in the medium at position x at time t.

Note that in a longitudinal wave the displacements are parallel to the direction of motion of the wave. So, x and y are measured parallel to each other, not perpendicular as in a transverse wave. The amplitude A is the maximum displacement of a particle in a medium from its equilibrium position.

These displacements are along the direction of the motion of the wave lead to **variations in the density and pressure** of air. Hence, a sound wave can also be described in terms of variations of pressure or density at various points. The pressure fluctuations are of the order of $1\ \text{Pa}\ (=1\ \text{N/m}^2)$, whereas atmospheric pressure is about 10^5 Pa. In a sinusoidal sound wave in air, the pressure fluctuates above and below atmospheric pressure p_0 with the same frequency as the motions of the air particles. Similarly, the density of air also vibrates sinusoidally above and below its normal level. So, we can express a sound wave either in terms of $y(x, t)$ or $\Delta p(x, t)$ or $\Delta\rho(x, t)$. All the three equations are related to one another. For example, amplitude of pressure variation $(\Delta p)_m$ is related to amplitude of displacement A by the equation,

$$\Delta p_m = BAk$$

where B is the bulk modulus of the medium. Moreover, pressure or density wave is 90° out of phase with the displacement wave, i.e. when the displacement is zero, the pressure and density changes are either maximum or minimum and when the displacement is a maximum or minimum, the pressure and density changes are zero. Now, let us find the relation between them.

Relation between Displacement Wave and Pressure Wave

Figure shows a harmonic displacement wave moving through air contained in a long tube of cross-sectional area S.

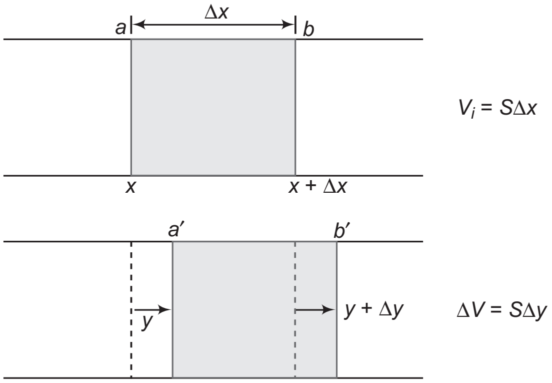


Fig. 19.1

The volume of gas that has a thickness Δx in the horizontal direction is $V_i = S\Delta x$. The change in volume ΔV is $S\,\Delta y$, where Δy is the difference between the value of y at $x + \Delta x$ and the value of y at x. From the definition of bulk modulus, the pressure variation in the gas is

$$\Delta p = -B\,\frac{\Delta V}{V_i} = -B\left(\frac{S\Delta y}{S\Delta x}\right) = -B\left(\frac{\Delta y}{\Delta x}\right)$$

As Δx approaches zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{\partial y}{\partial x}$. (The partial derivative indicates that we are interested in the variation of y with position at a fixed time).

Therefore,
$$\boxed{\Delta p = -B\,\frac{\partial y}{\partial x}} \qquad \ldots\text{(i)}$$

So, this is the equation which relates the displacement equation with the pressure equation. Suppose the displacement equation is

$$\boxed{y = A\cos(kx - \omega t)} \qquad \ldots\text{(ii)}$$

Then,
$$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t) \qquad \ldots\text{(iii)}$$

and from Eqs. (i) and (iii), we find that

$$\boxed{\begin{gathered}\Delta p = BAk\sin(kx - \omega t) = (\Delta p)_m \sin(kx - \omega t) \\ (\Delta p)_m = BAk\end{gathered}} \qquad \ldots\text{(iv)}$$

Here,

where Δp_m is the amplitude of pressure variation. From Eqs. (ii) and (iv), we see that pressure equation is 90° out of phase with displacement equation. When the displacement is zero, the pressure variation is either maximum or minimum and *vice-versa.* Fig. 19.2 (a) shows displacement from equilibrium of air molecules in a harmonic sound wave *versus* position at some instants. Points x_1 and x_3 are points of zero displacement.

Now, refer Fig. 19.2 (b) : Just to the left of x_1, the displacement

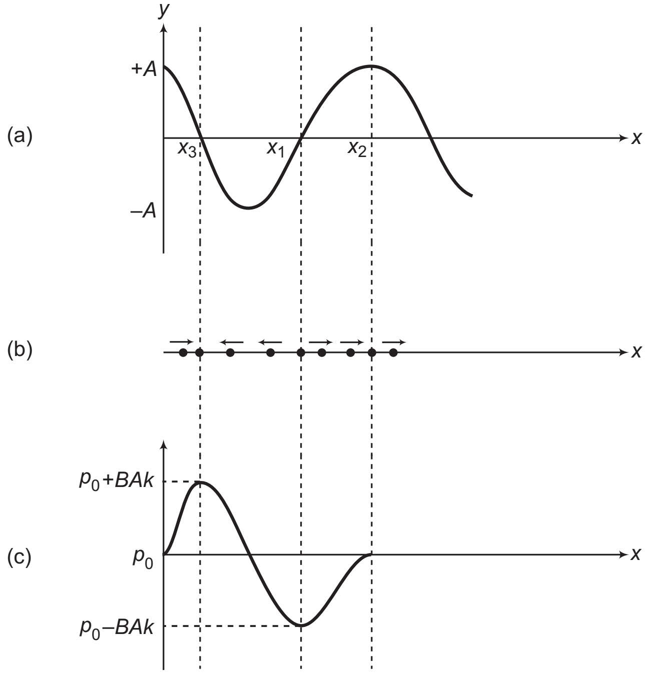


Fig. 19.2

is negative indicating that the gas molecules are displaced to left away from point x_1 at this instant. Just to the right of x_1, the displacement is positive indicating that the molecules are displaced to the right, which is again away from point x_1. So, at point x_1 the pressure of the gas is minimum. So, if p_0 is the atmospheric pressure (normal pressure), the pressure at x_1 will be

$$p(x_1) = p_0 - (\Delta p)_m = p_0 - BAk$$

At point x_3, the pressure (and hence the density also) is maximum because the molecules on both sides of that point are displaced toward point x_3. Hence,

$$p(x_3) = p_0 + (\Delta p)_m = p_0 + BAk$$

At point x_2, the pressure (and hence the density) does not change because the gas molecules on both sides of that point have equal displacements in the same direction or

$$p(x_2) = p_0$$

From Fig. 19.2 (a) and (c), we see that pressure change and displacement are 90° out of phase.

Relation between Pressure Wave and Density Wave

In this section, we will find the relation between pressure wave and density wave.

According to definition of bulk modulus (B),

$$B = \left(-\frac{dp}{dV/V} \right) \qquad \text{...(i)}$$

Further, $$\text{volume} = \frac{\text{mass}}{\text{density}} \quad \text{or} \quad V = \frac{m}{\rho}$$

or $$dV = -\frac{m}{\rho^2} \cdot d\rho = -\frac{V}{\rho} \cdot d\rho \quad \text{or} \quad \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho\,(dp)}{B} = \frac{dp}{v^2} \qquad \left(v = \sqrt{\frac{B}{\rho}} \Rightarrow \frac{\rho}{B} = \frac{1}{v^2} \right)$$

Or, this can be written as

$$\boxed{\Delta\rho = \frac{\rho}{B} \cdot \Delta p = \frac{1}{v^2} \cdot \Delta p}$$

So, this relation relates the pressure equation with the density equation. For example, if

$$\Delta p = (\Delta p)_m \sin\,(kx - \omega t)$$

then $$\Delta\rho = (\Delta\rho)_m \sin\,(kx - \omega t)$$

where, $$\boxed{(\Delta\rho)_m = \frac{\rho}{B}\,(\Delta p)_m = \frac{(\Delta p)_m}{v^2}}$$

Thus, density equation is in phase with the pressure equation and this is 90° out of phase with the displacement equation.

Example 19.1 *Corresponding to displacement equation,*

$$y = A \sin(kx + \omega t)$$

of a longitudinal wave make its pressure and density wave also. Bulk modulus of the medium is B and density is ρ.

Solution Given, $y(x,\ t) = A \sin\,(kx + \omega t)$

$$\frac{\partial y}{\partial x} = kA \cos\,(kx + \omega t) \qquad \ldots\text{(i)}$$

Now, pressure equation is given by

$$\Delta p = -B\left(\frac{\partial y}{\partial x}\right)$$

Substituting the value of $\frac{\partial y}{\partial x}$ from Eq. (i), we have

$$\Delta p = -BAk \cos\,(kx + \omega t) \qquad \textbf{Ans.}$$

Similarly, density equation is given by

$$(\Delta p) = \frac{\rho}{B}\,(\Delta p) = \frac{\rho}{B}\left(-B\frac{\partial y}{\partial x}\right) = -\rho\left(\frac{\partial y}{\partial x}\right)$$

Substituting the values of $\frac{\partial y}{\partial x}$ from Eq. (i), we have,

$$\Delta\rho = -\rho Ak \cos\,(kx + \omega t) \qquad \textbf{Ans.}$$

Example 19.2 *(a) What is the displacement amplitude for a sound wave having a frequency of* 100 *Hz and a pressure amplitude of* 10 *Pa?*

(b) The displacement amplitude of a sound wave of frequency 300 *Hz is* 10^{-7} *m. What is the pressure amplitude of this wave? Speed of sound in air is* 340 *m/s and density of air is* 1.29 *kg/m*^{3}.

Solution (a) $(\Delta p)_m = BAk$

Here, $$k = \frac{\omega}{v} = \frac{2\pi f}{v}$$

and $$B = \rho v^2 \qquad \left(\text{as } v = \sqrt{\frac{B}{\rho}}\right)$$

$$\therefore \qquad (\Delta p)_m = (\rho v^2)(A)\left(\frac{2\pi f}{v}\right)$$

$$\therefore \qquad A = \frac{(\Delta p)_m}{2\pi v \rho f} \qquad \ldots\text{(i)}$$

Substituting the values, we have

$$A = \frac{(10)}{2 \times 3.14 \times 340 \times 1.29 \times 100}$$

$$= 3.63 \times 10^{-5} \text{ m} \qquad \textbf{Ans.}$$

(b) From Eq. (i) $(\Delta p)_m = 2\pi f \rho v A$

Substituting the values, we have

$$(\Delta p)_m = 2 \times 3.14 \times 300 \times 1.29 \times 340 \times 10^{-7}$$

$$= 8.26 \times 10^{-2} \text{ N/m}^2 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 19.1

1. Calculate the bulk modulus of air from the following data for a sound wave of wavelength 35 cm travelling in air. The pressure at a point varies between $(10^5 \pm 14)$ Pa and the particles of the air vibrate in SHM of amplitude 5.5×10^{-6} m.

2. Find the minimum and maximum wavelengths of sound in water that is in the audible range for an average human ear. Speed of sound in water is 1450 m/s.

3. A typical loud sound wave with a frequency of 1 kHz has a pressure amplitude of about 10 Pa

(a) At $t = 0$, the pressure is a maximum at some point x_1. What is the displacement at that point at $t = 0$?

(b) What is the maximum value of the displacement at any time and place? Take the density of air to be 1.29 kg/m^3 and speed of sound in air is 340 m/s.

4. The pressure variation in a sound wave in air is given by

$$\Delta p = 12 \sin(8.18x - 2700t + \pi/4) \text{ N/m}^2$$

Find the displacement amplitude. Density of air = 1.29 kg/m^3.

19.3 Speed of a Longitudinal Wave

First we calculate the speed at which a longitudinal pulse propagates through a fluid. We will apply Newton's second law to the motion of an element of the fluid and from this we derive the wave equation.

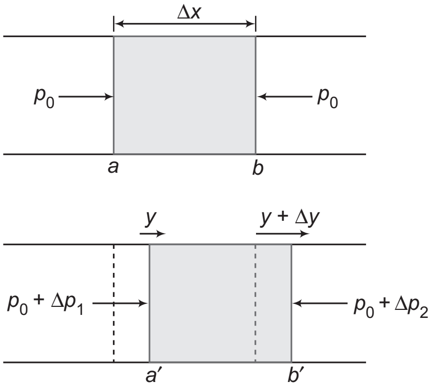


Fig. 19.3

Consider a fluid element *ab* confined to a tube of cross-sectional area *S* as shown in figure. The element has a thickness Δx. We assume that the equilibrium pressure of the fluid is p_0. Because of the disturbance, the section '*a*' of the element moves a distance *y* from its mean position and section '*b*' moves a distance $y + \Delta y$ to a new position b'. The pressure on the left side of the element becomes $p_0 + \Delta p_1$ and on the right side it becomes $p_0 + \Delta p_2$. If ρ is the equilibrium density, the mass of the element is $\rho S \Delta x$. (When the element moves its mass does not change, even though its volume and density do change).

The net force acting on the element is

$$F = (\Delta p_1 - \Delta p_2)\, S$$

and its acceleration is

$$a = \frac{\partial^2 y}{\partial t^2}$$

Thus, Newton's second law applied to the motion of the element is

$$(\Delta p_1 - \Delta p_2)\, S = ma = \rho S \Delta x \frac{\partial^2 y}{\partial t^2} \qquad \ldots\text{(i)}$$

Next we divide both sides by Δx and note that in the limit as $\Delta x \to 0$, we have $(\Delta p_1 - \Delta p_2)/\Delta x \to \partial p/\partial x$, Eq. (i) then take the form

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \qquad \ldots\text{(ii)}$$

The excess pressure Δp may be written as

$$\Delta p = -B \frac{\partial y}{\partial x} \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -B \frac{\partial^2 y}{\partial x^2}$$

When this is used in Eq. (ii), we obtain the wave equation.

$$\frac{\partial^2 y}{\partial x^2}=\frac{\rho}{B}\cdot\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2}=\frac{B}{\rho}\frac{\partial^2 y}{\partial x^2}$$

Comparing this equation with the wave equation,

$$\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}$$

We have $$\boxed{v=\sqrt{\frac{B}{\rho}}}$$ (speed of longitudinal wave in a fluid)

This is the speed of longitudinal waves within a **gas** or a **liquid.**

When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$\boxed{v=\sqrt{\frac{Y}{\rho}}}$$ (speed of a longitudinal wave in a solid rod)

19.4 Sound Waves in Gases

In the preceding section, we derived equation

$$v=\sqrt{B/\rho}$$

for the speed of longitudinal waves in a fluid of bulk modulus B and density ρ. We use this to find the speed of sound in an ideal gas. The bulk modulus of the gas, however depends on the process. When a wave travels through a gas, are the compressions and expansions adiabatic, or is there enough heat conduction between adjacent layers of gas to maintain a nearly constant temperature throughout?

Because thermal conductivities of gases are very small, it turns out that for ordinary sound frequencies (20 Hz to 20,000 Hz) propagation of sound is very nearly adiabatic. Thus, in the above equation, we use the **adiabatic bulk modulus (B_s)**, which is given by

$$B_s=\gamma p$$

Here, γ is the ratio of molar heat capacity C_p/C_V. Thus,

$$\boxed{v=\sqrt{\frac{\gamma p}{\rho}}}$$

We can get a useful alternative form of the above equation by substituting density ρ of an ideal gas

$$\rho=\frac{pM}{RT}$$

where, R is the gas constant, M the molecular mass and T is the absolute temperature combining all these equations, we can write

$$\boxed{v=\sqrt{\frac{\gamma RT}{M}}}$$ (speed of sound in an ideal gas)

Effect of Temperature, Pressure and Humidity on the Speed of Sound in Air

(i) **Effect of temperature** From the equation,

$$v = \sqrt{\frac{\gamma RT}{M}}$$

We can see that,

$$\boxed{v \propto \sqrt{T}} \quad \text{or} \quad \frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

At STP, the temperature is 0°C or 273 K. If the speed of sound at 0°C is v_0, its value at t°C will satisfy

$$\frac{v_t}{v_0} = \sqrt{\frac{273+t}{273}} = \left(1+\frac{t}{273}\right)^{\frac{1}{2}} = 1+\frac{t}{546}$$

$$\therefore \qquad v_t = v_0\left(1+\frac{t}{546}\right)$$

If the speed of sound in air at 0°C (v_0) be taken 332 m/s, then

$$v_t = 332\left(1+\frac{t}{546}\right)$$

or

$$\boxed{v_t = 332 + 0.61t}$$

Thus, the velocity of sound in air increases roughly by 0.61 m/s per degree centigrade rise in temperature.

(ii) **Effect of pressure** From the formula for the speed of sound in a gas $v = \sqrt{\frac{\gamma p}{\rho}}$, it appears that $v \propto \sqrt{p}$. But actually it is not so. Because

$$\frac{p}{\rho} = \frac{RT}{M} = \text{constant at constant temperature.}$$

That is, at constant temperature if p changes then ρ also changes in such a way that p/ρ remains constant. Hence, in the formula $v = \sqrt{\frac{\gamma p}{\rho}}$, the value of p/ρ does not change when p changes. From this it is clear that if the temperature of the gas remains constant, then there is no effect of the pressure change on the speed of sound.

(iii) **Effect of humidity** The density of moist air (i.e. air mixed with water-vapour) is less than the density of dry air. This is because in moist air heavy dust particles settle down due to condensation. Hence, density of air gets decreased thus increasing the speed of sound. Therefore, assuming the value of γ for moist air same as for dry air (which is actually slightly less than that for dry air) it is clear from the formula $v = \sqrt{\frac{\gamma p}{\rho}}$ that the speed of sound in moist air is slightly greater than in dry air.

Example 19.3 *Calculate the speed of longitudinal waves in the following gases at* $0°C$ *and* 1 *atm* $(=10^5$ *Pa):*

(a) *oxygen for which the bulk modulus is* 1.41×10^5 *Pa and density is* 1.43 *kg/m*^{3}.
(b) *helium for which the bulk modulus is* 1.7×10^5 *Pa and the density is* 0.18 *kg/m*^{3}.

Solution (a)

$$v_{O_2} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.41\times10^5}{1.43}} = 314 \text{ m/s}$$ **Ans.**

(b)

$$v_{He} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.7\times10^5}{0.18}} = 972 \text{ m/s}$$ **Ans.**

Example 19.4 *Find speed of sound in hydrogen gas at* $27°C$. *Ratio* C_p/C_V *for* H_2 *is 1.4. Gas constant* $R = 8.31$ *J/mol-K.*

Solution $\because \quad v = \sqrt{\frac{\gamma RT}{M}}$

Here, $T = 27 + 273 = 300\text{ K}$, $\gamma = 1.4$, $R = 8.31\text{ J/mol-K}$, $M = 2\times10^{-3}\text{ kg/mol}$

$$\therefore \quad v = \sqrt{\frac{1.4\times8.31\times300}{2\times10^{-3}}} = 1321\text{ m/s}$$ **Ans.**

Note *In the above formula, put* $M = 2\times10^{-3}$ *kg/mol. Don't put it 2 kg.*

Example 19.5 *At what temperature will the speed of sound in hydrogen be the same as in oxygen at* $100°C$*? Molar masses of oxygen and hydrogen are in the ratio* $16:1$.

Solution $v = \sqrt{\frac{\gamma RT}{M}}$

$$v_{H_2} = v_{O_2}$$

$$\therefore \quad \sqrt{\frac{\gamma_{H_2} RT_{H_2}}{M_{H_2}}} = \sqrt{\frac{\gamma_{O_2} RT_{O_2}}{M_{O_2}}}$$

$$\gamma_{H_2} = \gamma_{O_2}$$ (as both are diatomic)

$$\therefore \qquad T_{H_2} = \left(\frac{M_{H_2}}{M_{O_2}}\right)(T_{O_2}) = \left(\frac{1}{16}\right)(100 + 273)$$

$$= 23.31\,\text{K}$$

$$\approx -\,249.7^\circ\,\text{C}$$

Ans.

INTRODUCTORY EXERCISE 19.2

1. Calculate the temperature at which the velocity of sound in air is double its velocity at 0°C.
2. Calculate the difference in the speeds of sound in air at -3°C, 60 cm pressure of mercury and 30°C, 75 cm pressure of mercury. The speed of sound in air at 0°C is 332 m/s.
3. In a liquid with density $900\ \text{kg/m}^3$, longitudinal waves with frequency 250 Hz are found to have wavelength 8.0 m. Calculate the bulk modulus of the liquid.
4. Calculate the speed of sound in oxygen at 273 K.

19.5 Sound Intensity

Travelling sound waves, like all other travelling waves, transfer energy from one region of space to another. We define the **intensity** of a wave (denoted by I) to be the time average rate at which energy is transported by the wave, per unit area across a surface perpendicular to the direction of propagation. We have already derived an expression for the intensity of a mechanical wave

$$\boxed{I = \frac{1}{2}\rho A^2\omega^2 v} \qquad \text{...(i)}$$

For a sound wave,

$$(\Delta p)_m = BAk = BA\left(\frac{\omega}{v}\right) \quad \text{or} \quad \omega = \frac{(\Delta p)_m v}{BA}$$

Substituting this value in Eq. (i), we have

$$I = \frac{1}{2}\rho A^2 \frac{(\Delta p)_m^2 v^2}{B^2 A^2} v \qquad (\rho v^2 = B)$$

or

$$\boxed{I = \frac{v\,(\Delta p)_m^2}{2B}} \qquad \text{...(ii)}$$

Thus, intensity of a sound wave can be calculated by either of the Eqs. (i) or (ii).

Sound Intensity in Decibels

The physiological sensation of loudness is closely related to the intensity of wave producing the sound. At a frequency of 1 kHz people are able to detect sounds with intensities as low as $10^{-12}\ \text{W/m}^2$. On the other hand, an intensity of $1\ \text{W/m}^2$ can cause pain and prolonged exposure to sound at this level will damage a person's ears. Because the range in intensities over which people hear is so large, it is convenient to use a logarithmic scale to specify intensities. This scale is defined as follows.

If the intensity of sound in watts per square metre is I, then the intensity level β in decibels (dB) is given by

$$\boxed{\beta = 10 \log \frac{I}{I_0}} \qquad \text{...(iii)}$$

where the base of the logarithm is 10, and $I_0 = 10^{-12}\ \text{W/m}^2$ (roughly the minimum intensity that can be heard).

On the decibel scale, the pain threshold of $1\ \text{W/m}^2$ is then

$$\beta = 10 \log \frac{1}{10^{-12}} = 120\ \text{dB}$$

Note *For the comparison of two different sounds in dB ; we can modify Eq. (iii) as*

$$\boxed{\beta_2 - \beta_1 = 10 \log_{10} \frac{I_2}{I_1}} \qquad \text{...(iv)}$$

Table 19.1 gives typical values for the intensity levels of some of the common sounds.

Table 19.1 Sound intensity levels in decibels

(Threshold of hearing = 0 dB; threshold of pain = 120 dB)

Source of sound	dB
Rusting leaves	10
Whisper	20
Quiet room	30
Normal level of speech (inside)	65
Street traffic (inside car)	80
Riveting tool	100
Thunder	110
Indoor rock concert	120

Example 19.6 *For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about* 6.0×10^{-5} *Pa. Calculate the corresponding intensity in* W/m^2*. Take speed of sound in air as 344 m/s and density of air 1.2* kg/m^3.

Solution $\because \quad I = \dfrac{v(\Delta p)_m^2}{2B}$

Substituting $B = \rho v^2$, the above equation reduces to

$$I = \frac{(\Delta p)_m^2}{2\rho v} = \frac{(6.0 \times 10^{-5})^2}{2 \times 1.2 \times 344}$$

$$= 4.4 \times 10^{-12}\ \text{W/m}^2$$ **Ans.**

Example 19.7 *Find intensity of sound in dB if its intensity in Watt/m² is* 10^{-10}.

Solution Using the equation,

$$\beta = 10 \log_{10} \frac{I}{I_0}, \text{ we have}$$

$$\beta = 10 \log_{10} \left(\frac{10^{-10}}{10^{-12}} \right) = 10 \log_{10} (10^2)$$

$$= 20 \, \text{dB}$$ **Ans.**

Example 19.8 *A point source of sound emits a constant power with intensity inversely proportional to the square of the distance from the source. By how many decibels does the sound intensity level drops when you move from point* P_1 *to* P_2*? Distance of* P_2 *from the source is two times the distance of source from* P_1.

Solution We label the two points 1 and 2, and we use the equation $\beta = 10 \log \frac{I}{I_0}$ (dB) twice.

The difference in sound intensity level $\beta_2 - \beta_1$ is given by

$$\beta_2 - \beta_1 = (10 \, \text{dB}) \left(\log \frac{I_2}{I_0} - \log \frac{I_1}{I_0} \right)$$

$$= (10 \, \text{dB}) [(\log I_2 - \log I_0) - (\log I_1 - \log I_0)]$$

$$= (10 \, \text{dB}) \log \frac{I_2}{I_1}$$

Now, $$I \propto \frac{1}{r^2} \Rightarrow \therefore \frac{I_2}{I_1} = \left(\frac{r_1}{r_2} \right)^2 = \frac{1}{4} \quad \text{as} \quad r_2 = 2r_1$$

$\therefore$ $$\beta_2 - \beta_1 = (10 \, \text{dB}) \log \left(\frac{1}{4} \right) = -6.0 \, \text{dB}$$ **Ans.**

INTRODUCTORY EXERCISE 19.3

1. A sound wave in air has a frequency of 300 Hz and a displacement amplitude of 6.0×10^{-3} mm. For this sound wave calculate the
   (a) pressure amplitude (b) intensity (c) sound intensity level (in dB)
   Speed of sound = 344 m/s and density of air = 1.2 kg/m^3.
2. Most people interpret a 9.0 dB increase in sound intensity level as a doubling in loudness. By what factor must the sound intensity be increased to double the loudness?
3. A baby's mouth is 30 cm from her father's ear and 3.0 m from her mother's ear. What is the difference between the sound intensity levels heard by the father and by the mother.
4. The faintest sound that can be heard has a pressure amplitude of about 2×10^{-5} N/m^2 and the loudest that can be heard without pain has a pressure amplitude of about 28 N/m^2. Determine in each case (a) the intensity of the sound both in W/m^2 and in dB and (b) the amplitude of the oscillations if the frequency is 500 Hz. Assume an air density of 1.29 kg/m^3 and a velocity of sound is 345 m/s.

19.6 Interference in Sound Wave and Stationary Wave

In chapter 18, we have already discussed about the principle of superposition, interference and stationary wave. In the similar manner, we can make stationary waves with longitudinal (or sound) waves also.

When two identical sound waves travel in opposite directions, then by their superposition (and hence interference) longitudinal stationary waves are formed.

But, as we know that, with every longitudinal travelling wave three equations are associated $y(x, t)$, $\Delta P(x, t)$ and $\Delta\rho(x, t)$. So, with every longitudinal stationary wave three equations are associated, $y(x, t)$, $\Delta P(x, t)$ and $\Delta\rho(x, t)$.

Following three points are important with longitudinal stationary wave :

(i) If individual displacement amplitudes of constituent travelling waves are A_0 each, then displacement amplitude in stationary wave varies from 0 to $2A_0$.

(ii) If two identical (same frequency) longitudinal waves travel in opposite directions, standing waves are produced by their superposition. If the equations of the two waves are written as

$$\Delta p_1 = (\Delta p)_m \sin(kx - \omega t)$$

and
$$\Delta p_2 = (\Delta p)_m \sin(kx + \omega t)$$

then from the principle of superposition, the resultant wave is

$$\Delta p = \Delta p_1 + \Delta p_2 = 2(\Delta p)_m \sin kx \cos \omega t \qquad \ldots\text{(i)}$$

This equation is similar to the equation obtained in chapter 18 for standing waves on a string.

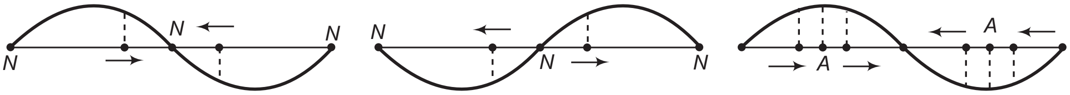
From Eq. (i), we can see that

$$\Delta p = 0 \quad \text{at } x = 0, \lambda/2, \lambda, \ldots, \text{etc.} \qquad \text{(pressure nodes)}$$

and
$$\Delta p = \text{maximum at } x = \lambda/4, 3\lambda/4, \ldots, \text{etc.} \qquad \text{(pressure antinodes)}$$

The distance between two adjacent nodes or between two adjacent antinodes is $\lambda/2$. Longitudinal standing waves can be produced in air columns trapped in tubes of cylindrical shape. Organ pipes are such vibrating air columns.

(iii) Because the pressure wave is 90° out of phase with the displacement wave. Consequently, the displacement node behaves as a pressure antinode and *vice-versa*.



(a) (b) (c)

Fig. 19.4

This can be explained by realizing that two small volume elements of fluid on opposite sides of a displacement node are vibrating in opposite phase. Hence, when they approach each other [see Fig. (a)] the pressure at this node is a maximum, and when they recede from each other, [see Fig. (b)] the pressure at this node is a minimum. Similarly, two small elements of fluid which are on opposite sides of a displacement antinode vibrate in phase and therefore give rise to no pressure variations at the antinode [see Fig. (c)].

Extra Points to Remember

- Most of the problems of interference can be solved by calculating the path difference Δx and then by putting

$$\Delta x = 0, \lambda, 2\lambda, \dots \qquad \text{(constructive interference)}$$

$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \dots \qquad \text{(destructive interference)}$$

provided the waves emitted from S_1 and S_2 are in phase.

- If the two waves emitted from S_1 and S_2 have already a phase difference of π, the conditions of maximas and minimas are interchanged, i.e. path difference

$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \dots \qquad \text{(for constructive interference)}$$

and $$\Delta x = \lambda, 2\lambda, \dots \qquad \text{(for destructive interference)}$$

Example 19.9 *Two sound sources S_1 and S_2 emit pure sinusoidal waves in phase. If the speed of sound is 350 m/s, then*

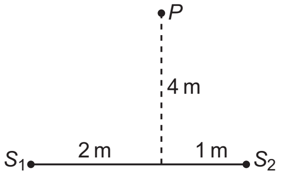


Fig. 19.5

(a) for what frequencies does constructive interference occur at P?

(b) for what frequencies does destructive interference occur at point P?

Solution Path difference, $\Delta x = S_1P - S_2P$

$$= \sqrt{(2)^2 + (4)^2} - \sqrt{(1)^2 + (4)^2}$$

$$= 4.47 - 4.12 = 0.35 \text{ m}$$

(a) Constructive interference occurs when the path difference is an integer multiple of wavelength.

or $$\Delta x = n\lambda = \frac{nv}{f} \quad \text{or} \quad f = \frac{nv}{\Delta x} \qquad \text{where, } n = 1, 2, 3, \dots$$

$$\therefore \quad f = \frac{350}{0.35}, \frac{2 \times 350}{0.35}, \frac{3 \times 350}{0.35}, \dots$$

$$f = 1000 \text{ Hz}, 2000 \text{ Hz}, 3000 \text{ Hz}, \dots, \text{etc.}$$ **Ans.**

(b) Destructive interference occurs when the path difference is a half-integer multiple of wavelengths

or $$\Delta x = (2n+1)\frac{\lambda}{2} \qquad n = 0, 1, 2, \dots$$

or $$\Delta x = (2n+1)\frac{v}{2f}$$

$$\therefore \quad f = \frac{(2n+1)\,v}{2\,\Delta x}$$

$$= \frac{350}{2 \times 0.35}, \frac{3 \times 350}{2 \times 0.35}, \frac{5 \times 350}{2 \times 0.35}, \dots$$

$$= 500 \text{ Hz}, 1500 \text{ Hz}, 2500 \text{ Hz}, \dots$$ **Ans.**

INTRODUCTORY EXERCISE 19.4

1. Two sound waves emerging from a source reach a point simultaneously along two paths. When the path difference is 12 cm or 36 cm, then there is a silence at that point. If the speed of sound in air be 330 m/s, then calculate maximum possible frequency of the source.
2. A wave of frequency 500 cycle/s has a phase velocity of 350 m/s.
   (a) How far apart are two points 60° out of phase?
   (b) What is the phase difference between two displacements at a certain point at time 10^{-3} s apart?

19.7 Standing Longitudinal Waves in Organ Pipes

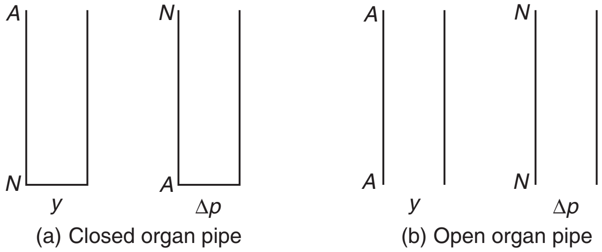
When longitudinal waves propagate in a fluid in a pipe with finite length, the waves are reflected from the ends in a same way the transverse waves on a string are reflected at its ends. The superposition of the waves travelling in opposite directions forms a longitudinal standing wave.

Transverse waves on a string, including standing waves, are usually described only in terms of the displacement of the string. But longitudinal standing waves in a fluid may be described either in terms of the displacement of the fluid or in terms of the pressure variation in the fluid. To avoid confusion we will use the terms **displacement node** and **displacement antinode** to refer to points where particles of the fluid have zero displacement and maximum displacement respectively. Similarly, **pressure node** and **pressure antinode** refer to the points where pressure and density variation in the fluid is zero or maximum, respectively.

Conditions at the Boundary of an Organ Pipe

Let us take an example of a closed organ pipe. In a closed pipe one end is closed and the other is open. When a longitudinal wave encounters the closed end of the pipe it gets reflected from this end. But the reflected wave is 180° out of phase with the incident wave, i.e. a compression is reflected as a compression and a rarefaction is reflected as a rarefaction. This is a necessary condition because the displacement of the small volume elements at the closed end must always be zero. Hence, a closed end is a displacement node.

A sound wave is also reflected from an open end. You may wonder how a sound wave can reflect from an open end, since there may not appear to be a change in the medium at this point. At the open end pressure is the same as atmospheric pressure and does not vary. Thus, there is a pressure node (or displacement antinode) at this end. A compression is therefore reflected as a rarefaction and rarefaction as a compression. Now let us see how this reflection takes place. When a rarefaction reaches an open end, the surrounding air rushes towards this region and creates a compression that travels back along the pipe. Similarly, when a compression reaches an open end, the air expands to form a rarefaction. This can be said in a different way as : at the open end of the tube, fluid elements are free to move, so there is a displacement antinode. Thus, in a nutshell we can say that closed end of an organ pipe is a displacement node or pressure antinode and open end of the pipe is displacement antinode or pressure node.



(a) Closed organ pipe (b) Open organ pipe

Fig. 19.6

Similarly, both ends of an open organ pipe (open at both ends) are displacement antinodes or pressure nodes.

Standing Waves in a Closed Organ Pipe

To get resonance in a closed organ pipe sound waves are sent in by a source (normally a tuning fork) near the open end. Resonance corresponds to a pressure antinode at the closed end and a pressure node at the open end. The standing wave patterns for the three lowest harmonics in this situation are shown in figure. Since the node-antinode separation is $\frac{\lambda}{4}$, the resonance condition for the first harmonic is, $l = \frac{\lambda_1}{4}$.

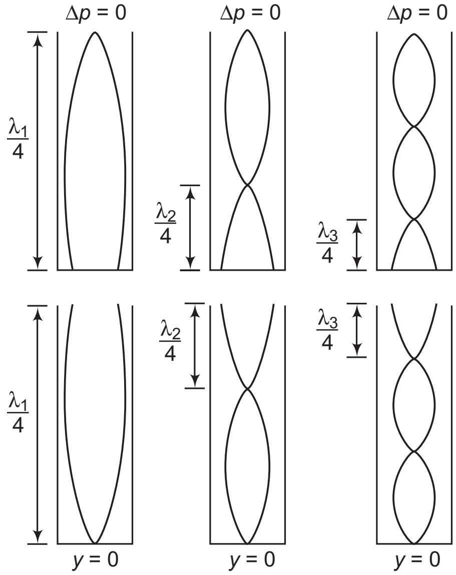


Fig. 19.7

Similarly, the resonance conditions for the higher harmonics are $l = \frac{3\lambda_2}{4}, \frac{5\lambda_3}{4}, \ldots$, etc. or, $\lambda_n = \frac{4l}{n}$ (where $n = 1, 3, 5\ldots$). The natural frequencies of oscillation of the air in the tube closed at one end and open at the other are, therefore,

$$\boxed{f_n = \frac{v}{\lambda_n} = n\frac{v}{4l} = nf_1 \qquad (n = 1, 3, 5, \dots)}$$

Here, v = speed of sound in the tube

$$f_1 = \frac{v}{4l} \qquad \text{(fundamental frequency or the first harmonic)}$$

$$f_3 = 3\frac{v}{4l} = 3f_1 \qquad \text{(first overtone or the third harmonic)}$$

and

$$f_5 = 5\frac{v}{4l} = 5f_1 \qquad \text{(second overtone or the fifth harmonic)}$$

and so on.

Thus, in a pipe closed at one end and open at the other, the natural frequencies of oscillation form a harmonic series that includes only odd integer multiples of the fundamental frequency.

Standing Waves in an Open Organ Pipe

Since both ends of the tube are open, there are pressure nodes (or displacement antinodes) at both ends. Figure shows the resulting standing waves for the three lowest resonant frequencies since the distance between pressure nodes is $\lambda/2$, the resonance condition is $l = n\left(\frac{\lambda_n}{2}\right)$ where $n = 1, 2, 3, \dots$ and l is the length of the tube. The resonant frequencies for a tube open at both ends are then :

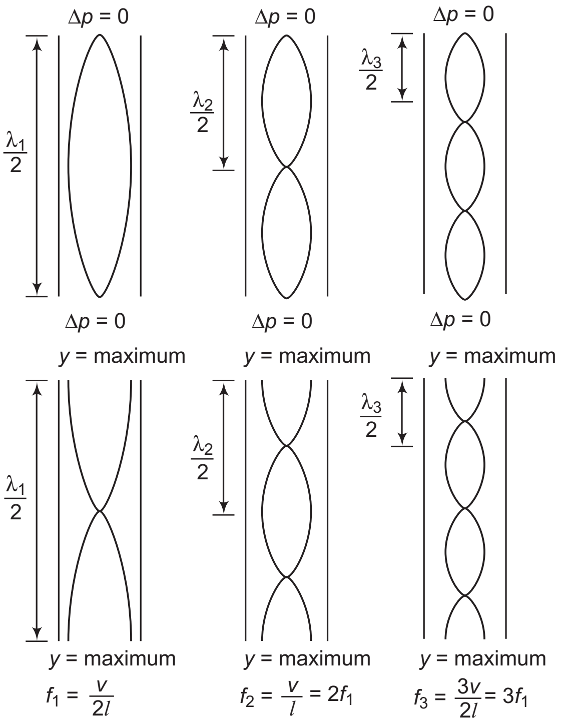


Fig. 19.8

$$\boxed{f_n = \frac{v}{\lambda_n} = n\frac{v}{2l} = nf_1 \quad (n = 1, 2, 3, \dots)}$$

Here,
$$f_1 = \frac{v}{2l}$$

(fundamental frequency or first harmonic)

$$f_2 = \frac{v}{l} = 2f_1$$

(first overtone or second harmonic)

$$f_3 = 3\left(\frac{v}{2l}\right) = 3f_1$$

(second overtone or third harmonic)

and so on.

Thus, in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.

✓ Extra Points to Remember

- **Laplace end correction** In practice, the pressure nodes lie slightly beyond the ends of the tube. A compression reaching an open end does not reflect until it passes beyond the end. For a thin walled tube of circular cross-section, this **end correction** is approximately 0.6 r where r is the tube's radius. Hence, the effective length of the tube is longer than the true length l.

  Therefore, Laplace correction $e = 0.6\,r$ (in closed pipe) and $2e = 1.2\,r$ (in open pipe)

  Hence, $$f = n\left[\frac{v}{2(l+1.2\,r)}\right] \quad \text{(in open pipe)}$$

  and $$f = n\left[\frac{v}{4(l+0.6\,r)}\right] \quad \text{(in closed pipe)}$$

  However, we neglect this small correction if the length of the tube is much larger than its diameter.
- If an open pipe and a closed pipe are of same lengths, then fundamental frequency of open pipe $\left(=\frac{v}{2l}\right)$ is two times the fundamental frequency of closed pipe $\left(=\frac{v}{4l}\right)$.
- In the above equations v is the speed of constituent longitudinal waves by which stationary wave is formed. This speed in air is given by

  $$v = \sqrt{\frac{\gamma RT}{M}}$$

  If temperature is increased or some light gas is filled in the pipe, then v will increase.So, this set of frequencies will also increase.
- As n increases, v remains unchanged, f increases. Therefore, $\lambda (= v/f)$ and the loop size $\left(=\frac{\lambda}{2}\right)$ decrease.

Example 19.10 *Third overtone of a closed organ pipe is in unison with fourth harmonic of an open organ pipe. Find the ratio of the lengths of the pipes.*

Solution Third overtone of closed organ pipe means seventh harmonic. Given

$$(f_7)_{\text{closed}} = (f_4)_{\text{open}}$$

$$7\left(\frac{v}{4l_c}\right) = 4\left(\frac{v}{2l_o}\right)$$

$$\therefore \quad \frac{l_c}{l_o} = \frac{7}{8} \qquad \textbf{Ans.}$$

Example 19.11 *An open organ pipe has a fundamental frequency of 300 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of this open pipe. How long is each pipe?*
(Speed of sound in air = 330 m/s)

Solution Fundamental frequency of an open organ pipe,

$$f_1 = \frac{v}{2l_o} \Rightarrow l_o = \frac{v}{2f_1} = \frac{330}{2\times 300}$$

$$= 0.55\text{ m}$$

$$= 55\text{ cm} \qquad \textbf{Ans.}$$

Given, first overtone of closed organ pipe = first overtone of open organ pipe

Hence, $$3\left(\frac{v}{4l_c}\right) = 2\left(\frac{v}{2l_o}\right)$$

∴ $$l_c = \frac{3}{4} l_o = \left(\frac{3}{4}\right)(0.55)$$

$$= 0.4125\,\text{m}$$

$$= 41.25\,\text{cm}$$ **Ans.**

Example 19.12 *A cylindrical tube, open at both ends, has a fundamental frequency f in air. The tube is dipped vertically in water so that half of its length is in water. The fundamental frequency of the air column is now* **(JEE 1981)**

(a) $f/2$ *(b)* $3f/4$ *(c)* f *(d)* $2f$

Solution Initially, the tube was open at both ends and then it is closed.

$$f_o = \frac{v}{2l_o} \quad \text{and} \quad f_c = \frac{v}{4l_c}$$

Since, tube is half dipped in water, $l_c = \dfrac{l_o}{2}$

∴ $$f_c = \frac{v}{4\left(\dfrac{l_o}{2}\right)} = \frac{v}{2l_o} = f_o = f$$

Hence, the correct option is (c).

Example 19.13 *A closed organ pipe of length L and an open organ pipe contain gases of densities* ρ_1 *and* ρ_2*, respectively. The compressibility of gases are equal in both the pipes. Both the pipes are vibrating in their first overtone with same frequency. The length of the open organ pipe is* **(JEE 2004)**

(a) $\dfrac{L}{3}$ *(b)* $\dfrac{4L}{3}$ *(c)* $\dfrac{4L}{3}\sqrt{\dfrac{\rho_1}{\rho_2}}$ *(d)* $\dfrac{4L}{3}\sqrt{\dfrac{\rho_2}{\rho_1}}$

Solution. $f_c = f_o$ (both first overtone)

or $$3\left(\frac{v_c}{4L}\right) = 2\left(\frac{v_o}{2l_o}\right)$$

∴ $$l_o = \frac{4}{3}\left(\frac{v_o}{v_c}\right)L = \frac{4}{3}\frac{\sqrt{B/\rho_2}}{\sqrt{B/\rho_1}}L$$

$$= \frac{4}{3}\sqrt{\frac{\rho_1}{\rho_2}}L$$

Hence, the correct option is (c).

Note *Compressibility* $= \dfrac{1}{\text{Bulk modulus}}$. *Compressibility is same. Therefore, bulk modulus is also same.*

INTRODUCTORY EXERCISE 19.5

1. An open pipe is suddenly closed at one end with the result that the frequency of third harmonic of the closed pipe is found to be higher by 100 Hz than the fundamental frequency of the open pipe. The fundamental frequency of the open pipe is **(JEE 1996)**

(a) 200 Hz (b) 300 Hz (c) 240 Hz (d) 480 Hz

2. An organ pipe P_1 closed at one end vibrating in its first harmonic and another pipe P_2 open at both ends vibrating in its third harmonic are in resonance with a given tuning fork. The ratio of the length of P_1 and P_2 is **(JEE 1988)**

(a) 8/3 (b) 3/8 (c) 1/6 (d) 1/3

3. A tube closed at one end and containing air produces, when excited the fundamental note of frequency 512 Hz. If the tube is opened at both ends, the fundamental frequency that can be excited is (in Hz) **(JEE 1986)**

(a) 1024 (b) 512 (c) 256 (d) 128

4. The fundamental frequency of a closed organ pipe is 220 Hz.

(a) Find the length of this pipe.

(b) The second overtone of this pipe has the same frequency as the third harmonic of an open pipe. Find the length of this open pipe. Take speed of sound in air 345 m/s.

5. Standing sound waves are produced in a pipe that is 0.8 m long, open at one end, and closed at the other. For the fundamental and first two overtones, where along the pipe (measured from the closed end) are

(a) the displacement antinodes

(b) the pressure antinodes.

6. An organ pipe has two successive harmonics with frequencies 400 and 560 Hz. The speed of sound in air is 344 m/s.

(a) Is this an open or a closed pipe?

(b) What two harmonics are there?

(c) What is the length of the pipe?

19.8 Beats

When two wavetrains of the same frequency travel along the same line in opposite directions, standing waves are formed in accordance with the principle of superposition. In standing waves, amplitude is a function of distance. This illustrates a type of interference that we can call **interference in space.** The same principle of superposition leads us to another type of interference, which we can call **interference in time.** It occurs when two wavetrains of slightly different frequency travel through the same region.

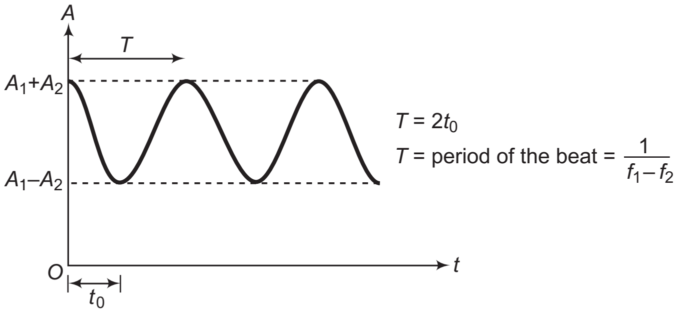


Fig. 19.9

If the waves are in phase at some time (say $t=0$) the interference will be constructive and the resultant amplitude at this moment will be A_1+A_2, where A_1 and A_2 are the amplitudes of individual wavetrains. But at some later time (say $t=t_0$), because the frequencies are different, the waves will be out of phase or the interference will be destructive and the resultant amplitude will be A_1-A_2 (if $A_1>A_2$). Later, we will see that the time t_0 is $\dfrac{1}{2(f_1-f_2)}$.

Thus, the resultant amplitude oscillates between A_1+A_2 and A_1-A_2 with a time period $T=2t_0=\dfrac{1}{f_1-f_2}$ or with a frequency $f=f_1-f_2$ known as **beat frequency.** Thus,

$$\boxed{\text{Beat frequency, } f=f_1-f_2}$$

Calculation of Beat Frequency

Suppose two waves of frequencies f_1 and f_2 $(<f_1)$ are meeting at some point in space. The corresponding periods are T_1 and T_2 $(>T_1)$. If the two waves are in phase at $t=0$, they will again be in phase when the first wave has gone through exactly one more cycle than the second. This will happen at a time $t=T$, the period of the beat. Let n be the number of cycles of the first wave in time T, then the number of cycles of the second wave in the same time is $(n-1)$. Hence,

$$T=nT_1 \qquad \ldots\text{(i)}$$

and

$$T=(n-1)\,T_2 \qquad \ldots\text{(ii)}$$

Eliminating n from these two equations, we have

$$T=\frac{T_1T_2}{T_2-T_1}=\frac{1}{\dfrac{1}{T_1}-\dfrac{1}{T_2}}=\frac{1}{f_1-f_2}$$

The reciprocal of the beat period is the beat frequency

$$f=\frac{1}{T}=f_1-f_2.$$

Alternate Method

Let the oscillations at some point in space (say $x=0$) due to two waves be

$$y_1=A_1\sin 2\pi f_1t \quad\text{and}\quad y_2=A_2\sin 2\pi f_2t \qquad (\omega=2\pi f)$$

If they are in phase at some time t, then

$$2\pi f_1t=2\pi f_2t \quad\text{or}\quad f_1t=f_2t \qquad \ldots\text{(i)}$$

They will be again in phase at time $(t+T)$ if,

$$2\pi f_1(t+T)=2\pi f_2(t+T)+2\pi$$

or

$$f_1(t+T)=f_2(t+T)+1 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get $T=\dfrac{1}{f_1-f_2}$

Note *If a tuning fork is loaded with wax, its frequency decreases. On the other hand, when tuning fork is filed, its frequency increases.*

Example 19.14 *Two tuning forks A and B produce* 6 *beats per second. Frequency of A is* 300 *Hz. When B is slightly loaded with wax, beat frequency decreases. Find original frequency of B.*

Solution Since, *A* and *B* produce 6 beats per second. Therefore, original frequency of *B* may be 306 Hz or 294 Hz.

When *B* is loaded with wax, its frequency will decrease (suppose it decreases by 1Hz). So, if it is 306 Hz, it will become 305 Hz. If it is 294 Hz, it will become 293 Hz.

Frequency of *A* is unchanged (= 300 Hz). If f_B becomes 305 Hz, then beat frequency will become 5 Hz. If f_B becomes 293 Hz, then it will become 7 Hz. But in the question it is given that beat frequency has decreased. So, the correct answer is 306 Hz.

Example 19.15 *The string of a violin emits a note of* 400 *Hz at its correct tension. The string is bit taut and produces* 5 *beats per second with a tuning fork of frequency* 400 *Hz. Find frequency of the note emitted by this taut string.*

Solution The frequency of vibration of a string increases with increase in the tension. Thus, the note emitted by the string will be a little more than 400 Hz. As it produces 5 beats per second with the 440 Hz tuning fork, the frequency will be 405 Hz.

Example 19.16 *Two tuning forks P and Q when set vibrating, give* 4 *beats per second. If a prong of the fork P is filed, the beats are reduced to* 2 *per second. Determine the original frequency of P, if that of Q is* 250 *Hz.*

Solution There are four beats between *P* and *Q*, therefore the possible frequencies of *P* are 246 or 254 (that is 250 ± 4) Hz.

When the prong of *P* is filed, its frequency becomes greater than the original frequency.

If we assume that the original frequency of *P* is 254, then on filing its frequency will be greater than 254. The beats between *P* and *Q* will be more than 4. But it is given that the beats are reduced to 2, therefore, 254 is not possible.

Therefore, the required frequency must be 246 Hz.

(This is true, because on filing the frequency may increase to 248, giving 2 beats with *Q* of frequency 250 Hz) **Ans.**

INTRODUCTORY EXERCISE 19.6

1. A tuning fork produces 4 beats per second with another tuning fork of frequency 256 Hz. The first one is now loaded with a little wax and the beat frequency is found to increase to 6 per second. What was the original frequency of the first tuning fork?
2. A tuning fork of unknown frequency makes three beats per second with a standard fork of frequency 384 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork?

19.9 The Doppler's Effect

If a wave source and a receiver are moving relative to each other, the frequency observed by the receiver (f') is different from the actual source frequency (f). This phenomenon is called the **Doppler's effect**, named after the Austrian physicist Christian Johann Doppler (1803–1853), who discovered it in light waves.

Perhaps you might have noticed how the sound of a vehicle's horn changes as the vehicle moves past you. The frequency (pitch) of the sound you hear as the vehicle approaches you is higher than the frequency you hear as it moves away from you. The Doppler's effect applies to waves in general. Let us apply it to sound waves. We consider the special case in which the source and observer move along the line joining them. We will use the following symbols,

v = speed of sound, v_s = speed of source and v_o = speed of observer

and v_m = velocity of medium in which sound travels.

For example, if Doppler's effect is observed in air, it is wind velocity. If Doppler's effect is observed inside a river, then it is river velocity.

The general formula of the changed frequency is

$$\boxed{f' = \left(\frac{v \pm v_m \pm v_o}{v \pm v_m \pm v_s}\right) f}$$

Sign Convention

v_m We are talking about that sound which is travelling from source to observer (S to O). If the medium is also travelling in the same direction, then it means medium is supporting the sound. So, take positive sign both in numerator and denominator. If the medium travels in opposite direction, then take negative sign. If nothing is given in the question, then take it zero.

v_o and v_s The concept is approaching nature always increases the frequency. So, take positive sign with v_o (because it is in numerator) and negative sign with v_s (as it is in denominator). On the other hand, receding nature decreases the frequency. So, take negative sign with v_o and positive sign with v_s. Now, let us make some different cases.

Case 1

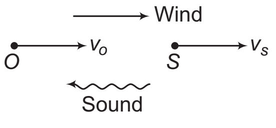


Fig. 19.10

In the given figure, sound is travelling from right to left but wind is blowing from left to right, so we will have to take negative sign with wind velocity. Observer is approaching towards source, so take positive sign with v_o. Source is receding from the observer, so take positive sign with v_s. The correct formula is given below

$$f' = \left(\frac{v - v_{\text{wind}} + v_o}{v - v_{\text{wind}} + v_s}\right) f$$

Case 2

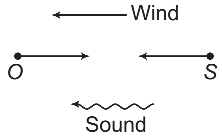


Fig. 19.11

Sound and wind both are travelling in the same direction (from S to O), so take positive sign with wind velocity. Source and observer both are approaching towards each other. So, take positive sign with v_o and negative sign with v_s. Therefore, the correct formula is

$$f' = \left(\frac{v + v_{\text{wind}} + v_o}{v + v_{\text{wind}} - v_s}\right) f$$

Now, let us derive two special cases when medium velocity is zero.

Source at Rest, Observer Moves

Suppose that the observer O moves towards the source S at speed v_o. The speed of the sound waves relative to O is $v_r = v + v_o$, but wavelength has its normal value $\lambda = \frac{v}{f}$. Thus, the frequency heard by O is

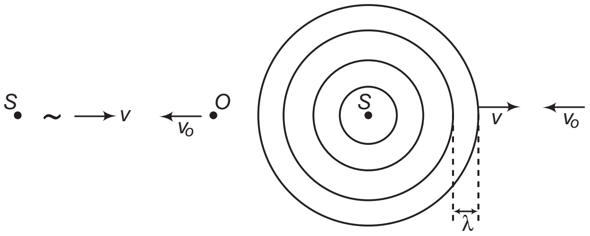


Fig. 19.12

$$f' = \frac{v_r}{\lambda} = \left(\frac{v + v_o}{v}\right) f$$

If O were moving away from S, the frequency heard by O would be

$$f' = \left(\frac{v - v_o}{v}\right) f$$

Combining these two expressions, we find

$$\boxed{f' = \left(\frac{v \pm v_o}{v}\right) f} \qquad \ldots\text{(i)}$$

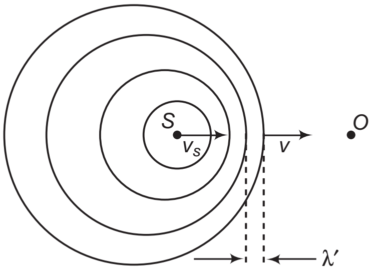
Source Moves, Observer at Rest

Fig. 19.13

Suppose that the source S moves towards O as shown in figure.

If S were at rest, the distance between two consecutive wave pulses emitted by sound would be $\lambda = \frac{v}{f} = vT$. However, in one time period S moves a distance v_sT before it emits the next pulse. As a result the wavelength is modified. Directly ahead of S the effective wavelength (for both S and O) is

$$\lambda' = vT - v_sT = (v - v_s)T = \left(\frac{v - v_s}{f}\right)$$



The wavelength in front of the source is less than the normal whereas in the rear it is larger than normal.

Fig. 19.14

The speed of sound waves relative to O is simply v. Thus, the frequency observed by O is

$$f' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right)f$$

If S were moving away from O, the effective wavelength would be $\lambda' = \left(\frac{v + v_s}{f}\right)$ and the apparent frequency would be

$$f' = \left(\frac{v}{v + v_s}\right)f$$

Combining these two results, we have

$$\boxed{f' = \left(\frac{v}{v \pm v_s}\right)f} \qquad \text{...(ii)}$$

All four possibilities can be combined into one equation

$$\boxed{f' = \left(\frac{v \pm v_o}{v \mp v_s}\right)f} \qquad \text{...(iii)}$$

where the upper signs (+ numerator, – denominator) correspond to the source and observer along the line joining the two in the direction toward the other, and the lower signs in the direction away from the other.

Alternate Method

The above formulae can be derived alternately as discussed below.

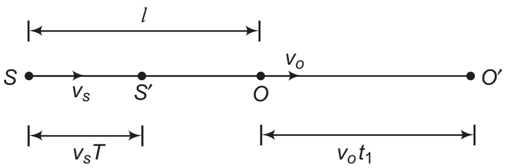


Fig. 19.15

Assume that the source and observer are moving along the same line and that the observer O is to the right of the source S. Suppose that at time $t=0$, when the source and the observer are separated by a distance $SO=l$, the source emits a wave pulse (say p_1) that reaches the observer at a later time t_1. In that time the observer has moved a distance $v_o t_1$ and the total distance travelled by p_1 in the time t_1 has been $l+v_o t_1$. If v is the speed of sound, this distance is also vt_1. Then,

$$vt_1 = l + v_0 t_1$$

or

$$t_1 = \frac{l}{v - v_0} \qquad \ldots\text{(iv)}$$

At time $t=T$, the source is at S' and the wave pulse (say p_2) emitted at this time will reach the observer at a time say t_2, measured from the same time $t=0$, as before. The total distance travelled by p_2 till it is received by the observer (measured from S') is $(l-v_sT)+v_o t_2$. The actual travel time for p_2 is (t_2-T) and the distance travelled is $v\,(t_2-T)$. Therefore,

$$v\,(t_2 - T) = (l - v_sT) + v_0 t_2$$

or

$$t_2 = \frac{l + (v - v_s)\,T}{v - v_o} \qquad \ldots\text{(v)}$$

The time interval reckoned by the observer between the two pulses emitted by the source at S and at S' is

$$T' = t_2 - t_1 = \left(\frac{v - v_s}{v - v_o}\right)T$$

This is really the changed time period as observed by the observer. Hence, the new frequency is

$$f' = \frac{1}{T'} = \left(\frac{v - v_o}{v - v_s}\right)\frac{1}{T}$$

or

$$f' = \left(\frac{v - v_o}{v - v_s}\right)f$$

This is a result which we can expect by using equation $f' = \left(\dfrac{v \pm v_o}{v \mp v_s}\right) f$ in the case when source is moving towards observer and observer is moving away from the source.

Extra Points to Remember

- We have derived equation number (iii) by assuming that v_o and v_s are along the line joining source and observer. If the motion is along some other direction, the components of velocities along the line joining source and observer are considered.

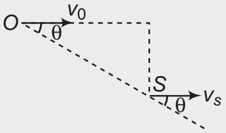


Fig. 19.16

For example in the figure shown,
$$f' = \left(\frac{v + v_o \cos\theta}{v + v_s \cos\theta}\right) f$$

- Change in frequency depends on the fact that whether the source is moving towards the observer or the observer is moving towards the source. But when the speed of source and observer are much lesser than that of sound, the change in frequency becomes independent of the fact whether the source is moving or the observer. This can be shown as under.

  Suppose a source is moving towards a stationary observer, with speed u and the speed of sound is v, then
$$f' = \left(\frac{v}{v-u}\right) f = \left(\frac{1}{1-\frac{u}{v}}\right) f = \left(1-\frac{u}{v}\right)^{-1} f$$
  Using the binomial expansion, we have
$$\left(1-\frac{u}{v}\right)^{-1} \approx 1+\frac{u}{v} \quad \text{if} \quad u << v$$
$$\therefore \qquad f' \approx \left(1+\frac{u}{v}\right) f \quad \text{if} \quad u << v$$
  On the other hand, if an observer moves towards a stationary source with same speed u, then
$$f' = \left(\frac{v+u}{v}\right) f = \left(1+\frac{u}{v}\right) f$$
  which is same as above.

- As long as v_s and v_o are along the line joining S and O, Doppler's effect (or change in frequency) does not depend on the distance between S and O. For example in the given figure,

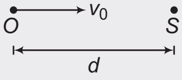


Fig. 19.17

$$f' = f\left(\frac{v+v_o}{v}\right)$$

  $f' > f$ but it is constant and independent of 'd'.

- Frequency is given by $f = \dfrac{v}{\lambda}$

  By the motion of source, λ changes, therefore frequency changes. But, from the motion of observer, relative velocity v between sound and observer changes, therefore frequency changes.

- Despite the motion of source or observer (or both), Doppler's effect is not observed (or $f' = f$) under the following four conditions.

***Condition* 1** $\mathbf{v}_s$ or $\mathbf{v}_o$ is making an angle of 90° with the line joining S and O. This is illustrated in the following figure.

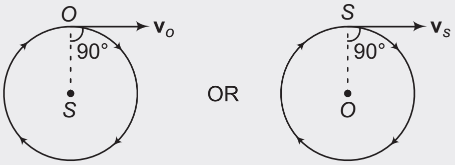


Fig. 19.18

***Condition* 2** Source and observer both are in motion but their velocities are equal or relative motion between them is zero.

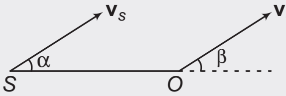


Fig. 19.19

In the figure shown, $\mathbf{v}_s = \mathbf{v}_o$ if $v_s = v_o$ and $\alpha = \beta$

Taking the components along SO, we have $f' = f\left(\dfrac{v - v_o \cos\beta}{v - v_s \cos\alpha}\right) \Rightarrow f' = f$ because $v_s = v_o$ and $\alpha = \beta$

***Condition* 3** Source and observer both are at rest. Only medium is in motion.

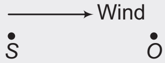


Fig. 19.20

In the figure shown, source and observer are at rest. Only wind is blowing in a direction from source to observer. The changed frequency is

$$f' = f\left(\frac{v + v_{\text{wind}}}{v + v_{\text{wind}}}\right) = f$$

***Condition* 4**

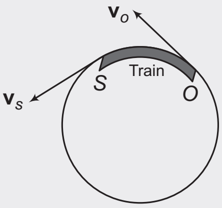


Fig. 19.21

A train is travelling on a circular track. Engine is the source of sound and guard is the observer. Although,

$$\mathbf{v}_s \neq \mathbf{v}_o$$

yet f' comes out to be f.

Exercise Derive the above result.

- **Doppler's Effect in Light**

Light waves also show Doppler's effect. If a light source is moving away from a stationary observer, then the frequency of light waves appear to be decreased and wavelength appear to be increased and *vice-versa*.

If the light source or the observer is moving with a velocity v such that the distance between them is decreasing, then the apparent frequency of the source will be given by

$$\boxed{f' = f\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}}$$

If the distance between the light source and the observer is increasing, then the apparent frequency of the source is given by

$$\boxed{f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}}$$

The change in wavelength can be determined by

$$\boxed{\Delta\lambda = \frac{v}{c}\cdot\lambda}$$

If the light source is moving away from the observer, the shift in the spectrum is towards red and if it is moving towards the observer the shift is towards the violet.

Note *The Doppler's effect in light depends only on the relative motion between the source and the observer while the Doppler's effect in sound also depends upon whether the source is moving or the observer is moving.*

Example 19.17 *A car approaching a crossing C at a speed of 20 m/s sounds a horn of frequency 500 Hz when 80 m from the crossing. Speed of sound in air is 330 m/s. What frequency is heard by an observer (at rest) 60 m from the crossing on the straight road which crosses car road at right angles?*

Solution The situation is as shown in figure.

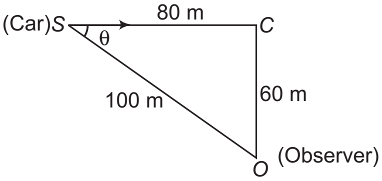


Fig. 19.22

$$\cos\theta = \frac{80}{100} = \frac{4}{5}$$

∴ Apparent frequency,

$$f_{\text{app}} = \left(\frac{v}{v - v_s\cos\theta}\right) f$$

$$= \left(\frac{330}{330 - 20\times\frac{4}{5}}\right)(500)$$

$$= 525.5\ \text{Hz}$$ **Ans.**

Example 19.18 *A siren emitting a sound of frequency 1000 Hz moves away from you towards a cliff at a speed of 10 m/s.*

(a) What is the frequency of the sound, you hear coming directly from the siren?

(b) What is the frequency of sound you hear reflected off the cliff. Speed of sound in air is 330 m/s?

Note *For reflected sound cliff can be assumed as a plane mirror.*

Solution (a) Sound heard directly

$$f_1 = f_0\left(\frac{v}{v+v_s}\right) \Rightarrow v_s = 10\ \text{m/s}$$

$$\therefore \qquad f_1 = \left(\frac{330}{330+10}\right)\times 1000$$

$$= 970.6\ \text{Hz}$$ **Ans.**

(b) The frequency of the reflected sound is given by

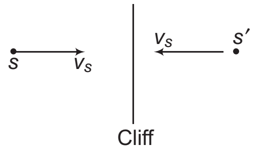


Fig. 19.23

$$f_2 = f_0\left(\frac{v}{v-v_s}\right)$$

$$f_2 = \left(\frac{330}{330-10}\right)\times 1000$$

$$= 1031.25\,\text{Hz}$$ **Ans.**

Example 19.19 *A whistle of frequency* 540 *Hz rotates in a circle of radius* 2 *m at a linear speed of* 30 *m/s. What is the lowest and highest frequency heard by an observer a long distance away at rest with respect to the centre of circle? Take speed of sound in air as* 330 *m/s. Can the apparent frequency be ever equal to actual?*

Solution Apparent frequency will be minimum when the source is at N and moving away from the observer.

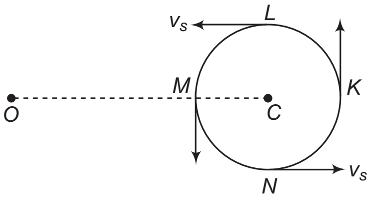


Fig. 19.24

$$f_{\min} = \left(\frac{v}{v+v_s}\right)f$$

$$= \left(\frac{330}{330+30}\right)(540) = 495\,\text{Hz}$$ **Ans.**

Frequency will be maximum when source is at L and approaching the observer.

$$f_{\max} = \left(\frac{v}{v-v_s}\right)f = \left(\frac{330}{330-30}\right)(540)$$

$$= 594\,\text{Hz}$$ **Ans.**

Further, when the source is at M and K, angle between velocity of source and line joining source and observer is 90° or $v_s \cos\theta = v_s \cos 90° = 0$. So, there will be no change in the apparent frequency.

Note *Although the source velocity v_s is not along the line joining S and O but at a long distance we can assume that it is along SO.*

Example 19.20 *Two tuning forks with natural frequencies* 340 *Hz each move relative to a stationary observer. One fork moves away from the observer while the other moves towards him at the same speed. The observer hears beats of frequency* 3 *Hz. Find the speed of the tuning fork (velocity of sound in air is* 340 *m/s).*

Note *The difference in apparent frequencies is very small (3 Hz). So, we may conclude that the speed of source (v_s) << speed of sound (v). Therefore, we can neglect the higher terms of $\frac{v_s}{v}$.*

Solution Given,

$$f_1 - f_2 = 3$$

or

$$\left(\frac{v}{v-v_s}\right)f - \left(\frac{v}{v+v_s}\right)f = 3$$

or $$\left[\frac{1}{\left(1-\frac{v_s}{v}\right)}-\frac{1}{\left(1+\frac{v_s}{v}\right)}\right]f=3$$

or $$\left[\left(1-\frac{v_s}{v}\right)^{-1}-\left(1+\frac{v_s}{v}\right)^{-1}\right]f=3$$

or $$\left[\left(1+\frac{v_s}{v}\right)-\left(1-\frac{v_s}{v}\right)\right]f=3$$

or $$\frac{2v_s f}{v}=3$$

or Speed of tuning fork, $v_s=\dfrac{3v}{2f}$

Substituting the values, we get

$$v_s=\frac{(3)(340)}{(2)(340)}=1.5\text{ m/s}$$ **Ans.**

INTRODUCTORY EXERCISE 19.7

1. A whistle giving out 450 Hz approaches a stationary observer at a speed of 33 m/s. The frequency heard by the observer (in Hz) is (Speed of sound = 330 m/s) **(JEE 2000)**

(a) 409 (b) 429

(c) 517 (d) 500

2. A train moves towards a stationary observer with speed 34 m/s. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to 17 m/s, the frequency registered is f_2. If the speed of sound is 340 m/s, then the ratio f_1/f_2 is **(JEE 1997)**

(a) 18/19 (b) 1/2

(c) 2 (d) 19/18

3. A siren placed at a railway platform is emitting sound of frequency 5 kHz. A passenger sitting in a moving train *A* records a frequency of 5.5 kHz, while the train approaches the siren. During his return journey in a different train *B* he records a frequency of 6.0 kHz while approaching the same siren. The ratio of the velocity of train *B* to that of train *A* is **(JEE 2002)**

(a) 242/252 (b) 2

(c) 5/6 (d) 11/6

4. A train is moving on a straight track with speed 20 ms^{-1}. It is blowing its whistle at the frequency of 1000 Hz. The percentage change in the frequency heard by a person standing near the track as the train passes him is close to (speed of sound = 320 ms^{-1}) **(JEE 2015 Main)**

(a) 12% (b) 6%

(c) 18% (d) 24%

Solved Examples

TYPED PROBLEMS

Type 1. *Based on Doppler's effect and Beats together*

Concept

(i) If there is a relative motion between source and observer, frequency changes.

(ii) Natural frequencies of a stretched wire are

$$f_n = n\left(\frac{v}{2l}\right) = n\left[\frac{\sqrt{T/\mu}}{2l}\right] \qquad \text{(where } n = 1, 2, \ldots)$$

If tension in the wire is changed, then frequency changes.

(iii) Natural frequencies of a closed pipe are

$$f_n = n\left(\frac{v}{4l}\right) \qquad \text{(where } n = 1, 3, \ldots)$$

and natural frequencies of an open pipe are

$$f_n = n\left(\frac{v}{2l}\right) \qquad \text{(where } n = 1, 2, \ldots)$$

Here, $v = \sqrt{\frac{\gamma RT}{M}}$

When temperature is increased, these sets of frequencies also increase.

(iv) Beat frequency is given by

$$f_b = f_1 - f_2 \qquad \text{(if } f_1 > f_2)$$

Example 1 *The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air $v = 330$ m/s.*

Solution Let l_1 and l_2 be the lengths of closed and open pipes respectively.

Fundamental frequency of closed organ pipe is given by

$$f_1 = \frac{v}{4l}$$

where v is speed of sound in air and 330 m/s equal to

But, $f_1 = 110$ Hz (given)

Therefore, $\frac{v}{4l_1} = 110$ Hz

$$\therefore \quad l_1 = \frac{v}{4 \times 110} = \frac{330}{4 \times 110} \text{ m} = 0.75 \text{ m}$$

First overtone of closed organ pipe will be

$$f_3 = 3f_1 = 3\ (110)\ \text{Hz} = 330\ \text{Hz}$$

This produces a beat frequency of 2.2 Hz with first overtone of open organ pipe.

Therefore, first overtone frequency of open organ pipe is either

$$(330 + 2.2)\ \text{Hz} = 332.2\ \text{Hz}$$

or $$(330 - 2.2)\ \text{Hz} = 327.8\ \text{Hz}$$

If it is 332.2 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 332.2\ \text{Hz}$$

or $$l_2 = \frac{v}{332.2} = \frac{330}{332.2}\ \text{m} = 0.99\ \text{m}$$

and if it is 327.8 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 327.8\ \text{Hz}$$

or $$l_2 = \frac{v}{327.8}\ \text{m} = \frac{330}{327.8}\ \text{m} = 1.0067\ \text{m}$$

Therefore, length of the closed organ pipe is $l_1 = 0.75$ m while length of open pipe is either $l_2 = 0.99$ m or 1.0067 m. **Ans.**

Example 2 *A vibrating string of certain length l under a tension T resonates with a mode corresponding to the first overtone (third harmonic) of an air column of length 75 cm inside a tube closed at one end. The string also generates 4 beats/s with a tuning fork of frequency n. Now when the tension of the string is slightly increased the number of beats reduces to 2 per second. Assuming the velocity of sound in air to be 340 m/s, the frequency n of the tuning fork in Hz is* **(JEE 2008)**

(a) 344 *(b)* 336

(c) 117.3 *(d)* 109.3

Solution With increase in tension, frequency of vibrating string will increase. Since, number of beats are decreasing. Therefore, frequency of vibrating string or third harmonic frequency of closed pipe should be less than the frequency of tuning fork by 4.

∴ Frequency of tuning fork = third harmonic frequency of closed pipe + 4

$$= 3\left(\frac{v}{4l}\right) + 4 = 3\left(\frac{340}{4 \times 0.75}\right) + 4$$

$$= 344\ \text{Hz}$$

∴ Correct option is (a).

Example 3 *Two identical straight wires are stretched so as to produce 6 beats/s when vibrating simultaneously. On changing the tension slightly in one of them, the beat frequency remains unchanged. Denoting by T_1, T_2 the higher and the lower initial tension in the strings, then it could be said that while making the above changes in tension* **(JEE 1991)**

(a) T_2 was decreased *(b) T_2 was increased*

(c) T_1 was decreased *(d) T_1 was increased*

Solution $T_1 > T_2$

∴ $v_1 > v_2$

or $f_1 > f_2$

and $f_1 - f_2 = 6$ Hz

Now, if T_1 is increased, f_1 will increase or $f_1 - f_2$ will increase. Therefore, (d) option is wrong.

If T_1 is decreased, f_1 will decrease and it may be possible that now $f_2 - f_1$ become 6 Hz.Therefore, (c) option is correct.

Similarly, when T_2 is increased, f_2 will increase and again $f_2 - f_1$ may become equal to 6 Hz. So, (b) is also correct. But (a) is wrong.

Example 4 *A sonometer wire under tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of the sonometer wire has a length of 10 cm and mass of 1 g. The vibrating tuning fork is now moved away from the vibrating wire with a constant speed and an observer standing near the sonometer hears one beat per second. Calculate the speed with which the tuning fork is moved, if the speed of sound in air is 300 m/s.* **(JEE 1983)**

Solution Fundamental frequency of sonometer wire,

$$f = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

$$= \frac{1}{2 \times 0.1}\sqrt{\frac{64 \times 0.1}{10^{-3}}}$$

$$= 400 \text{ Hz}$$

Given beat frequency, $f_b = f - f' = 1$ Hz

∴ $f' = 399$ Hz

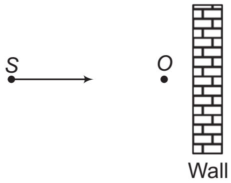
Using $$f' = f\left(\frac{v}{v + v_s}\right)$$

or $$399 = 400\left(\frac{300}{300 + v_s}\right)$$

Solving we get, $v_s = 0.75$ m/s **Ans.**

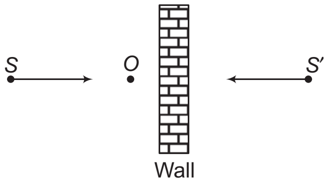
Type 2. *Reflection of a sound from a wall and beats*

Concept



A source of sound is approaching towards a wall. An observer hears two sounds. One is direct from the source and the other reflected from wall. If both frequencies are same, then the observer will hear no beats and if there is a frequency difference then beats are heard to the observer. This all depends on the position of observer. For finding the frequency of reflected sound we have the following two approaches.

Approach 1 Take wall as a plane mirror. Image of S on this plane mirror will behave like another source S' for the reflected sound.



For example, in the shown figure both direct sound source S and reflected sound source S' are approaching towards the observer.

Approach 2 First consider wall as an observer for the sound received to it from the source S. Now, consider this wall as a source for reflected sound.

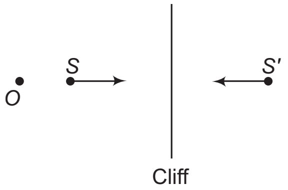
Example 5 *A siren emitting a sound of frequency* 1000 *Hz moves away from you toward a cliff at a speed of* 10 *m/s.*

(a) What is the frequency of the sound you hear coming directly from the siren?

(b) What is the frequency of the sound you hear reflected off the cliff?

(c) What beat frequency would you hear? Take the speed of sound in air as 330 *m/s.*

Solution The situation is as shown in figure.



(a) Frequency of sound reaching directly to us (by S)

$$f_1 = \left(\frac{v}{v + v_s}\right)$$

$$f = \left(\frac{330}{330 + 10}\right)(1000)$$

$$= 970.6 \text{ Hz}$$ **Ans.**

(b) Frequency of sound which is reflected from the cliff (from S')

$$f_2 = \left(\frac{v}{v - v_s}\right) f = \left(\frac{330}{330 - 10}\right)(1000)$$

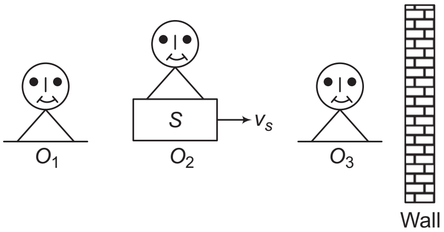
$$= 1031.3 \text{ Hz}$$ **Ans.**

(c) Beat frequency $= f_2 - f_1$

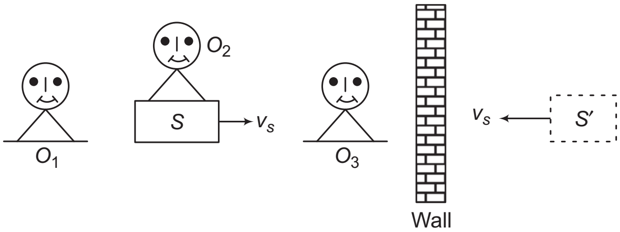
$$= 60.7 \text{ Hz}$$ **Ans.**

Note *Numerically the beat frequency comes out to be* 60.7 *Hz. But, beats between two tones can be detected by ear upto a frequency of about 7 per second. At higher frequencies beats cannot be distinguished in the sound produced. Hence, the correct answer of part (c) should be zero.*

Example 6 *A source of sound of frequency f is approaching towards a wall with speed v_s. Speed of sound is v. Three observers O_1, O_2 and O_3 are at different locations as shown. Find the beat frequency as observed by three different observers.*



Solution For reflected sound we can take wall as a plane mirror.



Observer O_1 S is receding from O_1 and S′ is approaching towards him. So,

$$f_{s'} > f_s$$

$$f_b = f_{s'} - f_s$$

$$= f\left(\frac{v}{v - v_s}\right) - f\left(\frac{v}{v + v_s}\right)$$

$$= \left(\frac{2vv_s}{v^2 - v_s^2}\right) f \qquad \textbf{Ans.}$$

Observer O_2 O_2 has no relative motion with S. Therefore, there will be no change in the frequency of direct sound from S. Further, O_2 and S′ are approaching towards each other. Therefore, the frequency of reflected sound will increase.

$$f_b = f_{s'} - f_s$$

$$= f\left(\frac{v + v_{o_2}}{v - v_s}\right) - f$$

But $v_{o_2} = v_s$

$$\therefore \qquad f_b = f\left(\frac{v + v_s}{v - v_s}\right) - f$$

$$= \left(\frac{2v_s}{v - v_s}\right) f \qquad \textbf{Ans.}$$

Observer O_3 S and S′ both are approaching towards O_3. So, both frequencies will increase. But after the increase they are equal. So, beat frequency will be zero.

$$f_s = f_{s'} = f\left(\frac{v}{v - v_s}\right)$$

$$\therefore \qquad f_b = f_s - f_{s'} = 0$$

Example 7 *A sound wave of frequency f travels horizontally to the right. It is reflected from a large vertical plane surface moving to left with a speed v. The speed of sound in medium is c* **(JEE 1995)**

(a) *The number of waves striking the surface per second is* $f\dfrac{(c+v)}{c}$

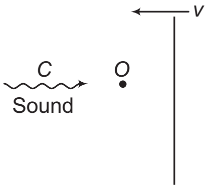
(b) *The wavelength of reflected wave is* $\dfrac{c(c-v)}{f(c+v)}$

(c) *The frequency of the reflected wave is* $f\dfrac{(c+v)}{(c-v)}$

(d) *The number of beats heard by a stationary listener to the left of the reflecting surface is* $\dfrac{vf}{c-v}$

Solution Moving plane is like a moving observer. Therefore, number of waves encountered by moving plane,

$$f_1 = f\left(\frac{v+v_0}{v}\right) = f\left(\frac{c+v}{c}\right)$$



Frequency of reflected wave, $f_2 = f_1\left(\dfrac{v}{v-v_s}\right) = f\left(\dfrac{c+v}{c-v}\right)$

Wavelength of reflected wave, $\lambda_2 = \dfrac{c}{f_2} = \dfrac{c}{f}\left(\dfrac{c-v}{c+v}\right)$

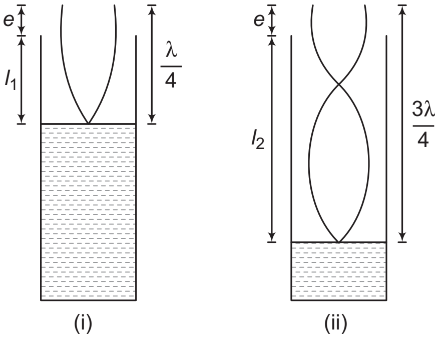
Beat frequency, $f_b = f_2 - f = f\left(\dfrac{c+v}{c-v}\right) - f = \dfrac{2fv}{c-v}$

Therefore, the correct options are (a), (b) and (c).

Type 3. *Based on the experiment of finding speed of sound using resonance tube*

Concept

(i) If a vibrating tuning fork (of known frequency) is held over the open end of a resonance tube, then resonance is obtained at some position as the level of water is lowered.



(i) (ii)

If e is the end correction of the tube and l_1 is the length from the water level to the top of the tube, then

$$l_1 + e = \frac{\lambda}{4} \qquad \ldots\text{(i)}$$

Now, the water level is further lowered until a resonance is again obtained. If l_2 is new length of the air column, then

$$l_2 + e = \frac{3\lambda}{4} \qquad \ldots\text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii),
we get

$$l_2 - l_1 = \frac{\lambda}{2}$$

or $$\lambda = 2(l_2 - l_1) = \frac{v}{f} \qquad \left(\lambda = \frac{v}{f}\right)$$

$\therefore$ $$\boxed{v = 2f(l_2 - l_1)}$$

(ii) The above result is independent of the end correction e.

(iii) In the above two figures v, λ, f and loop size $\left(= \frac{\lambda}{2}\right)$ are same in both figures. Length of closed organ pipe is different. In the second figure, length is more. So, the fundamental frequency $\left(f = \frac{v}{4l}\right)$ will be less.

Suppose frequency of given tuning fork is 300 Hz. Then, fundamental frequency of first pipe is also 300Hz. But fundamental frequency of the second pipe should be 100 Hz. So, that its next higher frequency (or first overtone frequency) which is three times the fundamental $(= 3 \times 100 = 300\,\text{Hz})$ should also be in resonance with the given tuning fork.

Example 8 *A tuning fork of 512 Hz is used to produce resonance in a resonance tube experiment. The level of water at first resonance is 30.7 cm and at second resonance is 63.2 cm. The error in calculating velocity of sound is* **(JEE 2005)**

(a) 204.1 *cm/s* *(b)* 110 *cm/s*

(c) 58 *cm/s* *(d)* 280 *cm/s*

Solution Actual speed of sound in air is 330 m/s

$$\frac{\lambda}{2} = (l_2 - l_1) = (63.2 - 30.7)\,\text{cm} = 32.5\,\text{cm} = 0.325\,\text{m}$$

or $$\lambda = 0.65\,\text{m}$$

$\therefore$ Speed of sound observed, $v_0 = f\lambda = 512 \times 0.65 = 332.8\,\text{m/s}$

$\therefore$ Error in calculating velocity of sound = 2.8 m/s = 280 cm/s

$\therefore$ The correct option is (d).

Example 9 *In the experiment for the determination of the speed of sound in air using the resonance column method, the length of the air column that resonates in the fundamental mode, with a tuning fork is 0.1 m. When this length is changed to 0.35 m, the same tuning fork resonates with the first overtone. Calculate the end correction.* **(JEE 2003)**

(a) 0.012 *m* *(b)* 0.025 *m* *(c)* 0.05 *m* *(d)* 0.024 *m*

Solution Let e be the end correction.

Given that, fundamental tone for a length 0.1m = first overtone for the length 0.35 m.

$$\frac{v}{4(0.1+e)} = \frac{3v}{4(0.35+e)}$$

Solving this equation, we get $e = 0.025\,\text{m} = 2.5\,\text{cm}$.

$\therefore$ The correct option is (b).

Alternate method

$$l_2 + e = 3\lambda/4 \quad \text{and} \quad l_1 + e = \lambda/4$$

Solving these two equations, we get

$$e = \frac{l_2 - 3l_1}{2} = \frac{0.35 - 3 \times 0.1}{2} = 0.025\ \text{m}$$

Example 10 *A student is performing the experiment of resonance column. The diameter of the column tube is 4 cm. The frequency of the tuning fork is 512 Hz. The air temperature is 38° C in which the speed of sound is 336 m/s. The zero of the meter scale coincides with the top end of the resonance column tube. When the first resonance occurs, the reading of the water level in the column is* **(JEE 2012)**

(a) 14.0 *cm* *(b)* 15.2 *cm*

(c) 16.4 *cm* *(d)* 17.6 *cm*

Solution With end correction,

$$f = n\left[\frac{v}{4(l+e)}\right], \qquad \text{(where, } n = 1, 3, \ldots\text{)}$$

$$= n\left[\frac{v}{4(l+0.6r)}\right]$$

Because, $e = 0.6\,r$, where r is radius of pipe.

For first resonance, $n = 1$

$$\therefore \quad f = \frac{v}{4(l+0.6r)} \quad \text{or} \quad l = \frac{v}{4f} - 0.6r = \left[\left(\frac{336 \times 100}{4 \times 512}\right) - 0.6 \times 2\right] \text{cm}$$

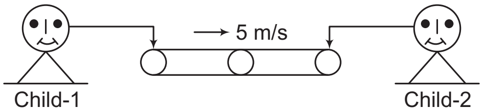
$$= 15.2\ \text{cm}$$

$\therefore$ The correct option is (b).

Type 4. *Based on a situation which is similar to Doppler's effect*

Concept

Let us take an example.



A belt shown in figure is moving with velocity 5 m/s. Child-1 is placing some coins over the belt at a regular interval of 1sec. Child -2 is receiving the coins on the other hand. Now, let us compare this situation with Doppler's effect.

Child-1 is source, child-2 is observer, velocity of belt is net velocity of sound ($= v \pm v_m$) by which sound travels from source to observer. Actually, in this case

$$v = \text{velocity of coin (similar to velocity of sound)}$$
$$= 0$$
$$v_m = \text{velocity of medium (or velocity of belt)}$$
$$= 5\,\text{m/s}$$
$$T = \text{time interval of placing two successive coins}$$
$$= 1\,\text{sec}$$
$$\therefore \quad f = \frac{1}{T} = \text{number of coins placed per second}$$

λ = distance between two coins = $v_m T = 5\,\text{m}$.

Now, let us consider the different cases.

Case 1 When source (child-1) and observer (child-2) both are at rest only belt (or medium) is in motion. In this case, distance between two coins moving towards child-2 with 5 m/s is 5m. If child-2 receives first coin at time $t_1 = 0$, then he will receive the second coin at $t_2 = \frac{5\,\text{m}}{5\,\text{m/s}} = 1\text{sec}$.

$\therefore$ T' = time interval of receival of two successive coins

$$= t_2 - t_1 = 1\text{ sec}$$
$$\therefore \quad f' = \frac{1}{T'} = 1\,\text{Hz}$$

Result By the motion of only medium, there is no change in frequency.

Case 2 If child-1 (or source) starts moving towards child-2 with 2 m/s. Then, in $T = 1$ sec, belt will move a distance of 5 m towards child-2 and child-1 will also travel a distance of 2m towards child-2 at the time of placing the second coin. Hence,

λ = new distance between two successive coins = $(5 - 2)\,\text{m} = 3\text{ m}$

Now, if child-2 receives the first coin at time $t_1 = 0$, then he will receive the second coin (at a distance of 3m from first coin) moving towards him with 5 m/s at time $t_2 = \frac{3}{5}$ sec.

$$\therefore \quad T' = t_2 - t_1 = \frac{3}{5}\text{sec} \quad \text{or} \quad f' = \frac{1}{T'} = \frac{5}{3}\text{Hz} \text{ or } f' > f$$

Result By the motion of source only wavelength changes therefore frequency changes.

Case 3 Child-1 (or source) is stationary and child-2 starts moving towards child-1 with 2 m/s. This time,

λ = distance between two successive coins is unchanged or 5 m.

But coins and child-2 are moving towards each other with 5 m/s and 2 m/s, respectively. Therefore, relative velocity between them is 7m/s. So, if child-2 receives the first coin at time $t_1 = 0$, then he will receive the second coin at time $t_2 = \frac{5}{7}$ sec

$$\therefore \quad T' = t_2 - t_1 = \frac{5}{7}\text{sec} \quad \text{or} \quad f' = \frac{1}{T'} = \frac{7}{5}\text{Hz} \text{ or } f' > f$$

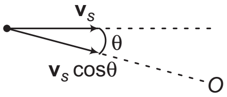
Result By the motion of observer relative velocity between sound and observer changes. Therefore, frequency changes.

Note *In case-2, $f' = \frac{5}{3}$ Hz and in case-3, $f' = \frac{7}{5}$ Hz. These two are not same, although speeds are same in both cases but change in frequency is different.*

Type 5. *When* $\mathbf{v}_s$ *or* $\mathbf{v}_o$ *are not along the line joining S and O but they are very far from each other.*

Concept

In this case, we take components of $\mathbf{v}_s$ and $\mathbf{v}_o$ along the line *SO*. For example in the figure shown below,



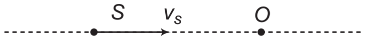
$$f' = f\left(\frac{v}{v - v_s\cos\theta}\right)$$

If source is very far from observer, then

$$\theta \to 0° \quad \text{or} \quad \cos\theta \to 1$$

$$\therefore \qquad f' = f\left(\frac{v}{v - v_s}\right)$$

Example 11 *A source of frequency f is moving towards the observer along the line SO with a constant velocity v_s as shown in figure. Plot f′ versus t graph. Where f′ is the changed frequency observed by the observer.*



Solution **During approach**

$$f' = f\left(\frac{v}{v - v_s}\right) = f_1 \qquad \text{(say)}$$

$$f' > f$$

but f' is constant with time.

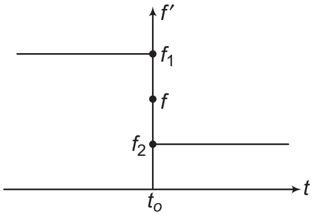
During recede

$$f' = f\left(\frac{v}{v + v_s}\right) = f_2 \ \text{(say)}$$

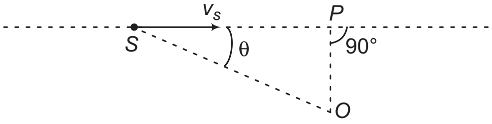
$$f' < f$$

but f' is constant with time. The correct f' *versus* t graph is as shown.

In the figure, t_o is the time when source is crossing the observer.



Example 12 *Repeat example-11 if source does not move along the line SO.*



Solution The changed frequency f' is now,

$$f' = f\left(\frac{v}{v - v_s\cos\theta}\right)$$

This time f' is not constant but it is a function of θ, which is variable. But, at a far distance, $\theta \to 0°$ or $\cos\theta \to 1$ or

$$\therefore \qquad f' = f\left(\frac{v}{v - v_s}\right)$$

and this is the maximum value of f' (say $f_{\max}$)

At point P, $\theta = 90°$

$$\therefore \qquad f' = f$$

During receding (to the right of P)

$$f' = f\left(\frac{v}{v + v_s\cos\theta}\right)$$

Again, at large distance from P,

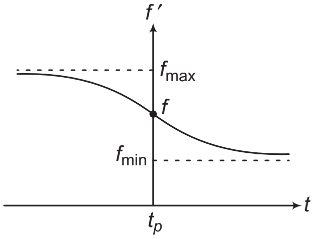
$\theta \to 0°$ or $\cos\theta \to 1$

$$\therefore \qquad f' = f\left(\frac{v}{v + v_s}\right)$$

This time, this is the minimum value of f' (say $f_{\min}$).

Hence, f' varies from $f_{\max}$ to $f_{\min}$, with $f' = f$ at the time of crossing P.

The correct f' *versus* t graph is as shown below.



Example 13 *A whistle emitting a sound of frequency 440 Hz is tied to a string of 1.5 m length and rotated with an angular velocity of 20 rad/s in the horizontal plane. Calculate the range of frequencies heard by an observer stationed at a large distance from the whistle. (Speed of sound = 330 m/s).* **(JEE 1996)**

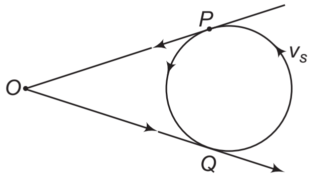
Solution v_s = Speed of source (whistle) $= R\omega = (1.5)\,(20)$ m/s, $v_s = 30$ m/s

Maximum frequency will be heard by the observer in position P and minimum in position Q.

Now,

$$f_{\max} = f\left(\frac{v}{v - v_s}\right)$$

where, v = speed of sound in air = 330 m/s



$$= (440)\left(\frac{330}{330-30}\right)\text{ Hz}$$

or $$f_{\max} = 484\text{ Hz}$$

and $$f_{\min} = f\left(\frac{v}{v + v_s}\right) = (440)\left(\frac{330}{330+30}\right)$$

$$f_{\min} = 403.33\text{ Hz}$$

Therefore, range of frequencies heard by observer is from 484 Hz to 403.33 Hz.

Miscellaneous Examples

Example 14 *The water level in a vertical glass tube* 1.0 *m long can be adjusted to any position in the tube. A tuning fork vibrating at* 660 *Hz is held just over the open top end of the tube. At what positions of the water level will there be in resonance? Speed of sound is* 330 *m/s.*

Solution Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\frac{\lambda}{4}$, the condition of resonance would be,

Length of air column $l = n\frac{\lambda}{4} = n\left(\frac{v}{4f}\right)$

Here, $n = 1, 3, 5, \ldots$

$$l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125\text{ m}$$

$$l_2 = 3l_1 = 0.375\text{ m}$$

$$l_3 = 5l_1 = 0.625\text{ m}$$

$$l_4 = 7l_1 = 0.875\text{ m}$$

$$l_5 = 9l_1 = 1.125\text{ m}$$

Since, $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only.

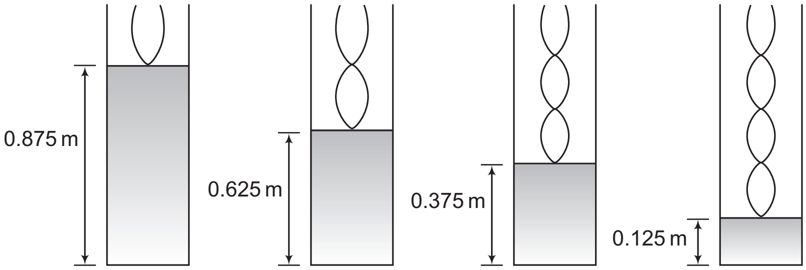
Therefore, level of water at resonance will be

$$(1.0 - 0.125)\ \text{m} = 0.875\ \text{m}$$

$$(1.0 - 0.375)\ \text{m} = 0.625\ \text{m}$$

$$(1.0 - 0.625)\ \text{m} = 0.375\ \text{m}$$

and $$(1.0 - 0.875)\ \text{m} = 0.125\ \text{m}$$ **Ans.**



In all the four cases shown in figure, the resonance frequency is 660 Hz but first one is the fundamental tone or first harmonic. Second is first overtone or third harmonic and so on.

Example 15 *A tube 1.0 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.3 m long and has a mass of 0.01 kg. It is held fixed at both ends and vibrates in its fundamental mode. It sets the air column in the tube into vibration at its fundamental frequency by resonance. Find*

(a) the frequency of oscillation of the air column and

(b) the tension in the wire.

Speed of sound in air = 330 m/s.

Solution (a) Fundamental frequency of closed pipe $= \dfrac{v}{4l}$

$$= \frac{330}{4 \times 1} = 82.5\ \text{Hz}$$ **Ans.**

(b) At resonance, given that :

fundamental frequency of stretched wire (at both ends)

= fundamental frequency of air column

$$\therefore \quad \frac{v}{2l} = 82.5\ \text{Hz}$$

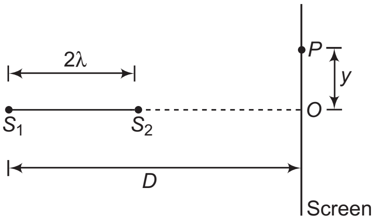
$$\therefore \quad \frac{\sqrt{T/\mu}}{2l} = 82.5$$

or $$T = \mu\ (2 \times 0.3 \times 82.5)^2$$

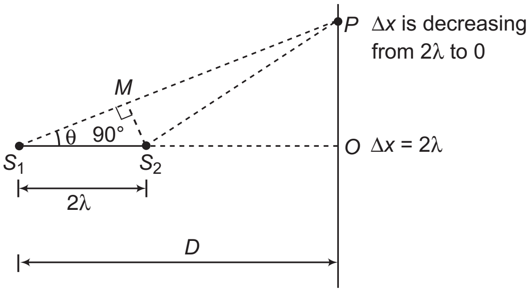
$$= \left(\frac{0.01}{0.3}\right) (2 \times 0.3 \times 82.5)^2$$

$$= 81.675\ \text{N}$$ **Ans.**

Example 16 *Two coherent narrow slits emitting sound of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The sound is detected by moving a detector on the screen at a distance $D (>> \lambda)$ from the slit S_1 as shown in figure. Find the distance y such that the intensity at P is equal to intensity at O.*



Solution At point O on the screen the path difference between the sound waves reaching from S_1 and S_2 is 2λ, i.e. constructive interference is obtained at O. At a very large distance from point O on the screen the path difference is zero.



Thus, we can conclude that as we move away from point O on the screen path difference decreases from 2λ to zero. At O constructive interference is obtained (where $\Delta x = 2\lambda$). So, next constructive interference will be obtained where $\Delta x = \lambda$. Hence,

$$S_1P - S_2P = \lambda$$

or
$$\sqrt{D^2 + y^2} - \sqrt{y^2 + (D - 2\lambda)^2} = \lambda$$

$\therefore$
$$\sqrt{D^2 + y^2} - \lambda = \sqrt{y^2 + (D - 2\lambda)^2}$$

On squaring both sides, we get

$$D^2 + y^2 + \lambda^2 - 2\lambda\sqrt{D^2 + y^2} = y^2 + D^2 + 4\lambda^2 - 4\lambda D$$

or
$$2\sqrt{D^2 + y^2} = 4D - 3\lambda$$

as $D >> \lambda$,
$$4D - 3\lambda \approx 4D$$

$\therefore$
$$2\sqrt{D^2 + y^2} = 4D$$

or
$$\sqrt{D^2 + y^2} = 2D$$

Again squaring both sides, we get

$$D^2 + y^2 = 4D^2$$

or
$$y = \sqrt{3}\,D$$ **Ans.**

Alternate method Let $\Delta x = \lambda$ at angle θ as shown. Path difference between the waves is $S_1M = 2\lambda\cos\theta$ because $S_2P \approx MP$

$\therefore$
$$2\lambda\cos\theta = \lambda \qquad (\Delta x = \lambda)$$

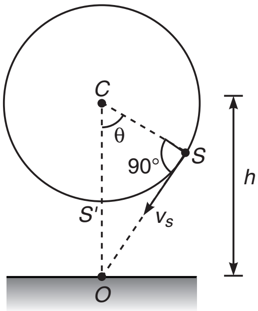
or
$$\theta = 60^\circ$$

Now,
$$PO = S_1O\tan\theta = S_1O\tan 60^\circ$$

or
$$y = \sqrt{3}\,D$$ **Ans.**

Example 17 *A fighter plane moving in a vertical loop with constant speed of radius R. The centre of the loop is at a height h directly overhead of an observer standing on the ground. The observer receives maximum frequency of the sound produced by the plane when it is nearest to him. Find the speed of the plane. Velocity of sound in air is v.*

Solution Let the speed of the plane (source) be v_s. Maximum frequency will be observed by the observer when v_s is along SO. The observer receives maximum frequency when the plane is nearest to him. That is as soon as the wave pulse reaches from S to O with speed v the plane reaches from S to S' with speed v_s. Hence,



$$t = \frac{SO}{v} = \frac{SS'}{v_s} \quad \text{or} \quad v_s = \left(\frac{SS'}{SO}\right) v$$

$$= \frac{R\theta}{\sqrt{h^2 - R^2}} v \qquad \textbf{Ans.}$$

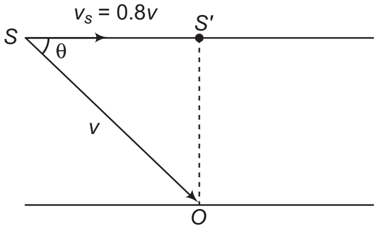
Here, $$\cos\theta = \frac{R}{h} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{R}{h}\right)$$

Example 18 *A source of sound of frequency* 1000 *Hz moves uniformly along a straight line with velocity* 0.8 *times velocity of sound. An observer is located at a distance* $l = 250$ *m from this line. Find*

(a) the frequency of the sound at the instant when the source is closest to the observer.

(b) the distance of the source when he observes no change in the frequency.

Solution (a) Suppose the pulse which is emitted when the source is at S reaches the observer O in the same time in which the source reaches from S to S', then



$$\cos\theta = \frac{SS'}{SO} = \frac{v_s t}{vt} = \frac{v_s}{v} = 0.8$$

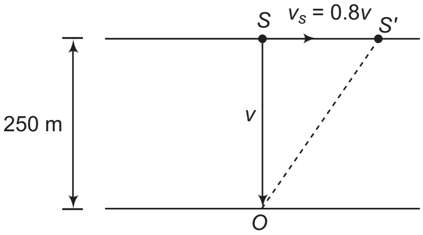
Now, $$f' = \left(\frac{v}{v - v_s \cos\theta}\right) f$$

$$= \left\{\frac{v}{v - (0.8v)(0.8)}\right\}(1000)$$

$$= \left(\frac{1}{1 - 0.64}\right)(1000)$$

$$= 2777.7 \text{ Hz}$$ **Ans.**

(b) The observer will observe no change in the frequency when the source is at S as shown in figure. In the time when the wave pulse reaches from S to O, the source will reach from S to S'. Hence,



$$t = \frac{SO}{v} = \frac{SS'}{v_s}$$

$\therefore$ $$SS' = \left(\frac{v_s}{v}\right) SO$$

$$= (0.8)(250) = 200 \text{ m}$$

Therefore, distance of observer from source at this instant is

$$S'O = \sqrt{(SO)^2 + (SS')^2}$$

$$= \sqrt{(250)^2 + (200)^2}$$

$$\approx 320 \text{ m}$$ **Ans.**

Example 19 *The air column in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency 440 Hz. The speed of sound in air is 330 m/s. End corrections may be neglected. Let p_0 denotes the mean pressure at any point in the pipe, and Δp_0 the maximum amplitude of pressure variation.*

(a) Find the length L of the air column.

(b) What is the amplitude of pressure variation at the middle of the column?

(c) What are the maximum and minimum pressures at the open end of the pipe?

(d) What are the maximum and minimum pressures at the closed end of the pipe?

Solution (a) Frequency of second overtone of the closed pipe

$$= 5\left(\frac{v}{4L}\right) = 440 \text{ Hz}$$ (Given)

$\therefore$ $$L = \frac{5v}{4 \times 440} \text{ m}$$

Substituting

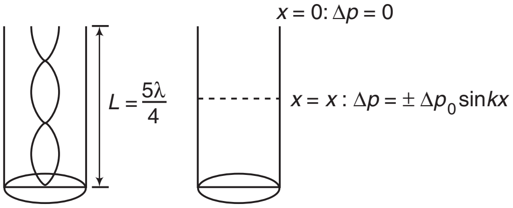
$$v = \text{speed of sound in air} = 330 \text{ m/s}$$

$$L = \frac{5 \times 330}{4 \times 440} = \frac{15}{16} \text{ m} \qquad \textbf{Ans.}$$

(b) $$\lambda = \frac{4L}{5} = \frac{4\ (15/16)}{5} = \frac{3}{4} \text{ m}$$

Open end is displacement antinode. Therefore, it would be a pressure node

or $$\text{at } x = 0; \quad \Delta p = 0$$



Pressure amplitude at $x = x$ can be written as

$$\Delta p = \pm\ \Delta p \sin kx$$

where, $$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3/4} = \frac{8\pi}{3} \text{ m}^{-1}$$

Therefore, pressure amplitude at $x = \frac{L}{2} = \frac{15/16}{2}$ m or (15/32) m will be

$$\Delta p = \pm\ \Delta p_0 \sin\left(\frac{8\pi}{3}\right)\left(\frac{15}{32}\right) = \pm\ \Delta p_0 \sin\left(\frac{5\pi}{4}\right)$$

$$\Delta p = \pm\ \frac{\Delta p_0}{\sqrt{2}} \qquad \textbf{Ans.}$$

(c) Open end is a pressure node, i.e. $\Delta p = 0$

Hence, $$p_{\max} = p_{\min} = \text{Mean pressure } (p_0)$$

(d) Closed end is a displacement node or pressure antinode.

Therefore, $$p_{\max} = p_0 + \Delta p_0$$

and $$p_{\min} = p_0 - \Delta p_0$$

Example 20 *At a distance* 20 *m from a point source of sound the loudness level is* 30 *dB. Neglecting the damping, find*

(a) the loudness at 10 *m from the source*

(b) the distance from the source at which sound is not heard.

Solution (a) Intensity due to a point source varies with distance r from it as

$$I \propto \frac{1}{r^2}$$

or $$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1^2}\right)^2$$

Now, $$L_1 = 10 \log \frac{I_1}{I_0}$$

and $$L_2 = 10 \log \frac{I_2}{I_0}$$

∴ $$L_1 - L_2 = 10 \left[\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0} \right]$$

$$= 10 \log \frac{I_1}{I_2} = 10 \log \left(\frac{r_2}{r_1} \right)^2$$

Substituting, $L_1 = 30\,\text{dB}$, $r_1 = 20\,\text{m}$ and $r_2 = 10\,\text{m}$

we have $$30 - L_2 = 10 \log \left(\frac{10}{20} \right)^2 = -6.0$$

or $$L_2 = 36\,\text{dB}$$ **Ans.**

(b) $L_1 - L_2 = 10 \log \left(\frac{r_2}{r_1} \right)^2$

Sound is not heard at a point where $L_2 = 0$

or $$30 = 10 \log \left(\frac{r_2}{r_1} \right)^2$$

∴ $$\left(\frac{r_2}{r_1} \right)^2 = 1000 \quad \text{or} \quad \frac{r_2}{r_1} = 31.62$$

∴ $$r_2 = (31.62)(20) \approx 632\,\text{m}$$ **Ans.**

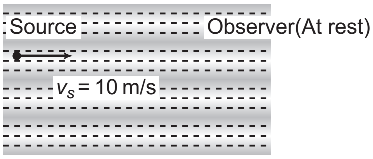
Example 21 *A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat, a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.*

(a) What will be the frequency detected by a receiver kept inside the river downstream?

(b) The transmitter and the receiver are now pulled up into air. The air is blowing with a speed 5 m/s in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.

(Temperature of the air and water = 20° C; Density of river water = 10^3 kg/m^3; Bulk modulus of the water = 2.088×10^9 Pa; Gas constant, R = 8.31 J/mol-K; Mean molecular mass of air = 28.8×10^{-3} kg/mol; C_p/C_V for air = 1.4) **(JEE 2001)**

Solution Velocity of sound in water is



$$v_w = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.088 \times 10^9}{10^3}}$$

$$= 1445\,\text{m/s}$$

Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45 \times 10^{-3}} \text{ Hz}$$

$$f_0 = 10^5 \text{ Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest would be

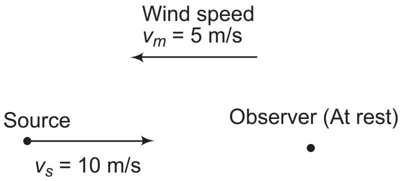
$$f_1 = f_0 \left(\frac{v_w + v_r}{v_w + v_r - v_s} \right)$$

$$= (10^5) \left(\frac{1445 + 2}{1445 + 2 - 10} \right) \text{ Hz}$$

$$f_1 = 1.0069 \times 10^5 \text{ Hz}$$ **Ans.**

(b) Velocity of sound in air is

$$v_a = \sqrt{\frac{\gamma RT}{M}}$$



$$= \sqrt{\frac{(1.4)\,(8.31)\,(20 + 273)}{28.8 \times 10^{-3}}}$$

$$= 344 \text{ m/s}$$

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

∴ Frequency of sound detected by receiver (observer) in air would be

$$f_2 = f_0 \left(\frac{v_a - v_w}{v_a - v_w - v_s} \right) = 10^5 \left[\frac{344 - 5}{344 - 5 - 10} \right] \text{Hz}$$

$$f_2 = 1.0304 \times 10^5 \text{ Hz}$$ **Ans.**

Example 22 *Three sound sources A, B and C have frequencies* 400, 401 *and* 402 *Hz, respectively. Calculate the number of beats noted per second.*

Solution Let us make the following table.

Group	Beat frequency $(f_1 - f_2)$ Hz	Beat time period $\left(\frac{1}{f_1 - f_2}\right)$ second
A and *B*	1	1
B and *C*	1	1
A and *C*	2	0.5

Beat time period for *A* and *B* is 1 s. It implies that if *A* and *B* are in phase at time $t = 0$, they are again in phase after 1 s. Same is the case with *B* and *C*. But beat time period for *A* and *C* is 0.5 s.

Therefore, beat time period for all together *A*, *B* and *C* will be 1 s. Because if, at $t = 0$, *A*, *B* and *C* all are in phase then after 1 s. (*A* and *B*) and (*B* and *C*) will again be in phase for the first time while (A and C) will be in phase for the second time. Or we can say that all A, B and C are again in phase after 1 s.

$\therefore$ Beat time period, $T_b = 1\text{ s}$

or Beat frequency, $f_b = \dfrac{1}{T_b} = 1\text{ Hz}$ **Ans.**

Alternate method Suppose at time t, the equations of waves are

$$y_1 = A_1 \sin 2\pi f_A t \qquad (\omega = 2\pi f)$$

$$y_2 = A_2 \sin 2\pi f_B t$$

and $$y_3 = A_3 \sin 2\pi f_C t$$

If they are in phase at some given instant of time t, then

$$2\pi f_A t = 2\pi f_B t = 2\pi f_C t \qquad \text{...(i)}$$

Let T_b be the beat time period, i.e. after time T_b they all are again in phase. As $f_C > f_B > f_A$, so

$$2\pi f_C\,(t + T_b) = 2\pi f_A\,(t + T_b) + 2m\pi \qquad \text{...(ii)}$$

and $$2\pi f_B\,(t + T_b) = 2\pi f_A\,(t + T_b) + 2n\pi \qquad \text{...(iii)}$$

Here, m and n ($< m$) are positive integers.

From Eqs. (i) and (ii),

$$(f_C - f_A)\,T_b = m \qquad \text{...(iv)}$$

Similarly, from Eqs. (i) and (iii)

$$(f_B - f_A)\,T_b = n \qquad \text{...(v)}$$

Dividing Eq. (iv) by Eq. (v),

$$\frac{m}{n} = \frac{f_C - f_A}{f_B - f_A} = \frac{402 - 400}{401 - 400} = \frac{2}{1}$$

Thus, letting $m = 2$ and $n = 1$

$$T_b = \frac{m}{f_C - f_A} \qquad \text{[from Eq. (iv)]}$$

$$= \frac{2}{2} = 1\text{ Hz}$$

$\therefore$ Beat frequency, $f_b = \dfrac{1}{T_b} = 1\text{ Hz}$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : A closed pipe and an open organ pipe are of same length. Then, neither of their frequencies can be same.

Reason : In the above case fundamental frequency of closed organ pipe will be two times the fundamental frequency of open organ pipe.

2. Assertion : A sound source is approaching towards a stationary observer along the line joining them. Then, apparent frequency to the observer will go on increasing.

Reason : If there is no relative motion between source and observer, apparent frequency is equal to the actual frequency.

3. Assertion : In longitudinal wave pressure is maximum at a point where displacement is zero.

Reason : There is a phase difference of $\frac{\pi}{2}$ between $y(x, t)$ and $\Delta P(x, t)$ equation in case of longitudinal wave.

4. Assertion : A train is approaching towards a hill. The driver of the train will hear beats.

Reason : Apparent frequency of reflected sound observed by driver will be more than the frequency of direct sound observed by him.

5. Assertion : Sound level increases linearly with intensity of sound.

Reason : If intensity of sound is doubled, sound level increases approximately 3 dB.

6. Assertion : Speed of sound in gases is independent of pressure of gas.

Reason : With increase in temperature of gas speed of sound will increase.

7. Assertion : Beat frequency between two tuning forks A and B is 4 Hz. Frequency of A is greater than the frequency of B. When A is loaded with wax, beat frequency may increase or decrease.

Reason : When a tuning fork is loaded with wax, its frequency decreases.

8. Assertion : Two successive frequencies of an organ pipe are 450 Hz and 750 Hz. Then, this pipe is a closed pipe.

Reason : Fundamental frequency of this pipe is 150 Hz.

9. **Assertion :** Fundamental frequency of a narrow pipe is more.

   Reason : According to Laplace end correction if radius of pipe is less, frequency should be more.

10. **Assertion :** In the experiment of finding speed of sound by resonance tube method, as the level of water is lowered, wavelength increases.

    Reason : By lowering the water level number of loops increases.

Objective Questions

1. Velocity of sound in vacuum is

(a) equal to 330 m/s
(b) greater than 330 m/s
(c) less than 330 m/s
(d) None of these

2. Longitudinal waves are possible in

(a) solids
(b) liquids
(c) gases
(d) All of these

3. If the fundamental frequency of a pipe closed at one end is 512 Hz. The frequency of a pipe of the same dimension but open at both ends will be

(a) 1024 Hz
(b) 512 Hz
(c) 256 Hz
(d) 128 Hz

4. The temperature at which the velocity of sound in oxygen will be the same as that of nitrogen at 15°C is

(a) 112°C
(b) 72°C
(c) 56°C
(d) 17°C

5. A closed organ pipe is excited to vibrate in the third overtone. It is observed that there are

(a) three nodes and three antinodes
(b) three nodes and four antinodes
(c) four nodes and three antinodes
(d) four nodes and four antinodes

6. When temperature is increased, the frequency of organ pipe

(a) increases
(b) decreases
(c) remains same
(d) Nothing can be said

7. When a sound wave travels from water to air, it

(a) bends towards normal
(b) bends away from normal
(c) may bend in any direction
(d) data insufficient

8. A closed organ pipe and an open organ pipe are tuned to the same fundamental frequency. The ratio of their lengths is

(a) 1 : 2
(b) 2 : 1
(c) 1 : 4
(d) 4 : 1

9. A sonometer wire under a tension of 10 kg weight is in unison with a tuning fork of frequency 320 Hz. To make the wire vibrate in unison with a tuning fork of frequency 256 Hz, the tension should be altered by

(a) 3.6 kg decreased
(b) 3.6 kg increased
(c) 6.4 kg decreased
(d) 6.4 kg increased

10. A tuning fork of frequency 256 Hz is moving towards a wall with a velocity of 5 m/s. If the speed of sound is 330 m/s, then the number of beats heard per second by a stationary observer lying between tuning fork and the wall is

(a) 2 (b) 4 (c) zero (d) 8

11. Two sound waves of wavelength 1 m and 1.01 m in a gas produce 10 beats in 3 s. The velocity of sound in the gas is

(a) 330 m/s (b) 337 m/s (c) 360 m/s (d) 300 m/s

12. When a source is going away from a stationary observer with the velocity equal to that of sound in air, then the frequency heard by observer is n times the original frequency. The value of n is

(a) 0.5 (b) 0.25
(c) 1.0 (d) No sound is heard

13. When interference is produced by two progressive waves of equal frequencies, then the maximum intensity of the resulting sound are N times the intensity of each of the component waves. The value of N is

(a) 1 (b) 2
(c) 4 (d) 8

14. A tuning fork of frequency 500 Hz is sounded on a resonance tube. The first and second resonances are obtained at 17 cm and 52 cm. The velocity of sound is

(a) 170 m/s (b) 350 m/s
(c) 520 m/s (d) 850 m/s

15. A vehicle, with a horn of frequency n is moving with a velocity of 30 m/s in a direction perpendicular to the straight line joining the observer and the vehicle. The observer perceives the sound to have a frequency $(n + n_1)$. If the sound velocity in air is 300 m/s, then

(a) $n_1 = 10\,n$ (b) $n_1 = 0$
(c) $n_1 = 0.1\,n$ (d) $n_1 = -0.1\,n$

16. How many frequencies below 1 kHz of natural oscillations of air column will be produced if a pipe of length 1 m is closed at one end? [velocity of sound in air is 340 m/s]

(a) 3 (b) 6
(c) 4 (d) 8

17. A sound source emits frequency of 180 Hz when moving towards a rigid wall with speed 5 m/s and an observer is moving away from wall with speed 5 m/s. Both source and observer moves on a straight line which is perpendicular to the wall. The number of beats per second heard by the observer will be [speed of sound = 355 m/s]

(a) 5 beats/s (b) 10 beats/s
(c) 6 beats/s (d) 8 beats/s

18. Two sound waves of wavelengths λ_1 and λ_2 $(\lambda_2 > \lambda_1)$ produce n beats/s, the speed of sound is

(a) $\dfrac{n\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$ (b) $n\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right)$
(c) $n(\lambda_2 - \lambda_1)$ (d) $n(\lambda_2 + \lambda_1)$

19. A, B and C are three tuning forks. Frequency of A is 350 Hz. Beats produced by A and B are 5/s and by B and C are 4/s. When a wax is put on A beat frequency between A and B is 2 Hz and between A and C is 6 Hz. Then, frequency of B and C respectively, are

(a) 355 Hz, 349 Hz (b) 345 Hz, 341 Hz
(c) 355 Hz, 341 Hz (d) 345 Hz, 349 Hz

20. The first resonance length of a resonance tube is 40 cm and the second resonance length is 122 cm. The third resonance length of the tube will be

(a) 200 cm (b) 202 cm (c) 203 cm (d) 204 cm

21. Two identical wires are stretched by the same tension of 100 N and each emits a note of frequency 200 Hz. If the tension in one wire is increased by 1 N, then the beat frequency is

(a) 2 Hz
(b) $\frac{1}{2}$ Hz
(c) 1 Hz
(d) None of these

22. A tuning fork of frequency 340 Hz is sounded above an organ pipe of length 120 cm. Water is now slowly poured in it. The minimum height of water column required for resonance is (speed of sound in air = 340 m/s)

(a) 25 cm
(b) 95 cm
(c) 75 cm
(d) 45 cm

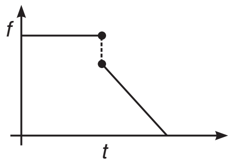
23. In a closed end pipe of length 105 cm, standing waves are set up corresponding to the third overtone. What distance from the closed end, amongst the following is a pressure node?

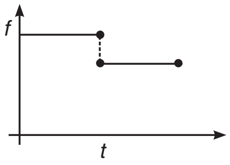
(a) 20 cm
(b) 60 cm
(c) 85 cm
(d) 45 cm

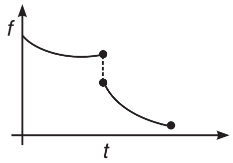
24. Oxygen is 16 times heavier than hydrogen. At NTP equal volume of hydrogen and oxygen are mixed. The ratio of speed of sound in the mixture to that in hydrogen is

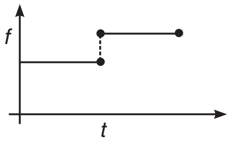
(a) $\sqrt{8}$
(b) $\sqrt{\frac{1}{8}}$
(c) $\sqrt{\frac{2}{17}}$
(d) $\sqrt{\frac{32}{17}}$

25. A train is moving towards a stationary observer. Which of the following curve best represents the frequency received by observer f as a function of time?

(a)


(b)


(c)


(d)


26. A closed organ pipe and an open organ pipe of same length produce 4 beats when they are set into vibrations simultaneously. If the length of each of them were twice their initial lengths, the number of beats produced will be

(a) 2
(b) 4
(c) 1
(d) 8

27. One train is approaching an observer at rest and another train is receding from him with the same velocity 4 m/s. Both trains blow whistles of same frequency of 243 Hz. The beat frequency in Hz as heard by the observer is (speed of sound in air = 320 m/ s)

(a) 10
(b) 6
(c) 4
(d) 1

28. Speed of sound in air is 320 m/s. A pipe closed at one end has a length of 1 m and there is another pipe open at both ends having a length of 1.6 m. Neglecting end corrections, both the air columns in the pipes can resonate for sound of frequency

(a) 80 Hz (b) 240 Hz (c) 320 Hz (d) 400 Hz

29. Four sources of sound each of sound level 10 dB are sounded together in phase, the resultant intensity level will be ($\log_{10} 2 = 0.3$)

(a) 40 dB (b) 26 dB (c) 22 dB (d) 13 dB

30. A longitudinal sound wave given by $p = 2.5 \sin \frac{\pi}{2}(x - 600\,t)$ (p is in N/m^2, x is in metre and t is in second) is sent down a closed organ pipe. If the pipe vibrates in its second overtone, the length of the pipe is

(a) 6 m (b) 8 m (c) 5 m (d) 10 m

31. Sound waves of frequency 600 Hz fall normally on perfectly reflecting wall. The distance from the wall at which the air particles have the maximum amplitude of vibration is

(speed of sound in air = 330 m/s)

(a) 13.75 cm (b) 40.25 cm (c) 70.5 cm (d) 60.75 cm

32. The wavelength of two sound waves are 49 cm and 50 cm, respectively. If the room temperature is 30°C, then the number of beats produced by them is approximately
(velocity of sound in air at 30°C = 332 m/s)

(a) 6 (b) 10 (c) 13 (d) 18

33. Two persons A and B, each carrying a source of frequency 300 Hz, are standing a few metre apart. A starts moving towards B with velocity 30 m/s. If the speed of sound is 300 m/s, which of the following is true?

(a) Number of beats heard by A is higher than that heard by B
(b) The number of beats heard by B are 30 Hz
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

34. A fixed source of sound emitting a certain frequency appears as f_a when the observer is approaching the source with speed v_0 and f_r when the observer recedes from the source with the same speed. The frequency of the source is

(a) $\frac{f_r + f_a}{2}$ (b) $\frac{f_r - f_a}{2}$ (c) $\sqrt{f_a f_r}$ (d) $\frac{2 f_r f_a}{f_r + f_a}$

Subjective Questions

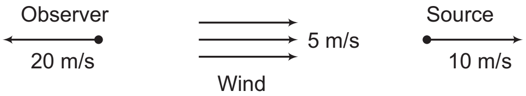
1. Determine the speed of sound waves in water, and find the wavelength of a wave having a frequency of 242 Hz. Take $B_{\text{water}} = 2 \times 10^9$ Pa.

2. If the source and receiver are at rest relative to each other but the wave medium is moving relative to them, will the receiver detect any wavelength or frequency shift.

3. Using the fact that hydrogen gas consists of diatomic molecules with $M = 2$ kg/K-mol. Find the speed of sound in hydrogen at 27° C.

4. About how many times more intense will the normal ear perceive a sound of 10^{-6} W/m^2 than one of 10^{-9} W/m^2?

5. A 300 Hz source, an observer and a wind are moving as shown in the figure with respect to the ground. What frequency is heard by the observer? Take speed of sound in air = 340 m/s.



6. A person standing between two parallel hills fires a gun. He hears the first echo after $\frac{3}{2}$ s, and a second echo after $\frac{5}{5}$ s. If speed of sound is 332 m/s, calculate the distance between the hills. When will he hear the third echo?

7. Helium is a monoatomic gas that has a density of 0.179 kg/m^3 at a pressure of 76 cm of mercury and a temperature of 0° C. Find the speed of compressional waves (sound) in helium at this temperature and pressure.

8. (a) In a liquid with density 1300 kg/m^3, longitudinal waves with frequency 400 Hz are found to have wavelength 8.00 m. Calculate the bulk modulus of the liquid.

(b) A metal bar with a length of 1.50 m has density 6400 kg/m^3. Longitudinal sound waves take 3.90×10^{-4} s to travel from one end of the bar to the other. What is Young's modulus for this metal?

9. What must be the stress (F/A) in a stretched wire of a material whose Young's modulus is Y for the speed of longitudinal waves equal to 30 times the speed of transverse waves?

10. A gas is a mixture of two parts by volume of hydrogen and one part by volume of nitrogen at STP. If the velocity of sound in hydrogen at 0° C is 1300 m/s. Find the velocity of sound in the gaseous mixture at 27° C.

11. The explosion of a fire cracker in the air at a height of 40 m produces a 100 dB sound level at ground below. What is the instantaneous total radiated power? Assuming that it radiates as a point source.

12. (a) What is the intensity of a 60 dB sound?

(b) If the sound level is 60 dB close to a speaker that has an area of 120 cm^2. What is the acoustic power output of the speaker?

13. (a) By what factor must the sound intensity be increased to increase the sound intensity level by 13.0 dB?

(b) Explain why you do not need to know the original sound intensity?

14. The speed of a certain compressional wave in air at standard temperature and pressure is 330 m/s. A point source of frequency 300 Hz radiates energy uniformly in all directions at the rate of 5 Watt.

(a) What is the intensity of the wave at a distance of 20 m from the source?

(b) What is the amplitude of the wave there? [Density of air at STP = 1.29 kg/m^3]

15. What is the amplitude of motion for the air in the path of a 60 dB, 800 Hz sound wave? Assume that $\rho_{\text{air}} = 1.29 \text{ kg/m}^3$ and $v = 330$ m/s.

16. A rock band gives rise to an average sound level of 102 dB at a distance of 20 m from the centre of the band. As an approximation, assume that the band radiates sound equally into a sphere. What is the sound power output of the band?

17. If it were possible to generate a sinusoidal 300 Hz sound wave in air that has a displacement amplitude of 0.200 mm. What would be the sound level of the wave? (Assume $v = 330$ m/s and $\rho_{\text{air}} = 1.29\ \text{kg/m}^3$)

18. (a) A longitudinal wave propagating in a water-filled pipe has intensity $3.00 \times 10^{-6}\ \text{W/m}^2$ and frequency 3400 Hz. Find the amplitude A and wavelength λ of the wave. Water has density $1000\ \text{kg/m}^3$ and bulk modulus 2.18×10^9 Pa.

(b) If the pipe is filled with air at pressure 1.00×10^5 Pa and density $1.20\ \text{kg/m}^3$, what will be the amplitude A and wavelength λ of a longitudinal wave with the same intensity and frequency as in part (a)?

(c) In which fluid is the amplitude larger, water or air?
What is the ratio of the two amplitudes? Why is this ratio so different from one? Consider air as diatomic.

19. For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. Calculate the corresponding intensity and sound intensity level at 20° C. (Assume $v = 330$ m/s and $\rho_{\text{air}} = 1.29\ \text{kg/m}^3$).

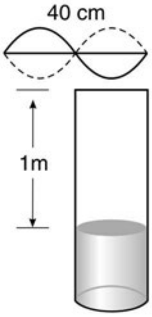
20. Find the fundamental frequency and the frequency of the first two overtones of a pipe 45.0 cm long. (a) If the pipe is open at both ends. (b) If the pipe is closed at one end. Use $v = 344$ m/s.

21. A uniform tube of length 60 cm stands vertically with its lower end dipping into water. First two air column lengths above water are 15 cm and 45 cm, when the tube responds to a vibrating fork of frequency 500 Hz. Find the lowest frequency to which the tube will respond when it is open at both ends.

22. Write the equation for the fundamental standing sound waves in a tube that is open at both ends. If the tube is 80 cm long and speed of the wave is 330 m/s. Represent the amplitude of the wave at an antinode by A.

23. A long glass tube is held vertically, dipping into water, while a tuning fork of frequency 512 Hz is repeatedly struck and held over the open end. Strong resonance is obtained, when the length of the tube above the surface of water is 50 cm and again 84 cm, but not at any intermediate point. Find the speed of sound in air and next length of the air column for resonance.

24. A wire of length 40 cm which has a mass of 4 g oscillates in its second harmonic and sets the air column in the tube to vibrations in its fundamental mode as shown in figure. Assuming the speed of sound in air as 340 m/s, find the tension in the wire.



25. In a resonance tube experiment to determine the speed of sound in air, a pipe of diameter 5 cm is used. The column in pipe resonates with a tuning fork of frequency 480 Hz when the minimum length of the air column is 16 cm. Find the speed of sound in air column at room temperature.

26. On a day when the speed of sound is 345 m/s, the fundamental frequency of a closed organ pipe is 220 Hz. (a) How long is this closed pipe? (b) The second overtone of this pipe has the same wavelength as the third harmonic of an open pipe. How long is the open pipe?

27. A closed organ pipe is sounded near a guitar, causing one of the strings to vibrate with large amplitude. We vary the tension of the string until we find the maximum amplitude. The string is 80% as long as the closed pipe. If both the pipe and the string vibrate at their fundamental frequency, calculate the ratio of the wave speed on the string to the speed of sound in air.

28. A police siren emits a sinusoidal wave with frequency $f_S = 300\,\text{Hz}$. The speed of sound is 340 m/s. (a) Find the wavelength of the waves if the siren is at rest in the air. (b) If the siren is moving at 30 m/s, find the wavelength of the waves ahead of and behind the source.

29. Two identical violin strings, when in tune and stretched with the same tension, have a fundamental frequency of 440.0 Hz. One of the string is retuned by adjusting its tension. When this is done, 1.5 beats per second are heard when both strings are plucked simultaneously. (a) What are the possible fundamental frequencies of the retuned string? (b) By what fractional amount was the string tension changed if it was (i) increased (ii) decreased?

30. A swimming duck paddles the water with its feet once every 1.6 s, producing surface waves with this period. The duck is moving at constant speed in a pond where the speed of surface waves is 0.32 m/s, and the crests of the waves ahead of the duck are spaced 0.12 m apart. (a) What is the duck's speed? (b) How far apart are the crests behind the duck?

31. A railroad train is travelling at 30.0 m/s in still air. The frequency of the note emitted by the train whistle is 262 Hz. What frequency is heard by a passenger on a train moving in the opposite direction to the first at 18.0 m/s and (a) approaching the first? (b) receding from the first? Speed of sound in air = 340 m/s.

32. A boy is walking away from a wall at a speed of 1.0 m/s in a direction at right angles to the wall. As he walks, he blows a whistle steadily. An observer towards whom the boy is walking hears 4.0 beats per second. If the speed of sound is 340 m/s, what is the frequency of the whistle?

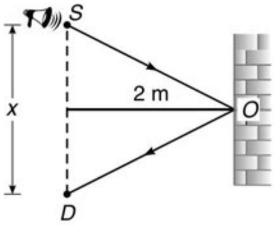
33. A tuning fork P of unknown frequency gives 7 beats in 2 seconds with another tuning fork Q. When Q runs towards a wall with a speed of 5 m/s it gives 5 beats per second with its echo. On loading P with wax, it gives 5 beats per second with Q. What is the frequency of P? Assume speed of sound = 332 m/s.

34. A stationary observer receives sonic oscillations from two tuning forks one of which approaches and the other recedes with the same velocity. As this takes place, the observer hears the beats of frequency $f = 2.0\,\text{Hz}$. Find the velocity of each tuning fork if their oscillation frequency is $f_0 = 680\,\text{Hz}$ and the velocity of sound in air is $v = 340$ m/s.

35. Sound waves from a tuning fork A reach a point P by two separate paths ABP and ACP. When ACP is greater than ABP by 11.5 cm, there is silence at P. When the difference is 23 cm the sound becomes loudest at P and when 34.5 cm there is silence again and so on. Calculate the minimum frequency of the fork if the velocity of sound is taken to be 331.2 m/s.

36. Two loudspeakers S_1 and S_2 each emit sounds of frequency 220 Hz uniformly in all directions. S_1 has an acoustic output of 1.2×10^{-3} W and S_2 has 1.8×10^{-3} W. S_1 and S_2 vibrate in phase. Consider a point P such that $S_1P = 0.75$ m and $S_2P = 3$ m. How are the phases arriving at P related? What is the intensity at P when both S_1 and S_2 are on? Speed of sound in air is 330 m/s.

37. A source of sound emitting waves at 360 Hz is placed in front of a vertical wall, at a distance 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Take speed of sound in air = 360 m/s. Assume that there is no phase change in reflected wave.



38. The atomic mass of iodine is 127 g/mol. A standing wave in iodine vapour at 400 K has nodes that are 6.77 cm apart when the frequency is 1000 Hz. At this temperature, is iodine vapour monoatomic or diatomic.

39. A tuning fork whose natural frequency is 440 Hz is placed just above the open end of a tube that contains water. The water is slowly drained from the tube while the tuning fork remains in place and is kept vibrating. The sound is found to be enhanced when the air column is 60 cm long and when it is 100 cm long. Find the speed of sound in air.

40. A piano wire *A* vibrates at a fundamental frequency of 600 Hz. A second identical wire *B* produces 6 beats per second with it when the tension in *A* is slightly increased. Find the ratio of the tension in *A* to the tension in *B*.

41. A tuning fork of frequency 256 Hz produces 4 beats per second with a wire of length 25 cm vibrating in its fundamental mode. The beat frequency decreases when the length is slightly shortened. What could be the minimum length by which the wire be shortened so that it produces no beats with the tuning fork?

42. Show that when the speed of the source and the observer are small compared to the speed of sound in the medium, the change in frequency becomes independent of the fact whether the source is moving or the observer.

43. A sound source moves with a speed of 80 m/s relative to still air toward a stationary listener. The frequency of sound is 200 Hz and speed of sound in air is 340 m/s.

(a) Find the wavelength of the sound between the source and the listener.

(b) Find the frequency heard by the listener.

44. A railroad train is travelling at 30 m/s in still air. The frequency of the note emitted by the locomotive whistle is 500 Hz. What is the wavelength of the sound waves :

(a) in front of the locomotive? (b) behind the locomotive?

What is the frequency of the sound heard by a stationary listener

(c) in front of the locomotive? (d) behind the locomotive?

Speed of sound in air is 344 m/s.

45. For a certain organ pipe, three successive resonance frequencies are observed at 425, 595 and 765 Hz respectively. Taking the speed of sound in air to be 340 m/s,

(a) explain whether the pipe is closed at one end or open at both ends.

(b) determine the fundamental frequency and length of the pipe.

46. Two tuning forks *A* and *B* sounded together give 8 beats per second. With an air resonance tube closed at one end, the two forks give resonances when the two air columns are 32 cm and 33 cm respectively. Calculate the frequencies of forks.

LEVEL 2

Single Correct Option

1. A plane wave of sound travelling in air is incident upon a plane water surface. The angle of incidence is 60°. If velocity of sound in air and water are 330 m/s and 1400 m/s, then the wave undergoes

(a) refraction only

(b) reflection only

(c) Both reflection and refraction

(d) neither reflection nor refraction

2. An organ pipe of $(3.9\,\pi)$ m long, open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillation is 1% of mean atmospheric pressure $[p_0 = 10^5 \text{ N/m}^2]$. The maximum displacement of particle from mean position will be

[Given, velocity of sound = 200 m/s and density of air = 1.3 kg/m^3]

(a) 2.5 cm (b) 5 cm

(c) 1 cm (d) 2 cm

3. A plane sound wave passes from medium 1 into medium 2. The speed of sound in medium 1 is 200 m/s and in medium 2 is 100 m/s. The ratio of amplitude of the transmitted wave to that of incident wave is

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$

(c) $\frac{5}{6}$ (d) $\frac{2}{3}$

4. Two sources of sound are moving in opposite directions with velocities v_1 and v_2 $(v_1 > v_2)$. Both are moving away from a stationary observer. The frequency of both the sources is 1700 Hz. What is the value of $(v_1 - v_2)$ so that the beat frequency observed by the observer is 10 Hz? $v_{\text{sound}} = 340$ m/s and assume that v_1 and v_2 both are very much less than v_{sound}.

(a) 1 m/s (b) 2 m/s

(c) 3 m/s (d) 4 m/s

5. A sounding body emitting a frequency of 150 Hz is dropped from a height. During its fall under gravity it crosses a balloon moving upwards with a constant velocity of 2 m/s one second after it started to fall. The difference in the frequency observed by the man in balloon just before and just after crossing the body will be (velocity of sound = 300 m/s, $g = 10$ m/s^2)

(a) 12 (b) 6

(c) 8 (d) 4

6. A closed organ pipe has length L. The air in it is vibrating in third overtone with maximum amplitude a. The amplitude at distance $\frac{L}{7}$ from closed end of the pipe is

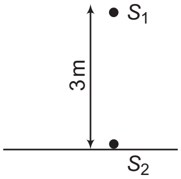
(a) 0

(b) a

(c) $\frac{a}{2}$

(d) Data insufficient

7. S_1 and S_2 are two coherent sources of sound having no initial phase difference. The velocity of sound is 330 m/s. No maxima will be formed on the line passing through S_2 and perpendicular to the line joining S_1 and S_2. If the frequency of both the sources is



(a) 330 Hz (b) 120 Hz (c) 100 Hz (d) 220 Hz

8. A source is moving with constant speed $v_s = 20$ m/s towards a stationary observer due east of the source. Wind is blowing at the speed of 20 m/s at 60° north of east. The source has frequency 500 Hz. Speed of sound = 300 m/s. The frequency registered by the observer is approximately

(a) 541 Hz (b) 552 Hz
(c) 534 Hz (d) 517 Hz

9. A car travelling towards a hill at 10 m/s sounds its horn which has a frequency 500 Hz. This is heard in a second car travelling behind the first car in the same direction with speed 20 m/s. The sound can also be heard in the second car by reflection of sound from the hill. The beat frequency heard by the driver of the second car will be (speed of sound in air = 340 m/s)

(a) 31 Hz (b) 24 Hz
(c) 21 Hz (d) 34 Hz

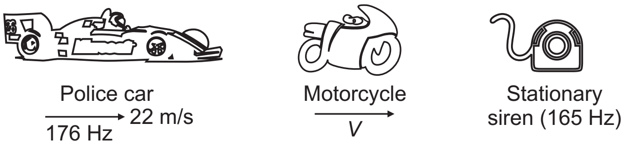
10. Two sounding bodies are producing progressive waves given by $y_1 = 2\sin(400\pi t)$ and $y_2 = \sin(404\pi t)$ where t is in second, which superpose near the ears of a person. The person will hear

(a) 2 beats/s with intensity ratio 9/4 between maxima and minima
(b) 2 beats/s with intensity ratio 9 between maxima and minima
(c) 4 beats/s with intensity ratio 16 between maxima and minima
(d) 4 beats/s with intensity ratio 16/9 between maxima and minima

11. The air in a closed tube 34 cm long is vibrating with two nodes and two antinodes and its temperature is 51°C. What is the wavelength of the waves produced in air outside the tube, when the temperature of air is 16°C?

(a) 42.8 cm (b) 68 cm (c) 17 cm (d) 102 cm

12. A police car moving at 22 m/s, chase a motorcyclist. The police man sounds his horn at 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of the motorcyclist, if he does not observe any beats. (velocity of sound in air = 330 m/s) **(JEE 2003)**

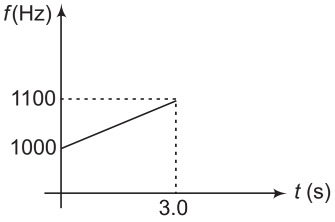


(a) 33 m/s (b) 22 m/s (c) zero (d) 11 m/s

13. A closed organ pipe resonates in its fundamental mode at a frequency of 200 Hz with O_2 in the pipe at a certain temperature. If the pipe now contains 2 moles of O_2 and 3 moles of ozone, then what will be the fundamental frequency of same pipe at same temperature?

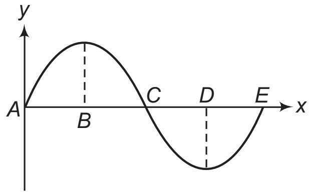
(a) 268.23 Hz (b) 175.4 Hz (c) 149.45 Hz (d) None of these

14. A detector is released from rest over a source of sound of frequency $f_0 = 10^3$ Hz. The frequency observed by the detector at time t is plotted in the graph. The speed of sound in air is ($g = 10\ \text{m/s}^2$)



(a) 330 m/s (b) 350 m/s (c) 300 m/s (d) 310 m/s

15. Sound waves are travelling along positive x-direction. Displacement of particle at any time t is as shown in figure. Select the wrong statement.



(a) Particle located at E has its velocity in negative x-direction
(b) Particle located at D has zero velocity
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

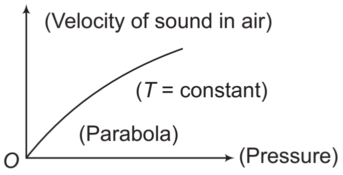
More than One Correct Options

1. An air column in a pipe, which is closed at one end, is in resonance with a vibrating tuning fork of frequency 264 Hz. If $v = 330$ m/s, the length of the column in cm is (are)

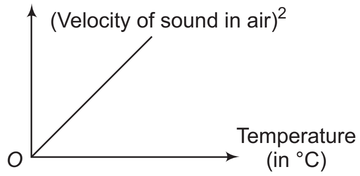
(a) 31.25 (b) 62.50
(c) 93.75 (d) 125

2. Which of the following is/are correct?

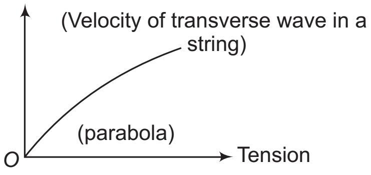
(a)



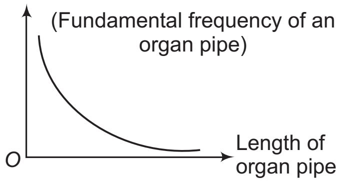
(b)



(c)



(d)



3. Choose the correct options for longitudinal wave

(a) maximum pressure variation is BAk
(b) maximum density variation is ρAk
(c) pressure equation and density equation are in phase
(d) pressure equation and displacement equation are out of phase

4. Second overtone frequency of a closed pipe and fourth harmonic frequency of an open pipe are same. Then, choose the correct options.

(a) Fundamental frequency of closed pipe is more than the fundamental frequency of open pipe
(b) First overtone frequency of closed pipe is more than the first overtone frequency of open pipe
(c) Fifteenth harmonic frequency of closed pipe is equal to twelfth harmonic frequency of open pipe
(d) Tenth harmonic frequency of closed pipe is equal to eighth harmonic frequency of open pipe

5. For fundamental frequency f of a closed pipe, choose the correct options.

(a) If radius of pipe is increased, f will decrease
(b) If temperature is increased, f will increase
(c) If molecular mass of the gas filled in the pipe is increased, f will decrease.
(d) If pressure of gas (filled in the pipe) is increased without change in temperature, f will remain unchanged

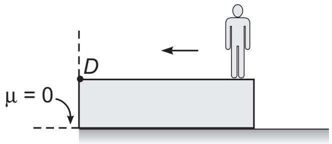
6. A source is approaching towards an observer with constant speed along the line joining them. After crossing the observer, source recedes from observer with same speed. Let f is apparent frequency heard by observer. Then,

(a) f will keep on increasing during approaching
(b) f will keep on decreasing during receding
(c) f will remain constant during approaching
(d) f will remain constant during receding

Comprehension Based Questions

Passage

A man of mass 50 *kg is running on a plank of mass* 150 *kg with speed of* 8 *m/s relative to plank as shown in the figure (both were initially at rest and the velocity of man with respect to ground any how remains constant). Plank is placed on smooth horizontal surface. The man, while running, whistles with frequency* f_0. *A detector* (D) *placed on plank detects frequency. The man jumps off with same velocity (w.r.t. to ground) from point D and slides on the smooth horizontal surface [Assume coefficient of friction between man and horizontal is zero]. The speed of sound in still medium is* 330 *m/s. Answer the following questions on the basis of above situations.*



1. The frequency of sound detected by detector D, before man jumps off the plank is

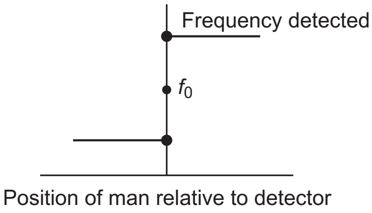
(a) $\frac{332}{324} f_0$ (b) $\frac{330}{322} f_0$
(c) $\frac{328}{336} f_0$ (d) $\frac{330}{338} f_0$

2. The frequency of sound detected by detector D, after man jumps off the plank is

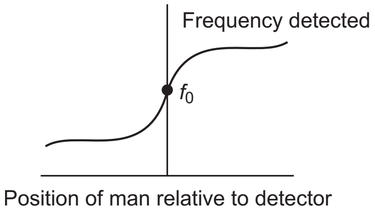
(a) $\frac{332}{324} f_0$ (b) $\frac{330}{322} f_0$
(c) $\frac{328}{336} f_0$ (d) $\frac{330}{338} f_0$

3. Choose the correct plot between the frequency detected by detector *versus* position of the man relative to detector.

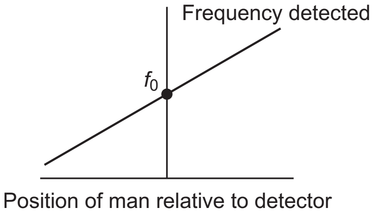
(a)



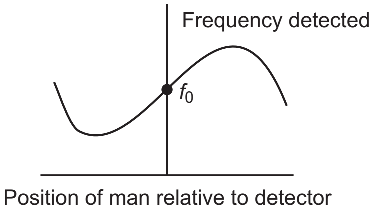
(b)



(c)



(d)

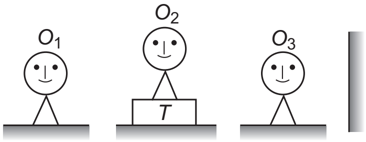


Match the Columns

1. Fundamental frequency of an open organ pipe is f. Match the following two columns for a closed pipe of double the length.

Column I	Column II
(a) Fundamental frequency	(p) $1.25\,f$
(b) Second overtone frequency	(q) f
(c) Third harmonic frequency	(r) $0.75\,f$
(d) First overtone frequency	(s) None of these

2. A train T horns a sound of frequency f. It is moving towards a wall with speed $\frac{1}{4}$th the speed of sound. There are three observers O_1, O_2 and O_3 as shown. Match the following two columns.



Column I	Column II
(a) Beat frequency observed to O_1	(p) $\frac{2}{3}f$
(b) Beat frequency observed to O_2	(q) $\frac{8}{15}f$
(c) Beat frequency observed to O_3	(r) None of these
(d) If train moves in opposite direction with same speed, then beat frequency observed to O_3	(s) Zero

3. A tuning fork is placed near a vibrating stretched wire. A boy standing near the two hears a beat frequency f. It is known that frequency of tuning fork is greater than frequency of stretched wire. Match the following two columns.

Column I	Column II
(a) If tuning fork is loaded with wax,	(p) beat frequency must increase.
(b) If prongs of tuning fork are filed,	(q) beat frequency must decrease.
(c) If tension in stretched wire is increased,	(r) beat frequency may increase.
(d) If tension in stretched wire is decreased,	(s) beat frequency may decrease.

4. I represents intensity of sound wave, A the amplitude and r the distance from the source. Then, match the following two columns.

Column I	Column II
(a) Intensity due to a point source.	(p) proportional to $r^{-1/2}$
(b) Amplitude due to a point source.	(q) proportional to r^{-1}
(c) Intensity due to a line source.	(r) proportional to r^{-2}
(d) Amplitude due to a line source.	(s) proportional to r^{-4}

5. The equation of longitudinal stationary wave in second overtone mode in a closed organ pipe is

$$y = (4 \text{ mm}) \sin \pi x \cos \pi t$$

Here, x is in metre and t in second. Then, match the following two columns.

Column I	Column II
(a) Length of pipe	(p) 1 m
(b) Wavelength	(q) 1.5 m
(c) Distance of displacement node from the closed end	(r) 2.0 m
(d) Distance of pressure node from the closed end	(s) None of these

Subjective Questions

1. A window whose area is $2\, m^2$ opens on a street where the street noise results at the window an intensity level of 60 dB. How much acoustic power enters the window through sound waves? Now, if a sound absorber is fitted at the window, how much energy from the street will it collect in a day?

2. A point A is located at a distance $r = 1.5$ m from a point source of sound of frequency 600 Hz. The power of the source is 0.8 W. Speed of sound in air is 340 m/s and density of air is 1.29 kg/m^3. Find at the point A,

(a) the pressure oscillation amplitude $(\Delta p)_m$

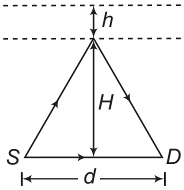
(b) the displacement oscillation amplitude A.

3. A flute which we treat as a pipe open at both ends is 60 cm long.

(a) What is the fundamental frequency when all the holes are covered?

(b) How far from the mouthpiece should a hole be uncovered for the fundamental frequency to be 330 Hz? Take speed of sound in air as 340 m/s.

4. A source S and a detector D of high frequency waves are a distance d apart on the ground. The direct wave from S is found to be in phase at D with the wave from S that is reflected from a horizontal layer at an altitude H. The incident and reflected rays make the same angle with the reflecting layer. When the layer rises a distance h, no signal is detected at D. Neglect absorption in the atmosphere and find the relation between d, h, H and the wavelength λ of the waves.



5. Two sound speakers are driven in phase by an audio amplifier at frequency 600 Hz. The speed of sound is 340 m/s. The speakers are on the y-axis, one at $y = +1.0$ m and the other at $y = -1.0$ m. A listener begins at $y = 0$ and walks along a line parallel to the y-axis at a very large distance x away.

(a) At what angle θ (between the line from the origin to the listener at the x-axis) will she first hear a minimum sound intensity?

(b) At what angle will she first hear a maximum (after $\theta = 0°$) sound intensity?

(c) How many maxima can she possibly hear if she keeps walking in the same direction?

6. Two speakers separated by some distance emit sound of the same frequency. At some point P the intensity due to each speaker separately is I_0. The path difference from P to one of the speakers is $\frac{1}{2}\lambda$ greater than that from P to the other speaker. What is the intensity at P if

(a) the speakers are coherent and in phase;

(b) the speakers are incoherent; and

(c) the speakers are coherent but have a phase difference of 180°?

7. Two loudspeakers radiate in phase at 170 Hz. An observer sits at 8 m from one speaker and 11m from the other. The intensity level from either speaker acting alone is 60 dB. The speed of sound is 340 m/ s.

(a) Find the observed intensity when both speakers are on together.

(b) Find the observed intensity level when both speakers are on together but one has its leads reversed so that the speakers are 180° out of phase.

(c) Find the observed intensity level when both speakers are on and in phase but the frequency is 85 Hz.

8. Two identical speakers emit sound waves of frequency 680 Hz uniformly in all directions with a total audio output of 1 mW each. The speed of sound in air is 340 m/s. A point P is a distance 2.00 m from one speaker and 3.00 m from the other.

(a) Find the intensities I_1 and I_2 from each speaker at point P separately.

(b) If the speakers are driven coherently and in phase, what is the intensity at point P?

(c) If they are driven coherently but out of phase by 180°, what is the intensity at point P?

(d) If the speakers are incoherent, what is the intensity at point P?

9. A train of length l is moving with a constant speed v along a circular track of radius R. The engine of the train emits a sound of frequency f. Find the frequency heard by a guard at the rear end of the train.

10. A 3 m long organ pipe open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillations is 1 per cent of mean atmospheric pressure ($p_0 = 10^5$ Nm^2). Find the amplitude of particle displacement and density oscillations. Speed of sound $v = 332$ m / s and density of air $\rho = 1.03$ kg / m^3.

11. A siren creates a sound level of 60 dB at a location 500 m from the speaker. The siren is powered by a battery that delivers a total energy of 1.0 kJ. Assuming that the efficiency of siren is 30%, determine the total time the siren can sound.

12. A cylinder of length 1 m is divided by a thin perfectly flexible diaphragm in the middle. It is closed by similar flexible diaphragms at the ends. The two chambers into which it is divided contain hydrogen and oxygen. The two diaphragms are set in vibrations of same frequency. What is the minimum frequency of these diaphragms for which the middle diaphragm will be motionless? Velocity of sound in hydrogen is 1100 m/ s and that in oxygen is 300 m/ s.

13. A conveyor belt moves to the right with speed $v = 300$ m/ min. A very fast pieman puts pies on the belt at a rate of 20 per minute and they are received at the other end by a pieeater.

(a) If the pieman is stationary find the spacing x between the pies and the frequency with which they are received by the stationary pieeater.

(b) The pieman now walks with speed 30 m/min towards the receiver while continuing to put pies on the belt at 20 per minute. Find the spacing of the pies and the frequency with which they are received by the stationary pieeater.

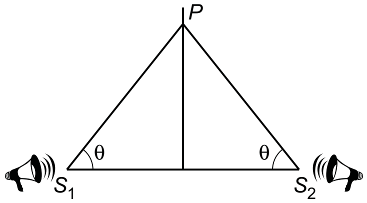
14. A point sound source is situated in a medium of bulk modulus 1.6×10^5 N/ m^2. An observer standing at a distance 10 m from the source writes down the equation for the wave as $y = A \sin (15\pi x - 6000\,\pi t)$. Here y and x are in metres and t is in second. The maximum pressure amplitude received to the observer's ear is (24π) Pa, then find.

(a) the density of the medium,

(b) the displacement amplitude A of the waves received by the observer and

(c) the power of the sound source.

15. Two sources of sound S_1 and S_2 vibrate at the same frequency and are in phase. The intensity of sound detected at a point P (as shown in figure) is I_0.



(a) If $\theta = 45°$ what will be the intensity of sound detected at this point if one of the sources is switched off ?

(b) What will be intensity of sound detected at P if $\theta = 60°$ and both the sources are now switched on?

16. Two narrow cylindrical pipes A and B have the same length. Pipe A is open at both ends and is filled with a monoatomic gas of molar mass M_A. Pipe B is open at one end and closed at the other end and is filled with a diatomic gas of molar mass M_B. Both gases are at the same temperature. **(JEE 2002)**

(a) If the frequency of the second harmonic of the fundamental mode in pipe A is equal to the frequency of the third harmonic of the fundamental mode in pipe B, determine the value of $\frac{M_A}{M_B}$.

(b) Now, the open end of pipe B is also closed (so that the pipe is closed at both ends.) Find the ratio of the fundamental frequency in pipe A to that in pipe B.

17. A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.

(a) What will be the frequency detected by a receiver kept inside the river downstream?

(b) The transmitter and the receiver are now pulled up into the air. The air is blowing with a speed 5 m/s in the direction opposite to the river stream. Determine the frequency of the sound detected by the receiver.

(Temperature of the air and water = 20° C; Density of river water = 10^3 kg/m^3; bulk modulus the water = 2.088×10^9 Pa; Gas constant $R = 8.31$ J/mol-K; Mean molecular mass of air = 28.8×10^{-3} kg per mol and C_p/C_V for air = 1.4)

18. A string 25 cm long and having a mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decrease the beat frequency. If the speed of sound in air is 320 m/s, find the tension in the string.

19. A source emits sound waves of frequency 1000 Hz. The source moves to the right with a speed of 32 m/s relative to ground. On the right a reflecting surface moves towards left with a speed of 64 m/s relative to the ground. The speed of sound in air is 332 m/s. Find

(a) the wavelength of sound in ahead of the source,

(b) the number of waves arriving per second which meets the reflecting surface,

(c) the speed of reflected waves and

(d) the wavelength of reflected waves.

Answers

Introductory Exercise 19.1

1. 1.4×10^5 N/m^2 **2.** 7.25 cm, 72.5 m **3.** (a) Zero (b) 3.63×10^{-6} m

4. 1.04×10^{-5} m

Introductory Exercise 19.2

1. 819°C **2.** 19.6 m/s **3.** 3.6×10^9 Pa **4.** 315 m/s

Introductory Exercise 19.3

1. (a) 4.67 Pa (b) 2.64×10^{-2} W/m^2 (c) 104 dB **2.** 7.9 **3.** 20 dB

4. Faintest (a) 4.49×10^{-13} W/m^2, –3.48 dB (b) 1.43×10^{-11} m

Loudest (a) 0.881 W/m^2, +119 dB (b) 2.01×10^{-5} m

Introductory Exercise 19.4

1. 1375 Hz **2.** (a) 11.7 cm (b) 180°

Introductory Exercise 19.5

1. (a) **2.** (c) **3.** (a) **4.** (a) 0.392 m (b) 0.470 m

5. (a) Fundamental 0.8 m, first overtone 0.267 m, 0.8 m, second overtone 0.16 m, 0.48 m, 0.8 m
(b) Fundamental 0, first overtone 0, 0.533 m. Second overtone 0, 0.32 m, 0.64 m

6. (a) closed (b) 5, 7 (c) 1.075 m

Introductory Exercise 19.6

1. 252 Hz **2.** 387 Hz

Introductory Exercise 19.7

1. (d) **2.** (d) **3.** (b) **4.** (a)

Exercises

LEVEL 1

Assertion and Reason

1. (c) **2.** (d) **3.** (d) **4.** (a) **5.** (d) **6.** (d) **7.** (b) **8.** (b) **9.** (a) **10.** (d)

Objective Questions

1.(d) **2.**(d) **3.**(a) **4.**(c) **5.**(d) **6.**(a) **7.**(a) **8.**(a) **9.**(a) **10.**(c)
11.(b) **12.**(a) **13.**(c) **14.**(b) **15.**(b) **16.**(b) **17.**(a) **18.**(a) **19.**(b) **20.**(d)
21.(c) **22.**(d) **23.**(d) **24.**(c) **25.**(b) **26.**(a) **27.**(b) **28.**(d) **29.**(c) **30.**(c)
31.(a) **32.**(c) **33.**(d) **34.**(a)

Subjective Questions

1. 1414 m/s, 5.84 m **2.** No **3.** 1321 m/s **4.** Two times **5.** 274 Hz

6. 664 m, 4 s **7.** 972 m/s **8.** (a) 1.33×10^{10} Pa (b) 9.47×10^{10} Pa **9.** $Y/900$

10. 591 m/s **11.** 201 W **12.** (a) 10^{-6} W/m^2 (b) 1.2×10^{-8} W **13.** (a) 20
14. (a) 9.95×10^{-4} W/m^2 (b) 1.15×10^{-6} m **15.** 13.6 nm **16.** 80 W **17.** 134.4 dB
18. (a) 9.44×10^{-11} m, 0.43 m (b) 5.66×10^{-9} m, 0.1 m (c) air, $\frac{A_{\text{air}}}{A_{\text{water}}} = 60$
19. 4.2×10^{-12} W/m^2, 6.23 dB **20.** (a) 382.2Hz, 764.4 Hz, 1146.6 Hz (b) 191.1 Hz, 573.3 Hz, 955.5 Hz
21. 250 Hz **22.** $y = A\cos(3.93x)\sin(1297t)$ **23.** 348.16 m/s, 118 cm **24.** 11.56 N
25. 336 m/s **26.** (a) 0.392 m (b) 0.470 m **27.** 0.40 **28.** (a) 1.13 m (b) 1.03 m, 1.23 m
29. (a) 441.5 Hz, 438.5 Hz (b) (i) + 0.68% (ii) –0.68% **30.** (a) 0.245 m/s (b) 0.904 m
31. (a) 302 Hz (b) 228 Hz **32.** 680 Hz **33.** 160 Hz **34.** 0.5 m/s
35. 1440 Hz **36.** $\Delta\phi_1 = \pi$, $\Delta\phi_2 = 4\pi$, Resultant intensity = 8.2×10^{-5} Wm^{-2} **37.** 7.5 m
38. Diatomic **39.** 352 m/s **40.** 1.02 **41.** 0.4 cm
43. (a) 1.3 m (b) 262 Hz **44.** (a) 0.628 m (b) 0.748 m (c) 548 Hz (d) 460 Hz
45. (a) closed (b) 85 Hz, 1 m **46.** f_1 = 264 Hz and f_2 = 256 Hz

LEVEL 2

Single Correct Option

1.(b) **2.**(a) **3.**(d) **4.**(b) **5.**(a) **6.**(b) **7.**(c) **8.**(c) **9.**(a) **10.**(b)
11.(a) **12.**(b) **13.**(b) **14.**(c) **15.**(c)

More than One Correct Options

1. (a,c) **2.** (c,d) **3.** (a,b,c,d) **4.** (b,c,d) **5.** (a,b,c,d) **6.** (c,d)

Comprehension Based Questions

1. (a) **2.** (c) **3.** (a)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → r
2. (a) → q (b) → p (c) → s (d) → s
3. (a) → r,s (b) → p (c) → r, s (d) → p
4. (a) → r (b) → q (c) → q (d) → p
5. (a) → s (b) → r (c) → p,r (d) → q

Subjective Questions

1. 2 μW, 0.173 J **2.** (a) 4.98 N/m^2 (b) 3.0×10^{-6} m **3.** (a) 283.33 Hz (b) 51.5 cm
4. $\lambda = 2\sqrt{4(H+h)^2 + d^2} - 2\sqrt{4H^2 + d^2}$
5. (a) 8.14° (b) 16.5° (c) Three maxima beyond the maximum corresponding to $\theta = 0°$
6. (a) 0 (b) $2I_0$ (c) $4I_0$ **7.** (a) 0 (b) 66 dB (c) 63 dB
8. (a) $I_1 = 19.9\,\mu$W/m^2, $I_2 = 8.84\,\mu$W/m^2 (b) 55.3 μW/m^2 (c) 2.2 μW/m^2 (d) 28.7 μW/m^2
9. f **10.** 0.28 cm, 9.0×10^{-3} kg/m^3 **11.** 95.5 s **12.** 1650 Hz
13. (a) 15 m, 20 min^{-1} (b) 13.5 m, 22.22 min^{-1} **14.** (a) 1 kg/m^3 (b) 10 μm (c) 288 π^3 W
15. (a) $\frac{I_0}{4}$ (b) I_0 **16.** (a) $\frac{400}{189}$ (b) $\frac{3}{4}$ **17.** (a) 1.0069×10^5 Hz (b) 1.0304×10^5 Hz
18. 27.04 N **19.** (a) 0.3 m (b) 1320 (c) 332 m/s (d) 0.2 m

20

Thermometry, Thermal Expansion and Kinetic Theory of Gases

Chapter Contents

20.1 Thermometers and The Celsius Temperature Scale
20.2 The Constant Volume Gas Thermometer and The Absolute Temperature Scale
20.3 Heat and Temperature
20.4 Thermal Expansion
20.5 Behaviour of Gases
20.6 Degree of Freedom
20.7 Internal Energy of an Ideal Gas
20.8 Law of Equipartition of Energy
20.9 Molar Heat Capacity
20.10 Kinetic Theory of Gases

20.1 Thermometers and The Celsius Temperature Scale

Thermometers are devices that are used to measure temperatures. All thermometers are based on the principle that some physical properties of a system change as the system's temperature changes. Some physical properties that change with temperature are

1. the volume of a liquid
2. the length of a solid
3. the pressure of a gas at constant volume
4. the volume of a gas at constant pressure and
5. the electric resistance of a conductor.

A common thermometer in everyday use consists of a mass of liquid, usually mercury or alcohol that expands in a glass capillary tube when heated. In this case, the physical property is the **change in volume of the liquid**. Any temperature change is proportional to the change in length of the liquid column. The thermometer can be calibrated accordingly. On the celsius temperature scale, a thermometer is usually calibrated between 0°C (called the ice point of water) and 100°C (called the steam point of water). Once the liquid levels in the thermometer have been established at these two points, the distance between the two points is divided into 100 equal segments to create the celsius scale.

Thus, each segment denotes a change in temperature of one celsius degree (1°C). A practical problem in this type of thermometer is that readings may vary for two different liquids. When one thermometer reads a temperature, for example 40°C the other may indicate a slightly different value. This discrepancies between thermometers are especially large at temperatures far from the calibration points. To surmount this problem we need a universal thermometer whose readings are independent of the substance used in it. The gas thermometer used in the next article meets this requirement.

20.2 The Constant Volume Gas Thermometer and The Absolute Temperature Scale

The physical property used by the constant volume gas thermometer is the **change in pressure of a gas at constant volume**.

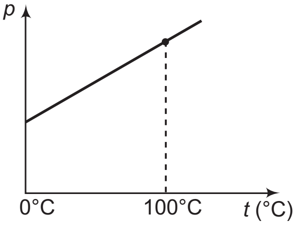


Fig. 20.1

The pressure *versus* temperature graph for a typical gas taken with a constant volume is shown in figure. The two dots represent the two reference temperatures namely, the ice and steam points of water. The line connecting them serves as a calibration curve for unknown temperatures. Experiments show that the thermometer readings are nearly independent of the type of gas used as long as the gas pressure is low and the temperature is well above the point at which the gas liquefies.

If you extend the curves shown in figure toward negative temperatures, you find, in every case, that the pressure is zero when the temperature is –273.15° C. This significant temperature is used as the basis for the **absolute temperature scale**, which sets –273.15° C as its zero point.

This temperature is often referred to as **absolute zero**. The size of a degree on the absolute temperature scale is identical to the size of a degree on the celsius scale. Thus, the conversion between these temperatures is

$$\boxed{T_C = T - 273.15} \qquad \ldots\text{(i)}$$

In 1954, by the International committee on weights and measures, the **triple point of water** was chosen as the reference temperature for this new scale. The triple point of water is the single combination of temperature and pressure at which liquid water, gaseous water and ice (solid water) coexist in equilibrium. This triple point occurs at a temperature of approximately 0.01° C and a pressure of 4.58 mm of mercury. On the new scale, which uses the unit **kelvin**, the temperature of water at the triple point was set at 273.16 kelvin, abbreviated as 273.16 K. (No degree sign is used with the unit kelvin).

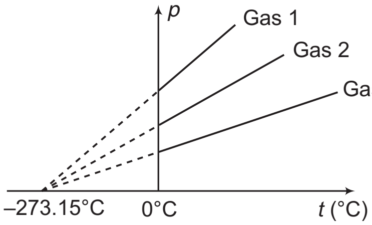


Fig. 20.2

This new absolute temperature scale (also called the kelvin scale) employs the SI unit of absolute temperature, the kelvin which is defined to be "$\frac{1}{273.16}$ of the difference between absolute zero and the temperature of the triple point of water".

The Celsius, Fahrenheit and Kelvin Temperature Scales

Eq. (i) shows the relation between the temperatures in celsius scales and kelvin scale. Because the size of a degree is the same on the two scales, a temperature difference of 10°C is equal to a temperature difference of 10 K. The two scales differ only in the choice of the zero point. The ice point temperature on the kelvin scale, 273.15 K, corresponds to 0.00°C and the kelvin steam point 373.15 K, is equivalent to 100.00°C.

A common temperature scale in everyday use in US is the **Fahrenheit scale**. The ice point in this scale is 32°F and the steam point is 212°F. The distance between these two points are divided in 180 equal parts. The relation between celsius scale and Fahrenheit scale is as derived below.

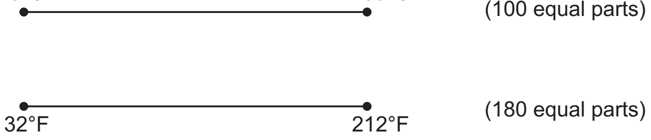
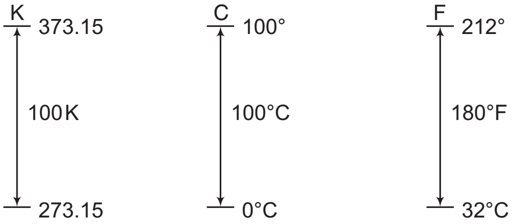


Fig. 20.3

$$100 \text{ parts of celsius scale} = 180 \text{ parts of Fahrenheit scale}$$

$$\therefore \quad 1 \text{ part of celsius scale} = \frac{9}{5} \text{ parts of Fahrenheit scale}$$



Relation among Kelvin, Celsius and Fahrenheit temperature scales

Fig. 20.4

Hence, $$T_F = 32 + \frac{9}{5} T_C \qquad \text{...(ii)}$$

Further, $$\Delta T_C = \Delta T = \frac{5}{9} \Delta T_F \qquad \text{...(iii)}$$

✓ Extra Points to Remember

- **Different Thermometers**

  Thermometric property It is the property that can be used to measure the temperature. It is represented by any physical quantity such as length, volume, pressure and resistance etc., which varies linearly with a certain range of temperature. Let X denotes the thermometric physical quantity and X_0, X_{100} and X_t be its values at 0°C, 100°C and t °C respectively. Then,

$$\boxed{t = \left(\frac{X_t - X_0}{X_{100} - X_0}\right) \times 100°\text{C}}$$

  (i) Constant volume gas thermometer The pressure of a gas at constant volume is the thermometric property. Therefore,

$$\boxed{t = \left(\frac{p_t - p_0}{p_{100} - p_0}\right) \times 100°\text{C}}$$

  (ii) Platinum resistance thermometer The resistance of a platinum wire is the thermometric property. Hence,

$$\boxed{t = \left(\frac{R_t - R_0}{R_{100} - R_0}\right) \times 100°\text{C}}$$

  (iii) Mercury thermometer In this thermometer, the length of a mercury column from some fixed point is taken as thermometric property. Thus,

$$\boxed{t = \left(\frac{l_t - l_0}{l_{100} - l_0}\right) \times 100°\text{C}}$$

- Two other thermometers, commonly used are **thermocouple thermometer** and **total radiation pyrometer.**

- **Total radiation pyrometer** is used to measure very high temperatures. When a body is at a high temperature, it glows brightly and the radiation emitted per second from unit area of the surface of the body is proportional to the fourth power of the absolute temperature of the body. If this radiation is measured by some device, the temperature of the body is calculated. This is the principle of a total radiation pyrometer. The main advantage of this thermometer is that the experimental body is not kept in contact with it. Hence, there is no definite higher limit of its temperature range. It can measure temperature from 800°C to 3000°C–4000°C. However, it cannot be used to measure temperatures below 800°C because at low temperatures the emission of radiation is so poor that it cannot be measured directly.
- **Ranges of different thermometers**

Table 20.1

Thermometer	Lower limit	Upper limit
Mercury thermometer	–30°C	300°C
Gas thermometer	–268°C	1500°C
Platinum resistance thermometer	–200°C	1200°C
Thermocouple thermometer	–200°C	1600°C
Radiation thermometer	800°C	No limit

- **Reaumur's scale** Other than Celsius, Fahrenheit and Kelvin temperature scales **Reaumur's scale** was designed by Reaumur in 1730. The lower fixed point is 0°R representing melting point of ice. The upper fixed point is 80°R, which represents boiling point of water. The distance between the two fixed points is divided into 80 equal parts. Each part represents 1°R. If T_C, T_F and T_R are temperature values of a body on Celsius scale, Fahrenheit scale and Reaumur scale respectively, then

$$\boxed{\frac{T_C - 0}{100} = \frac{T_F - 32}{180} = \frac{T_R - 0}{80}}$$

- A substance is found to exist in three states solid, liquid and gas. For each substance, there is a set of temperature and pressure at which all the three states may coexist. This is called **triple point** of that substance. For water, the values of pressure and temperature corresponding to triple point are 4.58 mm of Hg and 273.16°K.

Example 20.1 *Express a temperature of 60°F in degrees Celsius and in Kelvin.*

Solution Substituting $T_F = 60°$ F in Eq. (ii),

$$T_C = \frac{5}{9}(T_F - 32) = \frac{5}{9}(60° - 32°)$$

$$= 15.55° \text{ C}$$ **Ans.**

From Eq. (i),

$$T = T_C + 273.15 = 15.55° \text{ C} + 273.15$$

$$= 288.7 \text{ K}$$ **Ans.**

Example 20.2 *The temperature of an iron piece is heated from 30°C to 90°C. What is the change in its temperature on the Fahrenheit scale and on the kelvin scale?*

Solution $\Delta T_C = 90°\text{ C} - 30°\text{ C} = 60°\text{ C}$

Using Eq. (iii),

$$\Delta T_F = \frac{9}{5}\Delta T_C = \frac{9}{5}(60°\text{ C})$$

$$= 108°\text{ F}$$ **Ans.**

and $$\Delta T = \Delta T_C = 60\text{ K}$$ **Ans.**

INTRODUCTORY EXERCISE 20.1

1. What is the value of
   (a) 0°F in Celsius scale?
   (b) 0 K on Fahrenheit scale?
2. At what temperature is the Fahrenheit scale reading equal to
   (a) twice (b) half of Celsius?
3. A faulty thermometer reads 5°C in melting ice and 99°C in steam. Find the correct temperature in °F when this faulty thermometer reads 52°C.
4. At what temperature the Fahrenheit and Kelvin scales of temperature give the same reading?
5. At what temperature the Fahrenheit and Celsius scales of temperature give the same reading?

20.3 Heat and Temperature

The word heat is always used during transfer of thermal energy from hot body to cold body (due to temperature difference between them). Following are given some statements. Some of them are right and some are wrong. From those statements, you will be able to find the difference between heat and temperature.

Wrong Statements

(i) This body has large quantity of heat.
(ii) Temperature transfer is taking place from body A to body B.

Correct Statements

(i) Temperature of body A is more than temperature of body B.
(ii) Heat transfer is taking place from body A to body B because A is at higher temperature than body B.

Note *One calorie* (1 *cal*) *is defined as the amount of heat required to raise the temperature of one gram of water from* 14.5° *C to* 15.5° *C.*

Experiments have shown that

$$1\ cal = 4.186\ J$$

Similarly, $$1\ kcal = 1000\ cal = 4186\ J$$

The calorie is not a fundamental SI unit.

20.4 Thermal Expansion

Most substances expand when they are heated. Thermal expansion is a consequence of the change in average separation between the constituent atoms of an object. Atoms of an object can be imagined to be connected to one another by stiff springs as shown in Fig. 20.5. At ordinary temperatures, the atoms in a solid oscillate about their equilibrium positions with an amplitude of approximately 10^{-11} m. The average spacing between the atoms is about 10^{-10} m. As the temperature of solid increases, the atoms oscillate with greater amplitudes, as a result the average separation between them increases, consequently the object expands.

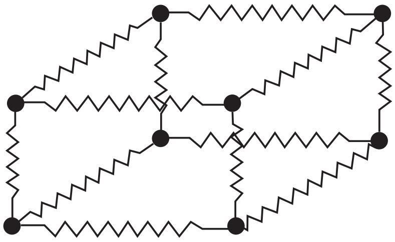


Fig. 20.5

More precisely, thermal expansion arises from the asymmetrical nature of the potential energy curve.

At the atomic level, thermal expansion may be understood by considering how the potential energy of the atoms varies with distance. The equilibrium position of an atom will be at the minimum of the potential energy well if the well is symmetric. At a given temperature, each atom vibrates about its equilibrium position and its average position remains at the minimum point. If the shape of the well is not symmetrical, as shown in figure, the average position of an atom will not be at the minimum point. When the temperature is raised the amplitude of the vibrations increases and the average position is located at a greater interatomic separation. This increased separation is manifested as expansion of the material.

Linear Expansion

Suppose that the temperature of a thin rod of length l is changed from T to $T+\Delta T$. It is found experimentally that, if ΔT is not too large, the corresponding change in length Δl of the rod is directly proportional to ΔT and l. Thus,

$$\Delta l \propto \Delta T$$

and $$\Delta l \propto l$$

Introducing a proportionality constant α (which is different for different materials) we may write Δl as

$$\Delta l = l\alpha\Delta T \qquad \text{...(i)}$$

Here, the constant α is called the **coefficient of linear expansion** of the material of the rod and its units are K^{-1} or $[(^\circ C)^{-1}]$. Remember that $\Delta T = \Delta T_C$.

Actually, α does depend slightly on the temperature, but its variation is usually small enough to be negligible, even over a temperature range of 100°C. We will always assume that α is a constant.

Volume Expansion

Because the linear dimensions of an object change with temperature, it follows that surface area and volume change as well. Just as with linear expansion, experiments show that if the temperature change ΔT is not too great (less than 100°C or so), the increase in volume ΔV is proportional to both, the temperature change ΔT and the initial volume V. Thus,

$$\Delta V \propto \Delta T$$

and

$$\Delta V \propto V$$

Introducing a proportionality constant γ, we may write ΔV as

$$\Delta V = V \times \gamma \times \Delta T \qquad \dots \text{(ii)}$$

Here, γ is called the **coefficient of volume expansion**. The units of γ are K^{-1} or $({}^\circ C)^{-1}$.

Relation Between γ and α

For an isotropic solid (which has the same value of α in all directions) $\gamma = 3\alpha$. To see that $\gamma = 3\alpha$ for a solid, consider a cube of length l and volume $V = l^3$.

When the temperature of the cube is increased by dT, the side length increases by dl and the volume increases by an amount dV given by

$$dV = \left(\frac{dV}{dl}\right) \cdot dl = 3l^2 \cdot dl$$

Now, $$dl = l\alpha dT$$

$\therefore$ $$dV = 3l^3 \alpha dT = (3\alpha) V dT$$

This is consistent with Eq. (ii),

$$dV = \gamma V dT, \text{ only if}$$

$$\gamma = 3\alpha \qquad \dots\text{(iii)}$$

Average values of α and γ for some materials are listed in Table 20.2. You can check the relation $\gamma = 3\alpha$, for the materials given in the table.

Table 20.2

Material	α [K^{-1} or $({}^\circ C)^{-1}$]	γ [K^{-1} or $({}^\circ C)^{-1}$]
Steel	1.2×10^{-5}	3.6×10^{-5}
Copper	1.7×10^{-5}	5.1×10^{-5}
Brass	2.0×10^{-5}	6.0×10^{-5}
Aluminium	2.4×10^{-5}	7.2×10^{-5}

The Anomalous Expansion of Water

Most liquids also expand when their temperatures increase. Their expansion can also be described by Eq. (ii). The volume expansion coefficients for liquids are about 100 times larger than those for solids.

Some substances contract when heated over a certain temperature range. The most common example is water.

Figure shows how the volume of 1 g of water varies with temperature at atmospheric pressure. The volume decreases as the temperature is raised from 0°C to about 4°C, at 4°C the volume is a minimum and the density is a maximum (1000 kg/m^3). Above 4° C, water expands with increasing temperature like most substances.

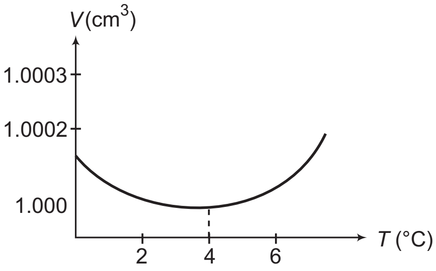


Fig. 20.6

This anomalous behaviour of water causes ice to form first at the surface of a lake in cold weather. As winter approaches, the water temperature decreases initially at the surface. The water there sinks because of its increased density. Consequently, the surface reaches 0°C first and the lake becomes covered with ice. Now ice is bad conductor of heat so water at the bottom remains at 4°C, the highest density of water. Aquatic life is able to survive the cold winter as the bottom of the lake remains unfrozen at a temperature of about 4°C.

✓ Extra Points to Remember

- If a solid object has a hole in it, what happens to the size of the hole, when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth is that if the object expands, the hole will expand too, because every linear dimension of an object changes in the same way when the temperature changes.

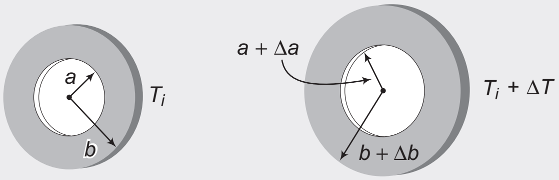


Fig. 20.7

- **Expansion of a bimetallic strip** As Table 20.2 indicates, each substance has its own characteristic average coefficient of expansion.

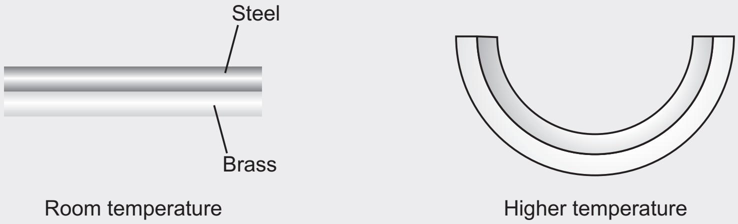


Fig. 20.8

For example, when the temperatures of a brass rod and a steel rod of equal length are raised by the same amount from some common initial value, the brass rod expands more than the steel rod because brass has a greater average coefficient of expansion than steel. Such type of bimetallic strip is found in practical devices such as **thermostats** to break or make electrical contact.

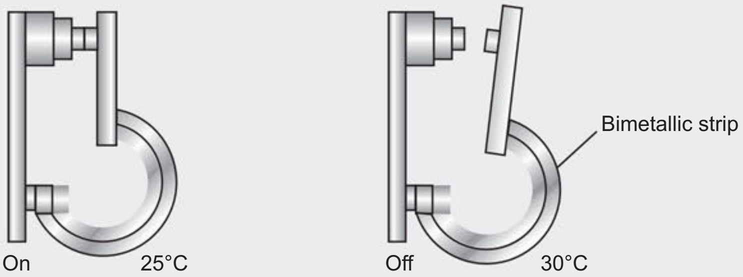


Fig. 20.9

- **Variation of density with temperature** Most substances expand when they are heated, i.e. volume of a given mass of a substance increases on heating, so the density should decrease $\left(\text{as } \rho \propto \frac{1}{V}\right)$. Let us see how the density (ρ) varies with increase in temperature.

$$\rho = \frac{m}{V}$$

or $$\rho \propto \frac{1}{V} \qquad \text{(for a given mass)}$$

∴ $$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + \Delta V} = \frac{V}{V + \gamma V \Delta T} = \frac{1}{1 + \gamma \Delta T}$$

∴ $$\boxed{\rho' = \frac{\rho}{1 + \gamma \Delta T}}$$

This expression can also be written as

$$\rho' = \rho (1 + \gamma \Delta T)^{-1}$$

If γ is very small, $$(1 + \gamma \Delta T)^{-1} \approx 1 - \gamma \Delta T$$

∴ $$\boxed{\rho' \approx \rho (1 - \gamma \Delta T)}$$

- **Effect of temperature on apparent weight when immersed in a liquid** When a solid body is completely immersed in a liquid, its apparent weight gets decreased due to an upthrust acting on it by the liquid. The apparent weight is given by

$$w_{\text{app}} = w - F$$

Here, F = upthrust = $V_S \rho_L g$

where, V_S = volume of solid and ρ_L = density of liquid

Now, as the temperature is increased V_S increases while ρ_L decreases. So, F may increase or decrease (or may remain constant also) depending upon the condition that which factor dominates on the other. We can write

$$F \propto V_S \rho_L$$

or $$\frac{F'}{F} = \frac{V_S'}{V_S} \cdot \frac{\rho_L'}{\rho_L} = \frac{(V_S + \Delta V_S)}{V_S} \cdot \left(\frac{1}{1 + \gamma_L \Delta T}\right)$$

$$= \left(\frac{V_S + \gamma_S V_S \Delta T}{V_S}\right)\left(\frac{1}{1 + \gamma_L \Delta T}\right)$$

or $$F' = F\left(\frac{1+\gamma_S \Delta T}{1+\gamma_L \Delta T}\right)$$

Now, if $\gamma_S > \gamma_L, F' > F$

or $w'_{\text{app}} < w_{\text{app}}$ and *vice-versa*.

And if $\gamma_S = \gamma_L, \quad F' = F$

or $w'_{\text{app}} = w_{\text{app}}$

- **Effect of temperature on immersed fraction of a solid in floating condition**

When a solid, whose density is less than the density of liquid is floating in it, then

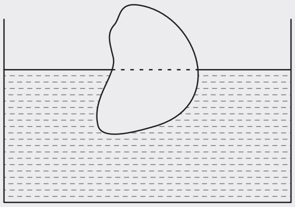


Fig. 20.10

Weight of the solid = Upthrust on solid from liquid

$$\therefore \quad V\rho_s g = V_i \rho_l g \qquad \text{...(i)}$$

Here,

ρ_s = density of solid

ρ_l = density of liquid

V_i = immersed volume of solid and

V = total volume of solid

From Eq. (i),

$$\frac{V_i}{V} = \frac{\rho_s}{\rho_l} = f \qquad \text{...(ii)}$$

where, f = immersed fraction of solid.

With increase in temperature, ρ_s and ρ_l both will decrease. Therefore, this fraction may increase, decrease or remain constant. At some higher temperature,

$$f' = \frac{\rho_s'}{\rho_l'} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we have

$$\frac{f'}{f} = \left(\frac{\rho_s'}{\rho_s}\right)\left(\frac{\rho_l}{\rho_l'}\right)$$

$$= \left(\frac{1}{1+\gamma_s \Delta\theta}\right)(1+\gamma_l \Delta\theta)$$

or $$\frac{f'}{f} = \left(\frac{1+\gamma_l \Delta\theta}{1+\gamma_s \Delta\theta}\right)$$

Now, if $\gamma_l > \gamma_s$, $f' > f$ or immersed fraction will increase.

If $\gamma_l = \gamma_s$, $f' = f$ or immersed fraction will remain unchanged and if,

$\gamma_l < \gamma_s$, then $f' < f$ or immersed fraction will decrease.

- **Effect of temperature on the time period of a pendulum** The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

or $$T \propto \sqrt{l}$$

As the temperature is increased, length of the pendulum and hence, time period gets increased or a pendulum clock becomes slow and it loses the time.

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{l + \Delta l}{l}}$$

Here, we put $\Delta l = l\alpha\Delta\theta$ in place of $l\alpha\Delta T$ so as to avoid the confusion with change in time period. Thus,

$$\frac{T'}{T} = \sqrt{\frac{l + l\alpha\Delta\theta}{l}} = (1 + \alpha\Delta\theta)^{1/2}$$

or $$T' \approx T\left(1 + \frac{1}{2}\alpha\Delta\theta\right) \qquad (\text{if } \alpha\Delta\theta << 1)$$

or $$\boxed{\Delta T = T' - T = \frac{1}{2}T\alpha\Delta\theta}$$

Time lost in time t (by a pendulum clock whose actual time period is T and the changed time period at some higher temperature is T') is

$$\Delta t = \left(\frac{\Delta T}{T'}\right)t$$

- At some higher temperature a scale will expand and scale reading will be lesser than true value. However, at lower temperatures scale reading will be more or true value will be less.
- When a rod whose ends are rigidly fixed such as to prevent from expansion or contraction undergoes a change in temperature, thermal stresses are developed in the rod. This is because, if the temperature is increased, the rod has a tendency to expand but since it is fixed at two ends it is not allowed to expand. So, the rod exerts a force on supports to expand.

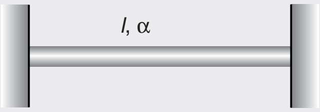


Fig. 20.11

Thermal strain $= \frac{\Delta l}{l} = \alpha\Delta T$

So thermal stress = (Y) (termal strain) = $Y\alpha\Delta T$

or force on supports $F = A$ (stress) = $YA\alpha\Delta T$

Here, Y = Young's modulus of elasticity of the rod.

$$\boxed{F = YA\alpha\Delta T}$$

- **Expansion of liquids** For heating a liquid it has to be put in some container. When the liquid is heated, the container will also expand. We define coefficient of apparent expansion of a liquid as the apparent increase in volume per unit original volume per °C rise in temperature. It is represented by γ_a. Thus,

$$\gamma_a = \gamma_r - \gamma_g$$

Here, γ_r = coefficient of volume expansion of liquid

and γ_g = coefficient of volume expansion of the container

Example 20.3 *A steel ruler exactly 20 cm long is graduated to give correct measurements at 20°C.*

(a) Will it give readings that are too long or too short at lower temperatures?

(b) What will be the actual length of the ruler when it is used in the desert at a temperature of 40° C ? $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution (a) If the temperature decreases, the length of the ruler also decreases through thermal contraction. Below 20°C, each centimetre division is actually somewhat shorter than 1.0 cm, so the steel ruler gives readings that are too long.

(b) At 40°C, the increase in length of the ruler is

$$\Delta l = l\alpha\Delta T$$

$$= (20)(1.2 \times 10^{-5})(40° - 20°)$$

$$= 0.48 \times 10^{-2}\ \text{cm}$$

∴ The actual length of the ruler is

$$l' = l + \Delta l$$

$$= 20.0048\ \text{cm}$$ **Ans.**

Example 20.4 *The scale on a steel meter stick is calibrated at 15°C. What is the error in the reading of 60 cm at 27°C?* $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution The error in the reading will be

$$\Delta l = (\text{scale reading})(\alpha)(\Delta T)$$

$$= (60)(1.2 \times 10^{-5})(27° - 15°)$$

$$= 0.00864\ \text{cm}$$ **Ans.**

Example 20.5 *A second's pendulum clock has a steel wire. The clock is calibrated at 20°C. How much time does the clock lose or gain in one week when the temperature is increased to 30°C?* $\alpha_{steel} = 1.2 \times 10^{-5} (°C)^{-1}$.

Solution The time period of second's pendulum is 2 second. As the temperature increases length and hence, time period increases. Clock becomes slow and it loses the time. The change in time period is

$$\Delta T = \frac{1}{2} T\alpha\Delta\theta$$

$$= \left(\frac{1}{2}\right)(2)(1.2 \times 10^{-5})(30° - 20°)$$

$$= 1.2 \times 10^{-4}\ \text{s}$$

∴ New time period is

$$T' = T + \Delta T = (2 + 1.2 \times 10^{-4})$$

$$= 2.00012\ \text{s}$$

∴ Time lost in one week

$$\Delta t = \left(\frac{\Delta T}{T'}\right) t$$

$$= \frac{(1.2 \times 10^{-4})}{(2.00012)} (7 \times 24 \times 3600)$$

$$= 36.28 \text{ s}$$ **Ans.**

Example 20.6 *A sphere of diameter 7 cm and mass 266.5 g floats in a bath of liquid. As the temperature is raised, the sphere just sinks at a temperature of $35° C$. If the density of the liquid at $0° C$ is 1.527 g/cm^3, find the coefficient of cubical expansion of the liquid. Ignore expansion of sphere.*

Solution The sphere will sink in the liquid at 35°C, when its density becomes equal to the density of liquid at 35° C.

The density of sphere,

$$\rho_{35} = \frac{266.5}{\frac{4}{3} \times \left(\frac{22}{7}\right) \times \left(\frac{7}{2}\right)^3}$$

$$= 1.483 \text{ g/cm}^3$$

Now,

$$\rho_0 = \rho_{35} \, [1 + \gamma \Delta T)$$

$$1.527 = 1.483 [1 + \gamma \times 35]$$

$$1.029 = 1 + \gamma \times 35$$

$$\gamma = \frac{1.029 - 1}{35}$$

$$= 0.00083/° \text{ C}$$ **Ans.**

Example 20.7 *A glass beaker holds exactly 1 L at 0°C*

(a) What is its volume at 50°C?

(b) If the beaker is filled with mercury at 0°C, what volume of mercury overflows when the temperature is 50°C ? $\alpha_g = 8.3 \times 10^{-6}$ per ° C and $\gamma_{Hg} = 1.82 \times 10^{-4}$ per ° C.

Solution (a) The volume of beaker after the temperature change is

$$V_{\text{beaker}} = V_0 \, (1 + 3\alpha_g \Delta\theta)$$

$$= (1) [1 + 3 \times 8.3 \times 10^{-6} \times 50]$$

$$= 1.001 \text{ L}$$ **Ans.**

(b) Volume of mercury at 50°C is

$$V_{\text{mercury}} = V_0 \, (1 + \gamma_{Hg} \Delta\theta)$$

$$= (1) [1 + 1.82 \times 10^{-4} \times 50]$$

$$= 1.009 \text{ L}$$

The overflow is thus $1.009 - 1.001 = 0.008$ L or 8 mL **Ans.**

INTRODUCTORY EXERCISE 20.2

Take the values of α from Table 20.2.

1. A pendulum clock of time period 2 s gives the correct time at 30°C. The pendulum is made of iron. How many seconds will it lose or gain per day when the temperature falls to 0°C? $\alpha_{Fe} = 1.2 \times 10^{-5} \, (^\circ\text{C})^{-1}$.
2. A block of wood is floating in water at 0°C. The temperature of water is slowly raised from 0°C to 10°C. How will the percentage of volume of block above water level change with rise in temperature?
3. A piece of metal floats on mercury. The coefficient of volume expansion of metal and mercury are γ_1 and γ_2, respectively. If the temperature of both mercury and metal are increased by an amount ΔT, by what factor does the fraction of the volume of the metal submerged in mercury changes?
4. A brass disc fits snugly in a hole in a steel plate. Should you heat or cool the system to losen the disc from the hole? Given that $\alpha_B > \alpha_{Fe}$.
5. An iron ball has a diameter of 6 cm and is 0.010 mm too large to pass through a hole in a brass plate when the ball and plate are at a temperature of 30° C. At what temperature, the same for ball and plate, will the ball just pass through the hole?
   Take the values of α from Table 20.2.
6. (a) An aluminium measuring rod which is correct at 5° C, measures a certain distance as 88.42 cm at 35 ° C. Determine the error in measuring the distance due to the expansion of the rod. (b) If this aluminium rod measures a length of steel as 88.42 cm at 35° C, what is the correct length of the steel at 35° C?
7. A steel tape is calibrated at 20° C . On a cold day when the temperature is –15° C, what will be the percentage error in the tape?

20.5 Behaviour of Gases

Gases are the most diffused form of matter. In a gas, the molecules are highly energetic and they can be widely separated from each other. The behaviour of gases and their properties derive from these facts.

Sometimes the term vapour is used to describe a gas. Strictly speaking, a gas is a substance at a temperature above its boiling point. A vapour is the gaseous phase of a substance that under ordinary conditions, exists as a liquid or solid.

One of the more obvious characteristics of gases is their ability to expand and fill any volume they are placed in. This contrasts with the behaviour exhibited by solids (fixed shape and volume) and liquids (fixed volume but indeterminate shape).

Ideal and Real Gas Laws

Gases, unlike solids and liquids have indefinite shape and indefinite volume. As a result, they are subjected to pressure changes, volume changes and temperature changes. Real gas behaviour is simpler. By understanding ideal gas behaviour, real gas behaviour becomes more tangible.

Ideal Gases

How do we describe an ideal gas? An ideal gas has the following properties :

1. An ideal gas is considered to be a "point mass". A point mass is a particle so small, its volume is very nearly zero. This means an ideal gas particle has virtually no volume.
2. Collisions between ideal gases are "elastic". This means that no attractive or repulsive forces are involved during collisions. Also, the kinetic energy of the gas molecules remains constant since these intermolecular forces are lacking.

   Volume and temperature are by now familiar concepts. Pressure, however, may need some explanation. Pressure is defined as a force per unit area. When gas molecules collide with the sides of a exert, they exert a force over that area of the container. This gives rise to the pressure inside the container.

Gas Laws

Assuming permanent (or real) gases to be ideal, through experiments, it was established that gases irrespective of their nature obey the following laws :

Boyle's Law

According to this law, for a given mass of a gas, the volume of a gas at constant temperature (called **isothermal process**) is inversely proportional to its pressure, i.e.

$$V \propto \frac{1}{p} \qquad (T = \text{constant})$$

or

$$pV = \text{constant}$$

or

$$p_i V_i = p_f V_f$$

Thus, p-V graph in an isothermal process is a rectangular hyperbola. Or pV *versus* p or V graph is a straight line parallel to p or V axis.

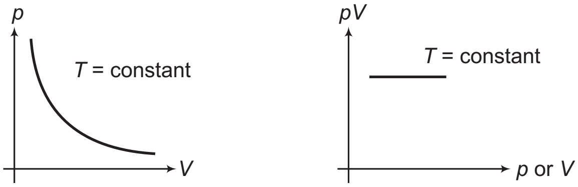


Fig. 20.12

Charles' Law

According to this law, for a given mass of a gas the volume of a gas at constant pressure (called **isobaric** process) is directly proportional to its absolute temperature, i.e.

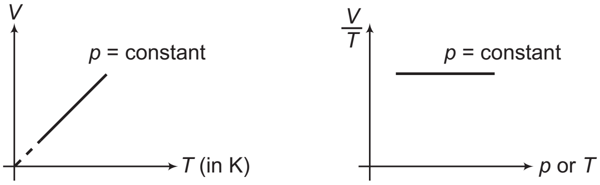


Fig. 20.13

$$V \propto T \qquad (p = \text{constant})$$

or $$\frac{V}{T} = \text{constant} \quad \text{or} \quad \frac{V_i}{T_i} = \frac{V_f}{T_f}$$

Thus, V-T graph in an isobaric process is a straight line passing through origin. Or V/T *versus* V or T graph is a straight line parallel to V or T axis.

Gay Lussac's Law or Pressure Law

According to this law, for a given mass of a gas the pressure of a gas at constant volume (called **isochoric process**) is directly proportional to its absolute temperature, i.e.

$$p \propto T \quad (V = \text{constant}) \quad \text{or} \quad \frac{p}{T} = \text{constant} \quad \text{or} \quad \frac{p_i}{T_i} = \frac{p_f}{T_f}$$

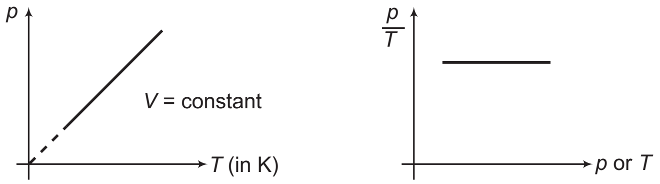


Fig. 20.14

Thus, p - T graph in an isochoric process is a straight line passing through origin or p/T *versus* p or T graph is a straight line parallel to p or T axis.

Ideal Gas Equation

All the above four laws can be written in one single equation known as ideal gas equation. According to this equation.

$$\boxed{pV = nRT = \frac{m}{M} RT}$$

In this equation, n = number of moles of the gas

$$= \frac{m}{M}$$

m = total mass of the gas

M = molecular mass of the gas

and R = universal gas constant

$= 8.31$ J/mol-K

$= 2.0$ cal/mol-K

The above three laws can be derived from this single equation. For example, for a given mass of a gas (m = constant)

$$pV = \text{constant at constant temperature} \qquad \text{(Boyle's law)}$$

$$\frac{p}{T} = \text{constant at constant volume} \qquad \text{(Pressure law)}$$

$$\frac{V}{T} = \text{constant at constant pressure} \qquad \text{(Charles' law)}$$

Avogadro's Law

The next empirical gas law, we will look at, is called Avogadro's law. This law deals with the relationship between the volume and moles of a gas at constant pressure and temperature. According to this law, at same temperature and pressure equal volumes of all gases contain equal number of molecules. Let's derive this law from the ideal gas law. Give the moles and volume subscripts, since their conditions will change.

$$pV_1 = n_1RT \quad \text{and} \quad pV_2 = n_2RT$$

Collect terms. Divide each equation by pressure, p and divide each equation by their respective mole term

$$\frac{n_1}{V_1} = \frac{n_2}{V_2} = \frac{p}{RT}$$

Thus, the number of molecules per unit volume is same for all gases at a fixed temperature and pressure. The number in 22.4 litres of any gas is 6.02×10^{23}. This is known as Avogadro number and is denoted by N_A. The mass of 22.4 litres of any gas is equal to its molecular weight in grams at S.T.P. (standard temperature 273 K and pressure 1 atm). This amount of substance is called a mole.

Note *$k = \frac{R}{N_A}$ is called **Boltzmann constant**. Its value in SI unit is* 1.38×10^{-23} *J/K.*

Extra Points to Remember

- In our previous discussion, we have discussed Charles' law and pressure law in absolute temperature scale. In centigrade scale, these laws are as under :

Charles' law When a given mass of a gas is heated at constant pressure then for each 1°C rise in temperature the volume of the gas increases by a fraction α of its volume at 0°C. Thus, if the volume of a given mass of a gas at 0°C is V_0, then on heating at constant pressure to t°C its volume will increase by $V_0\alpha t$. Therefore, if its volume at t°C be V_t, then

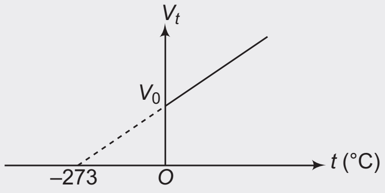


Fig. 20.15

$$V_t = V_0 + V_0\alpha t \quad \text{or} \quad V_t = V_0(1 + \alpha t)$$

Here, α is called the 'volume coefficient' of the gas. For all gases, the experimental value of α is nearly $\frac{1}{273}$ per °C.

$$\therefore \qquad V_t = V_0\left(1 + \frac{t}{273}\right)$$

Thus, V_t *versus* t graph is a straight line with slope $\frac{V_0}{273}$ and positive intercept V_0. Further $V_t = 0$ at

$$t = -273°\text{C}.$$

Pressure law According to this law, when a given mass of a gas is heated at constant volume then for each 1°C rise in temperature, the pressure of the gas increases by a fraction β of its pressure at 0°C. Thus, if the pressure of a given mass of a gas at 0°C be p_0, then on heating at constant volume to t°C, its pressure will increase by $p_0\beta t$. Therefore, if its pressure at t°C be p_t, then

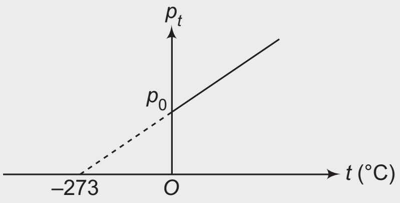


Fig. 20.16

$$p_t = p_0 + p_0\beta t \quad \text{or} \quad p_t = p_0(1 + \beta t)$$

Here, β is called the 'pressure coefficient' of the gas. For all gases the experimental value of β is also

$$\frac{1}{273} \text{ per } ^\circ\text{C} \quad \Rightarrow \quad p_t = p_0 \left(1 + \frac{t}{273}\right)$$

p_t *versus* t graph is as shown in Fig 20.16.

- The above forms of Charles' law and pressure law can be simply expressed in terms of absolute temperature.

  Let at constant pressure, the volume of a given mass of a gas at 0°C, t_1°C and t_2°C be V_0, V_1 and V_2 respectively. Then,

$$V_1 = V_0 \left(1 + \frac{t_1}{273}\right) = V_0 \left(\frac{273 + t_1}{273}\right)$$

$$V_2 = V_0 \left(1 + \frac{t_2}{273}\right) = V_0 \left(\frac{273 + t_2}{273}\right)$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{273 + t_1}{273 + t_2} = \frac{T_1}{T_2}$$

  where, T_1 and T_2 are the absolute temperatures corresponding to t_1°C and t_2°C . Hence,

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} \quad \text{or} \quad \frac{V}{T} = \text{constant} \quad \text{or} \quad V \propto T$$

  This is the form of Charles' law which we have already studied in article 20.5. In the similar manner, we can prove the pressure law.

- Under isobaric conditions (p = constant), V-T graph is a straight line passing through origin (where, T is in kelvin). The slope of this line is $\left(\frac{nR}{p}\right)$ as $V = \left(\frac{nR}{p}\right) T$ or slope of the line is directly proportional to $\frac{n}{p}$.

$$\text{slope} = \frac{nR}{p} \quad \text{or} \quad \text{slope} \propto \frac{n}{p}$$

  Similarly, under isochoric conditions (V = constant), p-T graph is a straight line passing through origin whose slope is $\frac{nR}{V}$ or slope is directly proportional to $\frac{n}{V}$.

- **Density of a gas** The ideal gas equation is

$$pV = nRT = \frac{m}{M} RT$$

$$\therefore \quad \frac{m}{V} = \rho = \frac{pM}{RT} \qquad (\rho = \text{density})$$

$$\therefore \quad \rho = \frac{pM}{RT}$$

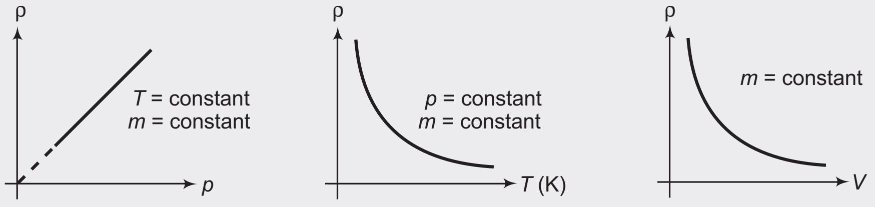


Fig. 20.17

From this equation, we can see that ρ-p graph is straight line passing through origin at constant temperature $(\rho \propto p)$ and ρ-T graph is a rectangular hyperbola at constant pressure $\left(\rho \propto \frac{1}{T}\right)$. Similarly, for a given mass of a gas ρ-V graph is a rectangular hyperbola $\left(\rho \propto \frac{1}{V}\right)$.

Example 20.8 *p-V diagrams of same mass of a gas are drawn at two different temperatures T_1 and T_2. Explain whether $T_1 > T_2$ or $T_2 > T_1$.*

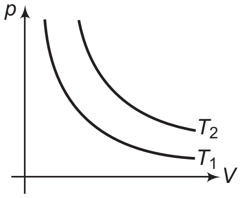


Fig. 20.18

Solution The ideal gas equation is

$$pV = nRT \quad \text{or} \quad T = \frac{pV}{nR}$$

$T \propto pV$ if number of moles of the gas are kept constant. Here, mass of the gas is constant, which implies that number of moles are constant, i.e. $T \propto pV$. In the given diagram, product of p and V for T_2 is more than T_1 at all points (keeping either p or V same for both graphs). Hence,

$$T_2 > T_1$$ **Ans.**

Example 20.9 *The p-V diagram of two different masses m_1 and m_2 are drawn (as shown) at constant temperature T. State whether $m_1 > m_2$ or $m_2 > m_1$.*

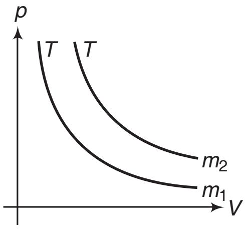


Fig. 20.19

Solution

$$pV = nRT = \frac{m}{M} RT$$

$$\therefore \quad m = (pV)\left(\frac{M}{RT}\right) \quad \text{or} \quad m \propto pV \text{ if } T = \text{constant}$$

From the graph, we can see that $p_2V_2 > p_1V_1$ (for same p or V). Therefore,

$$m_2 > m_1$$ **Ans.**

Example 20.10 *The p-T graph for the given mass of an ideal gas is shown in figure. What inference can be drawn regarding the change in volume (whether it is constant, increasing or decreasing)?*

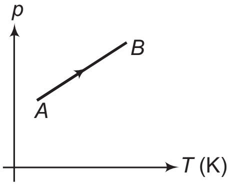


Fig. 20.20

How to Proceed *Definitely, it is not constant. Because when volume of the gas is constant p-T graph is a straight line passing through origin. The given line does not pass through origin, hence volume is not constant.*

$$V = (nR)\left(\frac{T}{p}\right)$$

Now, to see the volume of the gas we will have to see whether $\frac{T}{p}$ *is increasing or decreasing.*

Solution From the given graph, we can write the *p*-*T* equation as

$$p = aT + b \qquad (y = mx + c)$$

Here, a and b are positive constants. Further,

$$\frac{p}{T} = a + \frac{b}{T}$$

Now, $$T_B > T_A$$

$\therefore$ $$\frac{b}{T_B} < \frac{b}{T_A} \quad \text{or} \quad \left(\frac{p}{T}\right)_B < \left(\frac{p}{T}\right)_A$$

or $$\left(\frac{T}{p}\right)_B > \left(\frac{p}{T}\right)_A$$

or $$V_B > V_A$$ **Ans.**

Thus, as we move from A to B, volume of the gas is increasing.

Example 20.11 *A gas at* $27°\,C$ *in a cylinder has a volume of* 4 *litre and pressure* 100 Nm^{-2}.

(i) *Gas is first compressed at constant temperature so that the pressure is* 150 Nm^{-2}. *Calculate the change in volume.*

(ii) *It is then heated at constant volume so that temperature becomes* $127°\,C$. *Calculate the new pressure.*

Solution (i) Using Boyle's law for constant temperature,

$$p_1V_1 = p_2V_2$$

$\therefore$ $$V_2 = \frac{p_1V_1}{p_2} = \frac{100\times 4}{150} = 2.667\ \text{L}$$

$\therefore$ $$\text{Change in volume} = V_2 - V_1 = 2.667 - 4$$
$$= -1.333\ \text{L}$$

(ii) Using Gay Lussac's law for constant volume,

$$\frac{p_2}{p_1} = \frac{T_2}{T_1} \quad \text{or} \quad p_2 = \frac{T_2}{T_1}\times p_1 = \frac{(127+273)\times 150}{(27+273)}$$
$$= 200\ \text{Nm}^{-2}$$

Example 20.12 *A balloon partially filled with helium has a volume of* $30\ m^3$, *at the earth's surface, where pressure is* 76 *cm of Hg and temperature is* $27°\,C$. *What will be the increase in volume of gas if balloon rises to a height, where pressure is* 7.6 *cm of Hg and temperature is* $-54°\,C$?

Solution As $\dfrac{p_1V_1}{T_1} = \dfrac{p_2V_2}{T_2}$

$$\therefore \qquad V_2 = \frac{p_1V_1T_2}{T_1 p_2}$$

$$= \frac{76 \times 30 \times (-54 + 273)}{(27 + 273) \times 7.6}$$

$$= 219 \text{ m}^3$$

$\therefore$ Increase in volume of gas

$$= V_2 - V_1 = 219 - 30$$

$$= 189 \text{ m}^3$$

INTRODUCTORY EXERCISE 20.3

1. From the graph for an ideal gas, state whether m_1 or m_2 is greater.

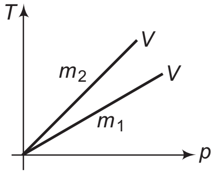


Fig. 20.21

2. A vessel is filled with an ideal gas at a pressure of 20 atm and is at a temperature of 27°C. One-half of the mass is removed from the vessel and the temperature of the remaining gas is increased to 87°C. At this temperature, find the pressure of the gas.
3. A vessel contains a mixture of 7 g of nitrogen and 11 g of carbon dioxide at temperature $T = 290$ K. If pressure of the mixture is 1 atm $(= 1.01 \times 10^5 \text{ N/m}^2)$, calculate its density $(R = 8.31 \text{J/mol-K})$.
4. An electric bulb of volume 250 cm^3 was sealed off during manufacture at a pressure of 10^{-3} mm of mercury at 27°C. Compute the number of air molecules contained in the bulb. Given that $R = 8.31 \text{J/mol-K}$ and $N_A = 6.02 \times 10^{23}$ per mol.
5. State whether $p_1 > p_2$ or $p_2 > p_1$ for given mass of a gas?

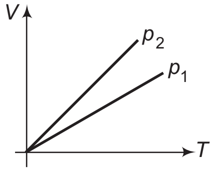


Fig. 20.22

6. For a given mass of a gas what is the shape of p *versus* $\dfrac{1}{V}$ graph at constant temperature?
7. For a given mass of a gas, what is the shape of pV *versus* T graph in isothermal process?

20.6 Degree of Freedom (f)

The minimum number of ways in which motion of a body (or a system) can be described completely is called its degree of freedom.

For example In Fig. (a), block has one degree of freedom, because it is confined to move in a straight line and has only one translational degree of freedom.

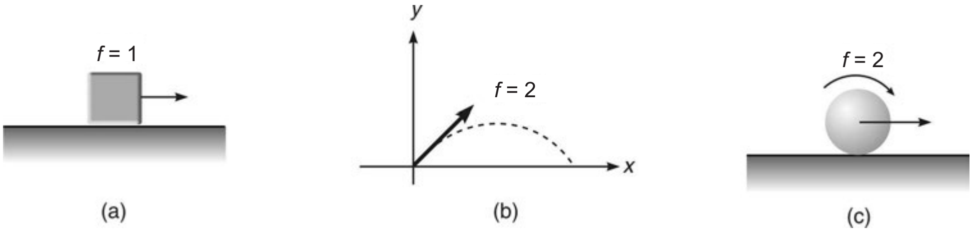


Fig. 20.23

In Fig. (b), the particle has two degrees of freedom because it is confined to move in a plane and so it has two translational degrees of freedom.

In Fig. (c), the sphere has two degrees of freedom one rotational and another translational.

Similarly, a particle free to move in space will have three translational degrees of freedom.

Degree of Freedom of Gas Molecules

A gas molecule can have, the following types of energies

(i) translational kinetic energy
(ii) rotational kinetic energy
(iii) vibrational energy (potential + kinetic)

Vibrational Energy

The forces between different atoms of a gas molecule (interatomic force) may be visualized by imagining every atom as being connected to its neighbours by springs. Each atom can vibrate along the line joining the atoms. Energy associated with this is called vibrational energy.

Degree of Freedom of Monoatomic Gas

A monoatomic gas molecule (like He) consists of a single atom. It can have translational motion in any direction in space. Thus, it has 3 translational degrees of freedom.

$$f = 3 \quad \text{(all translational)}$$

It can also rotate but due to its small moment of inertia, rotational kinetic energy is neglected.

Degree of Freedom of a Diatomic and Linear Polyatomic Gas

The molecules of a diatomic and linear polyatomic gas (like O_2, CO_2 and H_2) cannot only move bodily but also rotate about anyone of the three coordinate axes as shown in figure. However, its moment of inertia about the axis joining the two atoms (x-axis) is negligible. Hence, it can have only two rotational degrees of freedom. Thus, a diatomic molecule has 5 degrees of freedom : 3 translational and 2 rotational. At sufficiently high temperatures, it has vibrational energy as well providing it two more degrees of freedom (one vibrational kinetic energy and another vibrational potential energy). Thus, at high temperatures a diatomic molecule has 7 degrees of freedom, 3 translational, 2 rotational and 2 vibrational. Thus,

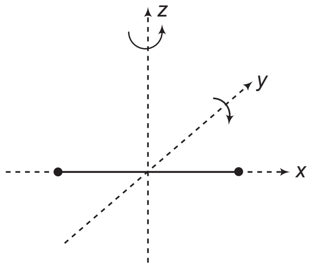


Fig. 20.24

$f = 5$ (3 translational + 2 rotational) at room temperatures and

$f = 7$ (3 translational + 2 rotational + 2 vibrational) at high temperatures.

Degree of Freedom of Non-linear Polyatomic Gas

A non-linear polyatomic molecule (such as NH_3) can rotate about any of three coordinate axes. Hence, it has 6 degrees of freedom 3 translational and 3 rotational. At room temperatures, a polyatomic gas molecule has insignificant vibrational energy. But at high enough temperatures it is also significant. So, it has 8 degrees of freedom 3 rotational, 3 translational and 2 vibrational. Thus,

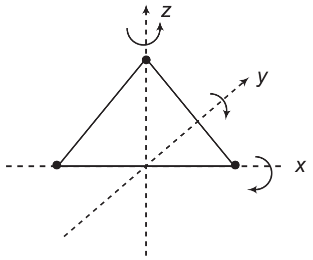


Fig. 20.25

$f = 6$ (3 translational + 3 rotational) at room temperatures and

$f = 8$ (3 translational + 3 rotational + 2 vibrational) at high temperatures.

Degree of Freedom of a Solid

An atom in a solid has no degrees of freedom for translational and rotational motion. At high temperatures, due to vibration along 3 axes it has $3 \times 2 = 6$ degrees of freedom.

$f = 6$ (all vibrational) at high temperatures

Note *(i) Degrees of freedom of a diatomic and polyatomic gas depends on temperature and since there is no clear cut demarcation line above which vibrational energy become significant. Moreover, this temperature varies from gas to gas. On the other hand, for a monoatomic gas there is no such confusion. Degree of freedom here is 3 at all temperatures. Unless and until stated in the question you can take $f = 3$ for monoatomic gas, $f = 5$ for a diatomic gas and $f = 6$ for a non-linear polyatomic gas.*

(ii) When a diatomic or polyatomic gas dissociates into atoms it behaves as a monoatomic gas. Whose degrees of freedom are changed accordingly.

20.7 Internal Energy of an Ideal Gas

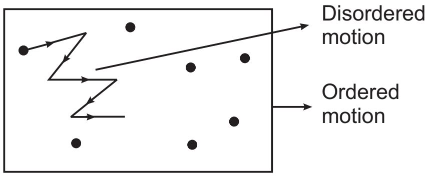


Fig. 20.26

Suppose a gas is contained in a closed vessel as shown in figure. If the container as a whole is moving with some speed, then this motion is called the **ordered motion** of the gas. Source of this motion is some external force. The *zig-zag* motion of gas molecules within the vessel is known as the **disordered motion**. This motion is directly related to the temperature of the gas. As the temperature is increased, the disordered motion of the gas molecules gets fast. The internal energy (U) of the gas is concerned only with its disordered motion. It is in no way concerned with its ordered motion. When the temperature of the gas is increased, its disordered motion and hence its internal energy is increased.

Intermolecular forces in an ideal gas is zero. Thus, PE due to intermolecular forces of an ideal gas is zero. A monoatomic gas is having a single atom. Hence, its vibrational energy is zero. For dia and polyatomic gases vibrational energy is significant only at high temperatures. So, they also have only translational and rotational KE. We may thus conclude that at room temperature the internal energy of an ideal gas (whether it is mono, dia or poly) consists of only translational and rotational KE. Thus,

U (of an ideal gas) $= K_T + K_R$ at room temperatures.

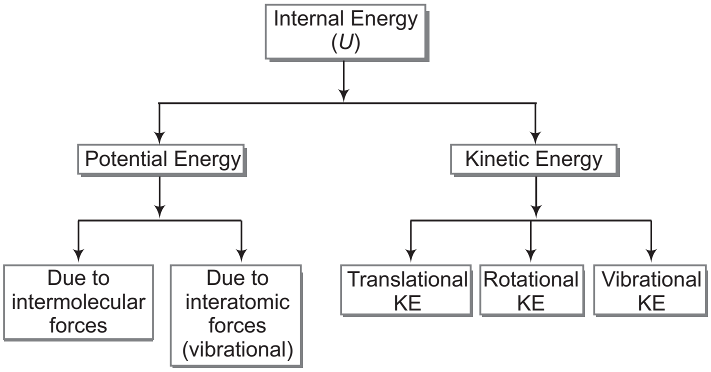


Fig. 20.27

Later in the next article, we will see that K_T (translational KE) and K_R (rotational KE) depends on T only. They are directly proportional to the absolute temperature of the gas. Thus, **internal energy of an ideal gas depends only on its absolute temperature (*T*) and is directly proportional to *T*.**

$$\boxed{U \propto T}$$

20.8 Law of Equipartition of Energy

An ideal gas is just like an ideal father. An ideal father distributes whole of its properties and assets equally among his children. Same is the case with an ideal gas. It distributes its internal energy equally in all degrees of freedom. In each degree of freedom energy of one mole of an ideal gas is

$\frac{1}{2}RT$, where T is the absolute temperature of the gas. Thus, if f be the number of degrees of freedom, the internal energy of 1 mole of the gas will be $\frac{f}{2}RT$ or internal energy of n moles of the gas will be $\frac{n}{2}fRT$. Thus,

$$\boxed{U = \frac{n}{2} fRT} \qquad \ldots\text{(i)}$$

For a monoatomic gas, $f = 3$.

Therefore, $$U = \frac{3}{2}RT \qquad \text{(for 1 mole of a monoatomic gas)}$$

For a dia and linear polyatomic gas at low temperatures, $f = 5$, so,

$$U = \frac{5}{2}RT \qquad \text{(for 1 mole)}$$

and for non-linear polyatomic gas at low temperatures, $f = 6$, so

$$U = \frac{6}{2}RT = 3RT \qquad \text{(for 1 mole)}$$

Note *(i) For one molecule, R (the gas constant) is replaced by k (the Boltzmann constant). For example, internal energy of one molecule in one degree of freedom will be* $\frac{1}{2}kT$.

(ii) Ignoring the vibrational effects we can summarise the above results in tabular form as below.

Table 20.3

Nature of gas	Degree of freedom			Internal energy of 1 mole			Internal energy of 1 molecule		
	Total	Translational	Rotational	Total	Translational	Rotational	Total	Translational	Rotational
Monoatomic	3	3	0	$\frac{3}{2}RT$	$\frac{3}{2}RT$	0	$\frac{3}{2}kT$	$\frac{3}{2}kT$	0
Dia or linear polyatomic	5	3	2	$\frac{5}{2}RT$	$\frac{3}{2}RT$	RT	$\frac{5}{2}kT$	$\frac{3}{2}kT$	kT
Non-linear polyatomic	6	3	3	$3RT$	$\frac{3}{2}RT$	$\frac{3}{2}RT$	$3kT$	$\frac{3}{2}kT$	$\frac{3}{2}kT$

(iii) From the above table, we can see that translational kinetic energy of all types of gases is same. The difference is in rotational kinetic energy.

Example 20.13 *Find total internal energy of 3 moles of hydrogen gas at temperature T.*

Solution Using the relation, $$U = \frac{nf}{2}RT$$

we have, $n = 3$, $f = 5$ for diatomic H_2 gas.

$$\therefore \qquad U = \frac{3 \times 5}{2}RT = 7.5\,RT$$ **Ans.**

Example 20.14 *Ten moles of O_2 gas are kept at temperature T. At some higher temperature 2T, forty percent of molecular oxygen breaks into atomic oxygen. Find change in internal energy of the gas.*

Solution **Initial energy** Using the relation, $U = \frac{nf}{2} RT$

we have, $n = 10$, $f = 5$ for diatomic O_2 gas

$$U_i = \frac{10 \times 5}{2} (RT) = 25\,RT$$

Final energy Forty percent means 4 moles O_2 breaks into O. So, it will become 8 moles of monoatomic gas O. Remaining 6 moles are of diatomic gas O_2. But now the new temperature is $2T$.

$$\therefore \qquad U_f = \frac{8 \times 3}{2} (R)(2T) + \frac{6 \times 5}{2} (R)(2T) = 54\,RT$$

So, change in internal energy, $\Delta U = U_f - U_i = 29\,RT$ **Ans.**

Example 20.15 *At a given temperature internal energy of a monoatomic, diatomic and non-linear polyatomic gas is U_0 each. Find their translational and rotational kinetic energies separately.*

Solution **Monoatomic gas** Translational degree of freedom of monoatomic gas is 3 and rotational degree of freedom is zero. Therefore, whole internal energy is in the form of translational kinetic energy.

$$\therefore \qquad K_T = U_0 \quad \text{and} \quad K_R = 0$$

Diatomic gas Translational degree of freedom of diatomic gas is 3 and rotational degree of freedom is 2. Hence, ratio of translational and rotational kinetic energy will be 3 : 2

$$\therefore \qquad K_T = \frac{3}{5} U_0 \quad \text{and} \quad K_R = \frac{2}{5} U_0$$

Non-linear polyatomic gas In non-linear polyatomic gas translational and rotational both degrees of freedom are 3 each. So, translational and rotational kinetic energy are equal. Hence,

$$K_T = K_R = \frac{U_0}{2}$$

INTRODUCTORY EXERCISE 20.4

1. A gas mixture consists of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting all vibrational modes, the total internal energy of the system is **(JEE 1999)**

   (a) $4\,RT$ (b) $15\,RT$ (c) $9\,RT$ (d) $11RT$

2. The average translational kinetic energy of O_2 (molar mass 32) molecules at a particular temperature is 0.048 eV. The translational kinetic energy of N_2 (molar mass 28) molecules in eV at the same temperature is **(JEE 1997)**

   (a) 0.0015 (b) 0.003 (c) 0.048 (d) 0.768

3. At a given temperature, rotational kinetic energy of diatomic gas is K_0. Find its translational and total kinetic energy.

20.9 Molar Heat Capacity

"Molar heat capacity C is the heat required to raise the temperature of 1 mole of a gas by 1°C (or 1 K)." Thus,

$$\boxed{C = \frac{\Delta Q}{n\Delta T} \quad \text{or} \quad \Delta Q = nC\Delta T}$$

For a gas, the value of C depends on the process through which its temperature is raised.

For example, in an isothermal process $\Delta T` = 0$ or $C_{\text{iso}} = \infty$. In an adiabatic process, (we will discuss it later) $\Delta Q = 0$. Hence, $C_{\text{adi}} = 0$. Thus, molar heat capacity of a gas varies from 0 to ∞ depending on the process. In general, experiments are made either at constant volume or at constant pressure. In case of solids and liquids, due to small thermal expansion, the difference in measured values of molar heat capacities is very small and is usually neglected. However, in case of gases molar heat capacity at constant volume C_V is quite different from that at constant pressure $C_{p.}$ Later in the next chapter, we will derive the following relations, (for an ideal gas)

$$\boxed{\begin{aligned} C_V &= \frac{dU}{dT} = \frac{f}{2}R = \frac{R}{\gamma - 1} \\ C_p &= C_V + R \\ \gamma &= \frac{C_p}{C_V} = 1 + \frac{2}{f} \end{aligned}}$$

Here, U is the internal energy of one mole of the gas. The most general expression for C in the process $pV^x =$ constant is

$$\boxed{C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}} \qquad \text{(we will derive it later)}$$

For example : For isobaric process $p =$ constant or $x = 0$ and

$$C = C_p = \frac{R}{\gamma - 1} + R = C_V + R$$

For an isothermal process, $pV =$ constant or $x = 1$

∴ $C = \infty$ and

For an adiabatic process $pV^{\gamma} =$ constant or $x = \gamma$

∴ $C = 0$

Values of f, U, C_V, C_p and γ for different gases are given in Table 20.4.

Table 20.4

Nature of gas	f	$U = \frac{f}{2}RT$	$C_V = dU/dT = \frac{f}{2}R$	$C_p = C_V + R$	$\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{f}$
Monoatomic	3	$\frac{3}{2}RT$	$\frac{3}{2}R$	$\frac{5}{2}R$	$\frac{5}{3} = 1.67$
Dia and linear polyatomic	5	$\frac{5}{2}RT$	$\frac{5}{2}R$	$\frac{7}{2}R$	$\frac{7}{5} = 1.4$
Non-linear polyatomic	6	$3RT$	$3R$	$4R$	$\frac{4}{3} = 1.33$

Extra Points to Remember

- Mixture of non-reactive gases

(a) $n = n_1 + n_2$

(b) $p = p_1 + p_2$

(c) $U = U_1 + U_2$

(d) $\Delta U = \Delta U_1 + \Delta U_2$

(e) $C_V = \dfrac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2}$

(f) $C_p = \dfrac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2} = C_V + R$

(g) $\gamma = \dfrac{C_p}{C_V}$ or $\dfrac{n}{\gamma - 1} \Rightarrow \dfrac{n_1 + n_2}{\gamma - 1} = \dfrac{n_1}{\gamma_1 - 1} + \dfrac{n_2}{\gamma_2 - 1}$

(h) $M = \dfrac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$

Example 20.16 *Two moles of helium (He) are mixed with four moles of hydrogen* (H_2)*. Find*

(a) C_V *of the mixture*

(b) C_p *of the mixture and*

(c) γ *of the mixture.*

Solution Helium is a monoatomic gas.

$$\therefore \qquad C_V = \frac{3}{2}R,\ C_P = \frac{5}{2}R \text{ and } \gamma = \frac{5}{3}$$

Hydrogen is a diatomic gas.

$$\therefore \qquad C_V = \frac{5}{2}R \text{ and } C_V = \frac{7}{2}R \text{ and } \gamma = \frac{7}{5}$$

(a) **C_V of the mixture**

$$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} = \frac{(2)\left(\frac{3}{2}R\right) + 4\left(\frac{5}{2}R\right)}{2+4}$$

$$= \frac{13}{6}R \qquad \textbf{Ans.}$$

(b) **C_p of the mixture**

$$C_p = C_V + R = \frac{13}{6}R + R$$

$$= \frac{19}{6}R \qquad \textbf{Ans.}$$

Alternate method

$$C_p = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2}$$

$$= \frac{(2)\left(\frac{5}{2}R\right) + (4)\left(\frac{7}{2}R\right)}{2+4}$$

$$= \frac{19}{6}R \qquad \textbf{Ans.}$$

(c) **γ of the mixture**

$$\gamma = \frac{C_p}{C_V} = \frac{(19/6)R}{(13/6)R}$$

$$= \frac{19}{13}$$ **Ans.**

Alternate method

$$\frac{n_1 + n_2}{\gamma - 1} = \frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}$$

$\therefore$ $$\frac{2+4}{\gamma - 1} = \frac{2}{\frac{5}{3} - 1} + \frac{4}{\frac{7}{5} - 1}$$

Solving this equation, we get

$$\gamma = \frac{19}{13}$$

Example 20.17 *Temperature of two moles of a monoatomic gas is increased by 300 K in the process $p \propto V$. Find*

(a) molar heat capacity of the gas in the given process

(b) heat required in the given process.

Solution (a) Molar heat capacity C is given by

$$C = C_V + \frac{R}{1-x} \qquad \text{...(i)}$$

in the process $pV^x =$ constant.

Now, $C_V = \frac{3}{2}R$ for monoatomic gas.

Further, the given process can be written as

$$PV^{-1} = \text{constant}$$

$\therefore$ $$x = -1$$

Substituting the values in Eq. (i), we have

$$C = \frac{3}{2}R + \frac{R}{1+1} = 2R$$ **Ans.**

(b) Q or $\Delta Q = nC\Delta T$

Substituting the values, we have

$$\Delta Q = (2)(2R)(300)$$

$$= 1200R$$ **Ans.**

20.10 Kinetic Theory of Gases

We have studied the mechanics of single particles. When we approach the mechanics associated with the many particles in systems such as gases, liquids and solids, we are faced with analyzing the dynamics of a huge number of particles. The dynamics of such many particle systems is called statistical mechanics.

The game involved in studying a system with a large number of particles is similar to what happens after every physics test. Of course we are interested in our individual marks, but we also want to know the class average.

The kinetic theory that we study in this article is a special aspect of the statistical mechanics of large number of particles. We begin with the simplest model for a monatomic ideal gas, a dilute gas whose particles are single atoms rather than molecules.

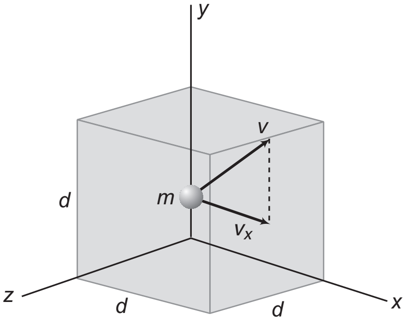
Macroscopic variables of a gas are pressure, volume and temperature and microscopic properties are speed of gas molecules, momentum of molecules, etc. Kinetic theory of gases relates the microscopic properties to macroscopic properties. Furthermore, the kinetic theory provides us with a physical basis for our understanding of the concept of pressure and temperature.

The Ideal Gas Approximation

We make the following assumptions while describing an ideal gas :

1. The number of particles in the gas is very large.
2. The volume V containing the gas is much larger than the total volume actually occupied by the gas particles themselves.
3. The dynamics of the particles is governed by Newton's laws of motion.
4. The particles are equally likely to be moving in any direction.
5. The gas particles interact with each other and with the walls of the container only *via* elastic collisions.
6. The particles of the gas are identical and indistinguishable.

The Pressure of an Ideal Gas



A cubical box with sides of length *d* containing an ideal gas. The molecule shown moves with velocity *v*.

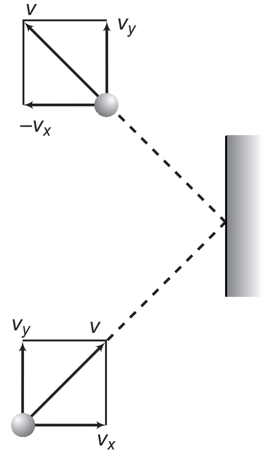
Fig. 20.28

Consider an ideal gas consisting of N molecules in a container of volume V. The container is a cube with edges of length d. Consider the collision of one molecule moving with a velocity **v** toward the right hand face of the cube. The molecule has velocity components v_x, v_y and v_z. Previously, we used m to represent the mass of a sample, but in this article we shall use m to represent the mass of one molecule. As the molecule collides with the wall elastically its x-component of velocity is reversed, while its y and z-components of velocity remain unaltered. Because the x-component of the momentum of the molecule is mv_x before the collision and $-mv_x$ after the collision, the change in momentum of the molecule is

$$\Delta p_x = -mv_x - (mv_x) = -2mv_x$$

Applying impulse = change in momentum to the molecule

$$F\Delta t = \Delta p_x = -2mv_x$$



A molecule makes an elastic collision with the wall of the container. Its *x*-component of momentum is reversed, while its *y*-component remains unchanged. In this construction, we assume that the molecule moves in the *x*-*y* plane.

Fig. 20.29

where, F is the magnitude of the average force exerted by the wall on the molecule in time Δt. For the molecules to collide second time with the same wall, it must travel a distance $2d$ in the x-direction. Therefore, the time interval between two collisions with the same wall is $\Delta t = \dfrac{2d}{v_x}$. Over a time interval that is long compared with Δt, the average force exerted on the molecules for each collision is

$$F = \frac{-2mv_x}{\Delta t} = \frac{-2mv_x}{2d/v_x} = \frac{-mv_x^2}{d}$$

According to Newton's third law, the average force exerted by the molecule on the wall is, $\dfrac{mv_x^2}{d}$.

Each molecule of the gas exerts a force on the wall. We find the total force exerted by all the molecules on the wall by adding the forces exerted by the individual molecules.

$$\therefore \qquad F_{\text{wall}} = \frac{m}{d}(v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2)$$

$$= \frac{mN}{d}\left(\frac{v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2}{N}\right)$$

This can also be written as

$$F_{\text{wall}} = \frac{Nm}{d}\bar{v}_x^2$$

where,
$$\bar{v}_x^2 = \frac{v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2}{N}$$

Since, the velocity has three components v_x, v_y and v_z, we can have

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2 \qquad (\text{as } v^2 = v_x^2 + v_y^2 + v_z^2)$$

Because the motion is completely random, the average values $\bar{v}_x^2$, $\bar{v}_y^2$ and $\bar{v}_z^2$ are equal to each other. So,

$$\bar{v}^2 = 3\bar{v}_x^2 \quad \text{or} \quad \bar{v}_x^2 = \frac{1}{3}\bar{v}^2$$

Therefore,
$$F_{\text{wall}} = \frac{N}{3}\left(\frac{m\bar{v}^2}{d}\right)$$

$\therefore$ Pressure on the wall

$$p = \frac{F_{\text{wall}}}{A} = \frac{F_{\text{wall}}}{d^2} = \frac{1}{3}\left(\frac{N}{d^3}m\bar{v}^2\right)$$

$$= \frac{1}{3}\left(\frac{N}{V}\right)m\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right)$$

$$\therefore \qquad p = \frac{1}{3}\frac{mN}{V}\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right) \qquad \ldots\text{(i)}$$

This result indicates that the pressure is proportional to the number of molecules per unit volume (N/V) and to the average translational kinetic energy of the molecules $\frac{1}{2}m\bar{v}^2$. This result relates the large scale quantity (macroscopic) of pressure to an atomic quantity (microscopic)—the average value of the square of the molecular speed. The above equation verifies some features of pressure with which you are probably familiar. One way to increase the pressure inside a container is to increase the number of molecules per unit volume in the container.

The Meaning of the Absolute Temperature

Rewriting Eq. (i) in the more familiar form

$$pV = \frac{2}{3}N\left(\frac{1}{2}m\bar{v}^2\right)$$

Let us now compare it with the ideal gas equation

$$pV = nRT$$

$$nRT = \frac{2}{3} N \left(\frac{1}{2} m\bar{v}^2 \right)$$

Here, $$n = \frac{N}{N_A} \qquad (N_A = \text{Avogadro number})$$

$\therefore$ $$T = \frac{2}{3} \left(\frac{N_A}{R} \right) \left(\frac{1}{2} m\bar{v}^2 \right)$$

or $$T = \frac{2}{3k} \left(\frac{1}{2} m\bar{v}^2 \right) \qquad \text{...(ii)}$$

where, k is **Boltzmann's constant** which has the value

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K}$$

By rearranging Eq. (ii) we can relate the translational molecular kinetic energy to the temperature

$$\frac{1}{2} mv^2 = \frac{3}{2} kT$$

That is, the average translational kinetic energy per molecule is $\frac{3}{2} kT$. Further,

$$\bar{v}_x^2 = \frac{1}{3} \bar{v}^2, \quad \text{it follows that}$$

$$\frac{1}{2} m\bar{v}_x^2 = \frac{1}{2} kT$$

In the similar manner, it follows that

$$\frac{1}{2} m\bar{v}_y^2 = \frac{1}{2} kT$$

and $$\frac{1}{2} m\bar{v}_z^2 = \frac{1}{2} kT$$

Thus, in each translational degree of freedom one gas molecule has an energy $\frac{1}{2} kT$. One mole of a gas has N_A number of molecules. Thus, one mole of the gas has an energy $\frac{1}{2} (kN_A) T = \frac{1}{2} RT$ in each degree of freedom. Which is nothing but the law of equipartition of energy. The total translational kinetic energy of one mole of an ideal gas is therefore, $\frac{3}{2} RT$.

$$\boxed{(\text{KE})_{\text{Trans}} = \frac{3}{2} RT \quad \text{(of one mole)}}$$

Root Mean Square Speed

The square root of $\overline{\mathbf{v}}^2$ is called the root mean square (rms) speed of the molecules. From Eq. (ii), we obtain, for the rms speed

$$v_{\text{rms}} = \sqrt{\overline{v}^2} = \sqrt{\frac{3kT}{m}}$$

Using $$k = \frac{R}{N_A}, \qquad mN_A = M$$

and $$\frac{RT}{M} = \frac{p}{\rho}$$

we can write,

$$\boxed{v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3p}{\rho}}}$$

Mean Speed or Average Speed

The particles of a gas have a range of speeds. The average speed is found by taking the average of the speeds of all the particles at a given instant. Remember that the speed is a positive scalar since it is the magnitude of the velocity.

$$v_{\text{av}} = \frac{v_1 + v_2 + \ldots + v_N}{N}$$

From Maxwellian speed distribution law, we can show that

$$\boxed{v_{\text{av}} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8p}{\pi\rho}}}$$

Most Probable Speed

This is defined as the speed which is possessed by maximum fraction of total number of molecules of the gas. For example, if speeds of 10 molecules of a gas are, 1, 2, 2, 3, 3, 3, 4, 5, 6, 6 km/s, then the most probable speed is 3 km/s, as maximum fraction of total molecules possess this speed. Again from Maxwellian speed distribution law :

$$\boxed{v_{\text{mp}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2p}{\rho}}}$$

Note 1. *In the above expressions of v_{rms}, v_{av} and v_{mp}, M is the molar mass in kilogram per mole. For example, molar mass of hydrogen is 2×10^{-3} kg/mol.*

2. $v_{rms} > v_{av} > v_{mp}$ *(RAM)*

3. $v_{rms} : v_{av} : v_{mp} = \sqrt{3} : \sqrt{\frac{8}{\pi}} : \sqrt{2}$

and since, $\frac{8}{\pi} \approx 2.5$, *we have* $v_{rms} : v_{av} : v_{mp} = \sqrt{3} : \sqrt{2.5} : \sqrt{2}$

Extra Points to Remember

- Pressure exerted by an ideal gas is numerically equal to two-third of the mean kinetic energy of translation per unit volume of (E) the gas. Thus,

$$\boxed{p = \frac{2}{3}E = \frac{1}{3}\frac{mN}{V}v_{\text{rms}}^2}$$

Here, m = mass of one gas molecule

N = total number of molecules

- **Mean Free Path** Every gas consists of a very large number of molecules. These molecules are in a state of continuous rapid and random motion. They undergo perfectly elastic collisions against one another. Therefore, path of a single gas molecule consists of a series of short *zig-zag* paths of different lengths. The mean free path of a gas molecule is the average distance between two successive collisions. It is represented by λ.

$$\boxed{\lambda = \frac{kT}{\sqrt{2}\pi\sigma^2 p}}$$

Here, σ = diameter of the molecule

k = Boltzmann's constant

- **Avogadro's Hypothesis** At constant temperature and pressure equal volumes of different gases contain equal number of molecules. In 1 g-mole of any gas there are 6.02×10^{23} molecules of that gas. This is called Avogadro's number. Thus,

$$\boxed{N = 6.02 \times 10^{23} \text{ per g-mole}}$$

Therefore, the number of molecules in mass m of the substance

$$\boxed{\text{Number of molecules} = nN = \frac{m}{M} \times N}$$

- **Dalton' Law of Partial Pressure** According to this law if the gases filled in a vessel do not react chemically, then the combined pressure of all the gases is due to the partial pressure of the molecules of the individual gases. If $p_1, p_2, \ldots$ represent the partial pressures of the different gases, then the total pressure is,

$$p = p_1 + p_2 \ldots$$

Example 20.18 *Find the rms speed of hydrogen molecules at room temperature (= 300 K).*

Solution Mass of 1 mole of hydrogen gas

$$= 2\text{g} = 2 \times 10^{-3} \text{ kg}$$

$$\Rightarrow \qquad v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

$$= \sqrt{\frac{3 \times 8.31 \times 300}{2 \times 10^{-3}}}$$

$$= 1.93 \times 10^3 \text{ m/s}$$

Ans.

Example 20.19 *A tank used for filling helium balloons has a volume of 0.3 m^3 and contains 2.0 mol of helium gas at 20.0°C. Assuming that the helium behaves like an ideal gas.*

(a) What is the total translational kinetic energy of the molecules of the gas?

(b) What is the average kinetic energy per molecule?

Solution (a) Using $(\text{KE})_{\text{Trans}} = \frac{3}{2} nRT$

with $n = 2.0$ mol and $T = 273 + 20 = 293$ K, we find that

$$(\text{KE})_{\text{Trans}} = \frac{3}{2}(2.0)(8.31)(293)$$

$$= 7.3 \times 10^3 \text{ J}$$ **Ans.**

(b) The average kinetic energy per molecule is $\frac{3}{2} kT$.

or $$\frac{1}{2} m\bar{v}^2 = \frac{1}{2} m\bar{v}^2_{\text{rms}} = \frac{3}{2} kT$$

$$= \frac{3}{2}(1.38 \times 10^{-23})(293)$$

$$= 6.07 \times 10^{-21} \text{ J}$$ **Ans.**

Example 20.20 *Consider an 1100 particles gas system with speeds distribution as follows :*

1000 particles each with speed 100 m/s

2000 particles each with speed 200 m/s

4000 particles each with speed 300 m/s

3000 particles each with speed 400 m/s and 1000 particles each with speed 500 m/s

Find the average speed, and rms speed.

Solution The average speed is

$$v_{\text{av}} = \frac{(1000)(100) + (2000)(200) + (4000)(300) + (3000)(400) + (1000)(500)}{1100}$$

$$= 309 \text{ m/s}$$ **Ans.**

The rms speed is

$$v_{\text{rms}} = \sqrt{\frac{(1000)(100)^2 + (2000)(200)^2 + (4000)(300)^2 + (3000)(400)^2 + (1000)(500)^2}{1100}}$$

$$= 328 \text{ m/s}$$ **Ans.**

Note *Here,* $\frac{v_{rms}}{v_{av}} \neq \sqrt{\frac{3}{8/\pi}}$ *as values and gas molecules are arbitrarily taken.*

Example 20.21 *Calculate the change in internal energy of 3.0 mol of helium gas when its temperature is increased by 2.0 K.*

Solution Helium is a monatomic gas. Internal energy of n moles of the gas is

$$U = \frac{3}{2} nRT$$

$$\therefore \qquad \Delta U = \frac{3}{2} nR(\Delta T)$$

Substituting the values,

$$\Delta U = \left(\frac{3}{2}\right)(3)(8.31)(2.0)$$

$$= 74.8 \text{ J}$$ **Ans.**

Example 20.22 *In a crude model of a rotating diatomic molecule of chlorine (Cl_2), the two Cl atoms are 2.0×10^{-10} m apart and rotate about their centre of mass with angular speed $\omega = 2.0 \times 10^{12}$ rad/s. What is the rotational kinetic energy of one molecule of Cl_2, which has a molar mass of 70.0 g/mol?*

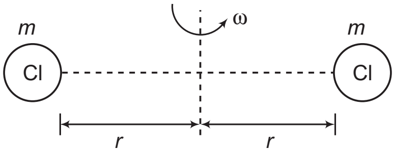


Fig. 20.30

Solution Moment of inertia,

$$I = 2(mr^2) = 2mr^2$$

Here,

$$m = \frac{70 \times 10^{-3}}{2 \times 6.02 \times 10^{23}}$$

$$= 5.81 \times 10^{-26} \text{ kg}$$

and

$$r = \frac{2.0 \times 10^{-10}}{2}$$

$$= 1.0 \times 10^{-10} \text{ m}$$

$$\therefore \qquad I = 2(5.81 \times 10^{-26})(1.0 \times 10^{-10})^2$$

$$= 1.16 \times 10^{-45} \text{ kg-m}^2$$

$$\therefore \qquad K_R = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \times (1.16 \times 10^{-45}) \times (2.0 \times 10^{12})^2$$

$$= 2.32 \times 10^{-21} \text{ J}$$ **Ans.**

Example 20.23 *Prove that the pressure of an ideal gas is numerically equal to two third of the mean translational kinetic energy per unit volume of the gas.*

Solution Translational KE per unit volume

$$E = \frac{1}{2}(\text{mass per unit volume})(\overline{v}^2)$$

$$= \frac{1}{2}(\rho)\left(\frac{3p}{\rho}\right) = \frac{3}{2}p$$

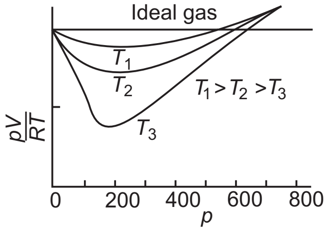
or $$p = \frac{2}{3}E$$ **Hence Proved**.

INTRODUCTORY EXERCISE 20.5

1. Calculate the root mean square speed of hydrogen molecules at 373.15 K.
2. Five gas molecules chosen at random are found to have speed of 500, 600, 700, 800 and 900 m/s. Find the rms speed. Is it the same as the average speed?
3. The average speed of all the molecules in a gas at a given instant is not zero, whereas the average velocity of all the molecules is zero. Explain why?
4. A sample of helium gas is at a temperature of 300 K and a pressure of 0.5 atm. What is the average kinetic energy of a molecule of a gas?
5. A sample of helium and neon gases has a temperature of 300 K and pressure of 1.0 atm. The molar mass of helium is 4.0 g/mol and that of neon is 20.2 g/mol.
   (a) Find the rms speed of the helium atoms and of the neon atoms.
   (b) What is the average kinetic energy per atom of each gas?
6. At what temperature will the particles in a sample of helium gas have an rms speed of 1.0 km/s?
7. For any distribution of speeds $v_{\text{rms}} \geq v_{\text{av}}$. Is this statement true or false?

Final Touch Points

1. **Departures from ideal gas behaviour for a real gas** An ideal gas is a simple theoretical model of a gas. No real gas is truly ideal. Figure shows departures from ideal gas behaviour for a real gas at three different temperatures. Notice that all curves approach the ideal gas behaviour for low pressures and high temperatures.



At low pressures or high temperatures the molecules are far apart and molecular interactions are negligible. Without interactions, the gas behaves like an ideal one .

2. **van der Waal's Equation** Experiments have proved that real gases deviate largely from ideal behaviour. The reason of this deviation is two wrong assumptions in the kinetic theory of gases.
   (i) The size of the molecules is much smaller in comparison to the volume of the gas, hence, it may be neglected.
   (ii) Molecules do not exert intermolecular force on each other.
   van der Waal made corrections for these assumptions and gave a new equation. This equation is known as van der Waal's equation for real gases.
   (a) **Correction for the finite size of molecules:** Molecules occupy some volume. Therefore, the volume in which they perform thermal motion is less than the observed volume of the gas. It is represented by $(V - b)$. Here, b is a constant which depends on the effective size and number of molecules of the gas. Therefore, we should use $(V - b)$ in place of V in gas equation.
   (b) **Correction for intermolecular attraction:** Due to the intermolecular force between gas molecules the molecules which are very near to the wall experiences a net inward force. Due to this inward force there is a decrease in momentum of the particles of a gas. Thus, the pressure exerted by real gas molecules is less than the pressure exerted by the molecules of an ideal gas. So, we use $\left(p + \frac{a}{V^2}\right)$ in place of p in gas equation. Here, again a is a constant.

   van der Waal's equation of state for real gases thus becomes,

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

3. **Critical Temperature, Pressure and Volume :** Gases can't be liquified above a temperature called critical temperature (T_C) however large the pressure may be. The pressure required to liquify the gas at critical temperature is called critical pressure (p_C) and the volume of the gas at critical temperature and pressure is called critical volume (V_C). Value of critical constants in terms of van der Waal's constants a and b are as under :

$$V_C = 3b, \qquad p_C = \frac{a}{27b^2} \qquad \text{and} \qquad T_C = \frac{8a}{27Rb}$$

Further, $\frac{RT_C}{p_C V_C} = \frac{8}{3}$ is called critical coefficient and is same for all gases.

4. Detailed Discussion on Molar Heat Capacity

For monatomic gases value of C_V is $\frac{3}{2}R$. No variation is observed in this. So, value of $C_V, C_p, C_p - C_V$ and γ comes out to be same for different monatomic gases (Table 20.5).

Table 20.5

Gas	C_p	C_V	$C_p - C_V$	$\gamma = C_p/C_V$
Monoatomic Gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Diatomic Gases				
H_2	28.8	20.4	8.33	1.41
N_2	29.1	20.8	8.33	1.40
O_2	29.4	21.1	8.33	1.40
CO_2	29.3	21.0	8.33	1.40
Cl_2	34.7	25.7	8.96	1.35
Polyatomic Gases				
CO_2	37.0	28.5	8.50	1.30
SO_2	40.4	31.4	9.00	1.29
H_2O	35.4	27.0	8.37	1.30
CH_2	35.5	27.1	8.41	1.31

*All values except that for water were obtained at 300 K. SI units are used for C_p and C_V.

For dia and polyatomic gases these values are not equal for different gases. These values vary from gas to gas. Even for one gas values are different at different temperatures.

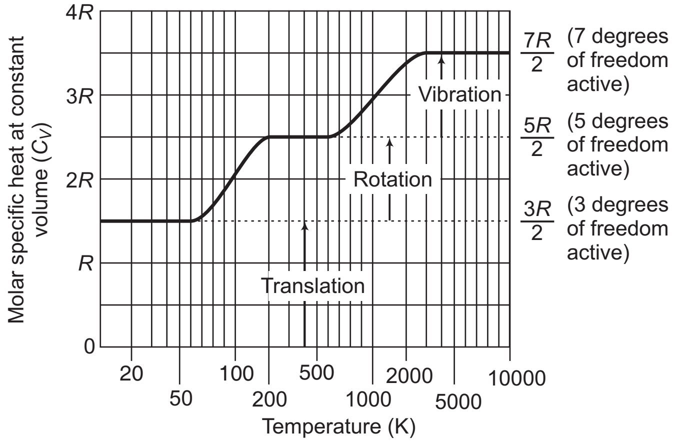
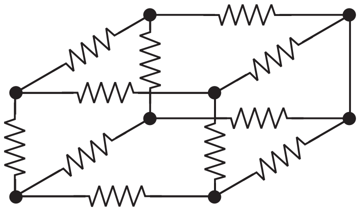


Figure illustrates the variation in the molar specific heat (at constant volume) for H_2 over a wide range in temperatures. (Note that T is drawn on a logarithmic scale). Below about 100 K, C_V is $\frac{3R}{2}$ which is characteristic of three translational degrees of freedom. At room temperature (300 K) it is $\frac{5R}{2}$ which includes the two rotational degrees of freedom. It seems, therefore, that at low temperatures, rotation is not allowed. At high temperatures, C_V starts to rise toward the value $\frac{7R}{2}$. Thus, the vibrational degrees of freedom contribute only at these high temperatures. In Table 20.4 the large values of C_V for some polyatomic molecules show the contributions of vibrational energy. In addition, a molecule with three or more atoms that are not in a straight line has three, not two, rotational degrees of freedom.

From this discussion, we expect heat capacities to be temperature-dependent, generally increasing with increasing temperature.

5. **Solids** In crystalline solids (monoatomic), the atoms are arranged in a three dimensional array, called a lattice. Each atom in a lattice can vibrate along three mutually perpendicular directions, each of which has two degrees of freedom. One corresponding to vibrations KE and the other vibrational PE. Thus, each atom has a total of six degrees of freedom. The volume of a solid does not change significantly with temperature, and so there is little difference between C_V and C_p for a solid. The molar heat capacity is expected to be

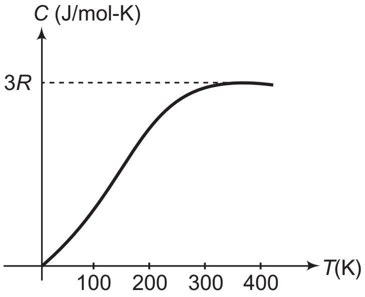


$$C = \frac{f}{2}R = \frac{6}{2}R$$

or $$C = 3R \qquad \text{(ideal monatomic solid)}$$

Its numerical value is $C \approx 25$ J/mol-K ≈ 6 cal/mol-K. This result was first found experimentally by **Dulong and Petit**.

Figure shows that the Dulong and Petit's law is obeyed quite well at high (> 250 K) temperatures. At low temperatures, the heat capacities decreases.



Solved Examples

TYPED PROBLEMS

Type 1. *Conversion of graph*

Concept

(i) In heat, we normally find the following graphs:

S.No	Equation type	Nature of graph
1.	$y \propto x$	Straight line passing through origin
2.	$y \propto \frac{1}{x}$	Rectangular hyperbola
3.	$y =$ constant	Straight line parallel to x-axis
4.	$x =$ constant	Straight line parallel to y-axis
5.	$x =$ constant and $y =$ constant	Dot

(ii) Density of any substance is given by

$$\rho = \frac{m}{V}$$

For given mass of any substance, density only depends on V. It varies with V as

$$\rho \propto \frac{1}{V}$$

If V increases, ρ decreases and *vice-versa*.

(iii) For an ideal gas, density is also given as

$$\rho = \frac{pM}{RT}$$

For a given gas,

$$\rho \propto \frac{p}{T}$$

(iv) For an ideal gas,

$$pV = nRT$$

and

$$U = \frac{nf}{2}RT$$

For given moles of the gas,

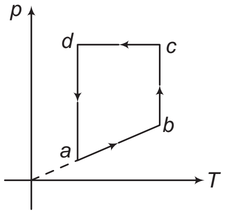
$$U \propto T \quad \text{and} \quad T \propto pV$$

If product of p and V is increasing, it means temperature is increasing. So, U is increasing.

For example, in isothermal process, T is constant.

Therefore, U is constant. Hence, U *versus* T graph is a dot.

Example 1



Corresponding to p-T graph as shown in figure, draw

(a) p-V graph *(b) V-T graph*

(c) ρ-T graph and *(d) U-T graph*

Solution ***ab*** **process** From the given graph, we can see that

$$p \propto T$$

$$\Rightarrow \quad V = \text{constant} \qquad \text{(isochoric)}$$

$$\text{or} \quad \rho = \text{constant} \qquad \left(\text{as } \rho \propto \frac{1}{V}\right)$$

p and T both are increasing.

Therefore, U is also increasing (as $U \propto T$)

Now,

(i) p-V graph is straight line parallel to p-axis as V is constant.

(ii) V-T graph is a straight line parallel to T-axis as V is constant.

(iii) ρ-T graph is a straight line parallel to T-axis as ρ is constant.

(iv) U-T graph is a straight line passing through origin as $U \propto T$.

bc **process** From the given graph, we can see that

$$T = \text{constant} \qquad \text{(isothermal)}$$

$$\Rightarrow \quad U = \text{constant}$$

$$\Rightarrow \quad pV = \text{constant}$$

$$\text{or} \quad p \propto \frac{1}{V}$$

p is increasing. Therefore, V will decrease.

Hence, ρ will increase.

Now,

(i) p-V graph is a rectangular hyperbola $\left(\text{as } P \propto \frac{1}{V}\right)$.

(ii) V-T graph is a straight line parallel to V-axis, as T = constant

(iii) ρ-T graph is a straight line parallel to ρ-axis, as T = constant

(iv) U-T graph is a dot, as U and T both are constants.

cd **process** From the given graph, we can see that

$$p = \text{constant} \qquad \text{(isobaric)}$$

$$\Rightarrow \quad V \propto T$$

Temperature is decreasing. So, volume will also decrease. But density will increase.

Now,

(i) p - V graph is a straight line parallel to V-axis because p is constant.

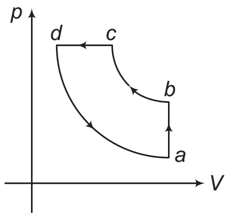
(ii) V - T graph is a straight line passing through origin, as $V \propto T$.

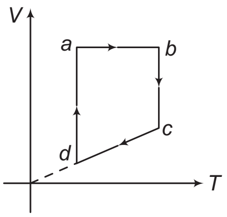
(iii) $\rho = \dfrac{pM}{RT} \Rightarrow \rho \propto \dfrac{1}{T}$ as p, M and R all are constants. Hence, ρ - T graph is rectangular hyperbola.

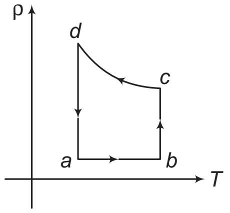
(iv) U - T graph is a straight line passing through origin as $U \propto T$.

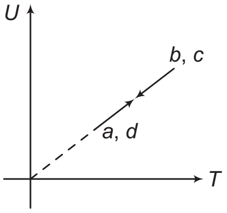
***da* process** This process is just inverse of *bc* process. So, this process will complete the cycle following the steps discussed in process *bc*.

The four graphs are as shown below.









Type 2. *When ordered motion of a gas converts into disordered motion*

Concept

In article 20.7, we have seen that if a container filled with a gas is in motion, then the gas molecules have some ordered motion also. When the container is suddenly stopped, this ordered motion of gas molecules converts into disordered motion. Therefore, internal energy (and hence the temperature) of the gas will increase. In such situation, we can apply the equation,

$$\frac{1}{2}mv^2 = \Delta U$$

where,

$$U = \frac{nf}{2}RT$$

$\therefore$

$$\Delta U = \frac{nf}{2}R\Delta T$$

where,

$$n = \frac{m}{M}$$

Example 2 *An insulated box containing a monoatomic gas of molar mass M moving with a speed v_0 is suddenly stopped. Find the increment in gas temperature as a result of stopping the box.* **(JEE 2003)**

Solution Decrease in kinetic energy = increase in internal energy of the gas

$$\therefore \qquad \frac{1}{2}mv_0^2 = \frac{nf}{2}R\Delta T = \left(\frac{m}{M}\right)\left(\frac{3}{2}R\right)\Delta T$$

$$\therefore \qquad \Delta T = \frac{Mv_0^2}{3R}$$

Type 3. *Based on the concept of pressure exerted by gas molecules striking elastically with walls of a container.*

Concept

(i) Gas molecules are assumed to be moving with v_{rms}.

(ii) They strike with the wall elastically. So, in each collision change in linear momentum is $2mv_{\text{rms}}$.

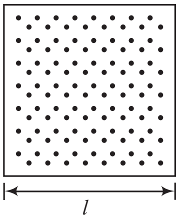
(iii) Time interval between two successive collisions is $\frac{2L}{v_{\text{rms}}}$ if we have a cubical box of side L and,

(iv) $F = \frac{\Delta p}{\Delta t}$, Pressure $= \frac{F}{A}$

Example 3 *A cubical box of side 1 m contains helium gas (atomic weight 4) at a pressure of $100\ N/m^2$. During an observation time of 1 second, an atom travelling with the root-mean-square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. Take $R = \frac{25}{3}$ J/mol-K and $k = 1.38 \times 10^{-23}$ J/K.*

(a) Evaluate the temperature of the gas.
(b) Evaluate the average kinetic energy per atom.
(c) Evaluate the total mass of helium gas in the box.

Solution Volume of the box = 1 m^3, pressure of the gas = 100 N/m^2. Let T be the temperature of the gas.



(a) Time between two consecutive collisions with one wall $= \frac{1}{500}$ s.

This time should be equal to $\frac{2l}{v_{\text{rms}}}$, where l is the side of the cube.

$$\frac{2l}{v_{\text{rms}}} = \frac{1}{500}$$

$$\text{or} \qquad v_{\text{rms}} = 1000 \text{ m/s} \qquad (\text{as } l = 1 \text{ m})$$

$$\therefore \qquad \sqrt{\frac{3RT}{M}} = 1000$$

or $$T = \frac{(1000)^2 M}{3R} = \frac{(10^6)\,(4 \times 10^{-3})}{3\left(\frac{25}{3}\right)}$$

$$= 160 \text{ K}$$ **Ans.**

(b) Average kinetic energy per atom $= \frac{3}{2} kT$

$$= \frac{3}{2}(1.38 \times 10^{-23})\,(160) \text{ J}$$

$$= 3.312 \times 10^{-21} \text{ J}$$ **Ans.**

(c) From $pV = nRT = \frac{m}{M} RT$

We get mass of helium gas in the box,

$$m = \frac{pVM}{RT}$$

Substituting the values, we get

$$m = \frac{(100)\,(1)\,(4 \times 10^{-3})}{\left(\frac{25}{3}\right)(160)}$$

$$= 3.0 \times 10^{-4} \text{ kg}$$ **Ans.**

Example 4 *1 g mole of oxygen at 27°C and 1 atmospheric pressure is enclosed in a vessel.*

(a) Assuming the molecules to be moving with v_{rms}, find the number of collisions per second which the molecules make with one square metre area of the vessel wall.

(b) The vessel is next thermally insulated and moved with a constant speed v_0. It is then suddenly stopped. The process results in a rise of temperature of the gas by 1°C. Calculate the speed v_0. $[k = 1.38 \times 10^{-23}\ J/K$ and $N_A = 6.02 \times 10^{23}/mol]$ **(JEE 2002)**

Solution (a) Mass of one oxygen molecule,

$$m = \frac{M}{N_A}$$

$$= \frac{32}{6.02 \times 10^{23}} \text{ g}$$

$$= 5.316 \times 10^{-23} \text{ g}$$

$$= 5.316 \times 10^{-26} \text{ kg}$$

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

$$= \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{5.316 \times 10^{-26}}}$$

$$= 483.35 \text{ m/s}$$

Change in momentum per collision,

$$\Delta p = mv_{\text{rms}} - (-mv_{\text{rms}}) = 2mv_{\text{rms}}$$

$$= (2)\,(5.316 \times 10^{-26})\,(483.35)$$

$$= 5.14 \times 10^{-23} \text{ kg-m/s}$$

Now, suppose n particles strike per second

$$F = n\Delta p = (n)\,(5.14\times 10^{-23})\ \text{N} \qquad \left(F_{\text{ext}} = \frac{dp}{dt}\right)$$

Now, as $p = \dfrac{F}{A}$, for unit area $F = p$

$\therefore \qquad (n)\,(5.14\times 10^{-23}) = 1.01\times 10^{5}$

or $\qquad n = 1.965\times 10^{27}$ per second **Ans.**

(b) When the vessel is stopped, the ordered motion of the vessel converts into disordered motion and temperature of the gas is increased.

$$\therefore \qquad \frac{1}{2} m v_0^2 = \Delta U \qquad \text{...(i)}$$

$$U = \frac{5}{2} RT \qquad \text{(for 1 g mole of } O_2\text{)}$$

$$\therefore \qquad \Delta U = \frac{5}{2} R\Delta T$$

Here, m is not the mass of one gas molecule but it is the mass of the whole gas.

$$m = \text{ mass of 1 mol} = 32\times 10^{-3}\ \text{kg}$$

Substituting these values in Eq. (i), we get

$$v_0 = \sqrt{\frac{5R\Delta T}{m}} = \sqrt{\frac{5\times 8.31\times 1}{32\times 10^{-3}}} = 36\ \text{m/s}$$ **Ans.**

Miscellaneous Examples

Example 5 *An ideal diatomic gas with $C_V = \dfrac{5R}{2}$ occupies a volume V_i at a pressure p_i. The gas undergoes a process in which the pressure is proportional to the volume. At the end of the process, it is found that the rms speed of the gas molecules has doubled from its initial value. Determine the amount of energy transferred to the gas by heat.*

Solution Given that, $p \propto V$ or $pV^{-1} = \text{constant}$

As we know, molar heat capacity in the process $pV^x = \text{constant}$ is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x} = C_V + \frac{R}{1-x}$$

In the given problem,

$$C_V = \frac{5R}{2} \quad \text{and} \quad x = -1$$

$$\therefore \qquad C = \frac{5R}{2} + \frac{R}{2} = 3R \qquad \text{...(i)}$$

At the end of the process rms speed is doubled, i.e. temperature has become four times ($v_{\text{rms}} \propto \sqrt{T}$).

Now,
$$\begin{aligned}\Delta Q &= nC\Delta T\\ &= nC\,(T_f - T_i)\\ &= nC\,(4T_i - T_i)\\ &= 3T_i nC\\ &= (3T_i)\,(n)\,(3R)\\ &= 9\,(nRT_i)\end{aligned}$$

or
$$\Delta Q = 9P_iV_i$$ **Ans.**

Example 6 *Given, Avogadro's number $N = 6.02 \times 10^{23}$ and Boltzmann's constant $k = 1.38 \times 10^{-23}$ J/K.*

(a) Calculate the average kinetic energy of translation of the molecules of an ideal gas at 0°C and at 100°C.

(b) Also calculate the corresponding energies per mole of the gas.

Solution (a) According to the kinetic theory, the average kinetic energy of translation per molecule of an ideal gas at kelvin temperature T is $\left(\frac{3}{2}\right)kT$, where k is Boltzmann's constant.

At 0° C ($T = 273$ K), the kinetic energy of translation $= \frac{3}{2}kT$

$$= \frac{3}{2} \times (1.38 \times 10^{-23}) \times 273 = 5.65 \times 10^{-21} \text{ J/mole}$$

At 100° C ($T = 373$ K), the energy is

$$\frac{3}{2} \times (1.38 \times 10^{-23}) \times 373 = 7.72 \times 10^{-21} \text{ J/mole}$$

(b) 1 mole of gas contains N ($= 6.02 \times 10^{23}$) molecules. Therefore, at 0° C, the kinetic energy of translation of 1 mole of the gas is

$$= (5.65 \times 10^{-21})\,(6.02 \times 10^{23}) \approx 3401 \text{ J/mol}$$

and at 100° C

kinetic energy of translation of 1 mole of gas is

$$= (7.72 \times 10^{-21})\,(6.02 \times 10^{23}) \approx 4647 \text{ J/mol}$$

Example 7 *An air bubble starts rising from the bottom of a lake. Its diameter is 3.6 mm at the bottom and 4 mm at the surface. The depth of the lake is 250 cm and the temperature at the surface is 40°C. What is the temperature at the bottom of the lake? Given atmospheric pressure = 76 cm of Hg and $g = 980$ cm/s^2.*

Solution At the bottom of the lake, volume of the bubble

$$V_1 = \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi(0.18)^3 \text{ cm}^3$$

Pressure on the bubble p_1 = Atmospheric pressure + Pressure due to a column of 250 cm of water

$$\begin{aligned}&= 76 \times 13.6 \times 980 + 250 \times 1 \times 980\\ &= (76 \times 13.6 + 250)\,980 \text{ dyne/cm}^2\end{aligned}$$

At the surface of the lake, volume of the bubble

$$V_2 = \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi\ (0.2)^3\ \text{cm}^3$$

Pressure on the bubble, p_2 = atmospheric pressure

$$= (76 \times 13.6 \times 980)\ \text{dyne/cm}^2$$

$$T_2 = 273 + 40°\,\text{C}$$
$$= 313°\ \text{K}$$

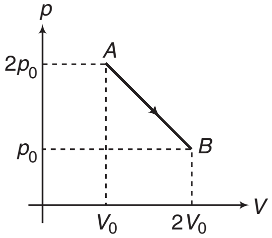
Now
$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

or
$$\frac{(76 \times 13.6 + 250)\,980 \times \left(\frac{4}{3}\right)\pi\,(0.18)^3}{T_1} = \frac{(76 \times 13.6) \times 980 \left(\frac{4}{3}\right)\pi\,(0.2)^3}{313}$$

or
$$T_1 = 283.37\ \text{K}$$

∴
$$T_1 = 283.37 - 273$$
$$= 10.37°\,\text{C}$$

Example 8 *p-V diagram of n moles of an ideal gas is as shown in figure. Find the maximum temperature between A and B.*



How to Proceed For given number of moles of a gas,

$$T \propto pV \qquad (pV = nRT)$$

Although $(pV)_A = (pV)_B = 2p_0V_0$ or $T_A = T_B$, yet it is not an isothermal process. Because in isothermal process *p-V* graph is a rectangular hyperbola while it is a straight line. So, to see the behaviour of temperature, first we will find either *T-V* equation or *T-p* equation and from that equation we can judge how the temperature varies. From the graph, first we will write *p-V* equation, then we will convert it either in *T-V* equation or in *T-p* equation with the help of equation, $pV = nRT$.

Solution From the graph the *p-V* equation can be written as

$$p = -\left(\frac{p_0}{V_0}\right)V + 3p_0 \qquad (y = -mx + c)$$

or
$$pV = -\left(\frac{p_0}{V_0}\right)V^2 + 3p_0V$$

or
$$nRT = 3p_0V - \left(\frac{p_0}{V_0}\right)V^2 \qquad (\text{as } pV = nRT)$$

or
$$T = \frac{1}{nR}\left[3p_0V - \left(\frac{p_0}{V_0}\right)V^2\right]$$

This is the required T-V equation. This is quadratic in V. Hence, T-V graph is a parabola. Now, to find maximum or minimum value of T, we can substitute.

$$\frac{dT}{dV}=0$$

or
$$3p_0-\left(\frac{2p_0}{V_0}\right)V=0 \quad \text{or} \quad V=\frac{3}{2}V_0$$

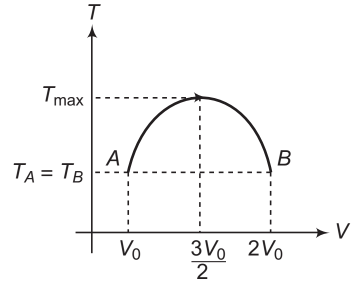
Further $\frac{d^2T}{dV^2}$ is negative at $V=\frac{3}{2}V_0$.

Hence, T is maximum at $V=\frac{3}{2}V_0$ and this maximum value is

$$T_{\max}=\frac{1}{nR}\left[(3p_0)\left(\frac{3V_0}{2}\right)-\left(\frac{p_0}{V_0}\right)\left(\frac{3V_0}{2}\right)^2\right]$$

or
$$T_{\max}=\frac{9p_0V_0}{4nR}$$ **Ans.**

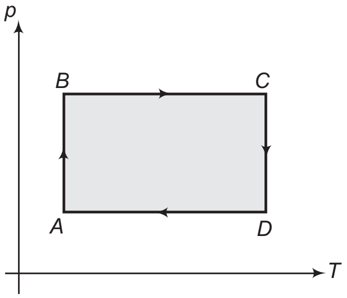
Thus, T-V graph is as shown in figure.



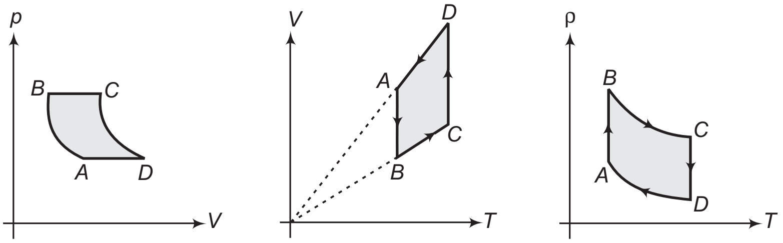
$$T_A=T_B=\frac{2p_0V_0}{nR} \quad \text{and} \quad T_{\max}=\frac{9p_0V_0}{4nR}$$
$$=2.25\,\frac{p_0V_0}{nR}$$

Note *Most of the problems of $T_{\max}$, $p_{\max}$ and $V_{\max}$ are solved by differentiation. Sometimes graph will be given and sometimes direct equation will be given. For $p_{\max}$ you will require either p-V or p-T equation.*

Example 9 *Plot p-V, V-T and ρ-T graph corresponding to the p-T graph for an ideal gas shown in figure.*



Solution ***Process AB*** *is an isothermal process with* T = constant *and* $p_B > p_A$.



***p-V* graph :** $p \propto \frac{1}{V}$ i.e. p-V graph is a rectangular hyperbola with $p_B > p_A$

and $$V_B < V_A.$$

***V-T* graph :** T = constant. Therefore, V-T graph is a straight line parallel to V-axis with $V_B < V_A$.

ρ-*T* graph : $$\rho = \frac{pM}{RT}$$

or $$\rho \propto p$$

As T is constant. Therefore, ρ-T graph is a straight line parallel to ρ-axis with $\rho_B > \rho_A$ as $p_B > p_A$.

Process *BC* is an isobaric process with P = constant

and $$T_C > T_B.$$

***p-V* graph :** As p is constant. Therefore, p-V graph is a straight line parallel to V-axis with $V_C > V_B$ (because $V \propto T$ in an isobaric process)

***V-T* graph :** In isobaric process $V \propto T$, i.e. V-T graph is a straight line passing through origin, with $T_C > T_B$

and $$V_C > V_B.$$

ρ-*T* graph : $\rho \propto \frac{1}{T}$ (when P = constant), i.e. ρ-T graph is a hyperbola with $T_C > T_B$

and $$\rho_C < \rho_B$$

There is no need of discussing C-D and D-A processes. As they are opposite to AB and BC respectively. The corresponding three graphs are shown above.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Straight line on p-T graph for an ideal gas represents isochoric process.

   Reason : If $p \propto T, V =$ constant.

2. **Assertion :** Vibrational kinetic energy is insignificant at low temperatures.

   Reason : Interatomic forces are responsible for vibrational kinetic energy.

3. **Assertion :** In the formula $p = \frac{2}{3}E$, the term E represents translational kinetic energy per unit volume of gas.

   Reason : In case of monoatomic gas, translational kinetic energy and total kinetic energy are equal.

4. **Assertion :** If a gas container is placed in a moving train, the temperature of gas will increase.

   Reason : Kinetic energy of gas molecules will increase.

5. **Assertion :** According to the law of equipartition of energy, internal energy of an ideal gas at a given temperature, is equally distributed in translational and rotational kinetic energies.

   Reason : Rotational kinetic energy of a monoatomic gas is zero

6. **Assertion :** Real gases behave as ideal gases most closely at low pressure and high temperature.

   Reason : Intermolecular force between ideal gas molecules is assumed to be zero.

7. **Assertion :** A glass of water is filled at 4°C. Water will overflow, if temperature is increased or decreased. (Ignore expansion of glass).

   Reason : Density of water is minimum at 4°C.

8. **Assertion :** If pressure of an ideal gas is doubled and volume is halved, then its internal energy will remain unchanged.

   Reason : Internal energy of an ideal gas is a function of only temperature.

9. **Assertion :** In equation $p = \frac{1}{3}\alpha\, v_{\text{rms}}^2$, the term α represents density of gas.

   Reason : $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$.

10. Assertion : In isobaric process, V-T graph is a straight line passing through origin. Slope of this line is directly proportional to mass of the gas. V is taken on y-axis.

Reason : $V = \left(\frac{nR}{p}\right)T$

$\therefore$ slope $\propto n$

or slope $\propto m$

Objective Questions

1. The average velocity of molecules of a gas of molecular weight M at temperature T is

(a) $\sqrt{\frac{3RT}{M}}$ (b) $\sqrt{\frac{8RT}{\pi M}}$

(c) $\sqrt{\frac{2RT}{M}}$ (d) zero

2. Four particles have velocities 1, 0, 2 and 3 m/s. The root mean square velocity of the particles (definition wise) is

(a) 3.5 m/s (b) $\sqrt{3.5}$ m/s

(c) 1.5 m/s (d) $\sqrt{\frac{14}{3}}$ m/s

3. The temperature of an ideal gas is increased from 27°C to 927°C. The rms speed of its molecules becomes

(a) twice (b) half

(c) four times (d) one-fourth

4. In case of hydrogen and oxygen at NTP, which of the following is the same for both?

(a) Average linear momentum per molecule (b) Average KE per molecule

(c) KE per unit volume (d) KE per unit mass

5. The average kinetic energy of the molecules of an ideal gas at 10°C has the value E. The temperature at which the kinetic energy of the same gas becomes $2E$ is

(a) 5°C (b) 10°C

(c) 40°C (d) None of these

6. A polyatomic gas with n degrees of freedom has a mean energy per molecule given by

(a) $\frac{n}{2}RT$ (b) $\frac{1}{2}RT$

(c) $\frac{n}{2}kT$ (d) $\frac{1}{2}kT$

7. In a process, the pressure of a gas remains constant. If the temperature is doubled, then the change in the volume will be

(a) 100% (b) 200%

(c) 50% (d) 25%

8. A steel rod of length 1 m is heated from 25° to 75°C keeping its length constant. The longitudinal strain developed in the rod is (Given, coefficient of linear expansion of steel $= 12 \times 10^{-6}/{}^\circ\text{C}$)

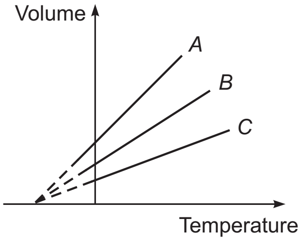
(a) 6×10^{-4} (b) -6×10^{-5}

(c) -6×10^{-4} (d) zero

9. The coefficient of linear expansion of steel and brass are $11 \times 10^{-6}/°C$ and $19 \times 10^{-6}/°C$, respectively. If their difference in lengths at all temperatures has to kept constant at 30 cm, their lengths at 0°C should be

(a) 71.25 cm and 41.25 cm	(b) 82 cm and 52 cm
(c) 92 cm and 62 cm	(d) 62.25 cm and 32.25 cm

10. The expansion of an ideal gas of mass m at a constant pressure p is given by the straight line B. Then, the expansion of the same ideal gas of mass $2m$ at a pressure $2p$ is given by the straight line



(a) C	(b) A
(c) B	(d) data insufficient

Subjective Questions

1. Change each of the given temperatures to the Celsius and Kelvin scales: 68° F, 5° F and 176° F.

2. Change each of the given temperatures to the Fahrenheit and Reaumur scales: 30° C, 5° C and −20° C.

3. At what temperature do the Celsius and Fahrenheit readings have the same numerical value?

4. You work in a materials testing lab and your boss tells you to increase the temperature of a sample by 40.0° C. The only thermometer you can find at your workbench reads in °F. If the initial temperature of the sample is 68.2 °F. What is its temperature in °F, when the desired temperature increase has been achieved?

5. The steam point and the ice point of a mercury thermometer are marked as 80° and 20°. What will be the temperature in centigrade mercury scale when this thermometer reads 32°?

6. A platinum resistance thermometer reads 0° C when its resistance is 80 Ω and 100° C when its resistance is 90 Ω. Find the temperature at which the resistance is 86 Ω.

7. The steam point and the ice point of a mercury thermometer are marked as 80° and 10°. At what temperature on centigrade scale the reading of this thermometer will be 59°?

8. Find the temperature at which oxygen molecules would have the same rms speed as of hydrogen molecules at 300 K.

9. Find the mass (in kilogram) of an ammonia molecule NH_3.

10. Three moles of an ideal gas having $\gamma = 1.67$ are mixed with 2 moles of another ideal gas having $\gamma = 1.4$. Find the equivalent value of γ for the mixture.

11. How many degrees of freedom have the gas molecules, if under standard conditions the gas density is $\rho = 1.3 \text{ kg/m}^3$ and velocity of sound propagation on it is $v = 330$ m/s?

12. 4 g hydrogen is mixed with 11.2 litre of He at STP in a container of volume 20 litre. If the final temperature is 300 K, find the pressure.

13. One mole of an ideal monoatomic gas is taken at a temperature of 300 K. Its volume is doubled keeping its pressure constant. Find the change in internal energy.

14. Two perfect monoatomic gases at absolute temperatures T_1 and T_2 are mixed. There is no loss of energy. Find the temperature of the mixture if the number of moles in the gases are n_1 and n_2.

15. If the water molecules in 1.0 g of water were distributed uniformly over the surface of earth, how many such molecules would there be in 1.0 cm^2 of earth's surface?

16. If the kinetic energy of the molecules in 5 litres of helium at 2 atm is E. What is the kinetic energy of molecules in 15 litres of oxygen at 3 atm in terms of E ?

17. At what temperature is the "effective" speed of gaseous hydrogen molecules (molecular weight = 2) equal to that of oxygen molecules (molecular weight = 32) at 47° C ?

18. At what temperature is v_{rms} of H_2 molecules equal to the escape speed from earth's surface. What is the corresponding temperature for escape of hydrogen from moon's surface? Given $g_m = 1.6 \text{ m/s}^2$, $R_e = 6367$ km and $R_m = 1750$ km.

19. The pressure of the gas in a constant volume gas thermometer is 80 cm of mercury in melting ice. When the bulb is placed in a liquid, the pressure becomes 160 cm of mercury. Find the temperature of the liquid.

20. The resistances of a platinum resistance thermometer at the ice point, the steam point and the boiling point of sulphur are 2.50, 3.50 and 6.50 Ω respectively. Find the boiling point of sulphur on the platinum scale. The ice point and the steam point measure 0° and 100°, respectively.

21. In a constant volume gas thermometer, the pressure of the working gas is measured by the difference in the levels of mercury in the two arms of a *U*-tube connected to the gas at one end . When the bulb is placed at the room temperature 27.0 °C, the mercury column in the arm open to atmosphere stands 5.00 cm above the level of mercury in the other arm. When the bulb is placed in a hot liquid, the difference of mercury levels becomes 45.0 cm. Calculate the temperature of the liquid.(Atmospheric pressure = 75.0 cm of mercury.)

22. A steel wire of 2.0 mm^2cross-section is held straight (but under no tension) by attaching it firmly to two points a distance 1.50 m apart at 30 ° C. If the temperature now decreases to –10° C and if the two points remain fixed, what will be the tension in the wire? For steel, Y = 20,0000 MPa.

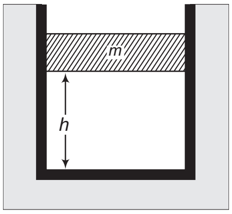
23. A metallic bob weighs 50 g in air. If it is immersed in a liquid at a temperature of 25° C, it weighs 45 g. When the temperature of the liquid is raised to 100° C, it weighs 45.1 g. Calculate the coefficient of cubical expansion of the liquid. Given that coefficient of cubical expansion of the metal is $12 \times 10^{-6}\ ^\circ C^{-1}$.

24. An ideal gas exerts a pressure of 1.52 MPa when its temperature is 298.15 K and its volume is $10^{-2} m^3$. (a) How many moles of gas are there? (b) What is the mass density if the gas is molecular hydrogen? (c) What is the mass density if the gas is oxygen?

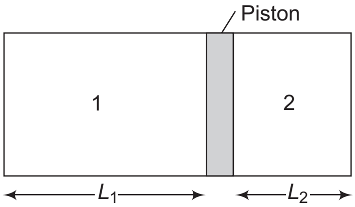
25. A compressor pumps 70 L of air into a 6 L tank with the temperature remaining unchanged. If all the air is originally at 1 atm. What is the final absolute pressure of the air in the tank?

26. A partially inflated balloon contains 500 m^3 of helium at 27° C and 1 atm pressure. What is the volume of the helium at an altitude of 18000 ft, where the pressure is 0.5 atm and the temperature is –3° C?

27. A cylinder whose inside diameter is 4.00 cm contains air compressed by a piston of mass m = 13.0 kg which can slide freely in the cylinder. The entire arrangement is immersed in a water bath whose temperature can be controlled. The system is initially in equilibrium at temperature $t_i = 20°$ C. The initial height of the piston above the bottom of the cylinder is $h_i = 4.00$ cm. The temperature of the water bath is gradually increased to a final temperature $t_f = 100°$ C. Calculate the final height h_f of the piston.



28. The closed cylinder shown in figure has a freely moving piston separating chambers 1 and 2. Chamber 1 contains 25 mg of N_2 gas and chamber 2 contains 40 mg of helium gas. When equilibrium is established what will be the ratio L_1 / L_2? What is the ratio of the number of moles of N_2 to the number of moles of He? (Molecular weights of N_2 and He are 28 and 4).



29. Two gases occupy two containers A and B. The gas in A of volume 0.11 m^3 exerts a pressure of 1.38 MPa. The gas in B of volume 0.16 m^3 exerts a pressure of 0.69 MPa. Two containers are united by a tube of negligible volume and the gases are allowed to intermingle. What is the final pressure in the container if the temperature remains constant ?

30. A glass bulb of volume 400 cm^3 is connected to another of volume 20 cm^3 by means of a tube of negligible volume. The bulbs contain dry air and are both at a common temperature and pressure of 20°C and 1.00 atm. The larger bulb is immersed in steam at 100°C and the smaller in melting ice at 0°C. Find the final common pressure.

31. The condition called standard temperature and pressure (STP) for a gas is defined as temperature of $0°\text{ C} = 273.15$ K and a pressure of $1 \text{ atm} = 1.013 \times 10^5$ Pa. If you want to keep a mole of an ideal gas in your room at STP, how big a container do you need?

32. A large cylindrical tank contains 0.750 m^3 of nitrogen gas at 27° C and 1.50×10^5 Pa (absolute pressure). The tank has a tightfitting piston that allows the volume to be changed. What will be the pressure if the volume is decreased to 0.480 m^3 and the temperature is increased to 157°C.

33. A vessel of volume 5 litres contains 1.4 g of N_2 and 0.4 g of He at 1500 K. If 30% of the nitrogen molecules are dissociated into atoms then find the gas pressure.

34. Temperature of diatomic gas is 300 K. If moment of inertia of its molecules is 8.28×10^{-38} g-cm^2. Calculate their root mean square angular velocity.

35. Find the number of degrees of freedom of molecules in a gas. Whose molar heat capacity

(a) at constant pressure $C_p = 29 \text{ J mol}^{-1}\text{K}^{-1}$

(b) $C = 29 \text{ J mol}^{-1}\text{K}^{-1}$ in the process pT = constant.

36. In a certain gas $\frac{2}{5}$th of the energy of molecules is associated with the rotation of molecules and the rest of it is associated with the motion of the centre of mass.

(a) What is the average translational energy of one such molecule when the temperature is 27°C?

(b) How much energy must be supplied to one mole of this gas at constant volume to raise the temperature by 1°C?

37. A mixture contains 1 mole of helium ($C_p = 2.5\,R, C_V = 1.5\,R$) and 1 mole of hydrogen ($C_p = 3.5R, C_V = 2.5\,R$). Calculate the values of C_p, C_V and γ for the mixture.

38. An ideal gas $(C_p / C_V = \gamma)$ is taken through a process in which the pressure and the volume vary as $P = aV^b$. Find the value of b for which the specific heat capacity in the process is zero.

39. An ideal gas is taken through a process in which the pressure and the volume are changed according to the equation $p = kV$. Show that the molar heat capacity of the gas for the process is given by $C = C_V + \frac{R}{2}$.

40. The pressure of a gas in a 100 mL container is 200 kPa and the average translational kinetic energy of each gas particle is 6×10^{-21} J. Find the number of gas particles in the container. How many moles are there in the container?

41. One gram mole NO_2 at 57° C and 2 atm pressure is kept in a vessel. Assuming the molecules to be moving with rms velocity. Find the number of collisions per second which the molecules make with one square metre area of the vessel wall.

42. A 2.00 mL volume container contains 50 mg of gas at a pressure of 100 kPa. The mass of each gas particle is 8.0×10^{-26} kg. Find the average translational kinetic energy of each particle.

43. Call the rms speed of the molecules in an ideal gas v_0 at temperature T_0 and pressure p_0. Find the speed if (a) the temperature is raised from $T_0 = 293$ K to 573 K (b) the pressure is doubled and $T = T_0$ (c) the molecular weight of each of the gas molecules is tripled.

44. (a) What is the average translational kinetic energy of a molecule of an ideal gas at temperature of 27° C?

(b) What is the total random translational kinetic energy of the molecules in one mole of this gas?

(c) What is the rms speed of oxygen molecules at this temperature?

45. At 0°C and 1.0 atm $(= 1.01 \times 10^5\ \text{N/m}^2)$ pressure the densities of air, oxygen and nitrogen are $1.284\ \text{kg/m}^3$, $1.429\ \text{kg/m}^3$ and $1.251\ \text{kg/m}^3$ respectively. Calculate the percentage of nitrogen in the air from these data, assuming only these two gases to be present.

46. An air bubble of $20\ \text{cm}^3$ volume is at the bottom of a lake 40 m deep where the temperature is 4°C. The bubble rises to the surface which is at a temperature of 20°C. Take the temperature to be the same as that of the surrounding water and find its volume just before it reaches the surface.

47. For a certain gas the heat capacity at constant pressure is greater than that at constant volume by 29.1 J/K.

(a) How many moles of the gas are there?

(b) If the gas is monatomic, what are heat capacities at constant volume and pressure?

(c) If the gas molecules are diatomic which rotate but do not vibrate, what are heat capacities at constant volume and at constant pressure?

48. The heat capacity at constant volume of a sample of a monatomic gas is 35 J/K. Find

(a) the number of moles

(b) the internal energy at 0°C

(c) the molar heat capacity at constant pressure.

LEVEL 2

Single Correct Option

1. Two thermally insulated vessels 1 and 2 are filled with air at temperatures (T_1, T_2), volumes (V_1, V_2) and pressures (p_1, p_2) respectively. If the valve joining the two vessels is opened, the temperature inside the vessel at equilibrium will be (P = common pressure)

(a) $T_1 + T_2$

(b) $(T_1 + T_2)/2$

(c) $\dfrac{T_1T_2p(V_1 + V_2)}{p_1V_1T_2 + p_2V_2T_1}$

(d) $\dfrac{T_1T_2(p_1V_1 + p_2V_2)}{p_1V_1T_2 + p_2V_2T_1}$

2. Two marks on a glass rod 10 cm apart are found to increase their distance by 0.08 mm when the rod is heated from 0°C to 100°C. A flask made of the same glass as that of rod measures a volume of 100 cc at 0°C. The volume it measures at 100°C in cc is

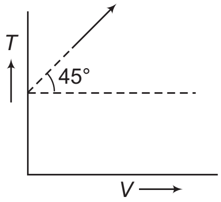
(a) 100.24

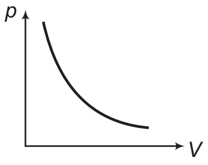
(b) 100.12

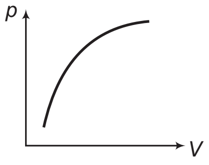
(c) 100.36

(d) 100.48

3. The given curve represents the variation of temperature as a function of volume for one mole of an ideal gas. Which of the following curves best represents the variation of pressure as a function of volume?



(a)


(b)


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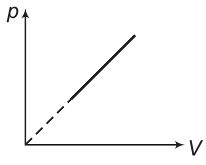

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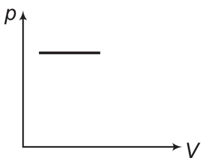


4. A gas is found to be obeyed the law p^2V = constant. The initial temperature and volume are T_0 and V_0. If the gas expands to a volume $3V_0$, then the final temperature becomes

(a) $\sqrt{3}\, T_0$

(b) $\sqrt{2}\, T_0$

(c) $\dfrac{T_0}{\sqrt{3}}$

(d) $\dfrac{T_0}{\sqrt{2}}$

5. Air fills a room in winter at 7°C and in summer at 37°C. If the pressure is the same in winter and summer, the ratio of the weight of the air filled in winter and that in summer is

(a) 2.2

(b) 1.75

(c) 1.1

(d) 3.3

6. Three closed vessels A, B and C are at the same temperature T and contain gases which obey Maxwell distribution law of velocities. Vessel A contains O_2, B only N_2 and C mixture of equal quantities of O_2 and N_2. If the average speed of the O_2 molecules in vessel A is v_1 that of N_2 molecules in vessel B is v_2, then the average speed of the O_2 molecules in vessel C is

(a) $\frac{(v_1 + v_2)}{2}$ (b) v_1

(c) $\sqrt{v_1 v_2}$ (d) None of these

7. In a very good vacuum system in the laboratory, the vacuum attained was 10^{-13} atm. If the temperature of the system was 300 K, the number of molecules present in a volume of 1 cm^3 is

(a) 2.4×10^6 (b) 24

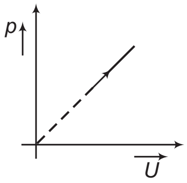
(c) 2.4×10^9 (d) zero

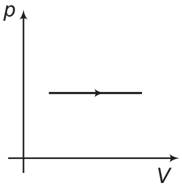
8. If nitrogen gas molecule goes straight up with its rms speed at 0°C from the surface of the earth and there are no collisions with other molecules, then it will rise to an approximate height of

(a) 8 km (b) 12 km

(c) 12 m (d) 8 m

9. The given p-U graph shows the variation of internal energy of an ideal gas with increase in pressure. Which of the following pressure-volume graph is equivalent to this graph?



(a)


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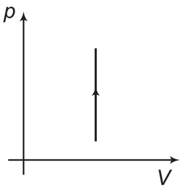

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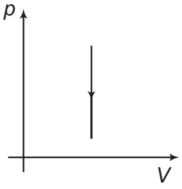
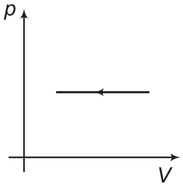


(d)


10. 28 g of N_2 gas is contained in a flask at a pressure of 10 atm and at a temperature of 57°C. It is found that due to leakage in the flask, the pressure is reduced to half and the temperature to 27°C. The quantity of N_2 gas that leaked out is

(a) 11/20 g (b) 20/11 g

(c) 5/63 g (d) 63/5 g

11. A mixture of 4 g of hydrogen and 8 g of helium at NTP has a density about

(a) 0.22 kg/m^3 (b) 0.62 kg/m^3

(c) 1.12 kg/m^3 (d) 0.13 kg/m^3

12. The pressure (p) and the density (ρ) of given mass of a gas expressed by Boyle's law, $p = K\rho$ holds true

(a) for any gas under any condition

(b) for same gas under any condition

(c) only if the temperature is kept constant

(d) None of the above

More than One Correct Options

1. During an experiment, an ideal gas is found to obey a condition $\frac{p^2}{\rho}$ = constant. (ρ = density of the gas). The gas is initially at temperature T, pressure p and density ρ. The gas expands such that density changes to $\rho/2$.

(a) The pressure of the gas changes to $\sqrt{2}\ p$
(b) The temperature of the gas changes to $\sqrt{2}\ T$
(c) The graph of the above process on p-T diagram is parabola
(d) The graph of the above process on p-T diagram is hyperbola

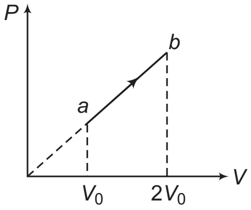
2. During an experiment, an ideal gas is found to obey a condition Vp^2 = constant. The gas is initially at a temperature T, pressure p and volume V. The gas expands to volume $4V$.

(a) The pressure of gas changes to $\frac{p}{2}$
(b) The temperature of gas changes to $4T$
(c) The graph of the above process on p-T diagram is parabola
(d) The graph of the above process on p-T diagram is hyperbola

3. Find the correct options.

(a) Ice point in Fahrenheit scale is 32°F
(b) Ice point in Fahrenheit scale is 98.8°F
(c) Steam point in Fahrenheit scale is 212°F
(d) Steam point in Fahrenheit scale is 252°F

4. In the P-V diagram shown in figure, choose the correct options for the process a-b :

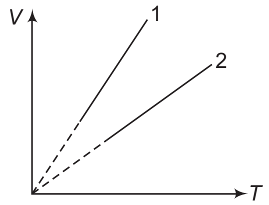


(a) density of gas has reduced to half
(b) temperature of gas has increased to two times
(c) internal energy of gas has increased to four times
(d) T-V graph is a parabola passing through origin

5. Choose the **wrong** options

(a) Translational kinetic energy of all ideal gases at same temperature is same
(b) In one degree of freedom all ideal gases has internal energy = $\frac{1}{2}RT$
(c) Translational degree of freedom of all ideal gases is three
(d) Translational kinetic energy of one mole of all ideal gases is $\frac{3}{2}RT$

6. Along the line-1, mass of gas is m_1 and pressure is p_1. Along the line-2 mass of same gas is m_2 and pressure is p_2. Choose the correct options.



(a) m_1 may be less than m_2
(b) m_2 may be less than m_1
(c) p_1 may be less than p_2
(d) p_2 may be less than p_1

7. Choose the correct options.

(a) In $p = \frac{m}{M} RT$, m is mass of gas per unit volume

(b) In $pV = \frac{m}{M} RT$, m is mass of one molecule of gas

(c) In $p = \frac{1}{3} \frac{mN}{V} v_{\text{rms}}^2$, m is total mass of gas.

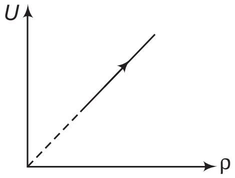
(d) In $v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$, m is mass of one molecule of gas

Match the Columns

1. Match the following two columns for 2 moles of an ideal diatomic gas at room temperature T.

Column I	Column II
(a) Translational kinetic energy	(p) $2RT$
(b) Rotational kinetic energy	(q) $4RT$
(c) Potential energy	(r) $3RT$
(d) Total internal energy	(s) None of these

2. In the graph shown, U is the internal energy of gas and ρ the density. Corresponding to given graph, match the following two columns.



Column I	Column II
(a) Pressure	(p) is constant
(b) Volume	(q) is increasing
(c) Temperature	(r) is decreasing
(d) Ratio T/V	(s) data insufficient

3. At a given temperature T,

$v_1 = \sqrt{\frac{x_1 RT}{M}}$ = rms speed of gas molecules, $v_2 = \sqrt{\frac{x_2 RT}{M}}$ = average speed of gas molecules

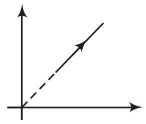
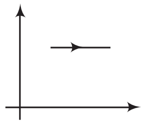
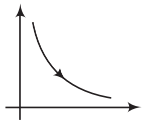
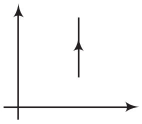
$v_3 = \sqrt{\frac{x_3 RT}{M}}$ = most probable speed of gas molecules, $v_4 = \sqrt{\frac{x_4 RT}{M}}$ = speed of sound

Column I	Column II
(a) x_1	(p) 1.5
(b) x_2	(q) 2.0
(c) x_3	(r) 3.0
(d) x_4	(s) data insufficient

4. With increase in temperature, match the following two columns.

Column I	Column II
(a) Density of water	(p) will increase
(b) Fraction of a solid floating in a liquid	(q) will decrease
(c) Apparent weight of a solid immersed in water	(r) will remain unchanged
(d) Time period of pendulum	(s) may increase or decrease

5. Corresponding to isobaric process, match the following two columns.

Column I	Column II
(a) p-T graph	(p) 
(b) U-ρ graph	(q) 
(c) T-V graph	(r) 
(d) T-ρ graph	(s) 

Note *First physical quantity is along y-axis.*

Subjective Questions

1. Show that the volume thermal expansion coefficient for an ideal gas at constant pressure is $\frac{1}{T}$.

2. The volume of a diatomic gas ($\gamma = 7/5$) is increased two times in a polytropic process with molar heat capacity $C = R$. How many times will the rate of collision of molecules against the wall of the vessel be reduced as a result of this process?

3. A perfectly conducting vessel of volume $V = 0.4\ \text{m}^3$ contains an ideal gas at constant temperature $T = 273\ \text{K}$. A portion of the gas is let out and the pressure of the gas falls by $\Delta p = 0.24$ atm. (Density of the gas at STP is $\rho = 1.2\ \text{kg/m}^3$). Find the mass of the gas which escapes from the vessel.

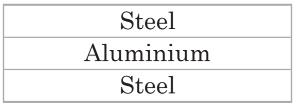
4. A thin-walled cylinder of mass m, height h and cross-sectional area A is filled with a gas and floats on the surface of water. As a result of leakage from the lower part of the cylinder, the depth of its submergence has increased by Δh. Find the initial pressure p_1 of the gas in the cylinder if the atmospheric pressure is p_0 and the temperature remains constant.

5. Find the minimum attainable pressure of an ideal gas in the process $T = T_0 + \alpha V^2$, where T_0 and α are positive constants and V is the volume of one mole of gas.

6. A solid body floats in a liquid at a temperature $t = 50° \text{C}$ being completely submerged in it. What percentage of the volume of the body is submerged in the liquid after it is cooled to $t_0 = 0° \text{C}$, if the coefficient of cubic expansion for the solid is $\gamma_s = 0.3 \times 10^{-5}\ °\text{C}^{-1}$ and of the liquid $\gamma_l = 8 \times 10^{-5}\ °\text{C}^{-1}$.

7. Two vessels connected by a pipe with a sliding plug contain mercury. In one vessel, the height of mercury column is 39.2 cm and its temperature is 0° C, while in the other, the height of mercury column is 40 cm and its temperature is 100° C. Find the coefficient of cubical expansion for mercury. The volume of the connecting pipe should be neglected.

8. Two steel rods and an aluminium rod of equal length l_0 and equal cross-section are joined rigidly at their ends as shown in the figure below. All the rods are in a state of zero tension at 0° C.



Find the length of the system when the temperature is raised to θ. Coefficient of linear expansion of aluminium and steel are α_a and α_s, respectively. Young's modulus of aluminium is Y_a and of steel is Y_s.

9. A metal rod A of 25 cm length expands by 0.050 cm when its temperature is raised from 0° C to 100° C. Another rod B of a different metal of length 40 cm expands by 0.040 cm for the same rise in temperature. A third rod C of 50 cm length is made up of pieces of rods A and B placed end to end expands by 0.03 cm on heating from 0° C to 50° C. Find the lengths of each portion of the composite rod.

Answers

Introductory Exercise 20.1

1. (a) –17.8° C (b) – 459.67°F **2.** (a) 160°C (b) – 24.6°C **3.** 122°F
4. 574.25 **5.** –40°C

Introductory Exercise 20.2

1. Gains, 15.55 s **2.** It will first increase and then decrease **3.** $(\gamma_2 - \gamma_1)\,\Delta T$
4. Cool the system **5.** 50.8°C **6.** (a) 0.064 cm (b) 88.48 cm
7. – 0.042%

Introductory Exercise 20.3

1. $m_1 > m_2$ **2.** 12 atm **3.** 1.5 kg/m^3 **4.** 8×10^{15} **5.** $p_1 > p_2$
6. Straight line passing through origin **7.** A dot

Introductory Exercise 20.4

1. (d) **2.** (c) **3.** $\frac{3}{2}K_0$, $\frac{5}{2}K_0$

Introductory Exercise 20.5

1. 2.15 km/s **2.** $v_{\text{rms}} = 714$ m/s, $v_{\text{av}} = 700$ m/s
3. Speed is a scalar quantity while velocity is a vector quantity.
4. 6.21×10^{-21} J **5.** (a) 1368 m/s, 609 m/s (b) 6.21×10^{-21} J **6.** 160 K
7. True

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (b) **7.** (c) **8.** (b) **9.** (b) **10.** (a)

Objective Questions

1. (b) **2.** (b) **3.** (a) **4.** (b) **5.** (d) **6.** (c) **7.** (a) **8.** (c) **9.** (a) **10.** (c)

Subjective Questions

1. 20°C, –15°C, 80°C, 293K, 258K, 353K **2.** 86°F, 41°F, – 4°F, 546°R, 501°R, 456°R
3. – 40°F = – 40°C **4.** 140.2°F **5.** 20°C **6.** 60° C **7.** 70° C
8. 4800 K **9.** 2.82×10^{-26} kg **10.** 1.53 **11.** 5
12. 3.12×10^5 N/m^2 **13.** 450 R **14.** $T = \dfrac{n_1T_1 + n_2T_2}{n_1 + n_2}$ **15.** 6.5×10^3
16. 7.5 E **17.** –253°C **18.** $T_E = 10059$ K, $T_M = 449$ K **19.** 546.30 K **20.** 400°
21. 177.07°C **22.** 192 N **23.** 3.1×10^{-4} per °C
24. (a) 6.135 mol (b) 1.24 kg/m^3 (c) 19.6 kg/m^3 **25.** 11.7 atm absolute pressure
26. 900 m^3 **27.** 5.09 cm **28.** 0.089, 0.089 **29.** 0.97 MPa

30. 1.13 atm **31.** 22.4 L **32.** 3.36×10^5 Pa **33.** $4.1 \times 10^5 \text{N/m}^2$ **34.** 10^{13} rad/s

35. (a) 5 (b) 3 **36.** (a) 6.21×10^{-21} J (b) 20.8 J **37.** $3R$, $2R$, 1.5

38. $-\gamma$ **40.** 5×10^{21}, 8.3×10^{-3} mol **41.** 3.1×10^{27} **42.** 4.8×10^{-22} J

43. (a) 1.40 v_0 (b) v_0 (c) 0.58 v_0 or $v_0/\sqrt{3}$ **44.** (a) 6.21×10^{-21} J (b) 3740 J (c) 484 m/s

45. 76.5% by mass **46.** 105 cm^3

47. (a) 3.5 mol (b) 43.65 J/K, 72.75 J/K (c) 72.75 J/K, 101.85 J/K

48. (a) 2.81 mol (b) 9.56 kJ (c) 20.8 J/mol-K

LEVEL 2

Single Correct Option

1.(c) **2.**(a) **3.**(a) **4.**(a) **5.**(c) **6.**(b) **7.**(a) **8.**(b) **9.**(b) **10.**(d)

11.(d) **12.**(c)

More than One Correct Options

1. (b,d) **2.** (a,d) **3.** (a,c) **4.** (a,c,d) **5.** (a,b) **6.** (a,b,c,d) **7.** (a,d)

Match the Columns

1. (a) → r (b) → p (c) → s (d) → s

2. (a) → q (b) → r (c) → q (d) → q

3. (a) → r (b) → s (c) → q (d) → s

4. (a) → s (b) → s (c) → s (d) → p

5. (a) → q (b) → r (c) → p (d) → r

Subjective Questions

2. $(2)^{4/3}$ times **3.** 115.2 g **4.** $p_1 = \left(p_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right)$

5. $2R\sqrt{\alpha T_0}$ **6.** 99.99% **7.** 2.0×10^{-4} per °C

8. $l_0\left[1+\left(\frac{\alpha_a Y_a + 2\alpha_s Y_s}{Y_a + 2Y_s}\right)\theta\right]$ **9.** 10 cm, 40 cm

21

Laws of Thermodynamics

Chapter Contents

21.1 The First Law of Thermodynamics

21.2 Further Explanation of Three Terms Used in First Law

21.3 Different Thermodynamic Processes

21.4 Heat Engine and its Efficiency

21.5 Refrigerator

21.6 Zeroth Law of Thermodynamics

21.7 Second Law of Thermodynamics

21.1 The First Law of Thermodynamics

The first law of thermodynamics is basically law of conservation of energy. This law can be applied for any type of system like solid, liquid and gas. But, in most of the cases the system will be an ideal gas. A process in which there are changes in the state of a thermodynamic system (like p, V, T, U and ρ etc.) is called a **thermodynamic process.**

We now come to the first law

Suppose Q heat is given to a system, then part of it is used by the system in doing work W against the surroundings (like atmosphere) and part is used by the system in increasing its internal energy ΔU. Thus,

$$\boxed{Q = W + \Delta U} \qquad \ldots\text{(i)}$$

Let us take a real life situation similar to first law.

Consider a person X. Suppose his monthly income is Rs. 50,000 (Q). He spends Rs. 30,000 (W) as his monthly expenditure. Then, obviously the remaining Rs. 20,000 goes to his savings (ΔU). In some month it is also possible that he spends more than his income. In that case he will withdraw it from his bank and his savings will get reduced ($\Delta U < 0$). In the similar manner, other combinations can be made.

Sign Convention

(i) ***Q*** If heat is given to the system, then Q is positive and if heat is taken from the system, then it is negative.

(ii) ***W*** Work done used in Eq. (i) is the work done by the system (not work done on the system). This work done is positive if volume of the system increases. Sign of work done in different situations is given in tabular form as below

Table 21.1

S.No	Volume of the system	Work done by the system used in Eq. (i)	Work done on the system
1.	is increasing	positive	negative
2.	is decreasing	negative	positive
3.	is constant	zero	zero

(iii) Δ***U*** Internal energy of a system is due to disordered motion of its constituent particles. It mainly depends on state (solid, liquid and gas) and temperature. For example, two different states at same temperature will have different internal energy. Ice at 0°C and water at 0°C have different energies. Similarly, same state at different temperatures will have different energies. For example, internal energy of an ideal gas is given by

$$U = \frac{nf}{2} RT$$

or

$$U \propto T$$

If temperature increases, internal energy also increases and $\Delta U = U_f - U_i$ will be positive.

Extra Points to Remember

- If a thermodynamic system changes from an initial equilibrium state A to final equilibrium state B through three different paths 1, 2 and 3, then Q and W will be different along these three paths. But $Q - W$ or ΔU will be same along all three paths. This is because, Q and W are path functions. But U is a state function. Thus,

$$\boxed{Q_1 - W_1 = Q_2 - W_2 = Q_3 - W_3 = \Delta U}$$

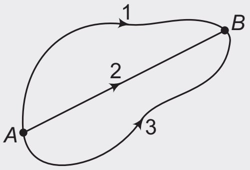


Fig. 21.1

In a closed path, $\Delta U = 0$

Example 21.1 *When a system goes from state A to state B, it is supplied with 400 J of heat and it does 100 J of work.*

(a) For this transition, what is the system's change in internal energy?

(b) If the system moves from B to A, what is the change in internal energy?

(c) If in moving from A to B along a different path in which $W'_{AB} = 400$ J *of work is done on the system, how much heat does it absorb?*

Solution (a) From the first law,

$$\Delta U_{AB} = Q_{AB} - W_{AB} = (400 - 100)\text{ J} = 300\text{ J}$$

(b) Consider a closed path that passes through the state A and B. Internal energy is a state function so ΔU is zero for a closed path.

Thus, $\Delta U = \Delta U_{AB} + \Delta U_{BA} = 0$ or $\Delta U_{BA} = - \Delta U_{AB} = - 300$ J

(c) The change in internal energy is the same for any path, so

$$\Delta U_{AB} = \Delta U'_{AB} = Q'_{AB} - W'_{AB}$$

$$300\text{ J} = Q'_{AB} - (-400\text{ J})$$

and the heat exchanged is $Q'_{AB} = - 100$ J

The negative sign indicates that the system loses heat in this transition.

INTRODUCTORY EXERCISE 21.1

1. The quantities in the following table represent four different paths for the same initial and final states. Find *a, b, c, d, e, f* and *g.*

Table 21.2

Q(J)	W(J)	ΔU(J)
–80	–120	*d*
90	*c*	*e*
a	40	*f*
b	–40	*g*

2. In a certain chemical process, a lab technician supplies 254 J of heat to a system. At the same time, 73 J of work are done on the system by its surroundings. What is the increase in the internal energy of the system?

21.2 Further Explanation of Three Terms Used in First Law

First law of thermodynamics basically revolves round the three terms $Q, \Delta U$ and W. If you substitute these three terms correctly with proper signs in the equation $Q = \Delta U + W$, then you are able to solve most of the problems of first law. Let us take each term one by one. Here, we are taking the system an ideal gas.

(i) **Heat Transfer (Q or ΔQ)** There are two methods of finding Q or ΔQ.

***Method* 1.**
$$\boxed{Q = nC\Delta T}$$

or
$$\boxed{\Delta Q = nC\Delta T}$$

where, C is the molar heat capacity of the gas and n is the number of moles of the gas. Always take,

$$\boxed{\Delta T = T_f - T_i}$$

where, T_f is the final temperature and T_i the initial temperature of the gas. Further, we have discussed in chapter 20, that molar heat capacity of an ideal gas in the process $pV^x =$ constant is

$$\boxed{C = \frac{R}{\gamma - 1} + \frac{R}{1-x} = C_V + \frac{R}{1-x}}$$

$C = C_V = \dfrac{R}{\gamma - 1}$ in isochoric process and

$C = C_p = C_V + R$ in isobaric process

Mostly C_p and C_V are used.

Note *For finding C, nature of gas and process should be known. Nature of gas will give us C_V and process the value of x.*

***Method* 2.** We can also find Q by finding ΔU and W by the equation

$$Q = W + \Delta U$$

(ii) **Change in internal energy (ΔU)** There are two methods of finding ΔU.

***Method* 1.** For change in internal energy of the gas

$$\Delta U = nC_V \Delta T$$

Students are often confused that the result $\Delta U = nC_V \Delta T$ can be applied only in case of an isochoric process (as C_V is here used). However, it is not so. It can be applied in any process, whether it is isobaric, isothermal, adiabatic or else.

Note *In the above expression, value of C_V depends on the nature of gas.*

***Method* 2.** We can also find ΔU from the basic equation of first law of thermodynamics,

$$\Delta U = Q - W$$

(iii) **Work done (W)** This is the most important of the three.

Work Done During Volume Changes

A gas in a cylinder with a movable piston is a simple example of a thermodynamic system.

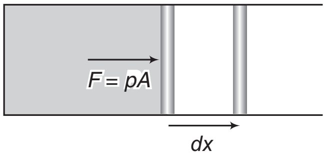


Fig. 21.2

Figure shows a gas confined to a cylinder that has a movable piston at one end. If the gas expands against the piston, it exerts a force and does work on the piston. If the piston compresses the gas as it is moved inward, work is done on the gas. The work associated with such volume changes can be determined as follows.

Let the gas pressure on the piston face be p. Then, the force on the piston due to the gas is pA, where A is the area of the face.

When the piston is pushed outward an infinitesimal distance dx, the work done by the gas is

$$dW = Fdx = pA\,dx$$

which, since the change in volume of the gas is $dV = Adx$, becomes

$$\boxed{dW = pdV}$$

For a finite change in volume from V_i to V_f, this equation is then integrated between V_i to V_f to find the net work

$$\boxed{W = \int dW = \int_{V_i}^{V_f} pdV}$$

Now, there are five methods of finding work done by a gas.

***Method* 1.** This is used when p - V equation is known to us. Suppose p as a function of V is known to us.

$$p = f(V)$$

then work done can be found by

$$W = \int_{V_i}^{V_f} f(V)\,dV$$

***Method* 2.** The work done by a gas is also equal to the area under p - V graph. Following different cases are possible :

Case 1 When volume is constant

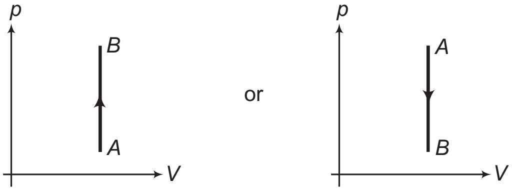


Fig. 21.3

$$V = \text{constant}$$

$$\therefore \qquad W_{AB} = 0$$

Case 2 When volume is increasing

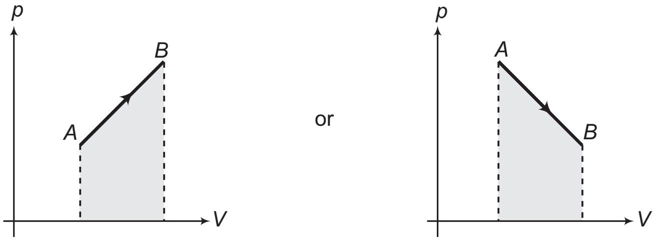


Fig. 21.4

$$V \text{ is increasing}$$

$$\therefore \qquad W_{AB} > 0$$

$$W_{AB} = \text{Shaded area}$$

Case 3 When volume is decreasing

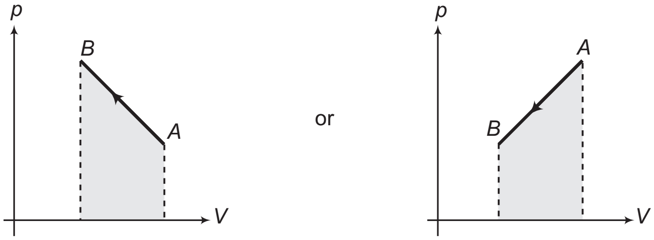


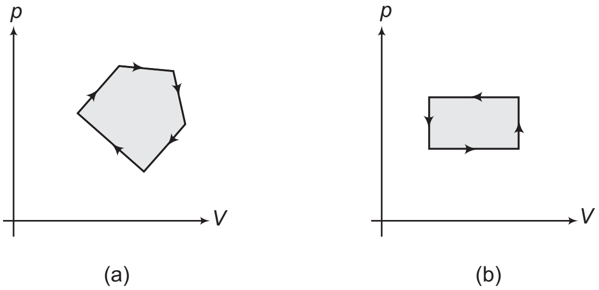
Fig. 21.5

$$V \text{ is decreasing}$$

$$\therefore \qquad W_{AB} < 0$$

$$W_{AB} = -\text{ Shaded area}$$

Case 4 Cyclic process



(a) (b)

Fig. 21.6

$$W_{\text{clockwise cycle}} = +\text{Shaded area} \qquad \text{[in figure (a)]}$$

$$W_{\text{anticlockwise cycle}} = -\text{ Shaded area} \qquad \text{[in figure (b)]}$$

Case 5 Incomplete cycle

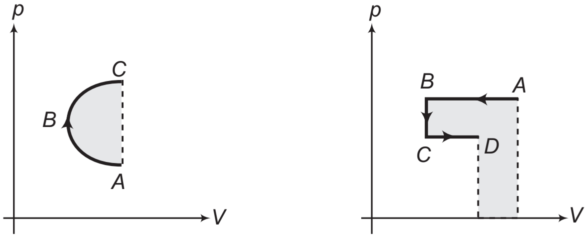


Fig. 21.7

$$W_{ABC} = +\text{Shaded area}$$
$$W_{ABCD} = -\text{Shaded area.}$$

***Method* 3.** Sometimes work done by the gas is also obtained by finding the forces against which work is done by the gas.

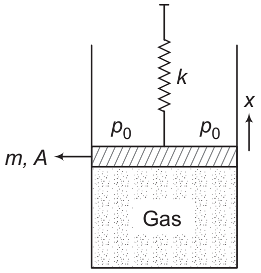


Fig. 21.8

For example, in the figure shown, work is done by the gas against the following forces (when the piston is displaced upwards)

(i) Against gravity force mg. Since, mg is a constant force.

$\therefore \quad W_1 = \text{Force} \times \text{displacement} = mgx$

(ii) Against the force $p_0 A$. Further, this is also a constant force.

$\therefore \quad W_2 = \text{Force} \times \text{displacement} = p_0 Ax$

But, $\quad Ax = \Delta V$

$\therefore \quad W_2 = p_0 \Delta V$

(iii) Against spring force kx. This is a variable force.

Hence,

$$W_3 = \int_0^x (kx)\,dx = \frac{1}{2}kx^2$$

Note *While calculating W_3, we have assumed that initially spring is in its natural length.*

***Method* 4.** In some cases, one process and limits of temperature (or temperature change) is given. In those cases, with the help of given process and the standard ideal gas equation $pV = nRT$, first we convert pdV into $f(T)dT$ and then integrate this expression with the limits of temperature. Thus,

$$W = \int pdV = \int_{T_i}^{T_f} f(T)\,dT$$

***Method* 5.** The last method of finding work done is from the fundamental equation of first law.

or
$$W = Q - \Delta U$$

Now, let us take example of each one of them.

Example 21.2 *Method 1 of Q*

Temperature of two moles of a monoatomic gas is increased by 300 K in the process $p \propto V$.

(a) Find molar heat capacity of the gas in the given process.

(b) Find heat given to the gas in that.

Solution (a) $p \propto V \quad \Rightarrow \quad pV^{-1} = \text{constant}$

If we compare with pV^x = constant, then

$$x = -1$$

Now,

$$C = C_V + \frac{R}{1-x}$$

$$C_V = \frac{3}{2}R \text{ for a monoatomic gas}$$

$$\therefore \qquad C = \frac{3}{2}R + \frac{R}{1-(-1)} = 2R$$ **Ans.**

(b) $Q = nC\Delta T$

Substituting the values, we get

$$Q = (2)(2R)(300)$$
$$= 1200R$$ **Ans.**

Example 21.3 *Method 2 of Q*

In a given process work done on a gas is 40 J and increase in its internal energy is 10 J. Find heat given or taken to/from the gas in this process.

Solution Given, $\Delta U = +10$ J

Work done on the gas is 40 J. Therefore, work done by the gas used in the equation, $Q = W + \Delta U$ will be –40 J. Now, putting the values in the equation,

$$Q = W + \Delta U$$

We have,

$$Q = -40 + 10$$
$$= -30 \text{ J}$$ **Ans.**

Here, negative sign indicates that heat is taken out from the gas.

Example 21.4 *Method 1 of ΔU*

Temperature of two moles of a monoatomic gas is increased by 600 K in a given process. Find change in internal energy of the gas.

Solution Using the equation,

$$\Delta U = nC_V \Delta T \qquad \text{for change in internal energy}$$

$$C_V = \frac{3}{2}R \qquad \text{for monoatomic gas}$$

$$\therefore \quad \Delta U = (2)\left(\frac{3}{2}R\right)(600)$$

$$= 1800\,R$$ **Ans.**

Example 21.5 *Method 2 of ΔU*

Work done by a gas in a given process is –20 J. Heat given to the gas is 60 J. Find change in internal energy of the gas.

Solution $\Delta U = Q - W$

Substituting the values we have,

$$\Delta U = 60 - (-20)$$

$$= 80\text{ J}$$ **Ans.**

ΔU is positive. Hence, internal energy of the gas is increasing.

Example 21.6 *Method 1 of W*

By integration, make expressions of work done by gas in

(a) Isobaric process (p = constant)

(b) Isothermal process (pV = constant)

(c) Adiabatic process (pV^{γ} = constant)

Solution (a) **Isobaric process**

$$W = \int_{V_i}^{V_f} p\,dV = p\int_{V_i}^{V_f} dV \qquad (\text{as } p = \text{constant})$$

$$= p\,[V]_{V_i}^{V_f} = p\,(V_f - V_i)$$

$$= p\Delta V$$ **Ans.**

Note *Any process freely taking place in atmosphere is considered isobaric. For example, melting of ice, boiling of water etc. Here, the constant pressure is p_0. Therefore,*

$$W = p_0\Delta V = p_0(V_f - V_i)$$

(b) **Isothermal process**

$$W = \int_{V_i}^{V_f} p\,dV = \int_{V_i}^{V_f}\left(\frac{nRT}{V}\right)dV \qquad \left(\text{as } p = \frac{nRT}{V}\right)$$

$$= nRT\int_{V_i}^{V_f}\frac{dV}{V} \qquad (\text{as } T = \text{constant})$$

$$= nRT\ln\left(\frac{V_f}{V_i}\right) = nRT\ln\left(\frac{p_i}{p_f}\right) \qquad \left(\text{as } p_iV_i = p_fV_f \text{ So, } \frac{V_f}{V_i} = \frac{p_i}{p_f}\right)$$

(c) **Adiabatic process**

$$pV^{\gamma} = \text{constant} = k \text{ (say)} = p_iV_i^{\gamma} = p_fV_f^{\gamma}$$

Further,
$$p = \frac{k}{V^{\gamma}} = kV^{-\gamma}$$

$$W = \int_{V_i}^{V_f} p\,dV = \int_{V_i}^{V_f} kV^{-\gamma}\,dV = \left[\frac{kV^{-\gamma+1}}{-\gamma+1}\right]_{V_i}^{V_f}$$

$$= \frac{kV_f^{-\gamma+1} - kV_i^{-\gamma+1}}{-\gamma+1} = \frac{p_fV_f^{\gamma}V_f^{-\gamma+1} - p_iV_i^{\gamma}V_i^{-\gamma+1}}{1-\gamma}$$

$$= \frac{p_fV_f - p_iV_i}{1-\gamma} = \frac{nRT_f - nRT_i}{1-\gamma} = \frac{nR\Delta T}{1-\gamma}$$

Example 21.7 Method 2 of W

In the given p-V diagram, find

(a) pressures at c and d

(b) work done in different processes separately

(c) work done in complete cycle abcd.

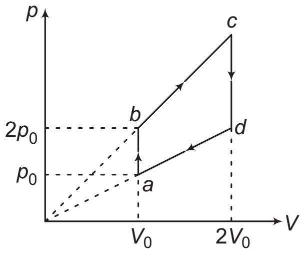


Fig. 21.9

Solution (a) Line bc is passing through origin. Hence,

$$p \propto V$$

From b to c, volume is doubled. Hence, pressure is also doubled.

$\therefore$
$$p_c = 2p_b = 4p_0$$ **Ans.**

Similarly, line ad is also passing through origin.

$\therefore$
$$p_d = 2p_a = 2p_0$$ **Ans.**

(b) ab and cd processes are isochoric ($V =$ constant). Hence,

$$W_{ab} = W_{cd} = 0$$

$$W_{bc} = \text{area under the line } bc$$
$$= \text{area of trapezium}$$
$$= \frac{1}{2}(p_b + p_c)(2V_0 - V_0)$$
$$= \frac{1}{2}(2p_0 + 4p_0)(V_0)$$
$$= 3\,p_0V_0$$ **Ans.**

Since, volume is increasing. Therefore, W_{bc} is positive.

$$W_{da} = -\text{area under the line } da$$
$$= -\text{area of trapezium}$$
$$= -\frac{1}{2}(p_a + p_d)(2V_0 - V_0)$$

$$= -\frac{1}{2}(p_0 + 2p_0)(V_0)$$

$$= -1.5\, p_0 V_0$$ **Ans.**

Volume is decreasing. Therefore, work done is negative.

(d) Work done in complete cycle

$$W_{\text{net}} = W_{ab} + W_{bc} + W_{cd} + W_{da}$$
$$= 0 + 3\, p_0 V_0 + 0 - 1.5\, p_0 V_0$$
$$= 1.5\, p_0 V_0$$ **Ans.**

Note *Net work done is also equal to the area between the cycle. Since, cycle is clockwise. Hence, net work done is positive.*

Example 21.8 *Method 3 of W*

Mass of a piston shown in Fig. 21.10 is m and area of cross-section is A. Initially spring is in its natural length. Find work done by the gas.

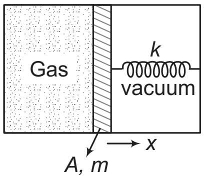


Fig. 21.10

Solution In the given condition, work is done by the gas only against spring force kx. This force is a variable force. Hence,

$$W = \int_0^x (kx)\, dx = \frac{1}{2}kx^2$$

Example 21.9 *Method 4 of W*

The temperature of n-moles of an ideal gas is increased from T_0 to $2T_0$ through a process $p = \frac{\alpha}{T}$. Find work done in this process.

Solution $$pV = nRT \quad \text{(ideal gas equation)} \qquad \text{...(i)}$$

and $$p = \frac{\alpha}{T} \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$V = \frac{nRT^2}{\alpha} \quad \text{or} \quad dV = \frac{2nRT}{\alpha}\, dT$$

$$\therefore \quad W = \int_{V_i}^{V_f} p\, dV = \int_{T_0}^{2T_0} \left(\frac{\alpha}{T}\right)\left(\frac{2nRT}{\alpha}\right) dT$$

$$= 2nRT_0$$ **Ans.**

Example 21.10 *Method 5 of W*

Heat taken from a gas in a process is 80 J and increase in internal energy of the gas is 20 J. Find work done by the gas in the given process.

Solution Heat is taken from the gas.

Therefore, Q is negative. Or, $Q = -80$ J

Internal energy of the gas is increasing.

Therefore, ΔU is positive. Or $\qquad \Delta U = +20\text{ J}$

Using the first law equation,

$$Q = W + \Delta U \quad \text{or} \quad W = Q - \Delta U = -80 - 20 = -100\text{ J}$$ **Ans.**

Here, negative sign indicates that volume of the gas is decreasing and work is done on the gas.

INTRODUCTORY EXERCISE 21.2

1. A gas in a cylinder is held at a constant pressure of 1.7×10^5 Pa and is cooled and compressed from 1.20 m^3 to 0.8 m^3. The internal energy of the gas decreases by 1.1×10^5 J.
  (a) Find the work done by the gas.
  (b) Find the magnitude of the heat flow into or out of the gas and state the direction of heat flow.
  (c) Does it matter whether or not the gas is ideal?

2. A thermodynamic system undergoes a cyclic process as shown in figure.

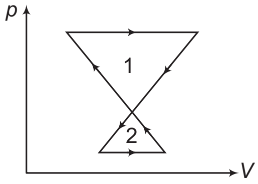


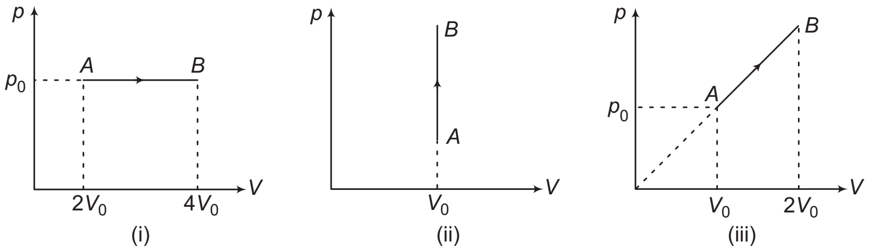
Fig. 21.12

  (a) over one complete cycle, does the system do positive or negative work.
  (b) over one complete cycle, does heat flow into or out of the system.
  (c) In each of the loops 1 and 2, does heat flow into or out of the system.

3. How many moles of helium at temperature 300 K and 1.00 atm pressure are needed to make the internal energy of the gas 100 J?

4. Temperature of four moles of a monoatomic gas is increased by 300 K in isochoric process. Find W, Q and ΔU.

5. Find work done by the gas in the process AB shown in the following figures.



(i) (ii) (iii)

Fig. 21.12

6. Temperature of two moles of an ideal gas is increased by 300 K in a process $V = \frac{a}{T}$, where a is positive constant. Find work done by the gas in the given process.

7. Pressure and volume of a gas changes from (p_0, V_0) to $\left(\frac{p_0}{4}, 2V_0\right)$ in a process $pV^2 =$ constant. Find work done by the gas in the given process.

21.3 Different Thermodynamic Processes

Different thermodynamic processes and their important points are given below in tabular form.

Table 21.3

S.No	Name of the process	Important points in the process	$Q = nC\Delta T = W + \Delta U$	$\Delta U = nC_V\Delta T$	W
1.	Isothermal	$T,\ pV,\ U = \text{constant}$ $\Delta T = \Delta(pV) = \Delta U = 0$ $p_1V_1 = p_2V_2$ or $p \propto \frac{1}{V}$	$Q = W$	0	$nRT\ln\left(\frac{V_f}{V_i}\right) = nRT\ln\left(\frac{p_i}{p_f}\right)$
2.	Isochoric	$V,\ \rho,\ \frac{p}{T} = \text{constant}$ $\Delta V = \Delta\rho = \Delta\left(\frac{p}{T}\right) = 0$ $\frac{p_1}{T_1} = \frac{p_2}{T_2}$ or $p \propto T$	$C = C_V$ $\therefore\ Q = nC_V\Delta T$	$nC_V\Delta T$	0
3.	Isobaric	$p,\ \frac{V}{T} = \text{constant}$ $\Delta p = \Delta\left(\frac{V}{T}\right) = 0$ $\frac{V_1}{T_1} = \frac{V_2}{T_2}$ or $V \propto T$	$C = C_P$ $\therefore\ Q = nC_P\Delta T$	$nC_V\Delta T$	$p\Delta V \rightarrow$ For any system $Q - \Delta U = n(C_P - C_V)\Delta T$ $= nR\Delta T \rightarrow$ for an ideal gas
4.	Adiabatic process	$pV^{\gamma} = \text{constant}$ $TV^{\gamma-1} = \text{constant}$ $T^{\gamma}p^{1-\gamma} = \text{constant}$	0	$nC_V\Delta T$	$W = -\Delta U$ $\therefore\ W = -nC_V\Delta T$ $= -n\left(\frac{R}{\gamma-1}\right)(T_f - T_i)$ $= \frac{p_iV_i - p_fV_f}{\gamma-1}$
5.	Cyclic process	$(p_i, V_i, T_i) = (p_f, V_f, T_f)$ Since $T_i = T_f$ $\Rightarrow\ U_i = U_f$ or $\Delta T = \Delta U = 0$	$Q_{\text{net}} = W_{\text{net}}$	0	W_{net} = area between cycle on p-V diagram
6.	Polytropic process $pV^X = \text{constant}$	$C = \frac{R}{\gamma-1} + \frac{R}{1-x}$ $= C_V + \frac{R}{1-x}$	$nC\Delta T$	$nC_V\Delta T$	$Q - \Delta U = \frac{nR\Delta T}{1-x}$ $= \frac{nR(T_i - T_f)}{1-x}$ $= \frac{(p_iV_i - p_fV_f)}{1-x}$
7.	Free expansion in vacuum	$\Delta U = 0$ $\Rightarrow\ U, T$ and $pV = \text{constant}$ or $p \propto \frac{1}{V}$	0	0	0

Important Points in the Above Table

In isobaric process In isobaric process, neither of the three terms (Q, ΔU and W) is zero but they have a constant ratio which depends on nature of gas (like monoatomic or diatomic etc.)

$$Q : \Delta U : W = nC_p \Delta T : nC_V \Delta T : nR\Delta T = C_p : C_V : R$$

So, this ratio is $C_p : C_V : R$. For example, $C_p = \frac{5}{2}R$ and $C_V = \frac{3}{2}R$ for a monoatomic gas. Therefore, for a monoatomic gas this ratio is $\frac{5}{2}R : \frac{3}{2}R : R$ or $5 : 3 : 2$. If Q is 50 J, then ΔU will be 30 J and W is 20 J.

In adiabatic process (i) In a thermodynamic process, there are three variables p, V and T. If relation between any two (p-V, V-T or p-T) are known, then other relations can be obtained using the ideal gas equation

$$pV = nRT$$

or $$pV \propto T \qquad \text{(For given value of } n\text{)}$$

or $$p \propto \frac{T}{V}$$

and $$V \propto \frac{T}{p}$$

For example, p-V equation $pV^{\gamma} = \text{constant}$ can be converted into p-T or V-T equation.

***p*-*T* equation** Replace V with $\frac{T}{p}$

$$\therefore \qquad p\left(\frac{T}{p}\right)^{\gamma} = \text{constant} \quad \text{or} \quad p^{1-\gamma}T^{\gamma} = \text{constant}$$

***V*-*T* equation** Replace p with $\frac{T}{V}$

$$\therefore \qquad \left(\frac{T}{V}\right)V^{\gamma} = \text{constant} \quad \text{or} \quad TV^{\gamma-1} = \text{constant}$$

(ii) An adiabatic process is defined as one with no heat transfer into or out of a system : $Q = 0$. We can prevent heat flow either by surrounding the system with thermally insulating material or by carrying out the process so **quickly** that there is not enough time for appreciable heat flow. From the first law, we find that for every adiabatic process,

$$W = -\Delta U \qquad (\text{as } Q = 0)$$

Therefore, if the work done by a gas is positive (i.e. volume of the gas is increasing), then ΔU will be negative. Hence, U and therefore T will decrease. The cooling of air can be experienced practically during bursting of a tyre. The process is so fast that it can be assumed as adiabatic. As the gas expands. Therefore, it cools. On the other hand, the compression stroke in an internal combustion engine is an approximately adiabatic process. The temperature rises as the air fuel mixture in the cylinder is compressed.

Note *Contrary to adiabatic process which is very fast an isothermal process is very slow. Because the system needs sufficient time to interact with surroundings to keep its temperature constant.*

In cyclic process In a cyclic process, initial and final points are same.

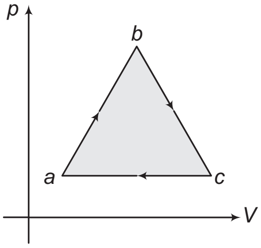


Fig. 21.13

Therefore,

$$(p_i, V_i, T_i) = (p_f, V_f, T_f)$$

Internal energy is a state function which only depends on temperature (in case of an ideal gas).

$$T_i = T_f$$

$\Rightarrow$ $$U_i = U_f$$

or $$\Delta U_{\text{net}} = 0$$

If there are three processes in a cycle *abc*, then

$$\Delta U_{ab} + \Delta U_{bc} + \Delta U_{ca} = 0$$

From first law of thermodynamics,

$$Q = W + \Delta U, \text{ if } \Delta U_{\text{net}} = 0, \text{ then}$$

$$Q_{\text{net}} = W_{\text{net}}$$

or $$Q_{ab} + Q_{bc} + Q_{ca} = W_{ab} + W_{bc} + W_{ca}$$

Further, W_{net} = area under p-V diagram. For example, W_{net} = + area of triangle '*abc*' in the shown diagram. Cycle is clockwise. So, work done will be positive.

Free expansion in vacuum A gas in a closed adiabatic chamber is taken to vacuum and then chamber is opened. So, the gas expands. Since, there is no external force which opposes this expansion. So, work done by gas

$$W = 0$$

Further no heat is supplied or taken from the gas. Therefore, heat exchange is also zero. Or

$$Q = 0$$

From first law of thermodynamics, $Q = W + \Delta U$ change in internal energy is also zero. Or

$$\Delta U = 0$$

or $$U = \text{constant}$$

$\Rightarrow$ $$T \text{ or } pV \text{ is also constant}$$

or $$p \propto \frac{1}{V}$$

This behaviour is similar to an isothermal process. The only difference is, in this process all three terms Q, ΔU and W are zero. But in isothermal process, only ΔU is zero.

✓ *Extra Points to Remember*

- **Slope of p-V diagram** In a general polytropic process,

$$pV^x = \text{constant}$$

or differentiating, we get

$$p(xV^{x-1})\,dV + V^x(dp) = 0 \quad \Rightarrow \quad \frac{dp}{dV} = -x\frac{p}{V}$$

or slope of p-V graph $= -x\dfrac{p}{V}$

In isobaric process $p =$ constant $\Rightarrow x = 0$, therefore slope $= 0$

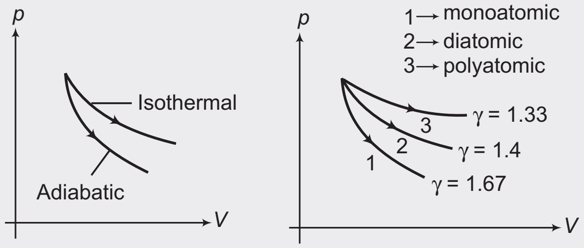
In isothermal process $pV =$ constant $\Rightarrow x = 1$, therefore slope $= -\dfrac{p}{V}$

In adiabatic process $pV^\gamma =$ constant $\Rightarrow x = \gamma$, therefore slope $= -\gamma\dfrac{p}{V}$

Thus, slope of an adiabatic graph is γ times slope of isothermal graph at that point.

Because $\gamma > 1$, the isothermal curve is not as steep as that for the adiabatic expansion.

Adiabatic and isothermal expansion of an ideal gas



Adiabatic expansion of mono,dia and polyatomic gases.

Fig. 21.14

p - V diagram of different processes is shown in one graph as below.

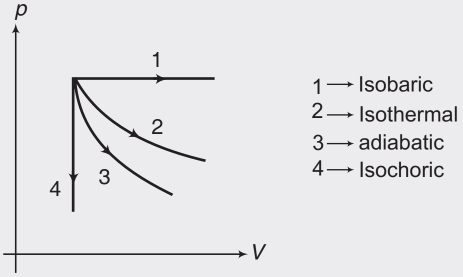


Fig. 21.15

Example 21.11 *p-V plots for two gases during adiabatic processes are shown in the figure. Plots 1 and 2 should correspond respectively to* **(JEE 2001)**

(a) He *and* O_2

(b) O_2 *and* He

(c) He *and* Ar

(d) O_2 *and* N_2

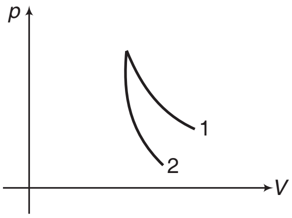


Fig. 21.16

Solution In adiabatic process :

slope of p-V graph, $$\frac{dp}{dV} = -\gamma \frac{p}{V}$$

$$\text{slope} \propto \gamma \qquad \text{(with negative sign)}$$

From the given graph,

$$(\text{slope})_2 > (\text{slope})_1$$

$$\therefore \qquad \gamma_2 > \gamma_1$$

Therefore, 1 should correspond to O_2 $(\gamma = 1.4)$ and 2 should correspond to He $(\gamma = 1.67)$.

Hence, the correct option is (b).

Example 21.12 *Starting with the same initial conditions, an ideal gas expands from volume V_1 to V_2 in three different ways, the work done by the gas is W_1 if the process is purely isothermal, W_2 if purely isobaric and W_3 if purely adiabatic, then* **(JEE 2000)**

(a) $W_2 > W_1 > W_3$

(b) $W_2 > W_3 > W_1$

(c) $W_1 > W_2 > W_3$

(d) $W_1 > W_3 > W_2$

Solution The corresponding p-V graphs (also called indicator diagram) in three different processes will be as shown below.

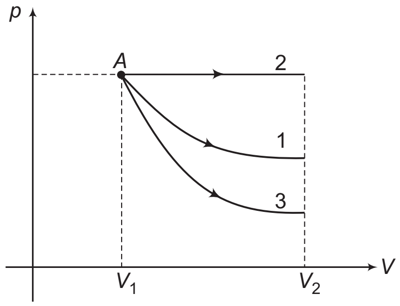


Fig. 21.17

Area under the graph gives the work done by the gas.

$$(\text{Area})_2 > (\text{Area})_1 > (\text{Area})_3 \quad \Rightarrow \quad W_2 > W_1 > W_3$$

Therefore, the correct option is (a).

Example 21.13 *When an ideal diatomic gas is heated at constant pressure, the fraction of the heat energy supplied, which increases the internal energy of the gas, is* **(JEE 1990)**

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{3}{7}$ (d) $\frac{5}{7}$

Solution The desired fraction is

$$f = \frac{\Delta U}{\Delta Q} = \frac{nC_V \Delta T}{nC_p \Delta T} = \frac{C_V}{C_p} = \frac{1}{\gamma} \quad \text{or} \quad f = \frac{5}{7} \qquad \left(\text{as } \gamma = \frac{7}{5}\right) \qquad \textbf{Ans.}$$

Therefore, the correct option is (d).

Example 21.14 *What is the heat input needed to raise the temperature of 2 moles of helium gas from 0°C to 100°C*

(a) at constant volume,

(b) at constant pressure?

(c) What is the work done by the gas in part (b)?

Give your answer in terms of R.

Solution Helium is monoatomic gas. Therefore,

$$C_V = \frac{3R}{2} \quad \text{and} \quad C_p = \frac{5R}{2}$$

(a) At constant volume,

$$\begin{aligned} Q &= nC_V \Delta T \\ &= (2)\left(\frac{3R}{2}\right)(100) \\ &= 300R \end{aligned}$$

(b) At constant pressure,

$$\begin{aligned} Q &= nC_p \Delta T \\ &= (2)\left(\frac{5R}{2}\right)(100) \\ &= 500R \end{aligned}$$

(c) At constant pressure,

$$\begin{aligned} W &= Q - \Delta U \\ &= nC_p \Delta T - nC_V \Delta T \\ &= nR\Delta T = (2)(R)(100) \\ &= 200R \end{aligned}$$

Example 21.15 *An ideal monoatomic gas at 300K expands adiabatically to twice its volume. What is the final temperature?*

Solution For an ideal monoatomic gas,

$$\gamma = \frac{5}{3}$$

In an adiabatic process, $TV^{\gamma - 1} = \text{constant}$

$\therefore$ $T_f V_f^{\gamma - 1} = T_i V_i^{\gamma - 1}$

or

$$\begin{aligned} T_f &= T_i \left(\frac{V_i}{V_f}\right)^{\gamma - 1} \\ &= (300)\left(\frac{1}{2}\right)^{\frac{5}{3} - 1} = 189\ \text{K} \end{aligned}$$

Example 21.16 *p-T graph of 2 moles of an ideal monoatomic gas* $\left(C_V = \frac{3}{2}R \text{ and } C_p = \frac{5}{2}R\right)$ *is as shown below. Find Q, W and ΔU for each of the four processes separately and then show that,*

$$\Delta U_{net} = 0$$

and

$$Q_{net} = W_{net}$$

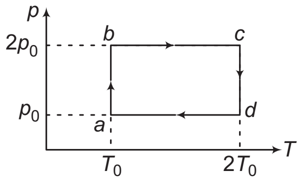


Fig. 21.18

Solution

Table 21.4

Process	Name of process	Q	ΔU	W
ab	Isothermal (as T = constant)	$Q = W = -2RT_0\ln(2)$	0	$nRT\ln\left(\frac{p_i}{p_f}\right)$ $= 2RT_0\ln\left(\frac{p_0}{2p_0}\right)$ $= -2RT_0\ln(2)$
bc	Isobaric (as p = constant)	$Q = nC_p\Delta T$ $= 2\left(\frac{5}{2}R\right)(2T_0 - T_0)$ $= 5RT_0$	$\Delta U = nC_V\Delta T$ $= 2\left(\frac{3}{2}R\right)(2T_0 - T_0)$ $= 3RT_0$	$Q - \Delta U$ or $nR\Delta T$ $= 2RT_0$
cd	Isothermal (as T = constant)	$Q = W = 4RT_0\ln(2)$	0	$nRT\ln\left(\frac{p_i}{p_f}\right)$ $= 2R(2T_0)\ln\left(\frac{2p_0}{p_0}\right)$ $= 4RT_0\ln(2)$
da	Isobaric (as p = constant)	$Q = nC_p\Delta T$ $= 2\left(\frac{5}{2}R\right)(T_0 - 2T_0)$ $= -5RT_0$	$\Delta U = nC_V\Delta T$ $= 2\left(\frac{3}{2}R\right)(T_0 - 2T_0)$ $= -3RT_0$	$Q - \Delta U$ or $nR\Delta T$ $= -2RT_0$
	Net values	$2RT_0\ln(2)$	0	$2RT_0\ln(2)$

In the given table, we can see that

$$\Delta U_{\text{net}} = 0$$

and

$$Q_{\text{net}} = W_{\text{net}}$$

INTRODUCTORY EXERCISE 21.3

1. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat are added as follows :

   (a) at constant volume (b) at constant pressure.

2. An ideal gas expands while the pressure is kept constant. During this process, does heat flow into the gas or out of the gas? Justify your answer.

3. Consider the cyclic process depicted in figure. If Q is negative for the process BC and if ΔU is negative for the process CA, what are the signs of Q, W and ΔU that are associated with each process?

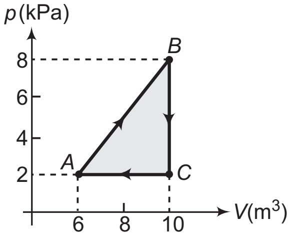


Fig. 21.19

4. A well insulated box contains a partition dividing the box into two equal volumes as shown in figure. Initially, the left hand side contains an ideal monoatomic gas and the other half is a vacuum. The partition is suddenly removed so that the gas expands throughout the entire box.

   (a) Does the temperature of the gas change?

   (b) Does the internal energy of the system change?

   (c) Does the gas work?

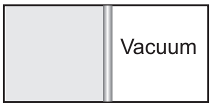


Fig. 21.20

5. Find the ratio of $\frac{\Delta Q}{\Delta U}$ and $\frac{\Delta Q}{\Delta W}$ in an isobaric process. The ratio of molar heat capacities $\frac{C_p}{C_V} = \gamma$.

6. A certain amount of an ideal gas passes from state A to B first by means of process 1, then by means of process 2. In which of the process is the amount of heat absorbed by the gas greater?

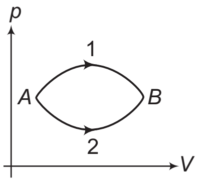


Fig. 21.21

7. A sample of ideal gas is expanded to twice its original volume of 1.00 m^3 in a quasi-static process for which $p = \alpha V^2$, with $\alpha = 5.00$ atm/m^6, as shown in Fig 21.22. How much work is done by the expanding gas?

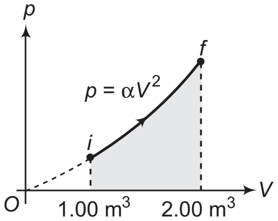


Fig. 21.22

8. As a result of the isobaric heating by $\Delta T = 72$ K, one mole of a certain ideal gas obtains an amount of heat $Q = 1.6$ kJ. Find the work performed by the gas, the increment of its internal energy and γ.

21.4 Heat Engine and its Efficiency

A heat engine is a device which converts heat energy into mechanical energy. In every heat engine, there are the following three components :

(i) Working substance (which is normally a gas in a cylinder)
(ii) Source (at temperature T_1)
(iii) Sink (at temperature T_2), $T_2 < T_1$ and sink is normally atmosphere.

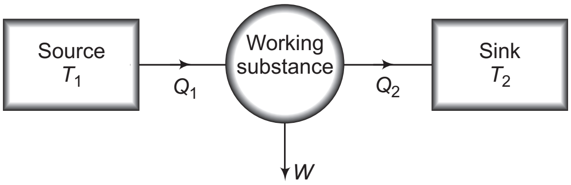


Fig. 21.23

The working substance absorbs some heat (Q_1) from the source, converts a part of it into work (W) and the rest (Q_2) is rejected to the sink.

From conservation of energy,

$$Q_1 = W + Q_2$$

Now, the above work done is used by us for different purposes.

It is just like a shopkeeper. He takes some money from you. (Suppose he takes Rs. 100/- from you). So, you are source. In lieu of this he provides services to you (suppose he provides services of worth Rs. 80/-). This is work done. The remaining Rs. 20/- is his profit which goes to his account and this is basically sink. Then, the efficiency of the shopkeeper is 80%. There can't be a shopkeeper whose efficiency is 100%.

Similarly, efficiency of a heat engine is defined as the ratio of net work done per cycle by the engine to the total amount of heat absorbed per cycle by the working substance from the source. It is denoted by η. Thus,

$$\boxed{\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}}$$

Work is done by the working substance in a cyclic process. In the above expression of efficiency, W is net work done in the complete cycle which should be positive. So, cycle should be clockwise on p-V diagram. Q_1 is the total heat given to the working substance or the total positive heat. Q_2 is total heat rejected by the working substance or it is the magnitude of total negative heat.

Thus, efficiency (η) of a cycle can also be defined as

$$\eta = \left(\frac{\text{Work done by the working substance (an ideal gas in our case) during a cycle}}{\text{Heat supplied to the gas during the cycle}} \right) \times 100$$

$$= \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100$$

$$= \frac{|Q_{+\text{ve}}| - |Q_{-\text{ve}}|}{|Q_{+\text{ve}}|} \times 100 = \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\} \times 100$$

Thus,
$$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100 = \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\} \times 100$$

Depending on the number of processes in a cycle, it may be called a two-stroke engine or four-stroke engine.

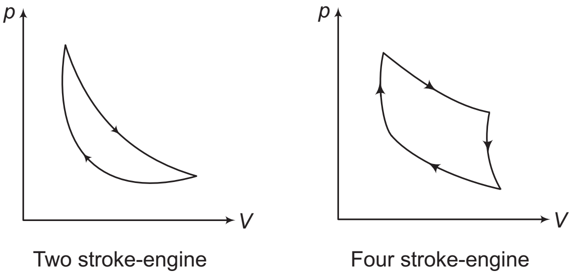


Fig. 21.24

Note *There cannot be a heat engine whose efficiency is* 100%. *It is always less than* 100%. *Thus,*

$$\eta \neq 100\%$$

or $W \neq Q_1$ *or* $W_{net} \neq Q_{+ve}$

or $Q_2 \neq 0$ *or* $|Q_{-ve}| \neq 0$

Efficiency of a Cycle

By the similar method discussed above, we can also find efficiency of a cycle provided net work done in the whole cycle comes out to be positive or it is clockwise cycle on p-V diagram. But always remember that, in a cyclic process,

$$\Delta U_{\text{net}} = 0$$

and
$$Q_{\text{net}} = W_{\text{net}}$$

If there are four processes in the cycle, then

$$\Delta U_1 + \Delta U_2 + \Delta U_3 + \Delta U_4 = 0 \text{ and}$$
$$Q_1 + Q_2 + Q_3 + Q_4 = W_1 + W_2 + W_3 + W_4$$

Carnot Engine

Carnot cycle consists of the following four processes :

(i) Isothermal expansion (process *AB*)
(ii) Adiabatic expansion (process *BC*)
(iii) Isothermal compression (process *CD*) and
(iv) Adiabatic compression (process *DA*)

The p-V diagram of the cycle is shown in the figure.

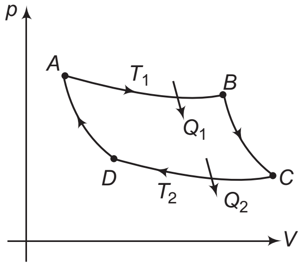


Fig. 21.25

Table 21.5

Process	Name of the process	Q	ΔU	W
AB	Isothermal expansion T_1 = constant	$Q = W$ = positive $= Q_1$	0	positive
BC	Adiabatic expansion	0	negative	positive
CD	Isothermal compression T_2 = constant	$Q = W$ = negative $= Q_2$	0	negative
DA	Adiabatic compression	0	positive	negative

In process AB, heat Q_1 is taken by the working substance at constant temperature T_1 and in process CD heat Q_2 is rejected from the working substance at constant temperature T_2. The net work done is area of graph $ABCD$.

Note *(i) In the whole cycle only Q_1 is the positive heat and Q_2 the negative heat. Thus,*

$$Q_{+ve} = Q_1$$

and

$$|Q_{-ve}| = Q_2$$

$$\therefore \quad \eta = \left(1 - \frac{Q_2}{Q_1}\right) \times 100$$

Specially for Carnot cycle, $\frac{Q_2}{Q_1}$ also comes out to be $\frac{T_2}{T_1}$.

$$\therefore \quad \eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

(ii) Efficiency of Carnot engine is maximum (not 100%) for given temperatures T_1 and T_2. But still Carnot engine is not a practical engine because many ideal situations have been assumed while designing this engine which can practically not be obtained.

21.5 Refrigerator

Refrigerator is an apparatus which takes heat from a cold body, work is done on it and the work done together with the heat absorbed is rejected to the source.

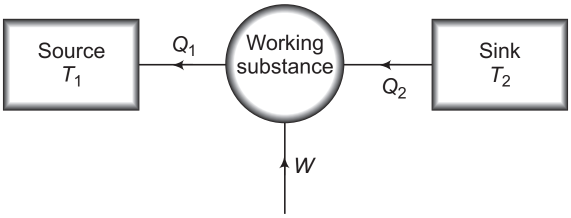


Fig. 21.26

An ideal refrigerator can be regarded as Carnot ideal heat engine working in the reverse direction.

Coefficient of Performance

Coefficient of performance (β) of a refrigerator is defined as the ratio of quantity of heat removed per cycle (Q_2) to the work done on the working substance per cycle to remove this heat. Thus,

$$\beta = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

We can also show that

$$\beta = \frac{T_2}{T_1 - T_2} = \frac{1 - \eta}{\eta}$$

Here, η is the efficiency of Carnot cycle.

Example 21.17 *An ideal gas is taken through a cyclic thermodynamic process through four steps. The amounts of heat involved in these steps are $Q_1 = 5960\,J$, $Q_2 = -5585\,J$, $Q_3 = -2980\,J$ and $Q_4 = 3645\,J$ respectively. The corresponding quantities of work involved are $W_1 = 2200\,J$, $W_2 = -825\,J$, $W_3 = -1100\,J$ and W_4 respectively.* **(JEE 1994)**

(a) Find the value of W_4.

(b) What is the efficiency of the cycle?

Solution (a) In a cyclic process, $\Delta U = 0$

Therefore, $$Q_{\text{net}} = W_{\text{net}}$$

or $$Q_1 + Q_2 + Q_3 + Q_4 = W_1 + W_2 + W_3 + W_4$$

Hence, $$W_4 = (Q_1 + Q_2 + Q_3 + Q_4) - (W_1 + W_2 + W_3)$$
$$= \{(5960 - 5585 - 2980 + 3645) - (2200 - 825 - 1100)\}$$

or $$W_4 = 765\text{ J}$$ **Ans.**

(b) Efficiency,

$$\eta = \frac{\text{Total work done in the cycle}}{\text{Heat absorbed (positive heat) by the gas during the cycle}} \times 100$$

$$= \left(\frac{W_1 + W_2 + W_3 + W_4}{Q_1 + Q_4}\right) \times 100$$

$$= \left\{\frac{(2200 - 825 - 1100 + 765)}{5960 + 3645}\right\} \times 100$$

$$= \frac{1040}{9605} \times 100$$

$$\eta = 10.82\%$$ **Ans.**

Note *From energy conservation,*

$$W_{\text{net}} = Q_{+\text{ve}} - Q_{-\text{ve}} \qquad \textit{(in a cycle)}$$

$$\therefore \quad \eta = \frac{W_{\text{net}}}{Q_{+\text{ve}}} \times 100 = \frac{(Q_{+\text{ve}} - Q_{-\text{ve}})}{Q_{+\text{ve}}} \times 100$$

$$= \left(1 - \frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right) \times 100$$

In the above question,

$$Q_{-\text{ve}} = |Q_2| + |Q_3| = (5585 + 2980)\text{ J} = 8565\text{ J}$$

and $$Q_{+\text{ve}} = Q_1 + Q_4 = (5960 + 3645)\ \text{J} = 9605\ \text{J}$$

$\therefore$ $$\eta = \left(1 - \frac{8565}{9605}\right) \times 100$$

$$\eta = 10.82\%$$

Example 21.18 *The density versus pressure graph of one mole of an ideal monatomic gas undergoing a cyclic process is shown in figure. The molecular mass of the gas is M.*

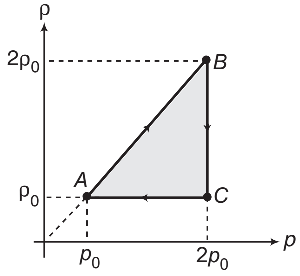


Fig. 21.27

(a) Find the work done in each process.
(b) Find heat rejected by gas in one complete cycle.
(c) Find the efficiency of the cycle.

Solution (a) Given, $n = 1$ $\therefore$ $m = M$

Process *AB* $\rho \propto p$, i.e. It is an isothermal process (T = constant), because $\rho = \dfrac{pM}{RT}$.

$\therefore$ $$W_{AB} = RT_A \ln\left(\frac{p_A}{p_B}\right) = RT_A \ln\left(\frac{1}{2}\right)$$

$$= -\frac{p_0 M}{\rho_0} \ln(2)$$

$$\Delta U_{AB} = 0$$

and $$Q_{AB} = W_{AB} = -\frac{p_0 M}{\rho_0} \ln(2)$$

Process *BC* is an isobaric process (p = constant)

$$W_{BC} = p_B (V_C - V_B) = 2p_0 \left(\frac{M}{\rho_C} - \frac{M}{\rho_B}\right) = 2p_0 \left(\frac{M}{\rho_0} - \frac{M}{2\rho_0}\right) = \frac{p_0 M}{\rho_0}$$

$$\Delta U_{BC} = C_V \Delta T$$

$$= \left(\frac{3}{2}R\right)\left[\frac{2p_0 M}{\rho_0 R} - \frac{2p_0 M}{2\rho_0 R}\right] = \frac{3p_0 M}{2\rho_0}$$

$$Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{5p_0 M}{2\rho_0}$$

Process *CA* As $\rho =$ constant

$$\therefore \qquad V = \text{constant}$$

So, it is an isochoric process.

$$W_{CA} = 0$$

$$\Delta U_{CA} = C_V \Delta T$$

$$= \left(\frac{3}{2}R\right)(T_A - T_C)$$

$$= \left(\frac{3}{2}R\right)\left[\frac{p_0 M}{\rho_0 R} - \frac{2p_0 M}{\rho_0 R}\right]$$

$$= -\frac{3p_0 M}{2\rho_0}$$

$$Q_{CA} = \Delta U_{CA} = -\frac{3p_0 M}{2\rho_0}$$

(b) Heat rejected by gas $= |Q_{AB}| + |Q_{CA}|$

$$= \frac{p_0 M}{\rho_0}\left[\frac{3}{2} + \ln(2)\right]$$ **Ans.**

(c) Efficiency of the cycle (in fraction)

$$\eta = \frac{\text{Total work done}}{\text{Heat supplied}} = \frac{W_{\text{Total}}}{Q_{+\text{ve}}}$$

$$= \frac{\frac{p_0 M}{\rho_0}[1 - \ln(2)]}{\frac{5}{2}\left(\frac{p_0 M}{\rho_0}\right)}$$

$$= \frac{2}{5}[1 - \ln(2)]$$ **Ans.**

Example 21.19 *Carnot engine takes one thousand kilo calories of heat from a reservoir at 827°C and exhausts it to a sink at 27°C. How, much work does it perform? What is the efficiency of the engine?*

Solution Given, $Q_1 = 10^6$ cal

$$T_1 = (827 + 273) = 1100\text{ K}$$

and $$T_2 = (27 + 273) = 300\text{ K}$$

as, $$\frac{Q_2}{Q_1} = \frac{T_2}{T_1} \Rightarrow Q_2 = \frac{T_2}{T_1}\cdot Q_1 = \left(\frac{300}{1100}\right)(10^6)$$

$$= 2.72 \times 10^5 \text{ cal}$$

$$W = Q_1 - Q_2 = 7.28 \times 10^5 \text{ cal}$$ **Ans.**

Efficiency of the cycle,

$$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100 \quad \text{or} \quad \eta = \left(1 - \frac{300}{1100}\right) \times 100$$

$$= 72.72\%$$ **Ans.**

Example 21.20 *In a refrigerator, heat from inside at 277 K is transferred to a room at 300 K. How many joules of heat shall be delivered to the room for each joule of electrical energy consumed ideally?*

Solution Coefficient of performance of a refrigerator,

$$\beta = \frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$$

$\therefore$
$$Q_2 = W \frac{T_2}{T_1 - T_2}$$

But W = Energy consumed by the refrigerator = 1 J, $T_1 = 300$ K, $T_2 = 277$ K

$\therefore$
$$Q_2 = 1 \times \frac{277}{300 - 277} = \frac{277}{23} = 12 \text{ J}$$

Heat rejected by the refrigerator,

$$Q_1 = W + Q_2 = 1 + 12$$
$$= 13 \text{ J}$$ **Ans.**

Example 21.21 *Calculate the least amount of work that must be done to freeze one gram of water at 0°C by means of a refrigerator. Temperature of surroundings is 27°C. How much heat is passed on the surroundings in this process? Latent heat of fusion L = 80 cal/g.*

Solution $Q_2 = mL = 1 \times 80 = 80 \text{ cal}$

$$T_2 = 0° \text{ C} = 273 \text{ K}$$

and
$$T_1 = 27° \text{ C} = 300 \text{ K}$$

Least amount of work will be needed for carnot's type of cycle.

$$\frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$$

$\therefore$
$$W = \frac{Q_2 (T_1 - T_2)}{T_2}$$
$$= \frac{80 (300 - 273)}{273}$$
$$= 7.91 \text{ cal}$$ **Ans.**

$$Q_1 = Q_2 + W$$
$$= (80 + 7.91)$$
$$= 87.91 \text{ cal}$$ **Ans.**

INTRODUCTORY EXERCISE 21.4

1. Carnot engine takes 1000 K cal of heat from a reservoir at 827°*C* and exhausts it to a sink at 27°*C*. How much heat is rejected to the sink? What is the efficiency of the engine?
2. One of the most efficient engines ever developed operated between 2100 K and 700 K. Its actual efficiency is 40%. What percentage of its maximum possible efficiency is this?
3. In a heat engine, the temperature of the source and sink are 500 K and 375 K. If the engine consumes 25×10^5 J per cycle, find (a) the efficiency of the engine, (b) work done per cycle, and (c) heat rejected to the sink per cycle.
4. A Carnot engine takes 3×10^6 cal of heat from a reservoir at 627 °C and gives it to a sink at 27 °C. Find the work done by the engine.
5. The efficiency of a Carnot cycle is 1/6. If on reducing the temperature of the sink by 65 °C, the efficiency becomes 1/3, find the source and sink temperatures between which the cycle is working.
6. Refrigerator *A* works between −10 °C and 27 °C, while refrigerator *B* works between −27 °C and 17 °C, both removing heat equal to 2000 J from the freezer. Which of the two is the better refrigerator?
7. A refrigerator has to transfer an average of 263 J of heat per second from temperature −10 °C to 25 °C. Calculate the average power consumed, assuming no energy losses in the process.
8. *n* moles of a monoatomic gas are taken around in a cyclic process consisting of four processes along *ABCDA* as shown. All the lines on the *p*-*V* diagram have slope of magnitude p_0 / V_0. The pressure at *A* and *C* is p_0 and the volumes at *A* and *C* are $V_0/2$ and $3V_0/2$, respectively. Calculate the percentage efficiency of the cycle.

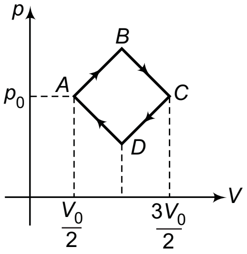


Fig. 21.28

21.6 Zeroth Law of Thermodynamics

The zeroth law of thermodynamics states that if two system *A* and *B* are in thermal equilibrium with a third system *C*, then *A* and *B* are also in thermal equilibrium with each other. It is analogous to the transitive property in math (if $A = C$ and $B = C$, then $A = B$). Another way of stating the zeroth law is that every object has a certain temperature, and when two objects are in thermal equilibrium, their temperatures are equal. It is called the zeroth law because it came to light after the first and second laws of thermodynamics had already been established and named, but was considered more fundamental and thus was given a lower number zero.

21.7 Second Law of Thermodynamics

The first law of thermodynamics is the principle of conservation of energy. Common experience shows that there are many conceivable processes that are perfectly allowed by the first law and yet are never observed. For example, nobody has ever seen a book lying on a table jumping to a height by itself. But such a thing would be possible if the principle of conservation of energy were the only restriction.

Thus, the second law of thermodynamics is a general principle which places constraints upon the direction of and the attainable efficiency of or coefficient of performance. It also imposes restrictions on entropy.

Second law of thermodynamics can be better understood by the following three statements

In Terms of Entropy

The second law can be expressed in several ways, the simplest being that heat will naturally flow from a hotter to a colder body. As its heat is a property of thermodynamic systems called entropy represented by "S" — in loose terms, a measure of the amount of disorder within a system. This can be represented in many ways, for example in the arrangement of the molecules — water molecules in an ice cube are more ordered than the same molecules after they have been heated into a gas. The entropy of the ice cube is, therefore, lower than that of the gas. Similarly, the entropy of a plate is higher when it is in pieces on the floor compared with when it is in one piece in the sink.

The second equation is a way to express the second law of thermodynamics in terms of entropy. The formula says that the entropy of an isolated natural system will always tend to stay the same or increase — in other words, the energy in the universe is gradually moving towards disorder.

Kelvin Planck's Statement

Kelvin-Planck's statement is based on the fact that the efficiency of the heat engine cycle is never 100%. This means that in the heat engine cycle some heat is always rejected to the low temperature reservoir. The heat engine cycle always operates between two heat reservoirs and produces work.

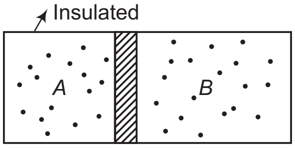
Clausius Statement

As we know from the previous statement, the natural tendency of the heat is to flow from high temperature reservoir to the low temperature reservoir.

The Clausius statement says that, to transfer the heat from low temperature to high temperature reservoir some external work should be done on the cycle. This statement has been the basis for the working for all refrigerators, heat pumps and air-conditioners. This work cannot be zero or coefficient of performance of a refrigerator cannot be infinite.

Final Touch Points

1. Adiabatic and Diathermic Wall

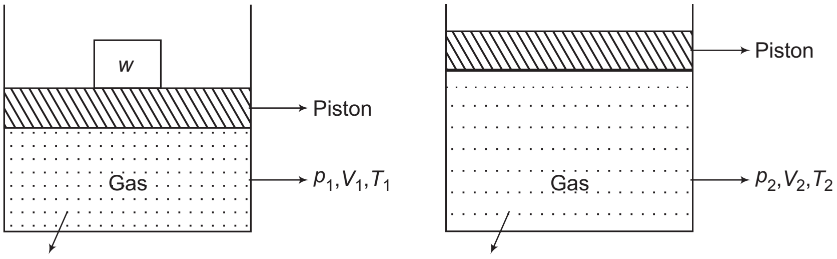


Adiabatic wall An insulating wall (can be movable also) that does not allow flow of energy (heat) from one chamber to another is called an adiabatic wall. If two thermodynamic systems *A* and *B* are separated by an adiabatic wall then the thermodynamic state of *A* will be independent of the state of *B* and *vice-versa* if wall is fixed otherwise if wall is movable, only pressure will be same on both sides.

Diathermic wall A conducting wall that allows energy flow (heat) from one chamber to another is called a diathermic wall. If two thermodynamic systems *A* and *B* are separated by a diathermic wall then thermal equilibrium is attained in due course of time (if wall is fixed). If wall is movable, then temperature and pressure on both sides will become same.

Note *In the above two cases, thermodynamic systems A and B are insulated from the external surroundings.*

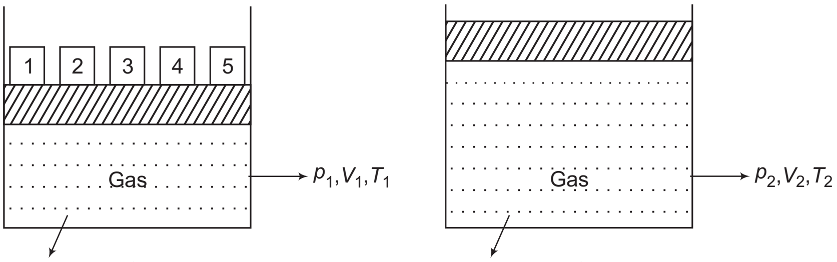
2. Quasi-Static Process Quasi means almost or near to.Quasi-static process means very nearly static process. Let us consider a system of gas contained in cylinder. The gas is held by a moving piston. A weight w is placed over the piston. Due to the weight , the gas in cylinder is compressed. After the gas reaches equilibrium, the properties of gas are denoted by p_1, V_1 and T_1. The weight placed over the piston is balanced by upward force exerted by the gas. If the weight is suddenly removed, then there will be an unbalanced force between the system and the surroundings. The gas under pressure will expand and push the piston upwards. The properties at this state after reaching equilibrium are P_2, V_2 and T_2. But the intermediate states passed through, by the system are non-equilibrium states which cannot be described by thermodynamic coordinates. In this case, we only have initial and final states and do not have a path connecting them.



Initial state of gas

Final state of gas

Sudden expansion of gas

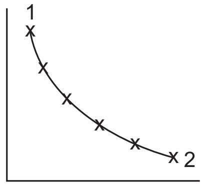


Initial state of gas

Final state of gas

Quasi-static expansion of gas

Suppose, the weight is made of large numbers of small weights. And one by one each of these small weights are removed and allowed the system to reach an equilibrium state. Then, we have intermediate equilibrium states and the path described by these states will not deviate much from the thermodynamic equilibrium state. Such a process, which is the locus of all the intermediate points passed by the system is known as quasi-static process. It means, this process is almost near to the thermodynamically equilibrium process. Infinite slowness is the characteristic feature of quasi-static process.

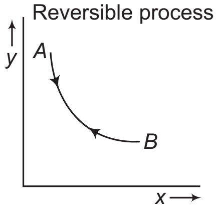


A quasi-static process

A quasi-static process is obviously a hypothetical concept. In practice, processes that are sufficiently slow and do not involve accelerated motion of the piston are reasonably approximation to an ideal quasi-static process. We shall from now onwards deal with quasi-static processes only, except when stated otherwise.

3. Reversible and Irreversible Process

Reversible process The process in which the system and surroundings can be restored to the initial state from the final state without producing any changes in the thermodynamic properties of the universe is called a reversible process. In the figure below, let us suppose that the system has undergone a change from state *A* to state *B*. If the system can be restored from state *B* to state *A*, and there is no change in the universe, then the process is said to be a reversible process. The reversible process can be reversed completely and there is no trace left to show that the system had undergone thermodynamic change.



For the system to undergo reversible change, it should occur infinitely slowly or it should be quasi-static process. During reversible process, all the changes in state that occur in the system are in thermodynamic equilibrium with each other.

Thus, there are two important conditions for the reversible process to occur. Firstly, the process should occur very slowly and secondly all of the initial and final states of the system should be in equilibrium with each other.

For example, a quasi-static isothermal expansion of an ideal gas in a cylinder fitted with a frictionless movable piston is a reversible process.

In actual practice, the reversible process never occurs, thus it is an ideal or hypothetical process.

Irreversible process The process is said to be an irreversible process if it cannot return the system and the surroundings to their original conditions when the process is reversed. The irreversible process is not at equilibrium throughout the process. Several examples can be cited. For examples,

(i) When we are driving the car uphill, it consumes a lot of fuel and this fuel is not returned when we are driving down the hill.

(ii) The base of a vessel on an oven is hotter than its other parts. When the vessel is removed, heat is transferred from the base to the other parts, bringing the vessel to a uniform temperature (which in due course cools to the temperature of the surroundings). The process cannot be reversed; a part of the vessel will not get cooler spontaneously and warm up the base. It will violate the second law of thermodynamics, if it did.

(iii) The free expansion of a gas is irreversible.

(iv) Cooking gas leaking from a gas cylinder in the kitchen diffuses to the entire room. The diffusion process will not spontaneously reverse and bring the gas back to the cylinder.

Many factors contribute in making any process irreversible. The most common of these are: friction, viscosity and other dissipative effects.

The irreversible process is also called the natural process because all the processes occurring in nature are irreversible processes.

4. $\boldsymbol{C_V = \frac{dU}{dT}}$ Let us derive the relation $C_V = \frac{dU}{dT}$, where U = internal energy of 1 mole of the gas.

Consider 1 mole ($n = 1$) of an ideal monoatomic gas which undergoes an isochoric process (V = constant).

From the first law of thermodynamics.

$$dQ = dW + dU \qquad \ldots\text{(i)}$$

Here, $dW = 0$ as V = constant

$$dQ = CdT = C_V dT \qquad (\text{In } dQ = nC_V dT,\ n = 1 \text{ and } C = C_V)$$

Substituting in Eq. (i), we have

$$C_V dT = dU$$

or $$C_V = \frac{dU}{dT}$$ **Hence proved.**

5. $\boldsymbol{C_p - C_V = R}$ To prove this relation (also known as **Mayor's formula**) let us consider 1 mole of an ideal gas which undergoes an isobaric (p = constant) process.

From first law of thermodynamics, $$dQ = dW + dU \qquad \ldots\text{(ii)}$$

Here, $$dQ = C_p dT \qquad (\text{as } n = 1 \text{ and } C = C_p)$$

$$dU = C_V dT$$

and $$dW = pdV = pd\left(\frac{RT}{p}\right) \qquad \left(\text{as } V = \frac{RT}{P}\right)$$

$$= d(RT) \qquad (\text{as } P = \text{constant})$$

$$= RdT$$

Substituting these values in Eq. (ii)

We have $$C_p dT = RdT + C_V dT \quad \text{or} \quad C_p - C_V = R$$ **Hence proved.**

6. $\boldsymbol{C_V = \frac{R}{\gamma - 1}}$ We have already derived,

$$C_p - C_V = R$$

Dividing this equation by C_V, we have

$$\frac{C_p}{C_V} - 1 = \frac{R}{C_V} \quad \text{or} \quad \gamma - 1 = \frac{R}{C_V} \qquad \left(\text{as } \frac{C_P}{C_V} = \gamma\right)$$

$\therefore$ $$C_V = \frac{R}{\gamma - 1}$$ **Hence proved.**

7. **Polytropic process** When p and V bear the relation pV^x = constant, where $x \neq 1$ or γ the process is called a polytropic one. In this process the molar heat capacity is,

$$C = C_V + \frac{R}{1-x} = \frac{R}{\gamma - 1} + \frac{R}{1-x}$$

Let us now derive this relation. The molar heat capacity is defined as

$$C = \frac{\Delta Q}{\Delta T} \quad \text{for 1 mole}$$

$$= \frac{\Delta U + \Delta W}{\Delta T}$$

$$= \frac{\Delta U}{\Delta T} + \frac{\Delta W}{\Delta T} \qquad \left(\frac{\Delta U}{\Delta T} = C_V\right)$$

$$\therefore \quad C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(iii)}$$

Here,

$$\Delta W = \int_{V_i}^{V_f} p dV = \int_{V_i}^{V_f} kV^{-x} dV = \left[\frac{kV^{-x+1}}{-x+1}\right]_{V_i}^{V_f}$$

$$= \frac{kV_f^{-x+1} - kV_i^{-x+1}}{-x+1}$$

$$= \frac{p_f V_f^x V_f^{-x+1} - p_i V_i^x V_i^{-x+1}}{1-x}$$

$$= \frac{p_f V_f - p_i V_i}{1-x} = \frac{RT_f - RT_i}{1-x}$$

$$= \frac{R\Delta T}{1-x}$$

$$\therefore \quad \frac{\Delta W}{\Delta T} = \frac{R}{1-x}$$

Substituting in Eq. (iii), we get the result i.e.

$$\therefore \quad C = C_V + \frac{R}{1-x} = \frac{R}{\gamma - 1} + \frac{R}{1-x}$$

Solved Examples

TYPED PROBLEMS

Type 1. *Based on first law of thermodynamics applied to a general system*

Concept

First law of thermodynamics is simply law of conservation of energy which can be applied for any system.

Example 1 ***Boiling water :*** *Suppose 1.0 g of water vaporizes isobarically at atmospheric pressure* $(1.01 \times 10^5\ Pa)$. *Its volume in the liquid state is* $V_i = V_{liquid} = 1.0\ cm^3$ *and its volume in vapour state is* $V_f = V_{vapour} = 1671\ cm^3$. *Find the work done in the expansion and the change in internal energy of the system. Ignore any mixing of the steam and the surrounding air. Take latent heat of vaporization* $L_v = 2.26 \times 10^6\ J/kg$.

Solution Because the expansion takes place at constant pressure, the work done is

$$W = \int_{V_i}^{V_f} p_0 dV = p_0 \int_{V_i}^{V_f} dV = p_0\ (V_f - V_i)$$

$$= (1.01 \times 10^5)\ (1671 \times 10^{-6} - 1.0 \times 10^{-6})$$

$$= 169\ \text{J}$$ **Ans.**

$$Q = mL_v = (1.0 \times 10^{-3})\ (2.26 \times 10^6)$$

$$= 2260\ \text{J}$$ **Ans.**

Hence, from the first law, the change in internal energy

$$\Delta U = Q - W = 2260 - 169$$

$$= 2091\ \text{J}$$ **Ans.**

Note *The positive value of ΔU indicates that the internal energy of the system increases. We see that most* $\left(\frac{2091\ J}{2260\ J} = 93\%\right)$ *of the energy transferred to the liquid goes into increasing the internal energy of the system only* $\frac{169\ J}{2260\ J} = 7\%$ *leaves the system by work done by the steam on the surrounding atmosphere.*

Example 2 *A metal of mass 1 kg at constant atmospheric pressure and at initial temperature 20°C is given a heat of 20000 J. Find the following* **(JEE 2005)**

(a) change in temperature,

(b) work done and

(c) change in internal energy.

(Given, specific heat = 400 J/kg-°C, coefficient of cubical expansion, $\gamma = 9 \times 10^{-5}/°C$, *density* $\rho = 9000\ kg/m^3$, *atmospheric pressure* $= 10^5\ N/m^2$*)*

Solution (a) From $\Delta Q = ms\Delta T$

$$\Delta T = \frac{\Delta Q}{ms} = \frac{20000}{1 \times 400} = 50°\text{C}$$

(b)

$$\Delta V = V\gamma \Delta T = \left(\frac{1}{9000}\right)(9 \times 10^{-5})(50)$$

$$= 5 \times 10^{-7} \text{ m}^3$$

$\therefore$

$$W = p_0 \cdot \Delta V = (10^5)(5 \times 10^{-7}) = 0.05 \text{ J}$$

(c)

$$\Delta U = \Delta Q - W = (20000 - 0.05) \text{ J}$$

$$= 19999.95 \text{ J}$$

Type 2. *To make p-V, V-T or p-T equation corresponding to a given process*

Concept

Suppose we wish to make p-V equation for a given process then with the help of equation of first law of thermodynamics and $pV = nRT$, first make an equation of type

$$f(p)\,dp + f(V)\,dV = 0$$

Now, integrating this equation we will get the desired p-V equation.

Example 3 *Make p-V equation for an adiabatic process.*

Solution In adiabatic process, $dQ = 0$

and

$$dW = -\,dU$$

$\therefore$

$$pdV = -\,C_V dT \qquad \text{(for } n = 1\text{)}$$

$\therefore$

$$dT = -\frac{pdV}{C_V} \qquad \text{...(i)}$$

Also, for 1 mole of an ideal gas,

$$d\,(pV) = d\,(RT)$$

or

$$pdV + Vdp = RdT$$

or

$$dT = \frac{pdV + Vdp}{R} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$C_V Vdp + (C_V + R)\,pdV = 0$$

or

$$C_V Vdp + C_p pdV = 0$$

Dividing this equation by PV, we are left with

$$C_V \frac{dp}{p} + C_p \frac{dV}{V} = 0$$

or

$$\frac{dp}{p} + \gamma \frac{dV}{V} = 0$$

or

$$\int \frac{dp}{p} + \gamma \int \frac{dV}{V} = 0$$

or

$$\ln(p) + \gamma \ln(V) = \text{constant}$$

We can write this in the form

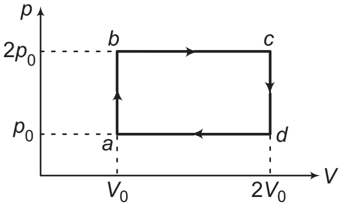
$$\boxed{pV^{\gamma} = \text{constant}}$$

Type 3. *To find values of all three terms of first law of thermodynamics from a p-V diagram provided nature of gas is given.*

Concept

Use the equation $nRT = pV$

Example 4 *A cyclic process abcd is given for a monoatomic gas* $\left(C_V = \frac{3}{2}R \text{ and } C_p = \frac{5}{2}R\right)$ *as shown in figure. Find Q, W and ΔU in each of the four processes separately. Also find the efficiency of cycle.*



Solution **Process ab**

$$V = \text{constant} \qquad (\because \text{ Isochoric process})$$

$$W_{ab} = 0$$

$$\therefore \quad Q_{ab} = \Delta U_{ab} = nC_V\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_b - T_a)$$

$$= \frac{3}{2}(nRT_b - nRT_a)$$

$$= \frac{3}{2}(p_bV_b - p_aV_a)$$

$$= \frac{3}{2}(2p_0V_0 - p_0V_0)$$

$$= 1.5\ p_0V_0$$

Process bc

$$p = \text{constant} \qquad (\because \text{ Isobaric process})$$

$$Q_{bc} = nC_p\Delta T$$

$$= n\left(\frac{5}{2}R\right)(T_c - T_b)$$

$$= \frac{5}{2}(nRT_c - nRT_b)$$

$$= \frac{5}{2}(p_cV_c - p_bV_b)$$

$$= \frac{5}{2}(4p_0V_0 - 2p_0V_0)$$

$$= 5\ p_0V_0$$

$$\begin{aligned}\Delta U_{bc} &= nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_c - T_b)\\ &= \frac{3}{2}(nRT_c - nRT_b)\\ &= \frac{3}{2}(p_cV_c - p_bV_b)\\ &= \frac{3}{2}(4p_0V_0 - 2p_0V_0)\\ &= 3p_0V_0\\ W_{bc} &= Q_{bc} - \Delta U_{bc} = 2p_0V_0\end{aligned}$$

Process *cd*

Again an isochoric process.

$\therefore$

$$\begin{aligned}W_{cd} &= 0\\ Q_{cd} &= \Delta U_{cd} = nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_d - T_c)\\ &= \frac{3}{2}(nRT_d - nRT_c)\\ &= \frac{3}{2}(p_dV_d - p_cV_c)\\ &= \frac{3}{2}(2p_0V_0 - 4p_0V_0)\\ &= -3p_0V_0\end{aligned}$$

Process *da*

This is an isobaric process.

$\therefore$

$$\begin{aligned}Q_{da} &= nC_p\Delta T\\ &= n\left(\frac{5}{2}R\right)(T_a - T_d)\\ &= \frac{5}{2}(nRT_a - nRT_d)\\ &= \frac{5}{2}(p_aV_a - p_dV_d)\\ &= \frac{5}{2}(p_0V_0 - 2p_0V_0)\\ &= -2.5\ p_0V_0\\ \Delta U_{da} &= nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_a - T_d)\\ &= \frac{3}{2}(nRT_a - nRT_d)\\ &= \frac{3}{2}(p_aV_a - p_dV_d)\end{aligned}$$

$$= \frac{3}{2}(p_0V_0 - 2p_0V_0)$$
$$= -1.5\ p_0V_0$$
$$W_{da} = Q_{da} - \Delta U_{da}$$
$$= -\ p_0V_0$$

Efficiency of cycle

In the complete cycle,

$$W_{\text{net}} = W_{ab} + W_{bc} + W_{cd} + W_{da}$$
$$= 0 + 2p_0V_0 + 0 - p_0V_0$$
$$= p_0V_0$$

Note *This W_{net} is also equal to area under the cycle.*

$$\Sigma Q_{+\text{ve}} = Q_{ab} + Q_{bc}$$
$$= 1.5p_0V_0 + 5p_0V_0$$
$$= 6.5p_0V_0$$

$$\therefore \qquad \eta = \frac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}} \times 100$$
$$= \left(\frac{p_0V_0}{6.5p_0V_0}\right) \times 100$$
$$= 15.38\ \%$$

Ans.

Miscellaneous Examples

Example 5 *For a Carnot cycle (or engine) discussed in article 21.4, prove that efficiency of cycle is given by*

$$\eta = \left(1 - \frac{T_2}{T_1}\right)$$

Solution Efficiency $= \dfrac{\text{net work done by gas}}{\text{heat absorbed by gas}} = \dfrac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|}$ ...(i)

Process 1 On this isothermal expansion process, the constant temperature is T_1 so work done by the gas

$$W_1 = nRT_1 \ln\left(\frac{V_b}{V_a}\right) \qquad \text{...(ii)}$$

Remember that $V_b > V_a$, so this quantity is positive, as expected. (In process 1, the gas does work by lifting something)

In isothermal process,

$$Q = W$$
$$\therefore \qquad |Q_1| = |W_1| \qquad \text{...(iii)}$$

Process 2 On this adiabatic expansion process, the temperature and volume are related through

$$TV^{\gamma-1} = \text{constant}$$

∴ $$T_1V_b^{\gamma-1} = T_2V_c^{\gamma-1}$$

or $$\frac{T_1}{T_2} = \left(\frac{V_c}{V_b}\right)^{\gamma-1} \qquad \text{...(iv)}$$

Work done by the gas in this adiabatic process is

$$W_2 = \frac{p_cV_c - p_bV_b}{1-\gamma} = \frac{nRT_c - nRT_b}{1-\gamma}$$

$$= \left(\frac{T_2-T_1}{1-\gamma}\right) nR = \left(\frac{T_1-T_2}{\gamma-1}\right) nR \qquad \text{...(v)}$$

Once again, as expected, this quantity is positive.

Process 3 In the isothermal compression process, the work done by the gas is

$$W_3 = nRT_2 \ln \frac{V_d}{V_c} \qquad \text{...(vi)}$$

Because $V_d < V_c$, the work done by the gas is negative. The work done on the gas is

$$|W_3| = -nRT_2 \ln \frac{V_d}{V_c} = nRT_2 \ln \frac{V_c}{V_d} \qquad \text{...(vii)}$$

Furthermore, just as in process 1,

$$|Q_3| = |W_3| \qquad \text{...(viii)}$$

Process 4 In the adiabatic compression process, the calculations are exactly the same as they were in process 2 but of course with different variables. Therefore,

$$\frac{T_2}{T_1} = \left(\frac{V_a}{V_d}\right)^{\gamma-1} \qquad \text{...(ix)}$$

and $$W_4 = \frac{nR}{\gamma-1}(T_2-T_1) \qquad \text{...(x)}$$

As expected, this quantity is negative and

$$|W_4| = -W_4 = \frac{nR}{\gamma-1}(T_1-T_2) \qquad \text{...(xi)}$$

We can now calculate the efficiency.

$$\text{Efficiency} = \frac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|}$$

$$= 1 + \frac{|W_2| - |W_3| - |W_4|}{|Q_1|} \qquad (\text{as } |W_1| = |Q_1|)$$

But our calculations show that $|W_2| = |W_4|$.

$$\text{Efficiency} = 1 - \frac{|W_3|}{|Q_1|} = 1 - \frac{nRT_2 \ln (V_c/V_d)}{nRT_1 \ln (V_b/V_a)} = 1 - \frac{T_2 \ln (V_c/V_d)}{T_1 \ln (V_b/V_a)} \qquad \text{...(xii)}$$

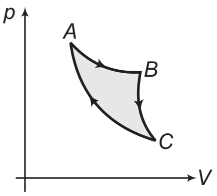
We have seen that

$$\frac{T_1}{T_2} = \left(\frac{V_c}{V_b}\right)^{\gamma-1} = \left(\frac{V_d}{V_a}\right)^{\gamma-1} \quad \text{or} \quad \frac{V_c}{V_b} = \frac{V_d}{V_a} \quad \text{or} \quad \frac{V_c}{V_d} = \frac{V_b}{V_a}$$

Substituting in Eq. (xii), we get

$$\boxed{\text{Efficiency} = 1 - \frac{T_2}{T_1}}$$

Example 6 *An ideal gas expands isothermally along AB and does 700 J of work.*



(a) *How much heat does the gas exchange along AB?*

(b) *The gas then expands adiabatically along BC and does 400 J of work. When the gas returns to A along CA, it exhausts 100 J of heat to its surroundings. How much work is done on the gas along this path?*

Solution (a) AB is an isothermal process. Hence,

$$\Delta U_{AB} = 0$$

and $$Q_{AB} = W_{AB} = 700 \text{ J}$$ **Ans.**

(b) BC is an adiabatic process. Hence,

$$Q_{BC} = 0$$

$$W_{BC} = 400 \text{ J}$$

$$\therefore \quad \Delta U_{BC} = -W_{BC} = -400 \text{ J}$$

ABC is a cyclic process and internal energy is a state function. Therefore,

$$(\Delta U)_{\text{whole cycle}} = 0 = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA}$$

and from first law of thermodynamics,

$$Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + W_{CA}$$

Substituting the values,

$$700 + 0 - 100 = 700 + 400 + \Delta W_{CA}$$

$$\therefore \quad \Delta W_{CA} = -500 \text{ J}$$ **Ans.**

Negative sign implies that work is done on the gas.

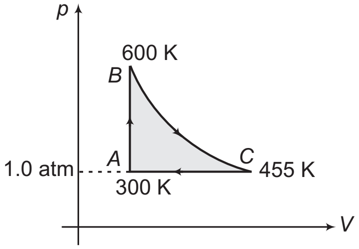
Table below shows different values in different processes.

Table 21.6

Process	Q (J)	W(J)	ΔU(J)
AB	700	700	0
BC	0	400	–400
CA	–100	–500	400
For complete cycle	600	600	0

Note *Total work done is 600 J, which implies that area of the closed curve is also 600 J.*

Example 7 *The p-V diagram of 0.2 mol of a diatomic ideal gas is shown in figure. Process BC is adiabatic. The value of γ for this gas is 1.4.*



(a) *Find the pressure and volume at points A, B and C.*

(b) *Calculate $\Delta Q, \Delta W$ and ΔU for each of the three processes.*

(c) *Find the thermal efficiency of the cycle.*

Take $1 \, atm = 1.0 \times 10^5 \, N/m^2$.

Solution (a) $p_A = p_C = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2$

Process AB is an isochoric process.

$$\therefore \quad p \propto T \quad \text{or} \quad \frac{p_B}{p_A} = \frac{T_B}{T_A}$$

$$\therefore \quad p_B = \left(\frac{T_B}{T_A}\right) p_A = \left(\frac{600}{300}\right)(1 \text{ atm}) = 2 \text{ atm}$$

$$= 2.02 \times 10^5 \text{ N/m}^2$$

From ideal gas equation $$V = \frac{nRT}{p}$$

$$\therefore \quad V_A = V_B = \frac{nRT_A}{p_A}$$

$$= \frac{(0.2)(8.31)(300)}{(1.01 \times 10^5)} \approx 5.0 \times 10^{-3} \text{ m}^3$$

$$= 5 \text{ L}$$

and $$V_C = \frac{nRT_C}{p_C} = \frac{(0.2)(8.31)(455)}{(1.01 \times 10^5)}$$

$$= 7.5 \times 10^{-3} \text{ m}^3$$

$$\approx 7.5 \text{ L}$$

Table 21.7

State	p	V
A	1 atm	5 L
B	2 atm	5 L
C	1 atm	7.5 L

(b) **Process *AB*** is an isochoric process. Hence,

$$\Delta W_{AB} = 0$$

$$\Delta Q_{AB} = \Delta U_{AB} = nC_V \Delta T = n\left(\frac{5}{2}R\right)(T_B - T_A)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)(600 - 300)$$

$$\approx 1246 \text{ J}$$

Process *BC* is an adiabatic process. Hence,

$$\Delta Q_{BC} = 0$$

$$\therefore \quad \Delta W_{BC} = -\Delta U_{BC}$$

$$\Delta U_{BC} = nC_V \Delta T = nC_V (T_C - T_B)$$

$$= (0.2)\left(\frac{5}{2}R\right)(455 - 600)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)(-145) \text{ J}$$

$$\approx -602 \text{ J}$$

$$\therefore \quad \Delta W_{BC} = -\Delta U_{BC} = 602 \text{ J}$$

Process *CA* is an isobaric process. Hence,

$$\Delta Q_{CA} = nC_p\Delta T = n\left(\frac{7}{2}R\right)(T_A - T_C)$$

$$= (0.2)\left(\frac{7}{2}\right)(8.31)(300 - 455)$$

$$\approx -902 \text{ J}$$

$$\Delta U_{CA} = nC_V\Delta T$$

$$= \frac{\Delta Q_{CA}}{\gamma} \qquad \left(\text{as } \gamma = \frac{C_p}{C_V}\right)$$

$$= -\frac{902}{1.4} \approx -644 \text{ J}$$

$$\therefore \qquad \Delta W_{CA} = \Delta Q_{CA} - \Delta U_{CA}$$

$$= -258 \text{ J}$$

Table 21.8

Process	ΔQ (in J)	ΔW (in J)	ΔU (in J)
AB	1246	0	1246
BC	0	602	– 602
CA	– 902	– 258	– 644
Total	344	344	0

(c) Efficiency of the cycle

$$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100 = \frac{344}{1246} \times 100$$

$$= 27.6\%$$

Example 8 *Find the molar specific heat of the process* $p = \frac{a}{T}$ *for a monoatomic gas, a being constant.*

Solution We know that

$$dQ = dU + dW$$

Specific heat,

$$C = \frac{dQ}{dT} = \frac{dU}{dT} + \frac{dW}{dT} \qquad \ldots\text{(i)}$$

Since,

$$dU = C_V dT \qquad \ldots\text{(ii)}$$

$$C = C_V + \frac{dW}{dT}$$

$$= C_V + \frac{pdV}{dT}$$

$$pV = RT$$

∴ For the given process,

$$V = \frac{RT}{p} = \frac{RT^2}{a}$$

$$\frac{dV}{dT} = \frac{2RT}{a}$$

$$\therefore \qquad C = C_V + p\left(\frac{2RT}{a}\right)$$

$$= C_V + 2R$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R$$

Example 9 *At 27° C two moles of an ideal monatomic gas occupy a volume V. The gas expands adiabatically to a volume $2V$. Calculate*

(a) final temperature of the gas

(b) change in its internal energy and

(c) the work done by the gas during the process.

[$R = 8.31$ *J/mol-K*]

Solution (a) In case of adiabatic change

$$TV^{\gamma-1} = \text{constant}$$

So that $$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \text{ with } \gamma = \left(\frac{5}{3}\right)$$

i.e. $$300 \times V^{2/3} = T(2V)^{2/3}$$

or $$T = \frac{300}{(2)^{2/3}} = 189 \text{ K}$$

(b) As $$\Delta U = nC_V\Delta T = n\left(\frac{3}{2}R\right)\Delta T$$

So, $$\Delta U = 2 \times \left(\frac{3}{2}\right) \times 8.31\ (189 - 300)$$

$$= -2767.23 \text{ J}$$

Negative sign means internal energy will decrease.

(c) According to first law of thermodynamics

$$Q = \Delta U + \Delta W$$

And as for adiabatic change $\Delta Q = 0$,

$$\Delta W = -\Delta U = 2767.23 \text{ J}$$

Example 10 *Two moles of a diatomic ideal gas is taken through pT = constant. Its temperature is increased from T to $2T$. Find the work done by the system?*

Solution pT = constant

$$\therefore \qquad p(pV) = \text{constant} \qquad (\text{as } T \propto pV)$$

or $$pV^{1/2} = \text{constant}$$

Comparing with $$pV^x = \text{constant}$$

We have, $$x = \frac{1}{2}$$

$$W = \frac{nR\Delta T}{1-x} = \frac{2R(2T-T)}{1-\frac{1}{2}}$$

$$= 4RT$$

Example 11 *An ideal monatomic gas at temperature* $27°C$ *and pressure* $10^6\ N/m^2$ *occupies* 10 *L volume.* 10,000 *cal of heat is added to the system without changing the volume. Calculate the change in temperature of the gas. Given :* $R = 8.31$ *J/mol-K and* $J = 4.18$ *J/cal.*

Solution For n moles of gas, we have $pV = nRT$

Here, $p = 10^6\ \text{N/m}^2$, $V = 10\ L = 10^{-2}\,\text{m}^3$ and $T = 27°\text{C} = 300\ \text{K}$

$$\therefore \qquad n = \frac{pV}{RT} = \frac{10^6 \times 10^{-2}}{8.31 \times 300} = 4.0$$

For monatomic gas, $C_V = \frac{3}{2}R$

$$\text{Thus,} \qquad C_V = \frac{3}{2} \times 8.31\ \text{J/mol-K}$$

$$= \frac{3}{2} \times \frac{8.31}{4.18} \approx 3\ \text{cal/mol-K}$$

Let ΔT be the rise in temperature when n moles of the gas is given Q cal of heat at constant volume. Then,

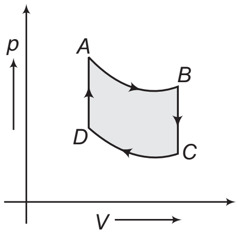
$$Q = nC_V\Delta T \quad \text{or} \quad \Delta T = \frac{Q}{nC_V}$$

$$= \frac{10000\ \text{cal}}{4.0\ \text{mole} \times 3\ \text{cal/mol-K}}$$

$$= 833\ \text{K}$$

Example 12 *One mole of a monoatomic ideal gas is taken through the cycle shown in figure.*

$A \to B$ Adiabatic expansion
$B \to C$ Cooling at constant volume
$C \to D$ Adiabatic compression.
$D \to A$ Heating at constant volume



The pressure and temperature at A, B etc., are denoted by p_A, T_A ; p_B, T_B *etc. respectively.*

Given, $T_A = 1000K$, $p_B = \left(\frac{2}{3}\right)p_A$ *and* $p_C = \left(\frac{1}{3}\right)p_A$. *Calculate*

(a) the work done by the gas in the process $A \to B$
(b) the heat lost by the gas in the process $B \to C$

Given, $\left(\frac{2}{3}\right)^{0.4} = 0.85$ *and* $R = 8.31$ *J/mol-K*

Solution (a) As for adiabatic change $pV^\gamma =$ constant

$$\text{i.e.} \qquad p\left(\frac{nRT}{p}\right)^\gamma = \text{constant} \qquad (\text{as } pV = nRT)$$

$$\text{i.e.} \qquad \frac{T^\gamma}{p^{\gamma-1}} = \text{constant so} \left(\frac{T_B}{T_A}\right)^\gamma = \left(\frac{p_B}{p_A}\right)^{\gamma-1}, \text{ where } \gamma = \frac{5}{3}$$

i.e. $$T_B = T_A \left(\frac{2}{3}\right)^{1-\frac{1}{\gamma}} = 1000\left(\frac{2}{3}\right)^{2/5} = 850\ \text{K}$$

So, $$W_{AB} = \frac{nR[T_F - T_I]}{[1-\gamma]} = \frac{1 \times 8.31\ [1000 - 850]}{\left[\left(\frac{5}{3}\right) - 1\right]}$$

i.e. $$W_{AB} = 1869.75\ \text{J}$$

(b) For $B \rightarrow C$, $V = \text{constant}$ so $\Delta W = 0$

So, from first law of thermodynamics

$$\Delta Q = \Delta U + \Delta W = nC_V \Delta T + 0$$

or $$\Delta Q = 1 \times \left(\frac{3}{2}R\right)(T_C - 850) \qquad \left(\text{as } C_V = \frac{3}{2}R\right)$$

Now, along path BC, $V = \text{constant}$; $p \propto T$

i.e. $$\frac{p_C}{p_B} = \frac{T_C}{T_B}$$

$$T_C = \frac{\left(\frac{1}{3}\right)p_A}{\left(\frac{2}{3}\right)p_A} \times T_B = \frac{T_B}{2} = \frac{850}{2} = 425\ \text{K} \qquad \text{...(ii)}$$

So, $$\Delta Q = 1 \times \frac{3}{2} \times 8.31\ (425 - 850) = -5297.625\ \text{J}$$

[Negative heat means, heat is lost by the system]

Example 13 *A gas undergoes a process such that* $p \propto \frac{1}{T}$. *If the molar heat capacity for this process is* $C = 33.24$ *J/mol-K, find the degree of freedom of the molecules of the gas.*

Solution As $$p \propto \frac{1}{T}$$

or $$pT = \text{constant} \qquad \text{...(i)}$$

We have for one mole of an ideal gas

$$pV = RT \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$p^2V = \text{constant}$$

or $$pV^{1/2} = K\ (\text{say}) = p_i V_i^{1/2} = p_f V_f^{1/2} \qquad \text{...(iii)}$$

From first law of thermodynamics,

$$\Delta Q = \Delta U + \Delta W$$

or $$C\Delta T = C_V \Delta T + \Delta W$$

or $$C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(iv)}$$

Here, $$\Delta W = \int p\,dV = K\int_{V_i}^{V_f} V^{-1/2} dV$$

$$= 2K\ [V_f^{1/2} - V_i^{1/2}] = 2\ [p_f V_f^{1/2} V_f^{1/2} - p_i V_i^{1/2} V_i^{1/2}]$$

$$= 2\ [p_f V_f - p_i V_i] = 2R\ [T_f - T_i]$$

$$= \frac{R\Delta T}{1/2} \Rightarrow \frac{\Delta W}{\Delta T} = 2R$$

Substituting in Eq. (iv), we have

$$C = C_V + 2R = \frac{R}{\gamma - 1} + 2R$$

Substituting the values, $33.24 = R\left(\frac{1}{\gamma - 1} + 2\right) = 8.31\left(\frac{1}{\gamma - 1} + 2\right)$

Solving this we get $\gamma = 1.5$

Now, $\gamma = 1 + \frac{2}{F}$

or degree of freedom $F = \frac{2}{\gamma - 1} = \frac{2}{1.5 - 1} = 4$

Alternate Solution In the process pV^x = constant, molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

The given process is $pV^{1/2}$ = constant or $x = \frac{1}{2}$

$$\therefore \qquad C = \frac{R}{\gamma - 1} + \frac{R}{1 - \frac{1}{2}} = \frac{R}{\gamma - 1} + 2R$$

Now, we may proceed in the similar manner.

Example 14 *A gaseous mixture enclosed in a vessel consists of one gram mole of a gas A with* $\gamma = \left(\frac{5}{3}\right)$ *and some amount of gas B with* $\gamma = \frac{7}{5}$ *at a temperature* T. *The gases A and B do not react with each other and are assumed to be ideal. Find the number of gram moles of the gas B if* γ *for the gaseous mixture is* $\left(\frac{19}{13}\right)$.

Solution As for an ideal gas, $C_p - C_V = R$ and $\gamma = \left(\frac{C_p}{C_V}\right)$

So, $C_V = \frac{R}{(\gamma - 1)}$

$$\therefore \qquad (C_V)_1 = \frac{R}{\left(\frac{5}{3}\right) - 1} = \frac{3}{2}R;$$

$$(C_V)_2 = \frac{R}{\left(\frac{7}{5}\right) - 1} = \frac{5}{2}R$$

and $(C_V)_{\text{mix}} = \frac{R}{\left(\frac{19}{13}\right) - 1} = \frac{13}{6}R$

Now, from conservation of energy,

i.e $\Delta U = \Delta U_1 + \Delta U_2$

$$(n_1 + n_2)\,(C_V)_{\text{mix}}\,\Delta T = [n_1 (C_V)_1 + n_2 (C_V)_2]\,\Delta T$$

i.e.
$$(C_V)_{\text{mix}} = \frac{n_1 (C_V)_1 + n_2 (C_V)_2}{n_1 + n_2}$$

We have
$$\frac{13}{6} R = \frac{1 \times \frac{3}{2} R + n_2 \frac{5}{2} R}{1 + n_2}$$

$$= \frac{(3 + 5n_2)R}{2\,(1 + n_2)}$$

or
$$13 + 13n_2 = 9 + 15n_2,$$

i.e.
$$n_2 = 2$$ **Ans.**

Example 15 *An ideal gas having initial pressure p, volume V and temperature T is allowed to expand adiabatically until its volume becomes $5.66\,V$, while its temperature falls to $T/2$.*

(a) How many degrees of freedom do the gas molecules have?

(b) Obtain the work done by the gas during the expansion as a function of the initial pressure p and volume V.

Given that $(5.66)^{0.4} = 2$

Solution (a) For adiabatic expansion,

$$TV^{\gamma - 1} = \text{constant}$$

i.e.
$$TV^{\gamma - 1} = T' V'^{\gamma - 1}$$

$$= \frac{T}{2} (5.66\,V)^{\gamma - 1}$$

i.e.
$$(5.66)^{\gamma - 1} = 2$$

i.e.
$$\gamma = 1.4$$

Using $\gamma = 1 + \frac{2}{F}$

We get degree of freedom,

$$F = 5$$ **Ans.**

(b) Work done during adiabatic process for one mole gas is

$$W = \frac{1}{1 - \gamma} [p' V' - pV]$$

From relation,
$$\frac{pV}{T} = \frac{p' V'}{T'}$$

We get
$$p' = \frac{T'}{T} \cdot \frac{pV}{V'} = \frac{1}{2} \times \frac{1}{5.66} p = \frac{p}{11.32}$$

∴
$$W = \frac{1}{1 - 1.4} \left[\frac{p}{11.32} \times \frac{V}{5.66} - pV \right]$$

$$= \frac{1}{0.4} \left[1 - \frac{1}{11.32 \times 5.66} \right] pV$$

$$= 2.461\,pV$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : In adiabatic expansion, temperature of gas always decreases.

Reason : In adiabatic process exchange of heat is zero.

2. Assertion : In a thermodynamic process, initial volume of gas is equal to final volume of gas. Work done by gas in this process should be zero.

Reason : Work done by gas in isochoric process is zero.

3. Assertion : First law of thermodynamics can be applied for ideal gases only.

Reason : First law is simply, law of conservation of energy.

4. Assertion : When ice melts, work is done by atmosphere on (ice + water) system.

Reason : On melting of ice volume of (ice + water) system decreases.

5. Assertion : Between two thermodynamic states, the value of $(Q - W)$ is constant for any process.

Reason : Q and W are path functions.

6. Assertion : Efficiency of a heat engine can't be greater than efficiency of Carnot engine.

Reason : Efficiency of any engine is never 100%.

7. Assertion : In the process $pT =$ constant, if temperature of gas is increased work done by the gas is positive.

Reason : For the given process, $V \propto T$.

8. Assertion : In free expansion of a gas inside an adiabatic chamber Q, W and ΔU all are zero.

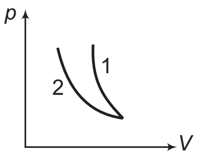
Reason : In such an expansion $p \propto \frac{1}{V}$.

9. Assertion : For an ideal gas in a cyclic process and in an isothermal process change in internal energy is zero.

Reason : In both processes there is no change in temperature.

10. Assertion : Isothermal and adiabatic, two processes are shown on p-V diagram. Process-1 is adiabatic and process-2 is isothermal.

Reason : At a given point, slope of adiabatic process $= \gamma \times$ slope of isothermal process.



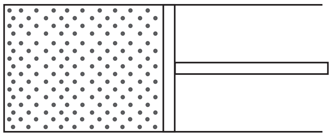
Objective Questions

1. In a process, the pressure of an ideal gas is proportional to square of the volume of the gas. If the temperature of the gas increases in this process, then work done by this gas

(a) is positive (b) is negative

(c) is zero (d) may be positive or negative

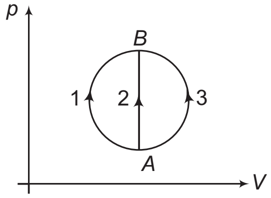
2. n moles of a gas are filled in a container at temperature T. If the gas is slowly and isothermally compressed to half its initial volume, the work done by the atmosphere on the gas is



(a) $\frac{nRT}{2}$ (b) $-\frac{nRT}{2}$

(c) $nRT \ln 2$ (d) $-nRT \ln 2$

3. A gas undergoes A to B through three different processes 1, 2 and 3 as shown in the figure. The heat supplied to the gas is Q_1, Q_2 and Q_3 respectively, then



(a) $Q_1 = Q_2 = Q_3$ (b) $Q_1 < Q_2 < Q_3$

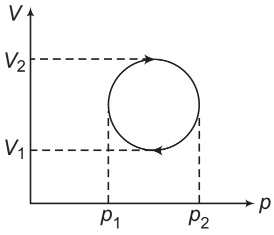
(c) $Q_1 > Q_2 > Q_3$ (d) $Q_1 = Q_3 > Q_2$

4. For an adiabatic compression the quantity pV

(a) increases (b) decreases

(c) remains constant (d) depends on γ

5. The cyclic process form a circle on a pV diagram as shown in figure. The work done by the gas is



(a) $\frac{\pi}{4}(p_2 - p_1)^2$ (b) $\frac{\pi}{4}(V_2 - V_1)^2$

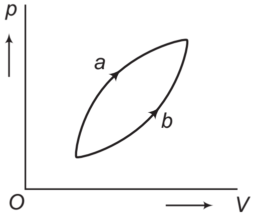
(c) $\frac{\pi}{2}(p_2 - p_1)(V_2 - V_1)$ (d) $\frac{\pi}{4}(p_2 - p_1)(V_1 - V_2)$

6. An ideal gas has initial volume V and pressure p. In doubling its volume the minimum work done will be in the process (of the given processes)

(a) isobaric process (b) isothermal process

(c) adiabatic process (d) same in all given processes

7. Figure shows two processes a and b for a given sample of a gas. If $\Delta Q_1, \Delta Q_2$ are the amounts of heat absorbed by the system in the two cases and $\Delta U_1, \Delta U_2$ are changes in internal energies respectively, then



(a) $\Delta Q_1 = \Delta Q_2; \Delta U_1 = \Delta U_2$
(b) $\Delta Q_1 > \Delta Q_2; \Delta U_1 > \Delta U_2$
(c) $\Delta Q_1 < \Delta Q_2; \Delta U_1 < \Delta U_2$
(d) $\Delta Q_1 > \Delta Q_2; \Delta U_1 = \Delta U_2$

8. A Carnot engine works between 600 K and 300 K. The efficiency of the engine is

(a) 50% (b) 70%
(c) 20% (d) 80%

9. Air in a cylinder is suddenly compressed by a piston which is then maintained at the same position. As the time passes pressure of the gas

(a) increases
(b) decreases
(c) remains the same
(d) may increase or decrease depending on the nature of the gas

10. A cycle pump becomes hot near the nozzle after a few quick strokes even if they are smooth because

(a) the volume of air decreases (b) the number of air molecules increases
(c) the compression is adiabatic (d) collision between air particles increases

11. In an adiabatic change, the pressure p and temperature T of a diatomic gas are related by the relation $p \propto T^{\alpha}$, where α equals

(a) 1.67 (b) 0.4
(c) 0.6 (d) 3.5

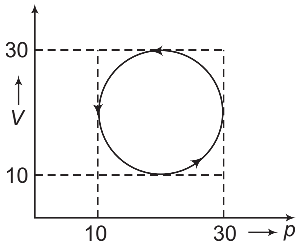
12. A diatomic gas obeys the law pV^x = constant. For what value of x, it has negative molar specific heat ?

(a) $x > 1.4$ (b) $x < 1.4$
(c) $1 < x < 1.4$ (d) $0 < x < 1$

13. The molar specific heat at constant volume of gas mixture is $\frac{13R}{6}$. The gas mixture consists of

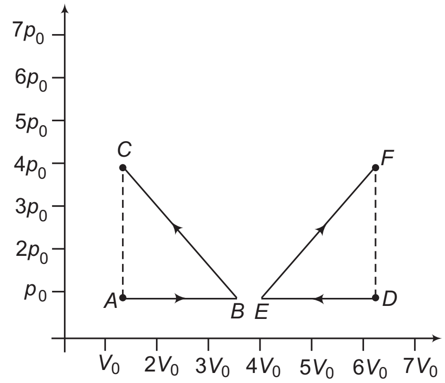
(a) 2 moles of O_2 and 4 moles of H_2 (b) 2 moles of O_2 and 4 moles of argon
(c) 2 moles of argon and 4 moles of O_2 (d) 2 moles of CO_2 and 4 moles of argon

14. Heat energy absorbed by a system in going through a cyclic process as shown in the figure [V in litres and p in kPa] is



(a) $10^7 \pi$J (b) $10^4 \pi$J (c) $10^2 \pi$J (d) $10^3 \pi$J

15. If W_{ABC} is the work done in process $A \to B \to C$ and W_{DEF} is work done in process $D \to E \to F$ as shown in the figure, then



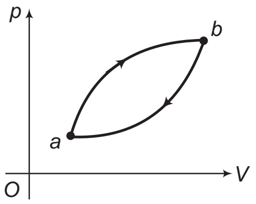
(a) $|W_{DEF}| > |W_{ABC}|$
(b) $|W_{DEF}| < |W_{ABC}|$
(c) $W_{DEF} = W_{ABC}$
(d) $W_{DEF} = -W_{ABC}$

Subjective Questions

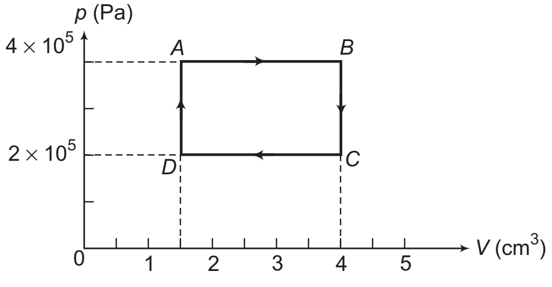
1. How many moles of helium at temperature 300 K and 1.00 atm pressure are needed to make the internal energy of the gas 100 J?

2. Show how internal energy U varies with T in isochoric, isobaric and adiabatic process?

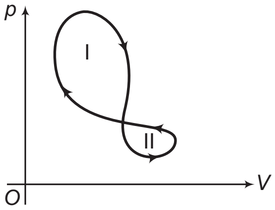
3. A system is taken around the cycle shown in figure from state a to state b and then back to state a. The absolute value of the heat transfer during one cycle is 7200 J. (a) Does the system absorb or liberate heat when it goes around the cycle in the direction shown in the figure? (b) What is the work W done by the system in one cycle? (c) If the system goes around the cycle in a counter-clock wise direction, does it absorb or liberate heat in one cycle? What is the magnitude of the heat absorbed or liberated in one counter-clockwise cycle?



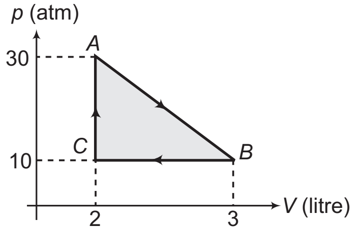
4. For the thermodynamic cycle shown in figure find (a) net output work of the gas during the cycle, (b) net heat flow into the gas per cycle.



5. A thermodynamic system undergoes a cyclic process as shown in figure. The cycle consists of two closed loops, loop I and loop II. (a) Over one complete cycle, does the system do positive or negative work? (b) In each of loops I and II, is the net work done by the system positive or negative ? (c) Over one complete cycle, does heat flow into or out of the system? (d) In each of loops I and II, does heat flow into or out of the system?



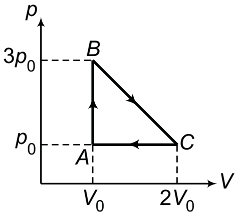
6. A gas undergoes the cycle shown in figure. The cycle is repeated 100 times per minute. Determine the power generated.



7. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat is added (a) at constant volume (b) at constant pressure.

8. A closed vessel 10 L in volume contains a diatomic gas under a pressure of 10^5N/m^2. What amount of heat should be imparted to the gas to increase the pressure in the vessel five times?

9. One mole of an ideal monatomic gas is taken round the cyclic process *ABCA* as shown in figure. Calculate



(a) the work done by the gas.
(b) the heat rejected by the gas in the path *CA* and heat absorbed in the path *AB*.
(c) the net heat absorbed by the gas in the path *BC*.
(d) the maximum temperature attained by the gas during the cycle.

10. A diatomic ideal gas is heated at constant volume until its pressure becomes three times. It is again heated at constant pressure until its volume is doubled. Find the molar heat capacity for the whole process.

11. Two moles of a certain gas at a temperature $T_0 = 300\,\text{K}$ were cooled isochorically so that the pressure of the gas got reduced 2 times. Then as a result of isobaric process, the gas is allowed to expand till its temperature got back to the initial value. Find the total amount of heat absorbed by gas in this process.

12. Five moles of an ideal monoatomic gas with an initial temperature of $127°C$ expand and in the process absorb 1200 J of heat and do 2100 J of work. What is the final temperature of the gas?

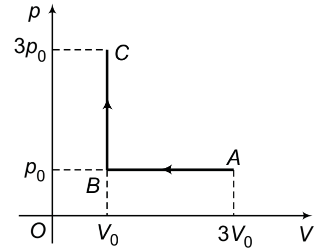
13. Find the change in the internal energy of 2 kg of water as it is heated from $0°C$ to $4°C$. The specific heat capacity of water is 4200 J/kg-K and its densities at $0°C$ and 4°C are 999.9 kg/m^3 and 1000 kg/m^3, respectively. Atmospheric pressure = 10^5 Pa.

14. Calculate the increase in the internal energy of 10 g of water when it is heated from $0°C$ to $100°C$ and converted into steam at 100 kPa. The density of steam = 0.6 kg/m^3. Specific heat capacity of water = 4200 J/kg-°C and the latent heat of vaporisation of water = 2.5×10^6 J/kg.

15. One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at a constant pressure of 1 atm (1.013×10^5 Pa). The heat of vaporization at this pressure is $L_v = 2.256 \times 10^6$ J/kg. Compute (a) the work done by the water when it vaporizes and (b) its increase in internal energy.

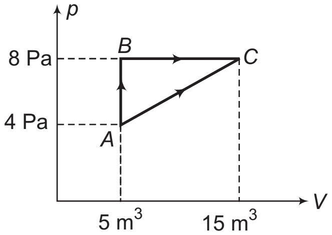
16. A gas in a cylinder is held at a constant pressure of 2.30×10^5 Pa and is cooled and compressed from 1.70 m^3 to 1.20 m^3. The internal energy of the gas decreases by 1.40×10^5 J. (a) Find the work done by the gas. (b) Find the absolute value $|Q|$ of the heat flow into or out of the gas and state the direction of the heat flow. (c) Does it matter whether or not the gas is ideal? Why or why not?

17. *p*-*V* diagram of an ideal gas for a process *ABC* is as shown in the figure.



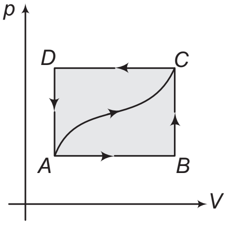
(a) Find total heat absorbed or released by the gas during the process *ABC*.
(b) Change in internal energy of the gas during the process *ABC*.
(c) Plot pressure *versus* density graph of the gas for the process *ABC*.

18. In the given graph, an ideal gas changes its state from *A* to *C* by two paths *ABC* and *AC*.



(a) Find the path along which work done is less.
(b) The internal energy of gas at *A* is 10 J and the amount of heat supplied in path *AC* is 200 J. Calculate the internal energy of gas at *C*.
(c) The internal energy of gas at state *B* is 20 J. Find the amount of heat supplied to the gas to go from *A* to *B*.

19. When a gas expands along *AB*, it does 500 J of work and absorbs 250 J of heat. When the gas expands along *AC*, it does 700 J of work and absorbs 300 J of heat.



(a) How much heat does the gas exchange along *BC*?

(b) When the gas makes the transition from *C* to *A* along *CDA*, 800 J of work are done on it from *C* to *D*. How much heat does it exchange along *CDA*?

20. A 1.0 kg bar of copper is heated at atmospheric pressure ($1.01 \times 10^5\ \text{N/m}^2$). If its temperature increases from 20°C to 50°C, calculate the change in its internal energy. $\alpha = 7.0 \times 10^{-6}\ /°\text{C}$, $\rho = 8.92 \times 10^3\ \text{kg/m}^3$ and $c = 387\ \text{J/kg-°C}$

21. One mole of an ideal monoatomic gas occupies a volume of $1.0 \times 10^{-2}\ \text{m}^3$ at a pressure of $2.0 \times 10^5\ \text{N/m}^2$.

(a) What is the temperature of the gas?

(b) The gas undergoes an adiabatic compression until its volume is decreased to $5.0 \times 10^{-3}\ \text{m}^3$. What is the new gas temperature?

(c) How much work is done on the gas during the compression?

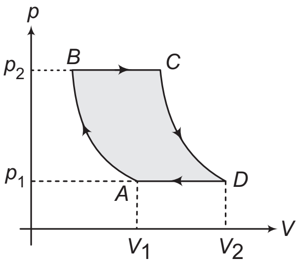
(d) What is the change in the internal energy of the gas?

22. A bullet of mass 10 g travelling horizontally at 200 m/s strikes and embeds in a pendulum bob of mass 2.0 kg.

(a) How much mechanical energy is dissipated in the collision?

(b) Assuming that C_v for the bob plus bullet is 3 *R*, calculate the temperature increase of the system due to the collision. Take the molecular mass of the system to be 200 g/mol.

23. An ideal gas is carried through a thermodynamic cycle consisting of two isobaric and two isothermal processes as shown in figure. Show that the net work done in the entire cycle is given by the equation. $W_{\text{net}} = p_1 (V_2 - V_1) \ln \frac{p_2}{p_1}$



24. An ideal gas is enclosed in a cylinder with a movable piston on top. The piston has mass of 8000 g and an area of 5.00 cm^2 and is free to slide up and down, keeping the pressure of the gas constant. How much work is done as the temperature of 0.200 mol of the gas is raised from 200°C to 300°C?

LEVEL 2

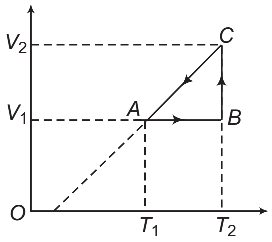
Single Correct Option

1. The equation of a state of a gas is given by $p(V - b) = nRT$. If 1 mole of a gas is isothermally expanded from volume V and $2V$, the work done during the process is

(a) $RT \ln \left|\frac{2V - b}{V - b}\right|$ (b) $RT \ln \left|\frac{V - b}{V}\right|$

(c) $RT \ln \left|\frac{V - b}{2V - b}\right|$ (d) $RT \ln \left|\frac{V}{V - b}\right|$

2. A cyclic process for 1 mole of an ideal gas is shown in the V-T diagram. The work done in AB, BC and CA respectively is



(a) $0, RT_2 \ln \left|\frac{V_2}{V_1}\right|, R(T_1 - T_2)$

(b) $R(T_1 - T_2), 0, RT_1 \ln \left|\frac{V_1}{V_2}\right|$

(c) $0, RT_1 \ln \left|\frac{V_1}{V_2}\right|, R(T_1 - T_2)$

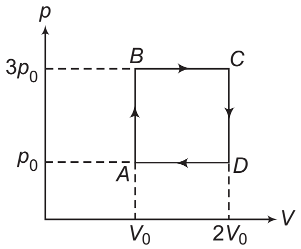
(d) $0, RT_2 \ln \left|\frac{V_2}{V_1}\right|, R(T_2 - T_1)$

3. Ten moles of a diatomic perfect gas are allowed to expand at constant pressure. The initial volume and temperature are V_0 and T_0, respectively. If $\frac{7}{2} RT_0$ heat is transferred to the gas, then the final volume and temperature are

(a) $1.1\, V_0, 1.1\, T_0$ (b) $0.9\, V_0, 0.9\, T_0$

(c) $1.1\, V_0, \frac{10}{11} T_0$ (d) $0.9\, V_0, \frac{10}{9} T_0$

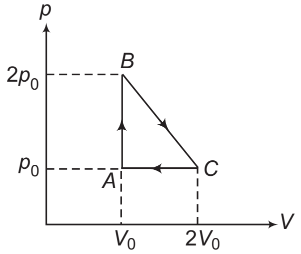
4. An ideal monoatomic gas is carried around the cycle $ABCDA$ as shown in the figure. The efficiency of the gas cycle is



(a) $\frac{4}{21}$ (b) $\frac{2}{21}$

(c) $\frac{4}{31}$ (d) $\frac{2}{31}$

5. In the process shown in figure, the internal energy of an ideal gas decreases by $\frac{3p_0V_0}{2}$ in going from point C to A. Heat transfer along the process CA is



(a) $(-3\ p_0V_0)$
(b) $(-5\ p_0V_0/2)$
(c) $(-3\ p_0V_0/2)$
(d) zero

6. One mole of an ideal monoatomic gas at temperature T_0 expands slowly according to the law $\frac{p}{V}$ = constant. If the final temperature is $2T_0$, heat supplied to the gas is

(a) $2RT_0$
(b) $\frac{3}{2}RT_0$
(c) RT_0
(d) $\frac{1}{2}RT_0$

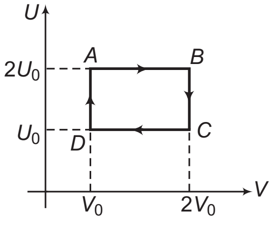
7. A mass of gas is first expanded isothermally and then compressed adiabatically to its original volume. What further simplest operation must be performed on the gas to restore it to its original state?

(a) An isobaric cooling to bring its temperature to initial value
(b) An isochoric cooling to bring its pressure to its initial value
(c) An isothermal process to take its pressure to its initial value
(d) An isochoric heating to bring its temperature to initial value

8. A monatomic ideal gas, initially at temperature T_1, is enclosed in a cylinder fitted with a frictionless piston. The gas is allowed to expand adiabatically to a temperature T_2 by releasing the piston suddenly. If L_1 and L_2 are the lengths of gas column before and after expansion respectively, then $\frac{T_1}{T_2}$ is given by

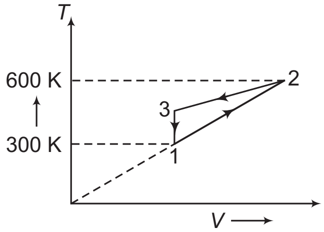
(a) $\left(\frac{L_1}{L_2}\right)^{\frac{2}{3}}$
(b) $\frac{L_1}{L_2}$
(c) $\frac{L_2}{L_1}$
(d) $\left(\frac{L_2}{L_1}\right)^{\frac{2}{3}}$

9. One mole of an ideal gas is taken through a cyclic process. The minimum temperature during the cycle is 300 K. Then, net exchange of heat for complete cycle is



(a) $600\ R \ln 2$
(b) $300\ R \ln 2$
(c) $-300\ R \ln 2$
(d) $900\ R \ln 2$

10. Two moles of an ideal gas are undergone a cyclic process 1-2-3-1. If net heat exchange in the process is 300 J, the work done by the gas in the process 2-3 is



(a) –500 J (b) – 5000 J (c) – 3000 J (d) None of these

11. Two cylinders fitted with pistons contain equal amount of an ideal diatomic gas at 300 K. The piston of *A* is free to move, while that of *B* is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in *A* is 30 K, then the rise in temperature of gas in *B* is

(a) 30 K (b) 18 K
(c) 50 K (d) 42 K

12. A gas follows a process TV^{n-1} = constant, where T = absolute temperature of the gas and V = volume of the gas. The bulk modulus of the gas in the process is given by

(a) $(n-1)p$ (b) $p/(n-1)$
(c) np (d) p/n

13. One mole of an ideal gas at temperature T_1 expands slowly according to the law $\frac{p}{V}$ = constant. Its final temperature is T_2. The work done by the gas is

(a) $R(T_2-T_1)$ (b) $2R(T_2-T_1)$
(c) $\frac{R}{2}(T_2-T_1)$ (d) $\frac{2R}{3}(T_2-T_1)$

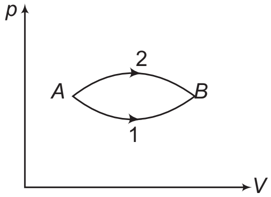
14. 600 J of heat is added to a monoatomic gas in a process in which the gas performs a work of 150 J. The molar heat capacity for the process is

(a) $3R$ (b) $4R$
(c) $2R$ (d) $6R$

15. The internal energy of a gas is given by $U = 2pV$. It expands from V_0 to $2V_0$ against a constant pressure p_0. The heat absorbed by the gas in the process is

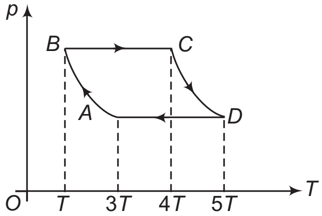
(a) $2p_0V_0$ (b) $4p_0V_0$
(c) $3p_0V_0$ (d) p_0V_0

16. The figure shows two paths for the change of state of a gas from *A* to *B*. The ratio of molar heat capacities in path 1 and path 2 is



(a) < 1 (b) > 1
(c) 1 (d) Data insufficient

17. *p*-*T* diagram of one mole of an ideal monatomic gas is shown. Processes *AB* and *CD* are adiabatic. Work done in the complete cycle is

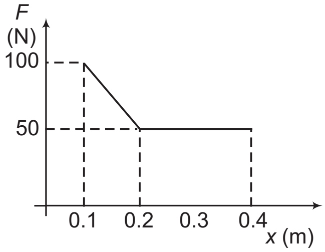


(a) 2.5 *RT*
(b) −2 *RT*
(c) 1.5 *RT*
(d) −3.5 *RT*

18. An ideal monoatomic gas undergoes a process in which its internal energy U and density ρ vary as $U\rho$ = constant. The ratio of change in internal energy and the work done by the gas is

(a) $\frac{3}{2}$
(b) $\frac{2}{3}$
(c) $\frac{1}{3}$
(d) $\frac{3}{5}$

19. The given figure shows the variation of force applied by ideal gas on a piston which undergoes a process during which piston position changes from 0.1 to 0.4 m. If the internal energy of the system at the end of the process is 2.5 J higher, then the heat absorbed during the process is



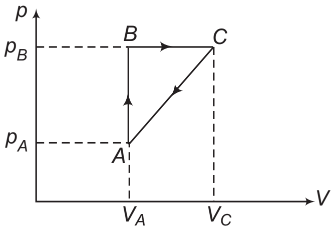
(a) 15 J
(b) 17.5 J
(c) 20 J
(d) 22.5 J

20. A gas can expand through two processes : (i) isobaric, (ii) $\frac{p}{V}$ = constant. Assuming that the initial volume is same in both processes and the final volume which is two times the initial volume is also same in both processes, which of the following is true?

(a) Work done by gas in process (i) is greater than the work done by the gas in process (ii)
(b) Work done by gas in process (i) is smaller than the work done by the gas in process (ii)
(c) Final pressure is greater in process (i)
(d) Final temperature is greater in process (i)

21. An ideal gas of adiabatic exponent γ is expanded so that the amount of heat transferred to the gas is equal to the decrease of its internal energy. Then, the equation of the process in terms of the variables T and V is

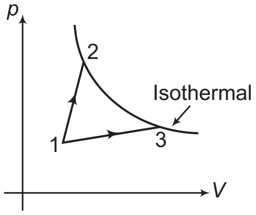
(a) $TV^{\frac{(\gamma-1)}{2}} = C$
(b) $TV^{\frac{(\gamma-2)}{2}} = C$
(c) $TV^{\frac{(\gamma-1)}{4}} = C$
(d) $TV^{\frac{(\gamma-2)}{4}} = C$



22. A thermodynamical process is shown in the figure with $p_A = 3 \times p_{\text{atm}}$, $V_A = 2 \times 10^{-4}\ \text{m}^3$, $p_B = 8 \times p_{\text{atm}}$, $V_C = 5 \times 10^{-4}\ \text{m}^3$. In the process AB and BC, 600 J and 200 J heat are added to the system. Find the change in internal energy of the system in the process CA. $[1\ p_{\text{atm}} = 10^5\ \text{N/m}^2]$

(a) 560 J (b) –560 J
(c) –240 J (d) +240 J

23. A gas takes part in two processes in which it is heated from the same initial state 1 to the same final temperature. The processes are shown on the p-V diagram by the straight lines 1-3 and 1-2. 2 and 3 are the points on the same isothermal curve. Q_1 and Q_2 are the heat transfer along the two processes. Then,

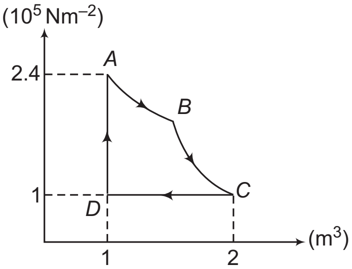


(a) $Q_1 = Q_2$ (b) $Q_1 < Q_2$
(c) $Q_1 > Q_2$ (d) Insufficient data

24. A closed system receives 200 kJ of heat at constant volume. It then rejects 100 kJ of heat while it has 50 kJ of work done on it at constant pressure. If an adiabatic process can be found which will restore the system to its initial state, the work done by the system during this process is

(a) 100 kJ (b) 50 kJ
(c) 150 kJ (d) 200 kJ

25. 100 moles of an ideal monatomic gas undergoes the thermodynamic process as shown in the figure

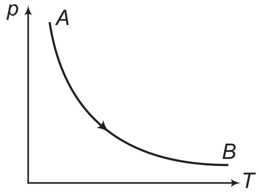


$A \to B$: isothermal expansion $B \to C$: adiabatic expansion
$C \to D$: isobaric compression $D \to A$: isochoric process

The heat transfer along the process AB is 9×10^4 J. The net work done by the gas during the cycle is [Take $R = 8\ \text{JK}^{-1}\ \text{mol}^{-1}$)

(a) -0.5×10^4 J (b) $+0.5 \times 10^4$ J
(c) -5×10^4 J (d) $+5 \times 10^4$ J

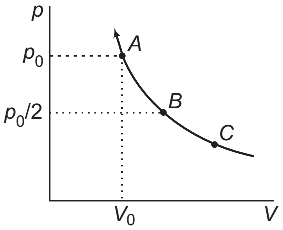
26. Two moles of an ideal monoatomic gas are expanded according to the equation pT = constant from its initial state (p_0, V_0) to the final state due to which its pressure becomes half of the initial pressure. The change in internal energy is



(a) $\frac{3p_0V_0}{4}$ (b) $\frac{3p_0V_0}{2}$

(c) $\frac{9p_0V_0}{2}$ (d) $\frac{5p_0V_0}{2}$

27. The state of an ideal gas is changed through an isothermal process at temperature T_0 as shown in figure. The work done by the gas in going from state B to C is double the work done by gas in going from state A to B. If the pressure in the state B is $\frac{p_0}{2}$, then the pressure of the gas in state C is



a) $\frac{p_0}{3}$ (b) $\frac{p_0}{4}$

(c) $\frac{p_0}{6}$ (d) $\frac{p_0}{8}$

More than One Correct Options

1. An ideal gas is taken from the state A (pressure p, volume V) to the state B (pressure $\frac{p}{2}$, volume $2V$) along a straight line path in the p-V diagram. Select the correct statement(s) from the following.

(a) The work done by the gas in the process A to B is negative
(b) In the T-V diagram, the path AB becomes a part of a parabola
(c) In the p-T diagram, the path AB becomes a part of a hyperbola
(d) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases

2. In the process pV^2 = constant, if temperature of gas is increased, then

(a) change in internal energy of gas is positive
(b) work done by gas is positive
(c) heat is given to the gas
(d) heat is taken out from the gas

3. T-V diagram of two moles of a monoatomic gas is as shown in figure.

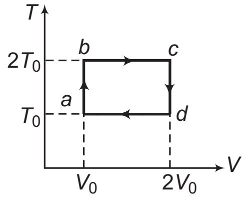
For the process *abcda* choose the correct options given below

(a) $\Delta U = 0$

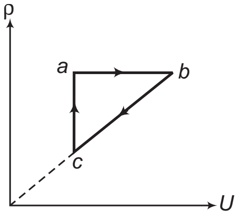
(b) work done by gas > 0

(c) heat given to the gas is $4RT_0$

(d) heat given to the gas is $2RT_0$



4. Density (ρ) *versus* internal energy (U) graph of a gas is as shown in figure. Choose the correct options.



(a) $Q_{bc} = 0$ (b) $W_{bc} = 0$ (c) $W_{ca} < 0$ (d) $Q_{ab} > 0$

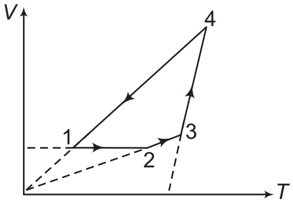
Here, W is work done by gas and Q is heat given to the gas.

5. Temperature of a monatomic gas is increased from T_0 to $2T_0$ in three different processes : isochoric, isobaric and adiabatic. Heat given to the gas in these three processes are Q_1, Q_2 and Q_3 respectively. Then, choose the correct option.

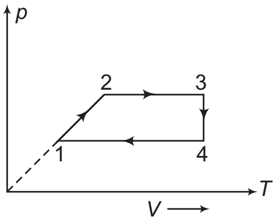
(a) $Q_1 > Q_3$ (b) $Q_2 > Q_1$

(c) $Q_2 > Q_3$ (d) $Q_3 = 0$

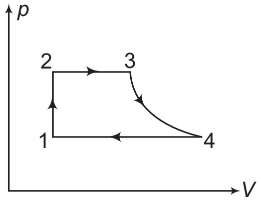
6. A cyclic process 1-2-3-4-1 is depicted on V-T diagram. The p-T and p-V diagrams for this cyclic process are given below. Select the correct choices (more than one options is/are correct)



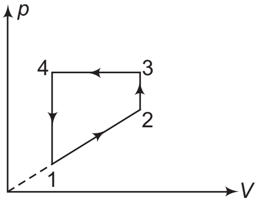
(a)



(b)



(c)

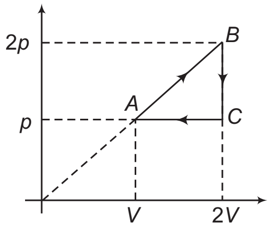


(d) None of these

Comprehension Based Questions

Passage I (Q. No. 1 and 2)

One mole of a monatomic ideal gas is taken along the cycle ABCA as shown in the diagram.



1. The net heat absorbed by the gas in the given cycle is

(a) pV (b) $\frac{pV}{2}$

(c) $2pV$ (d) $4pV$

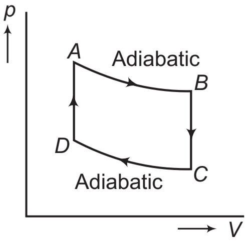
2. The ratio of specific heat in the process CA to the specific heat in the process BC is

(a) 2 (b) $\frac{5}{3}$

(c) 4 (d) None of these

Passage II (Q. No. 3 to 5)

One mole of a monatomic ideal gas is taken through the cycle ABCDA as shown in the figure. $T_A = 1000\,K$ *and* $2p_A = 3p_B = 6p_C$.



$$\left[Assume \left(\frac{2}{3}\right)^{0.4} = 0.85 \text{ and } R = \frac{25}{3} JK^{-1}\, mol^{-1} \right]$$

3. The temperature at B is

(a) 350 K (b) 1175 K

(c) 850 K (d) 577 K

4. Work done by the gas in the process $A \to B$ is

(a) 5312 J (b) 1875 J

(c) 6251 J (d) 8854 J

5. Heat lost by the gas in the process $B \to C$ is

(a) 5312 J (b) 1875 J

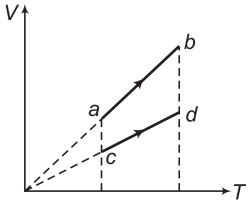
(c) 6251 J (d) 8854 J

Match the Columns

1. Temperature of 2 moles of a monatomic gas is increased from T to $2T$. Match the following two columns.

Column I	Column II
(a) Work done by gas in isobaric process	(p) $3RT$
(b) Change in internal energy in isobaric process	(q) $4RT$
(c) Work done by gas in adiabatic process	(r) $2RT$
(d) Change in internal energy in an adiabatic process	(s) None of these

2. For V-T diagrams of two processes a - b and c - d are shown in figure for same gas. Match the following two columns.



Column I	Column II
(a) Work done	(p) is more in process ab
(b) Change in internal energy	(q) is more in process ca
(c) Heat exchange	(r) is same in both processes
(d) Molar heat capacity	(s) Can't say anything

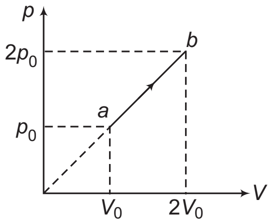
3. Temperature of a monatomic gas is increased by ΔT in process p^2V = constant. Match the following two columns.

Column I	Column II
(a) Work done by gas	(p) $2nR\Delta T$
(b) Change in internal energy of gas	(q) $5nR\Delta T$
(c) Heat taken by the gas	(r) $3nR\Delta T$
(d) Work done on the gas	(s) None of these

4. Match the following two columns.

Column I	Column II
(a) Isobaric expansion	(p) $W > \Delta U$
(b) Isochoric cooling	(q) $W < \Delta U$
(c) Adiabatic expansion	(r) $Q = \Delta U$
(d) Isothermal expansion	(s) $Q < \Delta U$

5. Heat taken by a gas in process a-b is $6p_0V_0$. Match the following columns.

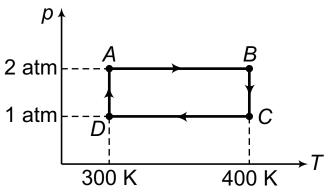


Column I	Column II
(a) W_{ab}	(p) $2p_0V_0$
(b) ΔU_{ab}	(q) $4p_0V_0$
(c) Molar heat capacity in given process	(r) $2R$
(d) C_V of gas	(s) None of these

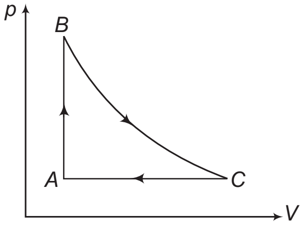
Subjective Questions

1. Two moles of helium gas undergo a cyclic process as shown in figure. Assuming the gas to be ideal, calculate the following quantities in this process.

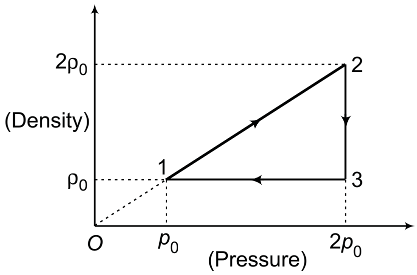
(a) The net change in the heat energy.
(b) The net work done.
(c) The net change in internal energy.



2. 1.0 k-mol of a sample of helium gas is put through the cycle of operations shown in figure. BC is an isothermal process and $p_A = 1.00$ atm, $V_A = 22.4$ m^3, $p_B = 2.00$ atm. What are T_A, T_B and V_C?



3. The density (ρ) *versus* pressure (p) graph of one mole of an ideal monoatomic gas undergoing a cyclic process is shown in figure. Molecular mass of gas is M.



(a) Find work done in each process.
(b) Find heat rejected by gas in one complete cycle.
(c) Find the efficiency of the cycle.

4. An ideal gas goes through the cycle *abc*. For the complete cycle 800 J of heat flows out of the gas. Process *ab* is at constant pressure and process *bc* is at constant volume . In process *c-a*, $p \propto V$. States *a* and *b* have temperatures $T_a = 200\,\text{K}$ and $T_b = 300\,\text{K}$. (a) Sketch the p-V diagram for the cycle. (b) What is the work done by the gas for the process *ca*?

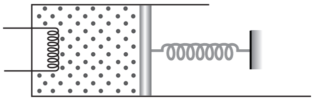
5. A cylinder of ideal gas is closed by an 8 kg movable piston of area $60\ \text{cm}^2$. The atmospheric pressure is 100 kPa. When the gas is heated from 30° C to 100° C, the piston rises 20 cm. The piston is then fastened in the place and the gas is cooled back to 30° C. If ΔQ_1 is the heat added to the gas during heating and ΔQ_2 is the heat lost during cooling, find the difference.

6. Three moles of an ideal gas $\left(C_p = \frac{7}{2} R\right)$ at pressure p_0 and temperature T_0 is isothermally expanded to twice its initial volume. It is then compressed at a constant pressure to its original volume.

(a) Sketch p-V and p-T diagram for complete process.
(b) Calculate net work done by the gas.
(c) Calculate net heat supplied to the gas during complete process.
(Write your answer in terms of gas constant = R)

7. Two moles of a gas ($\gamma = 5/3$) are initially at temperature 27°C and occupy a volume of 20 litres. The gas is first expanded at constant pressure until the volume is doubled. Then, it is subjected to an adiabatic change until the temperature returns to its initial value.

(a) Sketch the process on a p-V diagram.
(b) What are final volume and pressure of the gas?
(c) What is the work done by the gas?



8. An ideal monoatomic gas is confined by a spring loaded massless piston of cross-section $8.0 \times 10^{-3}\ \text{m}^2$. Initially, the gas is at 300 K and occupies a volume of $2.4 \times 10^{-3}\ \text{m}^3$ and the spring is in its relaxed state. The gas is heated by an electric heater until the piston moves out slowly without friction by 0.1 m. Calculate

(a) the final temperature of the gas and
(b) the heat supplied by the heater.
The force constant of the spring is 8000 N/m, atmospheric pressure is $1.0 \times 10^5\ \text{N/m}^2$. The cylinder and the piston are thermally insulated.

9. An ideal diatomic gas $\left(\gamma = \frac{7}{5}\right)$ undergoes a process in which its internal energy relates to the volume as $U = \alpha\sqrt{V}$, where α is a constant.

(a) Find the work performed by the gas to increase its internal energy by 100 J.
(b) Find the molar specific heat of the gas.

10. For an ideal gas the molar heat capacity varies as $C = C_V + 3aT^2$. Find the equation of the process in the variables (T, V) where a is a constant.

11. One mole of an ideal monatomic gas undergoes the process $p = \alpha T^{1/2}$, where α is a constant.

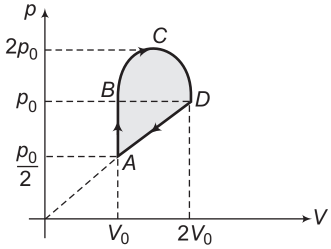
(a) Find the work done by the gas if its temperature increases by 50 K.
(b) Also, find the molar specific heat of the gas.

12. One mole of a gas is put under a weightless piston of a vertical cylinder at temperature T. The space over the piston opens into atmosphere. Initially, piston was in equilibrium. How much work should be performed by some external force to increase isothermally the volume under the piston to twice the volume? (Neglect friction of piston).

13. An ideal monatomic gas undergoes a process where its pressure is inversely proportional to its temperature.
(a) Calculate the molar specific heat for the process.
(b) Find the work done by two moles of gas if the temperature changes from T_1 to T_2.

14. The volume of one mole of an ideal gas with the adiabatic exponent γ is changed according to the relation $V = \frac{a}{T}$, where a is a constant. Find the amount of heat absorbed by the gas in the process, if the temperature is increased by ΔT.

15. Two moles of a monatomic ideal gas undergo a cyclic process *ABCDA* as shown in figure. *BCD* is a semicircle. Find the efficiency of the cycle.



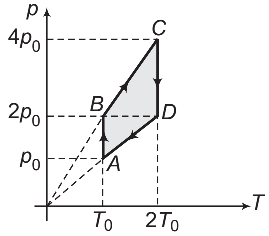
16. Pressure p, volume V and temperature T for a certain gas are related by

$$p = \frac{\alpha T - \beta T^2}{V}$$

where, α and β are constants. Find the work done by the gas if the temperature changes from T_1 to T_2 while the pressure remains the constant.

17. An ideal gas has a specific heat at constant pressure $C_p = \frac{5R}{2}$. The gas is kept in a closed vessel of volume V_0 at temperature T_0 and pressure p_0. An amount of 10 p_0V_0 of heat is supplied to the gas.
(a) Calculate the final pressure and temperature of the gas.
(b) Show the process on p-V diagram.

18. Pressure *versus* temperature (p-T) graph of n moles of an ideal gas is shown in figure. Plot the corresponding:
(a) density *versus* volume (ρ-V) graph,
(b) pressure *versus* volume (p-V) graph and
(c) density *versus* pressure (ρ-p) graph.



19. Three moles of an ideal gas being initially at a temperature $T_i = 273$ K were isothermally expanded 5 times its initial volume and then isochorically heated so that the pressure in the final state becomes equal to that in the initial state. The total heat supplied in the process is 80 kJ. Find $\gamma \left(= \frac{C_p}{C_V} \right)$ of the gas.

Answers

Introductory Exercise 21.1

1. $a = 80$ J, $b = 0$, $c = 50$ J, $d = 40$ J, $e = 40$ J, $f = 40$ J, $g = 40$ J **2.** 327 J

Introductory Exercise 21.2

1. (a) -6.8×10^4 J (b) 1.78×10^5 J, out of gas (c) No

2. (a) Positive (b) Into the system (c) In loop 1, into the system, In loop 2, out of the system

3. 2.67×10^{-2} mol **4.** $W = 0, Q = \Delta U = 1800\,R$ **5.** (i) $2\,p_0V_0$ (ii) zero (iii) $\frac{3}{2}\,p_0V_0$

6. $-600\,R$ **7.** $\frac{1}{2}\,p_0V_0$

Introductory Exercise 21.3

1. (a) 316 K (b) 309.6 K **2.** Into gas

3.

Process	Q	W	ΔU
BC	–	0	–
CA	–	–	–
AB	+	+	+

4. (a) No (b) No (c) No **5.** γ and $\frac{\gamma}{\gamma - 1}$ **6.** $Q_1 > Q_2$

7. 1.18 MJ **8.** 0.6 kJ, 1.0 kJ, 1.6

Introductory Exercise 21.4

1. 2.72×10^5 cal, 72.72 % **2.** 60 % **3.** (a) 25% (b) 6.25×10^5 J (c) 18.75×10^5 J

4. 2×10^6 cal **5.** 117°C and 52°C **6.** Refrigerator A **7.** 35 W **8.** 18.18%

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (d) **3.** (d) **4.** (a) **5.** (b) **6.** (d) **7.** (c) **8.** (b) **9.** (a) **10.** (a or b)

Objective Questions

1. (a) **2.** (c) **3.** (c) **4.** (a) **5.** (d) **6.** (c) **7.** (d) **8.** (a) **9.** (b) **10.** (c)
11 (d) **12.** (c) **13.** (c) **14.** (c) **15.** (d)

Subjective Questions

1. 2.67×10^{-2} mol **2.** $\Delta U = \frac{nR\Delta T}{\gamma - 1}$ for all processes

3. (a) absorbs (b) 7200 J (c) liberates, 7200 J **4.** (a) 0.50 J (b) 0.50 J

5. (a) positive (b) I→ positive, II→ negative (c) into the system
(d) I→ into the system II→ out of the system.

6. 1.68×10^3 W

7. (a) 316 K (b) 310 K

8. 10^4 J

9. (a) p_0V_0 (b) $\frac{5}{2}p_0V_0$, $3p_0V_0$ (c) $\frac{p_0V_0}{2}$ (d) $\frac{25p_0V_0}{8R}$

10. $C = 3.1R$

11. 2.49 kJ

12. 113°C

13. 33600.2 J

14. 2.75×10^4 J

15. (a) 169 J (b) 2087 J

16. (a) -1.15×10^5 J (b) 2.55×10^5 J, out of gas (c) No

17. (a) $Q_{ABC} = -2P_0V_0$ (b) $\Delta U_{ABC} = 0$

18. (a) AC (b) 150 J (c) 10 J

19. (a) – 150 J (b) – 400 J

20. 11609.99287 J

21. (a) 241 K (b) 383 K (c) 1770 J (d) 1770 J

22. (a) 200 J (b) 0.80°C

24. 166 J

LEVEL 2

Single Correct Option

1.(a) **2.**(a) **3.**(a) **4.**(a) **5.**(b) **6.**(a) **7.**(b) **8.**(d) **9.**(b) **10.**(d)
11(d) **12.**(c) **13.**(c) **14.**(c) **15.**(c) **16.**(a) **17.**(a) **18.**(a) **19.**(c) **20.**(b)
21.(a) **22.**(b) **23.**(c) **24.**(c) **25.**(d) **26.**(b) **27.**(d)

More than One Correct Options

1.(b,d) **2.**(a,c) **3.**(a,b) **4.**(c,d) **5.**(a,b,c,d) **6.**(a,b)

Comprehension Based Questions

1.(b) **2.**(b) **3.**(c) **4.**(b) **5.**(a)

Match the Columns

1. (a)→ r (b)→ p (c)→ s (d)→ p
2. (a)→ s (b)→ s (c)→ s (d)→ r
3. (a)→ p (b)→ s (c)→ s (d)→ s
4. (a)→ q (b)→ p,r (c)→ p (d)→ p
5. (a)→ s (b)→ s (c)→ r (d)→ s

Subjective Questions

1. (a) 1153 J (b) 1153 J (c) zero

2. $T_A = 273$ K, $T_B = 546$ K, $V_C = 44.8$ m^3

3. (a) $W_{12} = \frac{-p_0M}{\rho_0}\ln(2)$, $W_{23} = \frac{p_0M}{\rho_0}$, $W_{31} = 0$ (b) $\frac{p_0M}{\rho_0}\left(\frac{3}{2} + \ln 2\right)$ (c) $\frac{2}{5}(1 - \ln 2)$

4. (b) – 4000 J

5. 136 J

6. (b) $3RT_0\ln(2) - \frac{3}{2}RT_0$ (c) $3RT_0\ln(2) - \frac{21}{4}RT_0$

7. (b) 113.1 L, 0.44×10^5 N/m^2 (c) 12479 J

8. (a) 800 K (b) 720 J

9. (a) 80 J (b) $\frac{9R}{2}$

10. $Ve^{-\left(\frac{3a}{2R}\right)T^2}$ = constant

11. (a) 207.75 J (b) $2R$

12. $RT(1-\ln 2)$

13. (a) $\frac{7R}{2}$ (b) $4R(T_2-T_1)$

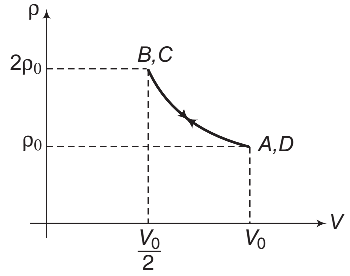
14. $\frac{(2-\gamma)R\Delta T}{(\gamma-1)}$

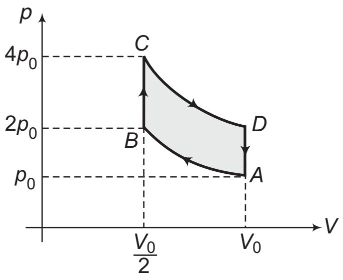
15. 25.8%

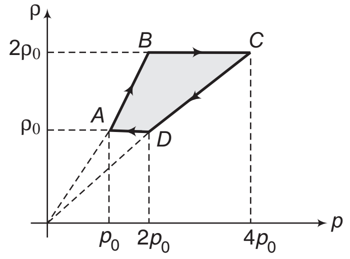
16. $\alpha(T_2-T_1)-\beta(T_2^2-T_1^2)$

17. (a) $\frac{23}{3}p_0, \frac{23}{3}T_0$

18.







19. 1.4

22

Calorimetary and Heat Transfer

Chapter Contents

22.1 Calorimetry
22.2 Heat Transfer

22.1 Calorimetry

Heat transfer between two substances, due to a temperature difference (may be in same or different states) comes under the topic calorimetry. Let us discuss this topic pointwise.

Specific Heat (c)

The amount of heat required to raise the temperature of unit mass of a substance by 1°C (or 1 K) is called its specific heat. Thus,

$$\boxed{c = \frac{Q}{m\Delta\theta} \quad \text{or} \quad \frac{Q}{m\Delta T}} \qquad \text{...(i)}$$

The SI unit of specific heat is J/kg-K. Because heat is so frequently measured in calories, the unit cal/g-°C is also used quite often.

Heat Required to Change The Temperature

From Eq. (i), we can write

$$\boxed{Q = mc\Delta\theta \quad \text{or} \quad mc\Delta T} \qquad \text{...(ii)}$$

Thus Eq. (ii) is used when temperature of a substance changes without change in state.

Note *(i) In general c is a function of temperature. But normally variation in c is small. So, it is assumed to be constant for wide range of temperature. For example*

"Specific heat of water is 1 cal/g-°C between 14.5 °C to 15.5 °C". This implies that 1 cal of heat will be required to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C. From 15.5 °C to 16.5 °C, a different amount of heat (≠ 1 cal) will be required. But normally this variation is very small. So, it is assumed constant (= 1 cal/g-°C) from `0 °C to 100 °C.

(ii) If c is given a function of temperature, then Eq. (ii) cannot be applied directly for calculation of Q. Rather integration will be used. Thus,

$$\boxed{Q = m\int_{T_i}^{T_f} c\,dT \quad \text{or} \quad m\int_{\theta_i}^{\theta_f} c\,d\theta} \qquad \text{...(iii)}$$

In the above expression, c is a function of T (or θ).

(iii) The specific heat of water is much larger than that of most other substances. Consequently, for the same amount of added heat, the temperature change of a given mass of water is generally less than that for the same mass of another substance. For this reason a large body of water moderates the climate of nearby land. In the winter, the water cools off more slowly than the surrounding land and tends to warm the land. In the summer, the opposite effect occurs as the water heats up more slowly than the land.

(iv) Specific heat is also represented by s or S. So, in different problems we have taken all notations for it.

Latent Heat (L)

The amount of heat required to change the state (or phase) of unit mass of a substance at constant temperature is called latent heat L. Thus,

$$\boxed{L = \frac{Q}{m}} \qquad \text{...(iv)}$$

For a solid-liquid transition, the latent heat is known as the **latent heat of fusion (L_f)** and for the liquid-gas transition, it is known as the **latent heat of vaporization (L_v).**

For water at 1 atmosphere, latent heat of fusion is 80.0 cal/g. This simply means 80.0 cal of heat are required to melt 1.0 g of water or 80.0 cal heat is liberated when 1.0 g of water freezes at 0°C. Similarly, latent heat of vaporization for water at 1 atmosphere is 539 cal/g.

Heat Required to Change the State (or Phase) at Constant Temperature

From Eq. (iv), we can see that

$$\boxed{Q = mL} \qquad \ldots\text{(v)}$$

So, this is the heat required to change the state (or phase) at constant temperature.

Extra Points to Remember

- In some problems, two or more than two substances at different temperatures are mixed. Heat is given by the hot bodies and taken by the cold bodies. Finally, all substances reach to a common equilibrium temperature. All such problems can be solved by a single equation,

  Heat given by hot bodies = Heat taken by cold bodies

  But note that, on both sides of this equation, we have to take the positive signs. To make them positive, always write

  $$\Delta\theta = \theta_{\text{Higher}} - \theta_{\text{Lower}}$$

  in the equation $Q = mc\Delta\theta$ when there is a change of temperature without change in state.
- **Water equivalent of a container** Normally, a liquid is heated in a container. So, some heat is wasted in heating the container also. Suppose water equivalent of a container is 10 g, then it implies that heat required to increase the temperature of this container is equal to heat required to increase the temperature of 10 g of water.

Example 22.1 *How much heat is required to convert 8.0 g of ice at – 15°C to steam at 100°C? (Given,* $c_{ice} = 0.53\ cal/g\text{-}°C$, $L_f = 80\ cal/g$ *and* $L_v = 539\ cal/g$, *and* $c_{water} = 1\ cal/g\text{-}°C$*)*

Solution

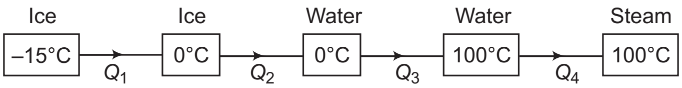


Fig. 22.1

$$Q_1 = mc_{\text{ice}}\,(T_f - T_i)$$
$$= (8.0)(0.53)[0 - (-15)] = 63.6\,\text{cal}$$
$$Q_2 = mL_f = (8)(80) = 640\,\text{cal}$$
$$Q_3 = mc_{\text{water}}\,(T_f - T_i)$$
$$= (8.0)(1.0)[100 - 0] = 800\,\text{cal}$$
$$Q_4 = mL_v = (8.0)(539) = 4312\,\text{cal}$$

∴ Net heat required,

$$Q = Q_1 + Q_2 + Q_3 + Q_4$$
$$= 5815.6\ \text{cal}$$

Ans.

Example 22.2 *In the above problem if heat is supplied at a constant rate of $q = 10$ cal/min, then plot temperature versus time graph.*

Solution

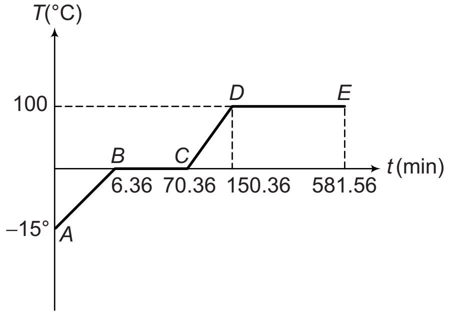


Fig. 22.2

From *A* to *B*

(i) Temperature of ice will increase from –15° C to 0°C.

(ii)
$$t_{AB} = \frac{\text{Total heat required}}{\text{Heat supplied per minute}} = \frac{Q_1}{q}$$
$$= \frac{63.6}{10} = 6.36 \text{ min}$$

(iii) Between *A* and *B* we will get only ice.

From *B* to *C*

(i) Temperature of (ice + water) mixture will remain constant at 0°C.

(ii)
$$t_{BC} = \frac{Q_2}{q} = \frac{640}{10} = 64 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} = 70.36 \text{ min}$$

(iii) Between *B* and *C* we will get both ice and water.

From *C* to *D*

(i) Temperature of water increases from 0°C to 100°C

(ii)
$$t_{CD} = \frac{Q_3}{q} = \frac{800}{10} = 80 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} + t_{CD} = 150.36 \text{ min}$$

(iii) Between *C* and *D* we will get only water.

From *D* to *E*

(i) Temperature of (water + steam) mixture will remain constant at 100°C.

(ii)
$$t_{DE} = \frac{Q_4}{q} = \frac{4312}{10} = 431.2 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} + t_{CD} + t_{DE} = 581.56 \text{ min}$$

(iii) Between *D* and *A* we will get both water and steam.

The corresponding graph is as shown in Fig. 22.2.

Exercise AB and CD correspond to temperature change without change in state. BC and DE correspond to state change without change in temperature.

In the given condition, prove that AB and CD are straight lines and slope of these lines is inversely proportional to specific heat in that state. Further, prove that lengths of lines BC and DE are proportional to latent heat in that state.

Example 22.3 *10 g of water at 70°C is mixed with 5 g of water at 30°C. Find the temperature of the mixture in equilibrium. Specific heat of water is 1 cal/g-°C*

Solution Let $t°$ C be the temperature of the mixture. From energy conservation,

$$\text{Heat given by 10 g of water} = \text{Heat taken by 5 g of water}$$

or $$m_1 c_{\text{water}} |\Delta t_1| = m_2 c_{\text{water}} |\Delta t_2|$$

$\therefore$ $$(10)(1)(70 - t) = 5(1)(t - 30)$$

$\therefore$ $$t = 56.67° \text{ C}$$ **Ans.**

Example 22.4 *The temperature of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C respectively. The temperature when A and B are mixed is 16°C and when B and C are mixed it is 23°C. What should be the temperature when A and C are mixed?*

Solution Let m be the mass of each liquid and s_A, s_B, s_C specific heats of liquids A, B and C respectively. When A and B are mixed. The final temperature is 16° C.

$\therefore$ $$\text{Heat gained by } A = \text{Heat lost by } B$$

i.e. $$ms_A(16 - 12) = ms_B(19 - 16)$$

i.e. $$s_B = \frac{4}{3} s_A \qquad \text{...(i)}$$

When B and C are mixed.

$$\text{Heat gained by } B = \text{Heat lost by } C$$

i.e. $$ms_B(23 - 19) = ms_C(28 - 23)$$

i.e. $$s_C = \frac{4}{5} s_B \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $$S_C = \frac{4}{5} \times \frac{4}{3} S_A = \frac{16}{15} S_A$$

When A and C are mixed, let the final temperature be θ

$$\text{Heat gained by } A = \text{Heat lost by } C$$

$\therefore$ $$ms_A(\theta - 12) = ms_C(28 - \theta)$$

i.e. $$\theta - 12 = \frac{16}{15}(28 - \theta)$$

By solving, we get

$$\theta = \frac{628}{31} = 20.26°\text{C}$$ **Ans.**

Example 22.5 *In a container of negligible mass 30 g of steam at 100°C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system? Also find masses of water and steam in equilibrium. Take* $L_v = 539$ *cal/g and* $c_{water} = 1$ *cal/g-° C.*

Solution Let Q be the heat required to convert 200 g of water at 40°C into 100°C, then

$$Q = mc\Delta T = (200)(1.0)(100 - 40) = 12000 \text{ cal}$$

Now, suppose m_0 mass of steam converts into water to liberate this much amount of heat, then

$$m_0 = \frac{Q}{L} = \frac{12000}{539} = 22.26 \text{ g}$$

Since, it is less than 30 g, the temperature of the mixture is 100°C. **Ans.**

Mass of steam in the mixture = 30 – 22.26 = 7.74 g **Ans.**

and mass of water in the mixture = 200 + 22.26 = 222.26 g **Ans.**

Example 22.6 *In an insulated vessel,* 0.05 *kg steam at* 373 *K and* 0.45 *kg of ice at* 253 *K are mixed. Find the final temperature of the mixture (in kelvin).*

Given, $L_{fusion} = 80 \text{ cal/g} = 336 \text{ J/g}$

$L_{vaporization} = 540 \text{ cal/g} = 2268 \text{ J/g}$

$s_{ice} = 2100 \text{ J/kg-K} = 0.5 \text{ cal/g-K}$

and $s_{water} = 4200 \text{ J/kg-K} = 1 \text{ cal/g-K}$ **(JEE 2006)**

Solution

$$0.05 \text{ kg steam at } 373 \text{ K} \xrightarrow{Q_1} 0.05 \text{ kg water at } 373 \text{ K}$$

$$0.05 \text{ kg water at } 373 \text{ K} \xrightarrow{Q_2} 0.05 \text{ kg water at } 273 \text{ K}$$

$$0.45 \text{ kg ice at } 253 \text{ K} \xrightarrow{Q_3} 0.45 \text{ kg ice at } 273 \text{ K}$$

$$0.45 \text{ kg ice at } 273 \text{ K} \xrightarrow{Q_4} 0.45 \text{ kg water at } 273 \text{ K}$$

$$Q_1 = (50)(540) = 27{,}000 \text{ cal} = 27 \text{ kcal}$$

$$Q_2 = (50)(1)(100) = 5000 \text{ cal} = 5 \text{ kcal}$$

$$Q_3 = (450)(0.5)(20) = 4500 \text{ cal} = 4.5 \text{ kcal}$$

$$Q_4 = (450)(80) = 36000 \text{ cal} = 36 \text{ kcal}$$

Now, since $Q_1 + Q_2 > Q_3$ but $Q_1 + Q_2 < Q_3 + Q_4$ ice will come to 273 K from 253 K, but whole ice will not melt. Therefore, temperature of the mixture is 273 K.

Example 22.7 *An ice cube of mass* 0.1 *kg at* 0°C *is placed in an isolated container which is at* 227°C. *The specific heat s of the container varies with temperature T according to the empirical relation* $s = A + BT$, *where* $A = 100$ *cal/kg-K and* $B = 2 \times 10^{-2}$ *cal/kg-K*^{2}. *If the final temperature of the container is* 27°C, *determine the mass of the container.*

(Latent heat of fusion for water $= 8 \times 10^4$ *cal/kg, specific heat of water* $= 10^3$ *cal/kg-K).* **(JEE 2001)**

Solution Let m be the mass of the container.

Initial temperature of container,

$$T_i = (227 + 273) = 500 \text{ K}$$

and final temperature of container,

$$T_f = (27 + 273) = 300 \text{ K}$$

Now, heat gained by the ice cube = heat lost by the container

$$\therefore \qquad (0.1)(8 \times 10^4) + (0.1)(10^3)(27) = -m \int_{500}^{300} (A + BT)\, dT$$

or

$$10700 = -m \left[AT + \frac{BT^2}{2} \right]_{500}^{300}$$

After substituting the values of A and B and the proper limits, we get

$$m = 0.495 \text{ kg}$$ **Ans.**

INTRODUCTORY EXERCISE 22.1

(Take $c_{\text{ice}} = 0.53$ cal/g-°C, $c_{\text{water}} = 1.0$ cal/g-°C, $(L_f)_{\text{water}} = 80$ cal/g and $(L_v)_{\text{water}} = 529$ cal/g unless given in the question.)

1. 10 g ice at 0°C is converted into steam at 100°C. Find total heat required. ($L_f = 80$ cal/g, $S_w = 1$ cal/g-°C, $L_v = 540$ cal/g)
2. Three liquids A, B and C of specific heats 1 cal/g -°C, 0.5 cal/g -°C and 0.25 cal/g -°C are at temperatures 20 °C, 40 °C and 60 °C respectively. Find temperature in equilibrium if they are mixed together. Their masses are equal.
3. Equal masses of ice (at 0°C) and water are in contact. Find the temperature of water needed to just melt the complete ice.
4. A lead bullet just melts when stopped by an obstacle. Assuming that 25 per cent of the heat is absorbed by the obstacle, find the velocity of the bullet if its initial temperature is 27°C. (Melting point of lead = 327°C, specific heat of lead = 0.03 cal/g-°C, latent heat of fusion of lead = 6 cal/g, $J = 4.2$ J/cal). **(JEE 1981)**
5. The temperature of 100 g of water is to be raised from 24°C to 90°C by adding steam to it. Calculate the mass of the steam required for this purpose. **(JEE 1996)**
6. 15 g ice at 0°C is mixed with 10 g water at 40°C. Find the temperature of mixture. Also, find mass of water and ice in the mixture.
7. Three liquids P, Q and R are given 4 kg of P at 60 °C and 1 kg of R at 50 °C when mixed produce a resultant temperature 55 °C. A mixture of 1 kg of P at 60 °C and 1 kg of Q at 50 °C shows a temperature of 55 °C. What will be the resulting temperature when 1 kg of Q at 60 °C is mixed with 1 kg of R at 50 °C?

22.2 Heat Transfer

Heat can be transferred from one place to the other by any of three possible ways : **conduction, convection** and **radiation.** In the first two processes, a medium is necessary for the heat transfer. Radiation, however, does no have this restriction. This is also the fastest mode of heat transfer, in which heat is transferred from one place to the other in the form of electromagnetic radiation.

Conduction

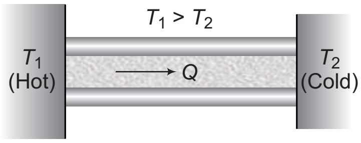


Fig. 22.3

Figure shows a rod whose ends are in thermal contact with a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. The sides of the rod are covered with insulating medium, so the transport of heat is along the rod, not through the sides. The molecules at the hot reservoir have greater vibrational energy. This energy is transferred by collisions to the atoms at the end face of the rod. These atoms in turn transfer energy to their neighbours further along the rod. Such transfer of heat through a substance in which heat is transported without direct mass transport is called conduction.

Most metals use another, more effective mechanism to conduct heat. The free electrons, which move throughout the metal can rapidly carry energy from the hotter to cooler regions, so metals are generally good conductors of heat. The presence of 'free' electrons also causes most metals to be good electrical conductors. A metal rod at 5°C feels colder than a piece of wood at 5°C because heat can flow more easily from your hand into the metal.

Heat transfer occurs only between regions that are at different temperatures, and the rate of heat flow is $\frac{dQ}{dt}$. This rate is also called the **heat current**, denoted by H. Experiments show that the heat current is proportional to the cross-section area A of the rod and to the temperature gradient $\frac{dT}{dx}$, which is the rate of change of temperature with distance along the bar. In general,

$$\boxed{H = \frac{dQ}{dt} = -KA\frac{dT}{dx}} \qquad \ldots\text{(i)}$$

The negative sign is used to make $\frac{dQ}{dt}$ a positive quantity since $\frac{dT}{dx}$ is negative. The constant K, called the **thermal conductivity** is a measure of the ability of a material to conduct heat. A substance with a large thermal conductivity k is a good heat conductor. The value of k depends on the temperature, increasing slightly with increasing temperature, but K can be taken to be practically constant throughout a substance if the temperature difference between its ends is not too great. Let us apply Eq. (i) to a rod of length l and constant cross-sectional area A in which a steady state has been reached. In a steady state, the temperature at each point is constant in time. Hence,

$$-\frac{dT}{dx} = T_1 - T_2$$

Therefore, the heat ΔQ transferred in time Δt is

$$\boxed{\Delta Q = KA\left(\frac{T_1 - T_2}{l}\right)\Delta t} \qquad \ldots\text{(ii)}$$

Thermal Resistance (*R*)

Eq. (ii) in differential form can be written as

$$\boxed{\frac{dQ}{dt} = H = \frac{\Delta T}{l/KA} = \frac{\Delta T}{R}} \qquad \ldots\text{(iii)}$$

Here, ΔT = temperature difference (TD) and $R = \frac{l}{KA}$ = thermal resistance of the rod.

Extra Points to Remember

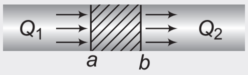
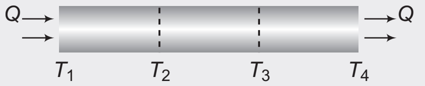


Fig. 22.4

- Consider a section *ab* of a rod as shown in figure. Suppose Q_1 heat enters into the section at '*a*' and Q_2 leaves at '*b*', then $Q_2 < Q_1$. Part of the energy $Q_1 - Q_2$ is utilized in raising the temperature of section *ab* and the remaining is lost to atmosphere through *ab*. If heat is continuously supplied from the left end of the rod, a stage comes when temperature of the section becomes constant. In that case, $Q_1 = Q_2$ if rod is insulated from the surroundings (or loss through *ab* is zero). This is called the **steady state** condition. Thus, in steady state temperature of different sections of the rod becomes constant (but not same).

  Hence, in the figure :



Insulated rod in steady state

Fig. 22.5

$$T_1 = \text{constant},\ T_2 = \text{constant etc. and } T_1 > T_2 > T_3 > T_4$$

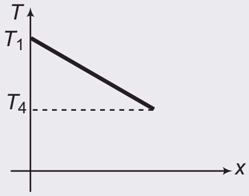


Fig. 22.6

Now, a natural question arises, why the temperature of whole rod not becomes equal when heat is being continuously supplied? The answer is : there must be a temperature difference in the rod for the heat flow, same as we require a potential difference across a resistance for the current flow through it.

In steady state, the temperature varies linearly with distance along the rod if it is insulated.

- Comparing equation number (iii), i.e. heat current

$$H = \frac{dQ}{dt} = \frac{\Delta T}{R} \qquad \left(\text{where, } R = \frac{l}{KA}\right)$$

with the equation, of current flow through a resistance,

$$i = \frac{dq}{dt} = \frac{\Delta V}{R} \qquad \left(\text{where, } R = \frac{l}{\sigma A}\right)$$

We find the following similarities in heat flow through a rod and current flow through a resistance.

Table 22.1

S.No	Heat flow through a conducting rod	Current flow through a resistance
1.	Conducting rod	Electrical resistance
2.	Heat flows	Charge flows
3.	TD is required	PD is required
4.	Heat current $H = \frac{dQ}{dt}$ = rate of heat flow	Electric current $i = \frac{dq}{dt}$ = rate of charge flow
5.	$H = \frac{\Delta T}{R} = \frac{\text{TD}}{R}$	$i = \frac{\Delta V}{R} = \frac{\text{PD}}{R}$
6.	$R = \frac{l}{KA}$	$R = \frac{l}{\sigma A}$
7.	K = thermal conductivity	σ = electrical conductivity

From the above table it is evident that flow of heat through rods in series and parallel is analogous to the flow of current through resistance in series and parallel. This analogy is of great importance in solving complicated problems of heat conduction.

Convection

Although conduction does occur in liquids and gases also, heat is transported in these media mostly by convection. In this process, the actual motion of the material is responsible for the heat transfer. Familiar examples include hot-air and hot-water home heating systems, the cooling system of an automobile engine and the flow of blood in the body.

You probably have warmed your hands by holding them over an open flame. In this situation, the air directly above the flame is heated and expands. As a result, the density of this air decreases and then air rises. When the movement results from differences in density, as with air around fire, it is referred to as **natural convection.** Air flow at a beach is an example of natural convection. When the heated substance is forced to move by a fan or pump, the process is called **forced convection.** If it were not for convection currents, it would be very difficult to boil water. As water is heated in a kettle, the heated water expands and rises to the top because its density is lowered. At the same time, the denser, cool water at the surface sinks to the bottom of the kettle and is heated. Heating a room by a radiator is an example of forced convection.

Radiation

The third means of energy transfer is radiation which does not require a medium. The best known example of this process is the radiation from sun. All objects radiate energy continuously in the form of electromagnetic waves.

Now, let us define few terms before studying the other topics.

Radiant Energy

All bodies radiate energy in the form of electromagnetic waves by virtue of their temperature. This energy is called the radiant energy.

Absorptive Power '*a*'

"It is defined as the ratio of the radiant energy absorbed by it in a given time to the total radiant energy incident on it in the same interval of time."

$$\boxed{a = \frac{\text{energy absorbed}}{\text{energy incident}}}$$

As a perfectly black body absorbs all radiations incident on it, the absorptive power of a perfectly black body is maximum and unity.

Spectral Absorptive Power 'a_λ'

The absorptive power '*a*' refers to radiations of all wavelengths (or the total energy) while the spectral absorptive power is the ratio of radiant energy absorbed by a surface to the radiant energy incident on it for a particular wavelength λ. It may have different values for different wavelengths for a given surface. Let us take an example, suppose $a = 0.6$, $a_\lambda = 0.4$ for 1000 Å and $a_\lambda = 0.7$ for 2000 Å for a given surface. Then, it means that this surface will absorb only 60% of the total radiant energy incident on it. Similarly, it absorbs 40% of the energy incident on it corresponding to 1000 Å and 70% corresponding to 2000 Å. The spectral absorptive power a_λ is related to absorptive power *a* through the relation

$$\boxed{a = \int_0^\infty a_\lambda \, d\lambda}$$

Emissive Power ‘*e*’

(Do not confuse it with the emissivity *e* which is different from it, although both have the same symbol *e*).

“For a given surface it is defined as the radiant energy emitted per second per unit area of the surface.” It has the units of W/m^2 or $J/s\text{-}m^2$. For a black body $e = \sigma T^4$.

Spectral Emissive Power ‘e_λ’

“It is emissive power for a particular wavelength λ.” Thus,

$$\boxed{e = \int_0^\infty e_\lambda \, d\lambda}$$

Stefan’s Law

The rate at which an object radiates energy is proportional to the fourth power of its absolute temperature. This is known as the **Stefan’s law** and is expressed in equation form as

$$\boxed{P = \sigma A e T^4}$$

Here, P is the power in watts (J/s) radiated by the object, A is the surface area in m^2, e is a fraction between 0 and 1 called the **emissivity** of the object and σ is a universal constant called **Stefan’s constant**, which has the value

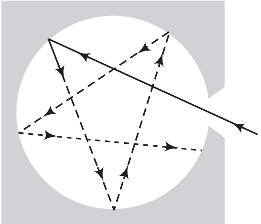
$$\boxed{\sigma = 5.67 \times 10^{-8} \ W/m^2\text{-}K^4}$$

Note *Emissivity e is also sometimes denoted by* e_r.

Perfectly Black Body

A body that absorbs all the radiation incident upon it and has an emissivity equal to 1 is called a perfectly black body. A black body is also an ideal radiator. It implies that if a black body and an identical another body are kept at the same temperature, then the black body will radiate maximum power as is obvious from equation $P = eA\sigma T^4$ also. Because $e = 1$ for a perfectly black body while for any other body $e < 1$.

Materials like black velvet or lamp black come close to being ideal black bodies, but the best practical realization of an ideal black body is a small hole leading into a cavity, as this absorbs 98% of the radiation incident on them.



Cavity approximating an ideal black body. Radiation entering the cavity has little chance of leaving before it is completely absorbed.

Fig. 22.7

Note *(i) Energy radiated by a body per unit area per unit time is called emissive power and given by*

$$\boxed{e = e_r \sigma T^4} \qquad (e_r = \text{emissivity})$$

It has the units of W/m^2 or $J/s\text{-}m^2$.

Energy radiated per unit time is called power and it is called power and it is given by

$$\boxed{P = e_r \sigma T^4 A}$$

It has the unit J/s or watt.

Total energy radiated by the body in time t is

$$\boxed{E = e_r \sigma T^4 A t}$$

(ii) *For a given body, values of e_r and a are same. But their meanings are different. For example if, $e_r = 0.6$ then a is also 0.6.*

Further, $e_r = 0.6$ means if this body and perfectly black body are kept under identical conditions then this body will radiate 60% of the energy radiated by a perfectly black body. But $a = 0.6$ means this body will absorb 60% of the total energy incident on its surface.

(iii) *Absorptive power (a), spectral absorptive power (a_λ) and emissivity (e_r) are the constants which are temperature independent. But emissive power (e) and spectral emissive power (e_λ) are temperature dependent. As,*

$$e = e_r \sigma T^4 \quad \Rightarrow \quad e \propto T^4$$

Kirchhoff's Law

"According to this law the ratio of emissive power to absorptive power is same for all surfaces at the same temperature."

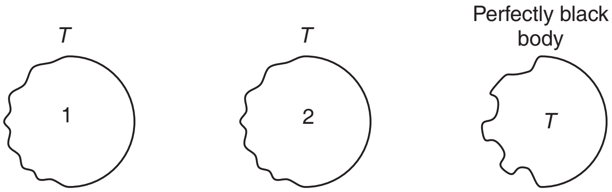


Fig. 22.8

Hence,
$$\frac{e_1}{a_1} = \frac{e_2}{a_2} = \left(\frac{e}{a}\right)_{\text{perfectly black body}}$$

but
$$(a)_{\text{black body}} = 1$$

and
$$(e)_{\text{black body}} = E \quad \text{(say)}$$

Then,
$$\boxed{\left(\frac{e}{a}\right)_{\text{for any surface}} = \text{constant} = E}$$

Similarly, for a particular wavelength λ,

$$\boxed{\left(\frac{e_\lambda}{a_\lambda}\right)_{\text{for any body}} = E_\lambda}$$

Here, E = emissive power of black body at temperature T

$= \sigma T^4$

From the above expression, we can see that

$$\boxed{e_\lambda \propto a_\lambda}$$

i.e. good absorbers for a particular wavelength are also good emitters of the same wavelength.

Note *According to Kirchhoff's law, at a given temperature, the ratio* $\frac{e}{a}$ *is constant for all bodies and this constant is equal to e of perfectly black body at that temperature. Similarly, the ratio* $\frac{e_\lambda}{a_\lambda}$ *is constant for all bodies at given temperature and value of this constant is* e_λ *of perfectly black body at that temperature.*

Cooling by Radiation

Consider a hot body at temperature T placed in an environment at a lower temperature T_0. The body emits more radiation than it absorbs and cools down while the surroundings absorb radiation from the body and warm up. The body is losing energy by emitting radiations at a rate

$$P_1 = eA\sigma T^4$$

and is receiving energy by absorbing radiations at a rate

$$P_2 = aA\sigma T_0^4$$

Here, 'a' is a pure number between 0 and 1 indicating the relative ability of the surface to absorb radiation from its surroundings. Note that this 'a' is different from the absorptive power 'a'. In thermal equilibrium, both the body and the surroundings have the same temperature (say T_c) and,

$$P_1 = P_2 \quad \text{or} \quad eA\sigma T_c^4 = aA\sigma T_c^4 \quad \text{or} \quad e = a$$

Thus, when $T > T_0$, the net rate of heat transfer from the body to the surroundings is

$$\frac{dQ}{dt} = eA\sigma\,(T^4 - T_0^4) \quad \text{or} \quad mc\left(-\frac{dT}{dt}\right) = eA\sigma\,(T^4 - T_0^4)$$

Rate of cooling

$$\boxed{\left(-\frac{dT}{dt}\right) = \frac{eA\sigma}{mc}(T^4 - T_0^4)} \quad \text{or} \quad -\frac{dT}{dt} \propto (T^4 - T_0^4)$$

Newton's Law of Cooling

According to this law, if the temperature T of the body is not very different from that of the surroundings T_0, then rate of cooling $-\frac{dT}{dt}$ is proportional to the temperature difference between them. To prove it, let us assume that

$$T = T_0 + \Delta T$$

So that

$$T^4 = (T_0 + \Delta T)^4 = T_0^4\left(1 + \frac{\Delta T}{T_0}\right)^4$$

$$\approx T_0^4\left(1 + \frac{4\Delta T}{T_0}\right) \qquad \text{(from binomial expansion)}$$

$$\therefore \qquad (T^4 - T_0^4) = 4T_0^3\,(\Delta T)$$

$$\text{or} \qquad (T^4 - T_0^4) \propto \Delta T \qquad (\text{as } T_0 = \text{constant})$$

Now, we have already shown that rate of cooling

$$\left(-\frac{dT}{dt}\right) \propto (T^4 - T_0^4)$$

and here we have shown that

$$(T^4 - T_0^4) \propto \Delta T$$

if the temperature difference is small.

Thus, rate of cooling

$$\boxed{-\frac{dT}{dt} \propto \Delta T} \quad \text{or} \quad \boxed{-\frac{d\theta}{dt} \propto \Delta\theta}$$

as $\qquad dT = d\theta \quad \text{or} \quad \Delta T = \Delta\theta$

Variation of Temperature of a Body According to Newton's Law

Suppose a body has a temperature θ_i at time $t = 0$. It is placed in an atmosphere whose temperature is θ_0. We are interested in finding the temperature of the body at time t. Assuming Newton's law of cooling to hold good or by assuming that the temperature difference is small. As per this law,

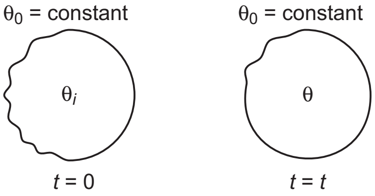


Fig. 22.9

Rate of cooling ∝ temperature difference

or
$$\left(-\frac{d\theta}{dt}\right) = \left(\frac{eA\sigma}{mc}\right)(4\theta_0^3)\,(\theta - \theta_0)$$

or
$$\left(-\frac{d\theta}{dt}\right) = \alpha\,(\theta - \theta_0)$$

Here,
$$\alpha = \left(\frac{4eA\sigma\theta_0^3}{mc}\right) \qquad \text{(is a constant)}$$

∴
$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta - \theta_0} = -\alpha \int_0^t dt$$

∴
$$\boxed{\theta = \theta_0 + (\theta_i - \theta_0)\,e^{-\alpha t}}$$

From this expression we see that $\theta = \theta_i$ at $t = 0$ and $\theta = \theta_0$ at $t = \infty$, i.e. temperature of the body varies exponentially with time from θ_i to θ_0 $(< \theta_i)$. The temperature *versus* time graph is as shown in Fig.22.10.

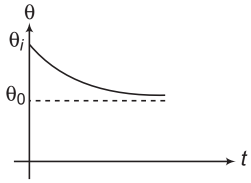


Fig. 22.10

Note *If the body cools by radiation from* θ_1 *to* θ_2 *in time* t, *then taking the approximation*

$$\left(-\frac{d\theta}{dt}\right) = \frac{\theta_1 - \theta_2}{t} \quad \text{and} \quad \theta = \theta_{\text{av}} = \left(\frac{\theta_1 + \theta_2}{2}\right)$$

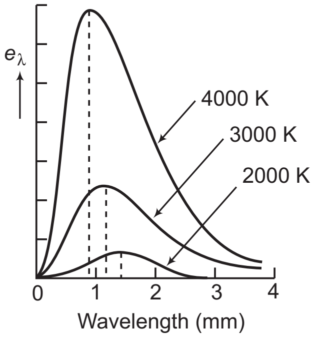
The equation $\left(-\frac{d\theta}{dt}\right) = \alpha\,(\theta - \theta_0)$ *becomes*

$$\boxed{\frac{\theta_1 - \theta_2}{t} = \alpha\left(\frac{\theta_1 + \theta_2}{2} - \theta_0\right)}$$

This form of the law helps in solving numerical problems related to ***Newton's law of cooling****.*

Wien's Displacement Law

At ordinary temperatures (below about 600°C) the thermal radiation emitted by a body is not visible, most of it is concentrated in wavelengths much longer than those of visible light.



Power of black body radiation *versus* wavelength at three temperatures. Note that the amount of radiation emitted (the area under a curve) increase with increasing temperature.

Fig. 22.11

Figure shows how the energy of a black body radiation varies with temperature and wavelength. As the temperature of the black body increases, two distinct behaviors are observed. The first effect is that the peak of the distribution shifts to shorter wavelengths. This shift is found to obey the following relationship called **Wien's displacement law.**

$$\boxed{\lambda_m T = b}$$

Here, b is a constant called Wien's constant. The value of this constant for perfectly black body in SI unit is 2.898×10^{-3} m-K. Thus,

$$\boxed{\lambda_m \propto \frac{1}{T}}$$

Here, λ_m is the wavelength corresponding to the maximum spectral emissive power e_λ.

The second effect is that the total amount of energy the black body emits per unit area per unit time $(=\sigma T^4)$ increases with fourth power of absolute temperature T. This is also known as the emissive power. We know

$$e = \int_0^\infty e_\lambda \, d\lambda = \text{Area under } e_\lambda \text{ - } \lambda \text{ graph} = \sigma T^4$$

or $\boxed{\text{Area} \propto T^4} \Rightarrow A_2 = (2)^4 A_1 = 16A_1$

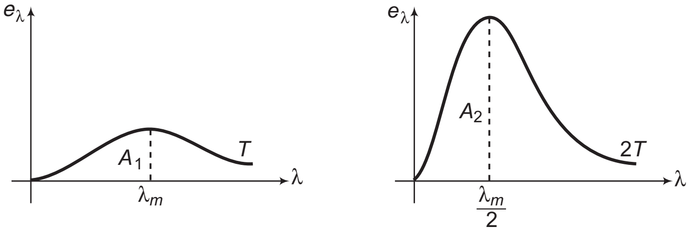


Fig. 22.12

Thus, if the temperature of the black body is made two fold, λ_m remains half while the area becomes 16 times.

Example 22.8 *In which of the following process, convection does not take place primarily?* (JEE 2005)

(a) Sea and land breeze
(b) Boiling of water
(c) Warming of glass of bulb due to filament
(d) Heating air around a furnace

Solution (c) Glass of bulb heats due to filament by radiation.

Example 22.9 *A copper rod 2 m long has a circular cross-section of radius 1 cm. One end is kept at 100°C and the other at 0°C. The surface is insulated so that negligible heat is lost through the surface. In steady state, find*

(a) the thermal resistance of the bar
(b) the thermal current H
(c) the temperature gradient $\frac{dT}{dx}$ *and*
(d) the temperature at a distance 25 cm from the hot end.

Thermal conductivity of copper is 401 W/m-K.

Solution (a) Thermal resistance, $R = \frac{l}{KA} = \frac{l}{K(\pi r^2)}$

or $$R = \frac{(2)}{(401)(\pi)(10^{-2})^2}$$

$$= 15.9\ \text{K/W}$$ **Ans.**

(b) Thermal current, $H = \frac{\Delta T}{R} = \frac{\Delta\theta}{R} = \frac{100}{15.9}$

or $$H = 6.3\ \text{W}$$ **Ans.**

(c) Temperature gradient

$$= \frac{0 - 100}{2} = -50\ \text{K/m}$$

$$= -50°\ \text{C/m}$$ **Ans.**

(d) Let θ be the temperature at 25 cm from the hot end, then

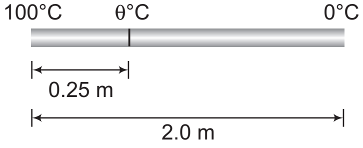


Fig. 22.13

$$(\theta - 100) = (\text{temperature gradient}) \times (\text{distance})$$

or $$\theta - 100 = (-50)(0.25)$$

or $$\theta = 87.5°\ \text{C}$$ **Ans.**

Example 22.10 *Three rods made of the same material and having the same cross-section have been joined as shown in the figure. Each rod is of the same length. The left and right ends are kept at 0°C and 90°C, respectively. The temperature of junction of the three rods will be* **(JEE 2001)**

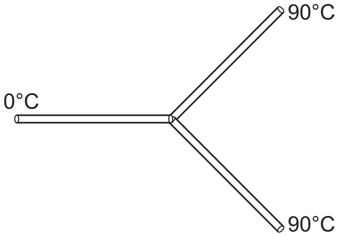


Fig. 22.14

(a) $45^\circ C$ *(b)* $60^\circ C$

(c) $30^\circ C$ *(d)* $20^\circ C$

Solution Let θ be the temperature of the junction (say B). Thermal resistance of all the three rods is equal. Rate of heat flow through AB + Rate of heat flow through CB = Rate of heat flow through BD

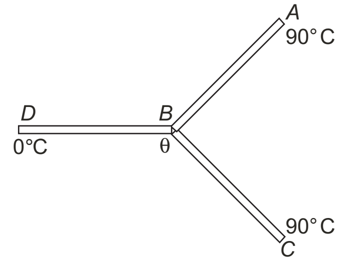


Fig. 22.15

$$\therefore \qquad \frac{90^\circ - \theta}{R} + \frac{90^\circ - \theta}{R} = \frac{\theta - 0}{R}$$

Solving this equation, we get $\theta = 60^\circ \mathrm{C}$

$\therefore$ The correct option is (b).

Here, R = Thermal resistance

Note *Rate of heat flow,* $H = \dfrac{\text{Temperature difference (TD)}}{\text{Thermal resistance } (R)}$

where, $R = \dfrac{l}{KA}$

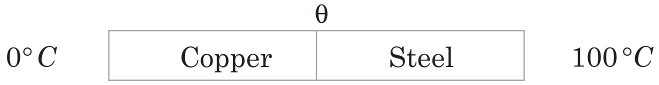
K = Thermal conductivity of the rod.

This is similar to the current flow through a resistance (R) where current (i) = Rate of flow of charge

$$= \frac{\text{Potential difference (PD)}}{\text{Electrical resistance } (R)}$$

Here, $R = \dfrac{l}{\sigma A}$ *where* σ *= Electrical conductivity.*

Example 22.11 *Figure shows a copper rod joined to a steel rod. The rods have equal length and equal cross- sectional area. The free end of the copper rod is kept at* $0^\circ C$ *and that of the steel rod is kept at* $100^\circ C$*. Find the temperature* θ *at the junction of the rods. Conductivity of copper* $= 390\ W/m\text{-}^\circ C$ *and that of steel* $= 46\ W/m\text{-}^\circ C$*.*



Solution $H_{\text{steel}} = H_{\text{copper}}$ (two rods are in series)

$$\therefore \qquad \frac{(\mathrm{TD})_S}{(l/KA)_S} = \frac{(\mathrm{TD})_C}{(l/KA)_C}$$

or $$K_S\ (\mathrm{TD})_S = K_C\ (\mathrm{TD})_C$$

or $$46\,(100 - \theta) = 390\,(\theta - 0)$$

Solving we get, $\theta = 10.6^\circ$ C **Ans.**

Note *In the above problem heat flows from right to left or from higher temperature to lower temperature.*

Example 22.12 *A spherical black body with a radius of 12 cm radiates 450 W power at 500 K. If the radius were halved and the temperature doubled, the power radiated in watt would be* **(JEE 1997)**

(a) 225 *(b)* 450

(c) 900 *(d)* 1800

Solution Power radiated $= e_r \sigma T^4 A$ or Power radiated $\propto$ (surface area) $(T)^4$. The radius is halved, hence, surface area will become $\frac{1}{4}$ times. Temperature is doubled, therefore, T^4 becomes 16 times.

$$\text{New power} = (450)\left(\frac{1}{4}\right)(16) = 1800\,\text{W}.$$

$\therefore$ The correct option is (d).

Example 22.13 *A cylinder of radius R made of a material of thermal conductivity K_1 is surrounded by a cylindrical shell of inner radius R and outer radius 2R made of a material of thermal conductivity K_2. The two ends of the combined system are maintained at two different temperatures. There is no loss of heat across the cylindrical surface and the system is in steady state. The effective thermal conductivity of the system is* **(JEE 1998)**

(a) $K_1 + K_2$ *(b)* $K_1K_2/(K_1 + K_2)$

(c) $(K_1 + 3K_2)/4$ *(d)* $(3K_1 + K_2)/4$

Solution Let R_1 and R_2 be the thermal resistances of inner and outer portions. Since, temperature difference at both ends is same, the resistances are in parallel. Hence,

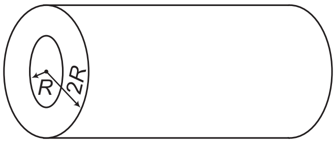


Fig. 22.16

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R = \frac{l}{KA}$$

or
$$\frac{1}{R} = \frac{KA}{l}$$

$$\therefore \quad \frac{K(4\pi R^2)}{l} = \frac{K_1(\pi R^2)}{l} + \frac{K_2(3\pi R^2)}{l}$$

$$\therefore \quad K = \frac{K_1 + 3K_2}{4}$$

$\therefore$ The correct option is (c).

Example 22.14 *A body cools in 10 minutes from 60°C to 40°C. What will be its temperature after next 10 minutes? The temperature of the surroundings is 10°C.*

Solution According to Newton's law of cooling,

$$\left(\frac{\theta_1-\theta_2}{t}\right)=\alpha\left[\left(\frac{\theta_1+\theta_2}{2}\right)-\theta_0\right]$$

For the given conditions,

$$\frac{60-40}{10}=\alpha\left[\frac{60+40}{2}-10\right] \qquad \text{...(i)}$$

Let θ be the temperature after next 10 minutes. Then,

$$\frac{40-\theta}{10}=\alpha\left[\frac{40+\theta}{2}-10\right] \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\theta = 28°\,\text{C}$$ **Ans.**

INTRODUCTORY EXERCISE 22.2

1. A rod is heated at one end as shown in figure. In steady state temperature of different sections becomes constant but not same. Why so?

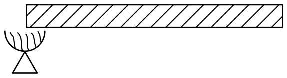


Fig. 22.17

2. Show that the SI units of thermal conductivity are W/m-K.
3. Find SI units of thermal resistance.
4. Suppose a liquid in a container is heated at the top rather than at the bottom. What is the main process by which the rest of the liquid becomes hot?
5. Three rods each of same length and cross-section are joined in series. The thermal conductivity of the materials are $K, 2K$ and $3K$ respectively. If one end is kept at 200 °C and the other at 100 °C. What would be the temperature of the junctions in the steady state? Assume that no heat is lost due to radiation from the sides of the rods.
6. A rod *CD* of thermal resistance 5.0 K/W is joined at the middle of an identical rod *AB* as shown in figure. The ends *A*, *B* and *D* are maintained at 100°C and 25°C respectively. Find the heat current in *CD*.

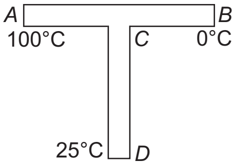


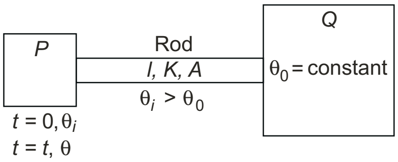
Fig. 22.18

7. A liquid takes 5 minutes to cool from 80 °C to 50 °C. How much time will it take to cool from 60 °C to 30°C? The temperature of surroundings is 20 °C.

Final Touch Points

1. Cooling by conduction or radiation

(i) **By conduction** A body P of mass m and specific heat c is connected to a large body Q (of specific heat infinite) through a rod of length l, thermal conductivity K and area of cross-section A. Temperature of Q is $\theta_0 (< \theta_i)$. This temperature will remain constant as its specific heat is very high. Heat will flow from P to Q through the rod. If we neglect the loss of heat due to radiation then due to this heat transfer, temperature of P will decrease but temperature of Q will remain almost constant. At time t, suppose temperature of P becomes θ then due to temperature difference heat transfer through the rod.



$$\frac{dQ}{dt} = H = \frac{\text{TD}}{R} = \frac{\theta - \theta_0}{R} \qquad \text{...(i)}$$

Here, $$R = \frac{l}{KA}$$

Now, if we apply equation of calorimetry in P, then

$$Q = mc(-\Delta\theta) \quad \text{or} \quad \frac{dQ}{dt} = mc\left(\frac{-d\theta}{dt}\right) \qquad \text{...(ii)}$$

Equating Eqs. (i) and (ii), we have

$$\boxed{-\frac{d\theta}{dt} = \frac{\text{TD}}{mcR} = \frac{\theta - \theta_0}{mcR} = \text{Rate of cooling}} \qquad \text{...(iii)}$$

So, this is the rate of cooling by conduction.

or $$\boxed{-\frac{d\theta}{dt} \propto \text{TD}} \qquad \text{...(iv)}$$

(ii) **By radiation** In article 22.2, we have already derived the expression of rate of cooling,

$$\boxed{-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4) \quad \text{or} \quad -\frac{dT}{dt} \propto (T^4 - T_0^4)} \qquad \text{...(v)}$$

Further in Newton's law of cooling, we have also seen that, if temperature difference between body and atmosphere is less than this rate of cooling,

$$\boxed{-\frac{dT}{dt} \propto \Delta T \quad \text{or} \quad \text{TD}} \qquad \text{...(vi)}$$

In Newton's law of cooling, we have also seen that temperature of the body falls exponentially, if

$$\text{rate of cooling, } -\frac{dT}{dt} \quad \text{or} \quad -\frac{d\theta}{dt} \propto \text{TD}$$

In Eq. (iv) of conduction and Eq. (vi) of radiation (special case when TD is small)

$$-\frac{dT}{dt} \quad \text{or} \quad -\frac{d\theta}{dt} \propto \text{TD}$$

So, in both cases temperature of the body will fall exponentially like,

$$\theta = \theta_0 + (\theta_i - \theta_0)\, e^{-\alpha t} \quad \text{or} \quad T = T_0 + (T_i - T_0)\, e^{-\alpha t}$$

In radiation, $\alpha = \dfrac{4eA\sigma T_0^3}{mc}$ and in conduction $\alpha = \dfrac{1}{mcR}$

2.

For emissivity, we have used the term e or e_r. This is sometimes confused with emissive power e. Emissivity is unitless and always less than or equal to one. But emissive power has the units J/s-m^2 or watt/m^2.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on calculation of thermal resistance*

Concept

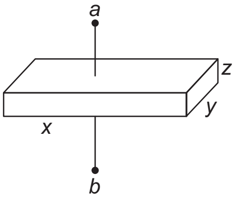
(i) A thermal resistance of a conducting rod is calculated/required between two points or two surfaces (say a and b). The formula of thermal resistance is

$$R = \frac{l}{KA}$$

Here, l is that dimension of conductor which is parallel to a and b and A is that cross-sectional area which is perpendicular to a and b.

(ii) From a to b if A is uniform at every point then direct formula $R = \frac{l}{KA}$ can be applied otherwise we will have to use integration.

Example 1 *Thermal conductivity of the conductor shown in figure is K. Find thermal resistance between points a and b.*

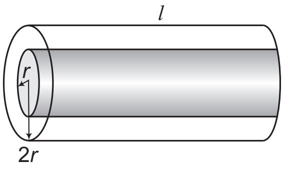


Solution From a to b area of cross-section perpendicular to ab is uniform ($A = xy$) and length along ab is z. Therefore, using the formula

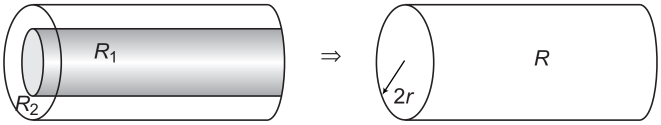
$$R = \frac{l}{KA}, \text{ we have}$$

$$R = \frac{z}{Kxy}$$ **Ans.**

Example 2 *Thermal conductivity of inner core of radius r is K and of the outer one of radius 2r is 2K. Find equivalent value of thermal conductivity between its two ends.*



Solution The meaning of equivalent value of thermal conductivity means, if a single material of K_e is used with same dimensions then value of thermal resistance should remain unchanged.



Between two ends R_1 and R_2 are in parallel. Hence,

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{R} \qquad \text{...(i)}$$

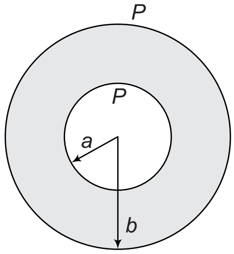
R is given by $\frac{l}{KA}$. Therefore, $\frac{1}{R}$ will be given by $\frac{KA}{l}$. Using this in Eq. (i), we have

$$\frac{K(\pi r^2)}{l}+\frac{2K[(\pi)(2r)^2-\pi r^2]}{l}=\frac{K_e(\pi)(2r)^2}{l}$$

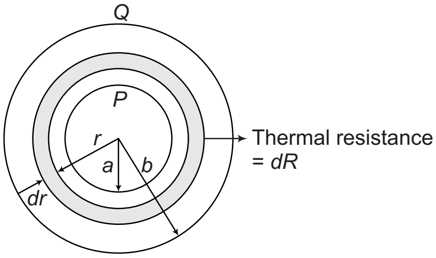
Solving this equation, we get

$$K_e=\frac{7}{4}K \qquad \textbf{Ans.}$$

Example 3 *A spherical body of radius 'b' has a concentric cavity of radius 'a' as shown. Thermal conductivity of the material is K. Find thermal resistance between inner surface P and outer surface Q.*



Solution As we move from P to Q surface perpendicular to PQ is spherical and its size keeps on increasing (just like different layers of a spherical onion). So, first we will calculate thermal resistance of one layer at a distance r from centre and thickness dr by using the formula $R=\frac{l}{KA}$.



In this formula, dimension of the layer along PQ is dr and the surface area perpendicular to PQ is $4\pi r^2$

$$\therefore \qquad dR = \frac{dr}{K(4\pi r^2)}$$

Now, if we integrate dR from $r = a$ to $r = b$, we will get the total thermal resistance between P and Q. Thus,

$$R = \int_a^b dR = \int_a^b \frac{dr}{K(4\pi r^2)}$$

Solving this expression, we get

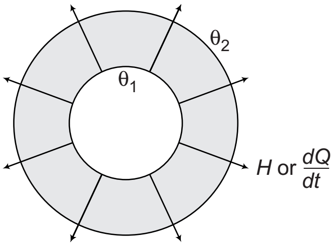
$$R = \frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad \textbf{Ans.}$$

Example 4 *In the above example, if temperature of inner surface P is kept constant at θ_1 and of the outer surface Q at $\theta_2 (< \theta_1)$. Then, Find*

(a) rate of heat flow or heat current from inner surface to outer surface.

(b) temperature θ at a distance $r (a < r < b)$ from centre.

Solution (a)

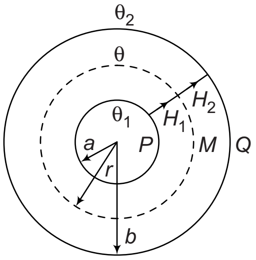


$$H \quad \text{or} \quad \frac{dQ}{dt} = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(\theta_1 - \theta_2)(4\pi K)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$= \frac{(\theta_1 - \theta_2)(ab)(4\pi K)}{(b - a)} \qquad \textbf{Ans.}$$

(b)



In the figure, we can see that

$$\text{Heat current } H_1 = \text{Heat current } H_2$$

$$\therefore \qquad \frac{(\text{TD})_{\text{PM}}}{R_{\text{PM}}} = \frac{(\text{TD})_{\text{MQ}}}{R_{\text{MQ}}} \qquad \text{...(i)}$$

Using the result obtained in Example-3 of thermal resistance, we can find

$$R_{\text{PM}} = \frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{r}\right)$$

and $$R_{\text{MQ}} = \frac{1}{4\pi K}\left(\frac{1}{r} - \frac{1}{b}\right)$$

Substituting the values in Eq. (i), we have

$$\frac{\theta_1 - \theta}{\frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{r}\right)} = \frac{\theta - \theta_2}{\frac{1}{4\pi K}\left(\frac{1}{r} - \frac{1}{b}\right)}$$

Solving this equation we can find θ.

Type 2. *Mixed problems of calorimetry and conduction*

Concept

(i) In calorimetry, we have two equations,

$$Q = mc\Delta\theta \qquad \text{(when temperature changes without change in state)}$$

and $$Q = mL \qquad \text{(when state changes without change in temperature)}$$

The above two equations in differential form can be written as

$$\frac{dQ}{dt} = mc\left(\pm\frac{d\theta}{dt}\right) \quad \text{or} \quad L\left(\frac{dm}{dt}\right) \qquad \text{...(i)}$$

In the above equation $\frac{dm}{dt}$ is rate of conversion of mass from one state to another state.

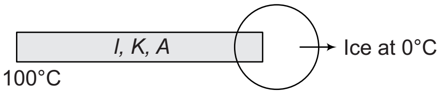
(ii) In conduction, we have the equation,

$$\frac{dQ}{dt} \quad \text{or} \quad H = \frac{\text{TD}}{R} \qquad \text{...(ii)}$$

where, $$R = \frac{l}{KA}$$

In these types of problems $\frac{dQ}{dt}$ of calorimetry is equated with $\frac{dQ}{dt}$ of conduction.

Example 5 *One end of the rod of length l, thermal conductivity K and area of cross-section A is maintained at a constant temperature of* 100 °C. *At the other end large quantity of ice is kept at* 0 °C. *Due to temperature difference, heat flows from left end to right end of the rod. Due to this heat ice will start melting. Neglecting the radiation losses find the expression of rate of melting of ice.*



Solution Putting,

$$\left(\frac{dQ}{dt}\right)_{\text{conduction}} = \left(\frac{dQ}{dt}\right)_{\text{calorimetry}}$$

$$\therefore \qquad \frac{\text{TD}}{R} = L \cdot \frac{dm}{dt}$$

$$\Rightarrow \qquad \frac{dm}{dt} = \frac{\text{TD}}{RL} \qquad \textbf{Ans.}$$

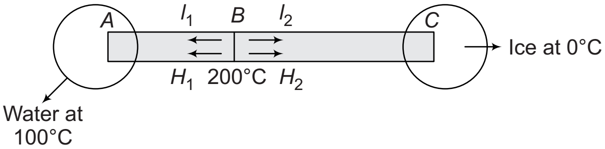
So, this is the desired expression of $\frac{dm}{dt}$.

In the above expression,

$$\text{TD} = 100°\text{C}, R = \frac{l}{KA}$$

and L = latent heat of fusion

Example 6 *B point of the rod shown in figure is maintained at 200°C. At left end A, there is water at 100°C and at right end C there is ice at 0°C. Heat currents H_1 and H_2 will flow on both sides. Due to H_1, water will convert into steam and due to H_2 ice will be melted. If latent heat of vaporization is 540 cal/g and latent heat of fusion is 80 cal/g then neglecting the radiation losses find $\frac{l_1}{l_2}$ so that rate of melting of ice is two times the rate of conversion of water into steam.*



Solution Using the relation of $\frac{dm}{dt}$ derived in above example,

$$\frac{dm}{dt} = \frac{\text{TD}}{RL} = \frac{(\text{TD})KA}{lL} \qquad \left(\text{as } R = \frac{l}{KA}\right)$$

Given that,

$$\left(\frac{dm}{dt}\right)_{\text{RHS}} = 2\left(\frac{dm}{dt}\right)_{\text{LHS}}$$

or

$$\left[\frac{(\text{TD})KA}{lL}\right]_{\text{RHS}} = 2\left[\frac{(\text{TD})KA}{lL}\right]_{\text{LHS}}$$

K and A are same on both sides. Hence,

$$\left(\frac{\text{TD}}{lL}\right)_{\text{RHS}} = 2\left(\frac{\text{TD}}{lL}\right)_{\text{LHS}}$$

Substituting the proper values, we have

$$\frac{200}{l_2 \times 80} = 2\left[\frac{100}{l_1 \times 540}\right]$$

$$\therefore \qquad \frac{l_1}{l_2} = \frac{80}{540} = \frac{4}{27} \qquad \textbf{Ans.}$$

Type 3. *Mixed problems of Stefan's law and Wien's displacement law*

Concept

Stefan's law

Energy radiated per unit time = emissive power = $e_r \sigma T^4$

Energy radiated per unit time called power = $e_r \sigma T^4 A$

Total energy radiated in time 't' = $e_r \sigma T^4 At$

Above all three terms $\propto T^4$

Wien's displacement law

$$\lambda_m \propto \frac{1}{T}$$

From Wien's law normally we find the ratio of temperatures of two bodies and then this ratio is used in Stefan's law.

Example 7 *Three discs, A, B and C having radii 2m, 4m and 6m respectively are coated with carbon black on their outer surfaces. The wavelengths corresponding to maximum intensity are 300 nm, 400 nm and 500 nm, respectively. The power radiated by them are Q_A, Q_B and Q_C respectively*

(JEE 2004)

(a) Q_A is maximum *(b) Q_B is maximum* *(c) Q_C is maximum* *(d) $Q_A = Q_B = Q_C$*

Solution (b) $Q \propto AT^4$ and $\lambda_m T$ = constant.

Hence,

$$Q \propto \frac{A}{(\lambda_m)^4} \quad \text{or} \quad Q \propto \frac{r^2}{(\lambda_m)^4}$$

$$Q_A : Q_B : Q_C = \frac{(2)^2}{(3)^4} : \frac{(4)^2}{(4)^4} : \frac{(6)^2}{(5)^4} = \frac{4}{81} : \frac{1}{16} : \frac{36}{625} = 0.05 : 0.0625 : 0.0576$$

i.e. Q_B is maximum.

Type 4. *Problems of cooling either by conduction or radiation*

Concept

In final touch points we have seen that if a body cools by conduction, then

$$\text{rate of cooling,} \quad -\frac{d\theta}{dt} = \frac{\text{TD}}{mcR} \qquad \text{...(i)}$$

Here, $mc = C$ = heat capacity of the body

This C is the heat required to raise the temperature of whole body by 1°C or 1K and R is the thermal resistance of conducting rod. So, Eq. (i) can also be written as

$$-\frac{d\theta}{dt} = \frac{\text{TD}}{CR}$$

If the body cools by radiation, then

$$\text{rate of cooling or } -\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4) \qquad \text{...(ii)}$$

For small temperature difference, Newton's law of cooling can be applied and

$$-\frac{dT}{dt} \propto \text{TD} \qquad \text{...(iii)}$$

Further, if rate of cooling ∝ TD as Eqs. (i) or (iii), temperature of the body falls exponentially.

Example 8 *Two spheres, one solid and other hollow are kept in atmosphere at same temperature. They are made of same material and their radii are also same. Which sphere will cool at a faster rate initially?*

Solution In the equation,

$$-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4)$$

Only mass is different. Therefore,

$$-\frac{dT}{dt} \propto \frac{1}{m}$$

Mass of hollow sphere is less. So, hollow sphere will cool at a faster rate initially.

Example 9 *The ratio of specific heats of two spheres is $2:3$, radii $1:2$, emissivity $3:1$ and density $1:1$. Initially, they are kept at same temperatures in atmosphere. Which sphere will cool at a faster rate initially.*

Solution In the equation,

$$-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4)$$

$$A = \text{surface area } = 4\pi R^2$$

$$m = \text{Volume} \times \text{density} = \frac{4}{3}\pi R^3 \rho$$

Substituting in the above equation, we have

$$-\frac{dT}{dt} = \frac{3e\sigma}{R\rho c}(T^4 - T_0^4) \quad \text{or} \quad -\frac{dT}{dt} \propto \frac{e}{R\rho c}$$

Now,

$$\left(\frac{e}{R\rho c}\right)_{\text{sphere-1}} = \frac{3}{1 \times 1 \times 2} = \frac{3}{2}$$

and

$$\left(\frac{e}{R\rho c}\right)_{\text{sphere-2}} = \frac{1}{2 \times 1 \times 3} = \frac{1}{6}$$

$$\left(\frac{e}{R\rho c}\right) \text{ for sphere-1 is more.}$$

So, first sphere will cool at faster rate initially.

Example 10 *Two metallic spheres S_1 and S_2 are made of the same material and have got identical surface finish. The mass of S_1 is thrice that of S_2. Both the spheres are heated to the same high temperature and placed in the same room having lower temperature but are thermally insulated from each other. The ratio of the initial rate of cooling of S_1 to that of S_2 is* **(JEE 1995)**

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{1}$ (d) $\left(\frac{1}{3}\right)^{1/3}$

Solution $-\dfrac{dT}{dt} = \dfrac{eA\sigma}{mc}(T^4 - T_o^4)$

$$m_1 = 3m_2$$

$$\therefore \qquad R_1 = (3)^{\frac{1}{3}} R_2$$

$$\left(-\frac{dT}{dt}\right) \propto \frac{A}{m} \propto \frac{R^2}{R^3} \propto \frac{1}{R}$$

$$\therefore \qquad \frac{(-dT/dt)_1}{(-dT/dt)_2} = \frac{R_2}{R_1} = \left(\frac{1}{3}\right)^{\frac{1}{3}}$$

∴ The correct option is (d).

Example 11 *Two identical conducting rods are first connected independently to two vessels, one containing water at 100°C and the other containing ice at 0°C. In the second case, the rods are joined end to end and connected to the same vessels. Let q_1 and q_2 gram per second be the rate of melting of ice in the two cases respectively. The ratio $\dfrac{q_1}{q_2}$ is* **(JEE 2004)**

(a) $\dfrac{1}{2}$ (b) $\dfrac{2}{1}$ (c) $\dfrac{4}{1}$ (d) $\dfrac{1}{4}$

Solution (c) $\dfrac{dQ}{dt} = L\left(\dfrac{dm}{dt}\right)$ or $\dfrac{\text{Temperature difference}}{\text{Thermal resistance}} = L\left(\dfrac{dm}{dt}\right)$

or $$\frac{dm}{dt} \propto \frac{1}{\text{Thermal resistance}} \quad \Rightarrow \quad q \propto \frac{1}{R}$$

In the first case rods are in parallel and thermal resistance is $\dfrac{R}{2}$ while in second case rods are in series and thermal resistance is $2R$.

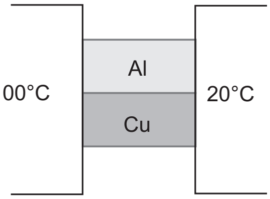
$$\frac{q_1}{q_2} = \frac{2R}{R/2} = \frac{4}{1}$$

Miscellaneous Examples

Example 12 *Two metal cubes with 3 cm-edges of copper and aluminium are arranged as shown in figure. Find*

(a) the total thermal current from one reservoir to the other

(b) the ratio of the thermal current carried by the copper cube to that carried by the aluminium cube. Thermal conductivity of copper is 401 W/m-K and that of aluminium is 237 W/m-K.



Solution (a) Thermal resistance of aluminium cube, $R_1 = \dfrac{l}{KA}$

or $$R_1 = \frac{(3.0 \times 10^{-2})}{(237)\,(3.0 \times 10^{-2})^2}$$

$$= 0.14 \text{ K/W}$$

and thermal resistance of copper cube, $R_2 = \dfrac{l}{KA}$

or $$R_2 = \frac{(3.0 \times 10^{-2})}{(401)\,(3.0 \times 10^{-2})^2} = 0.08 \text{ K/W}$$

As these two resistance are in parallel, their equivalent resistance will be

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{(0.14)\,(0.08)}{(0.14) + (0.08)} = 0.05 \text{ K/W}$$

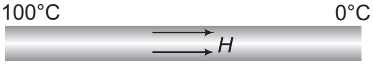
$\therefore$ $$\text{Thermal current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}} = \frac{(100 - 20)}{0.05} = 1.6 \times 10^3 \text{ W}$$ **Ans.**

(b) In parallel thermal current distributes in the inverse ratio of resistance. Hence,

$$\frac{H_{\text{Cu}}}{H_{\text{Al}}} = \frac{R_{\text{Al}}}{R_{\text{Cu}}} = \frac{R_1}{R_2} = \frac{0.14}{0.08} = 1.75$$ **Ans.**

Example 13 *One end of a copper rod of length* 1 *m and area of cross-section* 4.0×10^{-4} *m*^{2} *is maintained at* 100°C. *At the other end of the rod ice is kept at* 0°C. *Neglecting the loss of heat from the surroundings, find the mass of ice melted in* 1 *h. Given,* $K_{Cu} = 401$ *W/m-K and* $L_f = 3.35 \times 10^5$ *J/kg.*

Solution Thermal resistance of the rod,



$$R = \frac{l}{KA} = \frac{1.0}{(401)\,(4 \times 10^{-4})} = 6.23 \text{ K/W}$$

$\therefore$ $$\text{Heat current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}} = \frac{(100 - 0)}{6.23} = 16 \text{ W}$$

Heat transferred in 1 h,

$$Q = Ht \qquad \left(\because \; H = \frac{Q}{t}\right)$$

$$= (16)\,(3600) = 57600 \text{ J}$$

Now, let m mass of ice melts in 1 h, then

$$m = \frac{Q}{L} \qquad (Q = mL)$$

$$= \frac{57600}{3.35 \times 10^5}$$

$$= 0.172 \text{ kg}$$

or $= 172$ g **Ans.**

Example 14 *Two bodies A and B have thermal emissivities of* 0.01 *and* 0.81 *respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. The wavelength* λ_B *corresponding to maximum spectral radiancy from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A by* 1.0 μm. *If the temperature of A is* 5802 *K, calculate (a) the temperature of B, (b) wavelength* λ_B.

Solution (a)

$$P_A = P_B$$

∴ $$e_A \sigma A_A T_A^4 = e_B \sigma A_B T_B^4$$

∴ $$T_B = \left(\frac{e_A}{e_B}\right)^{1/4} T_A \qquad \text{(as } A_A = A_B\text{)}$$

Substituting the values,

$$T_B = \left(\frac{0.01}{0.81}\right)^{1/4} (5802) = 1934 \text{ K}$$ **Ans.**

(b) According to Wien's displacement law,

$$\lambda_A T_A = \lambda_B T_B$$

∴ $$\lambda_B = \left(\frac{T_A}{T_B}\right)\lambda_A = \left(\frac{5802}{1934}\right)\lambda_A$$

or $$\lambda_B = 3\lambda_A$$

Also, $$\lambda_B - \lambda_A = 1\ \mu\text{m}$$

or $$\lambda_B - \left(\frac{1}{3}\right)\lambda_B = 1\ \mu\text{m}$$

or $$\lambda_B = 1.5\ \mu\text{m}$$ **Ans.**

Example 15 5 *g of water at* 30°*C and* 5 *g of ice at* – 20°*C are mixed together in a calorimeter. Find the final temperature of mixture. Water equivalent of calorimeter is negligible, specific heat of ice* = 0.5 *cal/g-°C and latent heat of ice* = 80 *cal/g.*

Solution In this case heat is given by water and taken by ice

Heat available with water to cool from 30° C to 0° C

$$= ms\Delta\theta = 5 \times 1 \times 30$$

$$= 150 \text{ cal}$$

Heat required by 5 g ice to increase its temperature upto 0°C

$$ms\Delta\theta = 5 \times 0.5 \times 20$$

$$= 50 \text{ cal}$$

Out of 150 cal heat available, 50 cal is used for increasing temperature of ice from – 20° C to 0° C. The remaining heat 100 cal is used for melting the ice.

If mass of ice melted is m g, then

$$m \times 80 = 100$$

⇒ $$m = 1.25 \text{ g}$$

Thus, 1.25 g ice out of 5 g melts and mixture of ice and water is at 0° C.

Example 16 *A bullet of mass* 10 *g moving with a speed of* 20 *m/s hits an ice block of mass* 990 *g kept on a frictionless floor and gets stuck in it. How much ice will melt if* 50% *of the lost kinetic energy goes to ice?*
(Temperature of ice block $= 0°C$ *).*

Solution Velocity of bullet + ice block,

$$V = \frac{(10\ \text{g}) \times (20\ \text{m/s})}{1000\ \text{g}} = 0.2\ \text{m/s} \qquad (P_i = P_f)$$

$$\text{Loss of KE} = \frac{1}{2}mv^2 - \frac{1}{2}(m+M)V^2$$

$$= \frac{1}{2}[0.01 \times (20)^2 - 1 \times (0.2)^2]$$

$$= \frac{1}{2}[4 - 0.04] = 1.98\ \text{J}$$

$\therefore$ Heat received by ice block $= \dfrac{1.98}{4.2 \times 2}$ cal

$$= 0.24\ \text{cal}$$

$\therefore$ Mass of ice melted $= \dfrac{(0.24\ \text{cal})}{(80\ \text{cal/g})}$

$$= 0.003\ \text{g}$$ **Ans.**

Example 17 *At* 1 *atmospheric pressure,* 1.000 *g of water having a volume of* 1.000 cm^3 *becomes* 1671 cm^3 *of steam when boiled. The heat of vaporization of water at* 1 *atmosphere is* 539 *cal/g. What is the change in internal energy during the process?*

Solution Heat spent during vaporisation,

$$Q = mL = 1.000 \times 539 = 539\ \text{cal}$$

$$\text{Work done, } W = P(V_v - V_l)$$

$$= 1.013 \times 10^5 \times (1671 - 1.000) \times 10^{-6}$$

$$= 169.2\ \text{J} = \frac{169.2}{4.18}\ \text{cal}$$

$$= 40.5\ \text{cal}$$

$\therefore$ Change in internal energy,

$\therefore$ $$\Delta U = 539\ \text{cal} - 40.5\ \text{cal}$$

$$= 498.5\ \text{cal}$$ **Ans.**

Example 18 *At* 1 *atmospheric pressure,* 1.000 *g of water having a volume of* 1.000 cm^3 *becomes* 1.091 cm^3 *of ice on freezing. The heat of fusion of water at* 1 *atmosphere is* 80.0 *cal/g. What is the change in internal energy during the process?*

Solution Heat given out during freezing,

$$Q = -mL = -1 \times 80 = -80\ \text{cal}$$

External work done $$W = P(V_{\text{ice}} - V_{\text{water}})$$

$$= 1.013 \times 10^5 \times (1.091 - 1.000) \times 10^{-6}$$

$$= 9.22 \times 10^{-3}\ \text{J}$$

$$= \frac{9.22 \times 10^{-3}}{4.18}\ \text{cal}$$

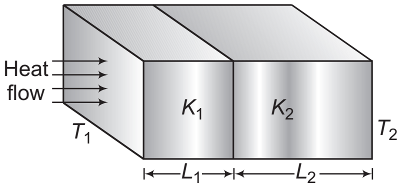
$$= 0.0022\ \text{cal}$$

∴ Change in internal energy,

$$\Delta U = Q - W = -80 - 0.0022$$

$$= -80.0022\ \text{cal}$$ **Ans.**

Example 19 *Two plates each of area A, thickness L_1 and L_2 thermal conductivities K_1 and K_2 respectively are joined to form a single plate of thickness $(L_1 + L_2)$. If the temperatures of the free surfaces are T_1 and T_2. Calculate*



(a) rate of flow of heat
(b) temperature of interface and
(c) equivalent thermal conductivity.

Solution (a) If the thermal resistance of the two plates are R_1 and R_2 respectively. Plates are in series.

$$R_S = R_1 + R_2 = \frac{L_1}{AK_1} + \frac{L_2}{AK_2}$$

as $$R = \frac{L}{KA}$$

and so $$H = \frac{dQ}{dt} = \frac{\Delta T}{R}$$

$$= \frac{(T_1 - T_2)}{(R_1 + R_2)} = \frac{A(T_1 - T_2)}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]}$$ **Ans.**

(b) If T is the common temperature of interface, then as in series rate of flow of heat remains same, i.e. $H = H_1\ (= H_2)$

$$\frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T}{R_1}$$

i.e. $$T = \frac{T_1R_2 + T_2R_1}{(R_1 + R_2)}$$

or $$T = \frac{\left[T_1\frac{L_2}{K_2} + T_2\frac{L_1}{K_1}\right]}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]} \qquad \left(\text{as } R = \frac{L}{KA}\right)$$

(c) If K is the equivalent conductivity of composite slab, i.e. slab of thickness $L_1 + L_2$ and cross-sectional area A, then as in series

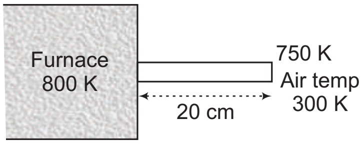
$$R_S = R_1 + R_2 \quad \text{or} \quad \frac{(L_1 + L_2)}{AK_{\text{eq}}} = R_1 + R_2$$

i.e.

$$K_{\text{eq}} = \frac{L_1 + L_2}{A(R_1 + R_2)} = \frac{L_1 + L_2}{\left[\dfrac{L_1}{K_1} + \dfrac{L_2}{K_2}\right]} \qquad \left(\text{as } R = \frac{L}{KA}\right)$$

Example 20 *One end of a rod of length 20 cm is inserted in a furnace at 800 K. The sides of the rod are covered with an insulating material and the other end emits radiation like a black body. The temperature of this end is 750 K in the steady state. The temperature of the surrounding air is 300 K. Assuming radiation to be the only important mode of energy transfer between the surroundings and the open end of the rod, find the thermal conductivity of the rod. Stefan constant* $\sigma = 6.0 \times 10^{-8}\ W/m^2\text{-}K^4$. *Take emissivity of the open end* $e = 1$

Solution Heat flowing through the rod per second in steady state,



$$\frac{dQ}{dt} = \frac{KAd\theta}{x} \qquad \text{...(i)}$$

Heat radiated from the open end of the rod per second in steady state,

$$\frac{dQ}{dt} = A\sigma\,(T^4 - T_0^4) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

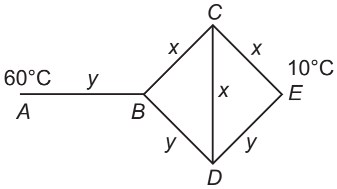
$$\frac{Kd\theta}{x} = \sigma\,(T^4 - T_0^4)$$

$$\frac{K \times 50}{0.2} = 6.0 \times 10^{-8}\ [(7.5)^4 - (3)^4] \times 10^8$$

or

$$K = 74 \text{ W/m-K}$$ **Ans.**

Example 21 *Three rods of material x and three rods of material y are connected as shown in figure. All the rods are of identical length and cross-sectional area. If the end A is maintained at 60°C and the junction E at 10°C, calculate temperature of junctions B, C and D. The thermal conductivity of x is 0.92 cal/cm-s°C and that of y is 0.46 cal/cm-s°C.*



Solution Thermal resistance, $R = \dfrac{l}{KA}$

$$\therefore \qquad \frac{R_x}{R_y} = \frac{K_y}{K_x} \qquad \text{(as } l_x = l_y \text{ and } A_x = A_y\text{)}$$

$$= \frac{0.46}{0.92} = \frac{1}{2}$$

So, if $R_x = R$ then $R_y = 2R$

CEDB forms a balanced Wheatstone bridge, i.e. $T_C = T_D$ and no heat flows through *CD*.

$$\therefore \qquad \frac{1}{R_{BE}} = \frac{1}{R+R} + \frac{1}{2R+2R}$$

or

$$R_{BE} = \frac{4}{3} R$$

The total resistance between *A* and *E* will be

$$R_{AE} = R_{AB} + R_{BE} = 2R + \frac{4}{3} R = \frac{10}{3} R$$

$\therefore$ Heat current between *A* and *E* is

$$H = \frac{(\Delta T)_{AE}}{R_{AE}} = \frac{(60-10)}{(10/3)\,R} = \frac{15}{R}$$

Now, if T_B is the temperature at *B*,

$$H_{AB} = \frac{(\Delta T)_{AB}}{R_{AB}}$$

or

$$\frac{15}{R} = \frac{60 - T_B}{2R}$$

or

$$T_B = 30°\,\text{C} \qquad \textbf{Ans.}$$

Further,

$$H_{AB} = H_{BC} + H_{BD}$$

or

$$\frac{15}{R} = \frac{30 - T_C}{R} + \frac{30 - T_D}{2R} \qquad [T_C = T_D = T \text{ (say)}]$$

or

$$15 = (30 - T) + \frac{(30 - T)}{2}$$

Solving this, we get $T = 20°\,\text{C}$

or

$$T_C = T_D = 20°\,\text{C} \qquad \textbf{Ans.}$$

Example 22 *A hollow sphere of glass whose external and internal radii are* 11 *cm and* 9 *cm respectively is completely filled with ice at* 0°C *and placed in a bath of boiling water. How long will it take for the ice to melt completely? Given that density of ice* $= 0.9\ g/cm^3$, *latent heat of fusion of ice* $= 80\ cal/g$ *and thermal conductivity of glass* $= 0.002\ cal/cm\text{-}s°C$.

Solution In steady state, rate of heat flow (result is taken from Example 4)

$$H = \frac{4\pi K r_1 r_2 \Delta T}{r_2 - r_1}$$

Substituting the values, $$H = \frac{(4)(\pi)(0.002)(11)(9)(100-0)}{(11-9)}$$

or $$\frac{dQ}{dt} = 124.4 \text{ cal/s}$$

This rate should be equal to, $L\dfrac{dm}{dt}$

$$\therefore \quad \left(\frac{dm}{dt}\right) = \frac{dQ/dt}{L} = \frac{124.4}{80} = 1.555 \text{ g/s}$$

Total mass of ice, $$m = \rho_{\text{ice}}(4\pi r_1^2) = (0.9)(4)(\pi)(9)^2 = 916 \text{ g}$$

∴ Time taken for the ice to melt completely

$$t = \frac{m}{(dm/dt)} = \frac{916}{1.555} = 589 \text{ s}$$ **Ans.**

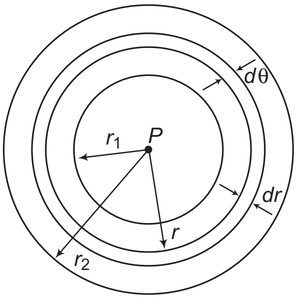
Example 23 *A point source of heat of power P is placed at the centre of a spherical shell of mean radius R. The material of the shell has thermal conductivity K. Calculate the thickness of the shell if temperature difference between the outer and inner surfaces of the shell in steady state is T.*

Solution Consider a concentric spherical shell of radius r and thickness dr as shown in figure. In steady state, the rate of heat flow (heat current) through this shell will be

$$H = \frac{\Delta T}{R} = \frac{(-d\theta)}{\dfrac{dr}{(k)(4\pi r^2)}} \qquad \left(R = \frac{l}{KA}\right)$$

or $$H = -(4\pi K r^2)\frac{d\theta}{dr}$$

Here, negative sign is used because with increase in r, θ decreases



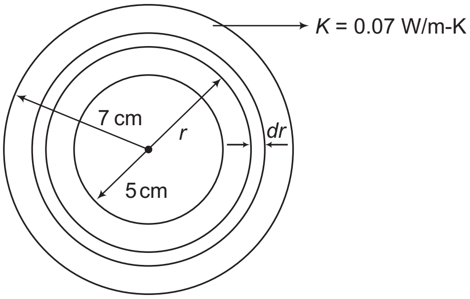
$$\therefore \quad \int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{4\pi K}{H}\int_{\theta_1}^{\theta_2} d\theta$$

This equation gives, $$H = \frac{4\pi K r_1 r_2 (\theta_1 - \theta_2)}{(r_2 - r_1)}$$

In steady state, $$H = P,\ r_1 r_2 \approx R^2 \text{ and } \theta_1 - \theta_2 = T$$

∴ Thickness of shell, $$r_2 - r_1 = \frac{4\pi K R^2 T}{P}$$ **Ans.**

Example 24 *A steam cylindrical pipe of radius* 5 *cm carries steam at* 100°*C. The pipe is covered by a jacket of insulating material* 2 *cm thick having a thermal conductivity* 0.07 *W/m-K. If the temperature at the outer wall of the pipe jacket is* 20°*C, how much heat is lost through the jacket per metre length in an hour?*



Solution Thermal resistance per metre length of an element at distance r of thickness dr is

$$dR = \frac{dr}{K\,(2\pi r)} \qquad \left(R = \frac{l}{KA}\right)$$

∴ Total resistance

$$R = \int_{r_1 = 5\text{ cm}}^{r_2 = 7\text{ cm}} dR$$

$$= \frac{1}{2\pi K}\int_{5.0\times10^{-2}\text{ m}}^{7.0\times10^{-2}\text{ m}} \frac{dr}{r}$$

$$= \frac{1}{2\pi K}\ln\left(\frac{7}{5}\right)$$

$$= \frac{1}{(2\pi)(0.07)}\ln(1.4)$$

$$= 0.765\text{ K/W}$$

$$\text{Heat current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(100-20)}{0.765} = 104.6\text{ W}$$

∴ Heat lost in one hour = Heat current × time

$$= (104.6)\,(3600)\text{ J}$$

$$= 3.76\times10^{5}\text{ J}$$

Ans.

Exercises

LEVEL 1

Take $c_{\text{ice}} = 0.53$ cal/g-°C, $c_{\text{water}} = 1.0$ cal/g-°C, $(L_f)_{\text{water}} = 80$ cal/g and $(L_v)_{\text{water}} = 529$ cal/g unless given in the question.

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Specific heat of any substance remains constant at all temperatures.

Reason : It is given by $$s = \frac{1}{m} \cdot \frac{dQ}{dT}$$

2. Assertion : When temperature of a body is increased, in radiant energy, number of low wavelength photons get increased.

Reason : According to Wien's displacement law $\lambda_m \propto \frac{1}{T}$.

3. Assertion : Warming a room by a heat blower is an example of forced convection.

Reason : Natural convection takes place due to gravity.

4. Assertion : A conducting rod is placed between boiling water and ice. If rod is broken into two equal parts and two parts are connected side by side, then rate of melting of ice will increase to four times.

Reason : Thermal resistance will become four times.

5. Assertion : A normal body can radiate energy more than a perfectly black body.

Reason : A perfectly black body is always black in colour.

6. Assertion : According to Newton's law, good conductors of electricity are also good conductors of heat.

Reason : At a given temperature, $e_\lambda \propto a_\lambda$ for any body.

7. Assertion : Good conductors of electricity are also good conductors of heat due to large number of free electrons.

Reason : It is easy to conduct heat from free electrons.

8. Assertion : Emissivity of any body (e) is always less than its absorptive power (a).

Reason : Both the quantities are dimensionless.

9. Assertion : Heat is supplied at constant rate from one end of a conducting rod. In steady state, temperature of all points of the rod becomes uniform.

Reason : In steady state, temperature of rod does not increase.

10. **Assertion :** A solid sphere and a hollow sphere of same material and same radius are kept at same temperatures in atmosphere. Rate of cooling of hollow sphere will be more.

Reason : If all other conditions are same, then rate of cooling is inversely proportional to the mass of body.

Objective Questions

1. For an enclosure maintained at 2000 K, the maximum radiation occurs at wavelength λ_m. If the temperature is raised to 3000 K, the peak will shift to

(a) $0.5\,\lambda_m$ (b) λ_m (c) $\frac{2}{3}\lambda_m$ (d) $\frac{3}{2}\lambda_m$

2. A substance cools from 75°C to 70°C in T_1 minute, from 70°C to 65°C in T_2 minute and from 65°C to 60°C in T_3 minute, then

(a) $T_1 = T_2 = T_3$ (b) $T_1 < T_2 < T_3$
(c) $T_1 > T_2 > T_3$ (d) $T_1 < T_2 > T_3$

3. Two liquids are at temperatures 20°C and 40°C. When same mass of both of them is mixed, the temperature of the mixture is 32°C. What is the ratio of their specific heats?

(a) 1/3 (b) 2/5
(c) 3/2 (d) 2/3

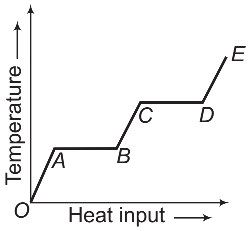
4. The specific heat of a metal at low temperatures varies according to $S = aT^3$, where a is a constant and T is absolute temperature. The heat energy needed to raise unit mass of the metal from temperature $T = 1$ K to $T = 2$ K is

(a) $3a$ (b) $\frac{15a}{4}$ (c) $\frac{2a}{3}$ (d) $\frac{13a}{4}$

5. The intensity of radiation emitted by the sun has its maximum value at a wavelength of 510 nm and that emitted by the North star has the maximum value at 350 nm. If these stars behave like black bodies, then the ratio of the surface temperatures of the sun and the north star is

(a) 1.46 (b) 0.69
(c) 1.21 (d) 0.83

6. A solid material is supplied heat at a constant rate. The temperature of material is changing with heat input as shown in the figure. What does the slope of DE represent?



(a) Latent heat of liquid (b) Latent heat of vaporization
(c) Heat capacity of vapour (d) Inverse of heat capacity of vapour

7. Two ends of rods of length L and radius R of the same material are kept at the same temperature. Which of the following rods conducts the maximum heat?

(a) $L = 50$ cm, $R = 1$ cm (b) $L = 100$ cm, $R = 2$ cm
(c) $L = 25$ cm, $R = 0.5$ cm (d) $L = 75$ cm, $R = 1.5$ cm

8. 1 g of ice at 0°C is mixed with 1 g of steam at 100°C. After thermal equilibrium is achieved, the temperature of the mixture is

(a) 100°C (b) 55°C (c) 75°C (d) 0°C

9. A wall has two layers *A* and *B* each made of different materials. The layer *A* is 10 cm thick and *B* is 20 cm thick. The thermal conductivity of *A* is thrice that of *B*. Under thermal equilibrium temperature difference across the wall is 35°C. The difference of temperature across the layer *A* is

(a) 30°C (b) 14°C (c) 8.75°C (d) 5°C

10. A wall has two layers *A* and *B* each made of different materials. Both the layers have the same thickness. The thermal conductivity of material *A* is twice of *B*. Under thermal equilibrium the temperature difference across the layer *B* is 36°C. The temperature difference across layer *A* is

(a) 6°C (b) 12°C (c) 18°C (d) 24°C

11. The end of two rods of different materials with their thermal conductivities, area of cross-section and lengths all in the ratio 1 : 2 are maintained at the same temperature difference. If the rate of flow of heat in the first rod is 4 cal/s. Then, in the second rod rate of heat flow in cal/s will be

(a) 1 (b) 2 (c) 8 (d) 16

Subjective Questions

1. A thin square steel plate 10 cm on a side is heated in a black smith's forge to temperature of $800° \text{C}$. If the emissivity is 0.60, what is the total rate of radiation of energy?

2. A lead bullet penetrates into a solid object and melts. Assuming that 50% of its kinetic energy was used to heat it, calculate the initial speed of the bullet. The initial temperature of the bullet is $27° \text{C}$ and its melting point is $327° \text{C}$. Latent heat of fusion of lead $= 2.5 \times 10^4$ J/kg and specific heat capacity of lead = 125 J/kg-K.

3. A ball is dropped on a floor from a height of 2.0 m . After the collision it rises upto a height of 1.5 m. Assume that 40% of the mechanical energy lost goes as thermal energy into the ball. Calculate the rise in the temperature of the ball in the collision. Specific heat of the ball is 800 J/K.

4. A nuclear power plant generates 500 MW of waste heat that must be carried away by water pumped from a lake. If the water temperature is to rise by 10°C, what is the required flow rate in kg/s?

5. The emissivity of tungsten is 0.4. A tungsten sphere with a radius of 4.0 cm is suspended within a large evacuated enclosure whose walls are at 300 K. What power input is required to maintain the sphere at a temperature of 3000 K if heat conduction along supports is neglected? Take, $\sigma = 5.67 \times 10^{-8}\ \text{Wm}^2\text{-K}^4$.

6. A pot with a steel bottom 1.2 cm thick rests on a hot stove. The area of the bottom of the pot is $0.150\ \text{m}^2$. The water inside the pot is at 100°C and 0.440 kg are evaporated every 5.0 minute. Find the temperature of the lower surface of the pot, which is in contact with the stove. Take $L_v = 2.256 \times 10^6$ J/kg and $k_{\text{steel}} = 50.2$ W/m-K

7. A carpenter builds an outer house wall with a layer of wood 2.0 cm thick on the outside and a layer of an insulation 3.5 cm thick as the inside wall surface. The wood has $K = 0.08$ W/m-K and the insulation has $K = 0.01$ W/m-K. The interior surface temperature is $19° \text{C}$ and the exterior surface temperature is $-10° \text{C}$.

(a) What is the temperature at the plane where the wood meets the insulation?

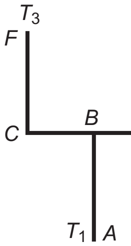
(b) What is the rate of heat flow per square metre through this wall?

8. A closely thermally insulated vessel contains 100 g of water at 0°C. If the air from this vessel is rapidly pumped out, intensive evaporation will produce cooling and as a result of this, water freeze. How much ice will be formed by this method? If latent heat of fusion is 80 cal/g and of evaporation 560 cal/g.
[**Hint** If m gram ice is formed, $mL_f = (100 - m) L_v$]

9. In a container of negligible mass 140 g of ice initially at $-15°$ C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system and masses of water and ice in mixture?

10. A certain amount of ice is supplied heat at a constant rate for 7 minutes. For the first one minute the temperature rises uniformly with time. Then, it remains constant for the next 4 minutes and again the temperature rises at uniform rate for the last two minutes. Calculate the final temperature at the end of seven minutes.
(Given, L of ice $= 336 \times 10^3$ J/ kg and specific heat of water = 4200 J/ kg-K)

11. Four identical rods AB, CD, CF and DE are joined as shown in figure. The length, cross-sectional area and thermal conductivity of each rod are l, A and K respectively. The ends A, E and F are maintained at temperatures T_1, T_2 and T_3 respectively. Assuming no loss of heat to the atmosphere. Find the temperature at B, the mid-point of CD.



12. The ends of a copper rod of length 1 m and area of cross-section 1 cm^2 are maintained at 0° C and 100° C. At the centre of the rod there is a source of heat of power 25 W. Calculate the temperature gradient in the two halves of the rod in steady state. Thermal conductivity of copper is 400 $\text{Wm}^{-1}\text{K}^{-1}$.

13. A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is maintained at a constant temperature of 500 K by heating it electrically . A total of 210 W of electric power is needed to do it. When the surface of the copper sphere is completely blackened, 700 W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.

LEVEL 2

Single Correct Options

1. A cylindrical rod with one end in a steam chamber and the other end in ice results in melting of 0.1 g of ice per second. If the rod is replaced by another with half the length and double the radius of the first and if the thermal conductivity of material of second rod is 1/4 that of first, the rate at which ice melts in g/s will be

(a) 0.4
(b) 0.05
(c) 0.2
(d) 0.1

2. Two sheets of thickness d and $3d$, are touching each other. The temperature just outside the thinner sheet is T_1 and on the side of the thicker sheet is T_3. The interface temperature is T_2. T_1, T_2 and T_3 are in arithmetic progression. The ratio of thermal conductivity of thinner sheet to thicker sheet is

(a) 1 : 3
(b) 3 : 1
(c) 2 : 3
(d) 3 : 9

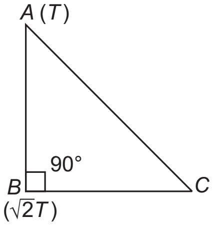
3. A long rod has one end at 0°C and other end at a high temperature. The coefficient of thermal conductivity varies with distance from the low temperature end as $K = K_0(1 + ax)$, where $K_0 = 10^2$ SI unit and $a = 1\ \text{m}^{-1}$. At what distance from the first end the temperature will be 100°C ? The area of cross-section is $1\ \text{cm}^2$ and rate of heat conduction is 1 W.

(a) 2.7 m
(b) 1.7 m
(c) 3 m
(d) 1.5 m

4. Two rods are of same material and having same length and area. If heat ΔQ flows through them for 12 min when they are joined side by side. If now both the rods are joined in parallel, then the same amount of heat ΔQ will flow in

(a) 24 min
(b) 3 min
(c) 12 min
(d) 6 min

5. Three rods of identical cross-sectional area and made from the same metal form the sides of an isosceles triangle ABC right angled at B as shown in the figure. The points A and B are maintained at temperatures T and $\sqrt{2}\,T$ respectively in the steady state. Assuming that only heat conduction takes place, temperature of point C will be

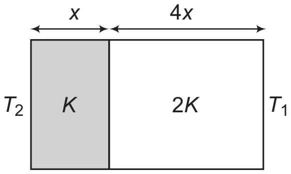


(a) $\dfrac{T}{\sqrt{2}-1}$
(b) $\dfrac{T}{\sqrt{2}+1}$
(c) $\dfrac{3T}{\sqrt{2}+1}$
(d) $\dfrac{T}{\sqrt{3}(\sqrt{2}-1)}$

6. A kettle with 2 litre water at 27°C is heated by operating coil heater of power 1 kW. The heat is lost to the atmosphere at constant rate 160 J/s, when its lid is open. In how much time will water heated to 77°C with the lid open? (specific heat of water = 4.2 kJ/° C-kg)

(a) 8 min 20 s
(b) 6 min 2 s
(c) 14 min
(d) 7 min

7. The temperature of the two outer surfaces of a composite slab consisting of two materials having coefficients of thermal conductivity K and $2K$ and thickness x and $4x$ respectively are T_2 and $T_1 (T_2 > T_1)$. The rate of heat transfer through the slab in steady state is $\dfrac{AK(T_2 - T_1)}{x} f$. where, f is equal to



(a) 1
(b) 1/2
(c) 2/3
(d) 1/3

More than One Correct Options

1. A solid sphere and a hollow sphere of the same material and of equal radii are heated to the same temperature

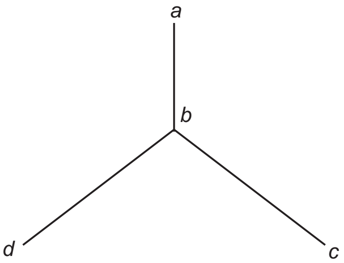
(a) both will emit equal amount of radiation per unit time in the beginning

(b) both will absorb equal amount of radiation per second from the surroundings in the beginning

(c) the initial rate of cooling will be the same for both the spheres

(d) the two spheres will have equal temperatures at any instant

2. Three identical conducting rods are connected as shown in figure. Given that $\theta_a = 40°\,\text{C}$, $\theta_c = 30°\,\text{C}$ and $\theta_d = 20°\,\text{C}$. Choose the correct options.



(a) temperature of junction *b* is 15°C

(b) temperature of junction *b* is 30°C

(c) heat will flow from *c* to *b*

(d) heat will flow from *b* to *d*

3. Two liquids of specific heat ratio 1 : 2 are at temperatures 2θ and θ

(a) if equal amounts of them are mixed, then temperature of mixture is $1.5\,\theta$

(b) if equal amounts of them are mixed, then temperature of mixture is $\frac{4}{3}\theta$

(c) for their equal amounts, the ratio of heat capacities is 1 : 1

(d) for their equal amounts, the ratio of their heat capacities is 1 : 2

4. Two conducting rods when connected between two points at constant but different temperatures separately, the rate of heat flow through them is q_1 and q_2

(a) When they are connected in series, the net rate of heat flow will be $q_1 + q_2$

(b) When they are connected in series, the net rate of heat flow is $\dfrac{q_1 q_2}{q_1 + q_2}$

(c) When they are connected in parallel, the net rate of heat flow is $q_1 + q_2$

(d) When they are connected in parallel, the net rate of heat flow is $\dfrac{q_1 q_2}{q_1 + q_2}$

5. Choose the correct options.

(a) Good absorbers of a particular wavelength are good emitters of same wavelength. This statement was given by Kirchhoff

(b) At low temperature of a body the rate of cooling is directly proportional to temperature of the body. This statement was given by the Newton

(c) Emissive power of a perfectly black body is 1

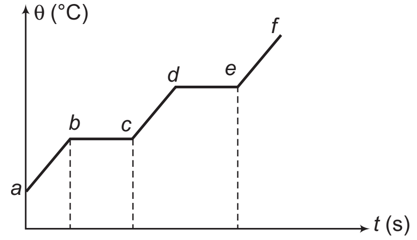
(d) Absorptive power of a perfectly black body is 1

Match the Columns

1. Match the following two columns.

Column I	Column II
(a) Stefan's constant	(p) $[L\theta]$
(b) Wien's constant	(q) $[ML^2T^{-3}\theta^{-2}]$
(c) Emissive power	(r) $[MT^{-3}]$
(d) Thermal resistance	(s) None of these

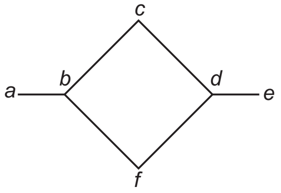
2. Heat is supplied to a substance in solid state at a constant rate. Its temperature varies with time as shown in figure. Match the following two columns.



Column I	Column II
(a) Slope of line *ab*	(p) *de*
(b) Length of line *bc*	(q) *cd*
(c) Solid + liquid state	(r) directly proportional to mass
(d) Only liquid state	(s) None of these

3. Six identical conducting rods are connected as shown in figure. In steady state temperature of point *a* is fixed at 100°C and temperature of *e* at – 80° C. Match the following two columns.

Column I	Column II
(a) Temperature of *b*	(p) 10°C
(b) Temperature of *c*	(q) 40°C
(c) Temperature of *f*	(r) – 20°C
(d) Temperature of *d*	(s) None of these



4. Three liquids A, B and C having same specific heats have masses m, $2m$ and $3m$. Their temperatures are, θ, 2θ and 3θ respectively. For temperature of mixture, match the following two columns.

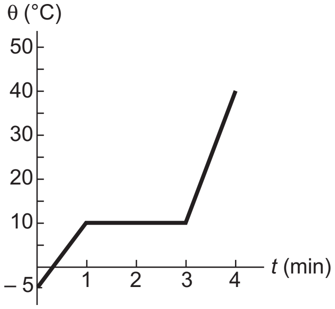
Column I	Column II
(a) When A and B are mixed	(p) $\frac{5}{2}\theta$
(b) When A and C are mixed	(q) $\frac{5}{3}\theta$
(c) When B and C are mixed	(r) $\frac{7}{3}\theta$
(d) When A, B and C all are mixed	(s) $\frac{13}{5}\theta$

5. Match the following two columns.

Column I	Column II
(a) Specific heat	(p) watt
(b) Heat capacity	(q) J/kg-°C
(c) Heat current	(r) J/s
(d) Latent heat	(s) None of these

Subjective Questions

1. As a physicist, you put heat into a 500 g solid sample at the rate of 10.0 kJ/ min, while recording its temperature as a function of time. You plot your data and obtain the graph shown in figure.



(a) What is the latent heat of fusion for this solid?
(b) What is the specific heat of solid state of the material?

2. A hot body placed in air is cooled according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surroundings. Starting from $t = 0$, find the time in which the body will lose half the maximum temperature it can lose.

3. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. The cross-sectional area of each rod is 4 cm^2. The end of copper rod is maintained at 100 °C and the ends of the brass and steel rods at 80 °C and 60 °C respectively. Assume that there is no loss of heat from the surfaces of the rods. The lengths of rods are : copper 46 cm, brass 13 cm and steel 12 cm.

(a) What is the temperature of the junction point?
(b) What is the heat current in the copper rod?
K (copper) = 0.92, K (steel) = 0.12 and K (brass) = 0.26 cal/ cm-s °C

4. Ice at 0°C is added to 200 g of water initially at 70°C in a vacuum flask. When 50 g of ice has been added and has all melted the temperature of the flask and contents is 40°C. When a further 80 g of ice has been added and has all melted the temperature of the whole becomes 10°C. Find the latent heat of fusion of ice.

5. A copper cube of mass 200 g slides down a rough inclined plane of inclination 37° at a constant speed. Assuming that the loss in mechanical energy goes into the copper block as thermal energy. Find the increase in temperature of the block as it slides down through 60 cm. Specific heat capacity of copper is equal to 420 J/ kg-K. (Take, $g = 10$ m/ s^2)

6. A cylindrical block of length 0.4 m and area of cross-section 0.04 m^2 is placed coaxially on a thin metal disc of mass 0.4 kg and of the same cross-section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 watt/m-K and the specific heat of the material of the disc is 600 J/kg-K, how long will it take for the temperature of the disc to increase to 350 K? Assume for purpose of calculation the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder.

7. A metallic cylindrical vessel whose inner and outer radii are r_1 and r_2 is filled with ice at 0° C. The mass of the ice in the cylinder is m. Circular portions of the cylinder is sealed with completely adiabatic walls. The vessel is kept in air. Temperature of the air is 50° C. How long will it take for the ice to melt completely. Thermal conductivity of the cylinder is K and its length is l. Latent heat of fusion is L.

8. An electric heater is placed inside a room of total wall area 137 m^2 to maintain the temperature inside at 20° C . The outside temperature is –10° C. The walls are made of three composite materials. The inner most layer is made of wood of thickness 2.5 cm the middle layer is of cement of thickness 1 cm and the exterior layer is of brick of thickness 2.5 cm. Find the power of electric heater assuming that there is no heat losses through the floor and ceiling . The thermal conductivities of wood, cement and brick are 0.125 W/m°-C, 1.5 W/m°-C and 1.0 W/m°-C respectively.

9. A 2 m long wire of resistance 4 Ω and diameter 0.64 mm is coated with plastic insulation of thickness 0.66 mm. A current of 5 A flows through the wire. Find the temperature difference across the insulation in the steady state. Thermal conductivity of plastic is 0.16×10^{-2} cal/s cm°-C.

10. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area A and fairly low conductivity K. The whole system is thermally insulated from the environment. At a moment $t = 0$, the temperature difference between two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time .

11. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as $k = a/T$, where a is a constant. The ends of the rod are kept at temperatures T_1 and T_2. Find the function $T(x)$, where x is the distance from the end whose temperature is T_1.

12. One end of a uniform brass rod 20 cm long and 10 cm^2 cross-sectional area is kept at 100° C. The other end is in perfect thermal contact with another rod of identical cross-section and length 10 cm. The free end of this rod is kept in melting ice and when the steady state has been reached, it is found that 360 g of ice melts per hour. Calculate the thermal conductivity of the rod, given that the thermal conductivity of brass is 0.25 cal/s cm° C and $L = 80$ cal/g.

13. Heat flows radially outward through a spherical shell of outside radius R_2 and inner radius R_1. The temperature of inner surface of shell is θ_1 and that of outer is θ_2. At what radial distance from centre of shell the temperature is just half way between θ_1 and θ_2?

14. A layer of ice of thickness y is on the surface of a lake. The air is at a constant temperature $-\theta°$ C and the ice water interface is at 0°C. Show that the rate at which the thickness increases is given by

$$\frac{dy}{dt} = \frac{K\theta}{L\rho y}$$

where, K is the thermal conductivity of the ice, L the latent heat of fusion and ρ is the density of the ice.

Answers

Introductory Exercise 22.1

1. 7200 cal **2.** 31.43 °C **3.** 80°C **4.** 12.96 m/s **5.** 12 g
6. 0°C, $m_w = 15$ g, $m_i = 10$ g **7.** 52°C

Introductory Exercise 22.2

1. For heat flow to take place, there must be a temperature difference along the rod.
3. KW^{-1} **4.** Conduction **5.** 145.5°C, 118.2°C **6.** 4 W **7.** 9 min

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (b) **4.** (c) **5.** (c) **6.** (d) **7.** (a) **8.** (d) **9.** (d) **10.** (a)

Objective Questions

1.(c) **2.**(b) **3.**(d) **4.**(b) **5.**(b) **6.**(d) **7.**(b) **8.**(a) **9.**(d) **10.**(c)
11.(c)

Subjective Questions

1. 900 W **2.** 500 m/s **3.** 2.5×10^{-3}°C **4.** 1.2×10^4 kg/s **5.** 3.7×10^4 W **6.** 105°C
7. (a) –8.1° C (b) 7.7 W/m^2 **8.** 87.5 g **9.** 0°C, mass of ice is 54 g and that of water is 286 g
10. 40°C **11.** $\dfrac{3T_1 + 2(T_2 + T_3)}{7}$ **12.** 424°C/m, 212°C/m **13.** 0.3

LEVEL 2

Single Correct Option

1.(c) **2.**(a) **3.**(b) **4.**(b) **5.**(c) **6.**(a) **7.**(d)

More than One Correct Options

1.(a,b) **2.**(b,d) **3.**(b,d) **4.**(b,c) **5.**(a,d)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → s **2.** (a) → s (b) → r (c) → s (d) → q
3. (a) → q (b) → p (c) → p (d) → r **4.** (a) → q (b) → p (c) → s (d) → r
5. (a) → q (b) → s (c) → p,r (d) → s

Subjective Questions

1. (a) 40 kJ/kg (b) 1.33 kJ/kg-° C **2.** $\ln(2)/k$ **3.** (a) 84°C (b) 1.28 cal/s **4.** 90 cal/g
5. 8.6×10^{-3}°C **6.** 166 s **7.** $t = mL\ln(r_2/r_1)/100\pi Kl$ **8.** 17647 W **9.** 2.23°C
10. $\Delta T = (\Delta T)_0 e^{-\alpha t}$, where $\alpha = \dfrac{KA(C_1 + C_2)}{lC_1C_2}$ **11.** $T = T_1\left(\dfrac{T_2}{T_1}\right)^{x/l}$ **12.** 0.222 cal/cm-s-°C **13.** $\dfrac{2R_1R_2}{R_1 + R_2}$

Hints & Solutions

17. Wave Motion

INTRODUCTORY EXERCISE 17.1

1. $$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a \sin \omega t$$

and $$\frac{\partial^2 y}{\partial x^2} = 0$$

Since, $$\frac{\partial^2 y}{\partial t^2} \neq \frac{\partial^2 y}{\partial x^2} \qquad \text{(constant)}$$

Hence, the given equation does not represent a wave equation.

2. Speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x} = \dfrac{c}{b}$

3. The converse is not true means if the function can be represented in the form $y = f(x \pm vt)$, it does not necessarily express a travelling wave. As the essential condition for a travelling wave is that the vibrating particle must have finite displacement value for all x and t.

4. (a) A has the dimensions of y.

$$\frac{x}{a} \text{ and } \frac{t}{T} \text{ are dimensionless.}$$

(b) $v = \dfrac{\text{coefficient of } t}{\text{coefficient of } x}$

(c) $\dfrac{x}{a}$ and $\dfrac{t}{T}$ are of same sign

(d) $y_{\max} = A$

INTRODUCTORY EXERCISE 17.2

1. Comparing the given equation with

$$y = \sin(kx - \omega t)$$

we find,

(a) Amplitude $A = 5$ mm

(b) Angular wave number $k = 1\ \text{cm}^{-1}$

(c) Wavelength, $\lambda = \dfrac{2\pi}{k} = 2\pi$ cm

(d) Frequency, $\nu = \dfrac{\omega}{2\pi} = \dfrac{60}{2\pi}$ Hz

$$= \frac{30}{\pi}\ \text{Hz}$$

(e) Time period, $T = \dfrac{1}{\nu} = \dfrac{\pi}{30}$ s

(f) Wave velocity, $v = \nu\lambda = 60\ \text{cm s}^{-1}$

2. (a) $T = \dfrac{2\pi}{\omega}$, $\lambda = \dfrac{2\pi}{k}$

(b) Find $\dfrac{\partial y}{\partial t}$ and then substitute the given values of x and t.

(c) and (d) Procedure is same.

INTRODUCTORY EXERCISE 17.3

1. (a) coefficient of x and t are of same sign.

(b) $v = \dfrac{314}{31.4} = 10\ \text{m/s} \Rightarrow \lambda = \dfrac{2\pi}{k},\ f = \dfrac{\omega}{2\pi}$

(c) $v_{\max} = \omega A$

2. (a) The velocity of the particle at x at time t is

$$v = \frac{\partial y}{\partial t} = (3.0\ \text{cm})(-314\ \text{s}^{-1}) \cos[(3.14\ \text{cm}^{-1}) x - (314\ \text{s}^{-1})t]$$

$$= (-9.4\ \text{ms}^{-1}) \cos[(3.14\text{cm}^{-1})x - (314\ \text{s}^{-1})t]$$

The maximum velocity of a particle will be

$$v = 9.4\ \text{ms}^{-1}$$

(b) The acceleration of the particle at x at time t is

$$a = \frac{\partial v}{\partial t} = -(9.4\ \text{ms}^{-1})(314\ \text{s}^{-1}) \sin[(3.14\ \text{cm}^{-1}) x - (314\ \text{s}^{-1})t]$$

$$= -(2952\ \text{ms}^{-1}) \sin[(3.14\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t].$$

The acceleration of the particle at $x = 6.0$ cm at time $t = 0.11$ s is

$$a = -(2952\ \text{ms}^{-2}) \sin[6\pi - 11\pi] = 0$$

3. (a) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{1/0.01}{1/0.05} = 5\ \text{m/s}$$

Since, coefficient of t and coefficient of x are of same sign. Hence, wave is travelling in negative x-direction.

or $v = -5$ m/s

(b) Velocity of particle,

$$v_P = \frac{\partial y}{\partial t} = 2\cos\left(\frac{x}{0.05} + \frac{t}{0.01}\right)$$

Substituting $x = 0.2$ and $t = 0.3$, we have

$$v_P = 2\cos(34) = 2(-0.85)$$

$$= -1.7\ \text{m/s}$$

4. (a)
$$\Delta\phi = \frac{2\pi}{\lambda}\,\Delta x$$

$$\therefore \quad \Delta x = \frac{\lambda\Delta\phi}{2\pi} = \frac{v\,(\Delta\phi)}{2\pi f}$$

$$= \frac{(350)\,(\pi/3)}{(2\pi)\,(500)}$$

$$= 0.116 \text{ m}$$

(b)
$$\Delta\phi = (\omega)\Delta t$$

$$= (2\pi f)\,(\Delta t)$$

$$= (2\pi)\,(500)(10^{-3})$$

$$= \pi \text{ or } 180^\circ$$

INTRODUCTORY EXERCISE 17.4

1. $f_{\max} = \dfrac{c}{\lambda_{\min}}$

$f_{\min} = \dfrac{c}{\lambda_{\max}}$

2. $\lambda = \dfrac{v}{f}$ for both

INTRODUCTORY EXERCISE 17.5

1. The tension in the string is $F = mg = 10$ N. The mass per unit length is $\mu = 1.0 \text{ g cm}^{-1} = 0.1 \text{ kg m}^{-1}$. The wave velocity is, therefore, $v = \sqrt{F\mu} = \sqrt{\dfrac{10 \text{ N}}{0.1 \text{ kg m}^{-1}}} = 10 \text{ ms}^{-1}$. The time taken by the pulse in travelling through 50 cm is, therefore 0.05 s.

2. $v = \sqrt{\dfrac{T}{\mu}}$

3. $T_{CD} = 3.2 \text{ g}, T_{AB} = 6.4 \text{ g}$

and $\qquad v = \sqrt{\dfrac{T}{\mu}}$

4. $v = \sqrt{\dfrac{T}{\mu}}, \quad \mu = \dfrac{m}{l} \quad$ and $\quad t = \dfrac{d}{v}$

5.
$$v = \sqrt{\frac{T}{\rho S}} = \sqrt{\frac{T}{\rho\,(\pi d^2/A)}}$$

$$= \sqrt{\frac{4T}{\rho\pi d^2}} \qquad (T = mg)$$

$$= \sqrt{\frac{4 \times 20}{(8920)\,(\pi)\,(2.4 \times 10^{-3})^2}}$$

$$= 22 \text{ m/s}$$

6. (a)
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg}{\mu}} \qquad \ldots\text{(i)}$$

$$= \sqrt{\frac{1.5 \times 9.8}{0.055}} = 16.3 \text{ m/s}$$

(b) $\lambda = \dfrac{v}{f} = \dfrac{16.3}{120} = 0.136$ m

(c) From Eq. (i), we can see that $v \propto \sqrt{m}$. If m is doubled, then v will become $\sqrt{2}$ times. Hence, from the relation $\lambda = \dfrac{v}{f} \cdot \lambda$ will also become $\sqrt{2}$ times. As f remains unchanged.

INTRODUCTORY EXERCISE 17.6

1. $I = \dfrac{P}{4\pi r^2} = \dfrac{1.0}{4\pi\,(1)^2} = \dfrac{1}{4\pi} \text{ W/m}^2$

2. Suppose power of line source is P. Then, at distance r, surface area is $2\pi rl$.

$$\therefore \quad I = \frac{P}{S} = \frac{P}{2\pi rl}$$

or
$$I \propto \frac{1}{r}$$

Further,
$$I \propto A^2$$

$$\therefore \quad A \propto \frac{1}{\sqrt{r}}$$

3. Energy density, $u = \dfrac{1}{2}\rho\omega^2 A^2$

Total energy $= (u)\,(V)$
$$= \frac{1}{2}(\rho V)\,(2\pi f)^2 A^2$$

$$= 2\pi^2\, m f^2 A^2$$

$$= (2)(\pi)^2(0.08)(120)^2(0.16 \times 10^{-3})^2$$

$$= 581 \times 10^{-3} \text{J}$$

$$= 0.58 \text{ mJ}$$

4. Power $= \dfrac{1}{2}\rho\omega^2 A^2 S v$

But, $\qquad \rho S = \mu$

and $\qquad v = \sqrt{\dfrac{T}{\mu}}$

$$\therefore \quad \rho S v = \sqrt{\mu T}$$

Hence, Power $= \dfrac{1}{2}\,(2\pi f)^2 A^2 \sqrt{T\mu}$
$$= 2\pi^2 f^2\, A^2 \sqrt{T\mu}$$

$$= (2)\,(\pi)^2\,(60)^2\,(0.06)^2 \sqrt{80 \times 5 \times 10^{-2}}$$

$$= 512 \text{W}$$

5. (a) As derived in above problem, power

$$= 2\pi^2 f^2 A^2 \sqrt{T\mu}$$

$$= (2)\,(\pi)^2\,(200)^2\,(10^{-3})^2\,\sqrt{60 \times (0.006)}$$

$$= 0.47 \text{ W}$$

(b) Energy density, $u = \frac{1}{2}\rho\omega^2 A^2$

$$\text{Total energy} = (u)\,(V)$$

$$= \left(\frac{1}{2}\rho\omega^2 A^2\right)(lS)$$

But, $\rho S = \mu$

$$\therefore \text{ Total energy} = \frac{1}{2}\mu l\,(2\pi f)^2\,A^2$$

$$= 2\pi^2 f^2 A^2 \mu l$$

$$= (2)\,(\pi)^2\,(200)^2 (10^{-3})^2\,(0.006)\,(2.0)$$

$$= 9.4 \times 10^{-3} \text{J} = 9.4 \text{ mJ}$$

6. $P = \frac{1}{2}\rho\omega^2 A^2 Sv$ ...(i)

But, $\rho S = \mu$

and $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho S}}$

$\therefore$ $\rho S = \frac{T}{v^2}$ or $\rho S v = \frac{T}{v}$

Substituting in Eq. (i), we have

$$P = \frac{1}{2}\,(2\pi f)^2 A^2 \frac{T}{v}$$

$$= \frac{2\pi^2 f^2 A^2 T}{v}$$

$$= \frac{(2)\,(\pi)^2\,(100)^2\,(0.0005)^2\,(100)}{100}$$

$$= 0.049 \text{ J} = 49 \text{ mJ}$$

Exercise

LEVEL 1

Assertion and Reason

3. If both waves ωt and kx are of opposite signs, then both are travelling in positive directions.

4. If f is doubled (which is source dependant), then λ will automatically become half, so that speed remains same. Because speed is only medium dependant.

5. Longitudinal or sound wave cannot travel in vacuum.

7. Electromagnetic wave which can travel with or without medium.

8. Just by observation we cannot say that $\mu_2 > \mu_1$ and hence $v_2 > v_1$.

If their densities are different, then μ_2 may be less than μ_1 also.

9. At mean position kinetic energy is maximum and there is a maximum stretch in string. Because one side particles are moving up and the other side particles are moving down. Hence, potential energy is also maximum.

10. In $v_P = -v\,\frac{\partial y}{\partial x}$,

v_P is negative and $\frac{\partial y}{\partial x}$ (the slope) is also negative.

Hence, the velocity v should also be negative.

Objective Questions

1. $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/9} = 18 \text{ m}$

2. $\Delta\phi = \frac{2\pi}{\lambda}\,\Delta x = k\Delta x$

$$= (10\pi \times 0.01)\,(10) = \pi$$

3. At $t = 0$, $y_{\max} = \frac{1}{2}$ at $x = 0$

At $t = 2$ s, $y_{\max} = \frac{1}{2}$ at $x = 2$ m

$\therefore$ In 2 s, $y_{\max}$ has travelled 2 m in positive direction

$$\therefore \quad v = +\frac{2}{3} = +1 \text{ m/s}$$

4. Wave velocity and particle velocity are two different things.

5. $\Delta\phi = \frac{2\pi}{\lambda}\,(\Delta x) = \frac{2\pi\,\phi(\Delta x)}{v}$ $\left(\text{as } \lambda = \frac{v}{f}\right)$

$$= \frac{2\pi\,(25)\,(16-10)}{300} = \pi$$

6. $v = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}}$

$$\therefore \quad T = \mu\left(\frac{\omega}{k}\right)^2 = (1.3 \times 10^{-4})\left(\frac{30}{1}\right)^2$$

$$= 0.12 \text{ N}$$

7. $\because \quad \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{(t/N)} = \dfrac{2\pi N}{t}$

$$= \frac{(2\pi)\,(150)}{60}$$

$$= (5\pi)\ \text{rad/s}$$

Subjective Questions

1. (b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{2\pi/28}$

$$= 28\ \text{cm}$$

(c) $f = \dfrac{\omega}{2\pi} = \dfrac{2\pi/0.036}{2\pi}$

$$= 27.8\ \text{Hz}$$

(d) $v = f\lambda = 778.4\ \text{cm/s}$

$$= 7.8\ \text{m/s}$$

Since, ωt and kx have opposite signs. Hence, wave is travelling in positive direction.

2. (a) Put $t = 0$, $x = 2\,\text{cm}$

(b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{(30\pi/240)} = 16\ \text{cm}$

(c) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{1}{1/240}$$

$$= 240\ \text{cm/s}$$

(d) $f = \dfrac{\omega}{2\pi} = \dfrac{30\pi}{2\pi} = 15\ \text{Hz}$

3. At $t = 0$, y is maximum at $x = 0$.

At $t = 2\,\text{s}$, y is maximum at $x = 1\,\text{m}$. Hence, in 2 s wave has travelled 2 m in positive x-direction

$\therefore \qquad v = +\dfrac{1}{2}\ \text{m/s}$

$$= +\ 0.5\ \text{m/s}$$

4. Since, coefficient of t and x are opposite signs, then wave is travelling along negative x-direction.

Further, speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{2}{1} = 2\ \text{m/s}$$

Amplitude = maximum value of y

$$= \frac{10}{5} = 2\ \text{m}$$

5. Wave speed, $\quad v = \dfrac{\omega}{k}$

and maximum particle speed,

$$(v_P)_{\max} = \omega A$$

From these two expressions, we can see that

$$(v_P)_{\max} = (kA)\,V$$

6. Mass per unit length, $\mu = \dfrac{m}{l}$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Tl}{m}} = \sqrt{\frac{500 \times 2}{0.06}}$$

$$= 129.1\ \text{m/s}$$

7. $\quad v = \sqrt{\dfrac{T}{\rho S}}$ as mass per unit length $= \rho S$

$\therefore \qquad v = \sqrt{\dfrac{0.98}{9.8 \times 10^{3} \times 10^{-6}}}$

$$= 10\ \text{m/s}$$

8. Since, wave is travelling along positive x-direction. Hence, coefficient of t and coefficient of x should have opposite signs. Further,

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$\therefore \qquad 2 = \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$\therefore$ Coefficient of $t = 2$ (coefficient of x)

$$= 2 \times 1 = 2\ \text{SI units}$$

$\therefore \qquad y = \dfrac{10}{(x - 2t)^2 + 2}$

9. (a) $\dfrac{\lambda}{2} = 2\ \text{cm}$

$\Rightarrow \qquad \lambda = 4\ \text{cm}$

$$f = \frac{v}{\lambda} = \frac{40}{4} = 10\ \text{Hz}$$

(b) $\Delta\phi = \dfrac{2\pi}{\lambda}\,\Delta x = \left(\dfrac{2\pi}{4}\right)(2.5) = \dfrac{5\pi}{4}$

(c) $\omega t = \theta$

or $\quad (2\pi/t) = \theta$

$\therefore \qquad t = \dfrac{\theta}{2\pi f} = \dfrac{\pi/3}{(2\pi)\,(10)}$

$$= \frac{1}{60}\ \text{s}$$

(d) At P, particle is at mean position. So,

$$v = \text{maximum velocity}$$

$$= \omega A$$

$$= 2\pi f A$$

$$= (2\pi)(10)(2)$$

$$= (40\pi)\ \text{cm/s}$$

$$= 125.7\ \text{cm/s} = 1.26\ \text{m/s}$$

Further, $$v_P = -(v)\left(\frac{\partial y}{\partial x}\right) \qquad \ldots\text{(i)}$$

Sign of v_1, the wave velocity is given positive.

Sign of $\frac{\partial y}{\partial x}$, slope of y - x graph is also positive.

Hence, from Eq. (i) particle velocity is negative.

$$\therefore \qquad v_P = -1.26\ \text{m/s}$$

10. (a) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{(1/0.01)}{(1/0.05)} = 5\ \text{m/s}$$

Since, coefficient of t and coefficient of x are of same sign. Hence, wave is travelling in negative x-direction.

or $$v = -5\ \text{m/s}$$

(b) Particle velocity, $v_P = \dfrac{\partial y}{\partial t}$

$$= 2\cos\left(\frac{x}{0.05} + \frac{t}{0.01}\right)$$

Substituting $x = 0.2$ and $t = 0.3$, we have

$$v_P = 2\cos 34$$
$$= (2)(-0.85)$$
$$= -1.7\ \text{m/s}$$

11. (a) $\omega = 2\pi f = 2\pi\left(\dfrac{v}{\lambda}\right) = 2\pi\left(\dfrac{12}{0.4}\right)$

$$= (60\,\pi)\ \text{rad/s}$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = (5\pi)\ \text{m}^{-1}$$

Since, wave is travelling along + ve x-direction, ωt and kx should have opposite sign. Further at $t = 0$, $x = 0$ the string has zero displacement and moving upward (in positive direction). Hence at $x = 0$, we should have $A\sin\omega t$ not $-A\sin\omega t$. Therefore, the correct expression is

$$y = A\sin(\omega t - kx)$$

or $$y = 0.05\sin(60\pi t - 5\pi x)$$

(b) Putting $x = 0.2$ m and $t = 0.15$ s in above equation we have,

$$y = -0.035\,4\ \text{m}$$
$$= -3.54\ \text{cm}$$

(c) In part (b), $y = A/\sqrt{2}$

From $y = \dfrac{A}{\sqrt{2}}$ to $y = 0$, time taken is

$$t = \frac{T}{8} = \frac{2\pi}{\omega\times 8}$$
$$= \frac{2\pi}{(60\pi)(8)}$$
$$= 4.2\times 10^{-3}\ \text{s}$$
$$= 4.2\ \text{ms}$$

12. (a) $\because \quad T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi/0.01} = 0.02$ s

$$= 20\ \text{ms}$$
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/2}$$
$$= 4.0\ \text{cm}$$

(b) $v_P = \dfrac{\partial y}{\partial x}$

$$= \left(-\frac{\pi}{0.01}\,\frac{\text{mm}}{\text{s}}\right)\cos\pi\left(\frac{x}{2.0\ \text{cm}} - \frac{t}{0.01\ \text{s}}\right)$$
$$= \left(-\frac{\pi}{10}\,\frac{\text{m}}{\text{s}}\right)\cos\pi\left(\frac{x}{2.0\ \text{cm}} - \frac{t}{0.01\ \text{s}}\right)$$

Put $x = 1$ cm and $t = 0.01$ s in above equation We get,

$$v_P = 0$$

(c) and (d) Putting the given values in the above equation, we get the answers.

13. (a) $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{40} = 0.157$ rad/cm

$$T = \frac{1}{f} = \frac{1}{8} = 0.125\ \text{s}$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.215} = 50.3\ \text{rad/s}$$
$$v = \frac{\omega}{k} = 320\ \text{cm/s}$$

(b) Amplitude is 15 cm.

At $t = 0$, $x = 0$, $y = +A$. Hence, equation should be a cos equation. Further, wave is travelling in positive x-direction. Hence, ωt and kx should have opposite signs.

14. (a) $T_x = (m_{L-x})\,g = \mu\,(L - x)\,g$

where, μ = mass per unit length

$$\therefore \qquad v_x = \sqrt{\frac{T_x}{\mu}} = \sqrt{g\,(L - x)}$$

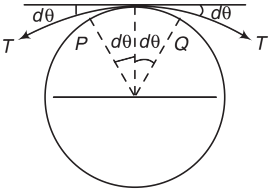
(b) $$v_x = -\frac{dx}{dt} = \sqrt{g\,(L - x)}$$

$$\therefore \quad \int_0^t dt = -\int_{x=L}^{x=0} \sqrt{g\,(L - x)}\,dx$$

Solving we get,

$$t = 2\sqrt{\frac{L}{g}}$$

15. (a) $T\cos d\theta$ components are cancelled. $T\sin d\theta$ components provide the necessary centripetal force to $P\theta$.



$$\therefore \qquad 2T\sin d\theta = m_{PQ}R\omega^2$$

For small angles $\sin d\theta \approx d\theta$

$$\therefore \qquad 2T\,d\theta = [\mu\,(2R)\,d\theta]\,R\omega^2$$

Solving the equation, we get

$$\sqrt{\frac{T}{\mu}} = v = R\omega$$

16. $$M = \int_0^L kx\,dx = \frac{kL^2}{2}$$

$$\therefore \qquad k = \frac{2M}{L^2}$$

$$v_x = \sqrt{\frac{T}{\mu_x}} = \sqrt{\frac{T}{kx}} = \sqrt{\frac{T}{(2M/L^2)\,x}}$$

$$= \left(L\sqrt{\frac{T}{2M}}\right)x^{-1/2} = \frac{dx}{dt}$$

$$\therefore \qquad \int_0^t dt = \frac{1}{L}\sqrt{\frac{2M}{T}}\int_0^L x^{1/2}\,dx$$

$$\text{or} \qquad t = \left(\frac{2}{3L}\sqrt{\frac{2M}{T}}\right)L^{3/2}$$

$$= \frac{2}{3}\sqrt{\frac{2ML}{T}}$$

LEVEL 2

Single Correct Option

1. $\because \quad v = \dfrac{\omega}{k}$

$$\therefore \qquad k = \frac{\omega}{v} = \frac{600\pi}{300}$$

$$= 2\pi\ \text{m}^{-1}$$

$$\therefore \qquad y = 0.04\sin(600\,\pi t - 2\pi x)$$

Now, put $t = 0.01$ s and $x = 0.75$ m

2. $\because \quad \omega = 2\pi f = (2\pi)(100)$

$$= (200\,\pi)\ \text{rad/s}$$

$$v = \frac{\omega}{k}$$

$$\therefore \qquad k = \frac{\omega}{v} = \omega\sqrt{\frac{\mu}{T}}$$

$$= (200\pi)\sqrt{\frac{3.5\times 10^{-3}}{35}}$$

$$= 2\pi\ \text{m}^{-1}$$

At zero displacement,

$$v_P = \omega A = -v\frac{\partial y}{\partial x}$$

$$\therefore \qquad |A| = \frac{v\left(\dfrac{\partial y}{\partial x}\right)}{\omega} = \frac{\left(\sqrt{\dfrac{T}{\mu}}\right)(\text{slope})}{\omega}$$

$$= \frac{\left(\sqrt{\dfrac{35}{3.5\times 10^{-3}}}\right)(\pi/20)}{200\pi}$$

$$= 0.025\ \text{m}$$

3. $\because \quad \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.25}$

$$= 8\pi\ \text{rad/s}$$

$$v = \frac{\omega}{k}$$

$$\therefore \qquad k = \frac{\omega}{v} = \frac{8\pi}{0.48} = \left(\frac{50}{3}\right)\pi\ \text{m}^{-1}$$

$$y = A\sin(\omega t - kx)$$

$$= A\sin\left(8\pi t - \frac{50}{3}\pi x\right)$$

Put $y = 3$ cm, $t = 1$ s, $x = 0.47$ m

showing we get $A = 6$ cm

4. $v = \sqrt{\dfrac{T}{\rho S}} = \sqrt{\dfrac{T}{\rho\,(\pi d^2/4)}}$

$$\therefore \qquad v \propto \sqrt{\frac{T}{d^2}}$$

5. $E \propto \omega^2 A^2$

$$\text{or} \qquad E \propto f^2 A^2$$

E is same

$$\therefore \qquad fA = \text{constant}$$

$$\text{or} \qquad A \propto \frac{1}{f}$$

6. The wave pulse is travelling along positive x-axis. Hence, at and bx should have opposite signs. Further, wave speed

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$$\therefore \qquad 4 = \frac{\text{Coefficient of } t}{1}$$

$\therefore$ Coefficient of $t = 4\text{ s}^{-1}$

More than One Correct Options

1. $\because$
$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$
$$= \frac{6.2}{2} = 3.1 \text{ m/s}$$
$$A = \frac{2}{20} = 0.1 \text{ m}$$

3.
$$y = A \sin(\pi x + \pi t)$$
$$v_P = \frac{\partial y}{\partial t} = \pi A \cos(\pi x + \pi t)$$
$$a_P = \frac{\partial^2 y}{\partial t^2} = -\pi^2 A \sin(\pi x + \pi t)$$

Now, substitute $t = 0$ and given value of x.

Since, ωt and kx are of same sign, hence the wave is travelling in negative x-direction.

4. Speed of wave
$$= \frac{\text{Coefficient of } t}{\text{Coefficient of } x} = \frac{b}{1} = b$$
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(2\pi/a)} = a$$

5. Speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$
$$= \frac{1/b}{1/a} = \frac{a}{b}$$
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{2\pi/a} = a$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi/b} = b$$

6. y - t graph is sine graph. Therefore, v - t graph is cos graph and a - t is − sine graph as
$$v_P = \frac{\partial y}{\partial t} \quad \text{and} \quad a_P = \frac{\partial^2 y}{\partial t^2}$$

Match the Columns

1. (a) Wave speed $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x} = \dfrac{b}{c}$

(b) Maximum particle speed $= \omega A = (b)(a)$

(c) $f = \dfrac{\omega}{2\pi} = \dfrac{b}{2\pi}$

(d) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{c}$

2. $v_P = \dfrac{\partial y}{\partial t} = (4\pi) \text{ cm/s} \cos[\pi t + 2\pi x]$

$$a_P = \frac{\partial^2 y}{\partial t^2} = (-4\pi^2) \text{ cm/s}^2 \sin[\pi t + 2\pi x]$$

Now substitute the values of t and x.

3. For velocity, $v_P = -v \dfrac{\partial y}{\partial x}$

$\dfrac{\partial y}{\partial x}$ = Slope of y-x graph.

Sign of v is not given in the question.

Hence, direction of v_P cannot be determined.

For particle acceleration,
$$a_P \propto -y$$
i.e. a_P and y are away in opposite directions

If $y = 0$, then $a_P = 0$

4. Energy density $= \dfrac{1}{2}\rho\omega^2 A^2$

= Energy per unit volume

Power = energy transfer per unit

Time $= \dfrac{1}{2}\rho\omega^2 A^2 S v$

Intensity = energy transfer per unit per unit area

$= \dfrac{1}{2}\rho\omega^2 A^2 v = \dfrac{P}{S}$

Wave number $= \dfrac{1}{\lambda}$

5. $v_P = \dfrac{\partial y}{\partial t}$, $a_P = \dfrac{\partial^2 y}{\partial t^2}$

If ωt and kx are of same sign, then wave travel in negative x-direction.

If they are of opposite signs, then wave travels in positive direction.

Subjective Questions

1. (a) $v_P = -v\left(\dfrac{dy}{dx}\right)$

As v_P and $(\text{slope})_P$ are both positive, v must be negative. Hence, the wave is moving in negative x-axis.

(b) $y = A \sin(\omega t - kx + \phi)$ ...(i)

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{2} \text{ cm}^{-1}$$
$$A = 4 \times 10^{-3} \text{ m} = 0.4 \text{ cm}$$

At $t = 0,\ x = 0,$ slope $\frac{dy}{dx} = +$ ve

$\therefore\quad v_P = -v(\text{slope}) = +$ ve

Further at $t = 0, x = 0,\ y = +$ ve

$$\therefore\quad \phi = \frac{\pi}{4}$$

Further, $20\sqrt{3} = -v \tan 60°$

$$\therefore\quad v = -20 \text{ cm/s}$$

$$f = \frac{v}{\lambda} = 5 \text{ Hz}$$

$$\therefore\quad \omega = 2\pi f = 10\pi$$

$$\therefore\quad y = (0.4 \text{ cm}) \sin\left(10\pi t + \frac{\pi}{2}x + \frac{\pi}{4}\right)$$ **Ans.**

(c) $P = 2\pi^2 A^2 f^2 \mu v$

$\therefore$ Energy carried per cycle

$$E = PT = \frac{P}{f} = 2\pi^2 A^2 f \mu v$$

Substituting the values, we have

$$E = 1.6 \times 10^{-5} \text{ J}$$ **Ans.**

2. (a) $v = \sqrt{\frac{T}{\rho S}}$

$$= \sqrt{\frac{64}{12.5 \times 10^3 \times 0.8 \times 10^{-6}}}$$

$= 80$ m/s **Ans.**

(b) $\omega = 2\pi f = 2\pi(20) = 40\pi$ rad/s

$$k = \frac{\omega}{v} = \frac{40\pi}{80} = \frac{\pi}{2} \text{ m}^{-1}$$

$$\therefore\quad y = (1.0 \text{ cm}) \cos\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}m^{-1}\right)x\right]$$

Ans.

(c) Substituting $x = 0.5$ m and $t = 0.05$ s, we get

$$y = \frac{1}{\sqrt{2}} \text{ cm}$$ **Ans.**

(d) Particle velocity at time t.

$$v_P = \frac{\partial y}{\partial t}$$

$$= -(40\pi \text{ cm/s}) \sin\left[(40\pi \text{ s}^{-1})t - \left(\frac{\pi}{2} \text{ m}^{-1}\right)x\right]$$

Substituting $x = 0.5$ m and $t = 0.05$ s, we get

$$v_P = 89 \text{ cm/s}$$ **Ans.**

3. $k_{\text{eff}} = 2k = 1.0$ N/m

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{10}{2\pi} \text{ s}^{-1}$$

$$v = 0.1 \text{ m/s},\quad A = 0.02 \text{ m}$$

$$\lambda = \frac{v}{f} = \frac{0.1}{\left(\frac{10}{2\pi}\right)} = \frac{2\pi}{100} \text{ m}$$

$$y = A \cos\left(\frac{2\pi}{\lambda}\right)(vt - x)$$

$$= 0.02 \cos 100(0.1t - x)$$

$$= 0.02 \cos (10t - 100x) \text{ m}$$ **Ans.**

The distance between two successive maxima

$$= \lambda = \frac{2\pi}{100} = 0.0628 \text{ m}$$ **Ans.**

4. (a) Dimensions of A and Y are same.

Similarly, dimensions of a and x are same.

(b) As the wave is travelling towards positive x-axis, there should be negative sign between term of x and term of t.

Further, speed of wave

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$\therefore$ Coefficient of $t = (v) \times$ coefficient of x

5. From the given figure, we can see that

(a) Amplitude, $A = 1.0$ mm

(b) Wavelength, $\lambda = 4$ cm

(c) Wave number, $k = \frac{2\pi}{\lambda} = 1.57 \text{ cm}^{-1} \approx 1.6 \text{ cm}^{-1}$

(d) Frequency, $f = \frac{v}{\lambda} = \frac{20}{4} = 5$ Hz

6. $v = \sqrt{\frac{T}{\rho S}}$ or $v \propto \frac{1}{\sqrt{\rho}}$

$$\therefore\quad \frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

$$\therefore\quad \frac{\rho_1}{\rho_2} = \left(\frac{v_2}{v_1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

7. $$v_1 = \sqrt{\frac{T_1}{\mu_1}}$$

$$= \sqrt{\frac{4.8}{1.2 \times 10^{-2}}} = 20 \text{ m/s}$$

$$v_2 = \sqrt{\frac{T_2}{\mu_2}} = \sqrt{\frac{7.5}{1.2 \times 10^{-2}}}$$

$$= 25 \text{ m/s}$$

Pulses will meet when $x_A = x_B$

or $\quad 20t = 25(t - 0.02)$

$\therefore\quad t = 0.1$ s

and $\quad x_A$ or $x_B = 20 \times 0.1 = 2$ m

8. (a) $P = \frac{1}{2}\rho\omega^2 A^2 S v$

or $$A = \frac{1}{\omega}\sqrt{\frac{2P}{\rho S v}} \qquad \ldots\text{(i)}$$

Here, $\rho S = \mu$ = mass per unit length

$$= \frac{6 \times 10^{-3}}{8} \text{ kg/m}$$

$$\omega = 2\pi f = \frac{2\pi v}{\lambda}$$

$$= \frac{2\pi \times 30}{0.2}$$

Substituting these values in Eq. (i), we have

$$A = \frac{0.2}{2\pi \times 30}\sqrt{\frac{2 \times 50 \times 8}{6 \times 10^{-3} \times 30}}$$

$$= 0.0707 \text{ m}$$

$$= 7.07 \text{ cm}$$

(b) $P \propto v\omega^2$

or $$P \propto v(v^2)$$

or $$P \propto v^3$$

When wave speed is doubled, then power will become eight times.

9. $-\,dT = (dm)x\omega^2 = \left(\frac{m}{L}dx\right)x\omega^2$

or $$-\int_0^T dT = \frac{m}{L}\omega^2 \int_{x=L}^{x} x\,dx$$

∴ $$-T = \frac{m}{L}\omega^2\left(\frac{x^2}{2} - \frac{L^2}{2}\right)$$

or $$T = \frac{m\omega^2}{2L}(L^2 - x^2)$$

$$v = \sqrt{\frac{T}{\mu}}$$

$$= \sqrt{\frac{\frac{m\omega^2}{2L}(L^2 - x^2)}{m/L}}$$

$$= \omega\sqrt{\frac{L^2 - x^2}{2}}$$

or $$\frac{dx}{dt} = \frac{\omega}{\sqrt{2}}\sqrt{L^2 - x^2}$$

∴ $$\int_0^t dt = \frac{\sqrt{2}}{\omega}\int_0^L \frac{dx}{\sqrt{L^2 - x^2}}$$

∴ $$t = \frac{\sqrt{2}}{\omega}\left[\sin^{-1}\left(\frac{x}{L}\right)\right]_0^L$$

$$= \frac{\sqrt{2}}{\omega}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}\omega}$$ **Ans.**

18. Superposition of Waves

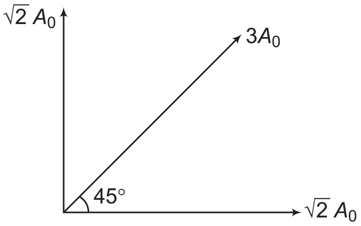
INTRODUCTORY EXERCISE 18.1

1. $\dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{\sqrt{I_1/I_2} + 1}{\sqrt{I_1/I_2} - 1}\right)^2$

2. (a) $\dfrac{A_{\max}}{A_{\min}} = \dfrac{A_1 + A_2}{A_1 - A_2}$

(b) $\dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{A_{\max}}{A_{\min}}\right)^2$

3. Path difference of $\dfrac{\lambda}{4}$ is equivalent to a phase difference of $\dfrac{\pi}{2}$.



$$A_{\text{net}} = 5A_0$$
$$I \propto A^2$$
$$\Rightarrow \qquad I_{\text{net}} = 25\,I_0$$

INTRODUCTORY EXERCISE 18.2

1. It is either 0 or π.

2. (a) $$v = \frac{\omega}{k} = \frac{40\pi}{\pi/3}$$
$$= 120 \text{ cm/s}$$

(b) Distance between adjacent nodes
$$= \frac{\lambda}{2} = \frac{\pi}{k} = \frac{\pi}{(\pi/3)}$$
$$= 3 \text{ cm}$$

(c) $v_P = \dfrac{\partial y}{\partial t} = (-\,200\,\pi)\sin\dfrac{\pi x}{3}\sin 40\pi t$

Now, substitute the given values of *x* and *t*.

3. In this case, node points will also oscillate, but standing waves are formed.

4. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{0.4} = 15.7 \text{ m}$
$$f = \frac{\omega}{2\pi} = \frac{200}{2\pi} = 31.8 \text{ Hz}$$
$$v = f\lambda = 500 \text{ m/s}$$

INTRODUCTORY EXERCISE 18.3

1. Wall will be a node (displacement). Therefore, shortest distance from the wall at which air particles have maximum amplitude of vibration (displacement antinode) should be $\lambda/4$.

Here, $\qquad \lambda = \dfrac{v}{f} = \dfrac{330}{660} = 0.5 \text{ m}$

$\therefore$ Desired distance is $\dfrac{0.5}{4} = 0.125$ m.

2. $5f - 2f = 54$
$$3f = 54$$
$$\therefore \qquad f = 18 \text{ Hz}$$

3. $$f \propto \sqrt{T}$$
$$\therefore \qquad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$
$$\text{or} \qquad \frac{220}{260} = \sqrt{\frac{2.2\text{g}}{(2.2 + M)\,\text{g}}}$$

Solving we get,
$$M = 0.873 \text{ kg}$$

4. (a) $\dfrac{f_1}{f_2} = \dfrac{250}{300} = \dfrac{5}{6}$

So, f_1 is 5th harmonic and f_2 is 6th harmonic.

(b) $f_1 = (5) = \dfrac{v}{2l} = \dfrac{2.5\sqrt{T/\mu}}{l}$
$$\therefore \qquad T = \frac{\mu\, f_1^2\, l^2}{(2.5)^2} = \frac{(0.036)\,(250)^2\,(1)^2}{(2.5)^2}$$
$$= 360 \text{ N}$$

5. $6\left[\dfrac{T/\mu}{2l_1}\right] = 2\left[\dfrac{T/\mu}{2l_2}\right]$
$$\therefore \qquad \frac{l_1}{l_2} = 3$$

INTRODUCTORY EXERCISE 18.4

1. At mean position, energy is in the form of kinetic energy.

2. (a) Assuming tension to be same
$$v \propto \frac{1}{\sqrt{\mu}}$$
$$\mu_{\text{RHS}} = \frac{\mu_{\text{RHS}}}{4}$$
$$\therefore \qquad v_{\text{RHS}} = 2v_{\text{LHS}} = 20 \text{ cm/s}$$

(b) $A_r = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right) A_i$

$$\therefore \quad \frac{A_r}{A_i} = \frac{20 - 10}{20 + 10} = \frac{1}{3}$$

$$\frac{A_t}{A_i} = \left(\frac{2v_2}{v_1 + v_2}\right) = \frac{2 \times 20}{20 + 10} = \frac{4}{5}$$

3. Speed of wave in first medium is,

$$v_1 = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$$= \frac{50}{2} = 25 \text{ m/s}$$

$$v_2 = 50 \text{ m/s}$$

$$A_i = 2 \times 10^{-3} \text{m}$$

$$A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i = \frac{2}{3} \times 10^{-3} \text{m}$$

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i = \frac{8}{3} \times 10^{-3} \text{m}$$

In second medium, speed becomes two times. Therefore, λ also becomes two times. So, k remains one-half, value of ω will remain unchanged. Further, second medium is rarer medium ($v_2 > v_1$). Hence, there is no change in phase angle anywhere.

$$\therefore \quad y_r = \frac{2}{3} \times 10^{-3} \cos \pi \ (2.0x + 50t)$$

$$\text{and} \quad y_t = \frac{8}{3} \times 10^{-3} \cos \pi \ (x - 50t)$$

Exercises

LEVEL 1

Assertion and Reason

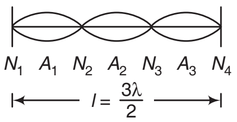
1. At $x = 0$, $y = y_1 + y_2 = A \sin \omega t + A \cos \omega t$. So, it is neither a node nor an antinode.

2. They are called stationary because net energy transfers from any section is zero, if amplitudes of constituent waves are zero.

3. $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

If $v_1 > v_2$ or 2 is rarer, then $A_t > A_i$

4.

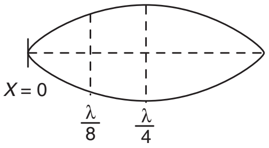


5. $f = \eta \left(\dfrac{v}{2l}\right)$

As η increases, frequency increases. Hence, λ decreases.

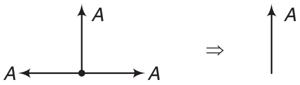
6. Amplitudes may be different also.

7.



Energy lying between $\dfrac{\lambda}{8}$ and $\dfrac{\lambda}{4}$ will be more.

9.



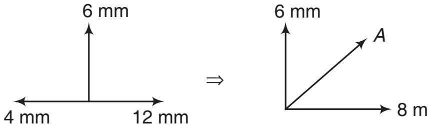
Resultant amplitude is A.

So, intensity will remain same.

10. Phase difference may be different but it should remain constant with time.

Objective Questions

2.



$$A = \sqrt{(8)^2 + (6)^2} = 10 \text{ mm}$$

3. (i) Two waves must travel in opposite directions.

(ii) At $x = 0$, $y = y_1 + y_2$ should be zero at all times.

6. $f_1 : f_2 : f_3 = 1 : 2 : 3$

$$\lambda = \frac{v}{f} \quad \text{or} \quad \lambda \propto \frac{1}{f}$$

$$\therefore \quad \lambda_1 : \lambda_2 : \lambda_3 = 1 : \frac{1}{2} : \frac{1}{3}$$

8. All frequencies are integral multiples of 35 Hz.

9. $3\left(\dfrac{v}{2l}\right) = 300$

$$\therefore \quad v = 200\, l = (200)\,(1)$$

$$= 200 \text{ m/s}$$

10. These are multiples of 30 Hz. Hence, fundamental frequency

$$f_0 = 30 \text{ Hz}$$

Now, $$f_0 = \frac{v}{2l}$$

$$\therefore \quad v = 2 f_0 l$$
$$= 2 \times 30 \times 0.8$$
$$= 48 \text{ m/s}$$

11. $$\because \quad \Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{v/f}\right)(\Delta x) = \left(\frac{2\pi}{vT}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{300 \times 0.04}\right)(16 - 10)$$

$$= \pi$$

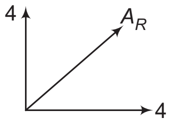
12. $$v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho S}}$$

$$\therefore \quad T = \rho S \left(\frac{\omega}{k}\right)^2 = (8000 \times 10^{-6})\left(\frac{30}{1}\right)^2$$

$$= 7.2 \text{ N}$$

Subjective Questions

1. $$A_R = 4\sqrt{2} \text{ cm}$$
$$= 5.66 \text{ cm}$$



2. $$v_2 = \frac{v_1}{2}$$

(a) $$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A$$

$$= \left[\frac{(v_1/2) - v_1}{v_1 + v_1/2}\right] A$$

$$= -\frac{A}{3}$$

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A$$

$$= \left[\frac{2 \times v_1/2}{v_1 + v_1/2}\right] A$$

$$= \frac{2}{3} A$$

(b) Since, reflected waves come in the same medium, we can say that

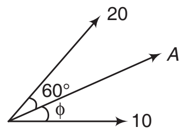
$$P \propto A^2$$

$$\frac{P_r}{P_i} = \left(\frac{A_r}{A_i}\right)^2 = \left(\frac{-A/3}{A}\right)^2 = \frac{1}{9}$$

$\therefore$ Fraction of power transmitted

$$= 1 - \frac{1}{9} = \frac{8}{9}$$

3. $$A = \sqrt{(10)^2 + (20)^2 + 2(10)(20)\cos 60^\circ}$$
$$= 26.46 \text{ cm}$$



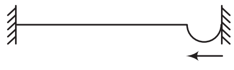
$$\tan\phi = \frac{20 \sin 60^\circ}{10 + 20 \cos 60^\circ}$$
$$= 0.866$$
$$\therefore \quad \phi = 40.89^\circ$$
$$= 0.714 \text{ rad}$$

4. Find y_1 and y_2 at given values of x and t and then simply add them to get net value of y.

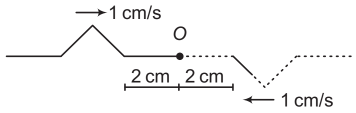
5. $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{16}{(0.4 \times 10^{-3})/(10^{-2})}} = 20 \text{ m/s}$$

(a) $$t = \frac{d}{v} = \frac{2l}{v} = \frac{0.4 \text{ m}}{20}$$
$$= 0.02 \text{ s}$$

(b) At half the time pulse is at other end and it gets a phase difference of π. Hence, the shape is as shown below.

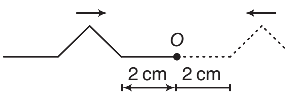


6. (a) **For fixed end.**



By the superposition of these two pulses we will get the resultant. But only to the left of point O, where string is actually present.

(b) **For free end**



7. $y = y_1 + y_2 = (6.0\text{ cm})\sin(\pi x)\cos(0.6\pi t)$

$\therefore \quad A_x = 6.0\text{ cm}\sin(\pi x) \qquad \ldots\text{(i)}$

In parts (*a*), (*b*) and (*c*) substitute the given values of *x* and find the displacement amplitude at these locations.

(d) At antinodes

$$A_x = \text{maximum} = \pm\ 6.0\text{ cm}$$

$$\therefore \quad \pi x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

or $\quad x = 0.5\text{ cm}, 1.5\text{ cm}, 2.5\text{ cm}$

8. (a) Distance between successive antinodes

$$= \frac{\lambda}{2} = \frac{\pi}{k} = \frac{\pi}{(\pi/2)} = 2\text{cm}$$

(b) $A_{\text{max}} = 2A = (2\pi)\text{ cm}$

If $x = 0$ is taken as node, we shall taken sin equation for A_x

$$\therefore \quad A_x = A_{\text{max}} = \sin kx$$

$$= (2\pi\text{ cm})\sin kx$$

Put $k = \dfrac{\pi}{2}$ and $x = 0.5$

9.
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{20}{9\times10^{-3}}} = 47.14\text{ m/s}$$

$$f_1 = \frac{v}{2l} = \frac{47.14}{2\times30}$$

$$= 0.786\text{ Hz}$$

Next three frequencies are $2f_1$, $3f_1$ and $4f_1$

10. (a)
$$f = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$\therefore \quad T = 4l^2f^2\mu$$

$$= (4)(0.7)^2(220)^2\left(\frac{1.2\times10^{-3}}{0.7}\right)$$

$$= 163\text{ N}$$

(b) $f_3 = 3f_1 = 3\times220 = 660\text{ Hz}$

11. Fundamental frequency,

$$f = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$= \frac{\sqrt{(50)/(0.1\times10^{-3}/10^{-2})}}{2\times0.6}$$

$$= 58.93\text{ Hz}$$

Let, *n* th harmonic is the highest frequency, then

$$(58.93)\,n = 20000$$

$$\therefore \quad n = 339.38$$

Hence, 339 is the highest frequency.

$$\therefore \quad f_{\text{max}} = (339)(58.93)\text{ Hz} = 19977\text{ Hz}$$

12. Fundamental frequency = (490− 420) Hz

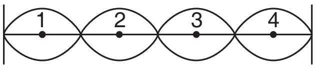
$$f_0 = 70\text{ Hz}$$

But, $\quad f_0 = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l}$

$$\therefore \quad l = \frac{\sqrt{T/\mu}}{2f_0} = \frac{\sqrt{(450)/0.005}}{2\times70}$$

$$= 2.142\text{ m}$$

13. $\lambda = \dfrac{v}{f} = 0.5\text{ m}$



$$l = 4\left(\frac{\lambda}{2}\right) = 1\text{ m}$$

14. (a) $f_4 = 4f_0$

$$\therefore \quad f_0 = \frac{f_4}{4} = \frac{400}{4} = 100\text{ Hz}$$

(b) $\quad f_7 = 7f_0 = 700\text{ Hz}$

15. Fundamental frequency $\propto \dfrac{1}{l}$

$$f_1 : f_2 : f_3 = \frac{1}{l_1} : \frac{1}{l_2} : \frac{1}{l_3}$$

$$= 1 : 2 : 3$$

$$\therefore \quad l_1 : l_2 : l_3 = \frac{1}{1} : \frac{1}{2} : \frac{1}{3} = 6 : 3 : 2$$

$$l_1 = \left(\frac{6}{11}\right)(1)\text{ m} = \frac{6}{11}\text{ m}$$

$$l_2 = \left(\frac{3}{11}\right)(1)\text{ m} = \frac{3}{11}\text{ m}$$

$$l_3 = \left(\frac{2}{11}\right)(1)\text{ m} = \frac{2}{11}\text{ m}$$

16. $f \propto \dfrac{1}{l}$

$$\therefore \quad \frac{f_1}{f_2} = \frac{l_2}{l_1} \quad \text{or} \quad l_2 = \left(\frac{f_1}{f_2}\right)l_1$$

$$= \left(\frac{124}{186}\right)(90\text{ cm}) = 60\text{ cm}$$

17. (a) $\dfrac{\lambda}{2} = 15\text{ cm}$

$$\therefore \quad \lambda = 30\text{ cm}$$

$$k = \frac{2\pi}{\lambda} = \left(\frac{\pi}{15}\right)\text{cm}^{-1}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.075}\text{ s}^{-1}$$

Since, $x = 0$ is a node we will write sin equation.

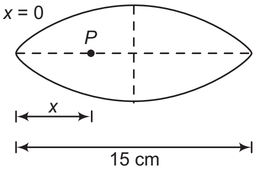
$$\therefore \quad A_x = A_{\max} \sin kx \qquad \text{...(i)}$$

and $y = A_x \sin \omega t$

(b) $v = \dfrac{\omega}{k} = \dfrac{(2\pi/0.075)}{(\pi/15)}$

$= 400 \text{ cm/s} = 4 \text{ m/s}$

(c) $x = 0$



$x = 7.5 - 3 = 4.5 \text{ cm}$

From Eq. (i),

$$A_x = (0.85 \text{ cm}) \sin\left(\frac{\pi}{15} \times 4.5\right)$$

$= 0.688 \text{ cm}$

18. (a) $f_1 = \dfrac{v}{2l} = \dfrac{48}{3}$

$= 16 \text{ Hz}$

$\lambda_1 = \dfrac{v}{f_1} = 3 \text{m}$

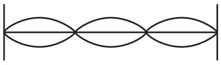
(b) Second overtone means f_3.

$f_3 = 3f_1$ and hence $\lambda_3 = \dfrac{\lambda_1}{3}$

(c) Forth harmonic means f_4.

$f_4 = 4f_1$ and $\lambda_4 = \dfrac{\lambda_1}{4}$

19. (a)



Third harmonic

(b) $A = \dfrac{5.6}{2} = 2.8 \text{ cm}$

(c) $k = 0.034 \text{ cm}^{-1}$

$\lambda = \dfrac{2\pi}{k} = 184.7 \text{ cm}$

$l = \dfrac{3\lambda}{2} = 277 \text{ cm}$

(d) $\lambda = 184.7 \text{ cm}$

$f = \dfrac{\omega}{2\pi} = \dfrac{50}{2\pi} = 7.96 \text{ Hz}$

$T = \dfrac{1}{f} = 0.126 \text{ s}$

$v = f\lambda = 1470 \text{ cm/s}$

(e) $v_{\max} = \omega A_{\max}$

$= (50)(5.60)$

$= 280 \text{ cm/s}$

(f) Frequency and hence ω will become $\dfrac{8}{3}$ times.

$\therefore \quad \omega' = 50 \times \dfrac{8}{3} = 133 \text{ rad/s}$

$k = \dfrac{\omega}{v}$ or $k \propto \omega$

Hence, k will become $\dfrac{8}{3}$ times.

$k' = \left(\dfrac{8}{3}\right)(0.034) = 0.0907 \text{ rad/cm}$

20. (a) $\dfrac{\lambda}{2} = l$ or $\lambda = 2l = 1.6 \text{ m}$

$v = f\lambda = 60 \times 1.6 = 96 \text{ m/s}$

(b) $v = \sqrt{\dfrac{T}{\mu}}$

$\therefore \quad T = \mu v^2 = \left(\dfrac{0.04}{0.8}\right)(96)^2$

$= 461 \text{ N}$

(c) $v_{\max} = \omega A_{\max}$

$= (2\pi f) A_{\max}$

$= (2\pi)(60)(0.003)$

$= 1.13 \text{ m/s}$

$a_{\max} = \omega^2 A_{\max}$

$= (2\pi f)^2 A_{\max}$

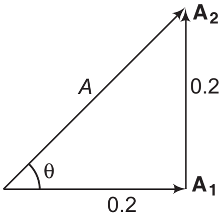
$= (2\pi \times 60)^2 (0.003)$

$= 426.4 \text{ m/s}^2$

21. (a) $y = y_1 + y_2$

$= 0.2 \sin(x - 3.0t) + 0.2 \sin\left(x - 3.0t + \dfrac{\pi}{2}\right)$

$= A \sin(x - 3.0t + \theta)$



Here, $A = \sqrt{(0.2)^2 + (0.2)^2}$

and $\theta = \dfrac{\pi}{4}$

$\therefore$ $y = 0.28 \sin \left(x - 3.0t + \dfrac{\pi}{4}\right)$ **Ans.**

(b) Since, the amplitude of the resulting wave is 0.32 m and $A = 0.2$ m, we have

$$0.32 = \sqrt{(0.2)^2 + (0.2)^2 + (2)(0.2)(0.2)\cos\phi}$$

Solving this, we get

$\phi = \pm 1.29$ rad **Ans.**

22. (i) $f_a = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l}$

$= \dfrac{\sqrt{(100/4 \times 10^{-2})}}{2 \times 2}$

$= 12.5$ Hz

$f_b = 3\left(\dfrac{v}{2l}\right) = 3\, fa = 37.5$ Hz

(ii) $\omega = 2\pi f$

$\therefore$ $\omega_a = 2\pi f_a = 25\pi$

$\omega_b = 2\pi f_b = 75\pi$

$v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{100}{4 \times 10^{-2}}}$

$= 50$ m/s

$k = \dfrac{\omega}{v}$

$\therefore$ $k_a = \dfrac{\omega a}{v} = \dfrac{25\pi}{50} = \dfrac{\pi}{2}$

$k_b = \dfrac{\omega_b}{v} = \dfrac{75\pi}{50} = \dfrac{3\pi}{2}$

23. (a) $l = \lambda_1/4$

$\therefore$ $\lambda_1 = 4l = 16$ m , $l = 3\lambda_2/4$

$\therefore$ $\lambda_2 = \dfrac{4l}{3} = 5.33$ m

$l = 5\lambda_3/4$

$\therefore$ $\lambda_3 = \dfrac{4l}{5} = 3.2$ m

(b) $v = \sqrt{\dfrac{T}{\mu}}$

$= \sqrt{\dfrac{400}{(0.16/4)}} = 100$ m/s

Now, $f_1 = \dfrac{v}{\lambda_1} = \dfrac{100}{16} = 6.25$ Hz

Similarly, f_2 and f_3.

24. (a) $\dfrac{n\lambda_1}{2} = l$

$\therefore$ $\dfrac{n\,(0.54)}{2} = l$

or $n = \dfrac{l}{0.27}$ ...(i)

Similarly, $\dfrac{(n+1)\lambda_2}{2} = l$

or $\dfrac{(n+1)\,(0.48)}{2} = l$

or $(n+1) = \dfrac{l}{0.24}$ ...(ii)

From Eqs. (i) and (ii) we have,

$\dfrac{n}{n+1} = \dfrac{8}{9}$

(b) From Eq. (i),

$l = (0.27)\,n$

$= 0.27 \times 8$

$= 2.16$ m

(c) $\dfrac{\lambda}{2} = l$ (Fundamental)

$\therefore$ $\lambda = 2l = 4.32$ m

25. From fixed end there will be a phase change of π. Further, wave will start travelling in opposite direction. Hence, ωt and kx both will now become position. From free end there is no change in phase.

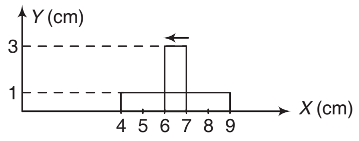
26. $t_1 = \dfrac{d_1}{v} = \dfrac{4+4}{1} = 8$ s

After one reflection from fixed boundary wave is inverted.

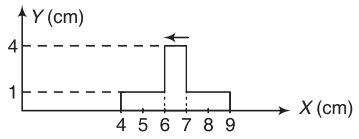
$t_2 = \dfrac{d_2}{v} = \dfrac{4+10+16}{1} = 20$ s

After two times reflection from fixed boundaries wave pulse will again become upright.

27. Let us plot at $t = 3$ s



Net $y = y_1 + y_2$



Similarly, we can draw at other times.

LEVEL 2

Single Correct option

1. $\because \quad f \propto \sqrt{T}$

$$\therefore \qquad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\therefore \qquad \frac{3}{2} = \sqrt{\frac{T + 2.5}{T}}$$

Solving, we get $T = 2\,\text{N}$

2. $f = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l} = \dfrac{\sqrt{T/\rho S}}{l}$

$$= \frac{\sqrt{T/\rho\,(\pi d^2/4)}}{l}$$

or $\qquad f \propto \dfrac{\sqrt{T}}{dl}$

$$\therefore \qquad \frac{f_1}{f_2} = \left(\sqrt{\frac{T_1}{T_2}}\right)\left(\frac{d_2}{d_1}\right)\left(\frac{l_2}{l_1}\right)$$

$$= \sqrt{\frac{1}{2}}\left(\frac{3}{1}\right)\left(\frac{2}{1}\right) = 3\sqrt{2}$$

3. $f \propto \dfrac{1}{l} \quad \Rightarrow \quad l = \dfrac{k}{f}$

Now, $\qquad l = l_1 + l_2 + l_3$

$$\therefore \qquad \frac{k}{f_0} = \frac{k}{f_1} + \frac{k}{f_2} + \frac{k}{f_3}$$

$$\therefore \qquad \frac{1}{f_0} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

4. $y_1 + y_2 = 2A \sin(\omega t - kx) = y_4$ (say)

Now, y_4 and y_3 produce standing waves where,

$A_{\max} = 2$ (Amplitude of constituent wave)

$= 2\,(2A) = 4A$

5. Tension in the string will be given by

$$T = \frac{YA\Delta l}{l} = \frac{YA}{\eta} \qquad \left(\text{as } \frac{\Delta l}{l} = \frac{1}{\eta}\right)$$

Now, $\qquad f \propto v$

$$\frac{f_1}{f_2} = \frac{v_1}{v_2} = \frac{\sqrt{T/\rho A}}{\sqrt{Y/\rho}}$$

$$= \sqrt{\frac{T}{YA}} = \frac{1}{\sqrt{\eta}}$$

6. $f_1 = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l}$

$$= \frac{\sqrt{100/0.01}}{2} = 50\text{ Hz}$$

$$f_2 = 2f_1 = 100\text{ Hz}$$

$$f_3 = 3f_1 = 150\text{ Hz}$$

$$n_1 = \frac{v}{4l} = 25\text{ Hz}$$

$$n_2 = 3n_1 = 75\text{ Hz}$$

$$n_3 = 5n_1 = 125\text{ Hz}$$

7. $\because \qquad f \propto \dfrac{1}{l}$

$$\therefore \qquad l_1 : l_2 : l_3 = \frac{1}{f_1} : \frac{1}{f_2} : \frac{1}{f_3}$$

$$= \frac{1}{1} : \frac{1}{3} : \frac{1}{4} = 12 : 4 : 3$$

$$\therefore \qquad l_1 = \left(\frac{12}{12 + 4 + 3}\right)(114)$$

$$= 72\text{ cm}$$

$$l_2 = \left(\frac{4}{12 + 4 + 3}\right)(114)$$

$$= 24\text{ cm}$$

$$l_3 = \left(\frac{3}{12 + 4 + 3}\right)(114)$$

$$= 18\text{ cm}$$

8. $f \propto \sqrt{T}$

$$\frac{f}{f/2} = \sqrt{\frac{T_1}{T_2}} = \sqrt{\frac{W}{W - V_1}}$$

$$2 = \sqrt{\frac{v\rho g}{V\rho g - V\rho_w g}} = \sqrt{\frac{\rho}{\rho - \rho_w}}$$

Solving $\quad \rho = \dfrac{4}{3}\rho_w$

$$\frac{f}{f/3} = \sqrt{\frac{W}{W - V_2}} = \sqrt{\frac{V\rho g}{V\rho g - V\rho_l g}}$$

$$9 = \frac{\rho}{\rho - \rho_l}$$

$$\therefore \qquad \rho_l = \frac{8}{9}\rho = \frac{32}{27}\rho_\omega$$

$\therefore$ Relative density of liquid $= \dfrac{\rho_l}{\rho_w}$

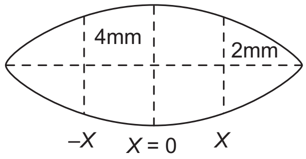
$$= \frac{32}{27} = 1.18$$

9. $\dfrac{\lambda}{2} = 1.5\text{ m}$

$$\therefore \qquad \lambda = 3.0\text{ m}$$

$$= k = \frac{2\pi}{\lambda} = \left(\frac{2\pi}{3}\right)\text{m}^{-1}$$

Let us take antinode at $x=0$, then



$$A_x = A_{\max} \cos kx$$

$$\therefore \quad (2\text{ mm}) = (4\text{ mm}) \cos\left(\frac{2\pi}{3}\right)x$$

$$\therefore \quad \left(\frac{2\pi}{3}\right)x = \frac{\pi}{3}$$

or $\quad x = 0.5$ m

The asked distance is $2x$ or 1.0 m.

10. $f = 3\text{Hz}$

$$\therefore \quad \omega = 2\pi f = (6\pi)\text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{6\pi}{3} = (2\pi)\text{ rad/m}$$

$$y = A\sin(kx - \omega t)$$

$$= A\sin(2\pi x - 6\pi t)$$

Putting $\quad y = \pm A$

and $x = 3$, we get

$$(2\pi)(3) - 6\pi\, t = \frac{\pi}{2}$$

$$\therefore \quad 6\pi t = \frac{11}{2}\pi$$

$$\therefore \quad t = \frac{11}{12}\text{ s}$$

11. $\Delta\phi_1 = \pi/2$

$$\Delta\phi_2 = \left(\frac{2\pi}{\lambda}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{\lambda}\right)(1.5\lambda)$$

$$= 3\pi$$

$$\therefore \quad \Delta\phi_{\text{net}} = \Delta\phi_2 - \Delta\phi_1 = \frac{5\pi}{2}$$

12. Fundamental frequency,

$$f_0 = 450 - 400$$

$$= 50\text{ Hz}$$

$$f_0 = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$\therefore \quad l = \frac{\sqrt{T/\mu}}{2f_0} = \frac{\sqrt{490/0.1}}{2 \times 50}$$

$$= 0.7\text{ m}$$

13. $\dfrac{f_5}{f_2} = \dfrac{5f_1}{2f_1} = \dfrac{5}{2}$

$$\therefore \quad f_2 = \frac{2}{5}f_5$$

$$= \frac{2}{5} \times 480$$

$$= 192\text{ Hz}$$

14. $I \propto A^2$

$$I_r = 0.64\, I_i$$

$$\therefore \quad A_r = 0.8\, A_i$$

$$= 0.8\text{A}$$

Reflected from a denser medium. Hence, a phase change of π will occur. Reflected wave will travel in opposite direction. Hence, ωt and kx will have positive signs.

15. $\eta\left(\dfrac{v}{2l}\right)_B = \left(\dfrac{v}{2l}\right)_A$

$$\eta = \left(\frac{\sqrt{T/\rho S}}{2l}\right)_B = \left(\frac{v}{2l}\right)_A$$

or $\quad \eta\left(\dfrac{\sqrt{4T/\rho\pi}}{2ld}\right)_B = \left(\dfrac{\sqrt{4T/\rho\pi}}{2ld}\right)_A \quad$ ...(i)

Given, $T_B = 2T_A,\ l_B = 2l_A$

$$d_B = 2d_A$$

and $\quad \rho_B = 2\rho_A$

Putting in Eq. (i), we get

$$\eta = 4$$

$\eta = 4$ means 4th harmonic or 3rd overtone.

More than One Correct Options

1. $f \propto \sqrt{T}$

$$\frac{f_2}{f_1} = \sqrt{\frac{T_2}{T_1}}$$

$$\therefore \quad \frac{f_1 + 15}{f_1} = \sqrt{\frac{1.21\,T_1}{T_1}}$$

$$= 1.1$$

Solving we get $f_1 = 150$ Hz

$$v \propto \sqrt{T}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

or $\quad v_2 = \left(\sqrt{\dfrac{1.21\,T_1}{T_1}}\right)v_1$

$$= 1.1\,v_1$$

Hence, increase in v is 10%

$$\frac{\lambda}{2} = l$$

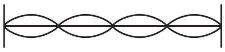
$\therefore \quad \lambda = 2l$

$\therefore$ Fundamental wavelength $= 2\lambda$ is unchanged.

4. $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

If $\quad v_2 > v_1,\ A_t > A_i$

5.



$$l = 4\frac{\lambda}{2} = 2\lambda = 2\left(\frac{2\pi}{k}\right)$$

$$= \frac{4\pi}{k}$$

6. Two identical waves should travel in opposite directions.

7. $y = y_1 + y_2 = (2A \cos kx) \sin \omega t$

$$= A_x \sin \omega t$$

Here, $A_x = 2A \cos kx$

At $x = 0$, A_x is maximum or $2A$.

So, it is an antinode. Next antinode will occur at

$$x = \frac{\lambda}{2}, \lambda \ \ldots \text{ etc.}$$

or $\quad x = \dfrac{\pi}{k}, \dfrac{2\pi}{k} \ldots$ etc.

Comprehension Based Questions

1. Reflected and incident rays are in the same medium. Hence,

$$I \propto A^2$$

I_r has become 64% or 0.64 times of I_i

$\therefore \quad A_r = 0.8\ A_i = 0.8\ A$

2. $y = y_i + y_r$

$$= A \sin (ax + bt + \pi/2) + 0.8\ A \sin (ax - bt + \pi/2)$$

$$= \left[0.8\ A \sin\left(ax + bt + \frac{\pi}{2}\right) + 0.8\ A \sin\left(ax - bt + \frac{\pi}{2}\right)\right] + 0.2A \sin\left(ax + bt + \frac{\pi}{2}\right)$$

$$= -1.6 \sin ax \sin bt + 0.2A \cos (bt + ax)$$

$\therefore \quad c = 0.2$

3. $A_x = -1.6 \sin ax$

$\therefore \quad x = 0$ is a node

Second antinode is at a distance.

$$x = \frac{\lambda}{4} + \frac{\lambda}{2} = \frac{3\lambda}{4} = \frac{3}{4}\left(\frac{2\pi}{k}\right)$$

But, $k = a$

$\therefore \quad x = \dfrac{3\pi}{2a}$

Match the Columns

1. $v \propto \dfrac{1}{\sqrt{\mu}}$

$$\mu_2 = 9\mu_1$$

Hence, $\quad v_2 = \dfrac{v_1}{3}$

So let $v_1 = 3$ units, then $v_2 = 1$ unit

(a) $\quad A_1 = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right) A_i$

and $\quad A_2 = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

$\therefore \quad \left|\dfrac{A_1}{A_2}\right| = \left|\dfrac{v_2 - v_1}{2v_2}\right| = \left|\dfrac{3-1}{2}\right| = 1$

(b) $\dfrac{v_1}{v_2} = 3$

(c) $\quad I = \dfrac{1}{2}\rho\omega^2 A^2 v$

ρ is not given, so we cannot find I_1/I_2.

(d) $P = \dfrac{1}{2}\rho\omega^2 A^2 S v$

But, $\quad S\rho = \mu$

$\therefore \quad P = \dfrac{1}{2}\omega^2 A^2 \mu v$

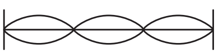
or $\quad P \propto A^2 \mu v \qquad$ (as $\omega \to$ same)

$$\frac{P_1}{P_2} = \left(\frac{A_1}{A_2}\right)^2 \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{v_1}{v_2}\right)$$

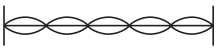
$$= (1)^2 \left(\frac{1}{9}\right)(3) = \frac{1}{3}$$

or $\quad \dfrac{P_2}{P_1} = 3$

2.



(i)



(ii)

(a) Second overtone mode = $f_3 = 3f_1$

Fifth harmonic mode = $f_5 = 5f_1$

So, the ratio is 3/5,

(d) $\lambda = \dfrac{v}{f}$ or $\lambda \propto \dfrac{1}{f}$

$$\therefore \quad \frac{\lambda_3}{\lambda_5} = \frac{f_5}{f_3} = \frac{5}{3}$$

3. (d) $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

If $v_2 > v_1$, i.e. 2 is rarer then $A_t > A_i$.

4. $A = 2A_0 \cos\dfrac{\phi}{2}$ and $I = 4I_0 \cos^2\dfrac{\phi}{2}$

5. Second overtone frequency means

$$f_3 = 3f_1 = 210 \text{ Hz}$$

$\therefore$ $f_1 = 70$ Hz = fundamental frequency

Third overtone frequency = $f_4 = 4f_1$

Second harmonic frequency = $f_2 = 2f_1$

Subjective Questions

1. (a) $v_1 = \sqrt{T/\mu_1}$

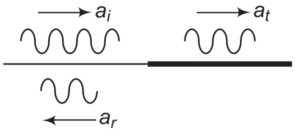
$$v_2 = \sqrt{F/4\mu_1} = \frac{1}{2}\sqrt{F/\mu_1},$$

$$v_3 = \sqrt{F/(\mu_1/4)} = 2\sqrt{F/\mu_1}$$

$$\therefore \quad t = t_1 + t_2 + t_3 = \frac{L}{v_1} + \frac{L}{v_2} + \frac{L}{v_3}$$

$$= \frac{7L}{2}\sqrt{\frac{\mu_1}{F}}$$ **Ans**

2. Let a_i and a_r be the amplitudes of incident and reflected waves.



Then,

$$\frac{a_i + a_r}{a_i - a_r} = 6 \qquad \text{(given)}$$

Hence, $$\frac{a_r}{a_i} = \frac{5}{7}$$

Now, $$\frac{E_r}{E_i} = \left(\frac{a_r}{a_i}\right)^2 = \left(\frac{5}{7}\right)^2$$

$$= 0.51$$

or percentage of energy reflected is

$$100 \times \frac{E_r}{E_i} = 51\%.$$

So, percentage of energy transmitted will be

$$(100 - 51)\% \text{ or } 49\%.$$ **Ans.**

3. Amplitude at a distance x is $A = a \sin kx$

First node can be obtained at $x = 0$,

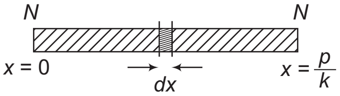
and the second at $x = \pi/k$

At position x, mass of the element PQ is

$$dm = (\rho S)dx$$

Its amplitude is $A = a \sin kx$

Hence mechanical energy stored in this element is



(energy of particle in SHM) $dE = \dfrac{1}{2}(dm)A^2\omega^2$

or $$dE = \frac{1}{2}(\rho S A^2 \omega^2)\, dx$$

$$= \frac{1}{2}(\rho S a^2 \omega^2 \sin^2 kx)\, dx$$

Therefore, total energy stored between two adjacent nodes will be

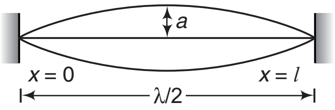
$$E = \int_{x=0}^{x=\pi/k} dE$$

Solving this, we get

$$E = \frac{\pi S \rho \omega^2 a^2}{4k}$$ **Ans.**

4. $l = \dfrac{\lambda}{2}$

or $$\lambda = 2l, k = \frac{2\pi}{\lambda} = \frac{\pi}{l}$$



The amplitude at a distance x from $x = 0$ is given by

$$A = a \sin kx$$

Total mechanical energy at x of length dx is

$$dE = \frac{1}{2}(dm)A^2\omega^2$$

$$= \frac{1}{2}(\mu dx)(a \sin kx)^2(2\pi f)^2$$

or $$dE = 2\pi^2 \mu f^2 a^2 \sin^2 kx\, dx \qquad \text{...(i)}$$

Here, $$f = \frac{v^2}{\lambda^2} = \frac{\left(\dfrac{T}{\mu}\right)}{(4l^2)} \text{ and } k = \frac{\pi}{l}$$

Substituting these values in Eq. (i) and integrating it from $x = 0$ to $x = l$, we get total energy of string.

$$E = \frac{\pi^2 a^2 T}{4l} \qquad \textbf{Ans.}$$

5. Tension, $T = 80$ N

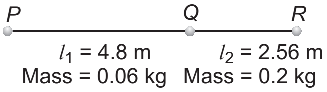
Amplitude of incident wave, $A_i = 3.5$ cm

Mass per unit length of wire PQ is

$$m_1 = \frac{0.06}{4.8} = \frac{1}{80} \text{ kg/m}$$

and mass per unit length of wire QR is

$$m_2 = \frac{0.2}{2.56} = \frac{1}{12.8} \text{ kg/m}$$



(a) Speed of wave in wire PQ is

$$v_1 = \sqrt{T/m_1} = \sqrt{\frac{80}{1/80}} = 80 \text{ m/s}$$

and speed of wave in wire QR is

$$v_2 = \sqrt{T/m_2}$$

$$= \sqrt{\frac{80}{1/12.8}} = 32 \text{ m/s}$$

$\therefore$ Time taken by the wave pulse to reach from P to R is

$$t = \frac{4.8}{v_1} + \frac{2.56}{v_2} = \left(\frac{4.8}{80} + \frac{2.56}{32}\right) \text{s}$$

$$= 0.14 \text{ s} \qquad \textbf{Ans.}$$

(b) The expressions for reflected and transmitted amplitudes (A_r and A_t) in terms of v_1, v_2 and A_i are as follows :

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i$$

and $$A_t = \frac{2v_2}{v_1 + v_2} A_i$$

Substituting the values, we get

$$A_r = \left(\frac{32 - 80}{32 + 80}\right)(3.5) = -1.5 \text{ cm}$$

i.e. the amplitude of reflected wave will be 1.5 cm. Negative sign of A_r indicates that there will be a phase change of π in reflected wave.

Similarly, $A_t = \left(\dfrac{2 \times 32}{32 + 80}\right)(3.5) = 2.0$ cm

i.e. the amplitude of transmitted wave will be 2.0 cm.

The expressions of A_r and A_t are derived as below.

Derivation

Suppose the incident wave of amplitude A_i and angular frequency ω is travelling in positive x-direction with velocity v_1, then we can write

$$y_i = A_i \sin \omega[t - x/v_1] \qquad \ldots\text{(i)}$$

In reflected as well as transmitted wave, ω will not change, therefore, we can write

$$y_r = A_r \sin \omega\,[t + x/v_1] \qquad \ldots\text{(ii)}$$

and $$y_t = A_t \sin \omega\,[t - x/v_2] \qquad \ldots\text{(iii)}$$

Now as wave is continuous, so at the boundary ($x = 0$).

Continuity of displacement requires

$$y_i + y_r = y_t \quad \text{for} \quad x = 0$$

Substituting from Eqs. (i), (ii) and (iii) in the above, we get

$$A_i + A_r = A_t \qquad \ldots\text{(iv)}$$

Also at the boundary, slope of wave will be continuous, i.e.

$$\frac{\partial y_i}{\delta x} + \frac{\partial y_r}{\delta x} = \frac{\partial y_t}{\partial x} \qquad [\text{for } x = 0]$$

which gives $$A_i - A_r = \left(\frac{v_1}{v_2}\right) A_t \qquad \ldots\text{(v)}$$

Solving Eqs. (iv) and (v) for A_r and A_t, we get the required equations, i.e.

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i \quad \text{and} \quad A_t = \frac{2v_2}{v_2 + v_1} A_i$$

6. **When A is a node** Suppose n_1 and n_2 are the complete loops formed on left and right side of point A. Then,

$$f_1 = f_2$$

or $$n_1\left(\frac{v_1}{2L}\right) = n_2\left(\frac{v_2}{2L}\right)$$

or $$\frac{n_1}{n_2} = \left(\frac{v_2}{v_1}\right) = \sqrt{\frac{\mu_1}{\mu_2}} = \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots, \text{etc}$$

$$\left(\text{as } v \propto \frac{1}{\sqrt{\mu}}\right)$$

$\therefore$ Possible frequencies are

$$\frac{v_1}{2L}, 2\left(\frac{v_1}{2L}\right), \frac{3v_1}{2L}, \ldots, \text{etc.} \qquad \left(v_1 = \sqrt{\frac{T}{\mu}}\right)$$

or $$\frac{1}{2L}\sqrt{\frac{T}{\mu}}, \frac{1}{L}\sqrt{\frac{T}{\mu}}, \frac{3}{2L}\sqrt{\frac{T}{2\mu}}, \ldots, \text{ etc.}$$

When *A* is an antinode Suppose n_1 and n_2 are complete loops on left and right side of point *A*,

$$n_1\frac{\lambda_1}{2}+\frac{\lambda_1}{4}=L$$

or $$f_1=\frac{v_1}{L}\left(\frac{n_1}{2}+\frac{1}{4}\right)$$

$$n_2\frac{\lambda_2}{2}+\frac{\lambda_2}{4}=L$$

or $$f_2=\frac{v_2}{L}\left(\frac{n_2}{2}+\frac{1}{4}\right)$$

Substituting $f_1=f_2$,

we get, $$\frac{2n_1+1}{2n_2+1}=\frac{1}{3}$$

For $$n_1=1,\quad n_2=4$$
$$n_1=2,\quad n_2=7$$
$$n_1=3,\quad n_2=10,\quad \text{etc.}$$
$$\ldots\quad\ldots\quad\ldots$$

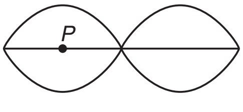
Therefore, the possible frequencies are

$$\frac{v_1}{L}\left(\frac{1}{2}+\frac{1}{4}\right), \frac{v_1}{L}\left(\frac{2}{2}+\frac{1}{4}\right), \frac{v_1}{L}\left(\frac{3}{2}+\frac{1}{4}\right), \ldots, \text{ etc.}$$

or $$\frac{3}{4L}\sqrt{\frac{T}{\mu}}, \frac{5}{4L}\sqrt{\frac{T}{\mu}}, \frac{7}{4L}\sqrt{\frac{T}{\mu}}, \ldots, \text{ etc.}$$

7. $$\frac{L}{4}=\frac{\lambda}{4}$$

$\therefore$ $L=\lambda$



In the next higher mode, there will be total 6 loops and the desired frequency is

$$\left(\frac{6}{2}\right)(100)=300\text{ Hz}$$ **Ans.**

8. $\because$ $$k=\frac{\omega}{v}=\frac{\pi}{3}$$

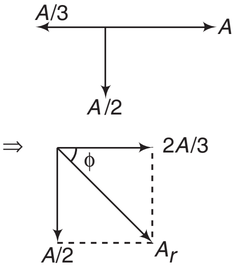
$$y_1=0.06\,(\pi t-kx)=0.06\sin\left(\pi t-\frac{\pi}{3}\times 12\right)$$
$$=0.06\sin(\pi t-4\pi)$$

Similarly,

$$y_2=0.02\sin(\pi t-kx')=0.02\sin\left(\pi t-\frac{\pi}{3}\times 8\right)$$
$$=0.02\sin\left(\pi t-\frac{8\pi}{3}\right)$$

$$y=y_1+y_2$$
$$=0.06\sin\pi t\cos 4\pi-0.06\cos\pi t\sin 4\pi$$
$$+\,0.02\sin\pi t\cos\frac{8\pi}{3}-0.02\cos\pi t\sin\frac{8\pi}{3}$$
$$=0.05\sin\pi t-0.0173\cos\pi t$$ **Ans.**

9. Resultant amplitude



$$A_r=\sqrt{\left(\frac{2A}{3}\right)^2+\left(\frac{A}{2}\right)^2}$$
$$=\frac{5}{6}A$$ **Ans.**

$$\tan\phi=-\frac{A/2}{2A/3}=-\frac{3}{4}$$

or $$\phi=-\tan^{-1}(3/4)$$ **Ans.**

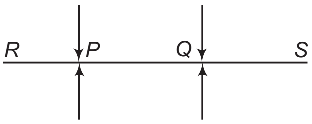
10. Speed of longitudinal waves in the rod,

$$v=\sqrt{\frac{Y}{\rho}}=\sqrt{\frac{1.6\times10^{11}}{2500}}=8000\text{ m/s}$$

At the clamped position nodes will be formed. Between the clamps integer number of loops will be formed. Hence,

$$n_1\frac{\lambda}{2}=80$$

or $$n_1\lambda=160 \qquad \ldots\text{(i)}$$



Between *P* and *R, P* is a fixed end and *R* is the free end. It means the number of loops between *P* and *R* will be odd multiple of $\frac{\lambda}{4}$. Then,

$$\frac{(2n_2-1)}{2}\frac{\lambda}{2}=5$$

or $$(2n_2-1)\lambda=20 \qquad \ldots\text{(ii)}$$

Also between Q and S,

$$(2n_3 - 1)\lambda = 60 \qquad \ldots\text{(iii)}$$

From Eqs. (i) and (ii), we get

$$\frac{n_1}{2n_2 - 1} = \frac{160}{20} = 8 \qquad \ldots\text{(iv)}$$

and from Eqs. (i) and (iii).

$$\frac{n_1}{2n_3 - 1} = \frac{160}{60} = \frac{8}{3} \qquad \ldots\text{(v)}$$

For minimum frequency n_1, n_2 and n_3 should be least from Eqs. (iv) and (v).

We get, $\quad n_1 = 8,\ n_2 = 1,\ n_3 = 2$

$$\lambda = \frac{20}{2n_2 - 1} = 20 \text{ cm} \qquad \text{[from Eq. (ii)]}$$

$$= 0.2 \text{ m}$$

$$\therefore \qquad f_{\min} = \frac{v}{\lambda} = \frac{8000}{0.2}$$

$$= 40 \text{ kHz}$$ **Ans.**

Next higher frequency corresponds to

$$n_1 = 24,\ n_2 = 2$$

and $\quad n_3 = 5$

$$f = 120 \text{ kHz}$$ **Ans.**

11. (a) Distance between two nodes is $\lambda/2$ or π/k. The volume of string between two nodes is

$$V = \frac{\pi}{k} s \qquad \ldots\text{(i)}$$

Energy density (energy per unit volume) of each wave will be

$$u_1 = \frac{1}{2}\rho\omega^2(8)^2 = 32\,\rho\omega^2$$

and $\quad u_2 = \frac{1}{2}\rho\omega^2(6)^2 = 18\,\rho\omega^2$

$\therefore$ Total mechanical energy between two consecutive nodes will be

$$E = (u_1 + u_2)V$$

$$= 50\frac{\pi}{k}\rho\omega^2 S$$

(b) $y = y_1 + y_2$

$$= 8\sin(\omega t - kx) + 6\sin(\omega t + kx)$$

$$= 2\sin(\omega t - kx) + \{6\sin(\omega t + kx) + 6\sin(\omega t + kx)\}$$

$$= 2\sin(\omega t - kx) + 12\cos kx \sin \omega t$$

Thus, the resultant wave will be a sum of standing wave and a travelling wave.

Energy crossing through a node per second = power or travelling wave

$$\therefore \qquad P = \frac{1}{2}\rho\omega^2(2)^2 Sv$$

$$= \frac{1}{2}\rho\omega^2(4)(S)\left(\frac{\omega}{k}\right)$$

$$= \frac{2\rho\omega^3 S}{k}$$

19. Sound Waves

INTRODUCTORY EXERCISE 19.1

1. $\Delta p_{\max} = BAk$

$$B = \frac{\Delta p_{\max}}{Ak} = \frac{\Delta p_{\max}}{A\,(2\pi/\lambda)}$$

$$= \frac{(\Delta p_{\max})\,\lambda}{2\pi A}$$

$$= \frac{(14)\,(0.35)}{(2\pi)\,(5.5 \times 10^{-6})}$$

$$= 1.4 \times 10^5\,\text{N/m}^2$$

2. $\lambda_{\min} = \dfrac{v}{f_{\max}} = \dfrac{1450}{20000}$

$$= 0.0725\text{ m} = 7.25\text{ cm}$$

$$\lambda_{\max} = \frac{v}{f_{\min}} = \frac{1450}{20} = 72.5\text{ m}$$

3. (a) Displacement is zero when pressure is maximum.

(b) $\Delta p_{\max} = BAk$

$$= (\rho v^2)\,A\left(\frac{\omega}{v}\right) = 2\pi f A \rho v$$

$$\therefore\quad A = \frac{\Delta p_{\max}}{2\pi f \rho v}$$

$$= \frac{10}{(2\pi)\,(10^3)\,(1.29)\,(340)}$$

$$= 3.63 \times 10^{-6}\text{ m}$$

4. In the above problem, we have found that

$$A = \frac{\Delta p_{\max}}{2\pi f \rho v}$$

$$= \frac{(\Delta p_{\max})k}{\rho \omega^2} \quad \left(\text{as } v = \frac{\omega}{k} \text{ and } 2\pi f = \omega\right)$$

Now substituting the value, we have

$$A = \frac{(12)\,(8.18)}{(1.29)\,(2700)^2} = 1.04 \times 10^{-5}\text{ m}$$

INTRODUCTORY EXERCISE 19.2

1. $v \propto \sqrt{T}$

$$\therefore\quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \quad \text{or } T_2 = \left(\frac{v_2}{v_1}\right)^2 T_1 = (2)^2\,(273)$$

$$= 1092\text{ K}$$

$$= 819°\text{C}$$

2. $\because\quad v \propto \sqrt{T}$

$$\therefore\quad v_2 = \left(\sqrt{\frac{T_2}{T_1}}\right) v_1$$

$$v_{-3°\text{C}} = \left(\sqrt{\frac{270}{273}}\right)(332)$$

$$= 330.17\text{ m/s}$$

$$v_{30°\text{C}} = \left(\sqrt{\frac{273 + 30}{273}}\right)(332)$$

$$= 349.77\text{ m/s}$$

The difference in these two speeds is approximately 19.6 m/s.

3. $\because\quad v = f\lambda = \sqrt{\dfrac{B}{\rho}}$

$$\therefore\quad B = \rho\ (f\lambda)^2$$

$$= (900)\,(250 \times 8)^2$$

$$= 3.6 \times 10^9\ \text{N/m}^2$$

4. $v = \sqrt{\dfrac{\gamma RT}{M}}$

$$= \sqrt{\frac{(7/5)\,(8.31)\,(273)}{(32 \times 10^{-3})}}$$

$$= 315\text{ m/s}$$

INTRODUCTORY EXERCISE 19.3

1. (a) $\Delta p_{\max} = BAk$

$$= (\rho v^2)\,(A)\left(\frac{\omega}{v}\right)$$

$$= (2\pi f A\,\rho v)$$

$$= (2\pi)\,(300)\,(6.0 \times 10^{-3})(1.2)\,(344)$$

$$= 4.67\text{ Pa}$$

(b) $I = \dfrac{v\,(\Delta p)^2_{\max}}{2B}$

$$= \frac{(\Delta p)^2_{\max}}{2\rho v} \qquad (\text{as } B = \rho v^2)$$

$$= \frac{(4.67)^2}{(2)\,(1.2)\,(344)}$$

$$= 0.0264\ \text{W/m}^2$$

$$= 2.64 \times 10^{-2}\ \text{W/m}^2$$

(c) $$L = \log_{10}\left(\frac{I}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{2.64\times10^{-2}}{10^{-12}}\right)$$

$$= 104 \text{ dB}$$

2. $L_2 - L_1 = 10\log_{10}\frac{I_1}{I_2}$

Given, $L_2 - L_1 = 9$ dB

Solving the equation, we get

$$\frac{I_1}{I_2} = 7.9$$

3. $I \propto \frac{1}{r^2}$

$$\therefore \quad \frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2 = \left(\frac{300}{30}\right)^2 = 100$$

Now, $L_1 - L_2 = 10\log_{10}\frac{I_1}{I_2}$

Substituting $\frac{I_1}{I_2} = 100$

We get, $L_1 - L_2 = 20$ dB

4. (a) $$I = \frac{v(\Delta p)^2_{\max}}{2B}$$

$$= \frac{(\Delta p)^2_{\max}}{2\rho v}$$

For finest sound,

$$I = \frac{(2\times10^{-5})^2}{2\times1.29\times345}$$

$$= 4.49\times10^{-13}\ \text{W/m}^2$$

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{4.49\times10^{-13}}{10^{-12}}\right)$$

$$= -3.48 \text{ dB}$$

Same formulae can be applied for loudest sound.

(b) $(\Delta p)_{\max} = BAk$

$$= (\rho v^2)(A)\left(\frac{\omega}{v}\right)$$

$$= 2\pi f\rho v$$

$$\therefore \quad A = \frac{(\Delta p)_{\max}}{2\pi f\rho v}$$

For finest sound,

$$A = \frac{2\times10^{-5}}{(2\pi)(500)(1.29)(345)}$$

$$= 1.43\times10^{-11} \text{ m}$$

INTRODUCTORY EXERCISE 19.4

1. $\Delta x = \lambda/2 = \frac{v}{2f}$

$$\therefore \quad f = \frac{v}{2\Delta x} = \frac{330}{2(0.12)} = 1375 \text{ Hz}$$

2. (a) $\Delta\phi = \frac{2\pi}{\lambda}(\Delta x)$

$$\therefore \quad \Delta x = \frac{\lambda\Delta\phi}{2\pi} = \frac{v(\Delta\phi)}{(f)(2\pi)}$$

$$= \frac{(350)(\pi/3)}{(500)(2\pi)}$$

$$= 0.1166 \text{ m} = 11.7 \text{ cm}$$

(b) $$\Delta\phi = \left(\frac{2\pi}{T}\right)\Delta t = (2\pi f)\,\Delta t$$

$$= (2\pi)(500)(10^{-3})$$

$$= \pi \text{ or } 180°$$

INTRODUCTORY EXERCISE 19.5

1. Length of the organ pipe is same in both the cases. Fundamental frequency of open pipe is $f_1 = \frac{v}{2l}$ and frequency of third harmonic of closed pipe will be

$$f_2 = 3\left(\frac{v}{4l}\right)$$

Given that, $f_2 = f_1 + 100$

or $f_2 - f_1 = 100$

or $$\frac{3}{4}\left(\frac{v}{l}\right) - \left(\frac{1}{2}\right)\left(\frac{v}{l}\right) = 100$$

$$\Rightarrow \quad \frac{v}{4l} = 100 \text{ Hz}$$

$$\therefore \quad \frac{v}{2l} \text{ or } f_1 = 200\,\text{Hz}$$

Therefore, fundamental frequency of the open pipe is 200 Hz.

2. First harmonic of closed pipe = Third harmonic of open pipe

$$\therefore \quad \frac{v}{4l_1} = 3\left(\frac{v}{2l_2}\right) \Rightarrow \therefore \quad \frac{l_1}{l_2} = \frac{1}{6}$$

3. $f_c = \frac{v}{4l} = 512\text{ Hz}$

$$f_o = \frac{v}{2l} = 2f_c = 1024\text{ Hz}$$

4. (a) $f_1 = \frac{v}{4l}$

$$\therefore \quad l = \frac{v}{4f_1} = \frac{345}{4 \times 220}$$

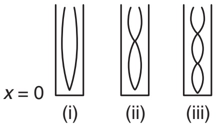
$$= 0.392\text{ m}$$

(b) $5\left(\frac{v}{4l_c}\right) = 3\left(\frac{v}{2l_0}\right)$

$$\therefore \quad l_0 = \frac{6}{5}l_c = \left(\frac{6}{5}\right)(0.392)$$

$$= 0.47\text{ m}$$

5.



(a) In second figure, $l = \frac{3\lambda}{4}$

$$\therefore \quad \frac{\lambda}{4} = \frac{l}{3} = \frac{0.8}{3}$$

$$= 0.267\text{ m}$$

Displacement antinode is at

$$x = \frac{\lambda}{4} = 0.267\text{ m}$$

and $\quad x = 3\frac{\lambda}{4} = 0.8\text{ m}$

(b) Pressure antinode is displacement node:

In third figure,

Pressure antinode or displacement node is at

$$x = 0, x = \frac{\lambda}{2}$$

and $\quad x = \lambda$

6. (a) $\frac{400}{560} = \frac{5}{7}$

Since, these are odd harmonics (5 and 7). Hence, pipe is closed.

(c) Given frequencies are integer multiples of 80 Hz. Hence, fundamental frequency is 80 Hz.

$$\frac{v}{4l} = 80$$

$$\therefore \quad l = \frac{v}{320} = \frac{344}{320}$$

$$= 1.075\text{ m}$$

INTRODUCTORY EXERCISE 19.6

1. Frequency of first will decrease by loading wax over it. Beat frequency is increasing. Hence,

$$f_2 > f_1$$

or $\quad f_2 - f_1 = 4$

$$\therefore \quad f_1 = f_2 - 4 = 256 - 4$$

$$= 252\text{ Hz}$$

2. By putting wax on first tuning fork, its frequency will decrease.

Beat frequency is also decreasing. Hence,

$$f_1 > f_2$$

$$\therefore \quad f_1 - f_2 = 3$$

or $\quad f_1 = 3 + f_2 = 3 + 384$

$$= 387\text{ Hz}$$

INTRODUCTORY EXERCISE 19.7

1. Source is moving towards the observer

$$f' = f\left(\frac{v}{v - v_s}\right) = 450\left(\frac{330}{330 - 33}\right)$$

$$f' = 500\text{ Hz}$$

2. $\quad f_1 = f\left(\frac{v}{v - v_s}\right)$

$$f_1 = f\left(\frac{340}{340 - 34}\right) = f\left(\frac{340}{306}\right)$$

and $\quad f_2 = f\left(\frac{340}{340 - 17}\right) = f\left(\frac{340}{323}\right)$

$$\therefore \quad \frac{f_1}{f_2} = \frac{323}{306} = \frac{19}{18}$$

3. Using the formula, $f' = f\left(\frac{v + v_0}{v}\right)$

We get, $\quad 5.5 = 5\left(\frac{v + v_A}{v}\right) \quad \ldots\text{(i)}$

and $\quad 6.0 = 5\left(\frac{v + v_B}{v}\right) \quad \ldots\text{(ii)}$

Here, $\quad v$ = speed of sound

v_A = speed of train *A*

v_B = speed of train *B*

Solving Eqs. (i) and (ii), we get $\frac{v_B}{v_A} = 2$

4. Observer is stationary and source is moving.

During approach, $f_1 = f\left(\frac{v}{v - v_s}\right)$

$$= 1000\left(\frac{320}{320-20}\right)$$

$$= 1066.67 \text{ Hz}$$

During recede, $f_2 = f\left(\dfrac{v}{v+v_s}\right)$

$$= 1000\left(\frac{320}{320+20}\right) = 941.18 \text{ Hz}$$

$$|\% \text{ change in frequency}| = \left(\frac{f_1 - f_2}{f_1}\right)\times 100$$

$$\approx 12\%$$

Exercise

LEVEL 1

Assertion and Reason

1. **Closed pipe**

Frequencies are, $\dfrac{v}{4l}, \dfrac{3v}{4l}, \dfrac{5v}{4l}$

Open pipe Frequencies are

$$\frac{v}{2l}, 2\left(\frac{v}{2l}\right), 3\left(\frac{v}{2l}\right)$$

They are never same.

2. $f' = f\left(\dfrac{v}{v-v_s}\right) \Rightarrow f' > f$ but $f' =$ constant.

3. Change in pressure is maximum at a point where displacement is zero.

5. $L = 10 \log_{10} \dfrac{I}{I_0}$

Increase in the value of L is not linear with I.

$$\Delta L = L_2 - L_1 = 10 \log \frac{I_2}{I_1}$$

$$\Delta L = 3\,\text{dB} \quad \text{if} \quad I_2 = 2I_1$$

6. It is independent of pressure as long as temperature remains constant.

7. $f_A - f_B = 4$ Hz

When A is loaded with wax f_A will decrease. So, $f_A - f_B$ will be less than 4 Hz till $f_A > f_B$. But $f_B - f_A$ may be greater than 4 Hz when f_B becomes greater than f_A.

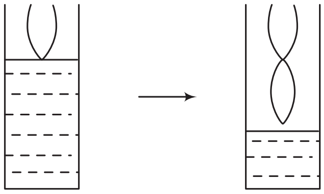
8. $\dfrac{450}{750} = \dfrac{3}{5}$

Successive harmonics are odd 3 and 5. Hence, it is a closed pipe.

9. $f_0 = \dfrac{v}{2\,(l + 1.2\,r)}$ of an open pipe and

$f_0 = \dfrac{v}{4\,(l + 0.6\,r)}$ for a closed pipe.

10. Wavelength remains same.



Objective Questions

1. Sound waves cannot travel in vacuum.

3. $f_0 = \dfrac{v}{2l}$ and $f_c = \dfrac{v}{4l}$

$$\therefore \quad f_0 = 2f_c$$

4. $v = \sqrt{\dfrac{\gamma RT}{M}} \propto \sqrt{\dfrac{T}{M}}$ ($\gamma = 1.4$ for both)

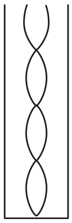
$$\therefore \quad \frac{T_{O_2}}{M_{O_2}} = \frac{T_{N_2}}{M_{N_2}}$$

$$\therefore \quad T_{O_2} = \left(\frac{M_{O_2}}{M_{N_2}}\right) T_{N_2}$$

$$= \left(\frac{32}{28}\right)(273 + 15)$$

$$= 329 \text{ K} = 56°\text{C}$$

5.



Third overtone

6. $f \propto v$ and $v \propto \sqrt{T}$

7. In air speed of sound is less, hence air is denser medium.

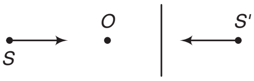
8. $\dfrac{v}{2l_0} = \dfrac{v}{4l_c} \Rightarrow l_c : l_0 = 1:2$

9. $f \propto v \propto \sqrt{T}$ ($T \to$ tension)

$$\therefore \quad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}} \Rightarrow T_2 = T_1\left(\frac{f_2}{f_1}\right)^2$$

$$= (10)\left(\frac{256}{320}\right)^2 = 6.4 \text{ kg}$$

10.



Both S and S' are moving toward observer. Hence,

$$f_S = f_{S'} \quad \text{or} \quad f_b = 0$$

11. $f_b = \frac{10}{3} = f_1 - f_2 = \frac{v}{1} - \frac{v}{1.01}$

Solving we get, $v = 337$ m/s

12. $f' = f\left(\frac{v}{v + v_s}\right) = f\left(\frac{v}{v + v}\right) = 0.5f$

13. $I_R = I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$

If $I_1 = I_2 = I_0$, then

$I_R = 4I_0$

14. $\frac{\lambda}{2} = 52 - 17 = 35$ cm

$$\therefore \quad \lambda = 70 \text{ cm} = 0.7 \text{ m}$$

$$v = f\lambda = 500 \times 0.7$$

$$= 350 \text{ m/s}$$

15. $f' = f\left(\frac{v}{v \pm v_s \cos\theta}\right)$

At $\theta = 90°$; $f' = f$

$\therefore \quad n_1 = 0$

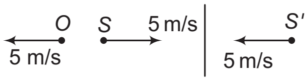
16. Fundamental frequency,

$$f_0 = \frac{v}{4l} = \frac{340}{4 \times 1} = 85 \text{ Hz}$$

Six frequencies can be produced below 1 kHz. Those six frequencies are, f_0, $3f_0$, $5f_0$, $7f_0$, $9f_0$ and $11f_0$.

As $13f_0 = 1105$ Hz > 1000 Hz or 1 kHz.

17. There is no relative motion between O and S'.



$$f_b = f_{S'} - f_S$$

$$= f - f_S = 180 - 180\left(\frac{355 - 5}{355 + 5}\right)$$

$$= 5 \text{ Hz}$$

18. $$n = n_1 - n_2 = \frac{v}{\lambda_1} - \frac{v}{\lambda_2}$$

$$\therefore \quad v = \frac{n\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$$

19. By putting wax on A. Its frequency will decrease. But beat frequency between A and B is also decreasing.

$\therefore \quad f_A > f_B$

or $\quad f_A - f_B = 5$ Hz ...(i)

$\therefore \quad f_B = f_A - 5 = 345$ Hz

Now, $\quad f_B \sim f_C = 4$ Hz

f_C is either 341 Hz or 349 Hz.

If it is 341 Hz, then beat frequency with A will be 9 Hz.

If it is 349 Hz, then beat frequency will be 1 Hz.

If wax is loaded on A, its frequency will decrease.

To produce 6 beats/s with C it should become either 347 Hz (if $f_C = 341$ Hz) or it should become 343 Hz (if $f_C = 349$ Hz).

If it becomes 347 Hz, then only it produces 2 beats/s with B, which is given in the question.

$\therefore \quad f_C = 341$ Hz **Ans.**

20. $\frac{\lambda}{2} = 122 - 40 = 82$ cm

$\therefore$ Next resonance length $= 122$ cm $+ 82$ cm

$= 204$ cm

21. $f \propto \sqrt{T}$

$$\frac{f'}{f} = \sqrt{\frac{T'}{T}} = \sqrt{\frac{101}{100}} = 1.0049$$

$$f' = (1.0049)(200)$$

$$\approx 201 \text{ Hz}$$

$$\therefore \quad f_b = f' - f = 1 \text{ Hz}$$

22. $\lambda = \frac{v}{f} = \frac{340}{340} = 1 \text{ m} = 100$ cm

$\frac{\lambda}{4} = 25$ cm

Air column lengths required are,

$$\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \text{ etc.}$$

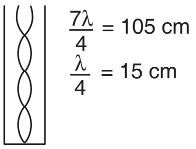
or 25 cm, 75 cm, 125 cm etc.

Maximum we can take 75 cm.

$\therefore$ Minimum water length

$= 120 - 75 = 45$ cm

23. Pressure node means displacement antinode



Displacement antinodes are at a distance $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ and $\frac{7\lambda}{4}$ from closed end or at a distance of 15 cm, 45 cm, 75 cm and 105 cm.

24. Number of moles ∝ Volume

$$M = \frac{n_1 M_{O_2} + n_2 M_{H_2}}{n_1 + n_2}$$

$$= \frac{(1)(32) + (1)(2)}{2} = 17$$

Now, $v = \sqrt{\frac{\gamma RT}{M}} \propto \frac{1}{\sqrt{M}}$

$$\frac{v_1}{v_2} = \sqrt{\frac{M_2}{M_1}} = \sqrt{\frac{2}{17}}$$

25. $f_1 = f\left(\frac{v}{v - v_s}\right) =$ constant. But $f_1 > f$

$f_2 = f\left(\frac{v}{v + v_s}\right) =$ constant, but $f_2 < f$

26. $f \propto \frac{1}{l}$

Both frequencies will becomes half. Hence, $f_b = f_1 - f_2$ will also become half.

27. $f_b = 243\left(\frac{320}{320 - 4}\right) - 243\left(\frac{320}{320 + 4}\right) = 6$ Hz

28. **Closed pipe**

Fundamental frequency is

$$f_1 = \frac{v}{4l} = \frac{320}{4 \times 1} = 80 \text{ Hz}$$

Other frequencies are

$3f_1 : 5f_1$ etc. or 240 Hz, 400 Hz etc.

Open pipe Fundamental frequency is

$$f_1 = \frac{v}{2l} = \frac{320}{2 \times 1.6} = 100 \text{ Hz}$$

Other frequencies are $2f_1 : 3f_1 : 4f_1$ etc. or 200 Hz, 300 Hz and 400 Hz etc. So, then resonate at 400 Hz.

29. Resultant amplitude will become 4 time.

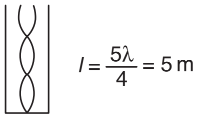
Therefore, resultant intensity is 16 times

$$L_2 - L_1 = 10 \log_{10} \frac{I_1}{I_2}$$

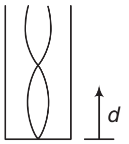
or $\quad L_2 - 10 = 10 \log_{10} (16)$

or $\quad L_2 = 22$ dB

30. $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/2} = 4$ m



31. $\lambda = \frac{v}{f} = \frac{330}{600} = 0.55$ m $= 55$ cm



The desired distance, $d = \frac{\lambda}{4} = 13.75$ cm

32. $f_b = f_1 - f_2 = \frac{v}{\lambda_1} - \frac{v}{\lambda_2}$

$$= \frac{332}{0.49} - \frac{332}{0.5} \approx 13 \text{ Hz}$$

33. $(f_b)_A = f_B - f_A$

$$= 300\left(\frac{300 + 30}{300}\right) - 300 = 30 \text{ Hz}$$

$(f_b)_B = f_A - f_B$

$$= 300\left(\frac{300}{300 - 30}\right) - 300 = 33.33 \text{ Hz}$$

34. $f_a = f\left(\frac{v + v_0}{v}\right)$

$$\therefore \quad \frac{v_0}{v} = \frac{f_a}{f} - 1 \quad \ldots\text{(i)}$$

$$f_r = f\left(\frac{v - v_0}{v}\right)$$

$$\therefore \quad \frac{v_0}{v} = 1 - \frac{f_r}{f} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$f = \frac{f_a + f_r}{2}$$

Subjective Questions

1. Speed of sound wave,

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{(2 \times 10^9)}{10^3}} = 1414 \text{ m/s}$$

Wavelength, $\lambda = \frac{v}{f} = 5.84$ m **Ans.**

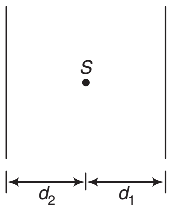
2. $f' = f\left(\dfrac{v \pm v_m}{v \pm v_m}\right) = f$

3.
$$v = \sqrt{\frac{\gamma RT}{M}}$$
$$= \sqrt{\frac{1.40 \times 8.31 \times 300}{2 \times 10^{-3}}} = 1321 \text{ m/s}$$

4.
$$L_1 = 10 \log_{10}\left(\frac{I_1}{I_0}\right)$$
$$= 10 \log_{10}\left(\frac{10^{-6}}{10^{-12}}\right) = 60 \text{ dB}$$
$$L_2 = 10 \log_{10}\left(\frac{I_2}{I_0}\right)$$
$$= 10 \log_{10}\left(\frac{10^{-9}}{10^{-12}}\right) = 30 \text{ dB}$$
$$L_1 = 2L_2$$

5.
$$f' = f\left(\frac{v - v_w - v_0}{v - v_w + v_S}\right)$$
$$= 300\left[\frac{340 - 5 - 20}{340 - 5 + 10}\right]$$
$$= 274 \text{ Hz}$$

6.



$$2d_1 = vt_1$$
$$\therefore \quad d_1 = \frac{vt_1}{2} = \frac{332 \times 3/2}{2}$$
$$= 249 \text{ m}$$
Similarly, $d_2 = \dfrac{vt_2}{2} = \dfrac{332 \times 5/2}{2}$
$$= 415 \text{ m}$$
$\therefore$ Total distance $= d_1 + d_2 = 664$ m

Next echo he will hear after time,
$$t = t_2 + t_1 = \frac{5}{2} + \frac{3}{2} = 4 \text{ s}$$

7.
$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{(5/3) \times 0.76 \times 13.6 \times 10^3 \times 9.8}{0.179}}$$
where, $p = h\rho g$

$\therefore \quad v = 972$ m/s

8. (a)
$$v = f\lambda = \sqrt{\frac{B}{\rho}}$$
$$\therefore \quad B = \rho\,(f\lambda)^2$$
$$= (1300)\,(400 \times 8)^2$$
$$= 1.33 \times 10^{10} \text{ N/m}^2$$
(b)
$$v = \frac{l}{t} = \sqrt{\frac{Y}{\rho}}$$
$$\therefore \quad Y = \rho\left(\frac{l}{t}\right)^2 = (6400)\left(\frac{1.5}{3.9 \times 10^{-4}}\right)^2$$
$$= 9.47 \times 10^{10} \text{ N/m}^2$$

9.
$$\sqrt{\frac{Y}{\rho}} = 30\sqrt{\frac{T}{\mu}} = 30\sqrt{\frac{F}{\rho A}} \qquad (\text{as } T = F \text{ and } \mu = \rho A)$$
$$\therefore \quad \frac{F}{A} = \frac{Y}{900}$$

10. Equal volume of different gases contains equal number of moles at STP.
$$\therefore \quad n \propto V$$
$$M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$$
$$= \frac{(2)\,(2) + (1)\,(28)}{2 + 1} = 10.67$$
$$v = \sqrt{\frac{\gamma RT}{M}}$$
$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1} \cdot \frac{M_1}{M_2}}$$
$$\text{or} \quad v_2 = \left(\sqrt{\frac{T_2}{T_1} \cdot \frac{M_1}{M_2}}\right) v_1$$
$$= \left(\sqrt{\frac{300}{273} \times \frac{2}{10.67}}\right)(1300)$$
$$\approx 591 \text{ m/s}$$

11.
$$L_1 = 10 \log_{10}\left(\frac{I}{I_0}\right)$$
$$100 = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$
Solving we get, $I = 10^{-2}$ W/m^2

Now, $I = \dfrac{P}{4\pi r^2}$
$$\therefore \quad P = I\,(4\pi r^2)$$
$$= (10^{-2})(4\pi)(40)^2$$
$$= 201 \text{ W}$$

12. (a) $$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$60 = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

Solving we get,

$$I = 10^{-6}\ \text{W/m}^2$$

(b) $I = \dfrac{P}{S}$

$$\therefore \quad P = (I)(S) = (10^{-6})(120 \times 10^{-4}) = 1.2 \times 10^{-8}\ \text{W}$$

13. (a) $$L_2 - L_1 = 10 \log_{10}\left(\frac{I_2}{I_1}\right)$$

$$\therefore \quad 13 = 10 \log_{10}\left(\frac{I_2}{I_1}\right)$$

Solving we get, $\dfrac{I_2}{I_1} = 20$

(b) From the above equation we can see that we do not need L_1 and L_2 separately.

14. (a) $$I = \frac{P}{4\pi r^2} = \frac{5}{4\pi\ (20)^2} = 9.95 \times 10^{-4}\ \text{W/m}^2$$

(b) $I = \dfrac{1}{2}\rho\omega^2 A^2 v$

$$\therefore \quad A = \sqrt{\frac{2I}{\rho\ (2\pi f)^2\ v}} = \sqrt{\frac{2 \times 9.95 \times 10^{-4}}{1.29\ (2\pi \times 300)^2\ (330)}} = 1.15 \times 10^{-6}\text{m}$$

15. $$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$60 = 10 \log_{10}\ (I/10^{-12})$$

Solving, we get $I = 10^{-6}\ \text{W/m}^2$

Now, using the result derived in above problem.

$$A = \sqrt{\frac{2I}{\rho\ (2\pi f)^2\ v}} = \sqrt{\frac{2 \times 10^{-6}}{1.29\ (2\pi \times 800)^2 (330)}} = 1.36 \times 10^{-8}\text{m} = 13.6\ \text{nm}$$

16. $$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$\therefore \quad 10^2 = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

Solving we get,

$$I = 1.59 \times 10^{-2}\ \text{W/m}^2$$

$$I = \frac{P}{4\pi r^2}$$

$$\therefore \quad P = (I)(4\pi r^2) = (1.59 \times 10^{-2})(4\pi)(20)^2 = 80\ \text{W}$$

17. $$I = \frac{1}{2}\rho\omega^2 A^2 v$$

$$= 2\rho\pi^2 f^2 A^2 v \qquad (\text{as } \omega = 2\pi f)$$

$$= (2)(1.29)(\pi)^2(300)^2\ (0.2 \times 10^{-3})^2\ (330)$$

$$= 30.27\ \text{W/m}^2$$

$$L = 10 \log_{10}\ (I/I_0)$$

where, $I_0 = 10^{-12}\ \text{W/m}^2$

Substituting the value we get,

$$L = 134.4\ \text{dB}$$

18. (a) $$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.18 \times 10^9}{1000}} = 1476\ \text{m/s}$$

$$\lambda = \frac{v}{f} = \frac{1476}{3400} = 0.43\ \text{m}$$

$$I = \frac{1}{2}\rho\omega^2 A^2 v$$

$$\therefore \quad A = \sqrt{\frac{2I}{\rho\ (2\pi f)^2 v}} = \sqrt{\frac{2 \times 3 \times 10^{-6}}{(1000)\ (2\pi \times 3400)^2\ (1476)}} = 9.44 \times 10^{-11}\text{m}$$

(b) $$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{(1.4)(10^5)}{(1.2)}} = 341.56\ \text{m/s}$$

$$\lambda = \frac{v}{f} = \frac{341.56}{3400} = 0.1\ \text{m}$$

$$A = \sqrt{\frac{2I}{\rho\ (2\pi f)^2\ v}}$$

$$= \sqrt{\frac{2 \times 3.0 \times 10^{-6}}{1.2 \times (2\pi \times 3400)^2\ (341.56)}}$$

$$= 5.66 \times 10^{-9}\ \text{m}$$

(c) $\dfrac{A_{\text{air}}}{A_{\text{water}}} = \dfrac{5.66 \times 10^{-9}}{9.44 \times 10^{-11}} = 60$

$$I = \frac{1}{2}\rho\omega^2 A^2 v$$

ρ and v are less in air. So, for same intensity A should be large.

19. $I = \dfrac{v\,(\Delta p)^2_{\text{max}}}{2B} \quad \Rightarrow \quad v = \sqrt{\dfrac{B}{\rho}}$

$$\therefore \qquad B = \rho v^2$$

$$\therefore \qquad I = \frac{(\Delta p)^2_{\text{max}}}{2\rho v}$$

$$= \frac{(6 \times 10^{-5})^2}{2 \times 1.29 \times 330}$$

$$= 4.2 \times 10^{-12}\ \text{W/m}^2$$

$$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$= 10 \log_{10}\left(\frac{4.2 \times 10^{-12}}{10^{-12}}\right)$$

$$= 6.23\ \text{dB}$$

20. (a) Fundamental, $f_1 = \dfrac{v}{2l} = \dfrac{344}{2 \times 0.45} = 382.2$ Hz

First two overtones are $2f_1$ and $3f_1$.

(b) Fundamental, $f_1 = \dfrac{v}{4l} = \dfrac{344}{4 \times 0.45} = 191.1$ Hz

First two overtones are $3f_1$ and $5f_1$.

21. $\dfrac{\lambda}{2} = (45 - 15)$ cm

$$\therefore \qquad \lambda = 60\ \text{cm}$$

$$v = f\lambda = (500)\,(0.6)$$

$$= 300\ \text{m/s}$$

Lowest frequency when open from both ends is $\dfrac{v}{2l}$.

$$= \frac{300}{2 \times 0.6} = 250\ \text{Hz}$$

22. Fundamental frequency $= \dfrac{v}{2l}$

$$f = \frac{330}{2 \times 0.8}$$

$$f = 206.25\ \text{Hz}$$

$$\omega = 2\pi f = 1297\ \text{rad/s}$$

$$k = \frac{\omega}{v} = \frac{1297}{330} = 3.93\ \text{rad/s}$$

Now, $x = 0$ (the open end) is an antinode. Hence, we can write the equation,

$$y = A \cos kx \sin \omega t$$

23. $\dfrac{\lambda}{2} = (84 - 50)$ cm $= 34$ cm

Next length $= 84 + \dfrac{\lambda}{2} = 118$ cm

$$\lambda = 68\ \text{cm} \quad \text{or} \quad 0.68\ \text{m}$$

$$v = f\lambda = (512)\,(0.68)$$

$$= 348.16\ \text{m/s}$$

24. $2\left(\dfrac{v}{2l_1}\right) = \left(\dfrac{v_2}{4l_2}\right)$

or $\dfrac{\sqrt{T/\mu}}{l_1} = \left(\dfrac{v_2}{4l_2}\right)$

$$\therefore \qquad T = \mu\left(\frac{v_2\, l_1}{4l_2}\right)^2$$

$$= \left(\frac{4 \times 10^{-3}}{0.4}\right)\left(\frac{340 \times 40}{4 \times 100}\right)^2$$

$$= 11.56\ \text{N}$$

25. $f = \dfrac{v}{4\,(l + 0.6\,\pi)}$

$$\therefore \qquad v = 4f\,(l + 0.6\,\pi)$$

$$= 4 \times 480\,(0.16 + 0.6 \times 0.025)$$

$$= 336\ \text{m/s}$$

26. (a) $f = \dfrac{v}{4l}$

$$\therefore \qquad l = \frac{v}{4f} = \frac{345}{4 \times 220}$$

$$= 0.392\ \text{m}$$

(b) $5\left(\dfrac{v}{4l_c}\right) = 3\left(\dfrac{v}{2l_0}\right)$

$$\therefore \qquad l_0 = \frac{6}{5}\, l_c = \frac{6}{5}\,(0.392)$$

$$= 0.470\ \text{m}$$

27. $\dfrac{v_1}{2\,(0.8l)} = \dfrac{v_2}{4l} \quad \Rightarrow \quad \dfrac{v_1}{v_2} = 0.4$

28. (a) $\lambda = \dfrac{v}{f} = \dfrac{340}{300} = 1.13$ m

(b) **Ahead**

$$f' = f\left(\frac{v}{v - v_s}\right) = 300\left(\frac{340}{340 - 30}\right)$$

$$= 329\ \text{Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{329} = 1.03 \text{ m}$$

Behind

$$f' = f\left(\frac{v}{v + v_s}\right) = 300\left(\frac{340}{340 + 30}\right)$$

$$= 275.67 \text{ Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{275.67}$$

$$= 1.23 \text{ m}$$

29. (a) Possible fundamental frequencies of retuned string is (440 ± 1.5) Hz.

(b) $f \propto \sqrt{T}$

$$\therefore \quad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\therefore \quad T_2 = \left(\frac{f_2}{f_1}\right)^2 T_1$$

(i) $T_2 = \left(\dfrac{441.5}{440}\right)^2$

$$T_1 = 1.0068\, T_1$$

$$\% \text{ change} = \frac{T_2 - T_1}{T_1} \times 100$$

$$= +\, 0.68\%$$

(ii) $T_2 = \left(\dfrac{438.5}{440}\right)^2 T_1 = 0.9932\, T_1$

$$\% \text{ change} = \frac{T_2 - T_1}{T_1} \times 100$$

$$= -\, 0.68\%$$

30. (a) $\lambda = 0.12$

Surface wave speed, $v = 0.32$ m/s

$$f' = \frac{v}{\lambda} = \frac{0.32}{0.12} = 2.66 \text{ Hz}$$

$$f = \frac{1}{T} = \frac{1}{1.6} = 0.625 \text{ Hz}$$

Now, using $f' = f\left(\dfrac{v}{v - v_s}\right)$

Here, $v_s =$ velocity of source or velocity of duck

$$\therefore \quad 2.66 = 0.625\left(\frac{0.32}{0.32 - v_s}\right)$$

Solving we get, $v_s = 0.245$ m/s

(b) **Behind the duck**

$$f' = f\left(\frac{v}{v + v_s}\right)$$

$$= 0.625\left(\frac{0.32}{0.32 + 0.245}\right)$$

$$= 0.354 \text{ Hz}$$

$$\lambda = \frac{v}{f'} = \frac{0.32}{0.354} = 0.904 \text{ m}$$

31. (a) $f' = f\left(\dfrac{v + v_0}{v - v_s}\right)$

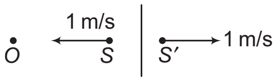
$$= 262\left(\frac{340 + 18}{340 - 30}\right)$$

$$= 302 \text{ Hz}$$

(b) $f' = f\left(\dfrac{v - v_0}{v + v_s}\right)$

$$= 262\left(\frac{340 - 18}{340 + 30}\right) = 228 \text{ Hz}$$

32.



$$f_b = f_S - f_{S'}$$

$$4 = f\left(\frac{340}{340 - 1}\right) - f\left(\frac{340}{340 + 1}\right)$$

or $\quad 4 = f\left(1 - \dfrac{1}{340}\right)^{-1} - f\left(1 + \dfrac{1}{340}\right)^{-1}$

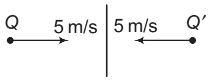
Applying Binomial, we get

$$4 = f\left(1 + \frac{1}{340}\right) - f\left(1 - \frac{1}{340}\right)$$

$$= \frac{f}{170} \quad \text{or} \quad f = 680 \text{ Hz}$$

33. Beat frequency between P and Q is $\dfrac{7}{2}$ or 3.5 Hz.

On loading P with wax, its frequency will decrease and beat frequency is increasing.



$$\therefore \quad f_Q > f_P$$

or $\quad f_Q - f_P = 3.5 \quad \ldots\text{(i)}$

Given, $\quad f'_Q - f_Q = 5$

or $\quad f_Q\left(\dfrac{332 + 5}{332 - 5}\right) - f_Q = 5$

Solving this equation, we get

$$f_Q = 163.5 \text{ Hz}$$

Now, from Eq. (i) we have

$$f_P = 160 \text{ Hz}$$

34. $f_b = f_1 - f_2$

$$2 = 680\left(\frac{340}{340 - v_s}\right) - 680\left(\frac{340}{340 + v_s}\right)$$

$$= 680\left(1 - \frac{v_s}{340}\right)^{-1} - 680\left(1 + \frac{v_s}{340}\right)^{-1}$$

Using Binomial expansion, we have

$$2 = 680\left[\frac{2v_s}{340}\right] \Rightarrow v_s = 0.5 \text{ m/s}$$

35. $\frac{\lambda}{2} = 11.5$ cm or $\lambda = 23$ cm or 0.23 m

$$f = \frac{v}{\lambda} = \frac{331.2}{0.23} = 1440 \text{ Hz}$$

36. $\lambda = \frac{v}{f} = \frac{330}{220} = 1.5$ m

$$\Delta\phi_1 = \left(\frac{2\pi}{\lambda}\right)(\Delta x_1) = \left(\frac{2\pi}{1.5}\right)(0.75) = \pi$$

$$\Delta\phi_2 = \left(\frac{2\pi}{\lambda}\right)(\Delta x_2) = \left(\frac{2\pi}{1.5}\right)(3) = 4\pi$$

Since, $\phi = \Delta\phi_2 - \Delta\phi_1 = 3\pi$

They will interfere destructively.

$\therefore$ $$I_R = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \text{…(i)}$$

where,

$$I_1 = \frac{P_1}{4\pi r_1^2} = \frac{1.2 \times 10^{-3}}{(4\pi)(0.75)^2}$$

$$= 16.97 \times 10^{-5} \text{ W/m}^2$$

$$I_2 = \frac{P_2}{4\pi r_2^2} = \frac{1.8 \times 10^{-3}}{(4\pi)(3)^2}$$

$$= 1.59 \times 10^{-5} \text{ W/m}^2$$

Substituting in Eq. (i), we get

$$I_R = 8.2 \times 10^{-5} \text{ W/m}^2$$

37. $SOD - SD = \lambda$

where, $\lambda = \frac{v}{f} = \frac{360}{360} = 1$ m

$\therefore$ $$2\sqrt{\frac{x^2}{4} + (2)^2} - x = \lambda = 1$$

Solving this equation, we get

$$x = 7.5 \text{ m}$$

38. $\because$ $v = f\lambda = (1000)(2 \times 6.77 \times 10^{-2})$

$$= 135.4 \text{ m/s} = \sqrt{\frac{\gamma RT}{M}}$$

$\therefore$ $$\gamma = (135.4)^2\left(\frac{M}{RT}\right)$$

$$= \frac{(135.4)^2 (0.127)}{(8.31 \times 400)}$$

$$= 0.7$$

Since, $\gamma \nless 1$

So, we will have to substitute,

$$M = 2 \times 0.127$$

The γ comes out to be 1.4 which is the γ of a diatomic gas.

39. $\frac{\lambda}{2} = (100 - 60) \times 10^{-2}$ m

$\therefore$ $\lambda = 0.8$ m

Now, $v = f\lambda = 440 \times 0.8$

$= 352$ m/s

40. When T_A is increased, f_A will increase. So, it will become (600 + 6) = 606 Hz

Now, $f \propto \sqrt{T}$

$\therefore$ $$\frac{T_A}{T_B} = \left(\frac{f_A}{f_B}\right)^2 = \left(\frac{606}{600}\right)^2$$

$$= 1.02$$

41. $f \propto \frac{1}{l}$

By decreasing the length, frequency of wire will increase. But beat frequency is decreasing. Hence, its original frequency was (256 – 4) Hz or 252 Hz. Now, we have to make it 256 Hz for no beats.

$$\frac{f_1}{f_2} = \frac{l_2}{l_1}$$

$\therefore$ $$l_2 = \left(\frac{f_1}{f_2}\right) l_1$$

$$= \left(\frac{252}{256}\right)(25)$$

$$= 24.6 \text{ cm}$$

$\therefore$ $\Delta l = l_1 - l_2 = 0.4$ cm

42. If observer is moving, then

$$f' = f\left(\frac{v \pm v_0}{v}\right) = f\left(1 \pm \frac{v_0}{v}\right)$$

If source is moving, then

$$f' = f\left(\frac{v}{v \mp v_s}\right)$$

$$= f\left(1 \mp \frac{v_s}{v}\right)^{-1} = f\left(1 \pm \frac{v_s}{v}\right)$$

If $v_s << v$

Both expressions come out to be of similar type.

43. $f' = f\left(\frac{v}{v - v_s}\right) = 200\left(\frac{340}{340 - 80}\right)$

$$= 261.53 \text{ Hz} \approx 262 \text{ Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{261.53} = 1.3 \text{ m}$$

44. In front of locomotive,

$$f' = f\left(\frac{v}{v - v_s}\right)$$

$$= 500\left(\frac{344}{344 - 30}\right) = 548 \text{ Hz}$$

$$\therefore \quad \lambda' = \frac{v}{f'} = \frac{344}{548} = 0.628 \text{ m}$$

Behind the locomotive,

$$f' = f\left(\frac{v}{v + v_s}\right)$$

$$= 500\left(\frac{344}{344 + 30}\right) = 460 \text{ Hz}$$

$$\therefore \quad \lambda' = \frac{v}{f'} = \frac{344}{460} = 0.748 \text{ cm}$$

45. (a) The given frequencies are in the ratio 5 : 7 : 9. As the frequencies are odd multiple of 85 Hz, the pipe must be closed at one end.

(b) Now, the fundamental frequency is the lowest, i.e. 85 Hz.

$$\therefore \quad 85 = \frac{v}{4l}$$

$$\Rightarrow \quad l = \frac{340}{4 \times 85} = 1 \text{ m} \qquad \textbf{Ans.}$$

46. Let the frequency of the first fork be f_1 and that of second be f_2.

We then have,

$$f_1 = \frac{v}{4 \times 32}$$

and $$f_2 = \frac{v}{4 \times 33}$$

We also see that $f_1 > f_2$

$$\therefore \quad f_1 - f_2 = 8 \qquad \ldots\text{(i)}$$

and $$\frac{f_1}{f_2} = \frac{33}{32} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$f_1 = 264 \text{ Hz}$$

and $$f_2 = 256 \text{ Hz} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. $\sin \theta_C = \frac{330}{1400} = 0.2357$

$\therefore \quad \theta_C = 13.6°$

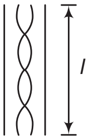
Since, $i > \theta_C$, therefore only reflection will take place.

2. $$\Delta p_{\max} = BAk \qquad \ldots\text{(i)}$$

$$v = \sqrt{\frac{B}{\rho}}$$

$$\therefore \quad B = \rho v^2$$

$$l = \frac{3\lambda}{2}$$



$$\therefore \quad \lambda = \frac{2l}{3}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2l/3} = \frac{3\pi}{l}$$

$$= \frac{3\pi}{3.9\,\pi} = \frac{1}{1.3} \text{ m}^{-1}$$

Substituting in Eq. (i), we have

$$A = \frac{\Delta p_{\max}}{Bk} = \frac{\Delta p_{\max}}{\rho v^2 k}$$

$$= \frac{(0.01 \times 10^5)}{1.3 \times (200)^2 \times (1/1.3)}$$

$$= 0.025 \text{ m} = 2.5 \text{ cm}$$

3. $$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

$$\therefore \quad \frac{A_t}{A_i} = \frac{2 \times 100}{200 + 100} = \frac{2}{3}$$

4. $f_b = f_1 - f_2$

or $$10 = 1700\left(\frac{340}{340 + v_2}\right) - 1700\left(\frac{340}{340 + v_1}\right)$$

$$= 1700\left(1 + \frac{v_2}{340}\right)^{-1} - 1700\left(1 + \frac{v_1}{340}\right)^{-1}$$

Using Binomial therefore, we get

$$10 = 1700\left(1 - \frac{v_2}{340}\right) - 1700\left(1 - \frac{v_1}{340}\right)$$

$$= (v_1 - v_2)\left(\frac{1700}{340}\right)$$

$$\therefore \quad v_1 - v_2 = 2 \text{ m/s}$$

5. Velocity of source after 1 s,

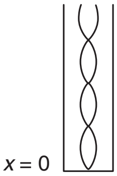
$$v_s = gt = 10 \times 1 = 10 \text{ m/s}$$

$$\Delta f = f\left(\frac{v+v_0}{v-v_s}\right) - f\left(\frac{v-v_0}{v+v_s}\right)$$

$$= 150\left(\frac{300+2}{300-10}\right) - 150\left(\frac{300-2}{300+10}\right)$$

$$= 12 \text{ Hz}$$

6. $L = \frac{7\lambda}{4}$



$\therefore \quad \lambda = \frac{4L}{7}$

$$k = \frac{2\pi}{\lambda}$$

$$= \frac{2\pi}{4L/7} = \frac{7\pi}{2L}$$

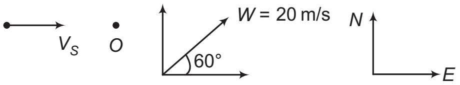
$x = 0$ is a node

$\therefore \quad A_x = a \sin kx = a \sin\left(\frac{7\pi}{2L}\right)\left(\frac{L}{7}\right) = a$

7. $\Delta x_{\max} = 3\text{m} = \lambda$

$\therefore \quad f = \frac{v}{\lambda} = \frac{330}{3} = 110$ Hz. For frequencies less than 100 Hz, λ will be more than 3m and no maximum will be obtained.

8.

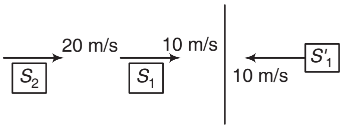


$$f' = f\left(\frac{v + w\cos 60^\circ}{v + w\cos 60^\circ - v_s}\right)$$

$$= 500\left(\frac{300+10}{300+10-20}\right)$$

$$= 534 \text{ Hz}$$

9.



$$f_b = f_{S'} - f_{S_1}$$

$$= 500\left[\frac{340+20}{340-10}\right] - 500\left[\frac{340+20}{340+10}\right]$$

$$\approx 31 \text{ Hz}$$

10. $f_1 = \frac{\omega_1}{2\pi} = \frac{(400\pi)}{2\pi} = 200$ Hz

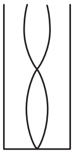
$$f_2 = \frac{\omega_2}{2\pi} = \frac{404\,\pi}{2\pi} = 202 \text{ Hz}$$

$\therefore \quad f_b = f_2 - f_1 = 2$ Hz

$A_1 = 2$ and $A_2 = 1$

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = \frac{9}{1}$$

11. $l = \frac{3\lambda}{4}$



$\therefore \quad \lambda = \frac{4l}{3} = \frac{4\,(34)}{3} = 45.33$ cm

$$\lambda = \frac{v}{f}$$

$\lambda \propto v$ (as $f =$ constant)

and $v \propto \sqrt{T}$ ($T =$ temperature)

$\therefore \quad \lambda \propto \sqrt{T}$

$\therefore \quad \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_2}{T_1}}$

$\therefore \quad \lambda_2 = \left(\sqrt{\frac{T_2}{T_1}}\right)\lambda_1$

$$= \left(\sqrt{\frac{16+273}{51+273}}\right)(45.33)$$

$$= 42.8 \text{ cm}$$

12. $\because \quad f_1 = f_2$

$\therefore \quad 176\left(\frac{330-v}{330-22}\right) = 165\left(\frac{330+v}{330}\right)$

Solving this equation we get,

$$v = 22 \text{ m/s}$$

13. $M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$

$$= \frac{(2)\,(32) + (3)\,(48)}{5} = 41.6$$

$$f = \frac{v}{4l} \quad \text{or} \quad f \propto v$$

But, $v = \sqrt{\frac{\gamma RT}{M}}$

or $v \propto \frac{1}{\sqrt{M}}$

$\therefore \quad f \propto \frac{1}{\sqrt{M}}$

$\therefore \quad \frac{f_2}{f_1} = \sqrt{\frac{M_1}{M_2}}$

or $f_2 = \left(\sqrt{\frac{M_1}{M_2}}\right) f_1$

$$= \left(\sqrt{\frac{32}{48}}\right)(200) = 175.4 \text{ Hz}$$

14. $$f = f_0\left(\frac{v + v_0}{v}\right)$$

$$= 10^3\left(1 + \frac{10t}{v}\right) \qquad (\text{as } v_0 = gt)$$

Hence, f *versus* t graph is a straight line of slope $\frac{10^4}{v}$.

$$\therefore \qquad \frac{10^4}{v} = \text{slope} = \frac{100}{3}$$

$$\therefore \qquad v = 300 \text{ m/s}$$

15. $\because \quad v_p = -v\left(\frac{\partial y}{\partial x}\right)$, where $v = +$ ve.

At E, $\frac{\partial y}{\partial x}$ or slope is positive.

Hence, v_p is negative.

At D, $\frac{\partial y}{\partial x}$ or slope is zero. Hence, v_p is zero.

More than One Correct Options

1. $\because \quad f = n\left(\frac{v}{4l}\right) \qquad (n = 1, 3, 5\ldots)$

$$\therefore \qquad l = \frac{nv}{4f} = n\left(\frac{330}{4 \times 264}\right)$$

$$= (0.3125\, n) \text{ m}$$

$$= (31.25\, n) \text{ cm}$$

Now, keep on substituting $n = 1, 3$, etc.

2. (a) Velocity of sound wave in air independent of pressure if T = constant

(b) $v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma R\,(t + 273)}{M}}$

or $\qquad v^2 \propto (t + 273)$

(c) $v = \sqrt{T/\mu}$

$\therefore \qquad v \propto \sqrt{T}$

or $\qquad v^2 \propto T$

(d) $f_0 = \frac{v}{2l}$ or $\frac{v}{4l}$

$\Rightarrow \qquad f_0 \propto \frac{1}{l}$

4. Let f_1 = fundamental frequency of closed pipe and

f_2 = fundamental frequency of open pipe.

Given, $\qquad 5 f_1 = 4 f_2$

or $\qquad f_2 = \frac{5}{4} f_1$

$\therefore \qquad f_2 > f_1$

(b) First overtone frequency of closed pipe = $3 f_1$ and first overtone frequency of open pipe

$$= 2 f_2 = 2\left(\frac{5}{4} f_1\right) = 2.5\ f_1$$

(c) Fifteenth harmonic of closed pipe = $15 f_1$

Twelfth harmonic of open pipe

$$= 12 f_2 = 12\left(\frac{5}{4} f_1\right) = 15\ f_1$$

(d) Tenth harmonic of closed pipe = $10\ f_1$

Eighth harmonic of open pipe

$$= 8 f_2 = 8\left(\frac{5}{4} f_1\right) = 10 f_1$$

5. $f = \frac{v}{4\,(l + 0.6\text{r})} = \frac{\sqrt{\gamma RT/M}}{4\,(l + 0.6\,r)}$

(d) v does not depend on pressure if temperature is kept constant.

6. During approach,

$$f = f_0\left(\frac{v}{v - v_s}\right)$$

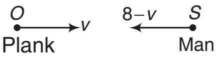
$> f_0$ but f is constant.

During receding.

$$f = f_0\left(\frac{v}{v - v_s}\right)$$

$< f_0$ but f is again constant.

Comprehension Based Questions

1. 

From conservation of linear momentum,

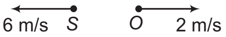
$$50\,(8 - v) = 150 v$$

$$\therefore \qquad v = 2 \text{ m/s}$$

$$8 - v = 6 \text{ m/s}$$

$$f_1 = f_0\left(\frac{v + v_0}{v - v_s}\right)$$

$$= f_0\left(\frac{330 + 2}{330 - 6}\right) = \frac{332}{324}\ f_0$$

2. 

$$f_2 = f_0\left(\frac{v - v_0}{v + v_s}\right)$$

$$= f_0\left(\frac{330 - 2}{330 + 6}\right) = \left(\frac{328}{336}\right) f_0$$

3. $f_1 > f_0$ but f_1 = constant.

Similarly, $f_2 < f_0$ but f_2 = constant

Match the Columns

1. (a) $f = \dfrac{v}{2l}$

$$f_c = \frac{v}{4(2l)} = \frac{v}{8l} = \frac{f}{4}$$

(b) Second overtone frequency

$$= 5f_c = 5\,(f/4) = 1.25f$$

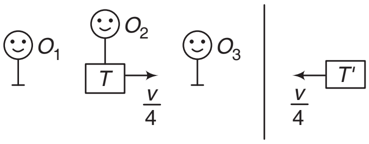
(c) Third harmonic frequency $= 3f_c$

$$= 3\,(f/4) = 0.75\ f$$

(d) First overtone frequency $= 3f_c$

$$= 3\,(f/4) = 0.75f$$

2.



(a) $f_b = f_{T'} - f_T$

$$= f\left(\frac{v}{v - v/4}\right) - f\left(\frac{v}{v + v/4}\right)$$

$$= \frac{8}{15}f$$

(b) $f_b = f_{T'} - f_T = f\left(\dfrac{v + v/4}{v - v/4}\right) - f$

$$= \frac{2}{3}f$$

(c) T and T' both are approaching towards O_3.

$\therefore \quad f_T = f_{T'}$ or $f_b = 0$

(d) Not T and T' both are receding from O_3. Hence, again

$f_T = f_{T'}$ or $f_b = 0$

3. Let $f_1 =$ frequency of tuning fork and

$f_2 =$ frequency of streamed wire

Given, $\quad f_1 > f_2$

$\therefore \quad f_b = f_1 - f_2$

(a) By loading wax on turning fork f_1 will decrease. Hence, $f_1 - f_2$ may be less than f_b or $f_2 - f_1$ may be greater than f_b.

(b) If prongs are filed, f_1 will increase. Hence,

$$f_1 - f_2 > f_b$$

(c) If tension in stretched wire is increased, f_2 will increase. Hence, $f_1 - f_2$ may be less than f_b or $f_2 - f_1$ may be greater than f_b.

(d) If tension in stretched wire is decreased, f_2 will decrease. Hence,

$$f_1 - f_2 > f_b$$

4. $I \propto A^2$ or $A \propto \sqrt{I}$

Due to a point source,

$$I = \frac{P}{4\pi r^2} \qquad (P = \text{power of source})$$

or $\quad I \propto \dfrac{1}{r^2}$ and $A \propto \dfrac{1}{r}$

Due to a line source

$$I = \frac{P}{2\pi l} \quad \text{or} \quad I \propto \frac{1}{r}$$

$$\therefore \qquad A \propto \frac{1}{\sqrt{r}}$$

5. $\because \quad k = \pi$

$$\lambda = \frac{2\pi}{k} = 2\text{ m}$$

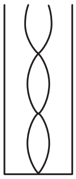
$$l = \frac{5\lambda}{4} = \frac{5}{4}(2)$$

$$= 2.5\text{ m}$$

Distance of displacement node from the closed end

$$= \frac{\lambda}{2} \quad \text{or} \quad \lambda$$

$$= 1\text{ m} \quad \text{or} \quad 2\text{m}$$



Pressure node means displacement antinode. Its distance from closed end

$$= \frac{\lambda}{4} \quad \text{or} \quad \frac{3\lambda}{4} \quad \text{or} \quad \frac{5\lambda}{4}$$

$$= 0.5\text{ m}, 1.5\text{ m and } 2.5\text{ m}$$

Subjective Questions

1. By definition sound level $= 10 \log \dfrac{I}{I_0} = 60$

or $\quad \dfrac{I}{I_0} = 10^6$

$\Rightarrow \quad I = 10^{-12} \times 10^6 = 1\,\mu\text{W/m}^2$

Power entering the room

$$= 1 \times 10^{-6} \times 2 = 2\,\mu\text{W}$$

Energy collected in a day $= 2 \times 10^{-6} \times 86400$

$= 0.173$ J **Ans.**

2. (a) At a distance r from a point source of power P, the intensity of the sound is

$$I = \frac{P}{4\pi r^2} = \frac{0.8}{(4\pi)(1.5)^2}$$

or $I = 2.83 \times 10^{-2}\ \text{W/m}^2$ ...(i)

Further, the intensity of sound in terms of $(\Delta p)_m$, ρ and v is given by

$$I = \frac{(\Delta p)_m^2}{2\rho v} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$(\Delta p)_m = \sqrt{2 \times 2.83 \times 10^{-2} \times 1.29 \times 340}$$

$= 4.98\ \text{N/m}^2$ **Ans.**

(b) Pressure oscillation amplitude $(\Delta p)_m$ and displacement oscillation amplitude A are related by the equation

$$(\Delta p)_m = BAk$$

Substituting $B = \rho v^2,\ k = \frac{\omega}{v}$

and $\omega = 2\pi f$

We get, $(\Delta p)_m = 2\pi A\rho v f$

$$\therefore \quad A = \frac{(\Delta p)_m}{2\pi\rho v f} = \frac{4.98}{(2\pi)(1.29)(340)(600)}$$

$= 3.0 \times 10^{-6}$ m **Ans.**

3. (a) Fundamental frequency when the pipe is open at both ends is

$$f_1 = \frac{v}{2l} = \frac{340}{2 \times 0.6}$$

$= 283.33$ Hz **Ans.**

(b) Suppose the hole is uncovered at a length l from the mouthpiece, the fundamental frequency will be

$$f_1 = \frac{v}{2l}$$

$$\therefore \quad l = \frac{v}{2f_1} = \frac{340}{2 \times 330}$$

$= 0.515$ m

$= 51.5$ cm **Ans.**

Note *Opening holes in the side effectively shortens the length of the resonance column, thus increasing the frequency.*

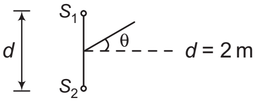
4. $2\sqrt{H^2 + d^2/4} - d = n\lambda$ ...(i)

Now, $2\sqrt{(H+h)^2 + d^2/4} - d = n\lambda + \frac{\lambda}{2}$ ...(ii)

Solving these two equations, we get

$$\lambda = 2\sqrt{4(H+h)^2 + d^2} - 2\sqrt{4H^2 + d^2}$$

5. (a) $\lambda = \frac{v}{f} = \frac{340}{600} = 0.57$ m



$$d \sin\theta = \frac{\lambda}{2}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{\lambda}{2d}\right)$$

$$= \sin^{-1}\left(\frac{0.57}{4}\right)$$

$= 8.14°$

(b) $d \sin\theta = \lambda$ or $\theta = \sin^{-1}\left(\frac{\lambda}{d}\right)$

$$= \sin^{-1}\left(\frac{0.57}{2}\right)$$

$= 16.5°$

(c) $d \sin\theta = n\lambda$

$$\therefore \quad n_{\text{max}} = \frac{d}{\lambda} \qquad (\text{When } \sin\theta = 1)$$

$$= \frac{2}{0.57} = 3.5$$

∴ Maximum 3 maximas can be heard corresponding to $n = 1, 2$ and 3 (beyond $\theta = 0°$)

6. $\Delta\phi_1 = \left(\frac{2\pi}{\lambda}\right)\Delta x = \left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2} = \pi$

(a) $\Delta\phi_{\text{net}} = \Delta\phi_1 + \Delta\phi_2$

$= \pi + 0$

$= \pi$

So, they will interfere destructively and

$$I_{\text{net}} = 0$$

(b) No interference will take place. Hence,

$$I_{\text{net}} = I_1 + I_2 = 2I_0$$

(c) $\Delta\phi_{\text{net}} = \Delta\phi_1 \pm \Delta\phi_2$

$= 2\pi$ or 0 (as $\Delta\phi_2 = \pi$)

In both conditions, they interfere constructively. Hence,

$$I = I_{\text{max}} = 4I_0$$

7. (a) $\Delta\phi = \frac{2\pi}{\lambda}(\Delta x) = \left(\frac{2\pi f}{v}\right)(\Delta x)$

$$= \frac{(2\pi)(170)}{340}(11 - 8)$$

$= 3\pi = \phi$ (say)

$$I = 4I_0 \cos^2\frac{\theta}{2} \qquad \text{...(i)}$$

Substituting value of ϕ, we get

$$I = 0$$

(b) Now, $\phi = 3\pi \pm \pi = 4\pi$ or 2π

Hence, from Eq. (i)

$$I = 4I_0$$

Now, $L_1 - L_2 = 10 \log_{10} \dfrac{I_1}{I_2}$

$$\therefore \quad L_1 - 60 = 10 \log_{10}\left(\frac{4I_0}{I_0}\right) = 6$$

$$\therefore \quad L_1 = 66 \text{ dB}$$

(c) Sources are incoherent.

Hence,

$$I = I_0 + I_0 = 2I_0$$

$$L_1 - L_2 = 10\log_{10} \frac{I_1}{I_2}$$

$$= 10\log_{10} \frac{2I_0}{I_0}$$

$$\therefore \quad L_1 - 60 = 10 \log_{10} 2 = 3$$

or $\quad L_1 = 63$ dB

8. (a) $I = \dfrac{P}{4\pi r^2}$

$$\therefore \quad I_1 = \frac{1 \times 10^{-3}}{4\pi\,(2)^2} = 19.9 \times 10^{-6} \text{ W/m}^2$$

$$= 19.9\,\mu\text{W/m}^2$$

$$I_2 = \frac{1 \times 10^{-3}}{4\pi\,(3)^2} = 8.84 \times 10^{-6} \text{ W/m}^2$$

$$= 8.84\,\mu\text{W/m}^2$$

(b) $I_p = I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$

$$= 55.3\,\mu\text{W/m}^2$$

(c) $I_p = I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$

$$= 2.2\,\mu\text{W/m}^2$$

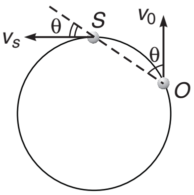
(d) $I_p = I_1 + I_2$

$$= 28.7\,\mu\text{W/m}^2$$

9. $v_s = v_0 = v$

Let u be the speed of sound.

Then,

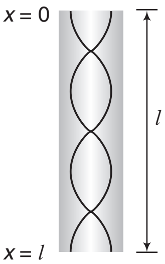


$$f' = f\left(\frac{u + v_0 \cos\theta}{u + v_s \cos\theta}\right)$$

$$= f\left(\frac{u + v \cos\theta}{u + v \cos\theta}\right)$$

or $\quad f' = f$ **Ans.**

10. Given length of pipe, $l = 3$ m



Third harmonic implies that

$$3\left(\frac{\lambda}{2}\right) = l$$

or $\quad \lambda = \dfrac{2l}{3} = \dfrac{2 \times 3}{3} = 2$ m

The angular frequency is

$\omega = 2\pi f.$

$= \dfrac{2\pi v}{\lambda} = \dfrac{(2\pi)(332)}{2}$ or

$\omega = 332\pi$ rad/s

The particle displacement $y(x, t)$ can be written as

$$y(x, t) = A \cos kx \sin \omega t$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2l/3)} = \frac{3\pi}{l}$$

and $\quad \omega = kv = \dfrac{3\pi v}{l} \qquad \left(v = \dfrac{\omega}{k}\right)$

$$\therefore \quad y(x, t) = A \cos\left(\frac{3\pi x}{l}\right) \cdot \sin\left(\frac{3\pi v}{l}\right) t$$

The longitudinal oscillations of an air column can be viewed as oscillations of particle displacement or pressure wave or density wave. Pressure variation is related to particle displacement as

$$p\,(x, t) = -B\frac{\partial y}{\partial x} \qquad (B = \text{Bulk modulus})$$

$$= \left(\frac{3BA\pi}{l}\right) \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi v}{l}\right) t$$

The amplitude of pressure variation is

$$p_{\max} = \frac{3BA\pi}{l} \Rightarrow v = \sqrt{\frac{B}{\rho}} \text{ or } B = \rho v^2$$

$$\therefore \quad p_{\max} = \frac{3\rho v^2 A\pi}{l} \quad \text{or} \quad A = \frac{p_{\max} l}{3\rho v^2 \pi}$$

Here, $p_{\max} = 1\%$ or $p_0 = 10^3$ N/m^2

Substituting the values,

$$A = \frac{(10^3)(3)}{(3)(1.03)(332)^2\pi} = 0.0028 \text{ m}$$

or $\quad A = 0.28$ cm **Ans.**

According to definition of Bulk modulus (B)

$$B = \frac{-dP}{(dV/V)} \qquad \ldots\text{(i)}$$

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}} \quad \text{or} \quad V = \frac{m}{\rho}$$

or $\quad dV = -\dfrac{m}{\rho^2}d\rho = -\dfrac{Vd\rho}{\rho}$

or $\quad \dfrac{dV}{V} = -\dfrac{d\rho}{\rho}$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho\,(dP)}{B}$$

or amplitude of density oscillation is

$$d\rho_{\max} = \frac{\rho}{B}\,p_{\max} = \frac{p_{\max}}{v^2} \qquad \left(\frac{\rho}{B} = \frac{1}{v^2}\right)$$

$$= \frac{10^3}{(332)^2} = 9 \times 10^{-3}\ \text{kg/m}^3$$

11. Sound level (in dB)

$$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

where, $I_0 = 10^{-12}\ \text{W/m}^2$

$L = 60$ dB

Hence, $I = (10^6) I_0 = 10^{-6}\ \text{W/m}^2$

Intensity, $I = \dfrac{\text{Power}}{\text{Area}} = \dfrac{P}{4\pi r^2}$

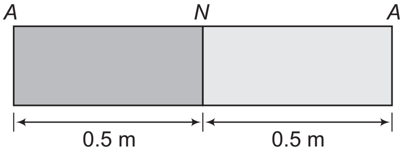
$\Rightarrow \quad P = I(4\pi r^2)$

$$P = (10^{-6})(4\pi)(500)^2 = 3.14\ \text{W}$$

$$\therefore \quad \text{Time, } t = \frac{(1.0 \times 10^3)\left(\dfrac{30}{100}\right)}{3.14}$$

$$t = 95.5\ \text{s}$$

12. Let n_1 harmonic of the chamber containing H_2 is equal to n_2 harmonic of the chamber containing O_2. Then,



$$n_1\left(\frac{v_{H_2}}{4l}\right) = n_2\left(\frac{v_{O_2}}{4l}\right)$$

$$\therefore \quad \frac{n_1}{n_2} = \frac{v_{O_2}}{v_{H_2}} = \frac{300}{1100} = \frac{3}{11}$$

$$\therefore \quad f_{\min} = 3\left(\frac{v_{H_2}}{4l}\right) = 3\left(\frac{1100}{4 \times 0.5}\right)$$

$= 1650$ Hz **Ans.**

13. This problem is a Doppler's effect analogy.

(a) Here, $f = 20\ \text{min}^{-1}$

$v = 300$ m/min

$v_s = 0$ and $v_0 = 0$

Spacing between the pies $= \dfrac{300}{20} = 15$ m **Ans.**

and $f' = f = 20\ \text{min}^{-1}$

(b) $v_s = 30$ m/min

Spacing between the pies will be

$$= \frac{300 - 30}{20} \quad \text{or} \quad 13.5\ \text{m}$$

and $$f' = f\left(\frac{v}{v - v_s}\right) = (20)\left(\frac{300}{300 - 30}\right)$$

$= 22.22\ \text{min}^{-1}$ **Ans.**

14. (a) Comparing with the equation of a travelling wave

$y = a \sin(kx - \omega t)$

$k = 15\pi$ and $\omega = 6000\pi$

$\therefore$ Velocity of the sound, $v = \dfrac{\omega}{k}$

$$= \frac{6000\pi}{15\pi} = 400\ \text{ms}^{-1} \quad \text{as} \quad v = \sqrt{\frac{B}{\rho}}$$

Hence, $\rho = \dfrac{B}{v^2} = \dfrac{1.6 \times 10^5}{(400)^2} = 1\ \text{kg/m}^3$ **Ans.**

(b) Pressure amplitude, $p_0 = BAk$

Hence, $A = \dfrac{p_0}{Bk} = \dfrac{24\pi}{1.6 \times 10^5 \times 15\pi}$

$= 10^{-5}\ \text{m} = 10\mu\text{m}$ **Ans.**

(c) Intensity received by the person,

$$I = \frac{W}{4\pi R^2} = \frac{W}{4\pi(10)^2}$$

Also, $I = \dfrac{p_0^2}{2\rho v} \Rightarrow \dfrac{W}{4\pi(10)^2} = \dfrac{(24\pi)^2}{2 \times 1 \times 400}$

$W = 288\pi^3$ W **Ans.**

15. (a) Path difference and hence phase difference at P from both the sources is 0°, whether $\theta = 45°$, or $\theta = 60°$. So, both the wave will interfere constructively. Or maximum intensity will be obtained. From

$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$ we have

$I_0 = (\sqrt{I} + \sqrt{I})^2 = 4I \quad (I_1 = I_2 = I,\ \text{say})$

$\therefore \quad I = I_0/4$

When one source is switched off, no interference will be obtained. Intensity will be due to a single source or $I_0/4$.

(b) At $\theta = 60°$, also maximum intensity or I_0 will be observed.

16. (a) Frequency of second harmonic in pipe A = frequency of third harmonic in pipe B

$$\therefore \quad 2\left(\frac{v_A}{2l_A}\right) = 3\left(\frac{v_B}{4l_B}\right)$$

or $$\frac{v_A}{v_B} = \frac{3}{4} \Rightarrow \frac{\sqrt{\dfrac{\gamma_A RT_A}{M_A}}}{\sqrt{\dfrac{\gamma_B RT_B}{M_B}}} = \frac{3}{4}$$

or $$\sqrt{\frac{\gamma_A}{\gamma_B}}\sqrt{\frac{M_B}{M_A}} = \frac{3}{4} \qquad (\text{as } T_A = T_B)$$

$$\therefore \quad \frac{M_A}{M_B} = \frac{\gamma_A}{\gamma_B}\left(\frac{16}{9}\right) = \left(\frac{5/3}{7/5}\right)\left(\frac{16}{9}\right)$$

or $$\frac{M_A}{M_B} = \left(\frac{25}{21}\right)\left(\frac{16}{9}\right) = \frac{400}{189}$$

(b) Ratio of fundamental frequency in pipe A and in pipe B is

$$\frac{f_A}{f_B} = \frac{v_A/2l_A}{v_B/2l_B} = \frac{v_A}{v_B} \qquad (\text{as } l_A = l_B)$$

$$= \sqrt{\frac{\dfrac{\gamma_A RT_A}{M_A}}{\dfrac{\gamma_B RT_B}{M_B}}} = \sqrt{\frac{\gamma_A}{\gamma_B}\cdot\frac{M_B}{M_A}} \qquad (\text{as } T_A = T_B)$$

Substituting $\frac{M_B}{M_A} = \frac{189}{400}$ from part (a), we get

$$\frac{f_A}{f_B} = \sqrt{\frac{25}{21}\times\frac{189}{400}} = \frac{3}{4}$$ **Ans.**

17. Velocity of sound in water is

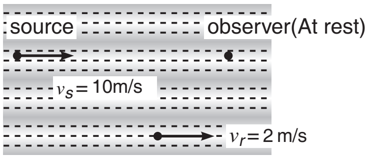
$$v_w = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.088\times10^9}{10^3}} = 1445 \text{ m/s}$$

Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45\times10^{-3}} \text{ Hz}$$

$$f_0 = 10^5 \text{ Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest would be

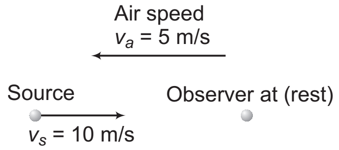


$$f_1 = f_0\left(\frac{v_w + v_r}{v_w + v_r - v_s}\right)$$

$$= 10^5\left(\frac{1445 + 2}{1445 + 2 - 10}\right) \text{Hz}$$

$$f_1 = 1.0069\times10^5 \text{ Hz}$$ **Ans.**

(b) Velocity of sound in air is



$$v_a = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{(1.4)(8.31)(20+273)}{28.8\times10^{-3}}}$$

$$= 344 \text{ m/s}$$

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

$\therefore$ Frequency of sound detected by receiver (observer) in air would be

$$f_2 = f_0\left(\frac{v_a - v_w}{v_a - v_w - v_s}\right)$$

$$= 10^5\left[\frac{344 - 5}{344 - 5 - 10}\right] \text{Hz}$$

$$f_2 = 1.0304\times10^5 \text{ Hz}$$ **Ans.**

18. Frequency of fundamental mode of closed pipe,

$$f_1 = \frac{v}{4l} = 200 \text{ Hz}$$

Decreasing the tension in the string decrease the beat frequency.

Hence, the first overtone frequency of the string should be 208 Hz (not 192 Hz)

$$\therefore \qquad 208 = \frac{1}{l}\sqrt{\frac{T}{\mu}}$$

$$\therefore \qquad T = \mu\cdot l^2(208)$$

$$= \left(\frac{2.5\times10^{-3}}{0.25}\right)(0.25)^2(208)^2$$

$$= 27.04 \text{ N}$$ **Ans.**

19. (a) Wavelength of sound ahead of the source is

$$\lambda' = \frac{v - v_s}{f} = \frac{332 - 32}{1000} = 0.3 \text{ m}$$ **Ans.**

(b) $$f' = f\left(\frac{v + v_0}{v - v_s}\right) = 1000\left(\frac{332 + 64}{332 - 32}\right)$$

$$= 1320 \text{ Hz}$$ **Ans.**

(c) Speed of reflected wave will remain 332 m/s. **Ans.**

(d) Wavelength of reflected wave.

$$\lambda'' = \frac{v - v_0}{f'} = \frac{332 - 64}{1320}$$

$$= 0.2 \text{ m}$$ **Ans.**

20. Thermometry, Thermal Expansion and Kinetic Theory of Gases

INTRODUCTORY EXERCISE 20.1

1. (a) $\frac{T_C-0}{100}=\frac{T_F-32}{180}$

Putting $T_F=0$, we get $T_C=-17.8°\text{C}$

(b) $\frac{T_C-0}{100}=\frac{T_F-32}{180}$ or $\frac{T_K-273}{100}=\frac{T_F-32}{180}$

Putting $T_K=0$, we get $T_F=-459.67°\text{F}$

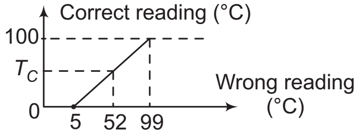
2. (a) $\frac{T_C-0}{100}=\frac{T_F-32}{180}$

Putting $T_F=2T_C$, we get $T_C=160°\text{C}$

(b) Putting $T_F=\frac{T_C}{2}$, we get $T_C=-24.6°\text{C}$

3. $\frac{T_C}{52-5}=\frac{100}{99-5}$

$\therefore\ T_C=50°\text{C}$



Now, using the equation, $\frac{T_C-0}{100}=\frac{T_F-32}{180}$

Putting $T_C=50°\text{C}$, we get $T_F=122°\text{F}$

4. $\frac{T_C-0}{100}=\frac{T_F-32}{180}$ or $\frac{T_K-273}{100}=\frac{T_F-32}{180}$

Putting $T_F=T_K$,

We get, T_F or $T_K=574.25$

5. $\frac{T_C-0}{100}=\frac{T_F-32}{180}$

Putting $T_C=T_F$, we get

T_C or $T_F=-40°\text{C}$ or $-40°\text{F}$

INTRODUCTORY EXERCISE 20.2

1. $\Delta T=\frac{1}{2}T\,\alpha\Delta\theta$

Temperature is decreased. Hence, l or T will decrease. So, it will gain time.

Time gained $=\frac{\Delta T}{T'}t\approx\frac{\Delta T}{T}t$ (as $T'\approx T$)

$$=\frac{1}{2}\alpha\Delta\theta t=\frac{1}{2}(1.2\times10^{-5})(30)(24\times60\times60)$$

$=15.55$ s

2. Density of water will increase by increasing the temperature from 0 to 4°C, then it will decrease. Fraction of volume immersed is given by

$$f_i=\frac{\rho_s}{\rho_l}$$

ρ_l is first increased the decreased first decreased the increased. Or percentage of volume above water level will first increased then decreased.

3. $f_i=\frac{\rho_s}{\rho_l}\ \Rightarrow\ f_i=\frac{\rho'_s}{\rho_l'}$

$$\therefore\ \frac{f_i'}{f_i}=\left(\frac{\rho'_s}{\rho_s}\right)\left(\frac{\rho_l}{\rho'_l}\right)=\left(\frac{1+\gamma_l\Delta T}{1+\gamma_s\,\Delta T}\right)=\left(\frac{1+\gamma_2\Delta T}{1+\gamma_1\Delta T}\right)$$

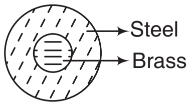
Now, $\frac{1}{1+\gamma_1\,\Delta T}\approx(1-\gamma_1\,\Delta T)$

$$\therefore\ \frac{f_i'}{f_i}=(1+\gamma_2\Delta T)(1-\gamma_1\Delta T)$$

Neglecting $\gamma_1\gamma_2\,(\Delta T)^2$ term, we get

$$\frac{f_i'}{f_i}\approx(\gamma_2-\gamma_1)\,\Delta T$$

4.



Since, $\alpha_{\text{B}}>\alpha_{\text{Fe}}$

On cooling brass will contract more.

5. $\Delta d=d\,(\alpha_{\text{B}}-\alpha_{\text{Fe}})\,\Delta\theta$

$$\therefore\ \Delta\theta=\frac{\Delta d}{d\,(\alpha_{\text{B}}-\alpha_{\text{Fe}})}$$

$$=\frac{0.01}{60\times0.8\times10^{-5}}=20.83°\text{C}$$

$\therefore$ New temperature $=(30+20.83)=50.83°\text{C}$

6. (a) $\Delta l=l\,\alpha\Delta\theta$

$=(88.42)(2.4\times10^{-5})(35-5)$

$=0.064$ cm

(b) At higher temperature, it measures less. Hence,

$l=l'+\Delta l$

$=(88.42)+(0.064)=88.484$ cm

7. $\frac{\Delta l}{l}\times100=\frac{-l\,\alpha\,\Delta\theta}{l}\times100$

$=-(\alpha\Delta\theta)\times100$

$=-(1.2\times10^{-5})(35)\times100$

$=-0.042\%$

INTRODUCTORY EXERCISE 20.3

1. $pV = nRT = \left(\frac{m}{M}\right) RT$

$\therefore \quad T = \left(\frac{VM}{mR}\right) p$

At constant volume T-p graph is a straight line of

slope, $= \frac{VM}{mR} \propto \frac{1}{m}$

Slope of m_1 is less. Hence, m_1 is greater.

2. $p=\frac{nRT}{V}$ or $p \propto nT$

$$\therefore \quad \frac{p_2}{p_1} = \frac{n_2 T_2}{n_1 T_1}$$

$$\text{or} \quad p_2 = \left(\frac{n_2}{n_1}\right)\left(\frac{T_2}{T_1}\right) p_1$$

$$= \left(\frac{1}{2}\right)\left(\frac{87 + 273}{27 + 273}\right) (20 \text{ atm})$$

$$= 12 \text{ atm}$$

3. $n_1 =$ number of moles of nitrogen $= \frac{7}{28} = \frac{1}{4}$

$n_2 =$ number of moles of $CO_2 = \frac{11}{44} = \frac{1}{4}$

$$M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$$

$$= \frac{\left(\frac{1}{4}\right)(28) + \left(\frac{1}{4}\right)(44)}{\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)} = 36$$

$$\text{Now,} \quad \rho = \frac{RM}{RT}$$

$$= \frac{(1.01 \times 10^5)(36 \times 10^{-3})}{(8.31)(290)}$$

$$= 1.5 \text{ kg/m}^3$$

4. $pV = nRT$

$$\therefore \quad n = \frac{pV}{RT} = \frac{(\rho g h) V}{RT}$$

$$= \frac{(13.6 \times 10^3)(9.8)(10^{-6})(250 \times 10^{-6})}{8.31 \times 300}$$

$$= 1.33 \times 10^{-8}$$

$\therefore$ Number of molecules $= (n) N_A$

$$= (1.33 \times 10^{-8})(6.02 \times 10^{23})$$

$$= 8 \times 10^{15}$$

5. $V = \frac{nRT}{p} = \left(\frac{nR}{p}\right) T$

For given mass, V - T graph is a straight line passing through origin having

$$\text{slope} = \frac{nR}{p} \propto \frac{1}{p}$$

Slope of p_2 is more. Hence, $p_2 < p_1$.

6. $p = (nRT) \frac{1}{V}$

i.e. p *versus* $\frac{1}{V}$ graph is a straight line passing through origin of slope nRT.

7. pV and T both are constants.

INTRODUCTORY EXERCISE 20.4

1. Internal energy of n moles of an ideal gas at temperature T is given by

$$U = n\left(\frac{f}{2} RT\right)$$

where, $f =$ degrees of freedom

$= 5$ for O_2 and 3 for Ar

Hence, $U = U_{O_2} + U_{Ar}$

$$= 2\left(\frac{5}{2} RT\right) + 4\left(\frac{3}{2} RT\right)$$

$$= 11 RT$$

2. Average translational kinetic energy of an ideal gas molecule is $\frac{3}{2} kT$ which depends on temperature only. Therefore, if temperature is same, translational kinetic energy of O_2 and N_2 both will be equal.

3. $\frac{f_T}{f_R} = \frac{3}{2} = \frac{K_T}{K_R} = \frac{K_T}{K_O}$

$$\therefore \quad K_T = \frac{3}{2} K_O$$

INTRODUCTORY EXERCISE 20.5

1. $v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.31 \times 373.15}{2 \times 10^{-3}}}$

$$= 2156 \text{ m/s} = 2.15 \text{ km/s}$$

2. $v_{\text{av}} = \frac{500 + 600 + 700 + 800 + 900}{5} = 700 \text{ m/s}$

$$v_{\text{rms}} = \sqrt{\frac{(500)^2 + (600)^2 + (700)^2 + (800)^2 + (900)^2}{5}} = 714 \text{ m/s}$$

4. Helium gas is monoatomic. So, its degree of freedom $f = 3$. Average kinetic energy of 1 molecule of gas

$$= \frac{f}{2} kT = \frac{3}{2} kT$$

$$= \frac{3}{2}(1.38 \times 10^{-23})(300)$$

$$= 6.21 \times 10^{-21} \text{ J}$$

5. (a) $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

For He gas : $v_{\text{rms}} = \sqrt{\frac{3 \times 8.31 \times 300}{4 \times 10^{-3}}}$

$$= 1368 \text{ m/s}$$

For Ne gas : $v_{\text{rms}} = \sqrt{\frac{3 \times 8.31 \times 300}{20.2 \times 10^{-3}}}$

$$= 609 \text{ m/s}$$

(b) Each gas is monoatomic for which degree of freedom $f = 3$. Hence, average kinetic energy of one atom

$$= \frac{f}{2} kT = \frac{3}{2} kT$$

$$= \left(\frac{3}{2}\right)(1.38 \times 10^{-23})(300)$$

$$= 6.21 \times 10^{-21} \text{ J}$$

6. $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

$\therefore \quad T = \frac{mv^2_{\text{rms}}}{3R} = \frac{(4 \times 10^{-3})(10^3)^2}{3 \times 8.31}$

$$= 160 \text{ K}$$

7. Suppose n_1 molecules have v_1 velocity and n_2 molecules have v_2 velocity. Then,

$$v_{\text{av}} = \frac{n_1 v_1 + n_2 v_2}{n_2 + n_2}$$

But, $\quad v_{\text{rms}} = \sqrt{\frac{n_1 v_1^2 + n_2 v_2^2}{n_1 + n_2}}$

Now, $v_{\text{rms}} \geq v_{\text{av}}$ because v_1 and v_2 may be in opposite directions also.

Exercises

LEVEL 1

Assertion and Reason

1. Straight line passing through origin represents isochoric process.

3. In monoatomic gas, there is no rotational degree of freedom.

4. Temperature of the gas is associated with random or disordered motion of the gas molecules. By the motion of container total kinetic energy of gas molecules will increase. But temperature will not increase.

5. Internal energy is equally distributed in all degrees of freedom. For example

$$\frac{K_T}{K_R} = \frac{3}{2} \text{ in diatomic gas}$$

As $\quad f_T = 3$ and $f_R = 2$

7. Density of water is maximum at 4°C. Hence, volume is minimum at 4°C.

8. $U \propto T \propto pV$

9. $p = \frac{1}{3}\left(\frac{mN}{V}\right) v^2_{\text{rms}}$

$$\frac{mN}{V} = \frac{\text{Total mass of gas}}{\text{Volume of gas}} = \text{density of gas}$$

Objective Questions

2. $v_{\text{rms}} = \sqrt{\frac{(1)^2 + (0)^2 + (2)^2 + (3)^4}{4}}$

$$= \sqrt{3.5} \text{ m/s}$$

3. $T_1 = 273 + 27 = 300 \text{ K}$

$T_2 = 273 + 927 = 1200 \text{ K}$

$v_{\text{rms}} \propto \sqrt{T}$

T has become four times.

Therefore, v_{rms} will become two times.

4. Hydrogen and oxygen both are diatomic gases. Therefore, average kinetic energy per molecule is

$$\frac{f}{2} kT$$

or $\quad \frac{5}{2} kT \qquad$ (as $f = 5$)

5. $E \propto T$

$\therefore \quad \frac{T_2}{T_1} = \frac{E_2}{E_1} = \frac{2E}{E} = 2$

$\therefore \quad T_2 = 2T_1 = 2(273 + 10)$

$$= 566 \text{ K} = 293°\text{C}$$

7. If $p = \text{constant}$, then $V \propto T$

8. It means rod is compressed from its natural length by Δl.

$$\therefore \quad \text{Strain} = -\frac{\Delta l}{l} = -\frac{l\Delta\alpha\Delta\theta}{l}$$

$$= (-\alpha d\theta) = -(12\times10^{-6})(50)$$

$$= -6\times10^{-4}$$

9. $\Delta l_1 = \Delta l_2$

$$\therefore \quad l_1\alpha_1\Delta\theta = l_2\alpha_2\Delta\theta$$

$$\text{or} \quad \frac{l_1}{l_2} = \frac{\alpha_2}{\alpha_1} = \frac{9}{11}$$

In option (a), ratio is $\frac{19}{11}$.

10. $V = \frac{nRT}{p} = \frac{mR\,(t+273)}{Mp}$

$\frac{m}{p}$ ratio is same. Therefore, slope and intercept of straight line between V and t should also remain same.

Subjective Questions

1. For Q.Nos. 1 and 2

Apply the formula,

$$\frac{T_C-0}{100} = \frac{T_K-273}{100} = \frac{T_F-32}{180} = \frac{T_R-0}{80}$$

3. In the above equation, putting $T_F = T_C$,

We get $T_F = -40°\text{F}$ or $T_C = -40°\text{C}$

4. Initially $T_C = \left(\frac{T_F-32}{180}\right)(100)$

$$= \left(\frac{68.2-32}{180}\right)(100)$$

$$= 20.11°\text{C}$$

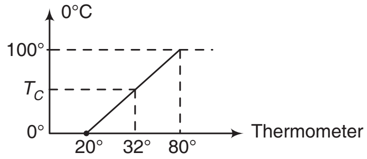
Finally $T_C = 20.11 + 40 = 60.11°\text{C}$

$$\therefore \quad T_F = 32 + \left(\frac{180}{100}\right)T_C$$

$$= 32 + \left(\frac{180}{100}\right)(60.11)$$

$$= 140.2°\text{F}$$

5.



$$\frac{T_C}{32-20} = \frac{100}{80-20} \quad \text{or} \quad T_C = 20°\text{C}$$

6. The temperature on the platinum scale is

$$t = \frac{R_t - R_0}{R_{100} - R_0}\times100°$$

$$= \frac{86-80}{90-80}\times100°\text{C}$$

$$= 60°\text{C}$$ **Ans.**

7. Let the relation between the thermometer reading and centigrade be $y = ax + b$

Given, at $x = 100$, $y = 80$ and at $x = 0$, $y = 10$

$\therefore \quad 80 = 100\,a + b, 10 = b$

$\Rightarrow \quad a = 0.7$

Now, we have to find x when $y = 59$

$\therefore \quad 59 = 0.7x + b \Rightarrow x = 70$

$\therefore$ The answer is 70°C. **Ans.**

8. If T be the corresponding temperature,

$$\sqrt{\frac{3RT}{M_O}} = \sqrt{\frac{3R(300)}{M_H}}$$

$$\therefore \quad T = (300)\left(\frac{M_O}{M_H}\right)$$

$$= 4800\text{ K}$$ **Ans.**

9. Mass of 6.02×10^{23} molecules is 17 g or 0.017 kg.

$\therefore$ Mass of one NH_3 molecule

$$= \frac{0.017}{6.02\times10^{23}}\text{ kg}$$

$$= 2.82\times10^{-26}\text{ kg}$$

10. $\frac{n_1+n_2}{\gamma-1} = \frac{n_1}{\gamma_1-1} + \frac{n_2}{\gamma_2-1}$

$$\therefore \quad \frac{3+2}{\gamma-1} = \frac{3}{1.67-1} + \frac{2}{1.4-1}$$

$$\frac{5}{\gamma-1} = 9.48$$

Solving, we get $\gamma = 1.53$

11. $v = \sqrt{\frac{\gamma p}{\rho}}$

$$\therefore \quad \gamma = \frac{\rho v^2}{p} = \frac{1.3\times(330)^2}{1.013\times10^5} = 1.4$$

$$\gamma = 1 + \frac{2}{f} \qquad (f = \text{degree of freedom})$$

$$1.4 = 1 + \frac{2}{f}$$

Solving, we get

$$f = 5$$

12. 4 g hydrogen = 2 moles hydrogen

11.2 litre He at STP $= \frac{1}{2}$ mole of He

$$p = p_{\text{H}} + p_{\text{He}} = (n_{\text{H}} + n_{\text{He}})\frac{RT}{V}$$

$$= \left(2 + \frac{1}{2}\right)\frac{8.31 \times (300\,\text{K})}{(20 \times 10^{-3})\text{m}^3}$$

$$= 3.12 \times 10^5\ \text{N/m}^2$$ **Ans.**

13. Since, pressure is constant.

$$\therefore \quad V \propto T$$

$$\therefore \quad \frac{V_i}{T_i} = \frac{V_f}{T_f}$$

$$\therefore \quad T_f = \frac{V_f}{V_i}T_i$$

$$\Rightarrow \quad T_f = 2T_i = 600\,\text{K}$$

$$\therefore \quad \Delta U = \frac{f}{2}nR\Delta T$$

$$= \frac{3}{2}R(600 - 300) = 450R$$ **Ans.**

14. From energy conservation principle,

$$E_i = E_f$$

or $$E_1 + E_2 = E$$

$$\therefore \ n_1\left(\frac{3}{2}RT_1\right) + n_2\left(\frac{3}{2}RT_2\right) = (n_1 + n_2)\left(\frac{3}{2}RT\right)$$

$$\therefore \quad T = \frac{n_1T_1 + n_2T_2}{n_1 + n_2}$$ **Ans.**

15. Number of gram moles $= \frac{1}{18} = n$

Avogadro number, $N_A = 6.02 \times 10^{23}$/g-mol

∴ Total number of molecules,

$$N = nN_A = 3.34 \times 10^{22}$$

∴ Number of molecules per cm²

$$= \frac{N}{4\pi R^2} \qquad (R \to \text{in cm})$$

$$= \frac{3.34 \times 10^{22}}{(4\pi)(6400 \times 10^5)^2} = 6500$$

16. $pV = nRT$

$$\therefore \quad nRT = pV$$

$$U = \frac{f}{2}(nRT) \quad (f = \text{degree of freedom})$$

$$\therefore \quad U = \frac{f}{2}(pV)$$

or $$U \propto fpV$$

f has become $\frac{5}{3}$ time (3 for He and 5 for O_2). p has become $\frac{3}{2}$ times and V has become 3 times.

∴ U will become $\frac{5}{3} \times \frac{3}{2} \times 3$ or 7.5 times.

17. $$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} \propto \sqrt{\frac{T}{M}}$$

v_{rms} is same. Hence,

$$\left(\frac{T}{M}\right)_{H_2} = \left(\frac{T}{M}\right)_{O_2}$$

or $$T_{H_2} = \left(\frac{M_{H_2}}{M_{O_2}}\right)T_{O_2}$$

$$= \left(\frac{2}{32}\right)(273 + 47)$$

$$= 20\ \text{K} = -253°\text{C}$$

18. (i) $$\sqrt{\frac{3RT}{M}} = 11.2 \times 10^3\ \text{m/s}$$

$$\therefore \quad T = \frac{(11.2 \times 10^3)^2 M}{3R} \qquad \text{...(i)}$$

$$= \frac{(11.2 \times 10^3)^2 \times 2 \times 10^{-3}}{3 \times 8.31}$$

$$= 10059\ \text{K}$$

(ii) Escape velocity from moon

$$= \sqrt{2g_m R_m}$$

$$= \sqrt{1.6 \times 2 \times 1750 \times 10^3}$$

$$= 2366\ \text{m/s}$$

Substituting in Eq. (i), we have

$$T = \frac{(2366)^2 \times 2 \times 10^{-3}}{3 \times 8.31}$$

$$= 449\ \text{K}$$

19. At constant volume,

$$p \propto T$$

$$\therefore \quad \frac{p_1}{T_1} = \frac{p_2}{T_2}$$

$$\therefore \quad T_2 = \left(\frac{p_2}{p_1}\right)T_1$$

$$= \left(\frac{160}{80}\right)(273) = 546\ \text{K}$$

20. $$t = \left(\frac{R_t - T_0}{R_{100} - R_0}\right) \times 100 = \left(\frac{6.5 - 2.5}{3.5 - 2.5}\right) \times 100$$

$$= 400°$$

21. At constant volume,

$$p \propto T$$

$$\therefore \quad \frac{p_2}{p_1} = \frac{T_2}{T_1} \quad \text{or} \quad T_2 = \left(\frac{p_2}{p_1}\right) T_1$$

$$= \left(\frac{45 + 75}{5 + 75}\right)(273 + 30)$$

$$= 450 \text{ K} = 450 - 273$$

$$= 177^\circ\text{C}$$

22. Strain $= \dfrac{\Delta l}{l} = (\alpha\Delta\theta)$

Stress $= Y \times$ strain $= \gamma\alpha\Delta\theta$

$\therefore$ Tension $= (A)$ (stress) $= YA\alpha\Delta\theta$

Substituting the value, we get

$$T = YA\alpha\Delta\theta$$

$$= (2.0 \times 10^{11})(2 \times 10^{-6})(1.2 \times 10^{-5})(40)$$

$$= 192 \text{ N}$$

23. Change in weight = upthrust on 100% volume

$$\therefore \quad \frac{\Delta W'}{\Delta W} = \frac{F'}{F} = \left(\frac{1 + \gamma_s\,\Delta\theta}{1 + \gamma_l\,\Delta\theta}\right)$$

$$\text{or} \quad \frac{(50 - 45.1)}{(50 - 45)} = \frac{1 + (12 \times 10^{-6})(75)}{1 + \gamma_l\,(75)}$$

Solving this equation, we get

$$\gamma_l = 3.1 \times 10^{-4} \text{ per}^\circ\text{C}$$

24. (a) $n = \dfrac{pV}{RT} = \dfrac{(1.52 \times 10^6)(10^{-2})}{(8.31)(298.15)}$

$$= 6.135$$

(b) $\rho = \dfrac{pM}{RT} = \dfrac{(1.52 \times 10^6)(2 \times 10^{-3})}{8.31 \times 298.15}$

$$= 1.24 \text{ kg/m}^3$$

(c) $\rho \propto M$ (if P and T are same)

$$\therefore \quad \frac{\rho_{O_2}}{\rho_{H_2}} = \frac{M_{O_2}}{M_{H_2}}$$

$$\therefore \quad \rho_{O_2} = \left(\frac{M_{O_2}}{M_{H_2}}\right)\rho_{H_2}$$

$$= \left(\frac{32}{2}\right)(1.23)$$

$$= 19.68 \text{ kg/m}^3$$

25. If T = constant, then

$$p_1V_1 = p_2V_2$$

$$\therefore \quad p_2 = p_1\left(\frac{V_1}{V_2}\right) = (1 \text{ atm}) = \left(\frac{70}{6}\right)$$

$$= 11.7 \text{ atm}$$

26. $n_1 = n_2$

$$\therefore \quad \frac{p_1V_1}{RT_1} = \frac{p_2V_2}{RT_2}$$

$$\therefore \quad V_2 = \left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)V_1 = \left(\frac{1}{0.5}\right)\left(\frac{270}{300}\right)(500)$$

$$= 900 \text{ m}^3$$

27. p = constant

$$\therefore \quad V \propto T$$

$$\text{or} \quad h \propto T$$

$$\therefore \quad \frac{h_f}{T_f} = \frac{h_i}{T_i}$$

$$\text{or} \quad h_f = \left(\frac{T_f}{T_i}\right)h_i$$

$$= \left(\frac{273 + 100}{273 + 20}\right)(4) = 5.09 \text{ cm}$$

28. $n_1 = \dfrac{0.025}{28} = 8.93 \times 10^{-4}$

$$n_2 = \frac{0.04}{4} = 0.01$$

$$\therefore \quad \frac{n_1}{n_2} = 0.089$$

In equilibrium p and T are same.

$$\therefore \quad pV = nRT$$

$$\therefore \quad V \propto n \quad \text{or} \quad L \propto n$$

$$\therefore \quad \frac{L_1}{L_2} = \frac{n_1}{n_2} = 0.089$$

29. $n = n_1 + n_2$

$$\frac{pV}{RT} = \frac{p_1V_1}{RT} + \frac{p_2V_2}{RT}$$

$$\therefore \quad p = \frac{p_1V_1 + p_2V_2}{V}$$

$$= \frac{(0.11)(1.38) + (0.16)(0.69)}{0.11 + 0.16}$$

$$= 0.97 \text{ MPa}$$

30. $(n_1 + n_2)_1 = (n_1 + n_2)f$

$$\therefore \quad \frac{p_iV_1}{RT_i} + \frac{p_iV_2}{RT_i} = \frac{pV_1}{RT_1} + \frac{pV_2}{RT_2}$$

$$\therefore \quad P = \frac{P\,(V_1 + V_2)}{T_i\,(V_1/T_1 + V_2/T_2)}$$

$$= \frac{(1 \text{ atm})(600)}{(293)\left(\dfrac{400}{373} + \dfrac{200}{273}\right)}$$

$$= 1.13 \text{ atm}$$

31. $V = \dfrac{nRT}{p} = \dfrac{(1)\,(8.31)\,(273.15)}{1.013\times 10^{5}}$

$$= 0.0224\ \text{m}^3 = 22.4\ \text{L}$$

32. $n_1 = n_2$

$$\therefore\quad \frac{p_1V_1}{RT_1} = \frac{p_2V_2}{RT_2} \quad\text{or}\quad p_2 = \left(\frac{V_1}{V_2}\right)\left(\frac{T_2}{T_1}\right)p_1$$

$$= \left(\frac{0.75}{0.48}\right)\left(\frac{273+157}{273+27}\right)(1.5\times 10^5)$$

$$= 3.36\times 10^5\ \text{Pa}$$

33. $n_1 =$ number of moles of $N_2 = \dfrac{1.4}{28} = 0.05$

30% of the gas dissociates into atoms.

$$\therefore\quad n_1' = 2\,(0.3)\,(0.05) + (0.7)\,(0.05) = 0.065$$

$$n_2 = \text{number of moles of He} = \frac{0.4}{4} = 0.1$$

$$\therefore\quad n = n_1' + n_2 = 0.165$$

Now, $P = \dfrac{nRT}{V} = \dfrac{(0.165)\,(8.31)\,(1500)}{(5\times 10^{-3})}$

$$= 4.1\times 10^5\ \text{N/m}^2$$

34. In diatomic gas rotational degree of freedom is two.

$\therefore$ Rotational KE of one molecule

$$\frac{1}{2}I\,\omega^2 = \frac{f}{2}kT = kT \qquad (\text{as } f = 2)$$

$$\therefore\quad \omega = \sqrt{\frac{2kT}{I}}$$

$$= \sqrt{\frac{2\times 1.38\times 10^{-23}\times 300}{8.28\times 10^{-38}\times 10^{-9}}}$$

$$= 10^{13}\ \text{rad/s}$$

35. (a) $C_p = (1 + f/2)\,R$

$$29 = (1 + f/2)\,8.31$$

Solving, we get

$$f = 5$$

(b) $pT =$ constant

$$\therefore\quad p\,(pV) = \text{constant}$$

$$\text{or}\quad pV^{1/2} = \text{constant}$$

C in the process $pV^x =$ constant is given by

$$C = C_V + \frac{R}{1-x} = \frac{f}{2}R + \frac{R}{1-x}$$

$$\therefore\quad 29 = 8.31\left(\frac{f}{2} + \frac{1}{1-1/2}\right)$$

Solving, we get $\quad f = 3$

36. (a) Average translational kinetic energy of any type of molecules is

$$\frac{3}{2}kT = \frac{3}{2}\,(1.38\times 10^{-23})\,(300)$$

$$= 6.21\times 10^{-21}\,\text{J}$$

(b) $\dfrac{2}{5}$ energy is associated with rotation. Hence, it is diatomic gas.

$$Q = nC_V\Delta T = (1)\left(\frac{5}{2}R\right)(1)$$

$$= \frac{5}{3}\times 8.31 = 20.8\ \text{J}$$

37. $C_p = \dfrac{n_1C_{p_1} + n_2C_{p_2}}{n_1 + n_2}$

$$= \frac{2.5R + 3.5R}{1+1} = 3R$$

$$C_V = \frac{n_1C_{V_1} + n_2C_{V_2}}{n_1 + n_2}$$

$$= \frac{1.5R + 2.5R}{2} = 2R$$

$$\gamma = \frac{C_p}{C_V} = \frac{3}{2} = 1.5$$

38. $pV^{-b} =$ constant

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-(-b)}$$

$$C = 0\ ,\ \text{if}\ \gamma - 1 = -\,(1 + b)$$

$$\therefore\quad b = -\,\gamma$$

39. $pV^{-1} =$ constant

$$\therefore\quad C = C_V + \frac{R}{1-x}$$

Here $\qquad x = -1$

$$\therefore\quad C = C_V + \frac{R}{2}$$

40. $K = \dfrac{3}{2}kT \;\Rightarrow\; T = \dfrac{2K}{3k}$

Now, $n = \dfrac{pV}{RT} = \dfrac{3kpV}{2KR}$

$$= \frac{3\,(200\times 10^3)\,(0.1\times 10^{-3})\,(1.38\times 10^{-23})}{2\times 6\times 10^{-21}\times 8.31}$$

$$= 8.3\times 10^{-3}\ \text{mol}$$

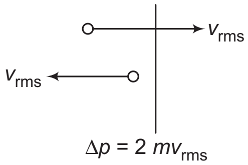
Now, $N = nN_A$

$$= (8.3\times 10^{-3})\,(6.02\times 10^{23})$$

$$= 5\times 10^{21}$$

41. $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times (273 + 57)}{46 \times 10^{-3}}}$

$= 423 \text{ m/s}$



Mass of one gas molecule,

$$m = \frac{46}{6.02 \times 10^{23}} \text{ g}$$

$$= 7.64 \times 10^{-26} \text{ kg}$$

Let n molecules strike per second per unit area. Then,

$$\text{Pressure} = \frac{F}{A} = \frac{\Delta P/\Delta t}{A} = 2mn\, v_{\text{rms}}$$

$$\therefore \quad n = \frac{\text{Pressure}}{(2m)\, v_{\text{rms}}} = \frac{2 \times 1.013 \times 10^5}{2 \times 7.64 \times 10^{-26} \times 423}$$

$$= 3.1 \times 10^{27}$$

42. $K_T = \dfrac{3}{2} kT = \dfrac{3}{2} k \left(\dfrac{pV}{nR}\right)$

$$= \frac{3}{2} k \frac{pV}{(M/mN_A)\, R}$$

$$= \frac{3kPV\ m\ N_A}{2MR}$$

$$= \frac{(3 \times 1.38 \times 10^{-23})\,(100 \times 10^3)\,(2 \times 10^{-6})\,(8.0 \times 10^{-23})\,(6.02 \times 10^{23})}{2 \times (50 \times 10^{-3})\,(8.31)}$$

$$= 4.8 \times 10^{-22} \text{J}$$

43. (a) $v_{\text{rms}} \propto \sqrt{T}$

$$\therefore \quad v_0' = \left(\sqrt{\frac{T'}{T}}\right) v_0 = \left(\sqrt{\frac{573}{293}}\right) v_0$$

$$= 1.40\, v_0$$

(b) v_{rms} does not depend on pressure, as long as temperature remains constant.

(c) $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}}$ or $v_{\text{rms}} \propto \dfrac{1}{\sqrt{M}}$

44. (a) $K_T = \dfrac{3}{2} kT$

$$= \frac{3}{2} (1.38 \times 10^{-23})\,(300)$$

$$= 6.21 \times 10^{-21} \text{J}$$

(b) K_T in one mole $= \dfrac{3}{2} RT$

$$= \frac{3}{2} \times 8.31 \times 300$$

$$= 3740 \text{ J}$$

(c) $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 300}{32 \times 10^{-3}}}$

$$= 484 \text{ m/s}$$

45. Let mass of nitrogen = (m) g. Then, mass of oxygen = $(100 - m)$ g. Number of moles of nitrogen, $n_1 = \dfrac{m}{28}$ and number of moles of oxygen

$$n_2 = \left(\frac{100 - m}{32}\right)$$

For air

$$\rho = \frac{pM}{RT} = \frac{p}{RT} \left(\frac{m_1 + m_2}{n_1 + n_2}\right)$$

$$\therefore \quad 1.284 = \frac{1.01 \times 10^5}{8.31 \times 273} \left[\frac{100 \times 10^{-3}}{(m/28) + (100 - m)/32}\right]$$

Solving this equation, we get

$$m = 76.5 \text{ g}$$

This is also percentage of N_2 by mass on air as total mass we have taken is 100 g.

46. $n_1 = n_2$

$$\therefore \quad \frac{p_1 V_1}{RT_1} = \frac{p_2 V_2}{RT_2}$$

$$\therefore \quad V_2 = \frac{p_1 V_1 T_2}{p_2 T_1} = \frac{(p_0 + \rho gh)\, V_1\, T_2}{p_0 T_1}$$

$$= \frac{(1.01 \times 10^5 + 10^3 \times 10 \times 40)\,(20)\,(273 + 20)}{1.01 \times 10^5\,(273 + 4)}$$

$$\approx 105 \text{ cm}^3$$

47. (a) Molar heat capacity $C_p = \dfrac{Q}{n\Delta T}$

and heat capacity $C_p = \dfrac{Q}{\Delta T}$

$$\therefore \quad C_p = nC_P$$

Similarly, $C_v = nC_V$

Now, $C_p - C_v = n\,(C_P - C_V) - nR$

$$\therefore \quad n = \frac{C_p - C_v}{R} = \frac{29.1}{8.31} = 3.5$$

(b) $C_p = nC_P = n\left(\dfrac{5}{2} R\right)$

$$= (3.5)\,(2.5)\,(8.31)$$

$$= 72.75 \text{ J/K}$$

$$C_v = nC_V = n\left(\frac{3}{2}R\right)$$

$$= (3.5)\,(1.5)\,(8.31)$$

$$= 43.65 \text{ J/K}$$

(c) $C_p = nC_P = n\left(\frac{7}{2}R\right)$

$$= (3.5)\,(3.5)\,(8.31)$$

$$= 101.8 \text{ J/ K}$$

$$C_v = nC_V = n\left(\frac{5}{2}R\right)$$

$$= (3.5)\,(2.5)\,(8.31)$$

$$= 72.75 \text{ J/K}$$

48. (a) As discussed in the above problem,

$$C_v = nC_V$$

$$\therefore \quad n = \frac{C_v}{C_V} = \frac{C_v}{(3/2)R}$$

$$= \frac{35}{1.5 \times 8.31} = 2.81$$

(b) Internal energy,

$$U = n\left(\frac{f}{2}RT\right)$$

$$f = \text{degree of freedom} = 3$$

$$\therefore \quad U = (2.81)\left(\frac{3}{2}\right)(8.31)\,(273)$$

$$= 9562 \text{ J} = 9.56 \text{ kJ}$$

(c) $C_P = \frac{5}{2}R = \left(\frac{5}{2}\right)(8.31)$

$$= 20.8 \text{ J/mol-K}$$

LEVEL 2

Single Correct Option

1. $(n_1 + n_2)_i = (n_1 + n_2)_f$

$$\therefore \quad \frac{p_1V_1}{RT_1} + \frac{p_2V_2}{RT_2} = \frac{p\,(V_1 + V_2)}{RT}$$

Solving, we get $\quad T = \dfrac{T_1T_2p\,(V_1 + V_2)}{p_1V_1T_2 + p_2V_2T_1}$

2. $\Delta l = l\alpha\Delta\theta$

$$\therefore \quad \alpha = \frac{\Delta l}{l\,\Delta\theta} = \frac{0.008}{(10)\,(100)} = 8 \times 10^{-6} \text{ per °C}$$

$$\gamma = 3\alpha = 2.4 \times 10^{-5} \text{ per°C}$$

$$\Delta V = V\gamma\,\Delta\theta = (100)\,(2.4 \times 10^{-5})\,(100)$$

$$= 0.24 \text{ cc}$$

$$\therefore \quad V' = V + \Delta V = 100.24 \text{ cc}$$

3. $T = mV + C$

$$\therefore \quad \frac{pV}{nR} = mV + C$$

or $\quad p = nR\left(m + \dfrac{C}{V}\right)$

$\therefore$ p-V graph is as shown in graph (a).

4. $p^2V = \text{constant}$

$$\therefore \quad \left(\frac{nRT}{V}\right)^2 V = \text{constant}$$

$$\therefore \quad T^2 \propto V$$

or $\quad T \propto \sqrt{V}$

V is made three times. So, T will become $\sqrt{3}$ times.

5. $n = \dfrac{pV}{RT}$

$$\therefore \quad n \propto \frac{1}{T} \quad \text{or} \quad m \propto \frac{1}{T}$$

$$\frac{m_1}{m_2} = \frac{T_2}{T_1} = \frac{273 + 37}{273 + 7}$$

$$= \frac{310}{280} = 1.1$$

6. $v_{\text{av}} = \sqrt{\dfrac{8RT}{\pi M}}$

T and M are same. Hence, average speed of O_2 molecules in both vessels will remain same or v_1.

7. $n = \dfrac{pV}{RT}$

$$\therefore \quad N = nN_A = \frac{pVN_A}{RT}$$

$$= \frac{(1.013 \times 10^5 \times 10^{-13})\,(10^{-6})\,(6.02 \times 10^{23})}{(8.31)\,(300)}$$

$$= 2.4 \times 10^6$$

8. $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 273}{28 \times 10^{-3}}}$

$$= 493 \text{ m/s}$$

Now, increase in gravitation PE

= decrease in kinetic energy

$$\therefore \quad \frac{(mg\,h)}{1 + \dfrac{h}{R}} = \frac{1}{2}\,m\,v_{\text{rms}}^2 \quad \text{or} \quad \frac{2gh}{1 + (h/R)} = v_{\text{rms}}^2$$

or $\quad \dfrac{2 \times 9.81 \times h}{1 + (h/6400 \times 10^3)} = (493)^2$

Solving this equation, we get

$$h \approx 12000 \text{ m}$$

or $\quad$ 12 km

9. $U \propto T$

According to given graph,

$$p \propto V$$

$$\therefore \quad p \propto T$$

Hence, $\quad V = \text{constant}$

10. 28 g of N_2 mean 1 g-mol

Now, $\quad n = \dfrac{pV}{RT} \quad$ or $\quad n \propto \dfrac{p}{T}$

$$\frac{n_f}{n_i} = \left(\frac{p_f}{p_i}\right)\left(\frac{T_i}{T_f}\right) = \left(\frac{1}{2}\right)\left(\frac{273+57}{273+27}\right) = \frac{11}{20}$$

or $\quad n_f = \left(\dfrac{11}{20}\right) n_i = \dfrac{11}{20}$ mol

$$\Delta n = n_i - n_f = 1 - \frac{11}{20} = \frac{9}{20} \text{ g-mol}$$

$$\Delta m = \frac{9}{20} \times 28 \text{ g} = \frac{63}{5} \text{ g}$$

11. 4 g H_2 means 2 g-moles and 8 g He means 2 g-moles.

Now, $\quad M = \dfrac{n_1 M_1 + n_2 M_2}{n_1 + n_2} = \dfrac{(2)(2) + (2)(4)}{2+2}$

$$= 3 \text{ g/mol}$$

$$\rho = \frac{PM}{RT} = \frac{(1.013 \times 10^5)(3 \times 10^{-3})}{(8.31)(273)} = 0.13 \text{ kg/m}^3$$

12. $\rho = \dfrac{1}{V}$

Given, $\quad p \propto K\rho$

$$\therefore \quad \rho \propto \frac{1}{V}$$

or $\quad pV = \text{constant}$

$$\therefore \quad T = \text{constant}$$

More than One Correct Options

1. $p \propto \sqrt{\rho}$ as per process. If ρ becomes half, then p will becomes $\sqrt{2}$ times.

Further, $\quad \rho \propto \dfrac{1}{V}$

$$\therefore \quad \frac{p^2}{(1/V)} = \text{constant}$$

or $\quad (pV)\,p = \text{constant}$

Hence, $\quad Tp = \text{constant} \qquad (\text{as } pV \propto T)$

$$\therefore \quad p \propto \frac{1}{T}$$

or P - T graph is a rectangular hyperbola.

2. $p \propto \dfrac{1}{\sqrt{V}}$

If volume becomes 4 times than p will remain half

$$V \propto \frac{T}{p}$$

$$\therefore \quad Vp^2 = \text{constant}$$

$$\therefore \quad \left(\frac{T}{P}\right) p^2 = \text{constant}$$

or $\quad pT = \text{constant}$

or $\quad p \propto \dfrac{1}{T}$

i.e. p - T graph is a rectangular hyperbola.

If p is halved than T will becomes two times.

4. $\rho \propto \dfrac{1}{V}$

V is doubled so ρ will remain half.

$$U \propto T \propto pV$$

According to given graph, $p \propto V$

$$\therefore \quad U \propto T \propto p^2 \text{ or } V^2$$

V is doubled, so U and T both will become four times.

$$p \propto V$$

$$\therefore \quad \frac{T}{V} \propto V \qquad \left(\text{as } p \propto \frac{T}{V}\right)$$

$$\therefore \quad T \propto V^2$$

or T - V graph is a parabola passing through origin.

6. $V = \dfrac{nRT}{p} = \dfrac{mRT}{Mp} = \left(\dfrac{mR}{Mp}\right) T$

i.e. V - T graph is a straight line passing through origin of slope

$$= \left(\frac{mR}{Mp}\right) \Rightarrow \text{Slope} \propto \frac{m}{p}$$

Hence, slope depends on both m and p.

7. (a) $p = \dfrac{nRT}{V} = \dfrac{(m/V)RT}{M}$

So, given m in the question is mass of gas per unit volume.

(b) m = total mass of gas

(c) m = mass of one molecule of gas.

Match the Columns

1. (a) $K_T = \dfrac{n f_T}{2} RT = \dfrac{(2)(3)\,RT}{2} = 3RT$

(b) $K_R = \dfrac{n f_R RT}{2} = \dfrac{(2)(2)\,RT}{2} = 2RT$

(c) Potential energy = 0

(d) $U = K_R + K_T = 5RT$

2. $U \propto T$ and $\rho \propto \frac{1}{V}$

According to given graph,

$$U \propto \rho \quad \Rightarrow \quad T \propto \frac{1}{V}$$

$$\therefore \quad pV \propto \frac{1}{V} \quad \text{or} \quad p \propto \frac{1}{V^2}$$

Density is increasing. So, volume is decreasing. Hence, pressure will increase.

U is increasing. So, temperature will also increase.

Now, $$\frac{T}{V} \propto \frac{1}{V^2}$$

Volume is decreasing. Hence, $\frac{T}{V}$ will increase.

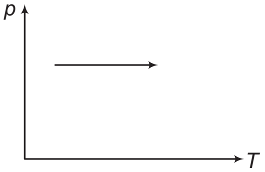
3. Speed of sound $= \sqrt{\frac{\gamma RT}{M}}$

4. (a) Density of water is maximum at 4°C. From 0°C to 4°C, density will first increase, then decrease.
(b) and (c) : Same logic as given in option (a).

(d) $T = 2\pi\sqrt{\frac{l}{g}}$ or $T \propto \sqrt{l}$

As temperature is increased, length will increase. So, time period will also increase.

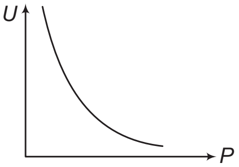
5. (a) $p =$ constant. Hence p - T graph is as shown below.



(b) $\rho = \frac{pM}{RT}$

If $p =$ constant, then

$$\rho \propto \frac{1}{T} \quad \text{or} \quad \rho \propto \frac{1}{U} \qquad (\text{as } U \propto T)$$



(c) If $p =$ constant, then

$$T \propto V$$

i.e. T - V graph is a straight line passing through origin.

(d) $\rho = \frac{pM}{RT}$

If $p =$ constant, then $\rho \propto \frac{1}{T}$

i.e. ρ - T graph is rectangular hyperbola.

Subjective Questions

1. $pV = nRT$

$$pdV = nRdT$$

$$\frac{dV}{dT} = \frac{nR}{P}, \; dV = V\gamma dT$$

$$\therefore \quad \gamma = \frac{dV/dT}{V} = \frac{nR}{PV} = \frac{1}{T}$$

2. In the process $pV^x =$ constant $C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$,

Given, $C = R$ and $\gamma = 7/5$

Substituting these values, we get $x = \frac{5}{3}$

Now, $pV^{5/3} =$ constant or $p \propto \frac{1}{(V)^{5/3}}$

By increasing volume to two times pressure will decrease $(2)^{5/3}$ times.

$$v_{\text{rms}} \propto \sqrt{T} \quad \text{or} \quad v_{\text{rms}} \propto \sqrt{pV}$$

or v_{rms} will become $\sqrt{\frac{(2)}{(2)^{5/3}}}$ times

or v_{rms} will become $(2)^{-1/3}$ times

or $\frac{1}{(2)^{1/3}}$ times.

Now, $p \propto$ (no. of collisions) (v_{rms})

$$\frac{1}{(2)^{5/3}} \propto (\text{no. of collisions}) \frac{1}{(2)^{1/3}}$$

or number of collisions will decrease $(2)^{4/3}$ times.

Ans.

3. $p_1V = n_1RT$

$$p_2V = n_2RT \qquad (T = 273 \text{ K})$$

$$\therefore \quad (p_1 - p_2)V = (n_1 - n_2)RT = \left(\frac{m_1 - m_2}{M}\right)RT$$

$$(\Delta p)V = \frac{\Delta m}{M}RT \qquad \text{...(i)}$$

In the initial condition (at STP)

$$\frac{RT}{M} = \frac{p_0}{\rho} \qquad \text{...(ii)}$$

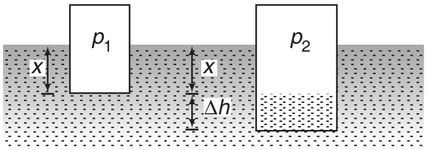
$(T = 273 \text{ K})$

From Eqs. (i) and (ii), we get

$$\Delta m = \frac{\rho V \Delta p}{P_0} = \frac{1.2 \times 0.4 \times 0.24}{1.0}$$

$$= 0.1152 \text{ kg} = 115.2 \text{ g} \qquad \textbf{Ans.}$$

4. $p_2 =$ pressure at depth x



$$p_1 V_1 = p_2 V_2$$

or $\quad p_1(Ah) = p_2(A)(h - \Delta h)$

$\therefore \quad p_1 = \dfrac{p_2(h - \Delta h)}{h} \qquad \ldots\text{(i)}$

Initially, $\quad mg = \rho g x A$

Now, $\quad p_2 = p_0 + \rho g x = p_0 + \dfrac{mg}{A}$

Substituting in Eq. (i), we have

$$p_1 = \left(p_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right) \qquad \textbf{Ans.}$$

5. $p = \dfrac{RT}{V} \qquad (n = 1)$

or $\quad p = \dfrac{R}{V}(T_0 + \alpha V^2) \qquad \ldots\text{(i)}$

For minimum attainable pressure

$$\frac{dp}{dV} = 0 \quad \text{or} \quad \frac{-RT_0}{V^2} + \alpha R = 0$$

or $\quad V = \sqrt{\dfrac{T_0}{\alpha}}$

At this volume we can see that $\dfrac{d^2p}{dV^2}$ is positive or p is minimum.

From Eq. (i)

$$p_{\min} = \frac{RT_0}{\sqrt{T_0/\alpha}} + \alpha R \sqrt{T_0/\alpha}$$

$$= 2R\sqrt{\alpha T_0} \qquad \textbf{Ans.}$$

6. At 50°C, density of solid = density of liquid.

At 0°C, fraction submerged $\left(\dfrac{\rho_s}{\rho_l}\right)_{0°\text{C}} \times 100 \qquad \ldots\text{(i)}$

$$(\rho)_{50} = \frac{\rho_0}{1 + 50\gamma}$$

$\therefore \quad \rho_0 = \rho_{50}(1 + 50\gamma)$

Substituting in Eq. (i), we have

% of fraction submerged

$$= \left(\frac{1 + 50\gamma_s}{1 + 50\gamma_l}\right) \times 100 = \left(\frac{1 + 0.3 \times 10^{-5}}{1 + 8 \times 10^{-5}}\right) \times 100$$

$$= 99.99\ \% \qquad \textbf{Ans.}$$

7. $\rho_{100} g h_{100} = \rho_0 g h_0 \quad \text{or} \quad \dfrac{\rho_{100}}{\rho_0} = \dfrac{h_0}{h_{100}}$

$$\left(\frac{1}{1 + 100\gamma}\right) = \frac{39.2}{40}$$

Solving this equation we get

$$\gamma = 2.0 \times 10^{-4} \text{ per °C} \qquad \textbf{Ans.}$$

8. Let Δl be the change in length. (let $\Delta l_s > \Delta l > \Delta l_a$)

$$\text{Strain in steel} = \frac{\Delta l_s - \Delta l}{l_0}$$

and strain in aluminium $= \dfrac{\Delta l - \Delta l_a}{l_0}$

In equilibrium, $\quad 2F_s = F_a$

$$2\left[\frac{\Delta l_s - \Delta l}{l_0}\right] Y_s A = \left[\frac{\Delta l - \Delta l_a}{l_0}\right] Y_a A$$

or $\quad 2(l_0 \alpha_s \theta - \Delta l) Y_s = (\Delta l - l_0 \alpha_a \theta) Ya$

Solving this equation, we get

$$\Delta l = \left(\frac{2\alpha_s Y_s + \alpha_a Y_a}{2Ys + Y_a}\right) l_0 \theta$$

$\therefore$ Total length

$$= l_0 + \Delta l = l_0 \left[1 + \left(\frac{\alpha_a Y_a + 2\alpha_s Y_s}{2Y_s + Y_a}\right)\theta\right] \qquad \textbf{Ans.}$$

9. From $\Delta l = l \alpha \Delta\theta$ we have,

$$0.05 = 25\alpha_A (100)$$

$\therefore \quad \alpha_A = 0.00002 \text{ per°C}$

$$0.04 = 40\,\alpha_B (100)$$

$$\alpha_B = 0.00001 \text{ per°C}$$

In third case let l is the length of rod A. Then length of rod B will be $(50 - l)$ cm.

$$\Delta l = \Delta l_1 + \Delta l_2$$

or $\quad 0.03 = l(0.00002)(50) + (50 - l)(0.00001)(50)$

Solving we get, $l = 10$ cm and $50 - l = 40$ cm

Ans.

21. Laws of Thermodynamics

INTRODUCTORY EXERCISE 21.1

1. ΔU will be same along all paths. Then, we can apply $Q = W + \Delta U$ for all.

2. $\Delta U = Q - W$

$$= 254 - (-73) = 327 \text{ J}$$

INTRODUCTORY EXERCISE 21.2

1. (a) Work done by gas under constant pressure is

$$W = p(V_f + V_i)$$
$$= (1.7 \times 10^5)(0.8 - 1.2)$$
$$= -6.8 \times 10^4 \text{ J}$$

(b) $$Q = W + \Delta U$$
$$= -6.8 \times 10^4 - 1.1 \times 10^5$$

or $|Q| = 1.78 \times 10^5 \text{J}$

As Q is negative. So, net heat flow is out of the gas.

2. (a) $W_1 = +\text{ve}$, as cycle is clockwise

$W_2 = -\text{ve}$ as cycle is anti-clockwise

Since, $|W_1| > |W_2|$

$\therefore$ Net work done by the system is positive.

(b) In cyclic process, $Q_{\text{net}} = W_{\text{net}}$

Since, W_{net} is positive. So, Q_{net} is also positive. So, heat flows into the system.

(c) $W_1 = Q_1 = +\text{ ve}$, so into the system.

$W_2 = Q_2 = -\text{ve}$, so out of the system.

3. Helium is monoatomic. So, degree of freedom $f = 3$.

$$U = \frac{nf}{2}RT = \frac{3n}{2}RT$$

or $$n = \frac{2U}{3RT} = \frac{2 \times 100}{3 \times 8.31 \times 300}$$

$$= 2.67 \times 10^{-2} \text{mol}$$ **Ans.**

4. In isochoric process,

$$W = 0$$
$$Q = \Delta U = nC_V \Delta T = 4\left(\frac{3}{2}R\right)(300)$$
$$= 1800\,R$$

5. W = Area under p-V diagram

Further $p \propto V$ along AB.

Therefore, $p_B = 2p_A = 2p_0$

6. VT = constant

or $V(pV)$ = constant (as $T \propto pV$)

$\therefore$ pV^2 = constant

Comparing with pV^x = constant, we have

$$x = 2$$
$$W = \frac{nR\Delta T}{1 - x} \quad \text{(Table 21.3)}$$
$$= \frac{(2)(R)(300)}{1 - 2}$$
$$= -600\,R$$

7. $pV^2 = K = p_0V_0^2$

$\therefore$ $$p = \frac{K}{V^2}$$

$$W = \int p\,dV = \int_{V_0}^{2V_0} \frac{K}{V^2}\,dV$$
$$= \left[-\frac{K}{V}\right]_{V_0}^{2V_0}$$
$$= \frac{K}{V_0} - \frac{K}{2V_0}$$
$$= \frac{K}{2V_0}$$

Substituting, $K = p_0V_0^2$, we have

$$W = \frac{1}{2}p_0V_0$$

INTRODUCTORY EXERCISE 21.3

1. (a) $Q = nC_V\,\Delta T$

$$200 = (1)\left(\frac{3}{2}R\right)(T_f - 300)$$

or $200 = 1.5 \times 8.31(T_f - 300)$

Solving we get,

$$T_f = 316 \text{ K}$$

(b) $Q = nC_p\,\Delta T$

$$200 = (1)\left(\frac{5}{2}R\right)(T_f - 300)$$

or $200 = 2.5 \times 8.31\,(T_f - 300)$

Solving we get,

$$T_f = 309.6 \text{ K}$$

2. p = constant

$\therefore$ $$T \propto V$$

As the gas expands V increases. So, T also increases. Hence, ΔT is positive. Therefore, in the expression,

$$Q = nC_p \Delta T$$

Q is positive.

3. ***AB***

V is increasing, so $W = +$ ve. Product of pV and therefore T and therefore U is increasing.

So, ΔU is also + ve.

$$Q = W + \Delta U$$

$\therefore$ Q is also positive.

BC $\quad V =$ constant

$\therefore \quad W = 0$

$\therefore \quad Q = \Delta U = -$ ve

CA

V is decreasing.

$\therefore \quad W = -$ ve

ΔU is also negative.

$\therefore \quad Q = W + \Delta U$ is also negative.

4. $Q = 0$ as the box is insulated.

$W_{\text{gas}} = 0$ as on the right hand side there is vacuum.

$\therefore \quad \Delta U = Q - W = 0$

$\therefore \quad \Delta T$ is also zero,

as $\quad U \propto T$

5. In an isobaric process $p =$ constant.

Therefore, $\quad C = C_p$.

Now, $$\frac{\Delta Q}{\Delta U} = \frac{nC_p\Delta T}{nC_V\Delta T} = \frac{C_p}{C_V} = \gamma$$

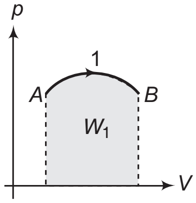
and $$\frac{\Delta Q}{\Delta W} = \frac{\Delta Q}{\Delta Q - \Delta U}$$

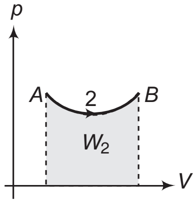
$$= \frac{nC_p\Delta T}{nC_p\Delta T - nC_V\Delta T}$$

$$= \frac{C_p}{C_p - C_V} = \frac{C_p/C_V}{C_p/C_V - 1}$$

$$= \frac{\gamma}{\gamma - 1}$$

6. $Q_1 = W_1 + \Delta U_1$ and $Q_2 = W_2 + \Delta U_2$





U is a state function. Hence, ΔU depends only on the initial and final positions. Therefore,

$$\Delta U_1 = \Delta U_2$$

But, $\quad W_1 > W_2$

as the area under 1 is greater than area under 2.

Hence,

$$Q_1 > Q_2$$

7. $$W = \int_{V_i}^{V_f} p\,dV = \int_1^2 (\alpha V^2)\,dV$$

$$= \int_1^2 (5 \times 1.01 \times 10^5)\,V^2 dV$$

Solving we get,

$$W = 1.18 \times 10^6 \text{ J}$$

$$= 1.18 \text{ MJ}$$

8. $W = nR\,\Delta T$ (in isobaric process)

$$= (1)(8.31)(72)$$

$$= 600 \text{ J} = 0.6 \text{ kJ}$$

$$\Delta U = Q - W = 1.0 \text{ kJ}$$

$$\gamma = \frac{C_p}{C_V} = \frac{n\,C_p\,\Delta T}{n\,C_V\,\Delta T} = \frac{Q}{\Delta U}$$

$$= \frac{1.6}{1.0} = 1.6$$

INTRODUCTORY EXERCISE 21.4

1. Given, $\quad Q_1 = 10^6$ cal

$$T_1 = (827 + 273) = 1100\text{K}$$

and $$T_2 = (27 + 273) = 300 \text{ K}$$

as, $$\frac{Q_2}{Q_1} = \frac{T_2}{T_1} \quad \text{(in Carnot engine)}$$

$\therefore$ $$Q_2 = \frac{T_2}{T_1} \cdot Q_1 = \left(\frac{300}{1100}\right)(10^6)$$

$$= 2.72 \times 10^5 \text{ cal}$$ **Ans.**

Efficiency of the cycle,

$$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

or $$\eta = \left(1 - \frac{300}{1100}\right) \times 100$$

$$= 72.72\%$$ **Ans.**

2. $T_1 = 2100$ K, $T_2 = 700$ K, $\eta_{\text{actual}} = 40\%$

Maximum possible efficiency of Carnot engine is

$$\eta_{\max} = 1 - \frac{T_2}{T_1} = 1 - \frac{700}{2100}$$

$$= 0.666 = 66.6\%$$

Actual efficiency as the percentage of the maximum possible efficiency is

$$\frac{\eta_{\text{actual}}}{\eta_{\max}}\times 100=\frac{40}{66.6}\times 100$$

$$= 60\% \qquad \textbf{Ans.}$$

3. (a) $\eta = 1-\frac{T_2}{T_1} = 1-\frac{375}{500} = 0.25 = 25\%$

(b) $W = \eta Q_1 = 0.25\times 25\times 10^5$

$= 6.25\times 10^5$ J

(c) $Q_2 = Q_1 - W = (25-6.25)\times 10^5$

$= 18.75\times 10^5$ J

4. $T_1 = 627+273 = 900$ K

$T_2 = 27+273 = 300$ K

$$Q_1 = 3\times 10^6 \text{ cal}$$

$$\eta = \left(1-\frac{T_2}{T_1}\right) = \left(1-\frac{Q_2}{Q_1}\right)$$

$$\therefore \quad \frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

$$\therefore \quad Q_2 = \frac{T_2}{T_1}\times Q_1$$

$$= \frac{3\times 10^6\times 300}{900} = 10^6 \text{ cal}$$

Work done by the engine,

$$W = Q_1 - Q_2 = 3\times 10^6 - 10^6 \text{ cal}$$

$$= 2\times 10^6 \text{ cal}$$

5. $\eta_1 = 1-\frac{T_2}{T_1}$ or $\frac{1}{6} = 1-\frac{T_2}{T_1}$ ...(i)

In the second case, the temperature of the sink is reduced by 65°C. Hence,

$$\eta_2 = 1-\frac{T_2-65}{T_1}$$

or $\frac{1}{3} = 1-\frac{T_2-65}{T_1}$ ...(ii)

Solving Eqs. (i) and (ii), we get

$T_1 = 390$ K $= 117$°C,

$T_2 = 325$ K

$= 52$°C

6. Coefficient of performance is given by

$$\beta = \frac{T_2}{T_1-T_2}$$

$$\beta_A = \frac{(-10+273)}{(27+273)-(-10+273)}$$

$$= 7.1$$

$$\beta_B = \frac{(-27+273)}{(17+273)-(-27+273)} = 5.6$$

$\beta_A > \beta_B$

Therefore, refrigerator A is better.

7. Here, $T_1 = 25+273 = 298$ K

$T_2 = -10+273 = 263$ K

$Q_2 = 263$ J/s

Coefficient of performance is given by

$$\beta = \frac{Q_2}{Q_1-Q_2} = \frac{T_2}{T_1-T_2}$$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

$$\therefore \quad Q_1 = \frac{T_1}{T_2}\times Q_2 = \frac{298}{263}\times 263$$

$$= 298 \text{ Js}^{-1}$$

Average power consumed,

$= 298-263 = 35$ J s^{-1} $= 35$ W

8. $\eta = \frac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}}\times 100$

W_{net} = Area under the cycle

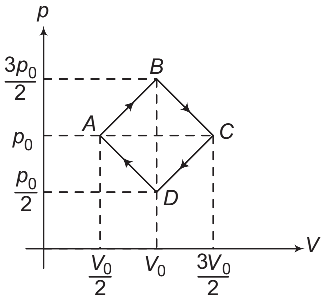
$$= \frac{p_0V_0}{2}$$

$$\Sigma Q_{+\text{ve}} = Q_{ABC}$$

$$= W_{ABC} + \Delta U_{ABC}$$

$$= (\text{Area under } ABC) + nC_V\,\Delta T$$

$$= \left(\frac{5}{4}p_0V_0\right) + n\left(\frac{3}{2}R\right)(T_C - T_A)$$



$$= \frac{5}{4}p_0V_0 + \frac{3}{2}(p_CV_C - p_AV_A)$$

$$= \frac{5}{4}p_0V_0 + \frac{3}{2}\left(\frac{3p_0V_0}{2} - \frac{p_0V_0}{2}\right)$$

$$= \frac{11}{4}p_0V_0$$

$$\therefore \quad \eta = \frac{(p_0V_0/2)}{(11p_0V_0/4)}\times 100 = 18.18\ \%$$

Exercises

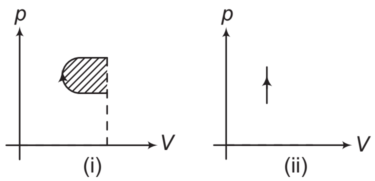
LEVEL 1

Assertion and Reason

1. $Q = 0, \Delta U = -W$

In expansion, W is positive. So, ΔU is negative. Hence, T also decreases.

2.



(i) (ii)

In both figures, $V_i = V_f$

but $W_1 =$ Hatched area

$W_2 = 0$

3. First law is just energy conservation law, which can be applied for any system.

5. $Q - W = \Delta U$ is state function.

So, it will remain constant for any two thermodynamic states.

6. For given temperatures T_1 and T_2 efficiency of a heat engine can't be greater than efficiency of Carnot engine.

7. $pT =$ constant

$$\therefore \quad \left(\frac{T}{V}\right) T = \text{constant}$$

$$\text{or} \quad V \propto T^2$$

If temperature is increased, then volume will also increase. So, work done is positive.

8. $Q = 0$ as chamber is adiabatic

$W = 0$ as expansion is taking place in vacuum.

$$\therefore \quad \Delta U = Q - W = 0$$

$$\therefore \quad U \text{ or } T = \text{constant}$$

$$\text{or} \quad pV = \text{constant}$$

$$\therefore \quad p \propto \frac{1}{V}$$

Objective Questions

1. $p \propto V^2$

$$\therefore \quad \frac{T}{V} \propto V \quad \text{or} \quad T \propto V^3 \Rightarrow V \propto T^{1/3}$$

T is increasing. So, V will also increase. Hence, work done will be positive.

2. $$W_{\text{gas}} = nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{1}{2}\right)$$

$$= -nRT \ln 2$$

$\therefore$ Work done on the gas $= +\, nRT \ln 2$

3. ΔU is same p.

$$Q = W + \Delta U$$

$$W_1 = +\text{ ve}, W_2 = 0$$

$$\text{and} \quad W_3 = -\text{ ve}$$

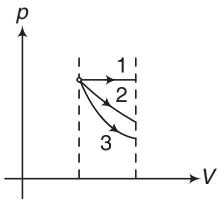
4. In adiabatic compression temperature increases. Hence, pV increases, as $T \propto pV$

5. $$W = -\pi(\text{Radius})^2$$

$$= -\pi\,(\text{Radius})\,(\text{Radius})$$

$$= -\pi\left(\frac{p_2 - p_1}{2}\right)\left(\frac{V_2 - V_1}{2}\right)$$

6.



1 → Isobaric

2 → Isothermal

3 → Adiabatic

Minimum area is under graph-3.

7. $W_1 > W_2$ as area under graph-a is more. Hence,

$$\Delta Q_1 > \Delta Q_2, \text{ as}$$

$$\Delta Q = W + \Delta U \text{ and } \Delta U_1 = \Delta U_2$$

8. $$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

$$T_2 = 300 \text{ K} \quad \text{and } T_1 = 600 \text{ K}$$

$$\therefore \quad \eta = 50\%$$

9. $$p = \frac{n\,RT}{V}$$

$V =$ constant, but temperature will decrease with time. So, p will decrease.

10. In adiabatic compression, internal energy of gas increases. So, temperature increases.

11. $$pT^{\frac{\gamma}{1-\gamma}} = \text{constant} \quad \text{or} \quad p \propto T^{\frac{\gamma}{\gamma-1}}$$

$$\therefore \quad \alpha = \frac{\gamma}{\gamma - 1} = \frac{1.4}{1.4 - 1} = 3.5$$

12. $$C = C_V + \frac{R}{1-x}$$
$$= \frac{5}{2}R + \frac{R}{1-x} = \frac{5}{2}R - \frac{R}{x-1}$$

C is negative if
$$x - 1 < \frac{2}{5}$$
$$\therefore \qquad x < 1.4$$

If x lies between 1 and 1.4, then also C is negative.

13. $$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2}$$

In option (c),
$$C_V = \frac{(2)\left(\frac{3}{2}R\right) + (4)\left(\frac{5}{2}R\right)}{6}$$
$$= \frac{13R}{6}$$

14. $Q_{\text{net}} = W_{\text{net}} =$ area under the graph
$$= \pi(\text{Radius})(\text{Radius})$$
$$= \pi(10 \times 10^3 \text{ Pa})(10 \times 10^{-3} \text{ m}^3)$$
$$= 10^2 \pi \text{ J}$$

15. Area is same, signs are opposite.

Subjective Questions

1. Helium is monoatomic. So, degree of freedom $f = 3$.
$$U = \frac{nf}{2}RT = \frac{3n}{2}RT$$
$$\text{or} \qquad n = \frac{2U}{3RT}$$
$$= \frac{2 \times 100}{3 \times 8.31 \times 300}$$
$$= 2.67 \times 10^{-2} \text{ mol}$$

2. For all process,
$$\Delta U = nC_V \Delta T$$
$$= \frac{nR\Delta T}{\gamma - 1} \qquad \left(\text{as } C_V = \frac{R}{\gamma - 1}\right)$$

3. (a) In a cyclic process,
$$Q_{\text{net}} = W_{\text{net}}$$
cyclic is clockwise. Hence, work done is positive. So, Q_{net} is also positive.

(b) $W_{\text{net}} = Q_{\text{net}} = 7200$ J

(c) In counter-clockwise direction,
$$W_{\text{net}} = Q_{\text{net}} = -\text{ ve}$$
$$|Q_{\text{net}}| = 7200 \text{ J}$$

4. $Q_{\text{net}} = W_{\text{net}} =$ area under the cycle
$$= (2 \times 10^5)(2.5 \times 10^{-6})$$
$$= 0.5 \text{ J}$$

5. (a) $W_{\text{I}} = +$ ve (as cycle is clockwise)

$W_{\text{II}} = -$ ve (as cycle is anti-clockwise)

But,
$$|W_{\text{I}}| > |W_{\text{II}}|$$
$$\therefore \qquad W_{\text{net}} = +\text{ ve}$$

(c) $$Q_{\text{net}} = W_{\text{net}} = +\text{ ve}$$

(d) In loop-I,
$$Q_{\text{I}} = W_{\text{I}} = +\text{ ve}$$
In loop-II,
$$Q_{\text{II}} = W_{\text{II}} = -\text{ ve}$$

6. Work done per cycle = area under the cycle
$$= \frac{1}{2}(3-2)(10^{-3})(30-10)(1.01 \times 10^5)$$
$$= 1010 \text{ J}$$

Total work done per second
$$= 1010 \times \frac{100}{60}$$
$$= 1683 \text{ J/s}$$

7. (a) $$Q = nC_V \,\Delta T = (1)\left(\frac{3}{2}R\right)(T_f - T_i)$$
$$\therefore \qquad 200 = 1.5 \times 8.31(T_f - 300)$$
Solving we get,
$$T_f = 316 \text{ K}$$

(b) $$Q = nC_p \,\Delta T = (1)\left(\frac{5}{2}R\right)(T_f - T_i)$$
$$\therefore \qquad 200 = 2.5 \times 8.31\,(T_f - 300)$$
$$\therefore \qquad T_f \approx 310 \text{ K}$$

8. $V =$ constant

From $pV = nRT$,
$$V\Delta p = nR(\Delta T)$$
$$Q = nC_V \,\Delta T$$
$$= n\left(\frac{5}{2}R\right)\Delta T = \frac{5}{2}(V\Delta p)$$
$$= \frac{5}{2} \times (10 \times 10^{-3})(4 \times 10^5)$$
$$= 10^4 \text{ J}$$ **Ans.**

9. (a) $W = +$ (Area of the cycle)
$$= +\frac{1}{2}(2V_0 - V_0)(3p_0 - p_0)$$
$$= p_0 V_0$$

(b) $Q_{CA} = nC_p \Delta T$

$$= n\left(\frac{5}{2}R\right)(T_A - T_C)$$

$$= \frac{5}{2}(p_AV_A - p_CV_C)$$

$$= \frac{5}{2}(p_0V_0 - 2p_0V_0) = -\frac{5}{2}\,p_0V_0$$

$$Q_{AB} = nC_V\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_B - T_A)$$

$$= \frac{3}{2}(p_BV_B - p_AV_A)$$

$$= 3p_0V_0$$

(c) $W_{BC} = +$ (Area under the graph)

$$= 2p_0V_0$$

$$\Delta U_{BC} = nC_V\,\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_C - T_B)$$

$$= \frac{3}{2}(p_CV_C - p_BV_B) = -\frac{3}{2}\,p_0V_0$$

$$\therefore \quad Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{p_0V_0}{2}$$

(d) p - V equation along path BC is

$$p = -\left(\frac{2p_0}{V_0}\right)V + 5p_0$$

or
$$pV = -\left(\frac{2p_0}{V_0}\right)V^2 + 5p_0V$$

or
$$RT = -\left(\frac{2p_0}{V_0}\right)V^2 + 5p_0V$$

$$\therefore \quad T = \frac{1}{R}\left[5p_0V - \frac{2p_0}{V_0}\cdot V^2\right] \qquad \text{...(i)}$$

For T to be maximum,

$$\frac{dT}{dV} = 0$$

$$\therefore \quad 5p_0 - \frac{4p_0}{V_0}V = 0$$

$$\therefore \quad V = \frac{5}{4}V_0$$

So at this volume, temperature is maximum. Substituting this value of V in Eq. (i), we get

$$T_{\max} = \frac{25p_0V_0}{8R}$$

10. **First process**

$$V = \text{constant}$$

$$\therefore \quad p \propto T$$

Pressure becomes three times. Therefore, temperature also becomes three times.

$$(T_f = 3T_i)$$

Second process

$$p = \text{constant}$$

$$\therefore \quad V \propto T$$

Volume is doubled. So, temperature also becomes two times. ($T_f' = 6T_i$)

Now,
$$C = \frac{Q}{n\,\Delta T} = \frac{Q_1 + Q_2}{n\,\Delta T}$$

$$= \frac{nC_V\,\Delta T_1 + nC_p\,\Delta T_2}{n\,\Delta T}$$

$$= \frac{\left(\frac{5}{2}R\right)(3T_i - T_i) + \left(\frac{7}{2}R\right)(6T_i - 3T_i)}{(6T_i - T_i)}$$

$$= 3.1\,R$$

11. **First Process**

$$V = \text{constant}$$

$$\therefore \quad T \propto p$$

Pressure becomes half. So, temperature also becomes half.

$$Q_1 = n_1C_V\Delta T = 2C_V(150 - 300)$$

$$= -300\,C_V$$

Second Process

$$p = \text{constant}$$

$$Q_2 = nC_p\,\Delta T$$

$$= 2\,C_p(300 - 150)$$

$$= 300\,C_p$$

$$Q = Q_1 + Q_2$$

$$= 300(C_p - C_V) = 300\,R$$

$$= 300 \times 8.31 = 2493 \text{ J}$$

12. $\Delta U = Q - W = nC_V\,\Delta T = n\left(\frac{3}{2}R\right)(T_f - T_i)$

$$\therefore \quad 1200 - 2100 = 5\left(\frac{3}{2} \times 8.31\right)(T_f - 400)$$

$$\therefore \quad T_f = 385 \text{ K}$$

$$\approx 113\ °\text{C}$$

13. $V_i = \dfrac{m}{\rho_i} = \dfrac{2}{999.9} = 2.0002 \times 10^{-3} \text{ m}^3$

$$V_f = \frac{m}{\rho_f} = \frac{2}{1000} = 2 \times 10^{-3}\text{m}^3$$

$$\Delta U = Q - W$$

$$= ms\,\Delta\theta - p_a\Delta V$$

$$= (2)(4200)(4) - (10)^5(-0.002 \times 10^{-3})$$

$$= 33600.2 \text{ J}$$

14. $V_i = \frac{m}{\rho_i} = \frac{0.01}{1000} = 10^{-5}\text{m}^3$

$V_f = \frac{m}{\rho_f} = \frac{0.01}{0.6} = 0.0167\text{ m}^3$

$\Delta U = Q - W$

$= ms\Delta\theta + mL - p_0\Delta V$

$= (0.01 \times 4200 \times 100) + (0.01)(2.5 \times 10)^6 - 10^5(0.0167 - 10^{-5})$

$= 27530\text{ J}$

15. (a) $W = p_0\Delta V$

$= (1.013 \times 10^5)(1671 - 1) \times 10^{-6}$

$= 169\text{ J}$

(b) $Q = mL = (10^{-3})(2.256 \times 10^6)$

$= 2256\text{ J}$

$\therefore \quad \Delta U = Q - W$

$= 2087\text{ J}$

16. (a) $W = p\Delta V = (2.30 \times 10^5)(1.2 - 1.7)$

$= -1.15 \times 10^5\text{ J}$

(b) $Q = W + \Delta U$

$= (-1.15 \times 10^5) + (-1.40 \times 10^5)$

$= -2.55 \times 10^5\text{J}$

17. (a) and (b) $(pV)_A = (pV)_C = 3p_0V_0$

$\therefore \quad T_A = T_C \quad \text{or} \quad \Delta T = 0$

$\therefore \quad \Delta U_{ABC} = 0$

$Q_{ABC} = W_{ABC} = \text{Area under the graph}$

$= -2p_0V_0$ **Ans.**

i.e. Heat is released during the process.

(c) **Process *AB***

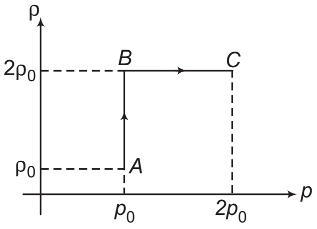
ρ - p graph will be a straight line parallel to ρ - axis because pressure is constant. Further, V is decreasing. Therefore, ρ will increase.

Process *BC*

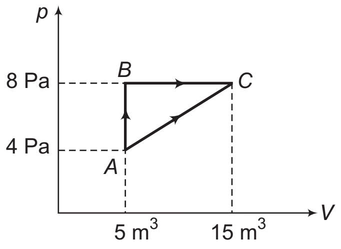
$V = \text{constant} \Rightarrow \rho = \text{constant}$

Therefore, ρ-p graph is a straight line parallel to p - axis.

ρ - p graph is as shown below.



18. (a) W_{AC} is less than W_{ABC} as area under graph is less.



(b) For process A to C,

$Q = 200\text{ J}$

Work done

$W_{AC} = \text{area under } AC$

$= \frac{1}{2}(8 + 4) \times 10$

$= 60\text{ J}$

From first law of thermodynamics,

$\Delta U = Q - W_{AC}$

$U_C - U_A = 200 - 60$

$\therefore \quad U_C = U_A + 140$

$= 10 + 140 = 150\text{ J}$ **Ans.**

(c) From A to B,

$U_B = 20\text{ J}$

$\therefore \quad \Delta U = Q - W_{AB}$

$U_B - U_A = Q - 0$

$20 - 10 = Q$

$\therefore \quad Q = 10\text{ J}$ **Ans.**

19. (a) $U_C - U_A$ along two paths should be same.

$\therefore \quad (Q_{AB} + Q_{BC}) - (W_{AB} + W_{BC}) = Q_{AC} - W_{AC}$

$\therefore \quad (250 + Q_{BC}) - (500 + 0) = 300 - 700$

Solving, we get

$Q_{BC} = -150\text{ J}$

(b) $Q_{CDA} = (W_{CD} + W_{DA}) + (U_A - U_C)$

$= (-800 + 0) - (U_C - U_A)$

$= -800 - (300 - 700)$

$= -400\text{ J}$

20. $V = \frac{m}{\rho} = \frac{1.0}{8.92 \times 10^3}$

$= 1.12 \times 10^{-4}\text{m}^3$

$\gamma = 3\alpha = 2.1 \times 10^{-5}\text{per °C}$

$\Delta V = V\gamma\,\Delta\theta$

$= (1.12 \times 10^{-4})(2.1 \times 10^{-5})(30)$

$= 7.056 \times 10^{-8}\text{ m}^3$

$$W = p\Delta V$$
$$= (1.01\times 10^{5})(7.056\times 10^{-8})$$
$$= 7.13\times 10^{-3}\,\text{J}$$
$$Q = mc\,\Delta Q$$
$$= (1)(387)(30)$$
$$\Delta U = Q - W$$
$$= 11609.99287\ \text{J}$$

21. (a) $T = \dfrac{pV}{nR} = \dfrac{(2\times 10^{5})(1.0\times 10^{-2})}{(1)(8.31)}$

$$= 240.7\ \text{K}$$

(b) In adiabatic process,

$$TV^{\gamma-1} = \text{constant}$$
$$\therefore\quad T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$
$$\therefore\quad T_f = T_i\left(\frac{V_i}{V_f}\right)^{\gamma-1}$$
$$= (240.7)\left[\frac{1.0\times 10^{-2}}{5.0\times 10^{-3}}\right]^{5/3-1}$$
$$\approx 383\ \text{K}$$

(c) Work done on the gas $= \Delta U$

$$= nC_V\,\Delta T$$
$$= (1)\left(\frac{3}{2}R\right)(T_f - T_i)$$
$$= 1.5\times 8.31(383 - 240.7)$$
$$\approx 1770\ \text{J}$$

(d) $\Delta U \approx 1770$ J

22. (a) From conservation of linear momentum,

$$0.01\times 200 = (2 + 0.01)\,v$$
$$\therefore\quad v \approx 1\ \text{m/s}$$

Mechanical energy dissipated in collision,

$$= K_i - K_f$$
$$= \frac{1}{2}\times(0.01)(200)^2 - \frac{1}{2}(2.01)(1)^2$$
$$\approx 199\ \text{J}$$

(b) $n = \dfrac{m}{M} = \dfrac{2000 + 10}{200} = 10.05$

Using,

$$Q = nC_V\,\Delta T$$
$$\therefore\quad \Delta T = \frac{Q}{nC_V} = \frac{199}{10.05\times 3\times 8.31}$$
$$= 0.80^\circ\text{C}$$

23. Along the process CD apply,

$pV =$ constant and find V_C

Similarly along path AB, again apply,

$pV =$ constant and find V_B.

Now, $W_{\text{net}} = W_{AB} + W_{BC} + W_{CD} + W_{DA}$

$$= nRT_A\ln\left(\frac{V_B}{V_A}\right) + p_B\,(V_C - V_B)$$
$$+ nRT_C\ln\left(\frac{V_D}{V_C}\right) + p_D(V_A - V_D)$$

Further, $nRT_A = p_A V_A$

and $nRT_C = p_C V_C$

24. $pV = nRT$

$$\therefore\quad p\Delta V = nR\Delta T$$

Work done under constant pressure is

$$W = p\Delta V = nR\Delta T$$
$$= (0.2)(8.31)(100) = 166.2\ \text{J}$$

LEVEL 2

Single Correct Option

1. $W = \displaystyle\int_V^{2V} p\,dV$

$$= \int_V^{2V}\left(\frac{n\,RT}{V - b}\right)dV$$
$$= nRT\,\ln\left(\frac{2V - b}{V - b}\right)$$

Put $n = 1$,

$$\therefore\quad W = RT\,\ln\left(\frac{2V - b}{V - b}\right)$$

2. **For *AB*** $\quad V = \text{constant}$

$$\therefore\quad W = 0$$

For *BC* $\quad W = n\,RT_2\,\ln\left(\dfrac{V_f}{V_i}\right)$

$$= RT_2\ln\left(\frac{V_2}{V_1}\right)$$

For *CA* $\quad V \propto T$

$$\therefore\quad p = \text{constant}$$
$$W = nR\Delta T$$
$$= R(T_1 - T_2)$$

3. $\because\quad Q = n\,C_p\,\Delta T$

$$\frac{7}{2}RT_0 = (10)\left(\frac{7}{2}R\right)(T_f - T_0)$$
$$T_f = 1.1\,T_0$$

If p = constant, then $V \propto T$

If temperature becomes 1.1 times, then volume will also become 1.1 times.

4. W_{net} = area under the cycle

$$= 2p_0V_0$$

$$Q_{+\text{ve}} = Q_{ABC} = W_{ABC} + \Delta U_{ABC}$$

$$= \text{area under the graph } + nC_V\,\Delta T$$

$$= (3p_0V_0) + n\left(\frac{3}{2}R\right)(T_C - T_A)$$

$$= (3p_0V_0) + \frac{3}{2}(p_CV_C - p_AV_A)$$

$$= 10.5\;p_0V_0$$

$$\eta = \frac{W_{\text{net}}}{Q_{+\text{ve}}} = \frac{4}{21}$$

5. $Q = W + \Delta U$

$$= \text{area under the graph} + \Delta U$$

$$= -\,p_0V_0 - \frac{3p_0V_0}{2}$$

$$= -\frac{5}{2}p_0V_0$$

6. pV^{-1} = constant

In the process, pV^x = constant

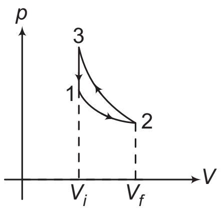
$$C = C_V + \frac{R}{1-x}$$

Here, $C = \frac{3}{2}R + \frac{R}{1+1} = 2R$

$$Q = n\,C\Delta T = (1)(2R)(2T_0 - T_0)$$

$$= 2RT_0$$

7.



In process 3 – 1

V = constant

∴ $T \propto p$

If p is decreasing, then T will also decrease.

8. $TV^{\gamma-1}$ = constant

$$\therefore \quad \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$= \left(\frac{L_2}{L_1}\right)^{5/3-1} = \left(\frac{L_2}{L_1}\right)^{2/3}$$

9. $T \propto U$

$$T_C = T_D = T_0 = 300\text{ K}$$

$$T_A = T_B = 2T_0 = 600\text{ K}$$

$$Q_{\text{net}} = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}$$

$$= nRT_A \ln\left(\frac{V_f}{V_i}\right) + nC_V(T_C - T_B)$$

$$+\, nRT_C \ln\left(\frac{V_f}{V_i}\right) + nC_V(T_A - T_D)$$

$$= (1)(R)(600)\ln(2) + (1)(C_V)(-300)$$

$$+ (1)(R)(300)\ln\left(\frac{1}{2}\right) + (1)(C_V)(300)$$

$$= 300\,R\ln 2$$

10. **1-2** $\quad T \propto V$

∴ p = constant

$$Q_{\text{net}} = W_{\text{net}} = W_{12} + W_{23} + W_{31}$$

∴ $W_{23} = Q_{\text{net}} - W_{12}$ (as $W_{31} = 0$)

$$= -\,300 - nR\Delta T$$

$$= -\,300 - 2 \times 8.31 \times 300$$

$$= -\,5286\text{ J}$$

11. $Q_A = Q_B$

$$nC_p\,\Delta T_A = nC_V\Delta T_B$$

$$\therefore \quad \Delta T_B = \frac{C_p}{C_V}\cdot\Delta T_A = \gamma\Delta T_A$$

$$= (1.4)(30) = 42\text{ K}$$

12. TV^{n-1} = constant

∴ $(pV)V^{n-1}$ = constant (as $T \propto pV$)

∴ pV^n = constant

Bulk modulus in the process pV^{γ} = constant is γp.

Hence, in the given process Bulk modulus is np.

13. pV^{-1} = constant

In the process, pV^x = constant

$$W = \left(\frac{R}{1-x}\right)\Delta T$$

Here, $x = -1$

$$\therefore \quad W = \frac{R}{2}\Delta T = \frac{R}{2}(T_2 - T_1)$$

14. $\Delta U = Q - W = 450\text{ J} = nC_V\Delta T$

$$= n\left(\frac{3}{2}R\right)\Delta T$$

$$\therefore \quad n\,\Delta T = \left(\frac{300}{R}\right)$$

Now, $$C = \frac{Q}{n\,\Delta T} = \frac{600}{300/R} = 2R$$

15. $W = p\Delta V = p_0(2V_0 - V_0) = p_0V_0$

$$\Delta U = U_f - U_i = 2p_0(2V_0) - 2p_0V_0 = 2p_0V_0$$

$\therefore \quad Q = W + \Delta U = 3p_0V_0$

16. $C = \frac{Q}{n\,\Delta T}$

or $\quad C \propto Q$

ΔU is same. But $W_1 < W_2$ as area under p-V graph is less.

Hence, $\quad Q_1 < Q_2$

or $\quad C_1 < C_2$.

$\therefore \quad \frac{C_1}{C_2} < 1$

17. $Q_{\text{net}} = W_{\text{net}}$

$Q_{AB} = Q_{CD} = 0$

$$Q_{BC} = n\,C_p\Delta T = (1)\left(\frac{5}{2}R\right)(4T - T) = 7.5RT$$

$$Q_{DA} = n\,C_p\Delta T = (1)\left(\frac{5}{2}R\right)(3T - 5T) = -5RT$$

$\therefore \quad W_{\text{net}} = Q_{\text{net}} = 7.5RT - 5RT = 2.5RT$

18. $U\rho =$ constant

$\therefore \quad T\left(\frac{1}{V}\right) =$ constant

or $\quad T \propto V$

$\therefore \quad p =$ constant

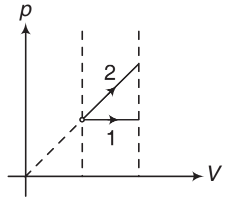
$$\frac{\Delta U}{W} = \frac{\Delta U}{Q - \Delta U} = \frac{n\,C_V\,\Delta T}{n\,C_p\Delta T - n\,C_V\Delta T} = \frac{1}{\gamma - 1} = \frac{1}{\frac{5}{3} - 1} = \frac{3}{2}$$

19. $W =$ Area under F-x graph

$= 17.5$ J

Now, $\quad Q = W + \Delta U = 20$ J

20. In second process, $p \propto V$ i.e. p-V graph is straight line passing through origin.



Area under graph-2 is more.

$\therefore \quad W_2 > W_1$

Further, $\quad T \propto pV$

$(pV)_2 > (pV)_1$

$\therefore \quad T_2 > T_1$

21. $dQ = -dU \quad$ or $\quad dU + dW = -dU$

or $\quad 2dU + dW = 0$

$\therefore \quad 2[n\,C_V dT] + pdV = 0$

or $\quad 2\left[n\left(\frac{R}{\gamma - 1}\right)dT\right] + pdV = 0$

or $\quad \frac{2nRdT}{\gamma - 1} + \left(\frac{nRT}{V}\right)dV = 0$

or $\quad \left(\frac{2}{\gamma - 1}\right)\frac{dT}{T} + \frac{dV}{V} = 0$

Integrating we get,

$$\left(\frac{2}{\gamma - 1}\right)\ln(T) + \ln(V) = \ln(C)$$

Solving we get

$$TV^{\frac{\gamma - 1}{2}} = \text{constant}$$

22. In a cyclic process,

$Q_{\text{net}} = W_{\text{net}}$

$\therefore \quad Q_{AB} + Q_{BC} + Q_{CA} =$ area under the graph

$\therefore \quad 600 + 200 + Q_{CA} = \frac{1}{2}(3 \times 10^{-4})(5 \times 10^5) = 75$ J

$\therefore\ Q_{CA} = -725\text{ J} = W_{CA} + \Delta U_{CA}$

$= -725 =$ (–Area under the graph) $+ \Delta U_{CA}$

$= -\left(\frac{1}{2} \times 11 \times 10^5\right)(3 \times 10^{-4}) + \Delta U_{CA}$

Solving we get, $\quad \Delta U_{CA} = -560$ J

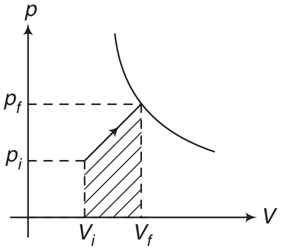
23. $T_2 = T_3 \quad$ (lying on same isotherm)

$\therefore \quad \Delta U_1 = \Delta U_2$

$W =$ Hatched area

$= \frac{1}{2}(p_i + p_f)(V_f - V_i)$

$$=\frac{1}{2}(p_iV_f - p_iV_i + p_fV_f - p_fV_i)$$



Now, p_iV_i and p_fV_f are same for both processes. Further p_i and V_i, area also same for both processes.

For $1\rightarrow 3$ V_f is more and p_f is less. Hence, W will be more.

Now, $$Q = W + \Delta U$$
$$W_{13} > W_{12}$$
$$\therefore \quad Q_{13} > Q_{12} \qquad \text{(as } \Delta U \text{ is same)}$$

24. $\Delta U_{\text{net}} = 0$

$$\therefore \quad \Delta U_1 + \Delta U_2 + \Delta U_3 = 0$$
$$\therefore \quad Q_1 + (Q_2 - W_2) + \Delta U_3 = 0$$
$$(200\text{ kJ}) + (-100 + 50)\text{ kJ} + \Delta U_3 = 0$$
$$\therefore \quad \Delta U_3 = -150\text{ kJ}$$

In adiabatic process,

$$W = -\Delta U$$
$$\therefore \quad W_3 = -\Delta U_3 = 150\text{ kJ}$$

25. $Q_{BC} = 0$

$$Q_{CD} = nC_p\Delta T$$
$$= n\left(\frac{5}{2}R\right)(T_D - T_C)$$
$$= \frac{5}{2}(p_DV_D - p_CV_C)$$
$$= \frac{5}{2}(10^5 - 2\times 10^5)$$
$$= -25\times 10^4\text{ J}$$
$$Q_{DA} = nC_V\,\Delta T$$
$$= n\left(\frac{3}{2}R\right)(T_A - T_D)$$
$$= \frac{3}{2}(p_AV_A - p_DV_D)$$
$$= \frac{3}{2}(2.4\times 10^5 - 10^5)$$
$$= 21\times 10^4\text{ J}$$

Now, $W_{\text{net}} = Q_{\text{net}}$
$$= (9 - 25 + 21)\times 10^4\text{ J} = 5\times 10^4\text{J}$$

26. Pressure becomes half. So, temperature is doubled. $\left(\text{as } T \propto \frac{1}{p}\right)$

$$\Delta U = nC_V\,\Delta T$$
$$T_i = \frac{p_0V_0}{nR}$$
$$T_f = \frac{2p_0V_0}{nR}$$
$$\therefore \quad \Delta T = T_f - T_i = \frac{p_0V_0}{nR}$$
$$\therefore \quad \Delta U = n\left(\frac{3}{2}R\right)\left(\frac{p_0V_0}{nR}\right) = \frac{3p_0V_0}{2}$$

27. $W = nRT_0\ln\left(\frac{p_i}{p_f}\right)$

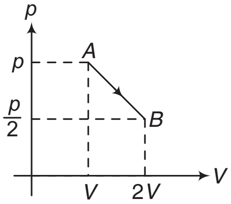
$$W_{BC} = 2\,W_{AB}$$
$$\therefore \quad nRT_0\ln\left(\frac{p_C}{p_0/2}\right) = 2nRT_0\ln\left(\frac{p_0/2}{p_0}\right)$$

Solving this equation we get,

$$p_C = \frac{p_0}{8}$$

More than One Correct Options

1.



Now, see the hint of Q-No 9 (d) of subjective questions for Level 1.

2. Temperature is increased. So, internal energy will also increase.

$$\therefore \quad \Delta U = +\text{ ve}$$

Further,

$$pV^2 = \text{constant}$$
$$\therefore \quad \left(\frac{T}{V}\right)V^2 = \text{constant} \quad \text{or} \quad V \propto \frac{1}{T}$$

Temperature is increased. So, volume will decrease and work done will be negative.

In the process pV^x = constant, molar heat capacity is given by

$$C = C_V + \frac{R}{1-x}$$

Here, $x = 2$

$$\therefore \quad C = C_V - R$$

C_V of any gas is greater than R.

So, C is positive. Hence, from the equation,

$$Q = nC\Delta T$$

Q is positive if T is increased.

or ΔT is positive.

3. ***ab*** $\quad W = 0 \qquad$ (as V = constant)

$$Q_1 = \Delta U_1 = nC_V\Delta T$$
$$= (2)\left(\frac{3}{2}R\right)(2T_0 - T_0) = 3RT_0$$

bc $\quad \Delta U = 0 \qquad$ (as T = constant)

$$\therefore \quad Q_2 = W_2 = nRT\ \ln\left(\frac{V_f}{V_i}\right)$$
$$= (2)(R)(2T_0)\ln(2)$$
$$= 4RT_0 \ln(2)$$

cd $\quad W = 0$

$$Q_3 = \Delta U_3 = nC_V\Delta T$$
$$= (2)\left(\frac{3}{2}R\right)(T_0 - 2T_0)$$
$$= -3RT_0$$

da $\quad \Delta U = 0$

$$Q_4 = W_4 = nRT\ \ln\left(\frac{V_f}{V_i}\right)$$
$$= (2)(R)(T_0)\ \ln\left(\frac{1}{2}\right)$$
$$= -2RT_0\ \ln(2)$$

Now in complete cycle,

$$\Delta U_{\text{net}} = 0$$
$$Q_{\text{net}} = W_{\text{net}} = 2RT_0\ \ln(2) = +\ \text{ve}$$

4. ***ab*** $\quad \rho$ = constant

$\therefore \quad V$ = constant

$\therefore \quad W = 0$

$$Q = \Delta U$$

ΔU is positive, as U is increasing.

Hence, Q is also positive.

bc $\quad \rho \propto U \ \Rightarrow\ \therefore\ \frac{1}{V} \propto T$

ρ is decreasing, so V is increasing. Hence, work done is positive.

Further, $\frac{1}{V} \propto T\ (T \propto pV)$

$\therefore \quad pV^2$ = constant

In the process, pV^x = constant,

Molar heat capacity is given by

$$C = C_V + \frac{R}{1-x}$$

Here, $x = 2$

$$\therefore \quad C = C_V - R$$

For any of the gas, $C_V \neq R$.

$$\therefore \quad C \neq 0$$

$\therefore\ Q = nC\Delta T \neq 0$ as $\Delta U \neq 0$ and $\Delta T \neq 0$

ca

ρ is increasing. Hence, V is decreasing. So, work done is negative.

5.
$$Q_1 = nC_V\Delta T$$
$$Q_2 = nC_p\Delta T$$
$$Q_3 = 0$$
$$C_p > C_V$$
$$\therefore \quad Q_2 > Q_1 > Q_3$$

6. **1-2** $\quad V$ = constant

$\therefore \quad p \propto T$

T is increasing. So, p will also increase.

2-3 $\quad V \propto T$

$\therefore \quad p$ = constant.

V and T both are increasing.

Process 4-1 is reverse of 2-3.

Comprehension Based Questions

1. $Q_{\text{net}} = W_{\text{net}}$ = area under the cycle $= \frac{pV}{2}$

2. $\frac{C_V}{C_p} = \frac{1}{\gamma} = \frac{5}{3}$

3. In adiabatic process,

$$p^{1-\gamma}T^{\gamma} = \text{constant} \quad \text{or} \quad T \propto p^{\frac{\gamma-1}{\gamma}}$$
$$\therefore \quad \frac{T_B}{T_A} = \left(\frac{p_B}{p_A}\right)^{\frac{\gamma-1}{\gamma}}$$
$$\therefore \quad T_B = (1000)\left(\frac{2}{3}\right)^{\frac{5/3-1}{5/3}}$$
$$= (1000)\left(\frac{2}{3}\right)^{0.4} = 850\ \text{K}$$

4. In adiabatic process,

$$W_{AB} = -\Delta U_{AB} \qquad (\text{as } Q = 0)$$
$$= nC_V(T_A - T_B)$$
$$= (1)\left(\frac{3}{2}R\right)(T_A - T_B)$$
$$= (1)\left(\frac{3}{2} \times \frac{25}{3}\right)(1000 - 850)$$
$$= 1875\ \text{J}$$

5. $W_{BC} = 0$ (as V = constant)

$\therefore \quad T \propto p$ (as V = constant)

$$p_C = p_{B/2}$$

$$\therefore \quad T_C = T_{B/2} = 425\text{ K}$$

$$Q = \Delta U = nC_V(\Delta T)$$

$$= (1)\left(\frac{3}{2}R\right)(T_C - T_B)$$

$$= \frac{3}{2} \times \frac{25}{3} \times (425 - 850)$$

$$= 5312.5\text{ J}$$

Match the Columns

1. (a) $W = nR\Delta T$

$$= (2)(R)(T) = 2RT$$

(b) $\Delta U = nC_V\Delta T$

$$= (2)\left(\frac{3}{2}R\right)(T) = 3RT$$

(c) $W = -\Delta U = -3RT$

(d) $\Delta U = 3RT$ (in all processes)

2. For both processes,

p = constant (as $V \propto T$)

$\therefore \quad C = C_p$ for same gases.

For W, ΔU and Q we will require number of moles also.

3. p^2V = constant or $pV^{1/2}$ = constant

Comparing with pV^x = constant, we get

$$x = \frac{1}{2}$$

$\therefore$ Molar heat capacity,

$$C = C_V + \frac{R}{1-x}$$

$$= \left(\frac{3}{2}R\right) + 2R = 3.5\,R$$

$$Q = nC\Delta T = n(3.5R)(\Delta T)$$

$$= 3.5\,nR\Delta T$$

$$\Delta U = nC_V\Delta T$$

$$= n\left(\frac{3}{2}R\right)\Delta T = 1.5\,nR\Delta T$$

$\therefore$ W by the gas $= Q - \Delta U = 2nR\Delta T$ and work done on the gas $= -2nR\Delta T$

4. (a) $Q = nC_p\Delta T$

$$\Delta U = nC_V\Delta T$$

$$W = Q - \Delta U = nR\Delta T$$

$$W < \Delta U \text{ as } C_V > R$$

(b) $W = 0$

$$\Delta U = -\text{ ve}$$

$$\therefore \quad Q = \Delta U = -\text{ ve}$$

(c) $Q = 0$

$W = +$ ve (as V is increasing)

$$\therefore \quad \Delta U = -W = -\text{ ve}$$

(d) $\Delta T = 0$

$$\Rightarrow \quad \Delta U = 0$$

and $Q = W = nRT\ln\left(\dfrac{V_f}{V_i}\right) = +$ ve

5. (a) W_{ab} = area under p-V graph

$$= \frac{3p_0V_0}{2} = 1.5\,p_0V_0$$

(b) $\Delta U_{ab} = Q_{ab} - W_{ab}$

$$= 6p_0V_0 - \frac{3p_0V_0}{2} = 4.5P_0V_0$$

(c) $C = \dfrac{Q}{n\Delta T} = \dfrac{W + \Delta U}{n\Delta T}$

$$= \frac{6p_0V_0}{n\left(\dfrac{p_bV_b}{nR} - \dfrac{p_aV_a}{nR}\right)} = 2R$$

(d) $C_V = \dfrac{\Delta U}{n\Delta T}$

$$= \frac{4.5p_0V_0}{n\left(\dfrac{p_bV_b}{nR} - \dfrac{p_aV_a}{nR}\right)} = 1.5R$$

Subjective Questions

1. In a cyclic process,

$$\Delta U = 0$$

$$Q_{\text{net}} = W_{\text{net}}$$

$$Q_{AB} = n\,C_p\,\Delta T$$

$$= (2)\left(\frac{5}{2}R\right)(400 - 300)$$

$$= 500R$$

$$Q_{BC} = n\,R\,T_B\ln\left(\frac{p_i}{p_f}\right)$$

$$= (2)(R)(400)\ln\left(\frac{2}{1}\right)$$

$$= 800R\ln(2)$$

$$Q_{CD} = nC_p\Delta T$$

$$= -500R$$

$$Q_{DA} = n\,RT_D\ln\left(\frac{p_i}{p_f}\right)$$

$$= (2)(R)(300)\ \ln\left(\frac{1}{2}\right)$$

$$= -600R\ \ln(2)$$

$$\therefore \quad Q_{\text{net}} = W_{\text{net}} = (200R)\ \ln(2)$$

$$= (200)(8.31)(0.693)$$

$$\approx 1153 \text{ J}$$

2. $$T_A = \frac{p_A V_A}{nR} = \frac{(1.013 \times 10^5)(22.4)}{(10^3)(8.31)}$$

$$= 273 \text{ K}$$

Along path *AB*

$$V = \text{constant}$$

$$\therefore \quad T \propto p$$

p is doubled. So, T is also doubled.

$$\therefore \quad T_B = 2T_A = 546 \text{ K.}$$

Further, $T_B = T_C$

$$\therefore \quad p_B V_B = p_C V_C$$

$$\therefore \quad V_C = \frac{(p_B)(V_A)}{(p_A)} = \frac{(2)(22.4)}{1}$$

$$= 44.8 \text{ m}^3$$

3. For Process 1 - 2 $\rho \propto p$

$$\therefore \quad \frac{1}{V} \propto p$$

$\therefore$ Process is isothermal.

$$\Delta U_1 = 0$$

$$Q_1 = W_1 = nRT\ \ln\left(\frac{p_i}{p_f}\right)$$

$$= \frac{p_0 M}{\rho_0} \ln\left(\frac{1}{2}\right) \quad (\text{as } n = 1 \text{ and } RT = \frac{PM}{\rho})$$

$$= -\frac{p_0 M}{\rho_0} \ln(2)$$

For Process 2-3

$$Q_2 = Q_p = nC_p\,\Delta T$$

$$= (1)\left(\frac{5}{2}R\right)(T_3 - T_2)$$

$$= \frac{5}{2}\left(\frac{p_3 M}{\rho_3} - \frac{p_2 M}{\rho_2}\right)$$

$$= \frac{5}{2}\left(\frac{2p_0 M}{\rho_0} - \frac{2p_0 M}{2\rho_0}\right)$$

$$= 2.5\,\frac{p_0 M}{\rho_0}$$

$$\Delta U_2 = nC_V \Delta T$$

Substituting the values like above we get,

$$\Delta U_2 = \frac{1.5 p_0 M}{\rho_0}$$

$$W_3 = Q_2 - \Delta U_2 = \frac{p_0 M}{\rho_0}$$

For Process 3-1

Density is constant. Hence,
volume is constant.

$$\therefore \quad W_3 = 0$$

$$\therefore \quad Q_3 = \Delta U_3 = nC_V \Delta T$$

$$= (1)\left(\frac{3}{2}R\right)(T_1 - T_3)$$

$$= \frac{3}{2}\left(\frac{p_0 M}{\rho_0} - \frac{2p_0 M}{\rho_0}\right)$$

$$= -\frac{1.5 p_0 M}{\rho_0}$$

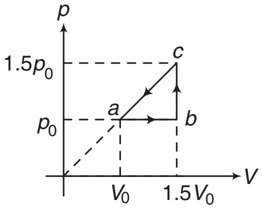
(b) $\Sigma Q_{-\text{ve}} = |Q_1 + Q_3|$

$$= \frac{p_0 M}{\rho_0}\left(\frac{3}{2} + \ln 2\right)$$

(c) $$\eta = \frac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}} = \frac{(+p_0 M/\rho_0) + \left(\dfrac{-p_0 M}{\rho_0} \ln 2\right)}{(2.5\ p_0 M/\rho_0)}$$

$$= \frac{2}{5}(1 - \ln 2)$$

4. (a)



$$W_{\text{net}} = Q_{\text{net}} \quad \text{in a cycle.}$$

Since, Q_{net} is negative (flows out of the gas). Hence, W_{net} should also be negative. Or, cycle should be anti-clockwise as shown in figure.

From *a* to *b*

$$p = \text{constant}$$

$$\therefore \quad V \propto T$$

T has become 1.5 times. Therefore,
V will also become 1.5 times.

From *c* to *a*

$$p \propto V$$

V has become $\frac{1}{1.5}$ times. Therefore,

p will also become $\frac{1}{1.5}$ times.

In a cycle

$$|Q_{\text{net}}| = |W_{\text{net}}| = \text{Area of cycle}$$

$$\therefore \quad 800 = \frac{1}{2}(0.5p_0)(0.5V_0)$$

$$\therefore \quad p_0V_0 = 6400 \text{ J}$$

$$W_{ca} = -\text{Area under the graph}$$

$$= -\frac{1}{2}(2.5p_0)(0.5V_0)$$

$$= -0.625p_0V_0$$

$$= -0.625 \times 6400 = -4000 \text{ J}$$

5. First process is isobaric.

$$\therefore \quad \Delta Q_1 = nC_V\Delta T + p\Delta V$$

Second process is isochoric

$$\therefore \quad \Delta Q_2 = nC_V\Delta T$$

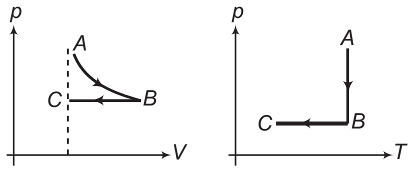
$$\Delta Q_1 - \Delta Q_2 = p\Delta V = \left[p_0 + \frac{mg}{A}\right][Ax]$$

$$= (p_0A + mg)x$$

$$= [10^5 \times 60 \times 10^{-4} + 8 \times 10](0.2)$$

$$= 136 \text{ J}$$ **Ans.**

6. (a)



(b) For the process AB,

$$\frac{p_0V_0}{T_0} = \frac{p_B(2V_0)}{T_0}$$

$$\therefore \quad p_B = \frac{p_0}{2}$$

$$\Delta U = 0$$

$$Q = W + \Delta U$$

$$= nRT \ln \frac{V_B}{V_A}$$

$$= 3RT_0 \ln 2$$

For the process BC,

$$\frac{2V_0}{T_0} = \frac{V_0}{T_C}$$

$$\therefore \quad T_C = \frac{T_0}{2}$$

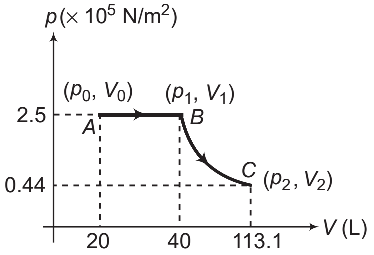
$$W = nR\Delta T = 3R\left(\frac{T_0}{2} - T_0\right) = -\frac{3}{2}RT_0$$

$$Q = nC_p\Delta T = -\frac{21}{4}RT_0$$

$$\therefore \quad W_{\text{Total}} = 3RT_0 \ln(2) - \frac{3}{2}RT_0$$

(c) $Q_{\text{Total}} = 3RT_0 \ln(2) - \dfrac{21}{4}RT_0$

7. (a)



(b) $p_0 = \dfrac{nRT_0}{V_0} = \dfrac{2 \times 8.31 \times 300}{20 \times 10^{-3}}$

$$= 2.5 \times 10^5 \text{ N/m}^2$$

$$p_1 = p_0 = 2.5 \times 10^5 \text{ N/m}^2$$

$$V_1 = 2V = 40 \times 10^{-3} \text{ m}^3$$

Process *AB*: $V \propto T$

$$\therefore \quad T_1 = 2T_0 = 600 \text{ K}$$

Process *BC*

Using $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$, we get

$$V_2 = 2\sqrt{2}V_1 = 113.1 \times 10^{-3} \text{ m}^3$$ **Ans.**

and $p_2 = \dfrac{nRT_2}{V_2} = \dfrac{(2)(8.31)(300)}{113.1 \times 10^{-3}}$

$$= 0.44 \times 10^5 \text{ N/m}^2$$ **Ans.**

(c) $W_{\text{Total}} = W_1 + W_2$

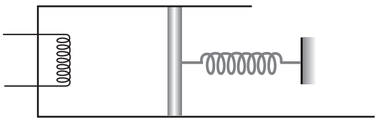
$$= p_0(V_1 - V_0) + \frac{nR}{\gamma - 1}(T_1 - T_2)$$

Substituting the values, we get

$$W_{\text{Total}} = 12479 \text{ J}$$ **Ans.**

8. (a) $\dfrac{p_1V_1}{T_1} = \dfrac{p_2V_2}{T_2}$

$$T_2 = \left(\frac{p_2V_2T_1}{p_1V_1}\right) \qquad \text{...(i)}$$



Here,

$$p_1 = 1.0 \times 10^5 \text{ N/m}^2$$

$$V_1 = 2.4 \times 10^{-3} \text{ m}^3$$

$$T_1 = 300 \text{ K}$$

$$p_2 = p_1 + \frac{kx}{A}$$

$$= 1.0 \times 10^5 + \frac{8000 \times 0.1}{8 \times 10^{-3}}$$

$$= 2.0 \times 10^5 \text{ N/m}^2$$

$$V_2 = V_1 + Ax$$

$$= 2.4 \times 10^{-3} + 8 \times 10^{-3} \times 0.1$$

$$= 3.2 \times 10^{-3} \text{ m}^3$$

Substituting in Eq. (i), we get

$$T_2 = 800 \text{ K}$$ **Ans.**

(b) Heat supplied by the heater,

$$Q = W + \Delta U$$

Here,

$$\Delta U = nC_V \, \Delta T$$

$$= \left(\frac{p_1V_1}{RT_1}\right)\left(\frac{3}{2}R\right)(800 - 300)$$

$$= \frac{(1.0 \times 10^5)(2.4 \times 10^{-3})(1.5)(500)}{(300)}$$

$$= 600 \text{ J}$$

$$W = \frac{1}{2}kx^2 + p_1\Delta V$$

$$= \frac{1}{2} \times (8000)(0.1)^2 + (1.0 \times 10^5)(0.1)(8 \times 10^{-3})$$

$$= (40 + 80) \text{ J} = 120 \text{ J}$$

$\therefore$ $Q = 600 + 120 = 720$ J **Ans.**

9. $U \propto \sqrt{V}$

As $U \propto T$

$\therefore$ $T \propto V^{1/2}$

or $TV^{-1/2} = \text{constant}$

or $pV^{1/2} = \text{constant}$

Comparing with $pV^x = \text{constant}$, we have

$$x = \frac{1}{2}$$

$\therefore$ Molar specific heat

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

$$= \frac{R}{7/5 - 1} + \frac{R}{1 - 1/2} \qquad (\gamma = 7/5)$$

$$= \frac{5}{2}R + 2R = \frac{9R}{2}$$ **Ans.**

$$Q = nC\Delta T$$

$$\Delta U = nC_V\Delta T$$

$\therefore$ $W = Q - \Delta U = n(C - C_V)\Delta T$

$$\frac{W}{\Delta U} = \frac{C - C_V}{C_V}$$

$\therefore$ $W = \left(\dfrac{C - C_V}{C_V}\right)\Delta U$

$$= \left(\frac{9/2 - 5/2}{5/2}\right)(100) = 80 \text{ J}$$ **Ans.**

10. $dQ = dU + dW$

$$CdT = C_V \, dT + pdV$$

$$(C_V + 3aT^2)\, dT = C_V \, dT + pdV$$

$\therefore$ $3aT^2 \, dT = pdV = \left(\dfrac{RT}{V}\right)dV$

$\therefore$ $\left(\dfrac{3a}{R}\right) TdT = \dfrac{dV}{V}$

Integrating, we get

$$\left(\frac{3aT^2}{2R}\right) = \ln V - \ln C$$

$$V = Ce^{\frac{3aT^2}{2R}} \quad \text{or} \quad Ve^{-\frac{3aT^2}{2R}} = \text{constant}$$

11. (a) $p = \alpha T^{1/2}$

or $pT^{-1/2} = \text{constant}$

$$p^{1/2}V^{-1/2} = \text{constant}$$

$$pV^{-1} = \text{constant}$$

$\therefore$ $x = -1$

$$\Delta W = \left(\frac{R}{1 - x}\right)(\Delta T)$$

$$= \left(\frac{8.31}{2}\right)(50)$$

$$= 207.75 \text{ J}$$ **Ans.**

(b) $C = C_V + \dfrac{R}{1 - x} = \dfrac{3}{2}R + \dfrac{R}{2} = 2R$ **Ans.**

12. $F + pA = p_0A$

$$F = (p_0 - p)A$$

$$W = \int_V^{2V} (p_0 - p)A \, dx$$

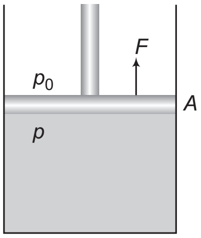
$$= \int_V^{2V} p_0 dV - \int_V^{2V} pdV$$

$$= p_0V - \int_V^{2V} RT\left(\frac{dV}{V}\right)$$

$$= p_0V - RT \ln (2)$$

$$= RT - RT \ln (2)$$

$$= RT(1 - \ln 2)$$ **Ans.**



13. (a) $p \propto \frac{1}{T}$ or $pT = \text{constant}$

$$\therefore \qquad p(pV)) = \text{constant}$$

$$\text{or} \qquad pV^{1/2} = \text{constant}$$

In the process, $pV^x = \text{constant}$,

Molar heat capacity is

$$C = C_V + \frac{R}{1-x}$$

$$\text{or} \qquad C = \frac{3}{2}R + \frac{R}{1-\frac{1}{2}}$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R$$ **Ans.**

(b) $W = Q - \Delta U = nC\Delta T - nC_V\Delta T$

$$= n(C - C_V)\Delta T$$

$$= 2\left[\frac{7}{2}R - \frac{3}{2}R\right](T_2 - T_1)$$

$$= 4R(T_2 - T_1)$$ **Ans.**

14. $V = \frac{a}{T}$

$$\text{or} \qquad VT = \text{constant}$$

$$\text{or} \qquad V(pV) = \text{constant}$$

$$\therefore \qquad pV^2 = \text{constant}$$

In the process $pV^x = \text{constant}$, molar heat capacity is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x}$$

Here, $x = 2$

$$\therefore \qquad C = \frac{R}{\gamma-1} + \frac{R}{1-2} = \left(\frac{2-\gamma}{\gamma-1}\right)R$$

Now, $Q = nC\Delta T$

$$= (1)\left(\frac{2-\gamma}{\gamma-1}\right)R\Delta T$$

$$= \left(\frac{2-\gamma}{\gamma-1}\right)R\Delta T$$ **Ans.**

15. Process AB is isochoric ($V = \text{constant}$).

Hence,

$$\Delta W_{AB} = 0$$

$$\Delta W_{BCD} = p_0V_0 + \frac{\pi}{2}(p_0)\left(\frac{V_0}{2}\right)$$

$$= \left(\frac{\pi}{4} + 1\right)p_0V_0$$

$$\Delta W_{DA} = -\frac{1}{2}\left(\frac{p_0}{2} + p_0\right)(2V_0 - V_0)$$

$$= -\frac{3}{4}p_0V_0$$

$$\Delta U_{AB} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_B - T_A)$$

$$\left(n = 2, C_V = \frac{3}{2}R\right)$$

$$= 3R\left(\frac{p_0V_0}{2R} - \frac{p_0V_0}{4R}\right)$$

$$= \frac{3}{4}p_0V_0 = \Delta Q_{AB} \qquad \left(T = \frac{pV}{nR}\right)$$

$$\Delta U_{BCD} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_D - T_B)$$

$$= (3R)\left(\frac{2p_0V_0}{2R} - \frac{p_0V_0}{2R}\right) = \frac{3}{2}p_0V_0$$

Hence, $\Delta Q_{BCD} = \Delta U_{BCD} + \Delta W_{BCD}$

$$= \left(\frac{\pi}{4} + \frac{5}{2}\right)p_0V_0$$

$$\Delta U_{DA} = nC_V\Delta T$$

$$= (2)\left(\frac{3}{2}R\right)(T_A - T_D)$$

$$= (3R)\left(\frac{p_0V_0}{4R} - \frac{2p_0V_0}{2R}\right)$$

$$= -\frac{9}{4}p_0V_0$$

$$\therefore \qquad \Delta Q_{DA} = \Delta U_{DA} + \Delta W_{DA}$$

$$= -\frac{9}{4}p_0V_0 - \frac{3}{4}p_0V_0$$

$$= -3p_0V_0$$

Net work done is,

$$W_{\text{net}} = \left(\frac{\pi}{4} + 1 - \frac{3}{4}\right)p_0V_0$$

$$= 1.04\, p_0V_0$$

and heat absorbed is

$$Q_{ab} = \Delta Q_{+\text{ve}}$$

$$= \left(\frac{3}{4} + \frac{\pi}{4} + \frac{5}{2}\right)p_0V_0 = 4.03p_0V_0$$

Hence, efficiency of the cycle is

$$\eta = \frac{W_{\text{net}}}{Q_{ab}} \times 100$$

$$= \frac{1.04\, p_0V_0}{4.03\, p_0V_0} \times 100$$

$$= 25.8\%$$ **Ans.**

16. $p = \dfrac{\alpha T - \beta T^2}{V}$ (p = constant)

Hence $V = \dfrac{\alpha T - \beta T^2}{p}$

or $dV = \left(\dfrac{\alpha - 2\beta T}{p}\right) dT$

$$W = \int p\,dV = \int_{T_1}^{T_2} p\left(\frac{\alpha - 2\beta T}{p}\right) dT$$

or $W = [\alpha T - \beta T^2]_{T_1}^{T_2}$

$= \alpha(T_2 - T_1) - \beta(T_2^2 - T_1^2)$ **Ans.**

17. (a) First law of thermodynamics for the given process from state 1 to state 2

$$Q_{12} - W_{12} = U_2 - U_1$$

Here, $Q_{12} = +\,10\,p_0V_0$ joule

$W_{12} = 0$ (Volume remains constant)

$$U_2 - U_1 = nC_V\,(T_2 - T_1)$$

$$nC_V\,(T_2 - T_1) = 10\,p_0V_0$$

For an ideal gas,

$$p_0V_0 = nRT_0$$

and $C_p - C_V = R$

$\therefore$ $C_V = C_p - R$

$= \dfrac{5R}{2} - R = \dfrac{3R}{2}$

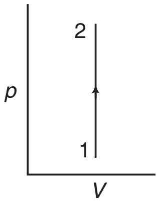
$\therefore$ $n\left(\dfrac{3R}{2}\right)(T_2 - T_0) = 10nRT_0$

$$T_2 = \frac{23}{3}T_0$$

As $p \propto T$ for constant volume

$$p_2 = \frac{23}{3}p_0$$

(b)



18. **Process *A*-*B*** is an isothermal process

i.e. T = constant

Hence, $p \propto \dfrac{1}{V}$

or p-V graph will be a rectangular hyperbola with increasing p and decreasing V.

$\rho \propto \dfrac{1}{V}$. Hence, ρ-V graph is also a rectangular hyperbola with decreasing V and hence increasing ρ.

$\rho \propto p$ $\left[\rho = \dfrac{pM}{RT}\right]$

Hence, ρ-p graph will be a straight line passing through origin, with increasing ρ and p.

Process *B*-*C* is an isochoric process, because p-T graph is a straight line passing through origin

i.e. V = constant

Hence, p-V graph will be a straight line parallel to p-axis with increasing p.

Since, V = constant hence ρ will also be constant

Hence ρ-V graph will be a dot.

ρ-p graph will be a straight line parallel to p-axis with increasing p, because

ρ = constant

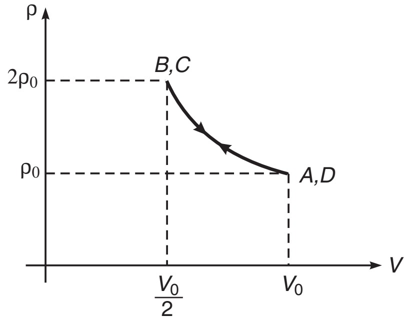
Process *C-D* is inverse of A-B and D-A is inverse of B-C.

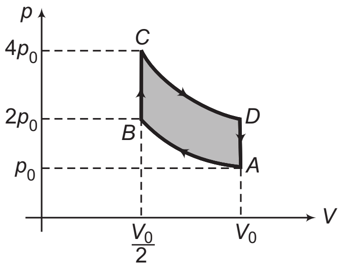
Different values of p, V, T and ρ in tabular form are shown below

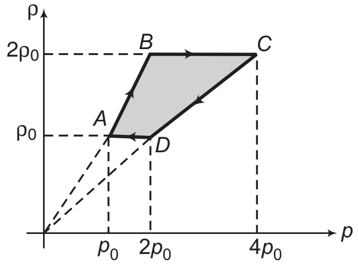
	p	**V**	**T**	ρ
A	p_0	V_0	T_0	ρ_0
B	$2p_0$	$\dfrac{V_0}{2}$	T_0	$2\rho_0$
C	$4p_0$	$\dfrac{V_0}{2}$	$2T_0$	$2\rho_0$
D	$2p_0$	V_0	$2T_0$	ρ_0

Here, $V_0 = nR\left(\dfrac{T_0}{P_0}\right)$ and $\rho_0 = \dfrac{P_0M}{RT_0}$

The corresponding graphs are as follows







19. First process

$$T = \text{constant}$$

$$\therefore \qquad p \propto \frac{1}{V}$$

V is made 5 times. Therefore, p will become $\frac{1}{5}$ th.

Second process

$$V = \text{constant} \quad \Rightarrow \quad T \propto p$$

Pressure again becomes first times. So, temperature will also become 5 times.

$$Q_1 + Q_2 = 80 \times 10^3$$

$$\therefore \; nRT \ln\left(\frac{V_f}{V_i}\right) + nC_V \, \Delta T = 80 \times 10^3$$

$$\therefore \; (3)(8.31)(273)\ln(5) + (3)\left(\frac{8.31}{\gamma - 1}\right)(5 \times 273 - 273)$$

$$= 80 \times 10^3$$

Solving we get, $\gamma = 1.4$

22. Calorimetry and Heat Transfer

INTRODUCTORY EXERCISE 22.1

1. $$Q = mL_f + ms_w\,\Delta\theta + mL_f$$
$$= m\,(L_f + s_w\Delta\theta + L_f)$$
$$= 10\,[80 + 1\times 100 + 540]$$
$$= 7200 \text{ cal}$$

2. Let θ $(> 60°)$ is the mixture temperature in equilibrium.

Heat given = Heat taken

$$0 = m(1)(\theta - 20) + m(0.5)(\theta - 40) + m\,(0.25)(\theta - 60)$$

Solving the equation, we get

$$\theta = 31.43°\text{C}$$

3. $$mL = ms\,(\theta - 0°)$$
$$\therefore \quad \theta = \frac{L}{s} = \frac{80}{1} = 80°\text{C}$$

4. 75% heat is retained by bullet

$$\frac{3}{4}\left[\frac{1}{2}mv^2\right] = ms\Delta\theta + mL$$

or $$v = \sqrt{\frac{(8s\Delta\theta + 8L)}{3}}$$

Substituting the values, we have

$$v = \sqrt{\frac{(8\times 0.03\times 4.2\times 300) + (8\times 6\times 4.2)}{3}}$$
$$= 12.96 \text{ m/s}$$

5. Let m be the mass of the steam required to raise the temperature of 100 g of water from 24°C to 90°C.

Heat lost by steam = Heat gained by water

$$\therefore \quad m\,(L + s\Delta\theta_1) = 100s\Delta\theta_2$$

or $$m = \frac{(100)(s)(\Delta\theta_2)}{L + s(\Delta\theta_1)}$$

Here, s = specific heat of water = 1 cal/g-°C,

L = latent heat of vaporization = 540 cal/g

$$\Delta\theta_1 = (100 - 90) = 10°\text{C}$$

and $$\Delta\theta_2 = (90 - 24) = 66°\text{C}$$

Substituting the values, we have

$$m = \frac{(100)(1)(66)}{(540) + (1)(10)} = 12 \text{ g}$$

$$\therefore \quad m = 12 \text{ g}$$

6. Heat liberated by 10 g water (at 40°C) when it converts into water at 0°C

$$Q = m_1 s\Delta\theta$$
$$= 10\times 1\times 40$$
$$= 400 \text{ cal}$$

Mass of ice melted by this heat,

$$m_2 = \frac{Q}{L} = \frac{400}{80} = 5 \text{ g} < 15 \text{ g}$$

Therefore, whole ice is not melted.

Temperature of mixture is 0°C.

Mass of water = $m_1 + m_2 = 15$ g

Mass of ice = $15 - m_2 = 10$ g

7. Similar to Example 22.4

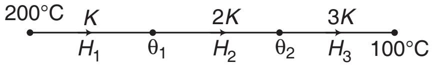
INTRODUCTORY EXERCISE 22.2

2. $$Q = \frac{KA\,(\theta_1 - \theta_2)t}{l}$$
$$\therefore \quad K = \frac{Ql}{A\,(\theta_1 - \theta_2)\,t} = \frac{\text{J - m}}{\text{m}^2\text{ K - s}}$$
$$= \frac{\text{W}}{\text{m-K}} \qquad \text{(as J/s = W)}$$

3. $$R = \frac{l}{KA} = \frac{\text{m}}{(\text{W/m -K})\text{ m}^2}$$
$$= \frac{\text{K}}{\text{W}} = \text{KW}^{-1}$$

4. In convection, liquid is heated from the bottom.

5.



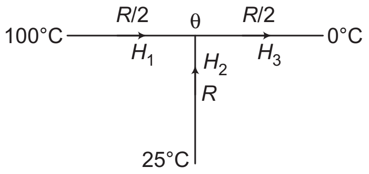
$$H_1 = H_2 = H_3$$
$$\therefore \quad \frac{200 - \theta_1}{(l/KA)} = \frac{\theta_1 - \theta_2}{(l/2KA)} = \frac{\theta_2 - 100}{(l/3KA)}$$

Solving these equations we get,

$$\theta_1 = 145.5°\text{C}$$

and $$\theta_2 = 118.2°\text{C}$$

6.



$$H_1 + H_2 = H_3$$
$$\frac{100 - \theta}{(R/2)} + \frac{25 - \theta}{R} = \frac{\theta - 0°}{R/2}$$

Solving we get, $\theta = 45°\text{C}$

$$H_{CD} = \frac{\text{TD}}{R} = \frac{45°\text{C} - 25°\text{C}}{5}$$
$$= 4\,\text{W}$$

7. Let us start with the assumption that

$$\left|\frac{\Delta\theta}{\Delta}\right| \propto \text{Temperature difference}$$

$$\therefore \quad \left(\frac{80-50}{5}\right) = \alpha\left[\frac{80+50}{2} - 20\right]$$

$$\text{and} \quad \left(\frac{60-30}{t}\right) = \alpha\left[\frac{60+30}{2} - 20\right]$$

Solving these two equations, we get

$$t = 9 \text{ min}$$

Exercises

LEVEL 1

Assertion and Reason

1. Specific heat is a function of temperature. But for most of the substances variation with temperature is almost negligible.

4. $R_1 = 2R$ and $R_2 = \dfrac{R}{2}$

Thermal resistance becomes $\dfrac{1}{4}$ th. Therefore heat current becomes four times.

5. If temperature of a normal body is more than a perfectly black body, it can radiate more energy.

6. This law was given by Kirchhoff.

9. In steady state, temperature of different sections (perpendicular to the direction of heat flow) becomes constant but not same.

10. $-\dfrac{dT}{dt} = \dfrac{eA\sigma}{mc}(T^4 - T_0^4)$ or $-\dfrac{dT}{dt} \propto \dfrac{1}{m}$

Objective Questions

1. $\lambda_m \propto \dfrac{1}{T}$

$$\therefore \quad \frac{(\lambda_m)_2}{(\lambda_m)_1} = \frac{T_1}{T_2} = \frac{2000}{3000} = \frac{2}{3}$$

$$\therefore \quad (\lambda_m)_2 = \frac{2}{3}(\lambda_m)_1$$

2. At higher temperature radiation is more. So, cooling is fast.

3. $Q_1 = Q_2$

$$\therefore \quad ms_1(32 - 20) = ms_2(40 - 32)$$

$$\therefore \quad \frac{s_1}{s_2} = \frac{8}{12} = \frac{2}{3}$$

4. $Q = \int dQ = \int_1^2 ms\,dT$

$$= \int_1^2 (1)(aT^3)\,dT = \frac{15a}{4}$$

5. $\lambda_m \propto \dfrac{1}{T}$

$$\therefore \quad \frac{(\lambda_m)_1}{(\lambda_m)_2} = \frac{T_2}{T_1} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{(\lambda_m)_2}{(\lambda_m)_1}$$

$$= \frac{350}{510} = 0.69$$

6. $dQ = CdT$

$$\therefore \quad \frac{dT}{dQ} = \frac{1}{C} = \text{Slope of } T\text{-}Q \text{ graph.}$$

7. Thermal resistance $= \dfrac{l}{KA} = \dfrac{l}{K(\pi R^2)} \propto \dfrac{l}{R^2}$

The rod for which $\dfrac{l}{R^2}$ is minimum will conduct maximum heat.

8. Heat taken by 1 g ice in transformation from ice at 0°C to water at 100°C is

$$Q = mL + ms\Delta\theta$$
$$= (1)(80) + (1)(1)(100)$$
$$= 180 \text{ cal}$$

Mass of steam condensed to give this much heat is

$$m = \frac{Q}{L} = \frac{180}{540} = \frac{1}{3}\text{ g}$$

This is less than 1g or total mass of steam. Therefore, whole steam is not condensed and mixture temperature is 100°C.

9. $H_A = H_{\text{Total}}$

$$\therefore \quad \frac{(\text{TD})_A}{R_A} = \frac{\text{TD}}{R_A + R_B}$$

$$\therefore \quad (\text{TD})_A = \left(\frac{R_A}{R_A + R_B}\right)\text{TD}$$

$$= \left(\frac{1}{1 + R_B/R_A}\right)\text{TD} \qquad \text{...(i)}$$

$$\frac{R_B}{R_A} = \left(\frac{l_B}{l_A}\right)\left(\frac{K_A A}{K_B A}\right) = \left(\frac{20}{10}\right)(3) = 6$$

Substituting in Eq. (i), we get

$$(\text{TD})_A = \left(\frac{1}{1+6}\right) \times 35 = 5^\circ\text{C}$$

10. Thermal resistance,

$$R = \frac{l}{KA} \propto \frac{1}{K} \quad \Rightarrow \quad K_A = 2\,K_B$$

$$\therefore \qquad R_A = \frac{R_B}{2}$$

So, let $R_A = R$ then $R_B = 2R$

Now, $\qquad H_A = H_B$

$$\therefore \quad \frac{(\text{TD})_A}{R_A} = \frac{(\text{TD})_B}{R_B} \quad \text{or} \quad (\text{TD})_A = \left(\frac{R_A}{R_B}\right)(\text{TD})_B$$

$$= \left(\frac{R}{2R}\right)(36^\circ\text{C}) = 18^\circ\text{C}$$

11. $R = \dfrac{l}{KA}$

$$\frac{R_2}{R_1} = \left(\frac{l_2}{l_1}\right)\left(\frac{K_1}{K_2} \cdot \frac{A_1}{A_2}\right)$$

$$= (2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

Therefore, thermal resistance of 2nd rod is half.
Hence, rate of heat flow will be twice.

Subjective Questions

1. Total rate of radiation of energy

$$= e_r\,\sigma T^4 A$$

$$= (0.6)(5.67 \times 10^{-8})(800 + 273)^4 (0.1)^2 (2)$$

$$\approx 900 \text{ W}$$

2. $\dfrac{1}{2}\left(\dfrac{1}{2}mv^2\right) = ms\Delta\theta + mL$

$$\therefore \qquad v = 2\sqrt{s\Delta\theta + L}$$

$$= 2\sqrt{(125)\,(327 - 27) + 2.5 \times 10^4}$$

$$= 500 \text{ m/s}$$

3. $0.4\,[mg\Delta h] = ms\Delta\theta$

$$\therefore \qquad \Delta\theta = \frac{0.4\,g\Delta h}{s} = \frac{0.4 \times 9.8 \times 0.5}{800}$$

$$= 2.5 \times 10^{-3}\,{}^\circ\text{C}$$

4. $P = \left(\dfrac{dm}{dt}\right) s \cdot \Delta\theta$

$$\therefore \qquad \frac{dm}{dt} = \frac{P}{s\Delta\theta} = \frac{500 \times 10^6}{(4200) \times 10}$$

$$= 11904 \text{ kg/s}$$

$$= 1.2 \times 10^4 \text{ kg/s}$$

5. $P = e_r\,\sigma A\,(T^4 - T_0^4)$

$$= (0.4)(5.67 \times 10^{-8})(4\pi)(4 \times 10^{-2})^2[(3000)^4 - (300)^4]$$

$$= 3.7 \times 10^4 \text{ W}$$

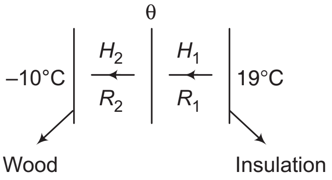
6. $L\left(\dfrac{dm}{dt}\right) = \dfrac{dQ}{dt} = H = \dfrac{\text{TD}}{R} = \dfrac{\theta - 100}{(l/KA)}$

$$\therefore \qquad \theta = 100 + \left(\frac{Ll}{KA}\right)\left(\frac{dm}{dt}\right)$$

$$= 100 + \left(\frac{2.256 \times 10^6 \times 1.2 \times 10^{-2}}{50.2 \times 0.15}\right)\left(\frac{0.44}{60 \times 5}\right)$$

$$= 105^\circ\text{C}$$

7. (a) $H_1 = H_2$



$$\therefore \qquad \frac{19 - \theta}{R_1} = \frac{\theta + 10}{R} \qquad \left(H = \frac{\text{TD}}{R}\right)$$

$$\text{or} \qquad \frac{19 - \theta}{(l_1/K_1 A)} = \frac{\theta + 10}{(l_2/K_2 A)}$$

$$\therefore \qquad \frac{K_1(19 - \theta)}{l_1} = \frac{K_2(\theta + 10)}{l_2}$$

$$\therefore \qquad \frac{0.01\,(19 - \theta)}{3.5} = \frac{0.08\,(\theta + 10)}{2.0}$$

Solving, we get

$$\theta = -8.1^\circ\text{C}$$

(b) $H = H_1 = \dfrac{19 - \theta}{(l_1/K_1 A)}$

$$= \frac{K_1 A\,(19 - \theta)}{l_1}$$

$$= \frac{(0.01)\,(19 + 8.1)\,(1)}{3.5 \times 10^{-2}}$$

$$= 7.7 \text{ W/m}^2$$

8. Hint is already given in the question

9. Let mixture is water at θ°C (where 0°C< θ <40°C)

Heat given by water = Heat taken by ice

$$\therefore \quad (200)\,(1)\,(40 - \theta) = (140)\,(0.53)\,(15) + (140)\,(80) + (140)\,(1)\,(\theta - 0)$$

Solving we get,

$$\theta = -12.7^\circ\text{C}$$

Since, $\theta < 0°\text{C}$ and we have assumed the mixture to be water whose temperature can't be less than $0°\text{C}$.

Hence, mixture temperature $\theta = 0°\text{C}$.

Heat given by water in reaching upto $0°\text{C}$ is,

$$\theta = (200)(1)(40-0) = 8000 \text{ cal.}$$

Let m mass of ice melts by this heat, then

$$8000 = (140)(0.53)(15) + (m)(80)$$

Solving we get $m = 86$ g

$\therefore$ Mass of water $= 200 + 86 = 286$ g

Mass of ice $= 140 - 86 = 54$ g

10. Let heat is supplied at a constant rate of $(\alpha)\dfrac{\text{J}}{\text{min}}$.

In 4 minutes, (when temperature remains constant) ice will be melting.

$$\therefore \} \qquad mL = (4)\alpha$$

$$\therefore \qquad \frac{\alpha}{m} = \frac{L}{4} = \frac{336 \times 10^3}{4}$$

$$= 84 \times 10^3 \text{ J/min - kg}$$

Now in last two minutes,

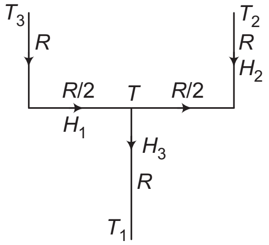
$$Q = ms\Delta\theta$$

$$\therefore \qquad (\alpha)(2) = (m)(4200)(\theta - 0°\text{C})$$

Substituting the value of $\dfrac{\alpha}{m}$ we get,

$$\theta = 40°\text{C}$$

11.

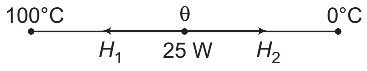


$$H_1 + H_2 = H_3$$

$$\left(\frac{T_3 - T}{3R/2}\right) + \left(\frac{T_2 - T}{3R/2}\right) = \frac{T - T_1}{R}$$

$$\therefore \qquad T = \frac{3T_1 + 2(T_2 + T_3)}{7}$$

12.



$$H_1 + H_2 = 25$$

$$\therefore \quad \frac{\theta - 100}{0.5/(400 \times 10^{-4})} + \frac{\theta - 0}{0.5/(400 \times 10^{-4})} = 25$$

Solving this equation, we get

$$\theta \approx 206°\text{C}$$

Now, temperature gradient on

$$\text{LHS} = \frac{\theta - 100}{0.5} = 212°\text{C/m}$$

$$\text{and on} \qquad \text{RHS} = \frac{\theta - 0}{0.5} = 424°\text{C/m}$$

13. Net power $= e_r\sigma(T^4 - T_0^4)A$

In first case,

$$210 = e_r\sigma\,[(500)^4 - (300)^4]\,A \qquad \text{...(i)}$$

In second case,

$$700 = \sigma\,[(500)^4 - (300)^4]\,A \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$e_r = \frac{210}{700} = 0.3$$

LEVEL 2

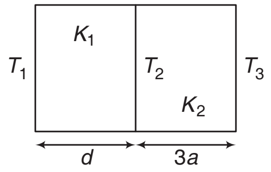
Single Correct Option

1. $R = \dfrac{l}{KA}$

l is halved, A is four times and K is $\dfrac{1}{4}$ times.

$\therefore$ R will become half. Hence, heat current will become two times. Therefore, rate of melting of ice will also become two times or 0.2 g/s.

2.



$$H_1 = H_2$$

$$\therefore \qquad \frac{T_1 - T_2}{(d/K_1A)} = \frac{T_2 - T_3}{(3d/K_2A)}$$

$$\therefore \qquad \frac{K_1}{K_2} = \left(\frac{T_2 - T_3}{T_1 - T_2}\right) \cdot \frac{1}{3}$$

But $T_1 : T_2$ and T_3 are in AP

$$\therefore \qquad T_2 - T_3 = T_1 - T_2$$

$$\therefore \qquad \frac{K_1}{K_2} = \frac{1}{3}$$

3. $H = \dfrac{\text{TD}}{R}$

$$\therefore \qquad R = \frac{\text{TD}}{H} = \frac{100 - 0}{1}$$

$$= 100 \text{ kW}^{-1}$$

Now, $R = \int_0^x dR = \int_0^x \frac{dx}{K_0 (1 + ax) A}$

or $100 = \int_0^x \frac{dx}{10^2 \, (1 + x) \, (10^{-4})}$

Solving this equation we get,

$$x = 1.7 \text{ m}$$

4. Let R = thermal resistance of each rod.

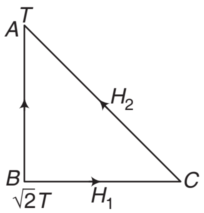
In first case, $R_{\text{net}} = R + R = 2R$ (in series)

In second case, $R_{\text{net}} = \frac{R}{2}$ (in parallel)

In second case thermal resistance has become $\frac{1}{4}$th.

So, heat current will become 4 times. So time taken to flow same amount of heat will be reduced to $\frac{1}{4}$th.

5.



$$H_1 = H_2$$

$$\therefore \quad \frac{\sqrt{2}T - T_C}{(l/KA)} = \frac{T_C - T}{(\sqrt{2}l/KA)}$$

Solving we get,

$$T_C = \frac{3T}{\sqrt{2} + 1}$$

6. Net power available per second,

$$P = 1000 - 160 = 840 \text{ J/s}$$

Heat required,

$$Q = ms\Delta\theta = (2)\,(4200)\,(77 - 27)$$
$$= 420000 \text{ J}$$

$\therefore$ Time required $= \frac{Q}{P} = 500 \text{ s} = 8 \text{ min } 20 \text{ s}$

7. $R_{\text{net}} = R_1 + R_2 = \left(\frac{X}{KA}\right) + \frac{4X}{2KA} = \frac{3X}{KA}$

Now, $H = \frac{dQ}{dt} = \frac{\text{TD}}{R_{\text{net}}}$

$$= \frac{(T_2 - T_1)}{(3X/KA)} = \left[\frac{KA\,(T_2 - T_1)}{x}\right]\left(\frac{1}{3}\right)$$

$$\therefore \quad f = \frac{1}{3}$$

More than One Correct Options

1. Total radiation per second is given by

$$P = e_r \sigma T^4 A$$

Here, $e_r : \sigma : T$ and A all are same.

Hence, P will be same.

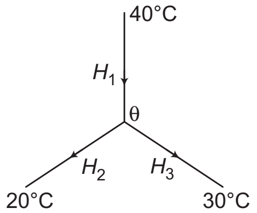
Same is the case with absorption per second.

Now, $P = ms\left(-\frac{dT}{dt}\right)$

or Rate of cooling $\left(-\frac{dT}{dt}\right) = \frac{P}{ms} \propto \frac{1}{m}$

Mass of hollow sphere is less. So, its initial rate of cooling will be more.

2.



$$H_1 = H_2 + H_3$$

$$\therefore \quad \frac{40 - \theta}{R} = \frac{\theta - 20}{R} + \frac{\theta - 30}{R}$$

where, R = thermal resistance of each rod

Solving this equation, we get

$$\theta = 30°\text{C}$$

Heat flows from higher temperature to lower temperature.

3. $ms\,(2\theta - \theta') = m\,(2s)\,(\theta' - \theta)$

Solving this equation we get,

$$\theta' = \frac{4}{3}\theta$$

Further, heat capacity

$C = ms$ or $C \propto s$ (as m is same)

$$\therefore \quad \frac{C_1}{C_2} = \frac{s_1}{s_2} = \frac{1}{2}$$

4. $q_1 = \frac{\text{TD}}{R_1}$

$$\therefore \quad R_1 = \frac{\text{TD}}{q_1}$$

Similarly, $R_2 = \frac{\text{TD}}{q_2}$

In series,

$$q_s = \frac{\text{TD}}{R_1 + R_2} = \frac{\text{TD}}{\frac{\text{TD}}{q_1} + \frac{\text{TD}}{q_2}}$$

$$= \frac{q_1 q_2}{q_1 + q_2}$$

In parallel, $q_p = \frac{\text{TD}}{R_{\text{net}}} = \text{TD}\left(\frac{1}{R_{\text{net}}}\right)$

$$= \text{TD}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \text{TD}\left(\frac{q_1}{\text{TD}} + \frac{q_2}{\text{TD}}\right) = q_1 + q_2$$

5. (b) If temperature difference is small, the rate of cooling is proportional to TD.

Match the Columns

1. (a) $E = \sigma T^4$ = energy radiated per unit surface are per unit time by a black body

$$\therefore \quad \sigma = \left[\frac{E}{T^4}\right] = \left[\frac{\text{ML}^2\text{T}^{-2}}{\text{L}^2\text{T}\theta^4}\right]$$

$$= [\text{MT}^{-3}\,\theta^{-4}]$$

(b) $$b = \lambda_m T$$

$$\therefore \quad [b] = [\text{L}\theta]$$

(c) Emissive power is energy radiated per unit time per unit surface area.

$$\therefore \quad [\text{E}] = \left[\frac{\text{ML}^2\,\text{T}^{-2}}{\text{L}^2\,\text{T}}\right] = [\text{MT}^{-3}]$$

(d) $H = \frac{dQ}{dt} = \frac{\text{TD}}{R}$

$$\therefore \quad R = \frac{\text{TD}}{(dQ/dt)} = \frac{\theta}{[\text{ML}^2\text{T}^{-2}/\text{T}]}$$

$$= [\text{M}^{-1}\text{L}^{-2}\text{T}^3\theta]$$

2. (a) $$\frac{dQ}{dt} = ms\frac{d\theta}{dt}$$

$$\therefore \quad \left(\frac{d\theta}{dt}\right) = \frac{dQ/dt}{ms}$$

$\therefore$ Slope of θ - t graph $\propto \frac{1}{m}$

(b) $$Q_{\text{Total}} = mL$$

or $\left(\frac{dQ}{dt}\right)$(time) $= mL$

or time $\propto m$ or length of line $bc \propto m$

(d) ab : only solid state

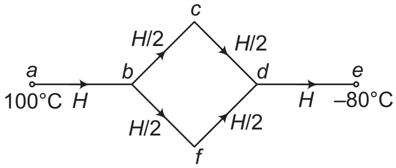
bc : solid + liquid state

cd : only liquid state

de : liquid + gaseous state

ef : only gaseous state.

3. If R = thermal resistance of one rod



Then, $R_{\text{net}} = 3R$

$$\therefore \quad H = \frac{100 - (-80)}{3R} = \frac{60}{R}$$

(a) ***ab*** $\quad H = \frac{60}{R} = \frac{100 - \theta_b}{R}$

$$\therefore \quad \theta_b = 40°\text{C}$$

(b) ***bc*** $\quad \frac{H}{2} = \frac{30}{R} = \frac{40 - \theta_c}{R}$

$$\therefore \quad \theta_C = 10°\text{C}$$

(c) Same as above

(d) ***de***

$$H = \frac{60}{R} = \frac{\theta_d - (-80)}{R}$$

$$\therefore \quad \theta_d = -20°\text{C}$$

4. (a) $m\,(s)(\theta' - \theta) = 2m\,(s)\,(2\theta - \theta')$

Solving we get,

$$\theta' = \frac{5}{3}\theta$$

(b) $ms\,(\theta' - \theta) = 3m\,(s)\,(3\theta - \theta')$

Solving we get,

$$\theta' = \frac{5}{2}\theta$$

(c) $2m\,(s)\,(\theta' - 2\theta) = 3m\,(s)\,(3\theta - \theta')$

Solving we get,

$$\theta' = \frac{13}{5}\theta$$

(d) $ms\,(\theta' - \theta) + 2\,ms\,(\theta' - 2\theta) + 3ms\,(\theta' - 3\theta) = 0$

Solving we get,

$$\theta' = \frac{7}{3}\theta$$

5. Unit of heat capacity is J/°C.

Unit of latent heat is J/kg.

Subjective Questions

1. (a) Between $t = 1$ min to $t = 3$ min, there is no rise in the temperature of substance. Therefore, solid melts in this time.

$$L = \frac{Q}{m} = \frac{Ht}{m} = \frac{10 \times 2}{0.5}$$
$$= 40 \text{ kJ/kg}$$

(b) From $Q = ms\Delta T$ or $s = \dfrac{Q}{m\Delta T}$

Specific heat in solid state

$$s = \frac{10 \times 1}{0.5 \times 15} = 1.33 \text{ kJ/kg-°C}$$

2. Let T_0 be the temperature of surrounding and T be the temperature of hot body at some instant. Then,

$$-\frac{dT}{dt} = K(T - T_0)$$

or $$\int_{T_m}^{T} \frac{dT}{T - T_0} = -K\int_0^t dT$$

(T_m = temperature at $t = 0$)

Solving this equation, we get

$$T = T_0 + (T_m - T_0)e^{-Kt} \qquad \text{(i)}$$

Maximum temperature it can lose is $(T_m - T_0)$

From Eq. (i),

$$T - T_0 = (T_m - T_0)e^{-Kt}$$

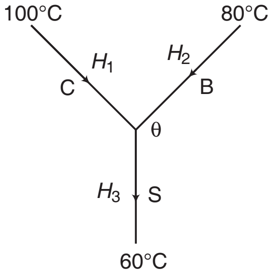
Given that

$$T - T_0 = \frac{T_m - T_0}{2} = (T_m - T_0)e^{-Kt}$$

Solving this equation we get,

$$t = \frac{\ln(2)}{K}$$

3. Let θ be the temperature of junction. Then,



$$H_1 + H_2 = H_3$$

$$\therefore \quad \frac{100 - \theta}{(46/0.92A)} + \frac{80 - \theta}{(13/0.26A)} = \frac{\theta - 60}{(12/0.12A)}$$

Solving this equation we have,

$$\theta = 84°\text{C}$$

(b) Heat current in copper rod,

$$H_1 = \frac{100 - \theta}{(46/0.92 \times 4)}$$
$$= 1.28 \text{ cal/s} \qquad \textbf{Ans.}$$

4. Let heat capacity of flask is C and latent heat of fusion of ice is L.

Then,

$$C(70 - 40) + 200 \times 1 \times (70 - 40) = 50\,L + 50 \times 1 \times (40 - 0)$$

or $$3C - 5L = -400 \qquad \text{...(i)}$$

Further, $$C(40 - 10) + 250 \times 1 \times (40 - 10) = 80L + 80 \times 1 \times (10 - 0)$$

or $$3C - 8L = -670 \qquad \text{...(ii)}$$

Solving Eq. (i) and (ii), we have

$$L = 90 \text{ cal/g} \qquad \textbf{Ans.}$$

5. $ms\Delta\theta$ = work done against friction

$$= (\mu mg \cos\theta)d \quad (\text{but } \mu mg \cos\theta = mg \sin\theta)$$

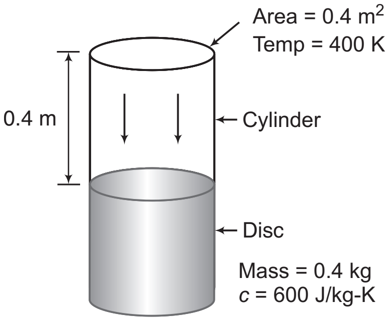
$$\Delta\theta = \frac{(\mu g \cos\theta)d}{s} = \frac{(g \sin\theta)d}{s}$$

$$= \frac{(10)\left(\frac{3}{5}\right)(0.6)}{420}$$

$$= 8.57 \times 10^{-3}\ °\text{C}$$

$$\approx 8.6 \times 10^{-3}\ °\text{C} \qquad \textbf{Ans.}$$

6. Using $$\frac{dQ}{dt} = mc\frac{d\theta}{dt} = \frac{KA(\theta_1 - \theta_2)}{l}$$



We have,

$$mc\left(\frac{d\theta}{dt}\right) = \frac{KA(400 - \theta)}{0.4}$$

$$(0.4)(600)\frac{d\theta}{dt} = \frac{(10)(0.04)(400 - \theta)}{0.4}$$

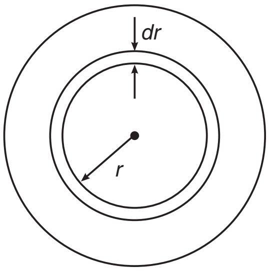
$$\therefore \quad \left(\frac{d\theta}{400 - \theta}\right) = \frac{1}{240}\,dt$$

$$\therefore \quad \int_0^t dt = 240\int_{300}^{350} \frac{d\theta}{400 - \theta}$$

Solving this, we get

$$t \approx 166 \text{ s} \qquad \textbf{Ans.}$$

7. Consider a differential cylinder



$$H = \frac{dQ}{dt} = KA\,\frac{d\theta}{dr} = (2\pi Krl)\,\frac{d\theta}{dr}$$

$$\therefore \qquad \frac{H}{2\pi Kl}\int_{r_1}^{r_2}\frac{dr}{r} = \int_0^{50} d\theta$$

$$\text{or} \qquad \frac{H}{2\pi Kl}\ln\left(\frac{r_2}{r_1}\right) = 50$$

$$\therefore \qquad H = \frac{100\pi Kl}{\ln(r_2/r_1)}$$

$$\text{Now,} \qquad Ht = mL$$

$$\therefore \qquad t = \frac{mL}{H} = \frac{mL\ln(r_2/r_1)}{100\pi Kl} \qquad \textbf{Ans.}$$

8. Three thermal resistances are in series. $\left(R = \frac{l}{KA}\right)$

$$\therefore \qquad R = R_1 + R_2 + R_3$$

$$= \frac{2.5\times 10^{-2}}{0.125\times 137} + \frac{1.0\times 10^{-2}}{1.5\times 137} + \frac{2.5\times 10^{-2}}{1.0\times 137}$$

$$= 0.0017\ ^\circ\text{C-s/J}$$

Now, heat current $H = \dfrac{\text{Temperature difference}}{\text{Net thermal resistance}}$

$$= \frac{30}{0.0017} = 17647\ \text{W}$$

9. Thermal resistance of plastic coating $\left(R_t = \frac{l}{KA}\right)$

$$R_t = \frac{t}{K(\pi d)l} \qquad (d = \text{diameter})$$

$$= \frac{0.06\times 10^{-3}}{(0.16\times 10^{-2}\times 4.18\times 10^2)(\pi)(0.64\times 10^{-3})(2)}$$

$$= 0.0223\ ^\circ\text{C-s/J}$$

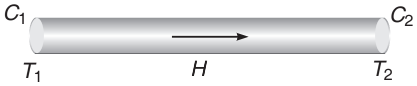
$$\text{Now,} \qquad i^2R_e = \frac{\text{TD}}{R_t}$$

$$\therefore \qquad \text{TD} = i^2R_eR_t$$

$$= (5)^2(4)(0.0223)$$

$$= 2.23^\circ\text{C} \qquad \textbf{Ans.}$$

10. Let T_1 be the temperature of C_1 and T_2 the temperature of C_2 at some instant of time. Further let T be the temperature difference at that instant.



Then,

$$C_1\left(-\frac{dT_1}{dt}\right) = H = \frac{T}{l/KA} = \frac{KA}{l}(T)$$

$$(\because T = T_1 - T_2)$$

$$\text{and} \quad C_2\left(+\frac{dT_2}{dt}\right) = H = \frac{T}{l/KA} = \frac{KA}{l}(T)$$

$$\therefore \qquad -\frac{dT}{dt} = -\frac{dT_1}{dt} + \frac{dT_2}{dt}$$

$$\therefore \qquad -\frac{dT_1}{dt} = \frac{KA}{lC_1}(T)$$

$$\text{and} \qquad +\frac{dT_2}{dt} = \frac{KA}{lC_2}$$

$$\text{Further,} \qquad -\frac{dT}{dt} = -\frac{dT_1}{dt} + \frac{dT_2}{dt}$$

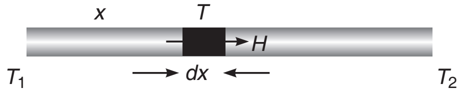
$$= \frac{KA(C_1 + C_2)}{lC_1C_1}T$$

$$\text{or} \qquad \int_{\Delta T_0}^{T}\frac{dT}{T} = -\frac{KA(C_1+C_2)}{lC_1C_2}\int_0^t dt$$

Solving, we get $T = \Delta T_0 e^{-\alpha t}$

$$\text{where,} \qquad \alpha = \frac{KA(C_1 + C_2)}{lC_1C_2}$$

11.



$$H = \frac{\text{TD}}{R} = \frac{(-dT)}{(dx)/KA}$$

$$= -\left(\frac{dT}{dx}\right)\left(\frac{a}{T}A\right) = \text{constant} \quad \ldots\text{(i)}$$

$$\therefore \qquad \int_{T_1}^{T_2} -\frac{dT}{T} = \frac{H}{aA}\int_0^l dx$$

$$\ln\left(\frac{T_1}{T_2}\right) = \frac{Hl}{aA}$$

$$\text{or} \qquad H = \frac{aA}{l}\ln(T_1/T_2)$$

Substituting in Eq. (i), we have

$$\frac{aA}{l}\ln\left(\frac{T_1}{T_2}\right) = -\left(\frac{dT}{dx}\right)\frac{aA}{T}$$

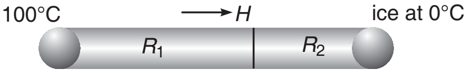
or
$$\int_{T_1}^{T}\frac{dT}{T} = -\frac{\ln\left(\dfrac{T_1}{T_2}\right)}{l}\int_0^x dx$$

∴
$$\ln\left(\frac{T}{T_1}\right) = -\frac{x}{l}\ln(T_1/T_2) = \ln\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}}$$

or
$$\frac{T}{T_1} = \left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}}$$

or
$$T = T_1\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}} = T_1\left(\frac{T_2}{T_1}\right)^{\frac{x}{l}}$$ **Ans.**

12. $L\left(\dfrac{dm}{dt}\right) = \dfrac{\text{TD}}{R_1 + R_2}$

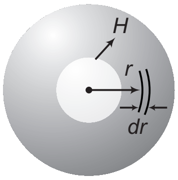


or
$$\frac{80\times 360}{3600} = \frac{100}{\dfrac{20}{0.25\times 10} + \dfrac{10}{K\times 10}}$$

Solving this equation, we get

$$K = 0.222 \text{ cal/cm-s-°C}$$ **Ans.**

13. $H = \dfrac{-d\theta}{dr/K(4\pi r^2)} = \left(-\dfrac{d\theta}{dr}\right)(4\pi K r^2)$ …(i)



or
$$\int_{R_1}^{R_2}\frac{dr}{r^2} = -\frac{4\pi K}{H}\int_{\theta_1}^{\theta_2} d\theta$$

or
$$\frac{R_2 - R_1}{R_1R_2} = \frac{4\pi K}{H}(\theta_1 - \theta_2)$$

∴
$$H = \frac{4\pi K(R_1R_2)(\theta_1 - \theta_2)}{R_2 - R_1}$$

Substituting this value of H in Eq. (i), we have

$$\frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2) = -\frac{d\theta}{dr}(r^2)$$

∴
$$\int_{\theta_1}^{\left(\frac{\theta_1+\theta_2}{2}\right)} d\theta = -\frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\int_{R_1}^{r}\frac{dr}{r^2}$$

or
$$\frac{\theta_1 + \theta_2}{2} - \theta_1 = \frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\left[\frac{1}{r} - \frac{1}{R_1}\right]$$

or
$$\frac{\theta_1 - \theta_2}{2} = \frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\left(\frac{1}{R_1} - \frac{1}{r}\right)$$

$$\frac{1}{R_1} - \frac{1}{r} = \frac{R_2 - R_1}{2R_1R_2}$$

$$\frac{1}{r} = \frac{1}{R_1} - \frac{R_2 - R_1}{2R_1R_2} = \frac{R_1 + R_2}{2R_1R_2}$$

or
$$r = \frac{2R_1R_2}{R_1 + R_2}$$ **Ans.**

14. See the extra points just before solved examples. Growth of ice on ponds. We have already derived that

$$t = \frac{1}{2}\frac{\rho L}{K\theta}y^2$$

∴
$$\frac{dt}{dy} = \frac{\rho L y}{K\theta}$$

∴
$$\frac{dy}{dt} = \frac{K\theta}{L\rho y}$$

JEE Main and Advanced

Previous Years' Questions (2018-13)

JEE Main

1. Two moles of an ideal monoatomic gas occupies a volume V at 27°C. The gas expands adiabatically to a volume $2V$. Calculate (i) the final temperature of the gas and (ii) change in its internal energy. **(2018)**

(a) (i) 189 K (ii) 2.7 kJ (b) (i) 195 K (ii) –2.7 kJ
(c) (i) 189 K (ii) –2.7 kJ (d) (i) 195 K (ii) 2.7 kJ

2. A silver atom in a solid oscillates in simple harmonic motion in some direction with a frequency of 10^{12} per second. What is the force constant of the bonds connecting one atom with the other? (Take, molecular weight of silver = 108 and Avogadro number = 6.02×10^{23} g mol^{-1}) **(2018)**

(a) 6.4 N/m (b) 7.1 N/m
(c) 2.2 N/m (d) 5.5 N/m

3. A granite rod of 60 cm length is clamped at its middle point and is set into longitudinal vibrations. The density of granite is 2.7×10^3 kg/m^3 and its Young's modulus is 9.27×10^{10} Pa. What will be the fundamental frequency of the longitudinal vibrations? **(2018)**

(a) 5 kHz (b) 2.5 kHz
(c) 10 kHz (d) 7.5 kHz

4. An observer is moving with half the speed of light towards a stationary microwave source emitting waves at frequency 10 GHz. What is the frequency of the microwave measured by the observer? (speed of light = 3×10^8 ms^{-1}) **(2017)**

(a) 12.1 GHz (b) 17.3 GHz
(c) 15.3 GHz (d) 10.1 GHz

5. C_p and C_V are specific heats at constant pressure and constant volume, respectively. It is observed that $C_p - C_V = a$ for hydrogen gas $C_p - C_V = b$ for nitrogen gas. The correct relation between a and b is **(2017)**

(a) $a = b$ (b) $a = 14b$
(c) $a = 28b$ (d) $a = \frac{1}{14}b$

6. A copper ball of mass 100 g is at a temperature T. It is dropped in a copper calorimeter of mass 100 g, filled with 170 g of water at room temperature. Subsequently, the temperature of the system is found to be 75°C. T is (Given, room temperature = 30°C, specific heat of copper = 0.1 cal/g°C) **(2017)**

(a) 885°C (b) 1250°C
(c) 825°C (d) 800°C

7. An external pressure p is applied on a cube at 0°C so that it is equally compressed from all sides. K is the bulk modulus of the material of the cube and α is its coefficient of linear expansion. Suppose we want to bring the cube to its original size by heating. The temperature should be raised by **(2017)**

(a) $\frac{p}{\alpha K}$ (b) $\frac{3\alpha}{pK}$
(c) $3pK\alpha$ (d) $\frac{p}{3\alpha K}$

8. The temperature of an open room of volume 30 m^3 increases from 17°C to 27°C due to the sunshine. The atmospheric pressure in the room remains 1×10^5 Pa. If n_i and n_f are the number of molecules in the room before and after heating, then $n_f - n_i$ will be **(2017)**

(a) 1.38×10^{23} (b) 2.5×10^{25}
(c) -2.5×10^{25} (d) -1.61×10^{23}

9. A uniform string of length 20 m is suspended from a rigid support. A short wave pulse is introduced at its lowest end. It starts moving up the string. The time taken to reach the support is (Take, $g = 10\ \text{ms}^{-2}$) **(2016)**

(a) $2\pi\sqrt{2}$ s (b) 2 s

(c) $2\sqrt{2}$ s (d) $\sqrt{2}$ s

10. A pipe open at both ends has a fundamental frequency f in air. The pipe is dipped vertically in water, so that half of it is in water. The fundamental frequency of the air column is now **(2016)**

(a) $\frac{f}{2}$ (b) $\frac{3f}{4}$

(c) $2f$ (d) f

11. A pendulum clock loses 12 s a day if the temperature is 40°C and gains 4 s a day if the temperature is 20°C. The temperature at which the clock will show correct time, and the coefficient of linear expansion α of the metal of the pendulum shaft are, respectively. **(2016)**

(a) 25°C, $\alpha = 1.85 \times 10^{-5}/°\text{C}$

(b) 60°C, $\alpha = 1.85 \times 10^{-4}/°\text{C}$

(c) 30°C, $\alpha = 1.85 \times 10^{-3}/°\text{C}$

(d) 55°C, $\alpha = 1.85 \times 10^{-2}/°\text{C}$

12. An ideal gas undergoes a quasi static, reversible process in which its molar heat capacity C remains constant. If during this process the relation of pressure p and volume V is given by pV^n = constant, then n is given by (Here, C_p and C_V are molar specific heat at constant pressure and constant volume, respectively) **(2016)**

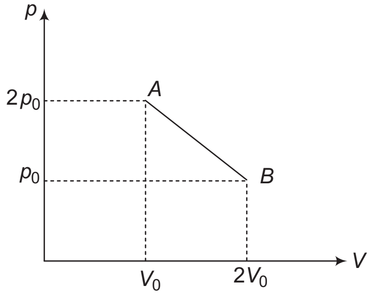
(a) $n = \frac{C_p}{C_V}$

(b) $n = \frac{C - C_p}{C - C_V}$

(c) $n = \frac{C_p - C}{C - C_V}$

(d) $n = \frac{C - C_V}{C - C_p}$

13. n moles of an ideal gas undergoes a process A and B as shown in the figure. The maximum temperature of the gas during the process will be **(2016)**



(a) $\frac{9}{4}\frac{p_0V_0}{nR}$ (b) $\frac{3}{2}\frac{p_0V_0}{nR}$ (c) $\frac{9}{2}\frac{p_0V_0}{nR}$ (d) $\frac{9p_0V_0}{nR}$

14. A train is moving on a straight track with speed $20\ \text{ms}^{-1}$. It is blowing its whistle at the frequency of 1000 Hz. The percentage change in the frequency heard by a person standing near the track as the train passes him is close to (speed of sound = $320\ \text{ms}^{-1}$) **(2015)**

(a) 12% (b) 6% (c) 18% (d) 24%

15. Consider a spherical shell of radius R at temperature T. The black body radiation inside it can be considered as an ideal gas of photons with internal energy per unit volume $u = \frac{U}{V} \propto T^4$ and pressure $p = \frac{1}{3}\left(\frac{U}{V}\right)$. If the shell now undergoes an adiabatic expansion, the relation between T and R is **(2015)**

(a) $T \propto e^{-R}$ (b) $T \propto \frac{1}{R}$ (c) $T \propto e^{-3R}$ (d) $T \propto \frac{1}{R^3}$

16. Consider an ideal gas confined in an isolated closed chamber. As the gas undergoes an adiabatic expansion, the average time of collision between molecules increases as V^q, where V is the volume of the gas. The value of q is $\left(\gamma = \frac{C_p}{C_V}\right)$ **(2015)**

(a) $\frac{3\gamma + 5}{6}$ (b) $\frac{\gamma + 1}{2}$ (c) $\frac{3\gamma - 5}{6}$ (d) $\frac{\gamma - 1}{2}$

17. A solid body of constant heat capacity 1 J/°C is being heated by keeping it in contact with reservoirs in two ways

(i) Sequentially keeping in contact with 2 reservoirs such that each reservoir supplies same amount of heat.

(ii) Sequentially keeping in contact with 8 reservoirs such that each reservoir supplies same amount of heat.

In both the cases, body is brought from initial temperature 100°C to final temperature 200°C. Entropy change of the body in the two cases respectively, is **(2015)**

(a) ln2, ln2 (b) ln2, 2ln2
(c) 2ln2, 8ln2 (d) ln2, 4ln2

18. A pipe of length 85 cm is closed from one end. Find the number of possible natural oscillations of air column in the pipe whose frequencies lie below 1250 Hz. The velocity of sound in air is 340 m/s. **(2014)**

(a) 12 (b) 8 (c) 6 (d) 4

19. Parallel rays of light of intensity $I = 912 \text{ Wm}^{-2}$ are incident on a spherical black body kept in surroundings of temperature 300 K. Take Stefan constant $\sigma = 5.7 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$ and assume that the energy exchange with the surroundings is only through radiation. The final steady state temperature of the black body is close to **(2014)**

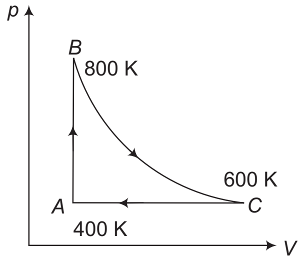
(a) 330 K (b) 660 K (c) 990 K (d) 1550

20. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. Area of cross-section of each rod is 4 cm^2. End of copper rod is maintained at 100°C whereas ends of brass and steel are kept at 0°C. Lengths of the copper, brass and steel rods are 46, 13 and 12 cm respectively.

The rods are thermally insulated from surroundings except at ends. Thermal conductivities of copper, brass and steel are 0.92, 0.26 and 0.12 in CGS units, respectively. Rate of heat flow through copper rod is **(2014)**

(a) 1.2 cal/s (b) 2.4 cal/s
(c) 4.8 cal/s (d) 6.0 cal/s

21. One mole of diatomic ideal gas undergoes a cyclic process *ABC* as shown in figure. The process *BC* is adiabatic. The temperatures at *A*, *B* and *C* are 400 K, 800 K and 600 K, respectively. Choose the correct statement. **(2014)**



(a) The change in internal energy in whole cyclic process is 250 *R*

(b) The change in internal energy in the process *CA* is 700 *R*

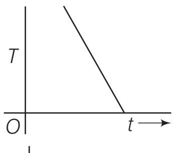
(c) The change in internal energy in the process *AB* is –350 *R*

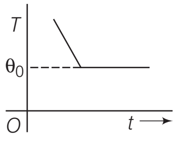
(d) The change in internal energy in the process *BC* is –500 *R*

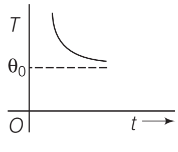
22. A sonometer wire of length 1.5 m is made of steel. The tension in it produces an elastic strain of 1%. What is the fundamental frequency of steel if density and elasticity of steel are $7.7 \times 10^3 \text{ kg/m}^3$ and $2.2 \times 10^{11} \text{ N/m}^2$ respectively?

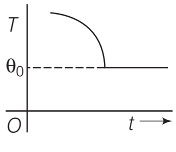
(a) 188.5 Hz **(2013)**
(b) 178.2 Hz
(c) 200.5 Hz
(d) 770 Hz

23. If a piece of metal is heated to temperature θ and then allowed to cool in a room which is at temperature θ_0. The graph between the temperature *T* of the metal and time *t* will be closed to **(2013)**

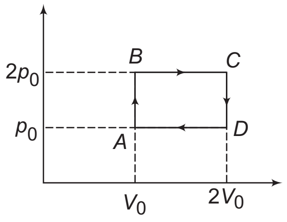
(a)


(b)


(c)


(d)


24. The shown p-V diagram represents the thermodynamic cycle of an engine, operating with an ideal monoatomic gas. The amount of heat, extracted from the source in a single cycle is **(2013)**



(a) p_0V_0 (b) $\left(\frac{13}{2}\right)p_0V_0$

(c) $\left(\frac{11}{2}\right)p_0V_0$ (d) $4p_0V_0$

25. An ideal gas enclosed in a vertical cylindrical container supports a freely moving piston of mass M. The piston and the cylinder have equal cross-sectional area A. When the piston is in equilibrium, the volume of the gas is V_0 and its pressure is p_0. The piston is slightly displaced from the equilibrium position and released. Assuming that the system is completely isolated from its surrounding, the piston executes a simple harmonic motion with frequency **(2013)**

(a) $\frac{1}{2\pi}\frac{A_\gamma p_0}{V_0M}$ (b) $\frac{1}{2\pi}\frac{V_0Mp_0}{A^2\gamma}$

(c) $\frac{1}{2\pi}\sqrt{\frac{A^2\gamma p_0}{MV_0}}$ (d) $\frac{1}{2\pi}\sqrt{\frac{MV_0}{A_\gamma p_0}}$

Answer *with* Explanations

1. **(c)** For adiabatic process relation of temperature and volume is,

$$T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1}$$

$$\Rightarrow \quad T_2(2V)^{2/3} = 300(V)^{2/3}$$

$$\left[\gamma = \frac{5}{3} \text{ for monoatomic gases}\right]$$

$$\Rightarrow \quad T_2 = \frac{300}{2^{2/3}} \approx 189\text{ K}$$

Also, in adiabatic process,

$\Delta Q = 0,\ \Delta U = -\Delta W$

$$\text{or} \quad \Delta U = \frac{-nR(\Delta T)}{\gamma - 1} = -2 \times \frac{3}{2} \times \frac{25}{3}(300 - 189)$$

$$\approx -2.7\text{ kJ}$$

$$T_2 \approx 189\text{ K},\ \Delta U \approx 2.7\text{ kJ}$$

2. **(b)** For a harmonic oscillator,

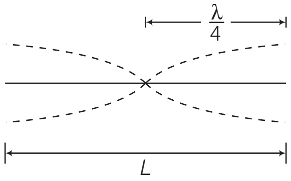
$T = 2\pi\sqrt{\frac{m}{k}}$, where k = force constant and $T = \frac{1}{f}$

$$\therefore \quad k = 4\pi^2f^2m$$

$$= 4 \times \left(\frac{22}{7}\right)^2 \times (10^{12})^2 \times \frac{108 \times 10^{-3}}{6.02 \times 10^{23}}$$

$$\Rightarrow \quad k = 7.1\text{ N / m}$$

3. **(a)**



From vibration mode,

$$\frac{\lambda}{2} = L \Rightarrow \lambda = 2L$$

$$\therefore \text{Wave speed, } v = \sqrt{\frac{Y}{\rho}}$$

$$\text{So, frequency } f = \frac{v}{\lambda}$$

$$\Rightarrow \quad f = \frac{1}{2L}\sqrt{\frac{Y}{\rho}}$$

$$= \frac{1}{2 \times 60 \times 10^{-2}}\sqrt{\frac{9.27 \times 10^{10}}{2.7 \times 10^3}}$$

$$\approx 5000\text{ Hz}$$

$$f = 5\text{ kHz}$$

4. **(b)** As observer is moving with relativistic speed; formula $\frac{\Delta f}{f} = \frac{v_{\text{radial}}}{c}$, does not apply here.

Relativistic doppler's formula is

$$f_{\text{observed}} = f_{\text{actual}} \cdot \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}$$

$$\text{Here,} \quad \frac{v}{c} = \frac{1}{2}$$

$$\text{So,} \quad f_{\text{observed}} = f_{\text{actual}}\left(\frac{3/2}{1/2}\right)^{1/2}$$

$$\therefore \quad f_{\text{observed}} = 10 \times \sqrt{3}$$

$$= 17.3\text{ GHz}$$

5. **(b)** By Mayor's relation, for 1 g mole of a gas,

$$C_p - C_V = R$$

So, when n gram moles are given,

$$C_p - C_V = \frac{R}{n}$$

As per given question,

$$a = C_p - C_V = \frac{R}{2}; \text{ for } H_2$$

$$b = C_p - C_V = \frac{R}{28}; \text{ for } N_2$$

$$a = 14b$$

6. (a) Heat gained (water + calorimeter) = Heat lost by copper ball

$$\Rightarrow \quad m_w s_w \Delta T + m_c s_c \Delta T = m_B s_B \Delta T$$

$$\Rightarrow \quad 170 \times 1 \times 45 + 100 \times 0.1 \times 45 = 100 \times 0.1 \times (T - 75)$$

$$\therefore \quad T = 885°\text{C}$$

7. (d) $K = \frac{p}{(-\Delta V/V)} \Rightarrow \frac{\Delta V}{V} = \frac{p}{K}$

$$\Rightarrow \quad -\Delta V = \frac{pV}{K} \Rightarrow \frac{pV}{K} = V(3\alpha)\,\Delta T \Rightarrow \Delta T = \frac{p}{3\alpha K}$$

8. (c) From $pV = nRT = \frac{N}{N_A} RT$

We have, $n_f - n_i = \frac{pVN_A}{RT_f} - \frac{pVN_A}{RT_i}$

$$\Rightarrow \quad n_f - n_i = \frac{10^5 \times 30}{8.3} \times 6.02 \times 10^{23} \cdot \left(\frac{1}{300} - \frac{1}{290}\right)$$

$$= -2.5 \times 10^{25}$$

$$\therefore \quad \Delta n = -2.5 \times 10^{25}$$

9. (c) At distance x from the bottom

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\left(\frac{mgx}{L}\right)}{\left(\frac{m}{L}\right)}} = \sqrt{gx} \Rightarrow \frac{dx}{dt} = \sqrt{x}\sqrt{g}$$

$$\Rightarrow \quad \int_0^L x^{-1/2} dx = \sqrt{g}\int_0^t dt \Rightarrow \left[\left.\frac{x^{1/2}}{(1/2)}\right|_0^L\right] = \sqrt{g}\cdot t$$

$$\Rightarrow \quad t = \frac{2\sqrt{L}}{\sqrt{g}} \Rightarrow t = 2\sqrt{\frac{20}{10}} = 2\sqrt{2}\text{ s}$$

10. (d) Fundamental frequency of open pipe.

$$f = \frac{v}{2l}$$

Now, after half filled with water it becomes a closed pipe of length $\frac{l}{2}$.

Fundamental frequency of this closed pipe,

$$f' = \frac{v}{4(l/2)} = \frac{v}{2l} = f$$

11. (a) $T_0 = 2\pi\sqrt{\frac{L}{g}}$

$$T' = T_0 + \Delta T = 2\pi\sqrt{\frac{L + \Delta L}{g}} \Rightarrow T' = T_0 + \Delta T$$

$$= 2\pi\sqrt{\frac{L(1 + \alpha\,\Delta\theta)}{g}} = \left\{2\pi\sqrt{\frac{L}{g}}\right\}(1 + \alpha\,\Delta\theta)^{\frac{1}{2}}$$

$$\approx T_0\left(1 + \frac{\alpha\,\Delta\theta}{2}\right)$$

$$\therefore \quad \Delta T = T' - T_0 = \frac{\alpha\,\Delta\theta T_0}{2} \qquad \ldots\text{(i)}$$

or $$\frac{\Delta T_1}{\Delta T_2} = \frac{\alpha\,\Delta\theta_1 T_0}{\alpha\,\Delta\theta_2 T_0} \Rightarrow \frac{12}{4} = \frac{40 - \theta}{\theta - 20}$$

$$\Rightarrow \quad 3(\theta - 20) = 40 - \theta \Rightarrow 4\theta = 100$$

$$\Rightarrow \quad \theta = 25°\text{C}$$

Time gained or lost is given by

$$\Delta T = \left(\frac{\Delta T}{T_0 + \Delta T}\right)t \approx \frac{\Delta t}{T_0}t$$

From Eq. (i),

$$\frac{\Delta T}{T_0} = \frac{\alpha\Delta\theta}{2} \Rightarrow \Delta t = \frac{\alpha\,(\Delta\theta)t}{2}$$

$$12 = \frac{\alpha\,(40 - 25)(24 \times 3600)}{2} \Rightarrow \alpha = 1.85 \times 10^{-5}/°\text{C}$$

12. (b) $\Delta Q = \Delta U + \Delta W$

In the process $pV^n =$ constant, molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - n} = C_V + \frac{R}{1 - n}$$

$$C - C_V = \frac{R}{1 - n} \Rightarrow 1 - n = \frac{C_p - C_V}{C - C_V}$$

$$\Rightarrow n = 1 - \left(\frac{C_p - C_V}{C - C_V}\right) = \frac{(C - C_V) - (C_p - C_V)}{C - C_V} = \frac{C - C_p}{C - C_V}$$

13. (a) p-V equation for path AB

$$p = -\left(\frac{p_0}{V_0}\right)V + 3p_0 \Rightarrow pV = 3p_0V - \frac{p_0}{V_0}V^2$$

or $$T = \frac{pV}{nR} = \frac{1}{nR}\left(3p_0V - \frac{p_0}{V_0}V^2\right)$$

For maximum temperature,

$$\frac{dT}{dV} = 0 \Rightarrow 3p_0 - \frac{2p_0V}{V_0} = 0$$

$$\Rightarrow \quad V = \frac{3}{2}V_0 \text{ and } p = 3p_0 - \frac{p_0}{V_0} = \frac{3p_0}{2}$$

Therefore, at these values :

$$\therefore \quad T_{\max} = \frac{\left(\frac{3p_0}{2}\right)\left(\frac{3V_0}{2}\right)}{nR} = \frac{9p_0V_0}{4nR}$$

14. (a) Observer is stationary and source is moving.

During approach, $f_1 = f\left(\frac{v}{v - v_s}\right)$

$$= 1000\left(\frac{320}{320 - 20}\right) = 1066.67\text{ Hz}$$

During recede, $f_2 = f\left(\frac{v}{v + v_s}\right)$

$$= 1000\left(\frac{320}{320 + 20}\right) = 941.18\text{ Hz}$$

$$|\%\text{ change in frequency}| = \left(\frac{f_1 - f_2}{f_1}\right) \times 100 \approx 12\%$$

15. **(b)** Given, $\frac{U}{V} \propto T^4$

$$\frac{U}{V} = \alpha T^4 \qquad \text{...(i)}$$

It is also given that, $P = \frac{1}{3}\left(\frac{U}{V}\right) \Rightarrow \frac{nR_0T}{V} = \frac{1}{3}(\alpha T^4)$

(R_0 = Gas constant)

or $VT^3 = \frac{3nR_0}{\alpha}$ = constant

$\therefore$ $\left(\frac{4}{3}\pi R^3\right)T^3$ = constant or RT = constant

$\therefore$ $T \propto \frac{1}{R}$

16. **(b)** Average time between two collisions is given by

$$\tau = \frac{1}{\sqrt{2}\,\pi n v_{\text{rms}} d^2} \qquad \text{...(i)}$$

Here, n = number of molecules per unit volume = $\frac{N}{V}$

and $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

Substituting these values in Eq.(i) we have,

$$\tau \propto \frac{V}{\sqrt{T}} \qquad \text{...(ii)}$$

For adiabatic process, $TV^{\gamma - 1}$ = constant

Substituting in Eq. (ii), we have $\tau \propto \frac{V}{\sqrt{\left(\frac{1}{V^{\gamma-1}}\right)}}$

or $\tau \propto V^{1+\left(\frac{\gamma-1}{2}\right)}$ or $\tau \propto V^{\left(\frac{1+\gamma}{2}\right)}$

17. **(a)** Entropy is a state functions. Therefore in both cases answer should be same.

18. **(c)** For closed organ pipe = $\frac{(2n+1)v}{4l}$ [n = 0, 1, 2 ......]

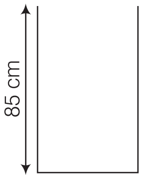
$$\frac{(2n+1)v}{4l} < 1250$$

$$(2n+1) < 1250 \times \frac{4 \times 0.85}{340}$$

$$(2n+1) < 12.52n < 11.50 \Rightarrow n < 5.25$$

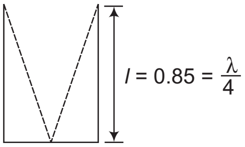
So, $n = 0, 1, 2, 3, \ldots, 5$

So, we have 6 possibilities.



Alternate method

In closed organ pipe, fundamental node



i.e. $\frac{\lambda}{4} = 0.85$

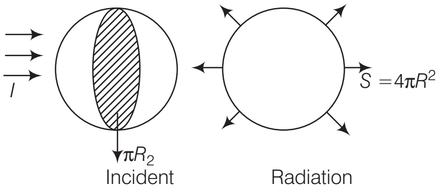
$\Rightarrow$ $\lambda = 4 \times 0.85$

As we know, $\nu = \frac{c}{\lambda}$

$\Rightarrow$ $\frac{340}{4 \times 0.85} = 100$ Hz

$\therefore$ Possible frequencies = 100 Hz, 300 Hz, 500 Hz, 700 Hz, 900 Hz, 1100 Hz below 1250 Hz.

19. **(a)** In steady state



Energy incident per second = Energy radiated per second

$\therefore$ $I\pi R^2 = \sigma(T^4 - T_0^4)\,4\pi R^2$

$\Rightarrow$ $I = \sigma(T^4 - T_0^4)\,4$

$\Rightarrow$ $T^4 - T_0^4 = 40 \times 10^8$

$\Rightarrow$ $T^4 - 81 \times 10^8 = 40 \times 10^8$

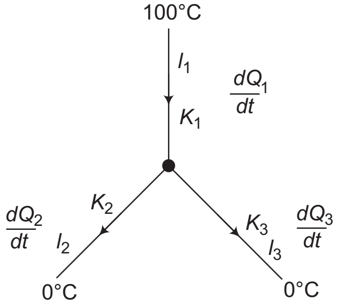
$\Rightarrow$ $T^4 = 121 \times 10^8$

$\Rightarrow$ $T \approx 330$ K

20. **(c)** In thermal conduction, it is found that in steady state the heat current is directly proportional to the area of cross-section A which is proportional to the change in temperature $(T_1 - T_2)$.

Then, $\frac{\Delta Q}{\Delta t} = \frac{KA(T_1 - T_2)}{x}$

According to thermal conductivity, we get



i.e. $$\frac{dQ_1}{dt}=\frac{dQ_2}{dt}+\frac{dQ_3}{dt}$$

$$\frac{0.92\,(100-T)}{46}=\frac{0.26\,(T-0)}{13}+\frac{0.12\,(T-0)}{12}$$

$$\Rightarrow \qquad T = 40°\,\text{C}$$

$$\therefore \qquad \frac{dQ_1}{dt}=\frac{0.92\times 4\,(100-40)}{40}=4.8\ \text{cal/s}$$

21. **(d)** According to first law of thermodynamics, we get

(i) Change in internal energy from A to B i.e. ΔU_{AB}

$$\Delta U_{AB} = nC_V(T_B - T_A)$$
$$= 1\times\frac{5R}{2}(800-400)=1000\,R$$

(ii) Change in internal energy from B to C

$$\Delta U_{BC} = nC_V(T_C - T_B)$$
$$= 1\times\frac{5R}{2}(600-800)$$
$$= -500\,R$$

(iii) $\Delta U_{\text{isothermal}} = 0$

(iv) Change in internal energy from C to A i.e. ΔU_{CA}

$$\Delta U_{CA} = nC_V(T_A - T_C)$$

$$= -500\,R$$

22. **(b)** Fundamental frequency of sonometer wire

$$f=\frac{v}{2l}=\frac{1}{2l}\sqrt{\frac{T}{\mu}}=\frac{1}{2l}\sqrt{\frac{T}{Ad}}$$

Here, μ = mass per unit length of wire.

Also, Young's modulus of elasticity $Y=\dfrac{Tl}{A\Delta l}$

$$\Rightarrow \qquad \frac{T}{A}=\frac{Y\Delta l}{l}\Rightarrow f=\frac{1}{2l}\sqrt{\frac{Y\Delta l}{ld}}$$

$$\Rightarrow \qquad l = 1.5\ \text{m},\ \frac{\Delta l}{l}=0.01$$

$$d = 7.7\times 10^3\ \text{kg/m}^3$$

$$\Rightarrow \qquad Y = 2.2\times 10^{11}\ \text{N/m}^2$$

After substituting the values we get,

$$f \approx 178.2\ \text{Hz}$$

23. **(c)** According to Newton's cooling law, option (c) is correct answer.

24. **(b)** Heat is extracted from the source means heat is given to the system (or gas) or Q is positive. This is positive only along the path ABC.

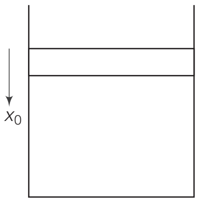
Heat supplied

$$\therefore Q_{ABC} = \Delta U_{ABC} + W_{ABC}$$
$$= nC_V\,(T_f - T_i) + \text{Area under } p\text{-}V \text{ graph}$$
$$= n\left(\frac{3}{2}R\right)(T_C - T_A) + 2p_0V_0$$
$$= \frac{3}{2}(nRT_C - nRT_A) + 2p_0V_0$$
$$= \frac{3}{2}(p_CV_C - p_AV_A) + 2p_0V_0$$
$$= \frac{3}{2}(4p_0V_0 - p_0V_0) + 2p_0V_0$$
$$= \frac{13}{2}p_0V_0$$

25. **(c)** In equilibrium,

$$p_0A = Mg \qquad \ldots\text{(i)}$$

when slightly displaced downwards,



$$dp = -\gamma\left(\frac{p_0}{V_0}\right)dV$$

$$\left(\text{As in adiabatic process, } \frac{dp}{dV}=-\gamma\frac{p}{V}\right)$$

$\therefore$ Restoring force,

$$F = (dp)A$$
$$= -\left(\frac{\gamma p_0}{V_0}\right)(A)(Ax)$$
$$F \propto -x$$

Therefore, motion is simple harmonic comparing with $F = -kx$ we have

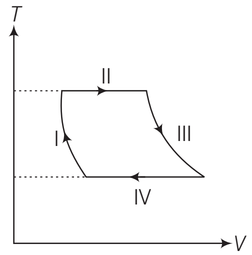
$$k=\frac{\gamma p_0A^2}{V_0}$$

$$\therefore \qquad f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$=\frac{1}{2\pi}\sqrt{\frac{\gamma p_0A^2}{MV_0}}$$

JEE Advanced

1. One mole of a monoatomic ideal gas undergoes a cyclic process as shown in the figure (where, V is the volume and T is the temperature). Which of the statements below is (are) true ?

(More than One Correct Option, 2018)



(a) Process I is an isochoric process

(b) In process II, gas absorbs heat

(c) In process IV, gas releases heat

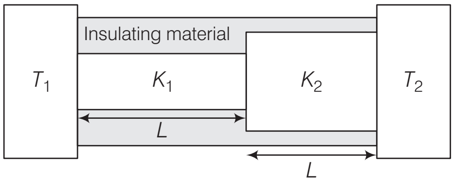
(d) Processes I and III are not isobaric

2. Two men are walking along a horizontal straight line in the same direction. The main in front walks at a speed 1.0ms^{-1} and the man behind walks at a speed 2.0 ms^{-1}. A third man is standing at a height 12 m above the same horizontal line such that all three men are in a vertical plane. The two walking men are blowing identical whistles which emit a sound of frequency 1430 Hz. The speed of sound in air 330 ms^{-1}. At the instant, when the moving men are 10 m apart, the stationary man is equidistant from them. The frequency of beats in Hz, heard by the stationary man at this instant, is ....

(Numerical Value, 2018)

3. Two conducting cylinders of equal length but different radii are connected in series between two heat baths kept at temperatures $T_1 = 300\text{K}$ and $T_2 = 100\text{K}$, as shown in the figure. The radius of the bigger cylinder is twice that of the smaller one and the thermal conductivities of the materials of the smaller and the larger cylinders are K_1 and K_2, respectively. If the temperature at the junction of the cylinders in the steady state is 200K, then $K_1/K_2 =$.......... .

(Numerical Value, 2018)



4. In an experiment to measure the speed of sound by a resonating air column, a tuning fork of frequency 500 Hz is used. The length of the air column is varied by changing the level of water in the resonance tube. Two successive resonances are heard at air columns of length 50.7 cm and 83.9 cm. Which of the following statements is (are) true?**(More than One Correct Option, 2018)**

(a) The speed of sound determined from this experiment is 332 m s^{-1}

(b) The end correction in this experiment is 0.9 cm

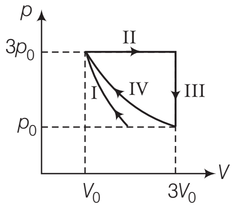
(c) The wavelength of the sound wave is 66.4 cm

(d) The resonance at 50.7 cm corresponds to the fundamental harmonic

5. One mole of a monoatomic ideal gas undergoes an adiabatic expansion in which its volume becomes eight times its initial value. If the initial temperature of the gas is 100 K and the universal gas constant $R = 8.0 \text{ j mol}^{-1} \text{ K}^{-1}$, the decrease in its internal energy in joule, is ....

(Numerical Value, 2018)

6. One mole of a monoatomic ideal gas undergoes four thermodynamic processes as shown schematically in the pV-diagram below.



Among these four processes, one is isobaric, one is isochoric, one is isothermal and one is adiabatic. Match the processes mentioned in List-l with the corresponding statements in List-II.

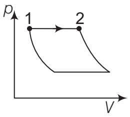
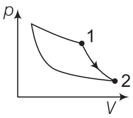
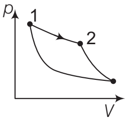
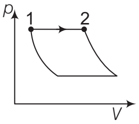
(Matching Type Question, 2018)

List-I	List-II
P. In process I	1. Work done by the gas is zero
Q. In process II	2. Temperature of the gas remains unchanged
R. In process III	3. No heat is exchanged between the gas and its surroundings
S. In process IV	4. Work done by the gas is $6p_0V_0$

(a) P → 4; Q → 3; R → 1; S → 2
(b) P → 1; Q → 3; R → 2; S → 4
(c) P → 3; Q → 4; R → 1; S → 2
(d) P → 3; Q → 4; R → 2; S → 1

Directions (Q.Nos. 7-8) Matching the information given in the three columns of the following table.

An ideal gas is undergoing a cyclic thermodynamic process in different ways as shown in the corresponding p-V diagrams in column 3 of the table. Consider only the path from state 1 to state 2. W denotes the corresponding work done on the system. The equations and plots in the table have standards notations and used in thermodynamic processes. Here γ is the ratio of heat capacities at constant pressure and constant volume. The number of moles in the gas is n. **(Matching Type, 2017)**

Column 1	Column 2	Column 3
(I) $W_{1\to 2} = \frac{1}{\gamma - 1}(p_2V_2 - p_1V_1)$	(i) Isothermal	(P) 
(II) $W_{1\to 2} = -pV_2 + pV_1$	(ii) Isochoric	(Q) 
(III) $W_{1\to 2} = 0$	(iii) Isobaric	(R) 
(IV) $W_{1\to 2} = -nRT \ln\left(\frac{V_2}{V_1}\right)$	(iv) Adiabatic	(S) 

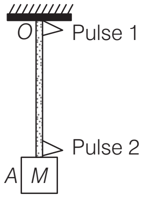
7. Which of the following options is the only correct representation of a process in which $\Delta U = \Delta Q - p\Delta V$?

(a) (II) (iii) (S)
(b) (II) (iii) (P)
(c) (III) (iii) (P)
(d) (II) (iv) (R)

8. Which one of the following options is the correct combination?

(a) (II) (iv) (P)
(b) (III) (ii) (S)
(c) (II) (iv) (R)
(d) (IV) (ii) (S)

9. A block M hangs vertically at the bottom end of a uniform rope of constant mass per unit length. The top end of the rope is attached to a fixed rigid support at O. A transverse wave pulse (Pulse 1) of wavelength λ_0 is produced at point O on the rope. The pulse takes time T_{OA} to reach point A. If the wave pulse of wavelength λ_0 is produced at point A (Pulse 2) without disturbing the position of M it takes time T_{AO} to reach point O. Which of the following options is/are correct? **(More than One Correct Option, 2017)**



(a) The time $T_{AO} = T_{OA}$
(b) The wavelength of Pulse 1 becomes longer when it reaches point A
(c) The velocity of any pulse along the rope is independent of its frequency and wavelength
(d) The velocities of the two pulses (Pulse 1 and Pulse 2) are the same at the mid-point of rope

10. A stationary source emits sound of frequency $f_0 = 492$ Hz. The sound is reflected by a large car approaching the source with a speed of $2\ \text{ms}^{-1}$. The reflected signal is received by the source and superposed with the original. What will be the beat frequency of the resulting signal in Hz? (Given that the speed of sound in air is $330\ \text{ms}^{-1}$ and the car reflects the sound at the frequency it has received). **(Single Integer Type, 2017)**

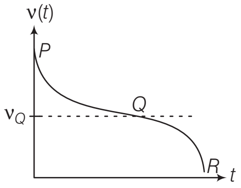
11. Two loudspeakers M and N are located 20 m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car in initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/h along the perpendicular bisector of MN.
It crosses Q and eventually reaches a point R, 1800 m away from Q. Let $v(t)$ represent the beat frequency measured by a person sitting in the car at time t. Let ν_P, ν_Q and ν_R be the beat frequencies measured at locations P, Q and R respectively.

The speed of sound in air is 330 ms^{-1}.

Which of the following statement(s) is (are) true regarding the sound heard by the person?

(More than One Correct Option, 2016)

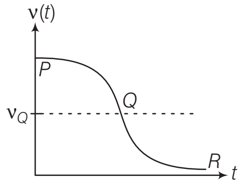
(a) The plot below represents schematically the variation of beat frequency with time



(b) The rate of change in beat frequency is maximum when the car passes through Q

(c) $\nu_P + \nu_R = 2\nu_Q$

(d) The plot below represents schematically the variations of beat frequency with time



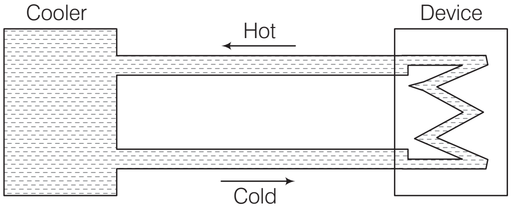
12. A gas is enclosed in a cylinder with a movable frictionless piston. Its initial thermodynamic state at pressure $P_i = 10^5$ Pa and volume $V_i = 10^{-3}\ \text{m}^3$ changes to a final state at $P_f = (1/32) \times 10^5$ Pa and $V_f = 8 \times 10^{-3}\ \text{m}^3$ in an adiabatic quasi-static process, such that $P^3V^5 =$ constant. Consider another thermodynamic process that brings the system from the same initial state to the same final state in two steps : an isobaric expansion at P_i followed by an isochoric (isovolumetric) process at volume V_f. The amount of heat supplied to the system in the two-step process is approximately

(Single Correct Option, 2016)

(a) 112 J (b) 294 J (c) 588 J (d) 813 J

13. A water cooler of storage capacity 120 litres can cool water at a constant rate of P watts. In a closed circulation system (as shown schematically in the figure), the water from the cooler is used to cool an external device that generates constantly 3 kW of heat (thermal load). The temperature of water fed into the device cannot exceed 30°C and the entire stored 120 litres of water is initially cooled to 10°C.

The entire system is thermally insulated. The minimum value of P (in watts) for which the device can be operated for 3 hours is **(Single Correct Option, 2016)**



(Specific heat of water is 4.2 kJ $\text{kg}^{-1}\text{K}^{-1}$ and the density of water is 1000 kg m^{-3})

(a) 1600 (b) 2067 (c) 2533 (d) 3933

14. A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated (P) by the metal. The sensor has a scale that displays $\log_2(P/P_0)$, where P_0 is a constant. When the metal surface is at a temperature of 487°C, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767°C? **(Single Integer Type, 2016)**

15. The ends Q and R of two thin wires, PQ and RS, are soldered (joined) together. Initially, each of the wire has a length of 1 m 10°C. Now, the end P is maintained at 10°C, while the end S is heated and maintained at 400°C. The system is thermally insulated from its surroundings. If the thermal conductivity of wire PQ is twice that of the wire RS and the coefficient of linear thermal expansion of PQ is $1.2 \times 10^{-5}\ \text{K}^{-1}$, the change in length of the wire PQ is

(Single Correct Option, 2016)

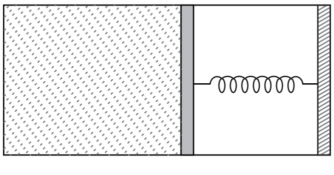
(a) 0.78 mm (b) 0.90 mm
(c) 1.56 mm (d) 2.34 mm

16. A container of fixed volume has a mixture of one mole of hydrogen and one mole of helium in equilibrium at temperature T. Assuming the gases are ideal, the correct statements is/are

(a) The average energy per mole of the gas mixture is $2RT$
(b) The ratio of speed of sound in the gas mixture to that in helium gas is $\sqrt{\frac{6}{5}}$
(c) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $\frac{1}{2}$
(d) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $\frac{1}{\sqrt{2}}$

(More than One Correct Option, 2015)

17. An ideal monoatomic gas is confined in a horizontal cylinder by a spring loaded piston (as shown in the figure). Initially the gas is at temperature T_1, pressure P_1 and volume V_1 and the spring is in its relaxed state. The gas is then heated very slowly to temperature T_2, pressure P_2 and volume V_2. During this process the piston moves out by a distance x.



Ignoring the friction between the piston and the cylinder, the correct statements is/are **(More than One Correct Option, 2015)**

(a) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the energy stored in the spring is $\frac{1}{4}P_1V_1$
(b) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is $3P_1V_1$
(c) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the work done by the gas is $\frac{7}{3}P_1V_1$
(d) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the heat supplied to the gas is $\frac{17}{6}P_1V_1$

18. Two spherical stars A and B emit blackbody radiation. The radius of A is 400 times that of B and A emits 10^4 times the power emitted from B. The ratio $\left(\frac{\lambda_A}{\lambda_B}\right)$ of their wavelengths λ_A and λ_B at which the peaks occur in their respective radiation curves is **(Single Integer Type, 2015)**

19. Four harmonic waves of equal frequencies and equal intensities I_0 have phase angles $0, \frac{\pi}{3}, \frac{2\pi}{3}$ and π. When they are superposed, the intensity of the resulting wave is nI_0. The value of n is

(Single Integer Type, 2015)

20. Two vehicles, each moving with speed u on the same horizontal straight road, are approaching each other. Wind blows along the road with velocity w. One of these vehicles blows a whistle of frequency f_1. An observer in the other vehicle hears the frequency of the whistle to be f_2. The speed of sound in still air is v. The correct statement(s) is (are)

(More than One Correct Option, 2014)

(a) If the wind blows from the observer to the source, $f_2 > f_1$.
(b) If the wind blows from the source to the observer, $f_2 > f_1$
(c) If the wind blows from the observer to the source, $f_2 < f_1$
(d) If the wind blows from the source to the observer, $f_2 < f_1$

21. One end of a taut string of length 3 m along the x-axis is fixed at $x = 0$. The speed of the waves in the string is $100\,\text{ms}^{-1}$. The other end of the string is vibrating in the y-direction so that stationary waves are set up in the string. The possible waveform(s) of these stationary wave is (are)

(More than One Correct Option, 2014)

(a) $y(t) = A \sin \frac{\pi x}{6} \cos \frac{50\pi t}{3}$

(b) $y(t) = A \sin \frac{\pi x}{3} \cos \frac{100\pi t}{3}$

(c) $y(t) = A \sin \frac{5\pi x}{2} \cos \frac{250\pi t}{3}$

(d) $y(t) = A \sin \frac{5\pi x}{2} \cos 250\pi t$

22. A student is performing an experiment using a resonance column and a tuning fork of frequency $244\,\text{s}^{-1}$. He is told that the air in the tube has been replaced by another gas (assume that the column remains filled with the gas). If the minimum height at which resonance occurs is (0.350 ± 0.005) m, the gas in the tube is (Useful information : $\sqrt{167RT} = 640\ \text{J}^{1/2}\ \text{mole}^{-1/2}$; $\sqrt{140RT} = 590\ \text{J}^{1/2}\ \text{mole}^{-1/2}$. The molar masses M in grams are given in the options. Take the value of $\sqrt{10/M}$ for each gas as given there.) **(Single Correct Option, 2014)**

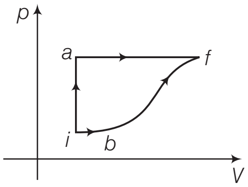
(a) Neon ($M = 20$, $\sqrt{10/20} = 7/10$)

(b) Nitrogen ($M = 28$, $\sqrt{10/28} = 3/5$)

(c) Oxygen ($M = 32$, $\sqrt{10/32} = 9/16$)

(d) Argon ($M = 36$, $\sqrt{10/36} = 17/32$)

23. A thermodynamic system is taken from an initial state i with internal energy $U_i = 100$ J to the final state f along two different paths iaf and ibf, as schematically shown in the figure. The work done by the system along the paths af, ib and bf are $W_{af} = 200$ J, $W_{ib} = 50$ J and $W_{bf} = 100$ J respectively. The heat supplied to the system along the path iaf, ib and bf are Q_{iaf}, Q_{ib} and Q_{bf} respectively. If the internal energy of the system in the state b is $U_b = 200$ J and $Q_{iaf} = 500$ J, the ratio Q_{bf} / Q_{ib} is



(Single Integer Type, 2014)

24. Parallel rays of light of intensity $I = 912\ \text{Wm}^{-2}$ are incident on a spherical black body kept in surroundings of temperature 300 K. Take Stefan constant $\sigma = 5.7 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$ and assume that the energy exchange with the surroundings is only through radiation. The final steady state temperature of the black body is close to

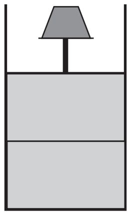
(Single Correct Option 2014)

(a) 330 K (b) 660 K

(c) 990 K (d) 1550 K

Passage (Q. Nos. 25-26)

In the figure a container is shown to have a movable (without friction) piston on top. The container and the piston are all made of perfectly insulating material allowing no heat transfer between outside and inside the container. The container is divided into two compartments by a rigid partition made of a thermally conducting material that allows slow transfer of heat.



The lower compartment of the container is filled with 2 moles of an ideal monoatomic gas at 700 K and the upper compartment is filled with 2 moles of an ideal diatomic gas at 400 K. The heat capacities per mole of an ideal monoatomic gas are $C_V = \frac{3}{2} R$, $C_p = \frac{5}{2} R$, and those for an ideal diatomic gas are $C_V = \frac{5}{2} R$, $C_p = \frac{7}{2} R$.

(Passage Type, 2014)

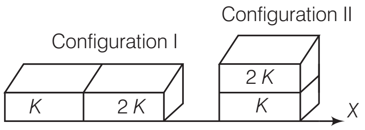
25. Consider the partition to be rigidly fixed so that it does not move. When equilibrium is achieved, the final temperature of the gases will be

(a) 550 K (b) 525 K (c) 513 K (d) 490 K

26. Now consider the partition to be free to move without friction so that the pressure of gases in both compartments is the same. Then total work done by the gases till the time they achieve equilibrium will be

(a) 250 R (b) 200 R (c) 100 R (d) –100 R

27. Two rectangular blocks, having indentical dimensions, can be arranged either in configuration I or in configuration II as shown in the figure. One of the blocks has thermal conductivity K and the other $2K$. The temperature difference between the ends along the x-axis is the same in both the configurations. It takes 9 s to transport a certain amount of heat from the hot end to the cold end in the configuration I. The time to transport the same amount of heat in the configuration II is **(Single Correct Option, 2013)**



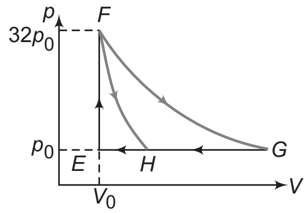
(a) 2.0 s (b) 3.0 s (c) 4.5 s (d) 6.0 s

28. Two non-reactive monoatomic ideal gases have their atomic masses in the ratio 2 : 3. The ratio of their partial pressures, when enclosed in a vessel kept at a constant temperature, is 4 : 3. The ratio of their densities is **(Single Correct Option, 2013)**

(a) 1 : 4 (b) 1 : 2 (c) 6 : 9 (d) 8 : 9

29. One mole of a monatomic ideal gas is taken along two cyclic processes $E \rightarrow F \rightarrow G \rightarrow E$ and $E \rightarrow F \rightarrow H \rightarrow E$ as shown in the p-V diagram.

The processes involved are purely isochoric, isobaric, isothermal or adiabatic.



Match the paths in Column I with the magnitudes of the work done in Column II and select the correct answer using the codes given below the lists.

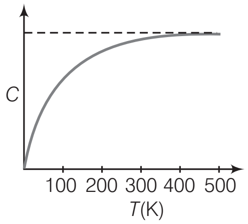
(Matching Type, 2013)

Column I	Column II
P. $G \rightarrow E$	1. 160 p_0V_0 ln 2
Q. $G \rightarrow H$	2. 36 p_0V_0
R. $F \rightarrow H$	3. 24 p_0V_0
S. $F \rightarrow G$	4. 31 p_0V_0

Codes

	P	Q	R	S		P	Q	R	S
(a)	2	3	1	4	(b)	1	2	4	3
(c)	4	3	2	1	(d)	2	3	4	1

30. The figure below shows the variation of specific heat capacity (C) of a solid as a function of temperature (T). The temperature is increased continuously from 0 to 500 K at a constant rate. Ignoring any volume change, the following statement(s) is (are) correct to reasonable approximation.



(More than One Correct Option, 2013)

(a) the rate at which heat is absorbed in the range 0-100 K varies linearly with temperature T

(b) heat absorbed in increasing the temperature from 0-100 K is less than the heat required for increasing the temperature from 400-500 K

(c) there is no change in the rate of heat absorbtion in the range 400-500 K

(d) the rate of heat absorption increases in the range 200-300 K

31. A horizontal stretched string, fixed at two ends, is vibrating in its fifth harmonic according to the equation, $y(x, t) = (0.01 \text{ m}) [\sin(62.8 \text{ m}^{-1})x] \cos[(628 \text{ s}^{-1})t]$.

Assuming $\pi = 3.14$, the correct statement(s) is (are)

(a) the number of nodes is 5

(b) the length of the string is 0.25 m

(c) the maximum displacement of the mid-point of the string from its equilibrium position is 0.01 m

(d) the fundamental frequency is 100 Hz

(More than One Correct Option, 2013)

Answer *with* Explanations

1. **(b,c,d)** (b) Process-II is isothermal expansion,

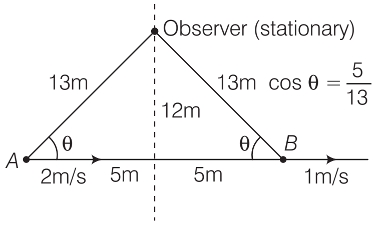
$$\Delta U = 0, W > 0$$
$$\Delta Q = W > 0$$

(c) Process-IV is isothermal compression,

$$\Delta U = 0, W < 0$$
$$\Delta Q = W < 0$$

(d) Process-I and III are not isobaric because in isobaric process $T \propto V$ hence, T-V graph should be a straight line passing through origin.

2. (5 **Hz**)



$$f_A = 1430\left[\frac{330}{330 - 2\cos\theta}\right]$$

$$= 1430\left[\frac{1}{1 - \frac{2\cos\theta}{330}}\right] \approx 1430\left[1 + \frac{2\cos\theta}{330}\right]$$

[from binomial expansion]

$$f_B = 1430\left[\frac{330}{330 + 1\cos\theta}\right]$$

$$\approx 1430\left[1 - \frac{\cos\theta}{330}\right]$$

$$\text{Beat frequency} = f_A - f_B = 1430\left[\frac{3\cos\theta}{330}\right] = 13\cos\theta$$

$$= 13\left(\frac{5}{13}\right) = 5.00 \text{ Hz}$$

3. **(4)** Rate of heat flow will be same,

$$\therefore \quad \frac{300 - 200}{R_1} = \frac{200 - 100}{R_2} \qquad \left(\text{as } H = \frac{dQ}{dt} = \frac{T \cdot D}{R}\right)$$

$$\therefore \quad R_1 = R_2 \Rightarrow \frac{L_1}{K_1 A_1} = \frac{L_2}{K_2 A_2}$$

$$\Rightarrow \quad \frac{K_1}{K_2} = \frac{A_2}{A_1} = 4$$

4. **(a, c)** Let n th harmonic is corresponding to 50.7 cm and $(n+1)$th harmonic is corresponding 83.9 cm.

$\therefore$ Their difference is $\frac{\lambda}{2}$.

$$\therefore \quad \frac{\lambda}{2} = (83.9 - 50.7) \text{ cm}$$

$$\text{or} \quad \lambda = 66.4 \text{ cm}$$

$$\therefore \quad \frac{\lambda}{4} = 16.6 \text{ cm}$$

Length corresponding to fundamental mode must be close to $\frac{\lambda}{4}$ and 50.7 cm must be an odd multiple of this length. $16.6 \times 3 = 49.8$ cm. Therefore, 50.7 is 3rd harmonic.

If end correction is e, then

$$e + 50.7 = \frac{3\lambda}{4}$$

$$e = 49.8 - 50.7 = -0.9 \text{ cm}$$

$\therefore$ Speed of sound, $v = f\lambda$

$$\Rightarrow \quad v = 500 \times 66.4 \text{ cm/s} = 332 \text{ m/s}$$

5. **(900)** Given, $n = 1$, $\gamma = \frac{5}{3}$

T-V equation in adiabatic process is

$$TV^{\gamma - 1} = \text{constant}$$

$$\therefore \quad T_1 V_1^{\gamma - 1} = T_2 V_2^{\gamma - 1}$$

$$\Rightarrow \quad T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma - 1} = 100 \times \left(\frac{1}{8}\right)^{\frac{2}{3}}$$

$$\Rightarrow \quad T_2 = 25 \text{ K}$$

$C_V = \frac{3}{2}R$ for monoatomic gas

$$\therefore \quad \Delta U = nC_V\Delta T = n \times \left(\frac{3R}{2}\right)(T_2 - T_1)$$

$$= 1 \times \frac{3}{2} \times 8 \times (25 - 100) = -900 \text{ J}$$

$\therefore$ Decrease in internal energy = 900 J

6. (c) $\left(\dfrac{dp}{dV}\right)_{\text{adiabatic}} = \gamma\left(\dfrac{dp}{dV}\right)_{\text{isothermal}}$

List-I

(P) Process I $\Rightarrow$ Adiabatic $\Rightarrow Q = 0$

(Q) Process II $\Rightarrow$ Isobaric

$\therefore \quad W = p\Delta V = 3p_0[3V_0 - V_0] = 6p_0V_0$

(R) Process III $\Rightarrow$ Isochoric $\Rightarrow W = 0$

(S) Process (IV) $\Rightarrow$ Isothermal

$\Rightarrow$ Temperature = Constant

7. **(b)** $\Delta U = \Delta Q - p\Delta V$

$$\Delta U + p\Delta V = \Delta Q$$

As $\Delta U \neq 0, W \neq 0, \Delta Q \neq 0$. The process represents, isobaric process

$$W_{\text{gas}} = -p(\Delta V) = -p(V_2 - V_1) = -pV_2 + pV_1$$

Graph '*P*' satisfies isobaric process.

8. **(b)** Work done in isochoric process is zero.

$$W_{12} = 0 \text{ as } \Delta V = 0$$

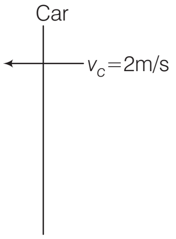
Graph '*S*' represents isochoric process.

9. **(a,c,d** or **a,c)** $v = \sqrt{\dfrac{T}{\mu}}$, so speed at any position will be same for both pulses, therefore time taken by both pulses will be same.

$$\lambda f = v \quad \Rightarrow \quad \lambda = \frac{v}{f}$$

since when pulse 1 reaches at *A* tension and hence speed decreases therefore λ decreases.

10. **(6)**



Frequency observed at car

$$f_1 = f_0\left(\frac{v + v_C}{v}\right) \quad (v = \text{speed of sound})$$

Frequency of reflected sound as observed at the source

$$f_2 = f_1\left(\frac{v}{v - v_C}\right) = f_0\left(\frac{v + v_C}{v - v_C}\right)$$

Beat frequency $= f_2 - f_0$

$$= f_0\left[\frac{v + v_C}{v - v_C} - 1\right] = f_0\left[\frac{2v_C}{v - v_C}\right]$$

$$= 492 \times \frac{2 \times 2}{328} = 6 \text{ Hz}$$

$$\Rightarrow \qquad \lambda \propto v \propto T$$

11. **(b,c,d)** Speed of car,

$$v = 60 \text{ km/h} = \frac{500}{3} \text{m/s}$$

At a point *S*, between *P* and *Q*

$$\nu'_M = \nu_M\left(\frac{C + v\cos\theta}{C}\right);$$

$$\nu'_N = \nu_N\left(\frac{C + v\cos\theta}{C}\right)$$

$$\Rightarrow \qquad \Delta v = (v_N - v_M)\left(1 + \frac{v\cos\theta}{C}\right)$$

Similarly, between *Q* and *R*

$$\Delta\nu = (\nu_N - \nu_M)\left(1 - \frac{v\cos\theta}{C}\right)$$

$$\frac{d(\Delta\nu)}{dt} = \pm(\nu_N - \nu_M)\frac{v}{C}\sin\theta\frac{d\theta}{dt}$$

$\theta \approx 0°$ at *P* and *R* as they are large distance apart.

$\Rightarrow$ Slope of graph is zero.

at *Q*, $\qquad \theta = 90°$

$\sin\theta$ is maximum also value of $\dfrac{d\theta}{dt}$ is maximum

as $\dfrac{d\theta}{dt} = \dfrac{v}{r}$, where *v* is its velocity and *r* is the length of the line joining *P* and *S*. and *r* is minimum at *Q*.

$\Rightarrow$ Slope is maximum at *Q*.

At *P*, $\nu_P = \Delta\nu = (\nu_N - \nu_M)\left(1 + \dfrac{V}{C}\right)$

$(\theta \approx 0°)$ At *R*, $\nu_R = \Delta\nu = (\nu_N - \nu_M)\left(1 - \dfrac{V}{C}\right)$

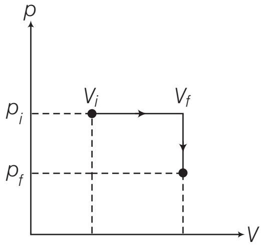
$(\theta \approx 0°)$ At *Q*,

$$\nu_Q = \Delta\nu = (\nu_N - \nu_M)$$

$$(\theta = 90°)$$

From these equations, we can see that $\nu_P + \nu_R = 2\nu_Q$

12. (c) In the first process : $p_iV_i^\gamma = p_fV_f^\gamma$



$$\Rightarrow \quad \frac{p_i}{p_f} = \left(\frac{V_f}{V_i}\right)^\gamma \quad \Rightarrow \quad 32 = 8^\gamma \quad \Rightarrow \quad \gamma = \frac{5}{3} \qquad \ldots\text{(i)}$$

For the two step process

$$W = p_i(V_f - V_i)$$

$$= 10^5(7 \times 10^{-3})$$

$$= 7 \times 10^2 \text{ J}$$

$$\Delta U = \frac{f}{2}(p_f V_f - p_i V_i)$$

$$= \frac{1}{\gamma - 1}\left(\frac{1}{4} \times 10^2 - 10^2\right)$$

$$\Delta U = -\frac{3}{2} \cdot \frac{3}{4} \times 10^2 = -\frac{9}{8} \times 10^2 \text{ J}$$

$$Q - W = \Delta U$$

$$\Rightarrow \quad Q = 7 \times 10^2 - \frac{9}{8} \times 10^2$$

$$= \frac{47}{8} \times 10^2 \text{ J} = 588 \text{ J}$$

13. **(b)** Heat generated in device in 3 h

$$= \text{time} \times \text{power} = 3 \times 3600 \times 3 \times 10^3$$

$$= 324 \times 10^5 \text{ J}$$

Heat used to heat water

$$= ms\,\Delta\theta = 120 \times 1 \times 4.2 \times 10^3 \times 20 \text{ J}$$

Heat absorbed by coolant

$$= Pt = 324 \times 10^5 - 120 \times 1 \times 4.2 \times 10^3 \times 20 \text{ J}$$

$$Pt = (325 - 100.8) \times 10^5 \text{ J} = 223.2 \times 10^5 \text{ J}$$

$$P = \frac{223.2 \times 10^5}{3600} = 2067 \text{ W}$$

14. **(9)** $\log_2 \frac{p_1}{p_0} = 1,$

Therefore, $\frac{p_1}{p_0} = 2$

According to Stefan's law, $p \propto T$

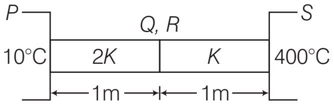
$$\Rightarrow \quad \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{2767 + 273}{487 + 273}\right)^4 = 4^4$$

$$\frac{p_2}{p_1} = \frac{p_2}{2p_0} = 4^4 \quad \Rightarrow \quad \frac{p_2}{p_0} = 2 \times 4^4$$

$$\log_2 \frac{p_2}{p_0} = \log_2[2 \times 4^4] = \log_2 2 + \log_2 4^4$$

$$= 1 + \log_2 2^8 = 1 + 8 = 9$$

15. **(a)**



Rate of heat flow from P to Q

$$\frac{dQ}{dt} = \frac{2KA(T - 10)}{1}$$

Rate of heat flow from Q to S

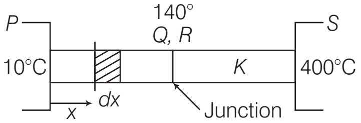
$$\frac{dQ}{dt} = \frac{KA(4000 - T)}{1}$$

At steady state, state rate of heat flow is same

$$\therefore \quad \frac{2KA(T - 10)}{1} = KA(400 - T)$$

or $\quad 2T - 20 = 400 - T$ or $3T = 420$

$\therefore \quad T = 140°$



Temperature of junction is 140°C

Temperature at a distance x from end P is

$$T_x = (130x + 10°)$$

Change in length dx is suppose dy

Then, $\quad dy = \alpha dx\,(T_x - 10)$

$$\int_0^{\Delta y} dy = \int_0^1 \alpha dx (130x + 10 - 10)$$

$$\Delta y = \left[\frac{\alpha x^2}{2} \times 130\right]_0^1$$

$$\Delta y = 1.2 \times 10^{-5} \times 65$$

$$\Delta y = 78.0 \times 10^{-5} \text{ m} = 0.78 \text{ mm}$$

16. **(a, b, d)** (a) Total internal energy

$$U = \frac{f_1}{2} nRT + \frac{f_2}{2} nRT$$

$$(U_{\text{ave}})_{\text{per mole}} = \frac{U}{2n} = \frac{1}{4}[5RT + 3RT] = 2RT$$

(b) $$\gamma_{\text{mix}} = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_2 C_{v_1} + n_2 C_{v_2}}$$

$$= \frac{(1)\frac{7R}{2} + (1)\frac{5R}{2}}{(1)\frac{5R}{2} + (1)\frac{3R}{2}} = \frac{3}{2}$$

$$M_{\text{mix}} = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2} = \frac{M_1 + M_2}{2} = \frac{2 + 4}{2} = 3$$

Speed of sound $V = \sqrt{\frac{\gamma RT}{M}}$

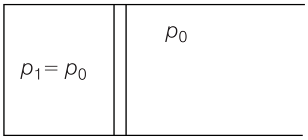
$$\Rightarrow \quad V \propto \sqrt{\frac{\gamma}{M}}$$

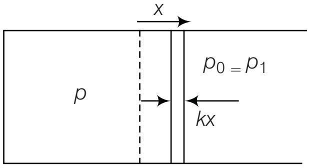
$$\frac{V_{\text{mix}}}{V_{\text{He}}} = \sqrt{\frac{\gamma_{\text{mix}}}{\gamma_{\text{He}}} \times \frac{M_{\text{He}}}{M_{\text{mix}}}} = \sqrt{\frac{3/2}{5/3} \times \frac{4}{3}} = \sqrt{\frac{6}{5}}$$

(d) $$V_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

$$\Rightarrow \quad V_{\text{rms}} \propto \frac{1}{\sqrt{M}}, \frac{V_{\text{He}}}{V_{\text{H}}} = \sqrt{\frac{M_{\text{H}}}{M_{\text{He}}}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$$

17. **(a, b, c)** **Note** *This question can be solved if right hand side chamber is assumed open, so that its pressure remains constant even if the piston shifts towards right.*





(a) $$pV = nRT$$

$$\Rightarrow \quad p \propto \frac{T}{V}$$

Temperature is made three times and volume is doubled

$$\Rightarrow \quad p_2 = \frac{3}{2} p_1$$

Further $$x = \frac{\Delta V}{A} = \frac{V_2 - V_1}{A} = \frac{2V_1 - V_1}{A} = \frac{V_1}{A}$$

$$p_2 = \frac{3p_1}{2} = p_1 + \frac{kx}{A} \Rightarrow kx = \frac{p_1 A}{2}$$

Energy of spring

$$\frac{1}{2} kx^2 = \frac{p_1 A}{4} x = \frac{p_1 V_1}{4}$$

(b) $\Delta U = nc_v \Delta T = n\left(\frac{3}{2}R\right)\Delta T$

$$= \frac{3}{2}(p_2 V_2 - p_1 V_1)$$

$$= \frac{3}{2}\left[\left(\frac{3}{2} p_1\right)(2V_1) - p_1 V_1\right] = 3\, p_1 V_1$$

(c) $p_2 = \dfrac{4p_1}{3}$

$$\Rightarrow \quad p_2 = \frac{4}{3} p_1 = p_1 + \frac{kx}{A}$$

$$\Rightarrow \quad kx = \frac{p_1 A}{3}$$

$$\Rightarrow \quad x = \frac{\Delta V}{A} = \frac{2V_1}{A}$$

$$W_{\text{gas}} = (p_0\, \Delta V + W_{\text{spring}})$$

$$= \left(p_1 Ax + \frac{1}{2} kx \cdot x\right)$$

$$= +\left(p_1 A \cdot \frac{2V_1}{A} + \frac{1}{2} \cdot \frac{p_1 A}{3} \cdot \frac{2V_1}{A}\right)$$

$$= 2 p_1 V_1 + \frac{p_1 V_1}{3} = \frac{7 p_1 V_1}{3}$$

(d) $\Delta Q = W + \Delta U$

$$= \frac{7 p_1 V_1}{3} + \frac{3}{2}(p_2 V_2 - p_1 V_1)$$

$$= \frac{7 p_1 V_1}{3} + \frac{3}{2}\left(\frac{4}{3} p_1 \cdot 3V_1 - p_1 V_1\right)$$

$$= \frac{7 p_1 V_1}{3} + \frac{9}{2} p_1 V_1 = \frac{41 p_1 V_1}{6}$$

Note $\Delta U = \frac{3}{2}(p_2 V_2 - p_1 V_1)$ *has been obtained in part (b).*

18. **(2)** Power, $P = (\sigma T^4 A) = \sigma T^4 (4\pi R^2)$

or, $$P \propto T^4 R^2 \qquad \ldots\text{(i)}$$

According to Wien's law,

$$\lambda \propto \frac{1}{T}$$

(λ is the wavelength at which peak occurs)

∴ Eq. (i) will become,

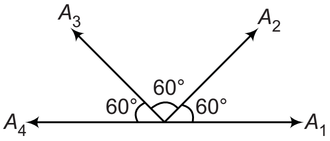
$$P \propto \frac{R^2}{\lambda^4}$$

or $$\lambda \propto \left[\frac{R^2}{P}\right]^{1/4}$$

$$\Rightarrow \quad \frac{\lambda_A}{\lambda_B} = \left[\frac{R_A}{R_B}\right]^{1/2} \left[\frac{P_B}{P_A}\right]^{1/4}$$

$$= [400]^{1/2} \left[\frac{1}{10^4}\right]^{1/4} = 2$$

19. **(3)** Let individual amplitudes are A_0 each. Amplitudes can be added by vector method.



$$A_1 = A_2 = A_3 = A_4 = A_0$$

Resultant of A_1 and A_4 is zero. Resultant of A_2 and A_3 is

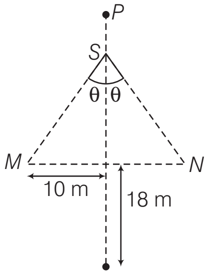
$$A = \sqrt{A_0^2 + A_0^2 + 2A_0 A_0 \cos 60°} = \sqrt{3} A_0$$

This is also the net resultant.

Now, $I \propto A^2$

∴ Net intensity will become $3I_0$.

∴ Answer is 3.



20. **(a,b)** When wind blows from S to O

$$f_2 = f_1\left(\frac{v + w + u}{v + w - u}\right)$$

or $\qquad f_2 > f_1$

when wind blows from O to S

$$f_2 = f_1\left(\frac{v - w + u}{v - w - u}\right) \Rightarrow f_2 > f_1$$

21. **(a,c,d)** There should be a node at $x = 0$ and antinode at $x = 3$ m. Also,

$$v = \frac{\omega}{k} = 100 \text{ m/s.}$$

$\therefore \qquad y = 0$ at $x = 0$

and $\qquad y = \pm\ A$ at $x = 3$ m.

Only (a), (c) and (d) satisfy the condition.

22. **(d)** Minimum length $= \dfrac{\lambda}{4}$

$\Rightarrow \qquad \lambda = 4l$

Now, $\qquad v = f\lambda = (244) \times 4 \times l$

as $\qquad l = 0.350 \pm 0.005$

$\Rightarrow v$ lies between 336.7 m/s to 346.5 m/s

Now, $v = \sqrt{\dfrac{\gamma RT}{M \times 10^{-3}}}$, here M is molecular mass in gram

$$= \sqrt{100\gamma RT} \times \sqrt{\frac{10}{M}}.$$

For monoatomic gas, $\gamma = 1.67$

$$\Rightarrow \qquad v = 640 \times \sqrt{\frac{10}{M}}$$

For diatomic gas,

$$\gamma = 1.4$$

$$\Rightarrow \qquad v = 590 \times \sqrt{\frac{10}{M}}$$

$$\therefore \qquad v_{\text{Ne}} = 640 \times \frac{7}{10} = 448 \text{ m/s}$$

$$v_{\text{Ar}} = 640 \times \frac{17}{32} = 340 \text{ m/s}$$

$$v_{\text{O}_2} = 590 \times \frac{9}{16} = 331.8 \text{ m/s}$$

$$v_{\text{N}_2} = 590 \times \frac{3}{5} = 354 \text{ m/s}$$

$\therefore$ Only possible answer is Argon.

23. **(2)** $\quad W_{ibf} = W_{ib} + W_{bf}$

$$= 50 \text{ J} + 100 \text{ J} = 150 \text{ J}$$

$$W_{iaf} = W_{ia} + W_{af}$$

$$= 0 + 200 \text{ J} = 200 \text{ J}$$

$$Q_{iaf} = 500 \text{ J}$$

So $\qquad \Delta U_{iaf} = Q_{iaf} - W_{iaf}$

$$= 500 \text{ J} - 200 \text{ J} = 300 \text{ J}$$

$$= U_f - U_i$$

So, $\qquad U_f = U_{iaf} + U_i$

$$= 300 \text{ J} + 100 \text{ J} = 400 \text{ J}$$

$$\Delta U_{ib} = U_b - U_i$$

$$= 200 \text{ J} - 100 \text{ J} = 100 \text{ J}$$

$$Q_{ib} = \Delta U_{ib} + W_{ib}$$

$$= 100 \text{ J} + 50 \text{ J} = 150 \text{ J}$$

$$Q_{ibf} = \Delta U_{ibf} + W_{ibf}$$

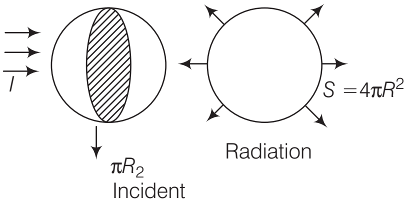
$$= \Delta U_{iaf} + W_{ibf}$$

$$= 300 \text{ J} + 150 \text{ J} = 450 \text{ J}$$

So, the required ratio $\dfrac{Q_{bf}}{Q_{ib}} = \dfrac{Q_{ibf} - Q_{ib}}{Q_{ib}}$

$$= \frac{450 - 150}{150} = 2$$

24. **(a)** In steady state



Energy incident per second = Energy radiated per second

$$\therefore \qquad I\pi R^2 = \sigma\,(T^4 - T_0^4)\,4\pi R^2$$

$$\Rightarrow \qquad I = \sigma\,(T^4 - T_0^4)\,4$$

$$\Rightarrow \qquad T^4 - T_0^4 = 40 \times 10^8$$

$$\Rightarrow \qquad T^4 - 81 \times 10^8 = 40 \times 10^8$$

$$\Rightarrow \qquad T^4 = 121 \times 10^8$$

$$\Rightarrow \qquad T \approx 330 \text{ K}$$

25. **(d)** Let final equilibrium temperature of gases is T

Heat rejected by gas by lower compartment

$$= nC_V\Delta T$$

$$= 2 \times \frac{3}{2} R\,(700 - T)$$

Heat received by the gas in above compartment

$$= nC_p\Delta T$$

$$= 2 \times \frac{7}{2} R\,(T - 400)$$

Equating the two, we get

$$2100 - 3T = 7T - 2800 \quad \Rightarrow \mathrm{T} = 490 \text{ K}$$

26. (d) $\Delta W_1 + \Delta U_1 = \Delta Q_1$ ...(i)

$\Delta W_2 + \Delta U_2 = \Delta Q_2$ ...(ii)

$\Delta Q_1 + \Delta Q_2 = 0$

$\therefore (nC_p\Delta T)_1 + (nC_p\Delta T)_2 = 0$

But $n_1 = n_2 = 2$

$$\therefore \quad \frac{5}{2}R\,(T - 700) + \frac{7}{2}R\,(T - 400) = 0$$

Solving, we get $T = 525$ K

Now, from equations (i) and (ii), we get

$\Delta W_1 + \Delta W_2 = -\Delta U_1 - \Delta U_2$

as $\Delta Q_1 + \Delta Q_2 = 0$

$$\therefore \quad \Delta W_1 + \Delta W_2 = -[(nC_V\Delta T)_1 + (nC_V\Delta T)_2]$$

$$= -\left[2 \times \frac{3}{2}R \times (525 - 700) + 2 \times \frac{5}{2}R \times (525 - 400)\right]$$

$$= -100R$$

27. (a) $R_{\text{I}} = R_1 + R_2 = \left(\frac{l}{KA}\right) + \left(\frac{l}{2KA}\right) = \frac{3}{2}\left(\frac{l}{KA}\right)$

$$\frac{1}{R_{\text{II}}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{KA}{l} + \frac{2KA}{l}$$

or $$R_{\text{II}} = \frac{l}{3KA} = \frac{R_{\text{I}}}{4.5}$$

Since thermal resistance R_{II} is 4.5 times less than thermal resistance R_{I}.

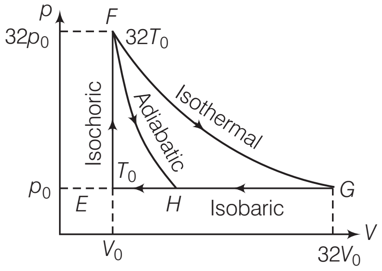
$$\therefore \quad t_{\text{II}} = \frac{t_{\text{I}}}{4.5} = \frac{9}{4.5}\text{ s} = 2\text{ s}$$

28. (d) $\rho = \frac{pM}{RT}$

$\therefore \quad \rho \propto pM$

or $$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)\left(\frac{M_1}{M_2}\right) = \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = \frac{8}{9}$$

29. (c)



In $F \to G$ work done in isothermal process is

$$nRT\ln\left(\frac{V_f}{V_i}\right) = 32\,p_0V_0\ln\left(\frac{32V_0}{V_0}\right)$$

$$= 32\,p_0V_0\ln 2^5$$

$$= 160p_0\,V_0\ln 2$$

In $G \to E$,

$$\Delta W = p_0\Delta V = p_0(31V_0) = 31\,p_0V_0$$

In $G \to H$ work done is less than $31\,p_0V_0$ *i.e.*, $24\,p_0V_0$

In $F \to H$ work done is $36\,p_0V_0$

30. (b, c, d) $Q = mCT$

$$\frac{dQ}{dt} = mc\frac{dT}{dt}$$

R = rate of absorption of heat $= \frac{dQ}{dt} \propto C$

(i) in 0 – 100 K

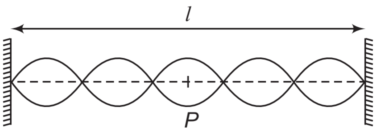
C increase, so R increases but not linearly

(ii) $\Delta Q = mC\Delta T$ as C is more in (400 K – 500 K) then (0 – 100 K) so heat is increasing

(iii) C remains constant so there no change in R from (400K – 500 K)

(iv) C is increases so R increases in range (200 K – 300K)

31. (b,c)



Number of nodes = 6

From the given equation, we can see that

$$k = \frac{2\pi}{\lambda} = 62.8\text{ m}^{-1}$$

$$\therefore \quad \lambda = \frac{2\pi}{62.8}\text{m} = 0.1\text{ m}$$

$$l = \frac{5\lambda}{2} = 0.25\text{ m}$$

The mid-point of the string is P, an antinode

$\therefore$ maximum displacement = 0.01 m

$$\omega = 2\pi f = 628\text{ s}^{-1}$$

$$\therefore \quad f = \frac{628}{2\pi} = 100\text{ Hz}$$

But this is fifth harmonic frequency.

$\therefore$ Fundamental frequency $f_0 = \frac{f}{5} = 20$ Hz.

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced, the NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. The exercises in this book have been divided into two sections viz., JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions. I am extremely thankful to (Dr.) Mrs. Sarita Pandey, Mr. Anoop Dhyani, Nisar Ahmad for their endless efforts during the project.

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

ELECTROSTATICS

Electric charges Conservation of charge, Coulomb's law-forces between two point charges, forces between multiple charges; Superposition principle and continuous charge distribution.

Electric field Electric field due to a point charge, Electric field lines, Electric dipole, Electric field due to a dipole, Torque on a dipole in a uniform electric field.

Electric flux, Gauss's law and its applications to find field due to infinitely long uniformly charged straight wire, Uniformly charged infinite plane sheet and uniformly charged thin spherical shell. Electric potential and its calculation for a point charge, Electric dipole and system of charges; Equipotential surfaces, Electrical potential energy of a system of two point charges in an electrostatic field.

Conductors and insulators, Dielectrics and electric polarization, Capacitor, Combination of Capacitors in series and in parallel, Capacitance of a parallel plate capacitor with and without dielectric medium between the plates, Energy stored in a capacitor.

CURRRENT ELECTRICITY

Electric current, Drift velocity, Ohm's law, Electrical resistance, Resistances of different materials, V-I characteristics of Ohmic and non-ohmic conductors, Electrical energy and power, Electrical resistivity, Colour code for resistors; Series and parallel combinations of resistors; Temperature dependence of resistance. Electric cell and its Internal resistance, Potential difference and emf of a cell, combination of cells in series and in parallel. Kirchhoff's laws and their applications. Wheatstone bridge, Meter bridge. Potentiometer – principle and its applications.

MAGNETIC EFFECTS OF CURRENT AND MAGNETISM

Biot-Savart law and its application to current carrying circular loop. Ampere's law and its applications to infinitely long current carrying straight wire and solenoid. Force on a moving charge in uniform magnetic and electric fields. Cyclotron.

Force on a current-carrying conductor in a uniform magnetic field. Force between two parallel current-carrying conductors-definition of ampere. Torque experienced by a current loop in uniform magnetic field; Moving coil galvanometer, its current sensitivity and conversion to ammeter and voltmeter.

Current loop as a magnetic dipole and its magnetic dipole moment. Bar magnet as an equivalent solenoid, magnetic field lines; Earth's magnetic field and magnetic elements. Para-, dia- and ferro- magnetic substances.

Magnetic susceptibility and permeability, Hysteresis, Electromagnets and permanent magnets.

ELECTROMAGNETIC INDUCTION AND ALTERNATING CURRENTS

Electromagnetic induction; Faraday's law, Induced emf and current; Lenz's law, Eddy currents. Self and mutual inductance. Alternating currents, Peak and *rms* value of alternating current/voltage; Reactance and impedance; LCR series circuit, Resonance; Quality factor, power in AC circuits, Wattless current. AC generator and transformer.

JEE Advanced

GENERAL

Verification of Ohm's law using voltmeter and ammeter. Specific resistance of the material of a wire using meter bridge and post office box.

ELECTRICITY AND MAGNETISM

Coulomb's law, Electric field and potential, Electrical potential energy of a system of point charges and of electrical dipoles in a uniform electrostatic field, Electric field lines, Flux of electric field, Gauss's law and its application in simple cases, such as, to find field due to infinitely long straight wire, Uniformly charged infinite plane sheet and uniformly charged thin spherical shell.

Capacitance, Parallel plate capacitor with and without dielectrics, Capacitors in series and parallel, Energy stored in a capacitor.

Electric current, Ohm's law, Series and parallel arrangements of resistances and cells, Kirchhoff's laws and simple applications, Heating effect of current.

Biot-Savart's law and Ampere's law, Magnetic field near a current-carrying straight wire, Along the axis of a circular coil and inside a long straight solenoid, Force on a moving charge and on a current-carrying wire in a uniform magnetic field.

Magnetic moment of a current loop, Effect of a uniform magnetic field on a current loop, Moving coil galvanometer, Voltmeter, Ammeter and their conversions.

ELECTROMAGNETIC INDUCTION

Faraday's law, Lenz's law, Self and mutual inductance, *RC, LR* and *LC* circuits with *DC* and AC sources.

23

Current Electricity

Chapter Contents

23.1 Introduction
23.2 Electric current
23.3 Electric currents in conductors
23.4 Drift velocity and Relaxation time
23.5 Resistance of a wire
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23.1 Introduction

An electrical circuit consists of some active and passive elements. The active elements such as a battery or a cell, supply electric energy to the circuit. On the contrary, passive elements consume or store the electric energy. The basic passive elements are resistor, capacitor and inductor.

A resistor opposes the flow of current through it and if some current is passed by maintaining a potential difference across it, some energy is dissipated in the form of heat. A capacitor is a device which stores energy in the form of electric potential energy. It opposes the variations in voltage. An inductor opposes the variations in current. It does not oppose the steady current through it. Fundamentally, electric circuits are a means for conveying energy from one place to another. As charged particles move within a circuit, electric potential energy is transferred from a source (such as a battery or a cell) to a device in which that energy is either stored or converted to another form, like sound in a stereo system or heat and light in a toaster or light bulb. Electric circuits are useful because they allow energy to be transported without any moving parts (other than the moving charged particles themselves).

In this chapter, we will study the basic properties of electric currents. We'll study the properties of batteries and how they cause current and energy transfer in a circuit. In this analysis, we will use the concepts of current, potential difference, resistance and electromotive force.

23.2 Electric Current

Flow of charge is called electric current. The direction of electric current is in the direction of flow of positive charge or in the opposite direction of flow of negative charge.

Current is defined quantitatively in terms of the rate at which net charge passes through a cross-section area of the conductor.

Thus,
$$\boxed{I = \frac{dq}{dt} \quad \text{or} \quad i = \frac{dq}{dt}}$$

We can have the following two concepts of current, as in the case of velocity, **instantaneous current** and **average current.**

Instantaneous current $= \dfrac{dq}{dt}$ = current at any point of time and

Average current $= \dfrac{q}{t}$

Hence-forth unless otherwise referred to, current would signify instantaneous current. By convention, the direction of the current is assumed to be that in which positive charge moves. In the SI system, the unit of current is ampere (A).

$$\boxed{1\ \text{A} = 1\ \text{C/s}}$$

Household currents are of the order of few amperes.

Flow of Charge

If current is passing through a wire then it implies that a charge is flowing through that wire. Further,

$$i = \frac{dq}{dt} \quad \Rightarrow \quad dq = i\,dt \qquad \ldots\text{(i)}$$

Now, three cases are possible :

Case 1 If given current is constant, then from Eq. (i) we can see that flow of charge can be obtained directly by multiplying that constant current with the given time interval. Or,

$$\boxed{\Delta q = i \times \Delta t}$$

Case 2 If given current is a function of time, then charge flow can be obtained by integration. Or,

$$\boxed{\Delta q = \int_{t_i}^{t_f} i\,dt}$$

Case 3 If current *versus* time is given, then flow of charge can be obtained by the area under the graph.

$$\boxed{\Delta q = \text{area under } i\text{-}t \text{ graph}}$$

✓ Extra Points to Remember

- The current is the same for all cross-sections of a conductor of non-uniform cross-section. Similar to the water flow, charge flows faster where the conductor is smaller in cross-section and slower where the conductor is larger in cross-section, so that charge rate remains unchanged.
- Electric current is very similar to water current, consider a water tank kept at some height and a pipe is connected to the water tank. The rate of flow of water through the pipe depends on the height of the tank. As the level of water in the tank falls, the rate of flow of water through the pipe also gets reduced. Just as the flow of water depends on the height of the tank or the level of water in the tank, the flow of current through a wire depends on the potential difference between the end points of the wire. As the potential difference is changed, the current will change. For example, during the discharging of a capacitor potential difference and hence, the current in the circuit decreases with time. To maintain a constant current in a circuit a constant potential difference will have to be maintained and for this a battery is used which maintains a constant potential difference in a circuit.
- Though conventionally a direction is associated with current (opposite to the motion of electrons), it is not a vector as the direction merely represents the sense of charge flow and not a true direction. Further, current does not obey the law of parallelogram of vectors, i.e. if two currents i_1 and i_2 reach a point we always have $i = i_1 + i_2$ whatever be the angle between i_1 and i_2.

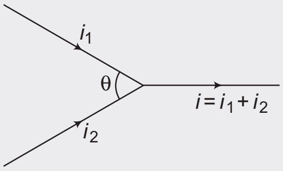


Fig. 23.1

- According to its magnitude and direction, current is usually divided into two types :
  - (i) **Direct current (DC)** If the magnitude and direction of current does not vary with time, it is said to be direct current (DC). Cell, battery or DC dynamo are its sources.
  - (ii) **Alternating current (AC)** If a current is periodic (with constant amplitude) and has half cycle positive and half negative, it is said to be alternating current (AC). AC dynamo is the source of it.
- If a charge q revolves in a circle with frequency f, the equivalent current,

$$\boxed{i = qf}$$

- In a conductor, normally current flow or charge flow is due to flow of free electrons.
- Charge is quantised. The quantum of charge is e. The charge on any body will be some integral multiple of e, i.e.

$$\boxed{q = \pm ne}$$

where, $n = 1, 2, 3 \ldots$

Example 23.1 *In a given time of* 10 s, 40 *electrons pass from right to left. In the same interval of time* 40 *protons also pass from left to right. Is the average current zero? If not, then find the value of average current.*

Solution No, the average current is not zero. Direction of current is the direction of motion of positive charge or in the opposite direction of motion of negative charge. So, both currents are from left to right and both currents will be added.

$$\therefore \quad I_{\text{av}} = I_{\text{electron}} + I_{\text{proton}}$$

$$= \frac{q_1}{t_1} + \frac{q_2}{t_2}$$

$$= \frac{40\,e}{10} + \frac{40\,e}{10} \qquad (q = ne)$$

$$= 8\,e$$

$$= 8 \times 1.6 \times 10^{-19}\ \text{A}$$

$$= 1.28 \times 10^{-18}\ \text{A}$$ **Ans.**

Example 23.2 *A constant current of* 4 *A passes through a wire for* 8 s. *Find total charge flowing through that wire in the given time interval.*

Solution Since, $i = \text{constant}$

$$\therefore \quad \Delta q = i \times \Delta t$$

$$= 4 \times 8$$

$$= 32\ \text{C}$$

Example 23.3 *A wire carries a current of* 2.0 *A. What is the charge that has flowed through its cross-section in* 1.0 *s? How many electrons does this correspond to?*

Solution $\because \quad i = \frac{q}{t}$

$$\therefore \quad q = it = (2.0\ \text{A})(1.0\ \text{s}) = 2.0\ \text{C}$$ **Ans.**

$$q = ne$$

$$\therefore \quad n = \frac{q}{e} = \frac{2.0}{1.6 \times 10^{-19}}$$

$$= 1.25 \times 10^{19}$$ **Ans.**

Example 23.4 *The current in a wire varies with time according to the relation*

$$i = (3.0\ A) + (2.0\ A/s)\,t$$

(a) How many coulombs of charge pass a cross-section of the wire in the time interval between $t = 0$ *and* $t = 4.0$ *s?*

(b) What constant current would transport the same charge in the same time interval?

Solution (a)
$$i = \frac{dq}{dt}$$

$$\therefore \quad \int_0^q dq = \int_0^4 idt$$

$$\therefore \quad q = \int_0^4 (3 + 2t)\, dt$$

$$= [3t + t^2]_0^4 = [12 + 16]$$

$$= 28\ \text{C}$$ **Ans.**

(b) $i = \frac{q}{t} = \frac{28}{4} = 7\ \text{A}$ **Ans.**

Example 23.5 *Current passing through a wire decreases linearly from* 10 *A to* 0 *in* 4 *s. Find total charge flowing through the wire in the given time interval.*

Solution Current *versus* time graph is as shown in figure.

Area under this graph will give us net charge flow.

Hence,

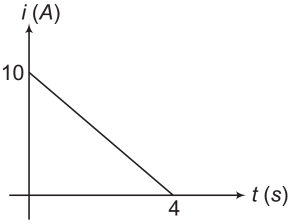


Fig. 23.2

$$\Delta q = \text{Area}$$

$$= \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 4 \times 10$$

$$= 20\ \text{C}$$ **Ans.**

INTRODUCTORY EXERCISE 23.1

1. How many electrons per second pass through a section of wire carrying a current of 0.7 A?
2. A current of 3.6 A flows through an automobile headlight. How many coulombs of charge flow through the headlight in 3.0 h?
3. A current of 7.5 A is maintained in wire for 45 s. In this time,
   (a) how much charge and
   (b) how many electrons flow through the wire?
4. In the Bohr model, the electron of a hydrogen atom moves in a circular orbit of radius 5.3×10^{-11} m with a speed of 2.2×10^6 m/s. Determine its frequency f and the current I in the orbit.
5. The current through a wire depends on time as, $i = (10 + 4t)$

   Here, i is in ampere and t in seconds. Find the charge crossed through a section in time interval between $t = 0$ to $t = 10$ s.
6. In an electrolyte, the positive ions move from left to right and the negative ions from right to left. Is there a net current? If yes, in what direction?

23.3 Electric Currents in Conductors

Conductors are those materials which can conduct electricity. Conductors can be broadly classified into two groups :

(i) Solid conductors

(ii) Electrolyte conductors

Normally in atoms and molecules, the negatively charged electrons and the positively charged nuclei are bound to each other and are thus not free to move. In solid conductors (notably metals), some of the electrons (called free electrons) are free to move within the bulk materials. In these conductors, current flow takes place due to these free electrons. Positive ions in these conductors are almost fixed. They do not move. So, they do not contribute in the current. In electrolyte solutions however, both positive and negative ions can move.

In our following discussions, we will focus only on solid conductors so that the current is carried by the negatively charged free electrons in the background of fixed positive ions.

Theory of Current Flow through Solid Conductors

At room temperature, the free electrons in a conductor move randomly with speeds of the order of 10^5 m/s. Since, the motion of the electrons is random, there is no net charge flow in any direction. For any imaginary plane passing through the conductor, the number of electrons crossing the plane in one direction is equal to the number crossing it in the other direction. Therefore, net current is zero from any section.

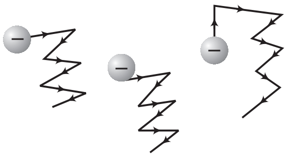


Fig. 23.3

When a constant potential difference V is applied between the ends of the conductor as shown in Fig. 23.4, an electric field $\mathbf{E}$ is produced inside the conductor. The conduction electrons within the conductor are then subjected to a force $-e\mathbf{E}$ and move overall in the direction of increasing potential.

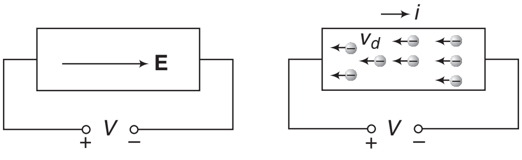


Fig. 23.4

However, this force does not cause the electrons to move faster and faster. Instead, a conduction electron accelerates through a very small distance (about 5×10^{-8} m) and then collides with fixed ions or atoms of the conductor. Each collision transfers some of the electron's kinetic energy to the ions (or atoms). Because of the collision, electron moves slowly along the conductor or we can say that it acquires a **drift velocity v_d** in the direction opposite to $\mathbf{E}$ (in addition to its random motion.)

The drift motion of free electrons produce an electric current in the opposite direction of this motion or in the direction of electric field (from higher potential to lower potential). It is interesting to note that the magnitude of the drift velocity is of the order of 10^{-4} m/s or about 10^9 times smaller than the average speed of the electrons of their random (or thermal) motion. The above discussion can be summarized as follows :

1. Free electrons inside a solid conductor can have two motions :
   (i) random or thermal motion (speed of the order of 10^5 m/s)
   (ii) drift motion (speed of the order of 10^{-4} m/s)
2. Net current due to random (or thermal motion) is zero from any section, whereas net current due to drift motion is non-zero.
3. In the absence of any electric field (or a potential difference across the conductor) free electrons have only random motion. Hence, net current from any section is zero.
4. In the presence of an electric field (or a potential difference across the conductor) free electrons have both motions (random and drift). Therefore, current is non-zero due to drift motion.
5. Drift motion of free electrons is opposite to the electric field. Therefore, direction of current is in the direction of electric field from higher potential to lower potential.

Example 23.6 *Electric field inside a conductor is always zero. Is this statement true or false?*

Solution False. Under electrostatic conditions when there is no charge flow (or no current) in the conductor, electric field is zero. If current is non-zero, then electric field is also non-zero. Because the drift motion (of free electrons) which produces a net current starts only due to electric force on them.

INTRODUCTORY EXERCISE 23.2

1. All points of a conductor are always at same potential. Is this statement true or false?

23.4 Drift Velocity and Relaxation Time

As discussed before, in the presence of electric field, the free electrons experience an electric force of magnitude.

$$F = qE \text{ or } eE \qquad (\text{as } q = e)$$

This will produce an acceleration of magnitude,

$$a = \frac{F}{m} = \frac{eE}{m} \qquad (m = \text{mass of electron})$$

Direction of force (and acceleration) is opposite to the direction of electric field.

After accelerating to some distance an electron will suffer collisions with the heavy fixed ions. The collisions of the electrons do not occur at regular intervals but at random times.

Relaxation time τ is the average time between two successive collisions. Its value is of the order of 10^{-14} second.

After every collision, let us assume that drift motion velocity of electron becomes zero. Then, it accelerates for a time interval τ, then again it collides and its drift motion velocity becomes zero and so on.

If v_d is the average constant velocity (called drift velocity) in the direction of drift motion, then relation between v_d and τ is given by

$$\boxed{v_d = \frac{eE\tau}{m}}$$

✓ *Extra Points to Remember*

- In some standard books of Indian authors, the relation is given as

$$v_d = \frac{eE\tau}{2m}$$

Initially, I was also convinced with this expression. But later on after consulting many more literatures in this. I found that $v_d = \frac{eE\tau}{m}$ is correct. But at this stage it is very difficult for me to give its correct proof. Because the correct proof requires a knowledge of high level of physics which is difficult to understand for a class XII student.

Current and Drift Velocity

Consider a cylindrical conductor of cross-sectional area A in which an electric field E exists. Drift velocity of free electrons is v_d and n is number of free electrons per unit volume (called free electron density).

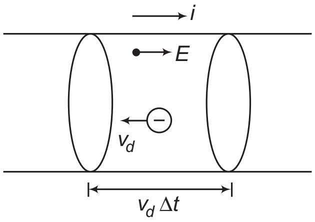


Fig. 23.5

Consider a length $v_d \Delta t$ of the conductor.

The volume of this portion is $Av_d \Delta t$.

Number of free electrons in this volume = (free electron density) × (volume)

$$= (n)\,(Av_d \Delta t)$$

$$= nAv_d \Delta t$$

All these electrons cross the area A in time Δt.

Thus, the charge crossing this area in time Δt is

$\therefore$ $$\Delta q = (nAv_d \Delta t)\,(e)$$

or $$i = \frac{\Delta q}{\Delta t} = neAv_d$$

or $$\boxed{i = neAv_d}$$

Thus, this is the relation between current and drift velocity.

Current Density

Current per unit area (taken normal to the current), i/A is called current density and is denoted by j. The SI units of current density are $\mathrm{A/m^2}$. Current density is a vector quantity $\mathbf{j}$ directed along $\mathbf{E}$.

$$\boxed{j = \frac{i}{A}}$$

But $i = neAv_d$, therefore

$$\boxed{j = nev_d}$$

Extra Points to Remember

- Drift velocity of electrons in a conductor is of the order of 10^{-4} m/s, then question arises in everybody's mind that why a bulb glows instantly when switched on? Reason is : when we close the circuit, electric field is set up in the entire closed circuit instantly (with the speed of light). Due to this electric field, the free electrons instantly get drift velocity in the entire circuit and a current is established in the circuit instantly. The current so set up does not wait for the electrons to flow from one end of the conductor to the other end.
- If a current i is flowing through a wire of non-uniform cross-section, then current will remain constant at all cross-sections. But drift velocity and current density are inversely proportional to the area of cross-section. This is because

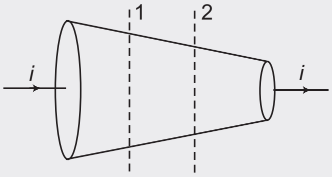


Fig. 23.6

$$i = neAv_d \quad \text{or} \quad v_d = \frac{i}{neA} \quad \text{or} \quad v_d \propto \frac{1}{A}$$

Further, $$j = \frac{i}{A} \quad \text{or} \quad j \propto \frac{1}{A}$$

So, in the figure $i_1 = i_2 = i$ but, $(v_d)_2 > (v_d)_1$ and $j_2 > j_1$ because $A_2 < A_1$

Note *Later we can also prove that electric field at 2 is also more than electric field at 1.*

Example 23.7 *An electron beam has an aperture of 1.0 mm^2. A total of 6.0×10^{16} electrons go through any perpendicular cross-section per second. Find (a) the current and (b) the current density in the beam.*

Solution (a) The current is given by

$$i = \frac{q}{t} = \frac{ne}{t}$$

Substituting the values we have,

$$i = \frac{(6.0 \times 10^{16})(1.6 \times 10^{-19})}{1}$$

$$= 9.6 \times 10^{-3}\ \mathrm{A}$$ **Ans.**

(b) The current density is

$$j = \frac{i}{A} = \frac{9.6 \times 10^{-3}}{10^{-6}}$$

$$= 9.6 \times 10^{3}\ \mathrm{A/m^2}$$ **Ans.**

Example 23.8 *Calculate the drift speed of the electrons when* $1\,A$ *of current exists in a copper wire of cross-section* $2\,mm^2$. *The number of free electrons in* $1\,cm^3$ *of copper is* 8.5×10^{22}.

Solution $n =$ free electron density,

$$= 8.5 \times 10^{22} \text{ per cm}^3 \qquad \text{(Given)}$$

$$= (8.5 \times 10^{22})(10^6) \text{ per m}^3$$

$$= 8.5 \times 10^{28} \text{ per m}^3$$

From $i = neAv_d$, we get

$$v_d = \frac{i}{neA}$$

Substituting the values in SI units we have,

$$v_d = \frac{1}{(8.5 \times 10^{28})(1.6 \times 10^{-19})(2 \times 10^{-6})}$$

$$= 3.6 \times 10^{-5} \text{ m/s}$$ **Answer**

INTRODUCTORY EXERCISE 23.3

1. When a wire carries a current of 1.20 A, the drift velocity is 1.20×10^{-4} m/s. What is the drift velocity when the current is 6.00 A?
2. Find the velocity of charge leading to 1 A current which flows in a copper conductor of cross-section 1cm^2 and length 10 km. Free electron density of copper is $8.5 \times 10^{28}/\text{m}^3$. How long will it take the electric charge to travel from one end of the conductor to the other?

23.5 Resistance of a Wire

Resistance of a wire is always required between two points or two surfaces (say P and Q).

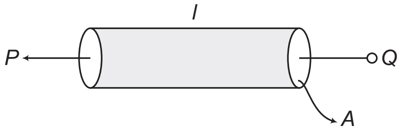


Fig. 23.7

Here, $l =$ length of wire, $A =$ area of cross-section

Now, $$R \propto l \qquad \text{...(i)}$$

and $$R \propto \frac{1}{A} \qquad \text{...(ii)}$$

Combining Eqs. (i) and (ii), we get

$$\boxed{R = \rho \frac{l}{A}} \qquad \text{...(iii)}$$

Here, ρ is called resistivity of the material of the wire. This depends on number of free electrons present in the material. With increase in number of free electrons the value of ρ decreases.

Note *(i)* $\rho = \frac{1}{\sigma}$, *where* σ *= conductivity.*

(ii) SI units of resistivity are Ω-m (ohm-metre).

(iii) SI units of conductivity are $(\Omega\text{-}m)^{-1}$.

(iv) In Eq. (iii), l is that dimension of conductor which is parallel to P and Q and A is that cross-sectional area, which is perpendicular to P and Q.

Example 23.9 *Two copper wires of the same length have got different diameters,*

(a) which wire has greater resistance?

(b) greater specific resistance?

Solution (a) For a given wire, $R = \rho \frac{l}{A}$, i.e. $R \propto \frac{1}{A}$

So, the thinner wire will have greater resistance.

(b) Specific resistance (ρ) is a material property. It does not depend on l or A.

So, both the wires will have same specific resistance.

Example 23.10 *A wire has a resistance R. What will be its resistance if it is stretched to double its length?*

Solution Let V be the volume of wire, then

$$V = Al$$

$$\therefore \qquad A = \frac{V}{l}$$

Substituting this in $R = \rho \frac{l}{A}$, we have $\quad R = \rho \frac{l^2}{V}$

So, for given volume and material (i.e. V and ρ are constants)

$$R \propto l^2$$

When l is doubled, resistance will become four times, or the new resistance will be $4R$.

Example 23.11 *The dimensions of a conductor of specific resistance* ρ *are shown below. Find the resistance of the conductor across AB, CD and EF.*

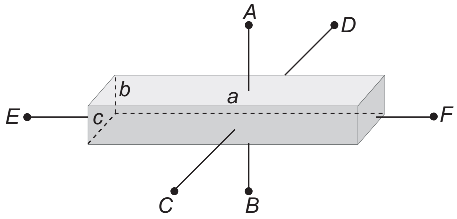


Fig. 23.8

Solution $R = \rho \dfrac{l}{A}$

Resistance across *AB, CD* and *EF* in tabular form is shown below.

Table 23.1

	l	*A*	*R*
AB	*c*	$a \times b$	$\rho \dfrac{c}{ab}$
CD	*b*	$a \times c$	$\rho \dfrac{b}{ac}$
EF	*a*	$b \times c$	$\rho \dfrac{a}{bc}$

Example 23.12 *A copper wire is stretched to make it* 0.1% *longer. What is the percentage change in its resistance?* **(JEE 1978)**

Solution
$$R = \rho \frac{l}{A} = \frac{\rho l}{V/l} \qquad (V = \text{volume of wire})$$
$$= \frac{\rho l^2}{V}$$
$$\therefore \qquad R \propto l^2 \qquad (\rho \text{ and } V = \text{constant})$$

For small percentage change

% change $R = 2$ (% change in l) $= 2\,(0.1\%) = 0.2\%$

Since $R \propto l^2$, with increase in the value of l, resistance will also increase.

INTRODUCTORY EXERCISE 23.4

1. In household wiring, copper wire 2.05 mm in diameter is often used. Find the resistance of a 35.0 m long wire. Specific resistance of copper is $1.72 \times 10^{-8}\ \Omega$-m.
2. The product of resistivity and conductivity of a conductor is constant. Is this statement true or false?
3. You need to produce a set of cylindrical copper wires 3.50 m long that will have a resistance of 0.125 Ω each. What will be the mass of each of these wires? Specific resistance of copper $= 1.72 \times 10^{-8}\ \Omega$-m, density of copper $= 8.9 \times 10^3$ kg/m^3.
4. Consider a thin square sheet of side L and thickness t, made of a material of resistivity ρ. The resistance between two opposite faces, shown by the shaded areas in the figure is **(JEE 2010)**
   (a) directly proportional to L
   (b) directly proportional to t
   (c) independent of L
   (d) independent of t

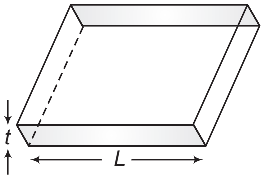


Fig. 23.9

23.6 Temperature Dependence of Resistance

If we increase the temperature of any material, the following two effects can be observed :

(i) Numbers of free electrons increase. Due to this effect conductivity of the material increases. So, resistivity or resistance decreases.

(ii) The ions of the material vibrate with greater amplitude and the collision between electrons and ions become more frequent. Due to this effect resistivity or resistance of the material increases.

In Conductors

There are already a large number of free electrons. So, with increase in temperature effect-(i) is not so dominant as effect-(ii). Hence, resistivity or resistance of conductors increase with increase in temperature.

Over a small temperature range (upto 100°C), the resistivity of a metal (or conductors) can be represented approximately by the equation,

$$\boxed{\rho(T)=\rho_0\,[1+\alpha\,(T-T_0)]} \qquad \text{...(i)}$$

where, ρ_0 is the resistivity at a reference temperature T_0 (often taken as 0°C or 20°C) and $\rho\,(T)$ is the resistivity at temperature T, which may be higher or lower than T_0. The factor α is called the **temperature coefficient of resistivity.**

The resistance of a given conductor depends on its length and area of cross-section besides the resistivity. As temperature changes, the length and area also change. But these changes are quite small and the factor l/A may be treated as constant.

Then, $\quad R\propto\rho \quad$ and hence, $\quad \boxed{R(T)=R_0\,[1+\alpha\,(T-T_0)]} \qquad$ ...(ii)

In this equation, $R\,(T)$ is the resistance at temperature T and R_0 is the resistance at temperature T_0, often taken to be 0°C or 20°C. The temperature coefficient of resistance α is the same constant that appears in Eq. (i), if the dimensions l and A in equation $R=\rho\dfrac{l}{A}$ do not change with temperature.

In Semiconductors

At room temperature, numbers of free electrons in semiconductors (like silicon, germanium etc.) are very less. So, with increase in temperature, effect-(i) is very dominant. Hence, resistivity or resistance of semiconductors decreases with increase in temperature or we can say that temperature coefficient of resistivity α for semiconductors is negative.

Example 23.13 *The resistance of a thin silver wire is* $1.0\ \Omega$ *at* $20°C$. *The wire is placed in a liquid bath and its resistance rises to* $1.2\ \Omega$. *What is the temperature of the bath?* α *for silver is* $3.8\times10^{-3}/°C$.

Solution $\quad R(T)=R_0[1+\alpha\,(T-T_0)]$

Here, $\quad R(T)=1.2\,\Omega,\quad R_0=1.0\,\Omega,\quad \alpha=3.8\times10^{-3}/°\text{C} \quad$ and $\quad T_0=20°\text{C}$

Substituting the values, we have $\quad 1.2=1.0\,[1+3.8\times10^{-3}\ (T-20)]$

or $\quad 3.8\times10^{-3}\ (T-20)=0.2$

Solving this, we get $\quad T=72.6°\text{C} \quad$ **Ans.**

Example 23.14 *Read the following statements carefully* **(JEE 1993)**

Y : The resistivity of semiconductor decreases with increase of temperature.

Z : In a conducting solid, the rate of collisions between free electrons and ions increases with increase of temperature.

Select the correct statement (s) from the following

(a) Y is true but Z is false *(b) Y is false but Z is true*

(c) Both Y and Z are true *(d) Y is true and Z is the correct reason for Y*

Solution Resistivity of conductors increases with increase in temperature because rate of collisions between free electrons and ions increase with increase of temperature. However, the resistivity of semiconductors decreases with increase in temperature, because more and more covalent bonds are broken at higher temperatures and free electrons increase with increase in temperature. Therefore, the correct option is (c).

Example 23.15 *An electric toaster uses nichrome for its heating element. When a negligibly small current passes through it. Its resistance at room temperature* $(27.0\ ^\circ C)$ *is found to be* $75.3\ \Omega$. *When the toaster is connected to a* $230\ V$ *supply, the current settles, after a few seconds, to a steady value of* $2.68\ A$. *What is the steady temperature of the nichrome element? The temperature coefficient of resistance of nichrome averaged over the temperature range involved, is* $1.70 \times 10^{-4}\ ^\circ C^{-1}$.

Solution Given, $T_0 = 27^\circ$ C and $R_0 = 75.3\ \Omega$

At temperature T, $$R_T = \frac{V_T}{i_T} \qquad \left(R = \frac{V}{i}\right)$$

$$= \frac{230}{2.68} = 85.82\ \Omega$$

Using the equation, $$R_T = R_0\ [1 + \alpha\ (T - T_0\ )]$$

We have $$85.82 = 75.3\ [1 + (1.70 \times 10^{-4}\ )(T - 27)]$$

Solving this equation, we get $T \approx 850\ ^\circ\text{C}$ **Ans.**

Thus, the steady temperature of the nichrome element is $850\ ^\circ$C.

INTRODUCTORY EXERCISE 23.5

1. A piece of copper and another of germanium are cooled from room temperature to 80 K. The resistance of **(JEE 1988)**
   (a) each of them increases
   (b) each of them decreases
   (c) copper increases and germanium decreases
   (d) copper decreases and germanium increases
2. The resistance of a copper wire and an iron wire at 20°C are $4.1\ \Omega$ and $3.9\ \Omega$, respectively. Neglecting any thermal expansion, find the temperature at which resistances of both are equal.
   $$\alpha_{\text{Cu}} = 4.0 \times 10^{-3}\ \text{K}^{-1} \text{ and } \alpha_{\text{Fe}} = 5.0 \times 10^{-3}\ \text{K}^{-1}.$$

23.7 Ohm's Law

The equation, $V = iR$ is not Ohm's law. It is a mathematical relation between current passing through a resistance, value of resistance R and the potential difference V across it.

According to Ohm's law, there are some of the materials (like metals or conductors) or some circuits for which, current passing through them is proportional to the potential difference applied across them or

$$i \propto V \quad \text{or} \quad V \propto i$$

$$\Rightarrow \qquad V = iR \quad \text{or} \quad \frac{V}{i} = R = \text{constant}$$

or V - i graph for such materials and circuits is a straight line passing through origin. Slope of this graph is called its resistance. The materials or circuits which follow this law are called **ohmic.**

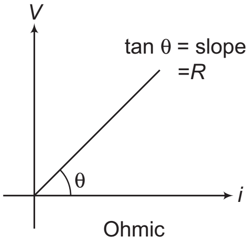


Fig. 23.10

The materials or circuits which do not follow this law are called **non-ohmic**. V - i graph for non-ohmic circuits is not a straight line passing through origin. $\frac{V}{i}$ or R is not constant and i is not proportional to V.

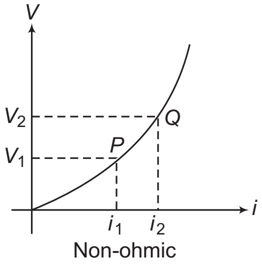


Fig. 23.11

Note *Equation $V = iR$ is applicable for even non-ohmic circuits also.*

For example, $\frac{V_1}{i_1} = R_1 =$ *resistance at P.*

$\frac{V_2}{i_2} = R_2 =$ *resistance at Q, but*

$R_1 \neq R_2$

Example 23.16 *The current-voltage graphs for a given metallic wire at two different temperatures T_1 and T_2 are shown in the figure. The temperature T_2 is greater than T_1. Is this statement true or false?* **(JEE 1985)**

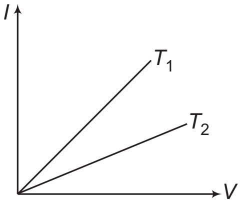


Fig. 23.12

Solution $\dfrac{I}{V} = \text{Slope of given graph } = \dfrac{1}{R}$

or $\quad R = \dfrac{1}{\text{Slope}} \quad \Rightarrow \quad (\text{Slope})_{T_2} < (\text{Slope})_{T_1} \quad \Rightarrow \quad \dfrac{1}{(\text{Slope})_{T_2}} > \dfrac{1}{(\text{Slope})_{T_1}}$

Resistance of a metallic wire increases with increase in temperature.

or $\quad R_{T_2} > R_{T_1} \quad \text{or} \quad T_2 > T_1$

Therefore, the statement is true.

23.8 The Battery and the Electromotive Force

Before studying the electromotive force (emf) of a cell let us take an example of a pump which is more easy to understand. Suppose we want to recycle water between a overhead tank and a ground water tank. Water flows from overhead tank to ground water tank by itself (by gravity). No external agent is required for this purpose. But to raise the water from ground water tank to overhead tank a pump is required or some external work has to be done. In an electric circuit, a battery or a cell plays the same role as the pump played in the above example.

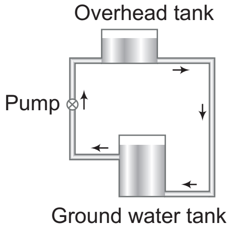


Fig. 23.13

Suppose a resistance (R) is connected across the terminals of a battery. A potential difference is developed across its ends. Current (or positive charge) flows from higher potential to lower potential across the resistance by itself. But inside the battery, work has to be done to bring the positive charge from lower potential to higher potential. The influence that makes current flow from lower to higher potential (inside the battery) is called **electromotive force** (abbreviated **emf**). If W work is done by the battery in taking a charge q from negative terminal to positive terminal, then work done by the battery per unit charge is called emf (E) of the battery.

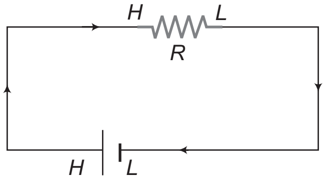


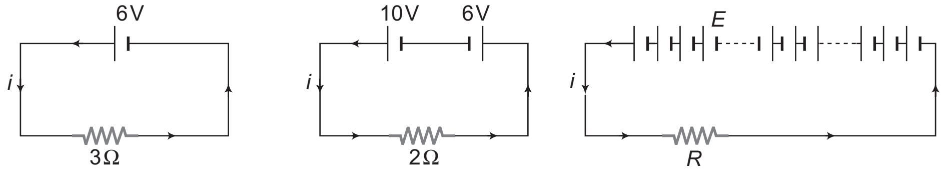
Fig. 23.14

Thus, $$\boxed{E = \frac{W}{q}}$$

The name electromotive force is misleading in the sense that emf is not a force it is work done per unit charge. The SI unit of emf is J/C or V ($1\ \text{V} = 1\ \text{J/C}$).

23.9 Direct Current Circuits, Kirchhoff's Laws

Single current in a simple circuit (single loop) can be found by the relation,



(a) (b) (c)

Fig. 23.15

$$\boxed{i = \frac{\text{net emf}}{\text{net resistance}} = \frac{E_{\text{net}}}{R_{\text{net}}}}$$

For example :

In Fig. (a) Net emf is 6 V and net resistance is 3 Ω. Therefore,

$$i = \frac{6}{3} = 2 \text{ A}$$

In Fig. (b) Net emf = (10 – 6) V = 4 V

and Net resistance = 2 Ω

Therefore, $i = \frac{4}{2} = 2 \text{ A}$

In Fig. (c) We have n cells each of emf E. Of these polarity of m cells (where $n > 2m$) is reversed. Then, net emf in the circuit is $(n - 2m)\,E$ and resistance of the circuit is R. Therefore,

$$\boxed{i = \frac{(n-2m)\,E}{R}}$$

Resistors in Series and in Parallel

In series :

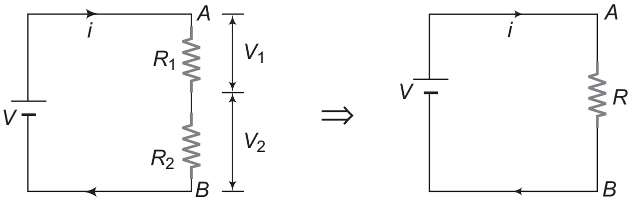


Fig. 23.16

Figure represents a circuit consisting of a source of emf and two resistors connected in series. We are interested in finding the resistance R of the network lying between A and B. That is, what single **equivalent resistor *R*** would have the same resistance as the two resistors linked together.

Because there is only one path for electric current to follow, i must have the same value everywhere in the circuit. The potential difference between A and B is V. This potential difference must somehow be divided into two parts V_1 and V_2 as shown,

∴ $$V = V_1 + V_2 = iR_1 + iR_2$$

or $$V = i\,(R_1 + R_2) \qquad \text{...(i)}$$

Let R be the equivalent resistance between A and B, then

$$V = iR \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\boxed{R = R_1 + R_2 \quad \text{for resistors in series}}$$

This result can be readily extended to a network consisting of n resistors in series.

∴ $$R = R_1 + R_2 + \ldots\ldots + R_n$$

In parallel :

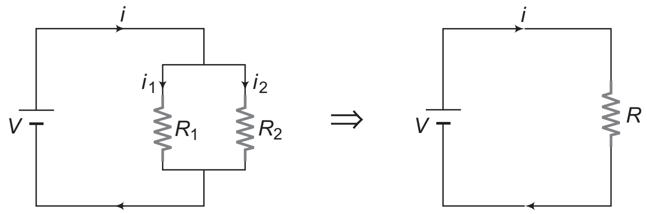


Fig. 23.17

In Fig. 23.17, the two resistors are connected in parallel. The voltage drop across each resistor is equal to the source voltage V. The current i, however, divides into two branches, which carry currents i_1 and i_2.

$$i = i_1 + i_2 \qquad \ldots\text{(iii)}$$

If R be the equivalent resistance, then

$$i = \frac{V}{R}, \quad i_1 = \frac{V}{R_1} \quad \text{and} \quad i_2 = \frac{V}{R_2}$$

Substituting in Eq. (iii), we get

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{for resistors in parallel}}$$

This result can also be extended to a network consisting of n resistors in parallel. The result is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots\ldots + \frac{1}{R_n}$$

Example 23.17 *Compute the equivalent resistance of the network shown in figure and find the current i drawn from the battery.*

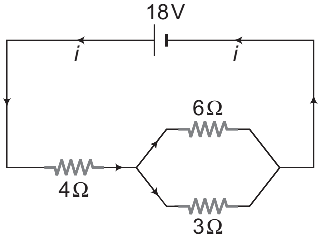


Fig. 23.18

Solution The 6 Ω and 3 Ω resistances are in parallel. Their equivalent resistance is

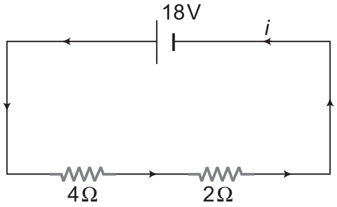


Fig. 23.19

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{3} \quad \text{or} \quad R = 2\,\Omega$$

Now, this 2 Ω and 4 Ω resistances are in series and their equivalent resistance is 4 + 2 = 6 Ω.

Therefore, equivalent resistance of the network = 6 Ω. **Ans.**

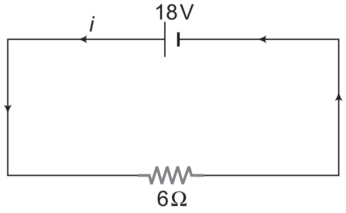


Fig. 23.20

Current drawn from the battery is

$$i = \frac{\text{net emf}}{\text{net resistance}} = \frac{18}{6}$$

$$= 3\text{ A}$$ **Ans.**

Kirchhoff's Laws

Many electric circuits cannot be reduced to simple series-parallel combinations. For example, two circuits that cannot be so broken down are shown in Fig. 23.21.

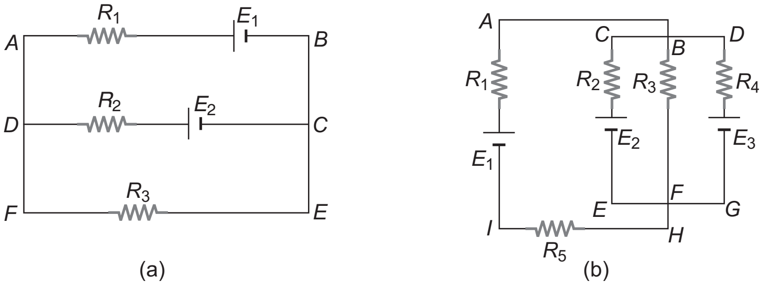


Fig. 23.21

However, it is always possible to analyze such circuits by applying two rules, devised by Kirchhoff in 1845 and 1846 when he was still a student.

First there are two terms that we will use often.

Junction

A junction in a circuit is a point where three or more conductors meet. Junctions are also called nodes or branch points.

For example, in Fig. (a) points *D* and *C* are junctions. Similarly, in Fig. (b) points *B* and *F* are junctions.

Loop

A loop is any closed conducting path. For example, in Fig. (a) *ABCDA*, *DCEFD* and *ABEFA* are loops. Similarly, in Fig. (b), *CBFEC*, *BDGFB* are loops.

Kirchhoff's rules consist of the following two statements :

Kirchhoff's Junction Rule

The algebraic sum of the currents into any junction is zero.

That is,
$$\boxed{\sum_{\text{junction}} i = 0}$$

This law can also be written as, "the sum of all the currents directed towards a point in a circuit is equal to the sum of all the currents directed away from that point."

Thus, in figure
$$i_1 + i_2 = i_3 + i_4$$

The junction rule is based on **conservation of electric charge.** No charge can accumulate at a junction, so the total charge entering the junction per unit time must equal to charge leaving per unit time. Charge per unit time is current, so if we consider the currents entering to be positive and those leaving to be negative, the algebraic sum of currents into a junction must be zero.

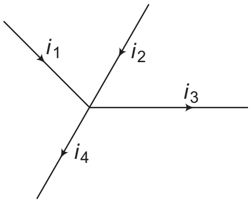


Fig. 23.22

Kirchhoff's Loop Rule

The algebraic sum of the potential differences in any loop including those associated emf's and those of resistive elements, must equal zero.

That is,
$$\boxed{\sum_{\text{closed loop}} \Delta V = 0}$$

Kirchhoff's second rule is based on the fact that the **electrostatic field is conservative** in nature. This result states that there is no net change in electric potential around a closed path. Kirchhoff's second rule applies only for circuits in which an electric potential is defined at each point. This criterion may not be satisfied if changing electromagnetic fields are present.

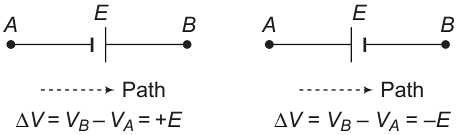


Fig. 23.23

In applying the loop rule, we need sign conventions. First assume a direction for the current in each branch of the circuit. Then starting at any point in the circuit, we imagine, travelling around a loop, adding emf's and *iR* terms as we come to them.

When we travel through a source in the direction from – to +, the emf is considered to be positive, when we travel from + to –, the emf is considered to be negative.

When we travel through a resistor in the same direction as the assumed current, the *iR* term is negative because the current goes in the direction of decreasing potential. When we travel through a resistor in the direction opposite to the assumed current, the *iR* term is positive because this represents a rise of potential.

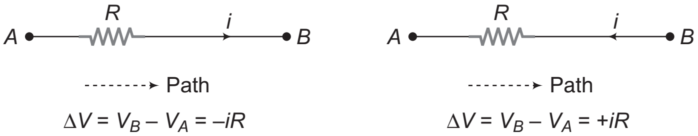


Fig. 23.24

Note *It is advised to write H (for higher potential) and L (for lower potential) across all the batteries and resistances of the loop under consideration while using the loop law. Then write – while moving from H to L and + for L to H. Across a battery write H on positive terminal and L on negative terminal. Across a resistance keep in mind the fact that current always flows from higher potential (H) to lower potential (L). For example, in the loop shown in figure we have marked H and L across all batteries and resistances. Now let us apply the second law in the loop ADCBA.*

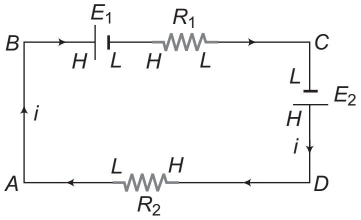


Fig. 23.25

The equation will be

$$+iR_2 - E_2 + iR_1 + E_1 = 0$$

Example 23.18 *Find currents in different branches of the electric circuit shown in figure.*

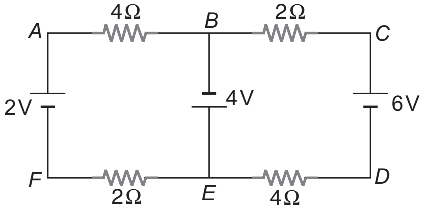


Fig. 23.26

HOW TO PROCEED *In this problem there are three wires EFAB, BE and BCDE. Therefore, we have three unknown currents i_1, i_2 and i_3. So, we require three equations. One equation will be obtained by applying Kirchhoff's junction law (either at B or at E) and the remaining two equations, we get from the second law (loop law). We can make three loops ABEFA, ACDFA and BCDEB. But we have to choose any two of them. Initially, we can choose any arbitrary directions of i_1, i_2 and i_3.*

Solution Applying Kirchhoff's first law (junction law) at junction *B*,

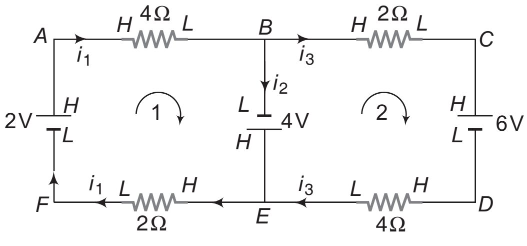


Fig. 23.27

$$i_1 = i_2 + i_3 \qquad \text{...(i)}$$

Applying Kirchhoff's second law in loop 1 (*ABEFA*),

$$-4i_1 + 4 - 2i_1 + 2 = 0 \qquad \text{...(ii)}$$

Applying Kirchhoff's second law in loop 2 ($BCDEB$),

$$-2i_3 - 6 - 4i_3 - 4 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$i_1 = 1\,\text{A}$$

$$i_2 = \frac{8}{3}\,\text{A}$$

$$i_3 = -\frac{5}{3}\,\text{A}$$

Ans.

Here, negative sign of i_3 implies that current i_3 is in opposite direction of what we have assumed.

Example 23.19 *In example 23.18, find the potential difference between points F and C.*

HOW TO PROCEED *To find the potential difference between any two points of a circuit you have to reach from one point to the other via any path of the circuit. It is advisable to choose a path in which we come across the least number of resistors preferably a path which has no resistance.*

Solution Let us reach from F to C *via* A and B,

$$V_F + 2 - 4i_1 - 2i_3 = V_C$$

$\therefore$
$$V_F - V_C = 4i_1 + 2i_3 - 2$$

Substituting, $i_1 = 1\,\text{A}$ and $i_3 = -\frac{5}{3}$ A, we get

$$V_F - V_C = -\frac{4}{3}\ \text{volt}$$

Ans.

Here, negative sign implies that $V_F < V_C$.

Internal Resistance (r) and Potential Difference (V) across the Terminals of a Battery

The potential difference across a real source in a circuit is not equal to the emf of the cell. The reason is that charge moving through the electrolyte of the cell encounters resistance. We call this the **internal resistance** of the source, denoted by r. If this resistance behaves according to Ohm's law r is constant and independent of the current i. As the current moves through r, it experiences an associated drop in potential equal to ir. Thus, when a current is drawn through a source, the potential difference between the terminals of the source is

$$\boxed{V = E - ir}$$

This can also be shown as below.

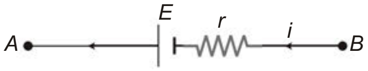


Fig. 23.28

$$V_A - E + ir = V_B$$

or
$$V_A - V_B = E - ir$$

The following three special cases are possible :

(i) If the current flows in opposite direction (as in case of charging of a battery), then $V = E + ir$

(ii) $V = E$, if the current through the cell is zero.

(iii) $V = 0$, if the cell is short circuited.

This is because current in the circuit

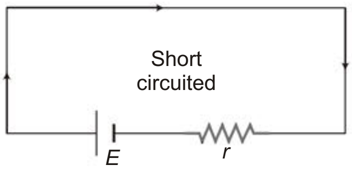


Fig. 23.29

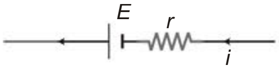
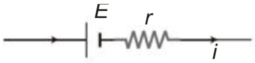
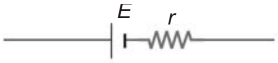
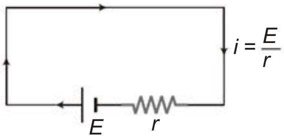
$$i = \frac{E}{r}$$

or $$E = ir$$

$\therefore$ $$E - ir = 0$$

or $$V = 0$$

Thus, we can summarise it as follows :

	$V = E - ir$	or	$V < E$
	$V = E + ir$	or	$V > E$
	$V = E$	if	$i = 0$
	$V = 0$	if short circuited	

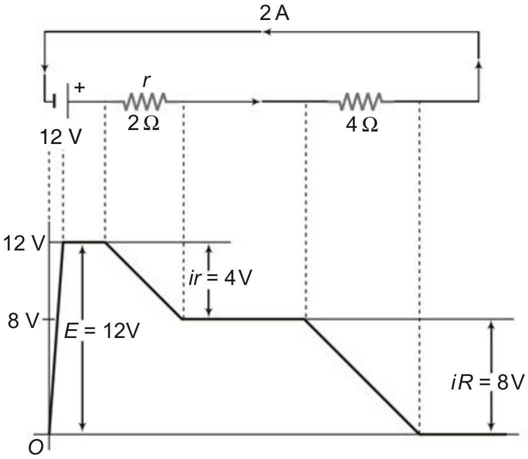


Fig. 23.30 Potential rise and fall in a circuit.

Extra Points to Remember

- **In figure (a) :** There are eight wires and hence, will have eight currents or eight unknowns. The eight wires are *AB*, *BC*, *CE*, *EA*, *AD*, *BD*, *CD* and *ED*. Number of independent loops are four. Therefore, from the second law we can make only four equations. Total number of junctions are five (*A, B, C, D* and *E*). But by using the first law, we can make only four equations (one less). So, the total number of equations are eight.

  In figure (b) : Number of wires are six (*AB, BC, CDA, BE, AE* and *CE*). Number of independent loops are three so, three equations can be obtained from the second law. Number of junctions are four (*A, B, C* and *E*) so, we can make only three (one less) equations from the first law. But total number of equations are again six.

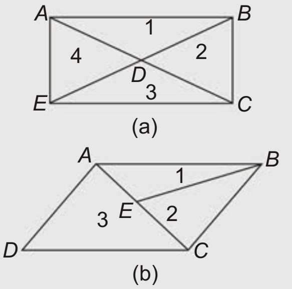


Fig. 23.31

- **Short circuiting :** Two points in an electric circuit directly connected by a conducting wire are called short circuited. Under such condition both points are at same potential.

  For example, resistance R_1 in the adjoining circuit is short circuited, i.e. potential difference across it is zero. Hence, no current will flow through R_1 and the current through R_2 is therefore, E/R_2.

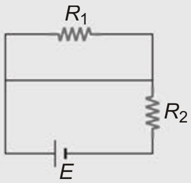


Fig. 23.32

- **Earthing :** If some point of a circuit is earthed, then its potential is taken to be zero.

  For example, in the adjoining figure,

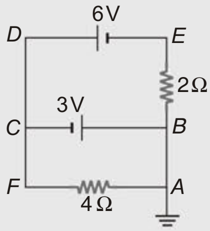


Fig. 23.33

$$V_A = V_B = 0$$

$$V_F = V_C = V_D = -3\text{ V}$$

$$V_E = -9\text{ V}$$

$$\therefore \qquad V_B - V_E = 9\text{ V}$$

or current through 2 Ω resistance is

$$\frac{V_B - V_E}{2} \quad \text{or} \quad \frac{9}{2}\text{ A} \qquad \text{(from } B \text{ to } E)$$

Similarly, $V_A - V_F = 3\text{ V}$

and the current through 4 Ω resistance is $\dfrac{V_A - V_F}{4}$ or $\dfrac{3}{4}$ A (from *A* to *F*)

- For a current flow through a resistance there must be a potential difference across it but between any two points of a circuit the potential difference may be zero.

  For example, in the circuit,

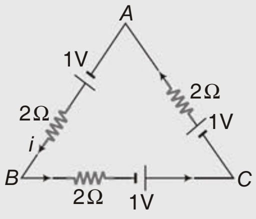


Fig. 23.34

$$\text{net emf } = 3\text{ V} \quad \text{and} \quad \text{net resistance } = 6\ \Omega$$

$$\therefore \quad \text{current in the circuit, } i = \frac{3}{6} = \frac{1}{2}\text{ A}$$

$\boldsymbol{V_A - V_B}$ $\qquad V_A + 1 - 2 \times \frac{1}{2} = V_B \quad \text{or} \quad V_A - V_B = 0$

or by symmetry, we can say that

$$V_A = V_B = V_C$$

So, the potential difference across any two vertices of the triangle is zero, while the current in the circuit is non-zero.

- **Distribution of current in parallel connections :** When more than one resistances are connected in parallel, the potential difference across them is equal and the current is distributed among them in inverse ratio of their resistance as

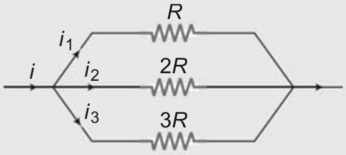


Fig. 23.35

$$i = \frac{V}{R}$$

or $$i \propto \frac{1}{R} \quad \text{for same value of } V$$

e.g. in the figure,

$$i_1 : i_2 : i_3 = \frac{1}{R} : \frac{1}{2R} : \frac{1}{3R} = 6:3:2$$

$\therefore$ $$i_1 = \left(\frac{6}{6+3+2}\right) i = \frac{6}{11} i$$

$$i_2 = \left(\frac{3}{6+3+2}\right) i = \frac{3}{11} i$$

and $$i_3 = \left(\frac{2}{6+3+2}\right) i = \frac{2}{11} i$$

Note *In case of only two resistances,* $\frac{i_1}{i_2} = \frac{R_2}{R_1}$

- **Distribution of potential in series connections :** When more than one resistances are connected in series, the current through them is same and the potential is distributed in the direct ratio of their resistance as

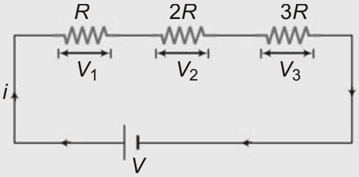


Fig. 23.36

$$V = iR \quad \text{or} \quad V \propto R \quad \text{for same value of } i.$$

For example in the figure,

$$V_1 : V_2 : V_3 = R : 2R : 3R = 1:2:3$$

$\therefore$ $$V_1 = \left(\frac{1}{1+2+3}\right) V = \frac{V}{6}$$

$$V_2 = \left(\frac{2}{1+2+3}\right) V = \frac{V}{3}$$

and $$V_3 = \left(\frac{3}{1+2+3}\right) V = \frac{V}{2}$$

Example 23.20 *In the circuit shown in figure,*

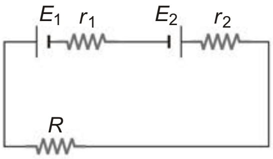


Fig. 23.37

$E_1 = 10\,V,\ E_2 = 4\,V,\ r_1 = r_2 = 1\,\Omega$ *and* $R = 2\,\Omega$.

Find the potential difference across battery 1 *and battery* 2.

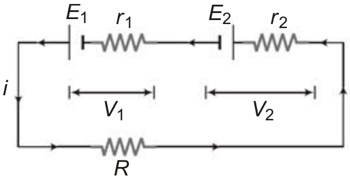


Fig. 23.38

Solution Net emf of the circuit $= E_1 - E_2 = 6\text{ V}$

Total resistance of the circuit $= R + r_1 + r_2 = 4\ \Omega$

$\therefore$ Current in the circuit, $i = \dfrac{\text{net emf}}{\text{total resistance}} = \dfrac{6}{4} = 1.5\text{ A}$

Now, $V_1 = E_1 - ir_1 = 10 - (1.5)(1)$

$= 8.5\text{ V}$ **Ans.**

and $V_2 = E_2 + ir_2 = 4 + (1.5)(1)$

$= 5.5\text{ V}$ **Ans.**

INTRODUCTORY EXERCISE 23.6

1. Find the current through 2 Ω and 4 Ω resistance.

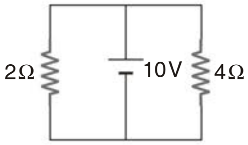


Fig. 23.39

2. In the circuit shown in figure, find the potentials of A, B, C and D and the current through 1 Ω and 2 Ω resistance.

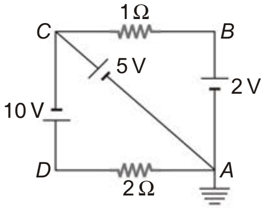


Fig. 23.40

3. For what value of E the potential of A is equal to the potential of B?

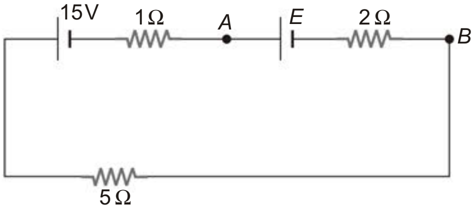


Fig. 23.41

4. Ten cells each of emf 1 V and internal resistance 1 Ω are connected in series. In this arrangement, polarity of two cells is reversed and the system is connected to an external resistance of 2 Ω. Find the current in the circuit.

5. In the circuit shown in figure, $R_1 = R_2 = R_3 = 10\ \Omega$. Find the currents through R_1 and R_2.

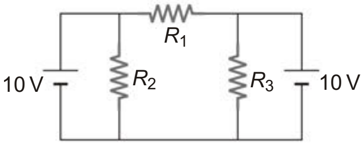


Fig. 23.42

23.10 Heating Effects of Current

An electric current through a resistor increases its thermal energy. Also, there are other situations in which an electric current can produce or absorb thermal energy.

Power Supplied or Power Absorbed by a Battery

When charges are transported across a source of emf, their potential energy changes. If a net charge Δq moves through a potential difference E in a time Δt, the change in electric potential energy of the charge is $E\,\Delta q$. Thus, the source of emf does work,

$$\Delta W = E\,\Delta q$$

Dividing both sides by Δt, then taking the limit as $\Delta t \to 0$, we find

$$\frac{dW}{dt} = E\,\frac{dq}{dt}$$

By definition, $\frac{dq}{dt} = i$, the current through the battery and $\frac{dW}{dt} = P$, the power output of (or input to) the battery. Hence,

$$\boxed{P = Ei}$$

The quantity P represents the rate at which energy is transferred from a discharging battery or to a charging battery.

In Fig. 23.43, energy is transferred from the source at a rate Ei

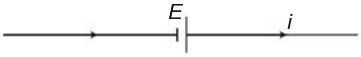


Fig. 23.43

In Fig. 23.44, energy is transferred to the source at a rate Ei

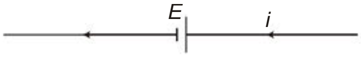


Fig. 23.44

Power dissipated across a resistance

Now, let's consider the power dissipated in a conducting element. Suppose it has a resistance R and the potential difference between its ends is V. In moving from higher to lower potential, a positive charge Δq loses energy $\Delta U = V\,\Delta q$. This electric energy is absorbed by the conductor through collisions between its atomic lattice and the charge carriers, causing its temperature to rise. This effect is commonly called **Joule heating.** Since, power is the rate at which energy is transferred, we have,

$$P = \frac{\Delta U}{\Delta t} = V \cdot \frac{\Delta q}{\Delta t} = Vi$$

$$\therefore \qquad \boxed{P = Vi}$$

which with the help of equation $V = iR$ can also be written in the forms,

$$\boxed{P = i^2 R} \quad \text{or} \quad \boxed{P = \frac{V^2}{R}}$$

Power is always dissipated in a resistance. With this rate, the heat produced in the resistor in time t is

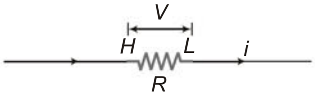


Fig. 23.45

$$H = Pt \quad \text{or} \quad \boxed{H = Vit = i^2Rt = \frac{V^2}{R}t}$$

Joule heating occurs whenever a current passes through an element that has resistance. To prevent the overheating of delicate electronic components, many electric devices like video cassette recorders, televisions and computer monitors have fans in their chassis to allow some of the heat produced to escape.

Extra Points to Remember

- We have seen above that power may be supplied or consumed by a battery. It depends on the direction of current.

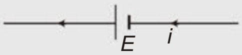


Fig. 23.46

In the above direction of current power is supplied by the battery (= Ei)

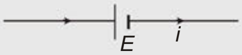


Fig. 23.47

In the opposite direction of current shown in Fig. 23.47, power is consumed by the battery. This normally happens during charging of a battery.

- A resistance always consumes power. It does not depend on the direction of current.

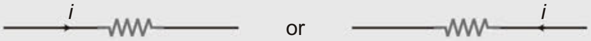


Fig. 23.48

In both cases shown in figure, power is only consumed and this power consumed is given by the formula.

$$P = i^2R = \frac{V^2}{R} = Vi$$

In the above equations V and i are the values across a resistance in which we wish to find the power consumed.

- In any electrical circuit, law of conservation of energy is followed.

Net power supplied by all batteries of the circuit = net power consumed by all resistors in the circuit.

Example 23.21 *In the circuit shown in figure, find*

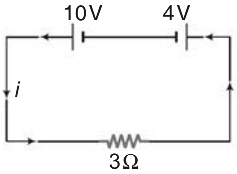


Fig. 23.49

(a) the power supplied by 10 *V battery*
(b) the power consumed by 4 *V battery and*
(c) the power dissipated in 3 Ω *resistance.*

Solution Net emf of the circuit = $(10-4)$ V = 6 V

$\therefore$ Current in the circuit

$$i=\frac{\text{net emf}}{\text{total resistance}}=\frac{6}{3}=2\text{ A}$$

(a) Power supplied by 10 V battery = $Ei=(10)(2)=20$ W **Ans.**

(b) Power consumed by 4 V battery = $Ei=(4)(2)=8$ W **Ans.**

(c) Power consumed by 3 Ω resistance = $i^2R=(2)^2(3)=12$ W **Ans.**

Note *Here, we can see that total power supplied by 10 V battery (i.e. 20 W) = power consumed by 4 V battery and 3 Ω resistance. Which proves that conservation of energy holds good in electric circuits also.*

Example 23.22 *In the circuit shown in figure, find the heat developed across each resistance in 2 s.*

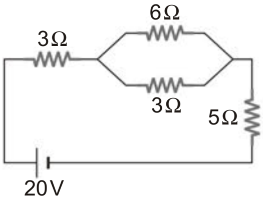


Fig. 23.50

Solution The 6 Ω and 3 Ω resistances are in parallel. So, their combined resistance is

$$\frac{1}{R}=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$$

or $$R=2\ \Omega$$

The equivalent simple circuit can be drawn as shown.

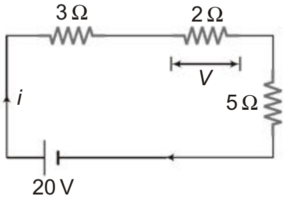


Fig. 23.51

Current in the circuit,

$$i=\frac{\text{net emf}}{\text{total resistance}}=\frac{20}{3+2+5}=2\text{ A}$$

$$V=iR=(2)(2)=4\text{ V}$$

i.e. Potential difference across 6 Ω and 3 Ω resistances are 4 V. Now,

$H_{3\Omega}$ (which is connected in series) = $i^2Rt=(2)^2\ (3)(2)=24$ J

$$H_{6\Omega}=\frac{V^2}{R}t=\frac{(4)^2}{6}(2)=\frac{16}{3}\text{ J}$$

$$H_{3\,\Omega} \text{ (which is connected in parallel)} = \frac{V^2}{R}t = \frac{(4)^2\,(2)}{3} = \frac{32}{3}\text{ J}$$

and $$H_{5\,\Omega} = i^2Rt = (2)^2\,(5)\,(2) = 40\text{ J}$$ **Ans.**

INTRODUCTORY EXERCISE 23.7

1. In the circuit shown in figure, a 12 V battery with unknown internal resistance *r* is connected to another battery with unknown emf *E* and internal resistance 1 Ω and to a resistance of 3 Ω carrying a current of 2 A. The current through the rechargeable battery is 1 A in the direction shown. Find the unknown current *i*, internal resistance *r* and the emf *E*.

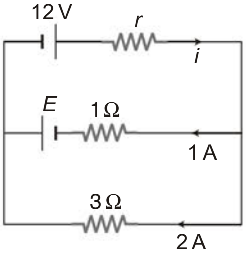


Fig. 23.52

2. In the above example, find the power delivered by the 12 V battery and the power dissipated in 3 Ω resistor.

23.11 Grouping of Cells

Cells are usually grouped in the following three ways :

Series Grouping

Suppose *n* cells each of emf *E* and internal resistance *r* are connected in series as shown in figure.

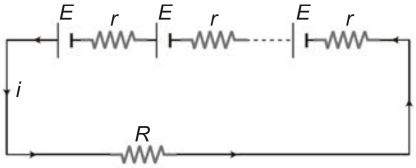


Fig. 23.53

Then, $$\text{Net emf} = nE$$

$$\text{Total resistance} = nr + R$$

∴ Current in the circuit, $i = \dfrac{\text{net emf}}{\text{total resistance}}$ or $\boxed{i = \dfrac{nE}{nr + R}}$

Note *If polarity of m cells is reversed, then equivalent emf = (n – 2m) E, while total resistance is still nr + R*

∴ $$i = \frac{(n-2m)\,E}{nr + R}$$

Parallel Grouping

Here, three cases are possible.

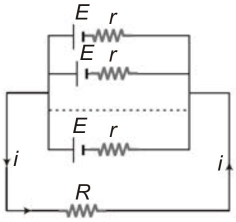


Fig. 23.54

Case 1 *When E and r of each cell has same value and positive terminals of all cells are connected at one junction while negative at the other.*

In this situation, the net emf is E. The net internal resistance is $\frac{r}{n}$ as n resistances each of r are in parallel. Net external resistance is R. Therefore, total resistance is $\left(\frac{r}{n}+R\right)$ and so the current in the circuit will be,

$$i=\frac{\text{Net emf}}{\text{Total resistance}} \quad \text{or} \quad \boxed{i=\frac{E}{R+\ r/n}}$$

Note *A comparison of series and parallel grouping reveals that to get maximum current, cells must be connected in series if effective internal resistance is lesser than external and in parallel if effective internal resistance is greater than external.*

Case 2 *If E and r of each cell are different but still the positive terminals of all cells are connected at one junction while negative at the other.*

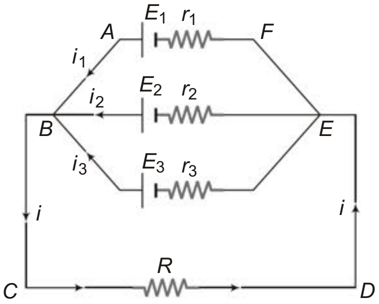


Fig. 23.55

Applying Kirchhoff's second law in loop $ABCDEFA$,

$$E_1-iR-i_1r_1=0 \quad \text{or} \quad i_1=-i\frac{R}{r_1}+\frac{E_1}{r_1} \qquad \ldots\text{(i)}$$

Similarly, we can write

$$i_2=-i\frac{R}{r_2}+\frac{E_2}{r_2} \qquad \ldots\text{(ii)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Adding all above equations, we have

$$(i_1+i_2+\ldots+i_n)=-iR\ \Sigma\left(\frac{1}{r}\right)+\Sigma\left(\frac{E}{r}\right)$$

But $\quad i_1+i_2+\ldots+i_n=i$

$\therefore \quad i=-iR\ \Sigma\left(\frac{1}{r}\right)+\Sigma\left(\frac{E}{r}\right)$

$$\therefore \qquad i = \frac{\Sigma\,(E/r)}{1 + R\,\Sigma\,(1/r)} = \frac{\Sigma\,(E/r)/\Sigma\,(1/r)}{\{1/\Sigma\,(1/r)\} + R} = \frac{E_{eq}}{R_{eq}}$$

where,
$$\boxed{E_{eq} = \frac{\Sigma\,(E/r)}{\Sigma\,(1/r)} \quad \text{and} \quad R_{eq} = R + \frac{1}{\Sigma\,(1/r)}}$$

From the above expression, we can see that $i = \dfrac{E}{R + r/n}$ if n cells of same emf E and internal resistance r are connected in parallel. This is because,

$$\Sigma\,(E/r) = nE/r \text{ and } \Sigma\,(1/r) = n/r$$

$$\therefore \qquad i = \frac{nE/r}{1 + nR/r}$$

Multiplying the numerator and denominator by r/n, we have

$$i = \frac{E}{R + r/n}$$

Exercise In parallel grouping (Case 2) prove that, $E_{eq} = E$ if $E_1 = E_2 = \ldots = E$ and $r_1 = r_2 = \ldots = r$

Case 3 This is the most general case of parallel grouping in which E and r of different cells are different and the positive terminals of few cells are connected to the negative terminals of the others as shown figure.

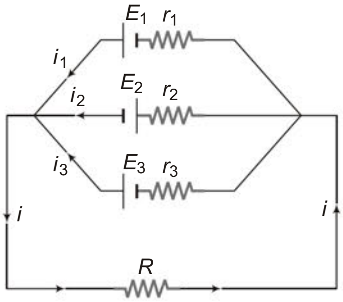


Fig. 23.56

Kirchhoff's second law in different loops gives the following equations :

$$E_1 - iR - i_1 r_1 = 0 \quad \text{or} \quad i_1 = \frac{E_1}{r_1} - \frac{iR}{r_1} \qquad \ldots\text{(i)}$$

$$-E_2 - iR - i_2 r_2 = 0 \quad \text{or} \quad i_2 = -\frac{E_2}{r_2} - \frac{iR}{r_2} \qquad \ldots\text{(ii)}$$

Similarly,
$$i_3 = \frac{E_3}{r_3} - \frac{iR}{r_3} \qquad \ldots\text{(iii)}$$

Adding Eqs. (i), (ii) and (iii), we get

$$i_1 + i_2 + i_3 = (E_1/r_1) - (E_2/r_2) + (E_3/r_3) - iR\,(1/r_1 + 1/r_2 + 1/r_3)$$

or
$$i\,[1 + R\,(1/r_1 + 1/r_2 + 1/r_3)] = (E_1/r_1) - (E_2/r_2) + (E_3/r_3)$$

$$\therefore \qquad \boxed{i = \frac{(E_1/r_1) - (E_2/r_2) + (E_3/r_3)}{1 + R\,(1/r_1 + 1/r_2 + 1/r_3)}}$$

or
$$i = \frac{(E_1/r_1 - E_2/r_2 + E_3/r_3)\,/\,(1/r_1 + 1/r_2 + 1/r_3)}{R + 1/\,(1/r_1 + 1/r_2 + 1/r_3)}$$

Mixed Grouping

The situation is shown in figure.

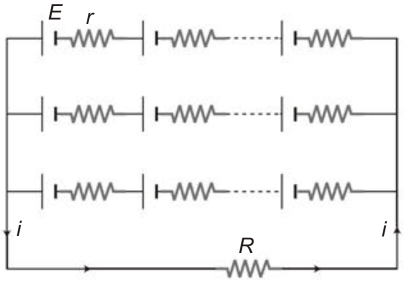


Fig. 23.57

There are n identical cells in a row and number of rows are m. Emf of each cell is E and internal resistance is r. Treating each row as a single cell of emf nE and internal resistance nr, we have

$$\text{Net emf} = nE$$

$$\text{Total internal resistance} = \frac{nr}{m}$$

$$\text{Total external resistance} = R$$

$\therefore$ Current through the external resistance R is

$$\boxed{i = \frac{nE}{R + \dfrac{nr}{m}}}$$

This expression after some rearrangements can also be written as

$$i = \frac{mnE}{(\sqrt{mR} - \sqrt{nr})^2 + 2\sqrt{mnrR}}$$

If total number of cells are given then mn is fixed. E, r and R are also given, we have liberty to arrange the given number of cells in different rows. Then in the above expression the numerator nmE and in the denominator $2\sqrt{mnrR}$ all are fixed. Only the square term in the denominator is variable. Therefore,

i is maximum when,

$$\sqrt{mR} = \sqrt{nr} \quad \text{or} \quad R = \frac{nr}{m}$$

or total external resistance = total internal resistance

Thus, we can say that the current and hence power transferred to the load is maximum when load resistance is equal to internal resistance. This is known as **maximum power transfer theorem.**

Extra Points to Remember

- Regarding maximum current in the circuit or maximum power consumed by the external resistance R, there are three special cases. One we have discussed above, where we have to arrange the cells in such a manner that current and power in the circuit should be maximum. And this happens when we arrange the cells in such a manner that total internal resistance comes out to be equal to total external resistance. Rest two cases are discussed below.

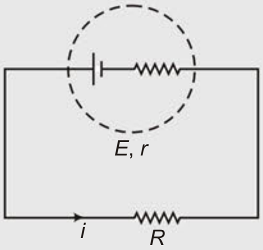


Fig. 23.58

In the figure shown,

$$i = \frac{E}{R + r}$$

$$P_R = i^2 R = \left(\frac{E}{R + r}\right)^2 R$$

(i) Now if r is variable. E and R are fixed, then i and P_R both are maximum when $r = 0$.

(ii) If R is variable. E and r are fixed. Then, current in the circuit is maximum when $R = 0$. But P_R will be maximum, when

$$R = r \quad \text{or} \quad \text{external resistance = internal resistance}$$

Example 23.23 *Find the emf and internal resistance of a single battery which is equivalent to a combination of three batteries as shown in figure.*

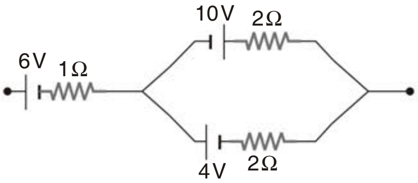


Fig. 23.59

Solution The given combination consists of two batteries in parallel and resultant of these two in series with the third one.

For parallel combination we can apply,

$$E_{eq} = \frac{\frac{E_1}{r_1} - \frac{E_2}{r_2}}{\frac{1}{r_1} + \frac{1}{r_2}} = \frac{\frac{10}{2} - \frac{4}{2}}{\frac{1}{2} + \frac{1}{2}} = 3\text{ V}$$

Further,

$$\frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore$

$$r_{eq} = 1\,\Omega$$

Now this is in series with the third one, i.e.

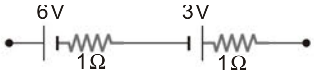


Fig. 23.60

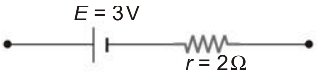
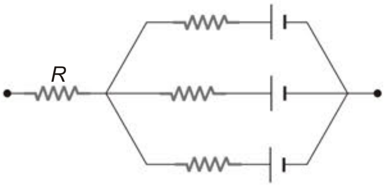


Fig. 23.61

Example 23.24 *In the circuit shown in figure* $E_1 = 3V$, $E_2 = 2V$, $E_3 = 1\ V$ *and* $R = r_1 = r_2 = r_3 = 1\ \Omega$. **(JEE 1981)**



(a) Find the potential difference between the points A and B and the currents through each branch.

(b) If r_2 *is short-circuited and the point A is connected to point B, find the currents through* E_1, E_2, E_3 *and the resistor R.*

Solution (a) Equivalent emf of three batteries would be

$$E_{\text{eq}} = \frac{\Sigma(E/r)}{\Sigma(1/r)} = \frac{(3/1 + 2/1 + 1/1)}{(1/1 + 1/1 + 1/1)} = 2\ \text{V}$$

– Ω as all three are in parallel.

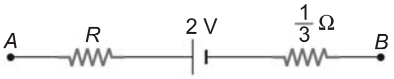


Fig. 23.63

The equivalent circuit is therefore shown in the figure.

Since, no current is taken from the battery.

$$V_{AB} = 2\ \text{V} \qquad \text{(From } V = E - i\,r\text{)}$$

Further, $V_A - V_B = E_1 - i_1 r_1$

$\therefore$

$$i_1 = \frac{V_B - V_A + E_1}{r_1} = \frac{-2 + 3}{1} = 1\ \text{A}$$

Similarly,

$$i_2 = \frac{V_B - V_A + E_2}{r_2} = \frac{-2 + 2}{1} = 0$$

and

$$i_3 = \frac{V_B - V_A + E_3}{r_3} = \frac{-2 + 1}{1} = -1\ \text{A}$$

(b) r_2 is short circuited means resistance of this branch becomes zero. Making a closed circuit with a battery and resistance R. Applying Kirchhoff's second law in three loops so formed.

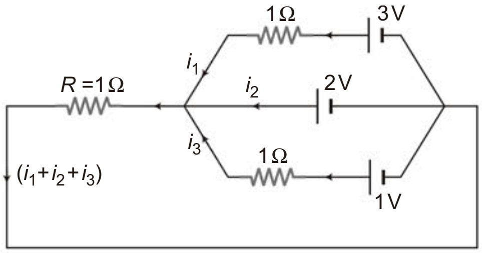


Fig. 23.64

$$3 - i_1 - (i_1 + i_2 + i_3) = 0 \qquad \ldots\text{(i)}$$

$$2 - (i_1 + i_2 + i_3) = 0 \qquad \ldots\text{(ii)}$$

$$1 - i_3 - (i_1 + i_2 + i_3) = 0 \qquad \ldots\text{(iii)}$$

From Eq. (ii) $\quad i_1 + i_2 + i_3 = 2\,\text{A}$

$\therefore$ Substituting in Eq. (i), we get $\quad i_1 = 1\,\text{A}$

Substituting in Eq. (iii), we get $\quad i_3 = -1\,\text{A}$

$\therefore \quad i_2 = 2\,\text{A}$

INTRODUCTORY EXERCISE 23.8

1. Find the emf (V) and internal resistance (r) of a single battery which is equivalent to a parallel combination of two batteries of emfs V_1 and V_2 and internal resistances r_1 and r_2 respectively, with polarities as shown in figure

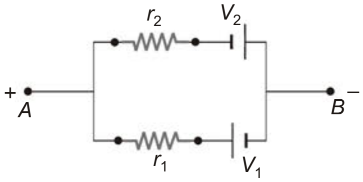


Fig. 23.65

2. Find the net emf of the three batteries shown in figure.

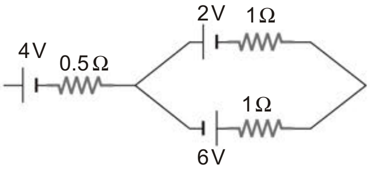


Fig. 23.66

3. Find the equivalent emf and internal resistance of the arrangement shown in figure.

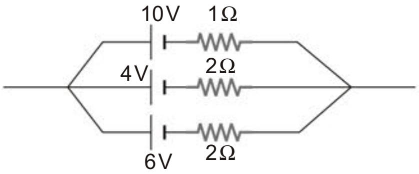


Fig. 23.67

23.12 Electrical Measuring Instruments

So far we have studied about current, resistance, potential difference and emf. Now, in this article we will study how these are measured. The basic measuring instrument is galvanometer, whose pointer shows a deflection when current passes through it. A galvanometer can easily be converted into an ammeter for measuring current, into a voltmeter for measuring potential difference. For accurate measurement of potential difference or emf, a potentiometer is more preferred. Resistances are accurately measured by using post office box or meter bridge which are based on the principle of "Wheatstone bridge". All these are discussed here one by one in brief.

Galvanometer

Many common devices including car instrument panels, battery chargers measure potential difference, current or resistance using d'Arsonval Galvanometer. It consists of a pivoted coil placed in the magnetic field of a permanent magnet. Attached to the coil is a spring. In the equilibrium position, with no current in the coil, the pointer is at zero and spring is relaxed. When there is a current in the coil, the magnetic field exerts a torque on the coil that is proportional to current. As the coil turns, the spring exerts a restoring torque that is proportional to the angular displacement. Thus, the angular deflection of the coil and pointer is directly proportional to the coil current and the device can be calibrated to measure current.

The maximum deflection, typically 90° to 120° is called full scale deflection. The essential electrical characteristics of the galvanometer are the current i_g required for full scale deflection (of the order of $10\,\mu\text{A}$ to 10 mA) and the resistance G of the coil (of the order of 10 to 1000 Ω).

The galvanometer deflection is proportional to the current in the coil. If the coil obeys Ohm's law, the current is proportional to potential difference. The corresponding potential difference for full scale deflection is

$$\boxed{V = i_g G}$$

Ammeter

A current measuring instrument is called an ammeter. A galvanometer can be converted into an ammeter by connecting a small resistance S (called shunt) in parallel with it.

Suppose we want to convert a galvanometer with full scale current i_g and coil resistance G into an ammeter with full scale reading i. To determine the shunt resistance S needed, note that, at full scale deflection the total current through the parallel combination is i, the current through the galvanometer is i_g and the current through the shunt is $i - i_g$. The potential difference $V_{ab}\ (= V_a - V_b)$ is the same for both paths, so

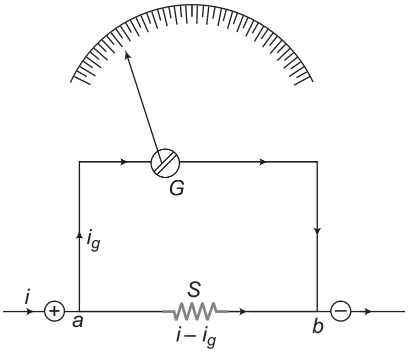


Fig. 23.68

$$i_g G = (i - i_g)\, S$$

$$\therefore \qquad \boxed{S = \left(\frac{i_g}{i - i_g}\right) G}$$

Voltmeter

A voltage measuring device is called a voltmeter. It measures the potential difference between two points.

A galvanometer can be converted into a voltmeter by connecting a high resistance (R) in series with it. The whole assembly called the voltmeter is connected in parallel between the points where potential difference has to be measured.

For a voltmeter with full scale reading V, we need a series resistor R such that

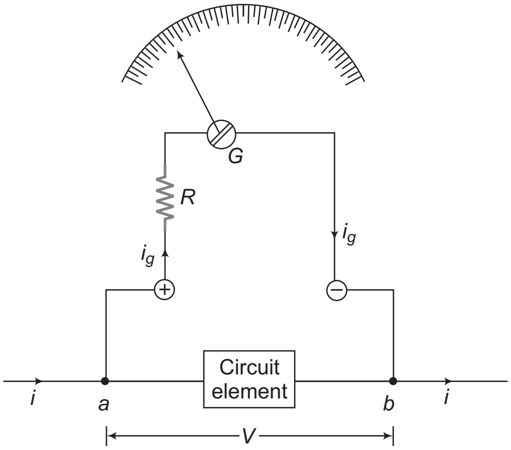


Fig. 23.69

$$V = i_g\,(G + R)$$

or
$$\boxed{R = \frac{V}{i_g} - G}$$

Extra Points to Remember

- **Conversion of galvanometer into an ammeter.**

  (i) A galvanometer is converted into an ammeter by connecting a low resistance (called shunt) in parallel with galvanometer. This assembly (called ammeter) is connected in series in the wire in which current is to be found. Resistance of an ideal ammeter should be zero.

  In parallel, current distributes in the inverse ratio of resistance. Therefore,

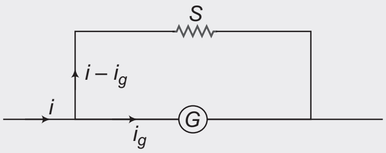


Fig. 23.70

$$\frac{S}{G} = \frac{i_g}{i - i_g}$$

$$\therefore \qquad S = \text{shunt} = \left(\frac{i_g}{i - i_g}\right)G$$

  (ii) Resistance of an ammeter is given by

$$\frac{1}{A} = \frac{1}{G} + \frac{1}{S} \quad \Rightarrow \quad A = \frac{GS}{G + S}$$

(iii) The reading of an ammeter is always lesser than actual current in the circuit.

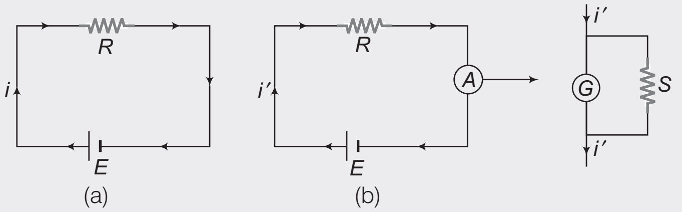


Fig. 23.71

For example, in Fig. (a), actual current through R is

$$i = \frac{E}{R} \qquad \ldots\text{(i)}$$

while the current after connecting an ammeter of resistance $A \left(= \frac{GS}{G+S}\right)$ in series with R is

$$i' = \frac{E}{R+A} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we see that $i' < i$ and $i' = i$ when $A = 0$.

i.e. resistance of an ideal ammeter should be zero.

- **Conversion of a galvanometer into a voltmeter**

(i) A galvanometer is converted into a voltmeter by connecting a high resistance in series with galvanometer. The whole assembly called voltmeter is connected in parallel across the two points between which potential difference is to be found. Resistance of an ideal voltmeter should be infinite.

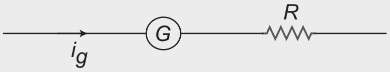


Fig. 23.72

$$V = I_g(G+R)$$

$\therefore$ R = high resistance required in series

$$= \frac{V}{i_g} - G$$

(ii) Resistance of a voltmeter is $R_V = R + G$

(iii) The reading of a voltmeter is always lesser than the true value.

For example, if a current i is passing through a resistance r, the actual value is

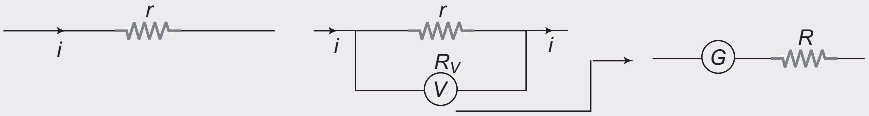


Fig. 23.73

$$V = ir \qquad \ldots\text{(i)}$$

Now, if a voltmeter of resistance R_V $(= G + R)$ is connected across the resistance r, the new value will be

$$V' = \frac{i \times (rR_V)}{r + R_V} \quad \text{or} \quad V' = \frac{ir}{1 + \frac{r}{R_V}} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we can see that, $V' < V$ and $V' = V$ if $R_V = \infty$

Thus, resistance of an ideal voltmeter should be infinite.

Example 23.25 *What shunt resistance is required to make the* $1.00\ mA, 20\ \Omega$ *galvanometer into an ammeter with a range of* 0 *to* $50.0\ mA$?

Solution Here, $i_g = 1.00\ \text{mA} = 10^{-3}\ \text{A}$, $G = 20\ \Omega$, $i = 50.0 \times 10^{-3}\ \text{A}$

Substituting in $S = \left(\dfrac{i_g}{i - i_g}\right) G = \dfrac{(10^{-3})(20)}{(50.0 \times 10^{-3}) - (10^{-3})}$

$$= 0.408\ \Omega$$ **Ans.**

Note *The resistance of ammeter is given by*

$$\frac{1}{A} = \frac{1}{G} + \frac{1}{S} = \frac{1}{20} + \frac{1}{0.408}$$

or $$A = 0.4\ \Omega$$

The shunt resistance is so small in comparison to the galvanometer resistance that the ammeter resistance is very nearly equal to the shunt resistance. This shunt resistance gives us a low resistance ammeter with the desired range of 0 *to* 50.0 *mA. At full scale deflection* $i = 50.0$ *mA, the current through the galvanometer is* 1.0 *mA while the current through the shunt is* 49.0 *mA. If the current i is less than* 50.0 *mA, the coil current and the deflection are proportionally less, but the ammeter resistance is still* $0.4\ \Omega$.

Example 23.26 *How can we make a galvanometer with* $G = 20\ \Omega$ *and* $i_g = 1.0\ mA$ *into a voltmeter with a maximum range of* $10\ V$?

Solution Using $R = \dfrac{V}{i_g} - G$,

We have, $$R = \frac{10}{10^{-3}} - 20$$

$$= 9980\ \Omega$$ **Ans.**

Thus, a resistance of $9980\ \Omega$ is to be connected in series with the galvanometer to convert it into the voltmeter of desired range.

Note *At full scale deflection current through the galvanometer, the voltage drop across the galvanometer*

$$V_g = i_g G = 20 \times 10^{-3}\ \text{volt} = 0.02\ \text{volt}$$

and the voltage drop across the series resistance R is

$$V = i_g R = 9980 \times 10^{-3}\ \text{volt} = 9.98\ \text{volt}$$

or we can say that most of the voltage appears across the series resistor.

Example 23.27 *Resistance of a milliammeter is* R_1 *of an ammeter is* R_2 *of a voltmeter is* R_3 *and of a kilovoltmeter is* R_4. *Find the correct order of* R_1, R_2, R_3 *and* R_4.

Solution To increase the range of an ammeter a low resistance has to be connected in parallel with galvanometer. Therefore, net resistance decreases. To increase the range of voltmeter, a high resistance has to be connected in series. So, net resistance further increases. Therefore, the correct order is

$$R_4 > R_3 > R_1 > R_2$$

Example 23.28 *A microammeter has a resistance of* 100 Ω *and full scale range of* 50 μA. *It can be used as a voltmeter or as a higher range ammeter provided a resistance is added to it. Pick the correct range and resistance combination (s)* **(JEE 1991)**

(a) 50 *V range with* 10 *k*Ω *resistance in series*

(b) 10 *V range with* 200 *k*Ω *resistance in series*

(c) 5 *mA range with* 1 Ω *resistance in parallel*

(d) 10 *mA range with* 1 Ω *resistance in parallel*

Solution To increase the range of ammeter a parallel resistance (called shunt) is required which is given by

$$S = \left(\frac{i_g}{i - i_g}\right) G$$

For option (c),

$$S = \left(\frac{50 \times 10^{-6}}{5 \times 10^{-3} - 50 \times 10^{-6}}\right)(100) \approx 1\,\Omega$$

To change it in voltmeter, a high resistance R is put in series, where R is given by $R = \frac{V}{i_g} - G$

For option (b),

$$R = \frac{10}{50 \times 10^{-6}} - 100 \approx 200\text{ k}\Omega$$

Therefore, options (b) and (c) are correct.

Example 23.29 *A galvanometer gives full scale deflection with* 0.006 *A current. By connecting it to a* 4990 Ω *resistance, it can be converted into a voltmeter of range* 0-30 *V. If connected to a* $\frac{2n}{249}$ Ω *resistance, it becomes an ammeter of range* 0-1.5 *A. The value of n is* **(JEE 2014)**

Solution

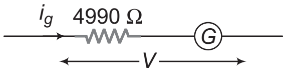


Fig. 23.74

$$i_g(G + 4990) = V$$

$$\Rightarrow \quad \frac{6}{1000}(G + 4990) = 30$$

$$\Rightarrow \quad G + 4990 = \frac{30000}{6} = 5000$$

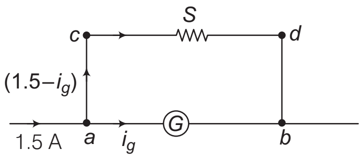


Fig. 23.75

$$\Rightarrow \qquad G = 10\,\Omega$$

$$V_{ab} = V_{cd}$$

$$\Rightarrow \qquad i_g G = (1.5 - i_g)\,S$$

$$\Rightarrow \qquad \frac{6}{1000} \times 10 = \left(1.5 - \frac{6}{1000}\right) S$$

$$\Rightarrow \qquad S = \frac{60}{1494} = \frac{2n}{249}$$

$$\Rightarrow \qquad n = \frac{249 \times 30}{1494}$$

$$= \frac{2490}{498} = 5$$

Ans.

INTRODUCTORY EXERCISE 23.9

1. The full scale deflection current of a galvanometer of resistance 1 Ω is 5 mA. How will you convert it into a voltmeter of range 5 V?
2. A micrometer has a resistance of 100 Ω and full scale deflection current of 50 μA. How can it be made to work as an ammeter of range 5 mA?
3. A voltmeter has a resistance G and range V. Calculate the resistance to be used in series with it to extend its range to nV.

Potentiometer

The potentiometer is an instrument that can be used to measure the emf or the internal resistance of an unknown source. It also has a number of other useful applications.

Principle of Potentiometer

The principle of potentiometer is schematically shown in figure.

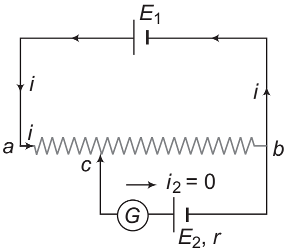


Fig. 23.76

A resistance wire ab of total resistance R_{ab} is permanently connected to the terminals of a source of known emf E_1. A sliding contact c is connected through the galvanometer G to a second source whose emf E_2 is to be measured. As contact c is moved along the potentiometer wire, the resistance R_{cb} between points c and b varies. If the resistance wire is uniform R_{cb} is proportional to the length of the wire between c and b. To determine the value of E_2, contact c is moved until a position is found at which the galvanometer shows no deflection. This corresponds to zero current passing through E_2. With $i_2 = 0$, Kirchhoff's second law gives

$$\boxed{E_2 = iR_{cb}}$$

With $i_2 = 0$, the current i produced by the emf E_1 has the same value no matter what the value of emf E_2. A potentiometer has the following applications.

To find emf of an unknown battery

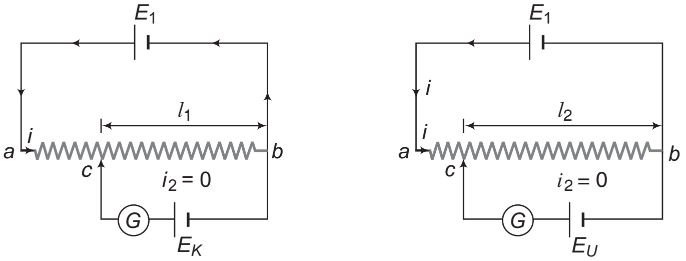


Fig. 23.77

We calibrate the device by replacing E_2 by a source of known emf E_K and then by unknown emf E_U. Let the null points are obtained at lengths l_1 and l_2. Then,

$$E_K = i(\rho l_1) \qquad \text{and} \qquad E_U = i(\rho l_2)$$

Here, ρ = resistance of wire ab per unit length.

$$\therefore \qquad \frac{E_K}{E_U} = \frac{l_1}{l_2} \qquad \text{or} \qquad \boxed{E_U = \left(\frac{l_2}{l_1}\right) E_K}$$

So, by measuring the lengths l_1 and l_2, we can find the emf of an unknown battery.

To find the internal resistance of an unknown battery

To find the internal resistance of an unknown battery let us derive a formula.

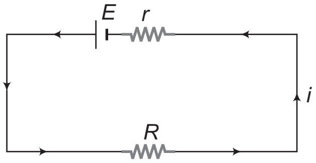


Fig. 23.78

In the circuit shown in figure,

$$i = \frac{E}{R + r} \qquad \text{...(i)}$$

and V = potential difference across the terminals of the battery

or $$V = E - ir = iR \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can prove that

$$\boxed{r = R\left(\frac{E}{V} - 1\right)}$$

Thus, if a battery of emf E and internal resistance r is connected across a resistance R and the potential difference across its terminals comes out to be V then the internal resistance of the battery is given by the above formula. Now, let us apply it in a potentiometer for finding the internal resistance of the unknown battery. The circuit shown in Fig. 23.79 is similar to the previous one.

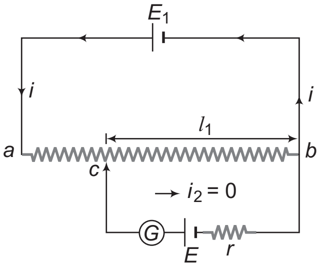


Fig. 23.79

Hence, $$E = i\rho l_1 \qquad \text{...(i)}$$

Now, a known resistance R is connected across the terminals of the unknown battery as shown in Fig. 23.80.

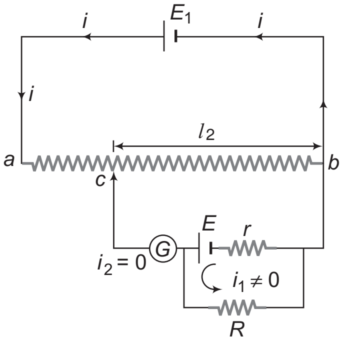


Fig. 23.80

This time $\quad V_{cb} \neq E, \quad$ but $\quad V_{cb} = V$

where, $\quad V =$ potential difference across the terminals of the unknown battery.

Hence, $$V = i\rho l_2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{E}{V} = \frac{l_1}{l_2}$$

Substituting in $r = R\left(\frac{E}{V} - 1\right)$, we get

$$r = R\left(\frac{l_1}{l_2} - 1\right)$$

So, by putting R, l_1 and l_2 we can determine the internal resistance r of unknown battery.

✓ Extra Points to Remember

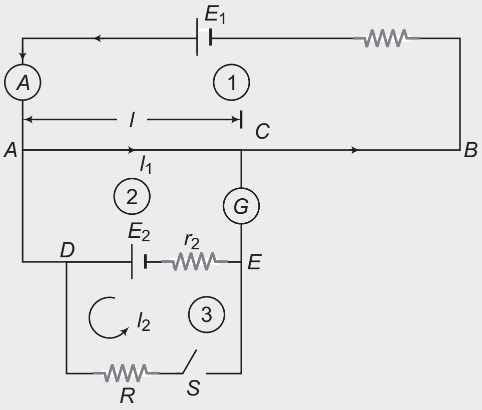
- 

Fig. 23.81

Under balanced condition (when $I_G = 0$) loop-1 and loop-3 are independent with each other. All problems in this condition can be solved by a single equation,

$$V_{AC} = V_{DE}$$

or $$i_1 R_{AC} = E_2 - i_2 r_2$$

or $$\boxed{i_1 \lambda l = E_2 - i_2 r_2} \qquad \ldots\text{(i)}$$

Here, λ is the resistance per unit length of potentiometer wire *AB*. Length *l* is called balance point length. Currents i_1 and i_2 are independent with each other. Current $i_2 = 0$, if switch is open.

- Under balanced condition, a part of potential difference of E_1 is balanced by the lower circuit. So, normally $E_2 < E_1$ for taking balance point length. Similarly, $V_{AC} = V_{DE} \Rightarrow V_A = V_D$ and $V_C = V_E$. Therefore, positive terminals of both batteries should be on same side and negative terminals on the other side.
- From Eq. (i), we can see that null point length is

$$l = \frac{E_2 - i_2 r_2}{i_1 \lambda}$$

Now, suppose E_1 is increased then i_1 will also increase and null point length *l* will decrease. Similarly, we can make some other cases also.

- If we do not get any balanced condition ($I_G \neq 0$), then the given circuit is simply a three loops problem, which can be solved with the help of Kirchhoff's laws.

Example 23.30 *A potentiometer wire of length* 100 cm *has a resistance of* 10 Ω. *It is connected in series with a resistance R and a cell of emf* 2 V *and of negligible internal resistance. A source of emf* 10 mV *is balanced against a length of* 40 cm *of the potentiometer wire. What is the value of R?*

Solution From the theory of potentiometer, $V_{cb} = E$, if no current is drawn from the battery

or $$\left(\frac{E_1}{R + R_{ab}}\right) R_{cb} = E$$

Here, $E_1 = 2\text{ V}$, $R_{ab} = 10\,\Omega$, $R_{cb} = \left(\frac{40}{100}\right) \times 10 = 4\,\Omega$

and $$E = 10 \times 10^{-3}\ \text{V}$$

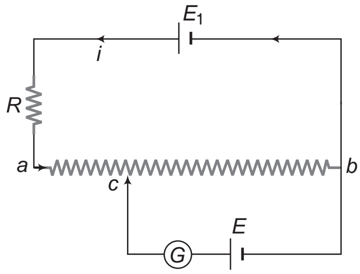


Fig. 23.82

Substituting in above equation, we get

$$R = 790\ \Omega$$ **Ans.**

Example 23.31 *When the switch is open in lowermost loop of a potentiometer, the balance point length is* 60 *cm. When the switch is closed with a known resistance of* $R = 4\ \Omega$, *the balance point length decreases to* 40 *cm. Find the internal resistance of the unknown battery.*

Solution Using the result,

$$r = R\left(\frac{l_1}{l_2} - 1\right)$$

$$= 4\left(\frac{60}{40} - 1\right)$$

$$= 2\ \Omega$$ **Ans.**

Example 23.32

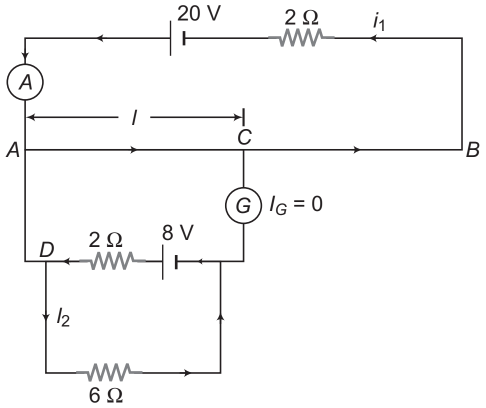


Fig. 23.83

In the figure shown, wire AB has a length of 100 *cm and resistance* $8\ \Omega$. *Find the balance point length l.*

Solution Using the equation, $i_1\lambda l = E_2 - i_2 r_2$

We have,

$$\left(\frac{20}{2+8}\right)\left(\frac{8}{100}\right) l = 8 - \left(\frac{8}{2+6}\right)(2)$$

Solving this equation, we get

$$l = 37.5 \text{ cm}$$ **Ans.**

INTRODUCTORY EXERCISE 23.10

1. In a potentiometer experiment it is found that no current passes through the galvanometer when the terminals of the cell are connected across 0.52 m of the potentiometer wire. If the cell is shunted by a resistance of 5 Ω a balance is obtained when the cell is connected across 0.4 m of the wire. Find the internal resistance of the cell.
2. The potentiometer wire *AB* is 600 cm long.

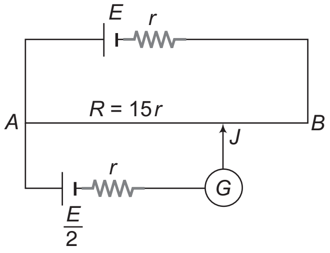


Fig. 23.84

(a) At what distance from *A* should the jockey *J* touch the wire to get zero deflection in the galvanometer.

(b) If the jockey touches the wire at a distance 560 cm from *A*, what will be the current through the galvanometer.

Principle of Wheatstone's Bridge

The scientist Wheatstone designed a circuit to find unknown resistance. Such a circuit is popularly known as Wheatstone's bridge. This is an arrangement of four resistances which can be used to measure one of them in terms of the rest. The figure shows the circuit designed by him. The bridge is said to be **balanced** when deflection in galvanometer is zero, i.e. $i_g = 0$, and hence,

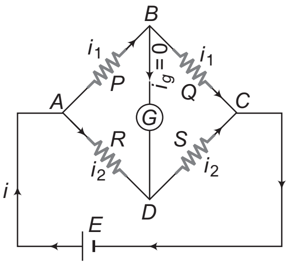


Fig. 23.85

$$V_B = V_D$$

Under this condition, $V_A - V_B = V_A - V_D$

or $i_1 P = i_2 R$

or $\dfrac{i_1}{i_2} = \dfrac{R}{P}$ …(i)

Similarly, $V_B - V_C = V_D - V_C$

or $$i_1Q = i_2S \quad \text{or} \quad \frac{i_1}{i_2} = \frac{S}{Q} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), $$\frac{R}{P} = \frac{S}{Q} \quad \text{or} \quad \boxed{\frac{P}{Q} = \frac{R}{S}}$$

So, this is a condition for which a Wheatstone's bridge is balanced.

To measure the resistance of an unknown resistor, it is connected as one of the four resistors in the bridge. One of the other three should be a variable resistor. Let us suppose P is the unknown resistance and Q is the variable resistance. The value of Q is so adjusted that deflection through the galvanometer is zero. In this case, the bridge is balanced and

$$P = \left(\frac{R}{S}\right) \cdot Q$$

Knowing R, S and Q, the value of P is calculated. Following two points are important regarding a Wheatstone's bridge.

(i) In Wheatstone's bridge, cell and galvanometer arms are interchangeable.

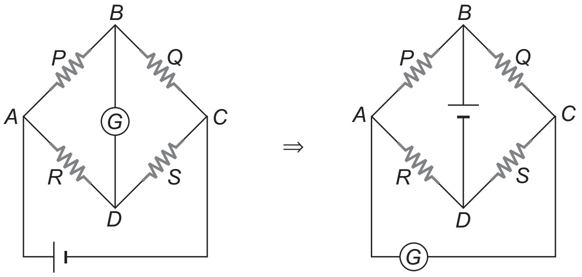


Fig. 23.86

In both the cases, condition of balanced bridge is

$$\frac{P}{Q} = \frac{R}{S}$$

(ii) If bridge is not balanced current will flow from D to B in Fig. 23.85 if,

$$PS > RQ$$

Exercise Try and prove the statements of both the points yourself.

Meter Bridge Experiment

Meter bridge works on Wheatstone's bridge principle and is used to find the unknown resistance (X) and its specific resistance (or resistivity).

Theory

As the meter bridge wire AC has uniform material density and area of cross-section, its resistance is proportional to its length. Hence, AB and BC are the ratio arms and their resistances correspond to P and Q respectively.

Thus, $$\frac{\text{Resistance of } AB}{\text{Resistance of } BC} = \frac{P}{Q} = \frac{\lambda l}{\lambda\,(100 - l)} = \frac{l}{100 - l}$$

Here, λ is the resistance per unit length of the bridge wire.

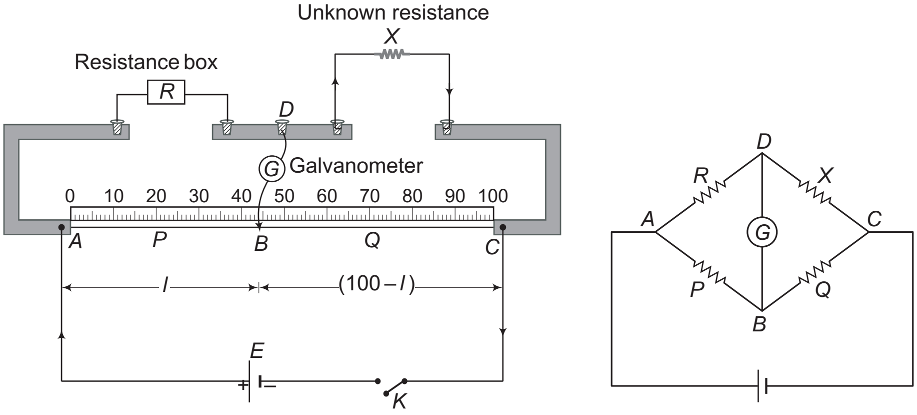


Fig. 23.87

Hence, according to Wheatstone's bridge principle,

When current through galvanometer is zero or bridge is balanced, then

$$\frac{P}{Q} = \frac{R}{X} \quad \text{or} \quad X = \frac{Q}{P}R$$

$$\therefore \qquad X = \left(\frac{100 - l}{l}\right)R \qquad \text{...(i)}$$

So, by knowing R and l unknown resistance X can be determined.

Specific Resistance From resistance formula,

$$X = \rho \frac{L}{A}$$

or

$$\rho = \frac{XA}{L}$$

For a wire of radius r or diameter $D = 2r$,

$$A = \pi r^2 = \frac{\pi D^2}{4}$$

or

$$\rho = \frac{X \pi D^2}{4L} \qquad \text{...(ii)}$$

By knowing X, D and L we can find specific resistance of the given wire by Eq. (ii).

Precautions

1. The connections should be clean and tight.
2. Null point should be brought between 40 cm and 60 cm.
3. At one place, diameter of wire (D) should be measured in two mutually perpendicular directions.
4. The jockey should be moved gently over the bridge wire so that it does not rub the wire.

End Corrections

In meter bridge, some extra length (under the metallic strips) comes at points A and C. Therefore, some additional length (α and β) should be included at the ends. Here, α and β are called the end corrections. Hence, in place of l we use $l+\alpha$ and in place of $100-l$ we use $100-l+\beta$.

To find α and β, use known resistors R_1 and R_2 in place of R and X and suppose we get null point length equal to l_1. Then,

$$\frac{R_1}{R_2}=\frac{l_1+\alpha}{100-l_1+\beta} \qquad \text{...(i)}$$

Now, we interchange the positions of R_1 and R_2 and suppose the new null point length is l_2. Then,

$$\frac{R_2}{R_1}=\frac{l_2+\alpha}{100-l_2+\beta} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\alpha=\frac{R_2l_1-R_1l_2}{R_1-R_2}$$

and

$$\beta=\frac{R_1l_1-R_2l_2}{R_1-R_2}-100$$

Example 23.33 *If resistance R_1 in resistance box is 300 Ω, then the balanced length is found to be 75.0 cm from end A. The diameter of unknown wire is 1 mm and length of the unknown wire is 31.4 cm. Find the specific resistance of the unknown wire.*

Solution $\dfrac{R}{X}=\dfrac{l}{100-l}$

$$\Rightarrow \qquad X=\left(\frac{100-l}{l}\right)R$$

$$=\left(\frac{100-75}{75}\right)(300)=100\,\Omega$$

Now,

$$X=\frac{\rho l}{A}=\frac{\rho l}{(\pi d^2/4)}$$

$$\therefore \qquad \rho=\frac{\pi d^2X}{4l}$$

$$=\frac{(22/7)(10^{-3})^2(100)}{(4)(0.314)}$$

$$=2.5\times10^{-4}\,\Omega\text{-m}$$ **Ans.**

Example 23.34 *In a meter bridge, null point is 20 cm, when the known resistance R is shunted by 10 Ω resistance, null point is found to be shifted by 10 cm. Find the unknown resistance X.*

Solution
$$\frac{R}{X}=\frac{l}{100-l}$$

∴
$$X=\left(\frac{100-l}{l}\right)R$$

or
$$X=\left(\frac{100-20}{20}\right)R=4R \qquad \text{...(i)}$$

When known resistance R is shunted, its net resistance will decrease. Therefore, resistance parallel to this (i.e. P) should also decrease or its new null point length should also decrease.

∴
$$\frac{R'}{X}=\frac{l'}{100-l'}$$

$$=\frac{20-10}{100-(20-10)}=\frac{1}{9}$$

or
$$X=9R' \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$4R=9R'=9\left[\frac{10R}{10+R}\right]$$

Solving this equation, we get

$$R=\frac{50}{4}\,\Omega$$

Now, from Eq. (i), the unknown resistance

$$X=4R=4\left(\frac{50}{4}\right)$$

or
$$X=50\,\Omega$$ **Ans.**

Note *R' is resultant of R and 10 Ω in parallel.*

∴
$$\frac{1}{R'}=\frac{1}{10}+\frac{1}{R}$$

or
$$R'=\frac{10R}{10+R}$$

Example 23.35 *If we use 100 Ω and 200 Ω in place of R and X we get null point deflection, $l=33$ cm. If we interchange the resistors, the null point length is found to be 67 cm. Find end corrections α and β.*

Solution
$$\alpha=\frac{R_2l_1-R_1l_2}{R_1-R_2}$$

$$=\frac{(200)(33)-(100)(67)}{100-200}$$

$$=1\,\text{cm}$$ **Ans.**

$$\beta = \frac{R_1 l_1 - R_2 l_2}{R_1 - R_2} - 100$$

$$= \frac{(100)(33) - (200)(67)}{100 - 200} - 100$$

$$= 1 \text{cm}$$ **Ans.**

INTRODUCTORY EXERCISE 23.11

1. A resistance of 2 Ω is connected across one gap of a meter bridge (the length of the wire is 100 cm) and an unknown resistance, greater than 2 Ω, is connected across the other gap. When these resistances are interchanged, the balance point shifts by 20 cm. Neglecting any corrections, the unknown resistance is **(JEE 2007)**

   (a) 3 Ω (b) 4 Ω
   (c) 5 Ω (d) 6 Ω

2. A meter bridge is setup as shown in figure, to determine an unknown resistance *X* using a standard 10 Ω resistor. The galvanometer shows null point when tapping key is at 52 cm mark. The end corrections are 1 cm and 2 cm respectively for the ends *A* and *B*. The determined value of *X* is

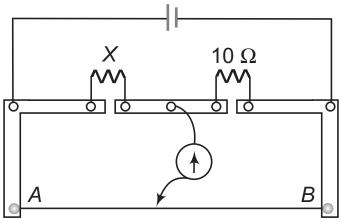


Fig. 23.88

(JEE 2011)

   (a) 10.2 Ω (b) 10.6 Ω
   (c) 10.8 Ω (d) 11.1 Ω

3. R_1, R_2, R_3 are different values of *R*. *A*, *B* and *C* are the null points obtained corresponding to R_1, R_2 and R_3 respectively. For which resistor, the value of *X* will be the most accurate and why? **(JEE 2005)**

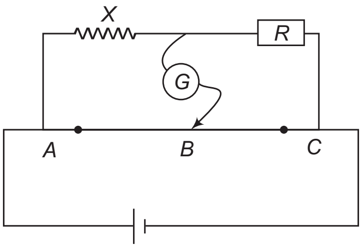


Fig. 23.89

Post Office Box

Post office box also works on the principle of Wheatstone's bridge.

In a Wheatstone's bridge circuit, if $\frac{P}{Q}=\frac{R}{X}$ then the bridge is balanced. So, unknown resistance $X=\frac{Q}{P}R$.

P and Q are set in arms AB and BC where we can have, $10\,\Omega, 100\,\Omega$ or $1000\,\Omega$ resistances to set any ratio $\frac{Q}{P}$.

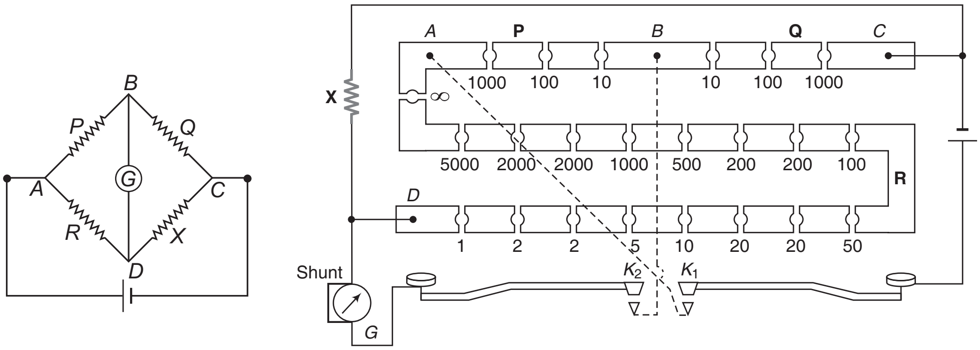


Fig. 23.90

These arms are called ratio arm, initially we take $Q=10\,\Omega$ and $P=10\,\Omega$ to set $\frac{Q}{P}=1$. The unknown resistance (X) is connected between C and D and battery is connected across A and C,

Now, adjust resistance in part A to D such that the bridge gets balanced. For this, keep on increasing the resistance with $1\,\Omega$ interval, check the deflection in galvanometer by first pressing key K_1 then galvanometer key K_2.

Suppose at $R=4\,\Omega$, we get deflection towards left and at $R=5\,\Omega$, we get deflection towards right. Then, we can say that for balanced condition, R should lie between 4 Ω to 5 Ω.

Now, $X=\frac{Q}{P}R=\frac{10}{10}R=R=4\,\Omega$ to $5\,\Omega$

To get closer value of X, in the second observation, let us choose $\frac{Q}{P}=\frac{1}{10}$ i.e. $\left(\frac{P=100}{Q=10}\right)$

Suppose, now at $R=42$ we get deflection towards left and at $R=43$ deflection is towards right. So $R\in(42, 43)$.

Now, $X=\frac{Q}{P}R=\frac{10}{100}R=\frac{1}{10}R$, where $R\in(4.2, 4.3\,\Omega)$. Now, to get further closer value take $\frac{Q}{P}=\frac{1}{100}$ and so on.

The observation table is shown below.

Table 23.2

S.No	Resistance in the ratio arm AB (P) (ohm)	Resistance in the ratio arm BC (Q) (ohm)	Resistance in arm (AD (R) (ohm)	Direction of deflection	Unknown resistance $X = \frac{Q}{P} \times R$ (ohm)
1	10	10	4	Left	4 to 5
			5	Right	
2	100	10	40	Left (large)	(4.2 to 4.3)
			50	Right (large)	
			42	Left	
			43	Right	
3	1000	10	420	Left	4.25
			424	Left	
			425	No deflection	
			426	Right	

So, the correct value of X is 4.25 Ω

Example 23.36 *To locate null point, deflection battery key* (K_1) *is pressed before the galvanometer key* (K_2). *Explain why?*

Solution If galvanometer key K_2 is pressed first then just after closing the battery key K_1 current suddenly increases.

So, due to self-induction, a large back emf is generated in the galvanometer, which may damage the galvanometer.

Example 23.37 *What are the maximum and minimum values of unknown resistance X, which can be determined using the post office box shown in the Fig.* 23.90*?*

Solution

$$X = \frac{QR}{P}$$

$$\therefore \quad X_{\max} = \frac{Q_{\max}\ R_{\max}}{P_{\min}}$$

$$= \frac{1000}{10}(11110)$$

$$= 1111\,\text{k}\Omega$$ **Ans.**

$$X_{\min} = \frac{Q_{\min}\ R_{\min}}{P_{\max}}$$

$$= \frac{(10)\,(1)}{1000}$$

$$= 0.01\,\Omega$$ **Ans.**

INTRODUCTORY EXERCISE 23.12

1. In post office box experiment, $\frac{Q}{P}=\frac{1}{10}$. In R if 142 Ω is used then we get deflection towards right and if $R=143\ \Omega$, then deflection is towards left. What is the range of unknown resistance?
2. What is the change in post office box experiment if battery is connected between B and C and galvanometer is connected across A and C?
3. For the post office box arrangement to determine the value of unknown resistance, the unknown resistance should be connected between **(JEE 2004)**

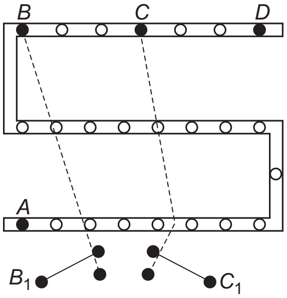


Fig. 23.91

(a) B and C (b) C and D (c) A and D (d) B_1 and C_1

Extra Topics For Other Examinations

23.13 Colour Codes for Resistors

Resistors are of the following two major types :

(i) wire bound resistors and (ii) carbon resistors

First type of resistors are made by winding the wires of an alloy like nichrome, manganin or constantan etc. Materials are so chosen that their resistivities are relatively less sensitive to temperature.

In carbon resistors, carbon with a suitable binding agent is molded into a cylinder. Wire leads are attached to this cylinder and the entire resistor is encased in a ceramic or plastic jacket. The two leads connect the resistor to a circuit. Carbon resistors are compact and inexpensive. Their values are given using a colour code.

Table 23.3

Colour	Number	Multiplier	Tolerance (%)
Black	0	1	
Brown	1	10^1	
Red	2	10^2	
Orange	3	10^3	
Yellow	4	10^4	

Colour	Number	Multiplier	Tolerance (%)
Green	5	10^5	
Blue	6	10^6	
Violet	7	10^7	
Gray	8	10^8	
White	9	10^9	
Gold		10^{-1}	5
Silver		10^{-2}	10
No colour			20

The resistors have a set of four (or three) co-axial coloured rings, whose significance are listed in above table. The colours are noted from left to right.

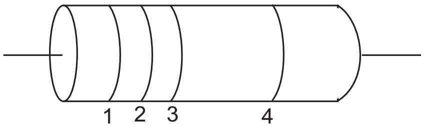


Fig. 23.92

Colour 1 → First significant figure

Colour 2 → Second significant figure

Colour 3 → Decimal multiplier

Colour 4 (or no colour) → Tolerance or possible variation in percentage.

Extra Points to Remember

- To remember the value of colour coding used for carbon resistor, the following sentences are found to be of great help (where bold letters stand for colours)
  B B ROY Great **B**ritain **V**ery **G**ood **W**ife wearing **Gold Silver** necklace.

  OR

  Black **B**rown **R**ods **O**f **Y**our **G**ate **B**ecome **V**ery **G**ood **W**hen **G**iven **S**ilver colour

Example 23.38 *The four colours on a resistor are : brown, yellow, green and gold as read from left to right. What is resistance corresponding to these colours.*

Solution From the table we can see that

$$\text{Brown colour} \rightarrow 1$$

$$\text{Yellow colour} \rightarrow 4$$

$$\text{Green colour} \rightarrow 10^5 \text{ and}$$

$$\text{Gold colour} \rightarrow 5\,\%$$

$$\therefore \qquad R = (14 \times 10^5 \pm 5\%)\,\Omega$$ **Ans.**

INTRODUCTORY EXERCISE 23.13

1. For the given carbon resistor, let the first strip be yellow, second strip be red, third strip be orange and fourth be gold. What is its resistance?
2. The resistance of the given carbon resistor is $(24 \times 10^6 \pm 5\%)\,\Omega$. What is the sequence of colours on the strips provided on resistor?

Final Touch Points

1. **Mobility** The physical significance of mobility means how mobile the charge carriers are for the current flow. If mobility of charge carriers is more than we can say that current flow will be more.

   In metals, the mobile charge carriers are electrons. In an ionised gas they are electrons and positive charged ions. In an electrolyte, these can be positive and negative ions. In semiconductors, charge carriers are electrons and holes. Later, we will see that mobility of electrons (in semiconductor) is more than the mobility of holes.

   Mobility (μ) is defined as the magnitude of drift velocity per unit electric field.

   Thus, $$\boxed{\mu = \frac{v_d}{E}}$$

   But, $$v_d = \frac{eE\tau}{m}$$

   ∴ $$\boxed{\mu = \frac{e\tau}{m}}$$

   The SI units of mobility are $m^2/V\text{-}s$. Therefore, practical units of mobility is $cm^2/V\text{-}s$. Mobility of any charge carrier (whether it is electron, ion or hole) is always positive.

2. **Deduction of Ohm's law** We know that

   $$i = neAv_d$$

   where, $v_d = \frac{e\tau E}{m}$

   ∴ $$i = \frac{ne^2\tau AE}{m}$$

   If V is the potential difference across the conductor and l is its length, then

   $$E = \frac{V}{l}$$

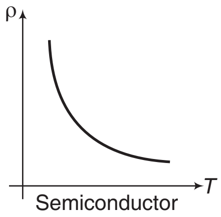
   ∴ $$i = \frac{ne^2\tau AV}{ml} \quad \text{or} \quad V = \left(\frac{m}{ne^2\tau}\right)\frac{l}{A}i$$

   Here, $$\boxed{\frac{m}{ne^2\tau} = \frac{1}{\sigma} \text{ or } \rho}$$

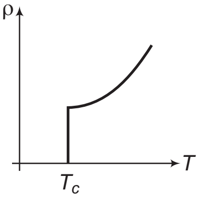
   ∴ $$V = \left(\frac{\rho l}{A}\right)\cdot i \quad \text{or} \quad V = Ri$$ **Hence proved.**

   where, $R = \rho\frac{l}{A}$ is the resistance of the conductor.

3. **Thermistor** The temperature coefficient of resistivity is negative for semiconductors. This means that the resistivity decreases as we raise the temperature of such a material. The magnitude of the temperature coefficient of resistivity is often quite large for a semiconducting material. This fact is used to construct thermometers to detect small changes in temperatures. Such a device is called a thermistor. The variation of resistivity of a semiconductor with temperature is shown in figure. A typical thermistor can easily measure a change in temperature of the order of 10^{-3} °C



Semiconductor



Superconductor

4. **Superconductors** Superconductivity was first discovered in 1911 by the Dutch physicist Heike Kamerlingh Onnes.

   There are certain materials, including several metallic alloys and oxides for which as the temperature decreases, the resistivity first decreases smoothly, like that of any metal. But then at a certain critical temperature T_c a phase transition occurs, and the resistivity suddenly drops to zero as shown in figure.

   Once, a current has been established in a superconducting ring, it continues indefinitely without the presence of any driving field. Possible applications of superconductors are ultrafast computer switches and transmission of electric power through superconducting power lines. However, the requirement of low temperature is posing difficulty. For instance the critical temperature for mercury is 4.2 K. Scientists are putting great effort to construct compounds and alloys which would be superconducting at room temperature (300 K). Superconductivity at around 125 K has already been achieved.

5. The ρ-T equation derived in article 23.6 can be derived from the relation,

$$\frac{d\rho}{dT} = \alpha\rho \quad \text{or} \quad \frac{d\rho}{\rho} = \alpha \, dT$$

$$\therefore \qquad \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = \alpha \int_{T_0}^{T} dT \qquad (\text{if } \alpha = \text{constant}) \quad \ldots\text{(i)}$$

$$\therefore \qquad \ln\left(\frac{\rho}{\rho_0}\right) = \alpha(T - T_0)$$

$$\therefore \qquad \rho = \rho_0 e^{\alpha(T - T_0)}$$

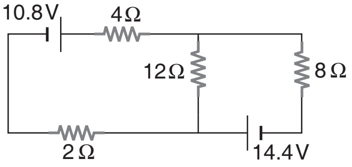
   If α is small, $e^{\alpha(T - T_0)}$ can approximately be written as $1 + \alpha(T - T_0)$. Hence,

$$\rho = \rho_0[1 + \alpha(T - T_0)]$$

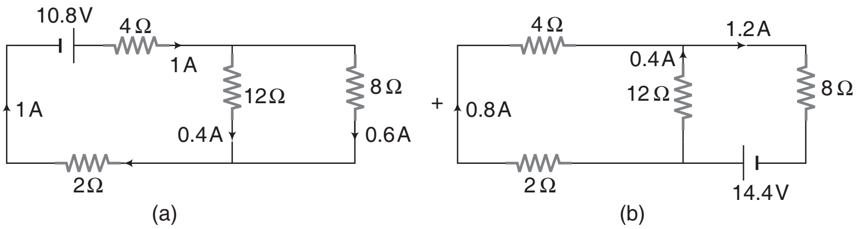
   Which is the same result as we have discussed earlier.

   In the above discussion, we have assumed α to be constant. If it is function of temperature it will come inside the integration in Eq. (i).

6. **The principle of superposition** Complex network problems can sometimes be solved easily by using the principle of superposition. This principle essentially states that when a number of emf's act in a network, the solution is the same as the superposition of the solutions for one emf acting at a time, the others being shorted.



7. Figure shows a network with two loops. The currents in various branches can be calculated using Kirchhoff's laws. We can get the same solution by considering only one battery at a time and then superposing the two solutions. If a battery has an internal resistance, it must be left in place when the emf of the battery is removed. Figure shows how the superposition principle can be applied to the present problem.



(a) (b)

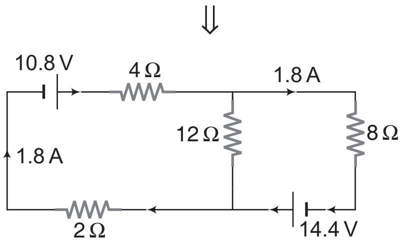
The current values in Fig. (a) and (b) are easily verified. For example when the 10.8 V battery alone is acting, the total resistance in the circuit is

$$4+\frac{12\times 8}{12+8}+2=10.8\ \Omega$$

This makes the total current $\frac{10.8\text{ V}}{10.8\ \Omega}=1$ A. This current splits between $8\ \Omega$ and $12\ \Omega$ in the ratio 3 : 2. Similarly, the total resistance when only the 14.4 V battery is acting is

$$8+\frac{12\times 6}{12+6}=12\ \Omega$$

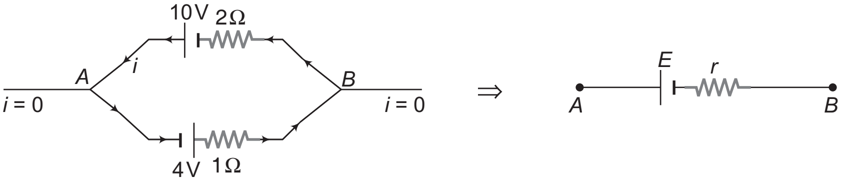
Therefore, the total current is $\frac{14.4\text{ V}}{12\ \Omega}=1.2$ A.



(c)

The superposition principle shows that there is no current in the $12\ \Omega$ resistance. Only a current of 1.8 A flows through the outer loop. All these conclusions can be verified by analyzing the circuit using Kirchhoff's laws.

8. The equivalent emf of a cell can also be found by the following method.



Suppose we wish to find the equivalent emf of the above circuit. We apply the fact that

$$E=V$$

When no current is drawn from the cell. But current in the internal circuit may be non-zero. This current is,

$$i=\frac{10+4}{2+1}=\frac{14}{3}\text{ A}$$

Now, $$V_A+4-1\times\frac{14}{3}=V_B$$

∴ $$V_A-V_B=\frac{14}{3}-4=\frac{2}{3}\text{ V}$$

∴ $$E=V_A-V_B=V=\frac{2}{3}\text{ V}$$

∴ $$E=\frac{2}{3}\text{ V}$$

Further, V_A-V_B is positive, i.e. $V_A>V_B$ or A is connected to the positive terminal of the battery and B to the negative.

Internal resistance of the equivalent battery is found by the normal procedure. For example, here $2\ \Omega$ and $1\ \Omega$ resistances are in parallel. Hence, their combined resistance is

$$\frac{1}{r}=\frac{1}{1}+\frac{1}{2}=\frac{3}{2}\quad\text{or}\quad r=\frac{2}{3}\ \Omega$$

Solved Examples

TYPED PROBLEMS

Type 1. *Based on potential difference across the terminals of a battery*

Concept

Potential difference V across the terminals of a battery is given by

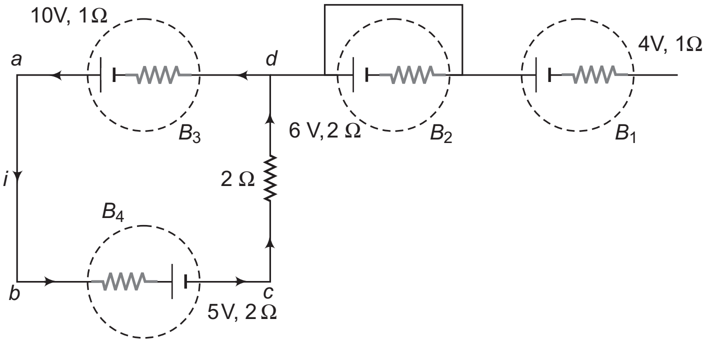
$V = E$ if $i = 0$

$V = 0$ if battery is short-circuited

$V = E - ir$ if normal current ($\leftarrow\!\mid\vdash\!\underset{i}{\leftarrow}$) flows through the battery and

$V = E + ir$ if current flows in the opposite direction ($\rightarrow\!\mid\vdash\!\underset{i}{\rightarrow}$)

Example 1 *Find the potential difference across each of the four batteries B_1, B_2, B_3 and B_4 as shown in the figure.*



Solution **Across B_1** This battery does not make any closed circuit.

$$\therefore \qquad i = 0 \quad \text{or} \quad V = E = 4 \text{ volt}$$ **Ans.**

Across B_2 This battery is short-circuited. Therefore,

$$V = 0$$ **Ans.**

Across B_3 and B_4 A current in anti-clockwise direction flows in the closed loop *abcda*. This current is

$$i = \frac{\text{net emf}}{\text{net resistance}} = \frac{10-5}{1+2+2}$$

$$= 1 \text{ A}$$

Now, current flows through B_3 in normal direction. Hence,

$$V = E - ir = 10 - 1 \times 1 = 9 \text{ volt}$$ **Ans.**

From B_4, current flows in opposite direction. Hence,

$$V = E + ir = 5 + 1 \times 2 = 7 \text{ volt}$$ **Ans.**

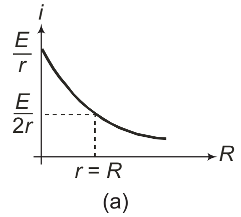
Example 2 *Draw (a) current versus load and (b) current versus potential difference (across its two terminals) graph for a cell.*

Solution (a) $i = \dfrac{E}{R + r}$

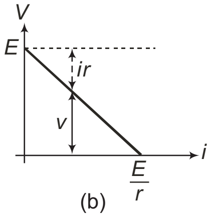
i versus R graph is shown in Fig. (a).

(b) $V = E - ir$

V versus i graph is shown in Fig. (b).



(a)



(b)

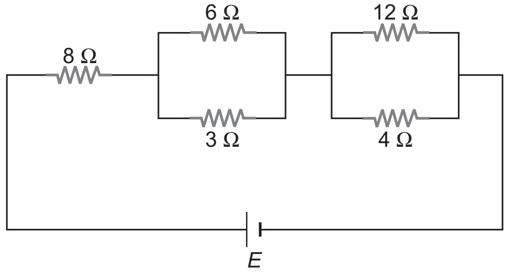
Type 2. *To find values of V, i and R across all resistors of a complex circuit if values across one resistance are known*

Concept

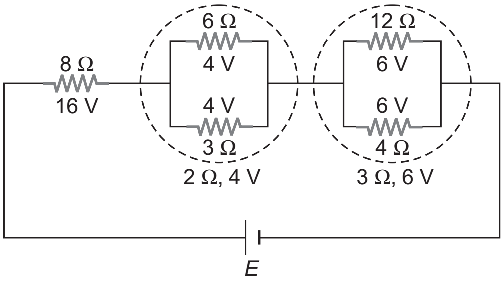
(i) In series, current remains same. But the potential difference distributes in the direct ratio of resistance.

(ii) In parallel, potential difference is same. But the current distributes in the inverse ratio of resistance.

Example 3 *In the circuit shown in figure potential difference across* $6\,\Omega$ *resistance is* 4 *volt. Find V and i values across each resistance. Also find emf E of the applied battery.*



Solution



$8\,\Omega, 2\,\Omega$ (Resultant of $6\,\Omega$ and $3\,\Omega$) and $3\,\Omega$ (resultant of $12\,\Omega$ and $4\,\Omega$) are in series. Therefore, potential drop across them should be in direct ratio of resistance. So, using this concept we can find the potential difference across other resistors. For example, potential across $2\,\Omega$ was 4 V. So, potential difference across $8\,\Omega$ (which is four times of $2\,\Omega$) should be 16 V. Similarly, potential difference across $3\,\Omega$ (which is 1.5 times of $2\,\Omega$) should be 1.5 times or 6 V.

Once V and R are known, we can find i across that resistance. For example,

$$i_{12\Omega} = \frac{6}{12} = \frac{1}{2}\text{ A} \qquad \left(i = \frac{V}{R}\right)$$

$$i_{8\Omega} = \frac{16}{8} = 2\text{ A etc}$$

Type 3. *To find equivalent value of temperature coefficient α if two or more than two resistors are connected in series or parallel*

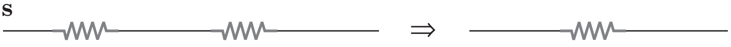
- This can be explained by the following example :

Example 4 *Two resistors with temperature coefficients of resistance α_1 and α_2 have resistances R_{01} and R_{02} at 0°C. Find the temperature coefficient of the compound resistor consisting of the two resistors connected*

(a) in series and

(b) in parallel.

Solution (a) **In Series**



At 0°C $\quad R_{01} \quad R_{02} \quad R_0 = R_{01} + R_{02}$

At t°C $\quad R_{01}(1+\alpha_1 t) \quad R_{02}(1+\alpha_2 t) \quad R_0(1+\alpha t)$

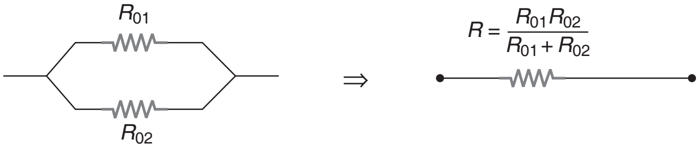
$$R_{01}(1+\alpha_1 t) + R_{02}(1+\alpha_2 t) = R_0(1+\alpha t)$$

or $$R_{01}(1+\alpha_1 t) + R_{02}(1+\alpha_2 t) = (R_{01}+R_{02})(1+\alpha t)$$

∴ $$R_{01} + R_{01}\alpha_1 t + R_{02} + R_{02}\alpha_2 t = R_{01} + R_{02} + (R_{01}+R_{02})\alpha t$$

or $$\alpha = \frac{R_{01}\alpha_1 + R_{02}\alpha_2}{R_{01}+R_{02}}$$ **Ans.**

(b) **In Parallel**



At t°C, $$\frac{1}{R_0(1+\alpha t)} = \frac{1}{R_{01}(1+\alpha_1 t)} + \frac{1}{R_{02}(1+\alpha_2 t)}$$

or $$\frac{R_{01}+R_{02}}{R_{01}R_{02}(1+\alpha t)} = \frac{1}{R_{01}(1+\alpha_1 t)} + \frac{1}{R_{02}(1+\alpha_2 t)}$$

Using the Binomial expansion, we have

$$\frac{1}{R_{02}}(1-\alpha t) + \frac{1}{R_{01}}(1-\alpha t) = \frac{1}{R_{01}}(1-\alpha_1 t) + \frac{1}{R_{02}}(1-\alpha_2 t)$$

i.e. $$\alpha t\left(\frac{1}{R_{01}} + \frac{1}{R_{02}}\right) = \frac{\alpha_1}{R_{01}}t + \frac{\alpha_2}{R_{02}}t$$

or $$\alpha = \frac{\alpha_1 R_{02} + \alpha_2 R_{01}}{R_{01}+R_{02}}$$ **Ans.**

Type 4. *Based on the verification of Ohm's law*

Concept

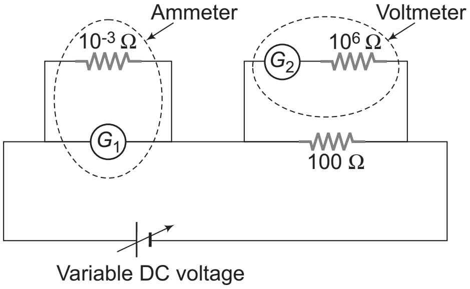
For verification of Ohm's law $\left(\frac{V}{i} = \text{constant} = R\right)$, we need an ohmic resistance, which follows this law. A voltmeter which will measure potential difference across this resistance, an ammeter which will measure current through this resistance and a variable battery which can provide a variable current in the circuit. Now, for different values of i, we have to measure different values of V and then prove that,

$$V \propto i \quad \text{or} \quad \frac{V}{i} = \text{constant}$$

and this constant is called resistance of that.

Example 5 *Draw the circuit for experimental verification of Ohm's law using a source of variable DC voltage, a main resistance of* $100\ \Omega$, *two galvanometers and two resistances of values* $10^6\ \Omega$ *and* $10^{-3}\ \Omega$ *respectively. Clearly show the positions of the voltmeter and the ammeter.* **(JEE 2004)**

Solution



Type 5. *Theory of bulbs or heater etc.*

Concept

(i) From the rated (written) values of power (P) and potential difference (V) we can determine resistance of filament of bulb.

$$P = \frac{V^2}{R}$$

$$\Rightarrow \qquad \boxed{R = \frac{V^2}{P}} \qquad \ldots\text{(i)}$$

For example, if rated values on a bulb are 220 V and 60 W, it means this bulb will consume 60 W of power (or 60 J in 1 s) if a potential difference of 220 V is applied across it. Resistance of this bulb will be

$$R = \frac{(220)^2}{60} = 806.67\ \Omega$$

(ii) Normally, rated value of V remains same in different bulbs. In India, it is 220 volt. Therefore, from Eq. (i)

$$R \propto \frac{1}{P} \quad \text{or} \quad R_{60\,\text{watt}} > R_{100\,\text{watt}}$$

(iii) Actual value of potential difference may be different from the rated value. Therefore, actual power consumption may also be different.

(iv) After finding resistance of the bulb using Eq. (i), we can apply normal Kirchhoff's laws for finding current passing through the bulb or actual power consumed by the bulb.

Example 6 *Prove that 60 W bulb glows more brightly than 100 W bulb if by mistake they are connected in series.*

Solution In series, we can use the formula

$$P = i^2R \quad \text{for the power consumption}$$

$$\Rightarrow \qquad P \propto R \qquad \text{(as } i \text{ is same in series)}$$

we have seen above that,

$$R_{60\,\text{W}} > R_{100\,\text{W}}$$

$$\therefore \qquad P_{60\,\text{W}} > P_{100\,\text{W}}$$

Hence Proved.

Note *In parallel 100 W bulb glows more brightly than 60 W bulb. Think why?*

Example 7 *The rated values of two bulbs are (P_1, V) and (P_2, V). Find actual power consumed by both of them if they are connected in*

(a) series

(b) parallel

and V potential difference is applied across both of them.

Solution (a) $\because \; R_1 = \dfrac{V^2}{P_1}$ and $R_2 = \dfrac{V^2}{P_2}$

In series,

$$P = \frac{V^2}{R_{\text{net}}} = \frac{V^2}{R_1 + R_2} = \frac{V^2}{\dfrac{V^2}{P_1} + \dfrac{V^2}{P_2}} = \frac{P_1P_2}{P_1 + P_2}$$

or

$$\frac{1}{P} = \frac{1}{P_1} + \frac{1}{P_2}$$

Ans.

(b) In parallel,

$$P = \frac{V^2}{R_{\text{net}}} = V^2\left(\frac{1}{R_{\text{net}}}\right) = V^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

or

$$P = V^2\left(\frac{P_1}{V^2} + \frac{P_2}{V^2}\right) \quad \text{or} \quad P = P_1 + P_2$$

Ans.

Example 8 *Heater-1, takes 3 minutes to boil a given amount of water. Heater-2 takes 6-minutes. Find the time taken if,*

(a) they are connected in series

(b) they are connected in parallel.

Potential difference V in all cases is same.

Solution Here, the heat required (say H) to boil water is same. Let P_1 and P_2 are the powers of the heaters. Then,

$$H = P_1 t_1 = P_2 t_2$$

$$\therefore \quad P_1 = \frac{H}{t_1} = \frac{H}{3}$$

$$P_2 = \frac{H}{t_2} = \frac{H}{6}$$

(a) **In series,**

$$P = \frac{P_1 P_2}{P_1 + P_2} = \frac{(H/3)(H/6)}{(H/3) + (H/6)} \qquad \text{(Refer Example 7)}$$

$$= \frac{H}{9}$$

Now,

$$t = \frac{H}{P} = \frac{H}{(H/9)} = 9 \text{ min}$$

Ans.

(b) **In parallel,**

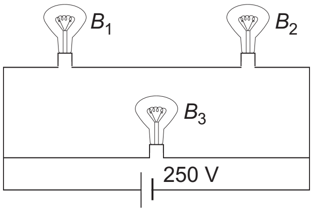
$$P = P_1 + P_2$$

$$= \frac{H}{3} + \frac{H}{6} = \frac{H}{2}$$

$$\therefore \quad t = \frac{H}{P} = \frac{H}{H/2} = 2 \text{ min}$$

Ans.

Example 9 *A 100 W bulb B_1, and two 60 W bulbs B_2 and B_3, are connected to a 250 V source as shown in the figure. Now W_1, W_2 and W_3 are the output powers of the bulbs B_1, B_2 and B_3 respectively. Then,* **(JEE 2002)**



(a) $W_1 > W_2 = W_3$ *(b) $W_1 > W_2 > W_3$* *(c) $W_1 < W_2 = W_3$* *(d) $W_1 < W_2 < W_3$*

Solution $P = \frac{V^2}{R}$ so, $R = \frac{V^2}{P}$

$$\therefore \quad R_1 = \frac{V^2}{100} \quad \text{and} \quad R_2 = R_3 = \frac{V^2}{60}$$

Now,

$$W_1 = \frac{(250)^2}{(R_1 + R_2)^2} \cdot R_1$$

$$W_2 = \frac{(250)^2}{(R_1 + R_2)^2} \cdot R_2 \quad \text{and} \quad W_3 = \frac{(250)^2}{R_3}$$

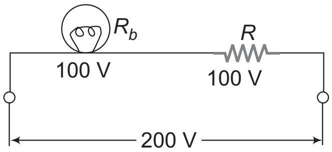
$$W_1 : W_2 : W_3 = 15 : 25 : 64 \quad \text{or} \quad W_1 < W_2 < W_3$$

$\therefore$ The correct option is (d).

Note *We have used $W = i^2 R$ for W_1 and W_2 and $W = \frac{V^2}{R}$ for W_3.*

Example 10 *An electric bulb rated for* 500 *W at* 100 *V is used in a circuit having a* 200 *V supply. The resistance R that must be put in series with the bulb, so that the bulb delivers* 500 *W is* ......Ω. **(JEE 1987)**

Solution Resistance of the given bulb

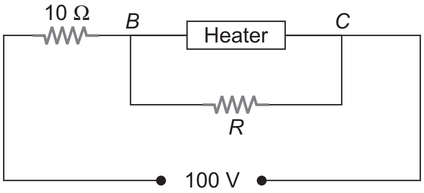


$$R_b = \frac{V^2}{P} = \frac{(100)^2}{500} = 20\,\Omega$$

To get 100 V out of 200 V across the bulb,

$$R = R_b = 20\,\Omega$$ **Ans.**

Example 11 *A heater is designed to operate with a power of* 1000 *W in a* 100 *V line. It is connected in combination with a resistance of* 10 Ω *and a resistance R, to a* 100 *V mains as shown in the figure. What will be the value of R so that the heater operates with a power of* 62.5 *W?* **(JEE 1978)**



Solution From $P = \frac{V^2}{R}$,

Resistance of heater, $$R = \frac{V^2}{P} = \frac{(100)^2}{1000} = 10\,\Omega$$

From $$P = i^2R$$

Current required across heater for power of 62.5 W,

$$i = \sqrt{\frac{P}{R}} = \sqrt{\frac{62.5}{10}} = 2.5\text{ A}$$

Main current in the circuit, $$I = \frac{100}{10 + \frac{10R}{10 + R}} = \frac{100\,(10 + R)}{100 + 20R} = \frac{10\,(10 + R)}{10 + 2R}$$

This current will distribute in inverse ratio of resistance between heater and R.

∴ $$i = \left(\frac{R}{10 + R}\right) I$$

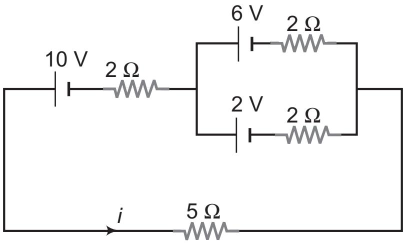
or $$2.5 = \left(\frac{R}{10 + R}\right)\left[\frac{10\,(10 + R)}{10 + 2R}\right] = \frac{10R}{10 + 2R}$$

Solving this equation, we get

$$R = 5\,\Omega$$ **Ans.**

Type 6. *To find current through a single external resistance in a complex circuit of batteries using the concepts of equivalent value of emf of battery*

Concept



In the above circuit, if we have to find only i then two parallel batteries may be converted into a single battery. Then, this battery is in series with the third battery of emf 10 volt. Now, we can find current i from the equation,

$$i = \frac{\text{Net emf}}{\text{Total resistance}}$$

Example 12 *Find the value of i in the circuit shown above.*

Solution Equivalent emf of the parallel combination is

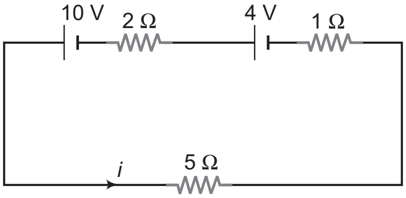
$$E = \frac{E_1/r_1 + E_2/r_2}{1/r_1 + 1/r_2}$$

$$= \frac{6/2 + 2/2}{1/2 + 1/2} = 4 \text{ volt}$$

Equivalent internal resistance of the parallel combination is

$$r = \frac{r_1 r_2}{r_1 + r_2} = \frac{(2)\,(2)}{2+2} = 1\ \Omega$$

Now, the equivalent simple circuit is as shown below



$$i = \frac{\text{Net emf}}{\text{Total resistance}}$$

$$= \frac{10+4}{2+1+5}$$

$$= 1.75 \text{ A}$$ **Ans.**

Note *By this concept, we can find only i. To find other currents (across 6V battery or 2V battery) we will have to apply Kirchhoff's laws.*

Miscellaneous Examples

Example 13 *Two sources of current of equal emf are connected in series and have different internal resistances* r_1 *and* r_2 $(r_2 > r_1)$*. Find the external resistance* R *at which the potential difference across the terminals of one of the sources becomes equal to zero.*

Solution $V = E - ir$

E and i for both the sources are equal. Therefore, potential difference (V) will be zero for a source having greater internal resistance, i.e. r_2.

$\therefore$ $$0 = E - ir_2$$

or $$E = ir_2 = \left(\frac{2E}{R + r_1 + r_2}\right) \cdot r_2$$

$\therefore$ $$2r_2 = R + r_1 + r_2$$

or $$R = r_2 - r_1$$ **Ans.**

Example 14

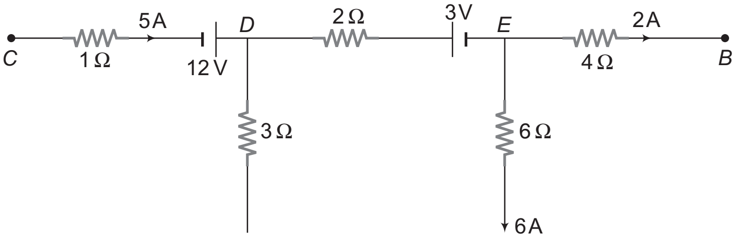
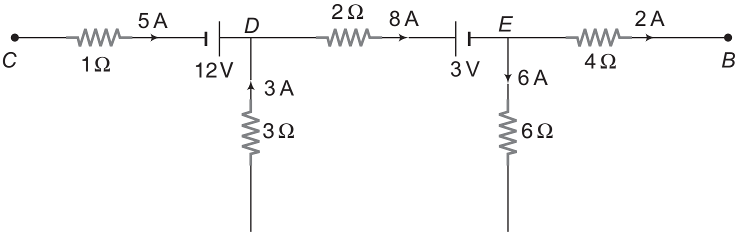


Figure shows the part of a circuit. Calculate the power dissipated in $3\ \Omega$ *resistance. What is the potential difference* $V_C - V_B$?

Solution Applying Kirchhoff's junction law at E current in wire DE is 8 A from D to E. Now further applying junction law at D, the current in $3\ \Omega$ resistance will be 3 A towards D.

Power dissipated in $3\ \Omega$ resistance $= i^2R = (3)^2\,(3) = 27$ W **Ans.**



$\boldsymbol{V_C - V_B}$ $$V_C - 5 \times 1 + 12 - 8 \times 2 - 3 - 4 \times 2 = V_B$$

$\therefore$ $$V_C - V_B = 5 - 12 + 16 + 3 + 8$$

or $$V_C - V_B = 20 \text{ V}$$ **Ans.**

Example 15 *The emf of a storage battery is* 90 *V before charging and* 100 *V after charging. When charging began the current was* 10 *A. What is the current at the end of charging if the internal resistance of the storage battery during the whole process of charging may be taken as constant and equal to* 2 Ω*?*

Solution The voltage supplied by the charging plant is here constant which is equal to,

$$V = E_i + i_i \cdot r = (90) + (10)(2)$$

$$= 110 \text{ V}$$

Let i_f be the current at the end of charging.

Then, $$V = E_f + i_f r$$

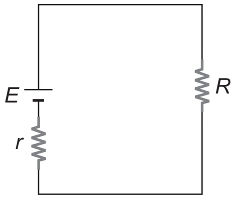
or $$i_f = \frac{V - E_f}{r}$$

$$= \frac{110 - 100}{2}$$

$$= 5 \text{ A}$$ **Ans.**

Example 16 *A battery has an open circuit potential difference of* 6 *V between its terminals. When a load resistance of* 60 Ω *is connected across the battery, the total power supplied by the battery is* 0.4 *W. What should be the load resistance R, so that maximum power will be dissipated in R. Calculate this power. What is the total power supplied by the battery when such a load is connected?*

Solution When the circuit is open, $V = E$



∴ $$E = 6 \text{ V}$$

Let r be the internal resistance of the battery.

Power supplied by the battery in this case is

$$P = \frac{E^2}{R + r}$$

Substituting the values, we have $$0.4 = \frac{(6)^2}{60 + r}$$

Solving this, we get $$r = 30 \ \Omega$$

Maximum power is dissipated in the circuit when net external resistance is equal to net internal resistance or

$$R = r$$

∴ $$R = 30 \ \Omega$$ **Ans.**

Further, total power supplied by the battery under this condition is

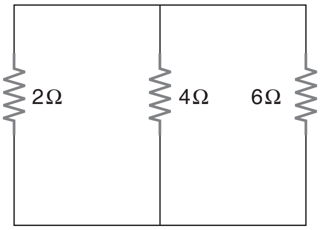
$$P_{\text{Total}} = \frac{E^2}{R + r} = \frac{(6)^2}{30 + 30}$$

$$= 0.6 \text{ W}$$ **Ans.**

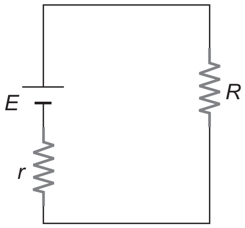
Of this 0.6 W half of the power is dissipated in R and half in r. Therefore, maximum power dissipated in R would be

$$\frac{0.6}{2} = 0.3 \text{ W}$$ **Ans.**

Example 17 *In which branch of the circuit shown in figure a* 11 *V battery be inserted so that it dissipates minimum power. What will be the current through the* 2 Ω *resistance for this position of the battery?*

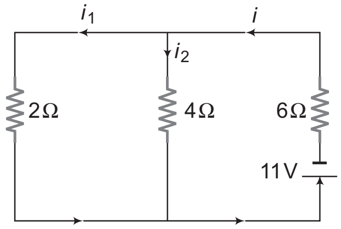


Solution Suppose, we insert the battery with 2 Ω resistance. Then, we can take 2 Ω as the internal resistance (r) of the battery and combined resistance of the other two as the external resistance (R). The circuit in that case is shown in figure,



Now power, $P = \dfrac{E^2}{R + r}$

This power will be minimum where $R + r$ is maximum and we can see that $(R + r)$ will be maximum when the battery is inserted with 6 Ω resistance as shown in figure.



Net resistance in this case is

$$6 + \frac{2 \times 4}{2 + 4} = \frac{22}{3}\ \Omega$$

$$\therefore \qquad i = \frac{11}{22/3} = 1.5\ \text{A}$$

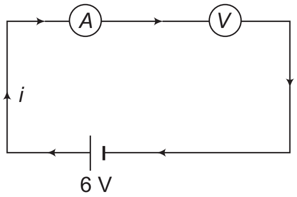
This current will be distributed in 2 Ω and 4 Ω in the inverse ratio of their resistances.

$$\therefore \qquad \frac{i_1}{i_2} = \frac{4}{2} = 2$$

$$\therefore \qquad i_1 = \left(\frac{2}{2+1}\right)(1.5) = 1.0\ \text{A}$$ **Ans.**

Example 18 *An ammeter and a voltmeter are connected in series to a battery of emf $E = 6.0\,V$. When a certain resistance is connected in parallel with the voltmeter, the reading of the voltmeter decreases two times, whereas the reading of the ammeter increases the same number of times. Find the voltmeter reading after the connection of the resistance.*

Solution Let $R =$ resistance of ammeter



Potential difference across voltmeter = 6 – potential difference across ammeter

In first case, $V = 6 - iR$ ...(i)

In second case, $\frac{V}{2} = 6 - (2i)\,R$ ...(ii)

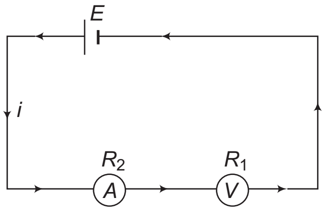
Solving these two equations, we get

$$V = 4 \text{ volt}$$

$\therefore$ $V/2 = 2$ volt **Ans.**

Example 19 *A voltmeter of resistance R_1 and an ammeter of resistance R_2 are connected in series across a battery of negligible internal resistance. When a resistance R is connected in parallel to voltmeter, reading of ammeter increases three times while that of voltmeter reduces to one third. Find R_1 and R_2 in terms of R.*

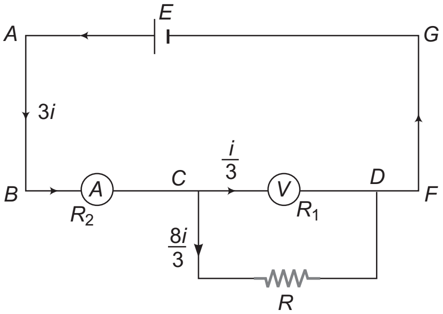
Solution Let E be the emf of the battery.



In the second case main current increases three times while current through voltmeter will reduce to $i/3$. Hence, the remaining $3i - i/3 = 8i/3$ passes through R as shown in figure.

$$V_C - V_D = \left(\frac{i}{3}\right) R_1 = \left(\frac{8i}{3}\right) R$$

or $R_1 = 8R$ **Ans.**



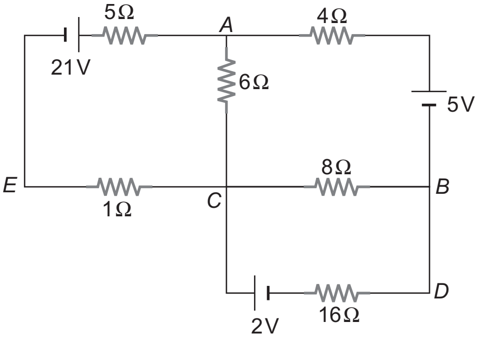
In the second case, main current becomes three times. Therefore, total resistance becomes $\frac{1}{3}$ times or

$$R_2 + \frac{RR_1}{R + R_1} = \frac{1}{3}(R_1 + R_2)$$

Substituting $R_1 = 8R$, we get

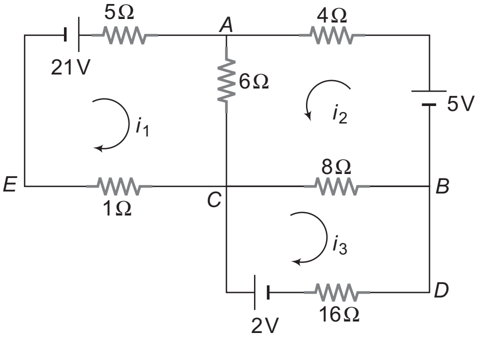
$$R_2 = \frac{8R}{3}$$ **Ans.**

Example 20 *Find the current in each branches of the circuit.*



Solution It is possible to use Kirchhoff's laws in a slightly different form, which may simplify the solution of certain problems. This method of applying Kirchhoff's laws is called the **loop current method.**

In this method, we assign a current to every closed loop in a network.



Suppose currents i_1, i_2 and i_3 are flowing in the three loops. The clockwise or anti-clockwise sense given to these currents is arbitrary. Applying Kirchhoff's second law to the three loops, we get

$$21 - 5i_1 - 6(i_1 + i_2) - i_1 = 0 \quad \text{...(i)}$$

$$5 - 4i_2 - 6(i_1 + i_2) - 8(i_2 + i_3) = 0 \quad \text{...(ii)}$$

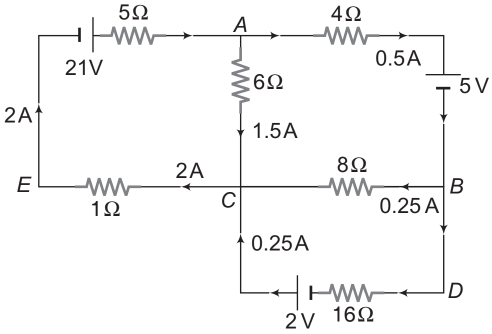
and

$$2 - 8(i_2 + i_3) - 16\,i_3 = 0 \quad \text{...(iii)}$$

Solving these three equations, we get

$$i_1 = 2\text{ A}, \qquad i_2 = -\frac{1}{2}\text{ A} \qquad \text{and} \qquad i_3 = \frac{1}{4}\text{ A}$$

Therefore, current in different branches are as shown in figure given below.



Note *In wire AC, current is $i_1 + i_2$ and in CB it is $i_2 + i_3$.*

Example 21 *What amount of heat will be generated in a coil of resistance R due to a charge q passing through it if the current in the coil*

(a) decreases down to zero uniformly during a time interval t_0?

(b) decreases down to zero halving its value every t_0 seconds?

HOW TO PROCEED *Heat generated in a resistance is given by*

$$H = i^2 Rt$$

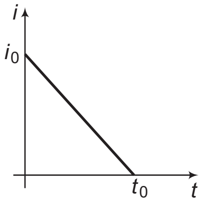
We can directly use this formula provided i is constant. Here, i is varying. So, first we will calculate i at any time t, then find a small heat dH in a short interval of time dt. Then by integrating it with proper limits we can obtain the total heat produced.

Solution (a) The corresponding *i-t* graph will be a straight line with *i* decreasing from a peak value (say i_0) to zero in time t_0.

i-t equation will be as

$$i = i_0 - \left(\frac{i_0}{t_0}\right) t \qquad (y = - mx + c) \qquad \text{...(i)}$$

Here, i_0 is unknown, which can be obtained by using the fact that area under *i-t* graph gives the flow of charge. Hence,



$$q = \frac{1}{2}(t_0)(i_0)$$

$$\therefore \qquad i_0 = \frac{2q}{t_0}$$

Substituting in Eq. (i), we get

$$i = \frac{2q}{t_0}\left(1 - \frac{t}{t_0}\right)$$

or
$$i = \left(\frac{2q}{t_0} - \frac{2qt}{t_0^2}\right)$$

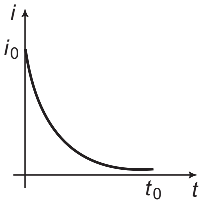
Now, at time t, heat produced in a short interval dt is

$$dH = i^2 R\,dt$$
$$= \left(\frac{2q}{t_0} - \frac{2qt}{t_0^2}\right)^2 R dt$$

$\therefore$ Total heat produced $= \int_0^{t_0} dH$

or
$$H = \int_0^{t_0}\left(\frac{2q}{t_0} - \frac{2qt}{t_0^2}\right)^2 R\,dt$$
$$= \frac{4}{3}\frac{q^2R}{t_0}$$ **Ans.**

(b) Here, current decreases from some peak value (say i_0) to zero exponentially with half life t_0.



i-t equation in this case will be

$$i = i_0\,e^{-\lambda t}$$

Here,
$$\lambda = \frac{\ln(2)}{t_0}$$

Now,
$$q = \int_0^{\infty} i\,dt = \int_0^{\infty} i_0\,e^{-\lambda t}\,dt = \left(\frac{i_0}{\lambda}\right)$$

$\therefore$ $i_0 = \lambda\,q$

$\therefore$ $i = (\lambda q)\,e^{-\lambda t}$

$\therefore$ $dH = i^2R\,dt = \lambda^2q^2e^{-2\lambda t}R\,dt$

or
$$H = \int_0^{\infty} dH = \lambda^2q^2R\int_0^{\infty} e^{-2\lambda t}\,dt = \frac{q^2\lambda R}{2}$$

Substituting $\lambda = \dfrac{\ln(2)}{t_0}$, we have $H = \dfrac{q^2R\ln(2)}{2t_0}$. **Ans.**

Note *In radioactivity, half-life is given by*

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

$\therefore$
$$\lambda = \frac{\ln 2}{t_{\frac{1}{2}}}$$

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

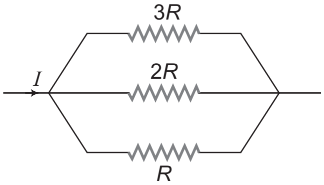
(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : If potential difference across two points is zero, current between these two points should be zero.

Reason : Current passing from a resistor

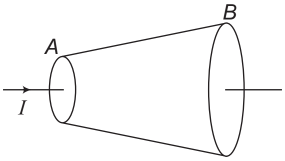
$$I = \frac{V}{R}$$

2. Assertion : In the part of the circuit shown in figure, maximum power is produced across R.



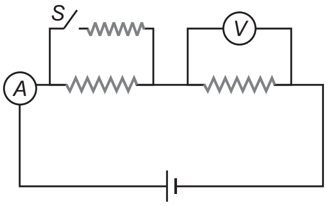
Reason : Power $P = \frac{V^2}{R}$

3. Assertion : Current I is flowing through a cylindrical wire of non-uniform cross-section as shown. Section of wire near A will be more heated compared to the section near B.



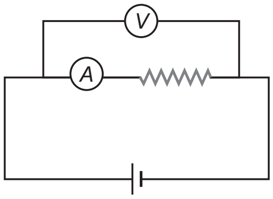
Reason : Current density near A is more.

4. Assertion : In the circuit shown in figure after closing the switch S reading of ammeter will increase while that of voltmeter will decrease.



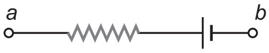
Reason : Net resistance decreases as parallel combination of resistors is increase`d.

5. **Assertion :** In the circuit shown in figure ammeter and voltmeter are non-ideal. When positions of ammeter and voltmeter are changed, reading of ammeter will increase while that of voltmeter will decrease.



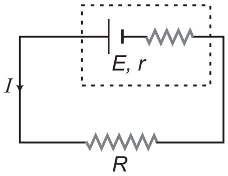
Reason : Resistance of an ideal ammeter is zero while that of an ideal voltmeter is infinite.

6. **Assertion :** In the part of a circuit shown in figure, given that $V_b > V_a$. The current should flow from b to a.



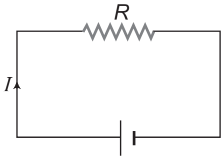
Reason : Direction of current inside a battery is always from negative terminal to positive terminal.

7. **Assertion :** In the circuit shown in figure R is variable. Value of current I is maximum when $R = r$.



Reason : At $R = r$, maximum power is produced across R.

8. **Assertion :** If variation in resistance due to temperature is taken into consideration, then current in the circuit I and power produced across the resistance P both will decrease with time.



Reason : $V = IR$ is Ohm's law.

9. **Assertion :** When a potential difference is applied across a conductor, free electrons start travelling with a constant speed called drift speed.

Reason : Due to potential difference an electric field is produced inside the conductor, in which electrons experience a force.

10. **Assertion :** When temperature of a conductor is increased, its resistance increases.

Reason : Free electrons collide more frequently.

11. **Assertion :** Two non-ideal batteries are connected in parallel with same polarities on same side. The equivalent emf is smaller than either of the two emfs.

Reason : Two non-ideal batteries are connected in parallel, the equivalent internal resistance is smaller than either of the two internal resistances.

Objective Questions

1. An ammeter should have very low resistance
(a) to show large deflection
(b) to generate less heat
(c) to prevent the galvanometer
(d) so that it may not change the value of the actual current in the circuit

2. A steady current flows in a metallic conductor of non-uniform cross-section. The quantity/quantities which remain constant along the length of the conductor is/are
(a) current, electric field and drift speed
(b) drift speed only
(c) current and drift speed
(d) current only

3. If M = mass, L = length, T = time and I = electric current, then the dimensional formula of resistance R will be given by
(a) $[R] = [ML^2T^{-3} I^{-2}]$
(b) $[R] = [ML^2T^{-3} I^2]$
(c) $[R] = [ML^2T^3 I^{-2}]$
(d) $[R] = [ML^2T^3 I^2]$

4. The unit of electrical conductivity is
(a) ohm-m^{-2}
(b) ohm × m
(c) ohm^{-1}-m^{-1}
(d) None of these

5. Through an electrolyte an electrical current is due to drift of
(a) free electrons
(b) positive and negative ions
(c) free electrons and holes
(d) protons

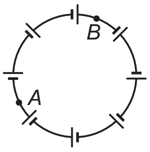
6. The current in a circuit with an external resistance of 3.75 Ω is 0.5 A. When a resistance of 1 Ω is introduced into the circuit, the current becomes 0.4 A. The emf of the power source is
(a) 1 V
(b) 2 V
(c) 3 V
(d) 4 V

7. The deflection in a galvanometer falls from 50 divisions to 20 divisions, when a 12 Ω shunt is applied. The galvanometer resistance is
(a) 18 Ω
(b) 24 Ω
(c) 30 Ω
(d) 36 Ω

8. If 2% of the main current is to be passed through the galvanometer of resistance G, the resistance of shunt required is
(a) $\frac{G}{49}$
(b) $\frac{G}{50}$
(c) $49\,G$
(d) $50\,G$

9. If the length of the filament of a heater is reduced by 10%, the power of the heater will
(a) increase by about 9%
(b) increase by about 11%
(c) increase by about 19%
(d) decrease by about 10%

10. N identical current sources each of emf E and internal resistance r are connected to form a closed loop as shown in figure. The potential difference between points A and B which divides the circuit into n and $(N - n)$ units is

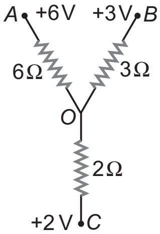


(a) NE
(b) $(N - n)E$
(c) nE
(d) zero

11. A 2.0 V potentiometer is used to determine the internal resistance of a 1.5 V cell. The balance point of the cell in the open circuit is 75 cm. When a resistor of 10 Ω is connected across the cell, the balance point shifts to 60 cm. The internal resistance of the cell is

(a) 1.5 Ω
(b) 2.5 Ω
(c) 3.5 Ω
(d) 4.5 Ω

12. Three resistances are joined together to form a letter *Y*, as shown in figure. If the potentials of the terminals *A*, *B* and *C* are 6 V, 3 V and 2 V respectively, then the potential of the point *O* will be



(a) 4 V
(b) 3 V
(c) 2.5 V
(d) 0 V

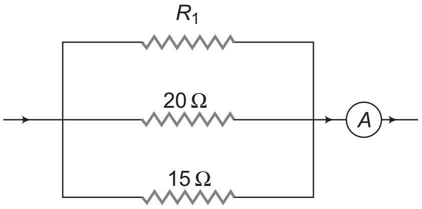
13. The drift velocity of free electrons in a conductor is v, when a current i is flowing in it. If both the radius and current are doubled, then the drift velocity will be

(a) v
(b) $v/2$
(c) $v/4$
(d) $v/8$

14. A galvanometer is to be converted into an ammeter or voltmeter. In which of the following cases the resistance of the device is largest?

(a) an ammeter of range 10 A
(b) a voltmeter of range 5 V
(c) an ammeter of range 5 A
(d) a voltmeter of range 10 V

15. In the given circuit the current flowing through the resistance 20 Ω is 0.3 A, while the ammeter reads 0.8 A. What is the value of R_1?



(a) 30 Ω
(b) 40 Ω
(c) 50 Ω
(d) 60 Ω

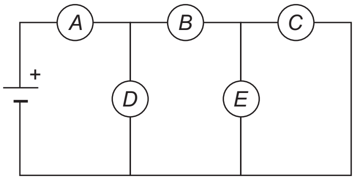
16. An ammeter and a voltmeter are joined in series to a cell. Their readings are *A* and *V* respectively. If a resistance is now joined in parallel with the voltmeter, then

(a) both *A* and *V* will increase
(b) both *A* and *V* will decrease
(c) *A* will decrease, *V* will increase
(d) *A* will increase, *V* will decrease

17. A resistor R has power of dissipation P with cell voltage E. The resistor is cut in n equal parts and all parts are connected in parallel with same cell. The new power dissipation is

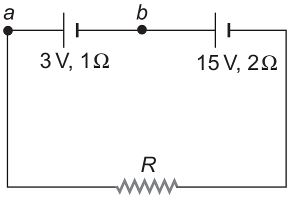
(a) nP	(b) nP^2
(c) n^2P	(d) n/P

18. In the circuit diagram shown in figure, a fuse bulb can cause all other bulbs to go out. Identify the bulb



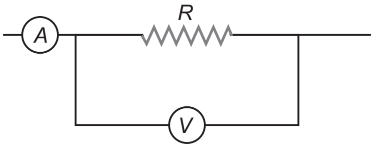
(a) B	(b) C
(c) A	(d) D or E

19. Two batteries one of the emf 3 V, internal resistance 1 Ω and the other of emf 15V, internal resistance 2 Ω are connected in series with a resistance R as shown. If the potential difference between points a and b is zero, the resistance R in Ω is



(a) 5	(b) 7
(c) 3	(d) 1

20. A part of a circuit is shown in figure. Here reading of ammeter is 5 A and voltmeter is 100 V. If voltmeter resistance is 2500 ohm, then the resistance R is approximately



(a) 20 Ω	(b) 10 Ω
(c) 100 Ω	(d) 200 Ω

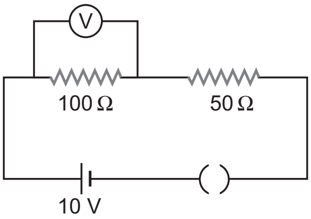
21. A copper wire of resistance R is cut into ten parts of equal length. Two pieces each are joined in series and then five such combinations are joined in parallel. The new combination will have a resistance

(a) R (b) $\frac{R}{4}$ (c) $\frac{R}{5}$ (d) $\frac{R}{25}$

22. Two resistances are connected in two gaps of a meter bridge. The balance point is 20 cm from the zero end. A resistance of 15 Ω is connected in series with the smaller of the two. The null point shifts to 40 cm. The value of the smaller resistance in Ω is

(a) 3	(b) 6
(c) 9	(d) 12

23. In the given circuit, the voltmeter records 5 volt. The resistance of the voltmeter in Ω is

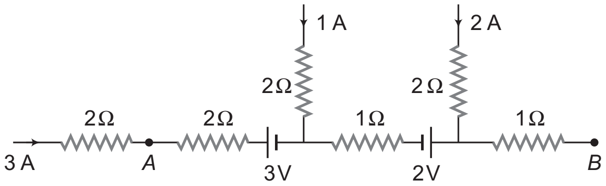


(a) 200
(b) 100
(c) 10
(d) 50

24. The wire of potentiometer has resistance 4 Ω and length 1 m. It is connected to a cell of emf 2 volt and internal resistance 1 Ω. If a cell of emf 1.2 volt is balanced by it, the balancing length will be

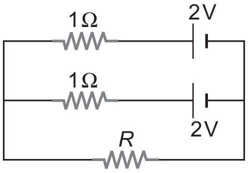
(a) 90 cm
(b) 60 cm
(c) 50 cm
(d) 75 cm

25. The potential difference between points *A* and *B*, in a section of a circuit shown, is



(a) 5 volt
(b) 1 volt
(c) 10 volt
(d) 17 volt

26. Two identical batteries, each of emf 2 V and internal resistance $r = 1\,\Omega$ are connected as shown. The maximum power that can be developed across *R* using these batteries is

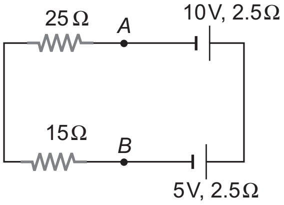


(a) 3.2 W
(b) 8.2 W
(c) 2 W
(d) 4 W

27. For a cell, the terminal potential difference is 2.2 V, when circuit is open and reduces to 1.8 V. When cell is connected to a resistance $R = 5\,\Omega$, the internal resistance of cell (r) is

(a) $\frac{10}{9}\,\Omega$
(b) $\frac{9}{10}\,\Omega$
(c) $\frac{11}{9}\,\Omega$
(d) $\frac{5}{9}\,\Omega$

28. The potential difference between points *A* and *B* in the circuit shown in figure, will be

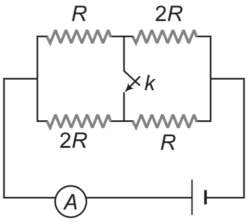


(a) 1 V
(b) 2 V
(c) – 3 V
(d) None of the above

29. Potentiometer wire of length 1 m is connected in series with 490 Ω resistance and 2 Vbattery. If 0.2 mV/cm is the potential gradient, then resistance of the potentiometer wire is approximately

(a) 4.9 Ω (b) 7.9 Ω (c) 5.9 Ω (d) 6.9 Ω

30. Find the ratio of currents as measured by ammeter in two cases when the key is open and when the key is closed



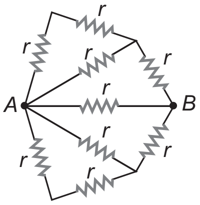
(a) 9/8 (b) 10/11 (c) 8/9 (d) None of these

31. A galvanometer has a resistance of 3663 Ω. A shunt S is connected across it such that (1/34) of the total current passes through the galvanometer. Then, the value of the shunt is

(a) 222 Ω (b) 111 Ω (c) 11 Ω (d) 22 Ω

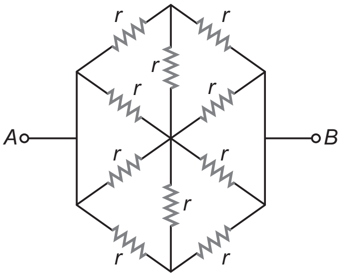
Note *Attempt the following questions after reading the chapter of capacitors.*

32. The network shown in figure is an arrangement of nine identical resistors. The resistance of the network between points A and B is 1.5 Ω. The resistance r is



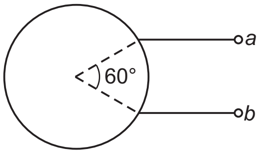
(a) 1.1 Ω
(b) 3.3 Ω
(c) 1.8 Ω
(d) 1.6 Ω

33. The equivalent resistance of the hexagonal network as shown in figure between points A and B is



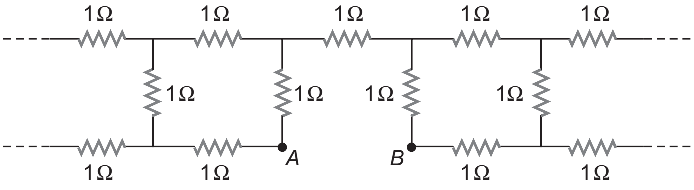
(a) r (b) 0.5 r (c) 2 r (d) 3 r

34. A uniform wire of resistance 18 Ω is bent in the form of a circle. The effective resistance across the points a and b is



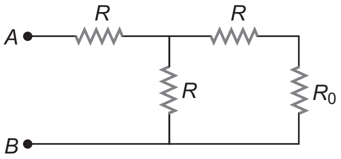
(a) 3 Ω
(b) 2 Ω
(c) 2.5 Ω
(d) 6 Ω

35. Each resistor shown in figure is an infinite network of resistance 1 Ω. The effective resistance between points *A* and *B* is



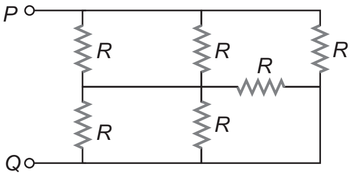
(a) less than 1 Ω
(b) 1 Ω
(c) more than 1 Ω but less than 3 Ω
(d) 3 Ω

36. In the circuit shown in figure, the total resistance between points *A* and *B* is R_0. The value of resistance *R* is



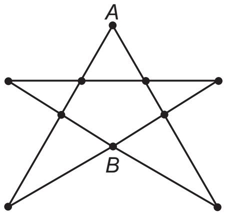
(a) R_0
(b) $\sqrt{3}\, R_0$
(c) $\frac{R_0}{2}$
(d) $\frac{R_0}{\sqrt{3}}$

37. In the circuit shown in the figure, $R = 55\ \Omega$, the equivalent resistance between the points *P* and *Q* is



(a) 30 Ω
(b) 35 Ω
(c) 55 Ω
(d) 25 Ω

38. The resistance of all the wires between any two adjacent dots is *R*. Then, equivalent resistance between *A* and *B* as shown in the figure is

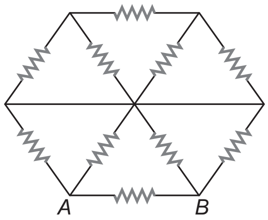


(a) (7/3) *R*
(b) (7/6) *R*
(c) (14/8) *R*
(d) None of the above

39. A uniform wire of resistance 4 Ω is bent into a circle of radius *r*. A specimen of the same wire is connected along the diameter of the circle. What is the equivalent resistance across the ends of this wire?

(a) $\frac{4}{(4+\pi)}\ \Omega$
(b) $\frac{3}{(3+\pi)}\ \Omega$
(c) $\frac{2}{(2+\pi)}\ \Omega$
(d) $\frac{1}{(1+\pi)}\ \Omega$

40. In the network shown in figure, each resistance is R. The equivalent resistance between points A and B is



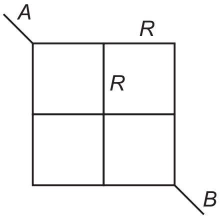
(a) $\frac{20}{11}R$

(b) $\frac{19}{20}R$

(c) $\frac{8}{15}R$

(d) $\frac{R}{2}$

41. The equivalent resistance between the points A and B is (R is the resistance of each side of smaller square)



(a) R (b) $\frac{3R}{2}$

(c) $2R$ (d) $\frac{R}{2}$

Subjective Questions

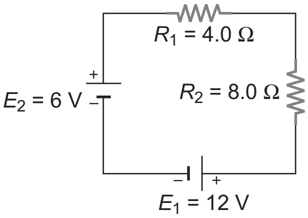
1. When a steady current passes through a cylindrical conductor, is there an electric field inside the conductor?

2. Electrons in a conductor have no motion in the absence of a potential difference across it. Is this statement true or false?

3. In the Bohr model of hydrogen atom, the electron is pictured to rotate in a circular orbit of radius 5×10^{-11} m, at a speed 2.2×10^{6} m/s. What is the current associated with electron motion?

4. A 120 V house circuit has the following light bulbs switched on : 40 W, 60 W and 75 W. Find the equivalent resistance of these bulbs.

5. Assume that the batteries in figure have negligible internal resistance. Find

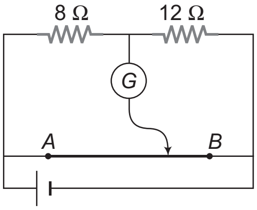


(a) the current in the circuit,

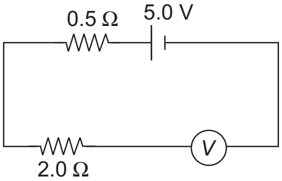
(b) the power dissipated in each resistor and

(c) the power of each battery, stating whether energy is supplied by or absorbed by it.

6. The potentiometer wire AB shown in figure is 40 cm long. Where the free end of the galvanometer should be connected on AB so that the galvanometer may show zero deflection?



7. An ideal voltmeter V is connected to a $2.0\,\Omega$ resistor and a battery with emf 5.0 V and internal resistance $0.5\,\Omega$ as shown in figure :

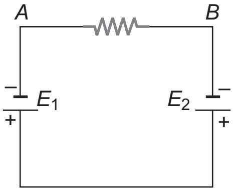


(a) What is the current in the $2.0\,\Omega$ resistor?

(b) What is the terminal voltage of the battery?

(c) What is the reading of the voltmeter?

8. In figure, $E_1 = 12$ V and $E_2 = 8$ V.

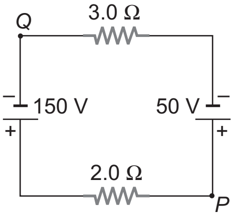


(a) What is the direction of the current in the resistor?

(b) Which battery is doing positive work?

(c) Which point, A or B, is at the higher potential?

9. In figure, if the potential at point P is 100 V, what is the potential at point Q?



10. Copper has one conduction electron per atom. Its density is 8.89 g/cm^3 and its atomic mass is 63.54 g/mol. If a copper wire of diameter 1.0 mm carries a current of 2.0 A, what is the drift speed of the electrons in the wire?

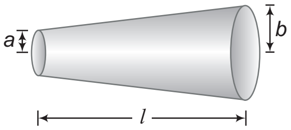
11. An aluminium wire carrying a current has diameter 0.84 mm. The electric field in the wire is 0.49 V/m. What is

(a) the current carried by the wire?

(b) the potential difference between two points in the wire 12.0 m apart?

(c) the resistance of a 12.0 m length of this wire?
Specific resistance of aluminium is $2.75 \times 10^{-8}\ \Omega$-m.

12. A conductor of length l has a non-uniform cross-section. The radius of cross-section varies linearly from a to b. The resistivity of the material is ρ. Find the resistance of the conductor across its ends.



13. If a battery of emf E and internal resistance r is connected across a load of resistance R. Show that the rate at which energy is dissipated in R is maximum when $R = r$ and this maximum power is $P = E^2/4r$.

14. Two identical batteries each of emf $E = 2$ volt and internal resistance $r = 1$ ohm are available to produce heat in an external resistance by passing a current through it. What is the maximum power that can be developed across an external resistance R using these batteries?

15. Two coils connected in series have resistance of $600\,\Omega$ and $300\,\Omega$ at $20°$ C and temperature coefficient of 0.001 and $0.004\,(°\text{C})^{-1}$ respectively. Find resistance of the combination at a temperature of 50° C. What is the effective temperature coefficient of combination?

16. An aluminium wire 7.5 m long is connected in parallel with a copper wire 6 m long. When a current of 5 A is passed through the combination, it is found that the current in the aluminium wire is 3 A. The diameter of the aluminium wire is 1 mm. Determine the diameter of the copper wire. Resistivity of copper is $0.017\,\mu\Omega\text{-m}$ and that of the aluminium is $0.028\,\mu\,\Omega\text{-m}$.

17. The potential difference between two points in a wire 75.0 cm apart is 0.938 V, when the current density is $4.40 \times 10^7\ \text{A/m}^2$. What is

(a) the magnitude of **E** in the wire?

(b) the resistivity of the material of which the wire is made?

18. A rectangular block of metal of resistivity ρ has dimensions $d \times 2d \times 3d$. A potential difference V is applied between two opposite faces of the block.

(a) To which two faces of the block should the potential difference V be applied to give the maximum current density? What is the maximum current density?

(b) To which two faces of the block should the potential difference V be applied to give the maximum current? What is this maximum current?

19. An electrical conductor designed to carry large currents has a circular cross-section 2.50 mm in diameter and is 14.0 m long. The resistance between its ends is $0.104\,\Omega$.

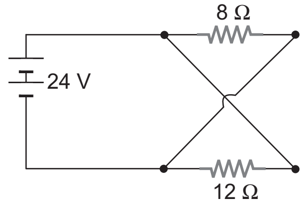
(a) What is the resistivity of the material?

(b) If the electric field magnitude in the conductor is 1.28 V/m, what is the total current?

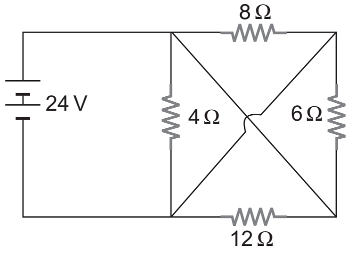
(c) If the material has 8.5×10^{28} free electrons per cubic metre, find the average drift speed under the conditions of part (b).

20. It is desired to make a $20.0\,\Omega$ coil of wire which has a zero thermal coefficient of resistance. To do this, a carbon resistor of resistance R_1 is placed in series with an iron resistor of resistance R_2. The proportions of iron and carbon are so chosen that $R_1 + R_2 = 20.00\,\Omega$ for all temperatures near 20° C. How large are R_1 and R_2? Given, $\alpha_C = -0.5 \times 10^{-3}\ \text{K}^{-1}$ and $\alpha_{Fe} = 5.0 \times 10^{-3}\ \text{K}^{-1}$.

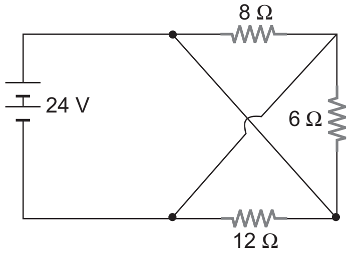
21. Find the current supplied by the battery in the circuit shown in figure.



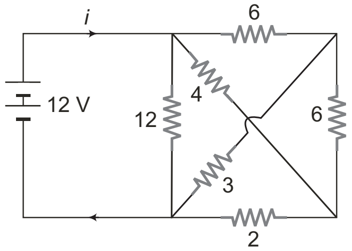
22. Calculate battery current and equivalent resistance of the network shown in figure.



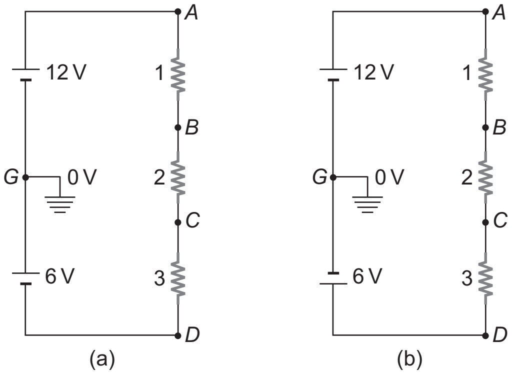
23. Compute total circuit resistance and battery current as shown in figure.



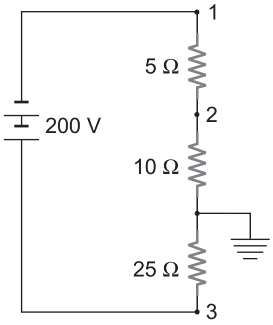
24. Compute the value of battery current i shown in figure. All resistances are in ohm.



25. Calculate the potentials of points A, B, C and D as shown in Fig. (a). What would be the new potential values if connections of 6 V battery are reversed as shown in Fig. (b)? All resistances are in ohm.

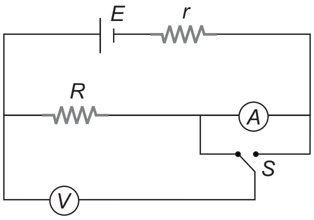


26. Give the magnitude and polarity of the following voltages in the circuit of figure :



(i) V_1	(ii) V_2	(iii) V_3	(iv) V_{3-2}
(v) V_{1-2}	(vi) V_{1-3}		

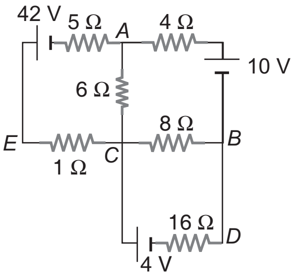
27. The emf E and the internal resistance r of the battery shown in figure are 4.3 V and 1.0 Ω respectively. The external resistance R is 50 Ω. The resistances of the ammeter and voltmeter are 2.0 Ω and 200 Ω, respectively.



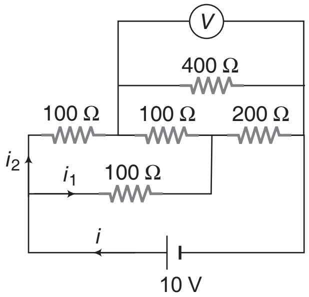
(a) Find the readings of the two meters.

(b) The switch is thrown to the other side. What will be the readings of the two meters now?

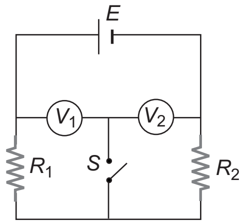
28. Find the current in each branch of the circuit shown in figure.



29. An electrical circuit is shown in figure. Calculate the potential difference across the resistor of 400 Ω as will be measured by the voltmeter V of resistance 400 Ω either by applying Kirchhoff's rules or otherwise. **(JEE 1996)**



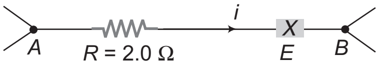
30. In the circuit shown in figure V_1 and V_2 are two voltmeters of resistances 3000 Ω and 2000 Ω, respectively. In addition $R_1 = 2000\ \Omega$, $R_2 = 3000\ \Omega$ and $E = 200$ V, then



(a) Find the reading of voltmeters V_1 and V_2 when
   (i) switch S is open
   (ii) switch S is closed

(b) Current through S, when it is closed
   (Disregard the resistance of battery)

31. In figure, circuit section AB absorbs energy at the rate of 5.0 W when a current $i = 1.0$ A passes through it in the indicated direction.



(a) What is the potential difference between points A and B?

(b) Emf device X does not have internal resistance. What is its emf?

(c) What is its polarity (the orientation of its positive and negative terminals)?

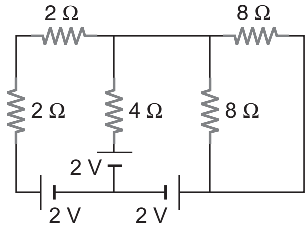
32. The potential difference across the terminals of a battery is 8.4 V when there is a current of 1.50 A in the battery from the negative to the positive terminal. When the current is 3.50 A in the reverse direction, the potential difference becomes 9.4 V.

(a) What is the internal resistance of the battery?

(b) What is the emf of the battery?

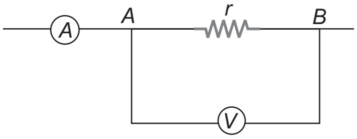
33. A battery of emf 2.0 V and internal resistance 0.10 Ω is being charged with a current of 5.0 A. Find the potential difference between the terminals of the battery?

34. Find the currents in different resistors shown in figure.

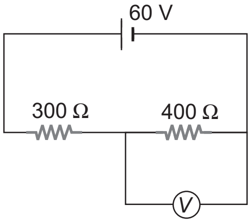


35. A resistance box, a battery and a galvanometer of resistance G ohm are connected in series. If the galvanometer is shunted by resistance of S ohm, find the change in resistance in the box required to maintain the current from the battery unchanged.

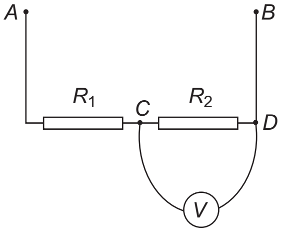
36. Determine the resistance r if an ammeter shows a current of $I = 5$ A and a voltmeter 100 V. The internal resistance of the voltmeter is $R = 2{,}500$ Ω.



37. In the circuit, a voltmeter reads 30 V when it is connected across 400 Ω resistance. Calculate what the same voltmeter will read when it is connected across the 300 Ω resistance?



38. Resistances R_1 and R_2, each 60 Ω, are connected in series. The potential difference between points A and B is 120 V. Find the reading of voltmeter connected between points C and D if its resistance $r = 120\ \Omega$.



39. A moving coil galvanometer of resistance 20 Ω gives a full scale deflection when a current of 1 mA is passed through it. It is to be converted into an ammeter reading 20 A on full scale. But the shunt of 0.005 Ω only is available. What resistance should be connected in series with the galvanometer coil?

40. A cell of emf 3.4 V and internal resistance 3 Ω is connected to an ammeter having resistance 2 Ω and to an external resistance of 100 Ω. When a voltmeter is connected across the 100 Ω resistance, the ammeter reading is 0.04 A. Find the voltage reading by the voltmeter and its resistance. Had the voltmeter been an ideal one what would have been its reading?

41. (a) A voltmeter with resistance R_V is connected across the terminals of a battery of emf E and internal resistance r. Find the potential difference measured by the voltmeter.

(b) If $E = 7.50$ V and $r = 0.45\ \Omega$, find the minimum value of the voltmeter resistance R_V so that the voltmeter reading is within 1.0% of the emf of the battery.

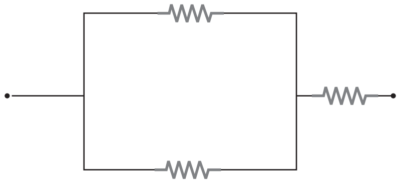
(c) Explain why your answer in part (b) represents a minimum value.

42. (a) An ammeter with resistance R_A is connected in series with a resistor R, a battery of emf ε and internal resistance r. The current measured by the ammeter is I_A. Find the current through the circuit if the ammeter is removed so that the battery and the resistor form a complete circuit. Express your answer in terms of I_A, r, R_A and R. Show that more "ideal" the ammeter, the smaller the difference between this current and the current I_A.

(b) If $R = 3.80\ \Omega$, $\varepsilon = 7.50$ V and $r = 0.45\ \Omega$, find the maximum value of the ammeter resistance R_A so that I_A is within 99% of the current in the circuit when the ammeter is absent.

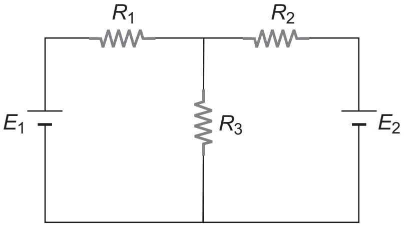
(c) Explain why your answer in part (b) represents a maximum value.

43. Each of three resistors in figure has a resistance of 2.4 Ω and can dissipate a maximum of 36 W without becoming excessively heated. What is the maximum power the circuit can dissipate?

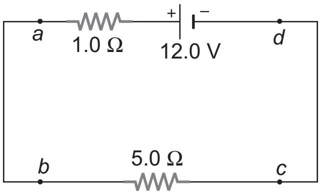


44. A storage battery with emf 2.6 V loaded with external resistance produces a current 1 A. In this case, the potential difference between the terminals of the storage battery equals 2 V. Find the thermal power generated in the battery and the net power supplied by the battery for external circuit.

45. In the circuit shown in figure $E_1 = 7$ V, $E_2 = 1$ V, $R_1 = 2\ \Omega$, $R_2 = 2\ \Omega$ and $R_3 = 3\ \Omega$ respectively. Find the power supplied by the two batteries.



46. In the circuit shown in figure, find



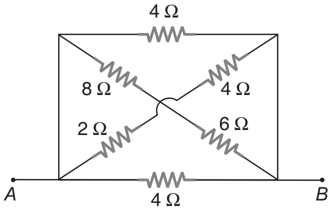
(a) the rate of conversion of internal (chemical) energy to electrical energy within the battery
(b) the rate of dissipation of electrical energy in the battery
(c) the rate of dissipation of electrical energy in the external resistor.

47. Three resistors having resistances of 1.60 Ω, 2.40 Ω and 4.80 Ω are connected in parallel to a 28.0 V battery that has negligible internal resistance. Find
(a) the equivalent resistance of the combination.
(b) the current in each resistor.
(c) the total current through the battery.
(d) the voltage across each resistor.
(e) the power dissipated in each resistor.
(f) which resistor dissipates the maximum power the one with the greatest resistance or the least resistance? Explain why this should be.

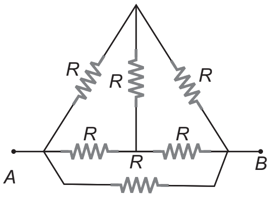
48. (a) The power of resistor is the maximum power the resistor can safely dissipate without too rise in temperature. The power rating of a 15 kΩ resistor is 5.0 W. What is the maximum allowable potential difference across the terminals of the resistor?
(b) A 9.0 kΩ resistor is to be connected across a 120 V potential difference. What power rating is required?

Note *Attempt the following questions after reading the chapter of capacitors.*

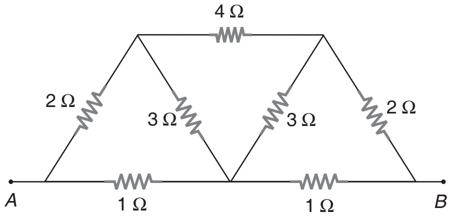
49. Find the equivalent resistance between points *A* and *B* in the following circuits :



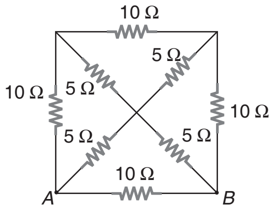
(a)



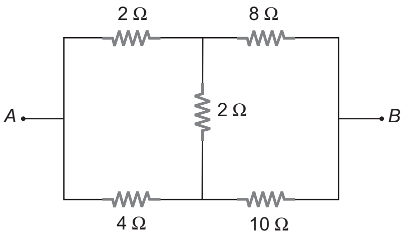
(b)



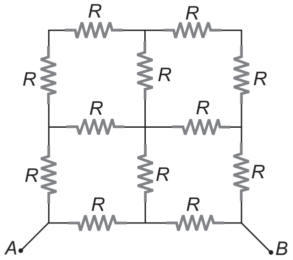
(c)



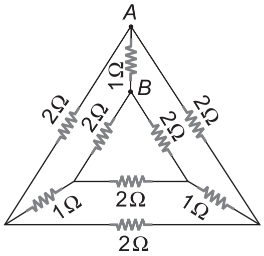
(d)



(e)

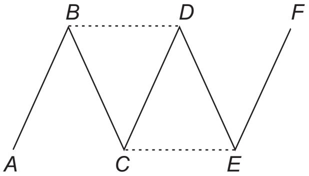


(f)

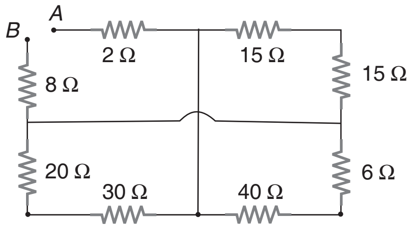


(g)

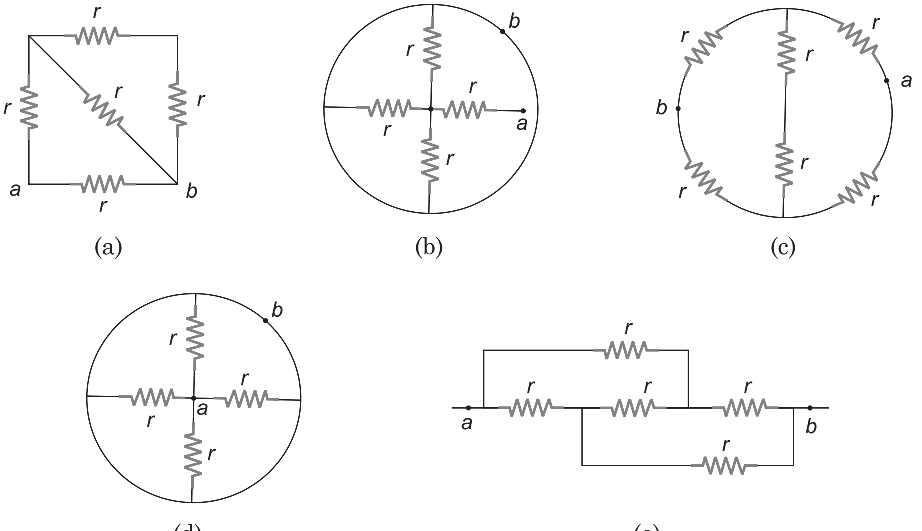
50. What will be the change in the resistance of a circuit between A and F consisting of five identical conductors, if two similar conductors are added as shown by the dashed line in figure?



51. Find R_{AB} in the circuit, shown in figure.



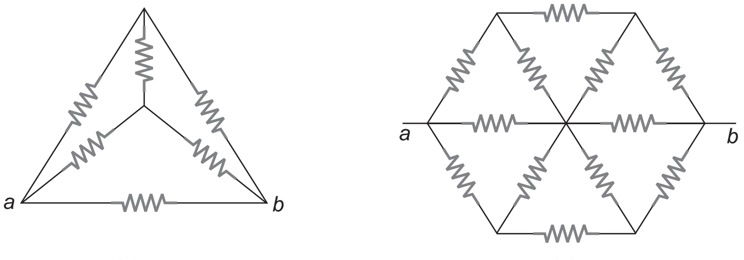
52. Find the equivalent resistance of the networks shown in figure between the points a and b.



(a) (b) (c)

(d) (e)

53. Find the equivalent resistance of the circuits shown in figure between the points a and b. Each resistor has a resistance r.

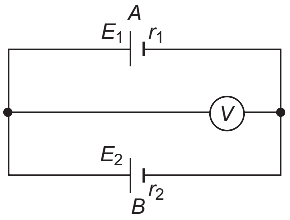


(a) (b)

LEVEL 2

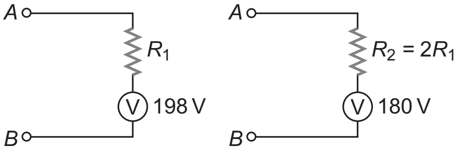
Single Correct Option

1. Two cells A and B of emf 1.3 V and 1.5 V respectively are arranged as shown in figure. The voltmeter reads 1.45 V. The voltmeter is assumed to be ideal. Then



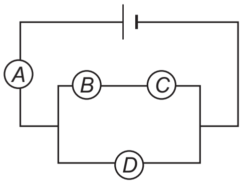
(a) $r_1 = 2r_2$ (b) $r_1 = 3r_2$ (c) $r_2 = 2r_1$ (d) $r_2 = 3r_1$

2. A voltmeter connected in series with a resistance R_1 to a circuit indicates a voltage $V_1 = 198$ V. When a series resistor $R_2 = 2R_1$ is used, the voltmeter indicates a voltage $V_2 = 180$ V. If the resistance of the voltmeter is $R_V = 900\ \Omega$, then the applied voltage across A and B is



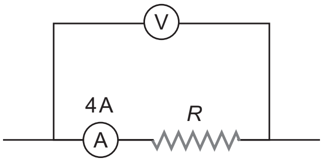
(a) 210 V (b) 200 V
(c) 220 V (d) 240 V

3. All bulbs in the circuit shown in figure are identical. Which bulb glows most brightly?



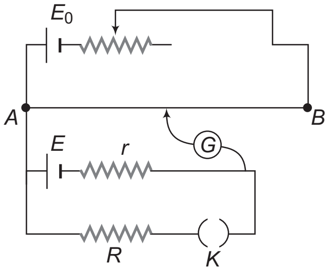
(a) *B* (b) *A*
(c) *D* (d) *C*

4. A student connects an ammeter *A* and a voltmeter *V* to measure a resistance *R* as shown in figure. If the voltmeter reads 20 V and the ammeter reads 4 A, then *R* is



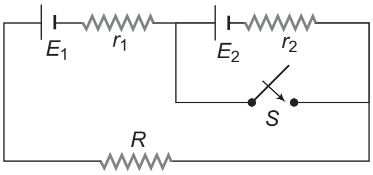
(a) equal to 5 Ω
(b) greater than 5 Ω
(c) less than 5 Ω
(d) greater or less than 5 Ω depending upon the direction of current

5. The given figure represents an arrangement of potentiometer for the calculation of internal resistance (*r*) of the unknown battery (*E*). The balance length is 70.0 cm with the key opened and 60.0 cm with the key closed. *R* is 132.40 Ω. The internal resistance (*r*) of the unknown cell will be



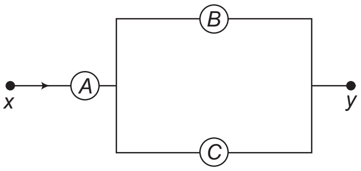
(a) 22.1 Ω (b) 113.5 Ω
(c) 154.5 Ω (d) 10 Ω

6. Switch S is closed at time $t = 0$. Which one of the following statements is correct?



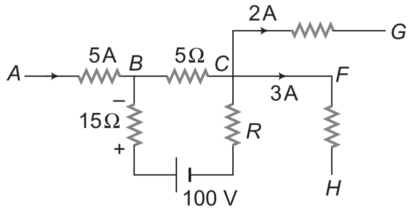
(a) Current in the resistance R increases if $E_1 r_2 < E_2(R + r_1)$
(b) Current in the resistance R increases if $E_1 r_2 > E_2(R + r_1)$
(c) Current in the resistance R decreases if $E_1 r_2 > E_2(R + r_1)$
(d) Current in the resistance R decreases if $E_1 r_2 = E_2(R + r_1)$

7. A, B and C are voltmeters of resistances $R, 1.5R$ and $3R$ respectively. When some potential difference is applied between x and y, the voltmeter readings are V_A, V_B and V_C, then



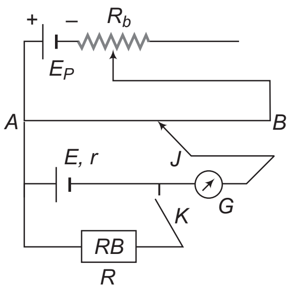
(a) $V_A = V_B = V_C$ (b) $V_A \neq V_B = V_C$ (c) $V_A = V_B \neq V_C$ (d) $V_A + V_B = V_C$

8. In the circuit shown, the voltage drop across the 15 Ω resistor is 30 V having the polarity as indicated. The ratio of potential difference across 5 Ω resistor and resistance R is



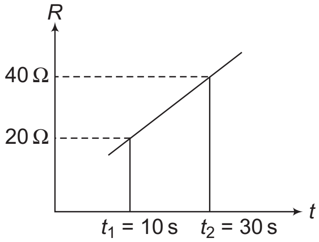
(a) 2/7 (b) 0.4 (c) 5/7 (d) 1

9. In an experiment on the measurement of internal resistance of a cell by using a potentiometer, when the key K is kept open then balancing length is obtained at y metre. When the key K is closed and some resistance R is inserted in the resistance box, then the balancing length is found to be x metre. Then, the internal resistance is



(a) $\frac{(x-y)}{y}R$ (b) $\frac{(y-x)}{x}R$ (c) $\frac{(y-x)}{y}R$ (d) $\frac{(x-y)}{x}R$

10. A source of emf $E = 10$ V and having negligible internal resistance is connected to a variable resistance. The resistance varies as shown in figure. The total charge that has passed through the resistor R during the time interval from t_1 to t_2 is

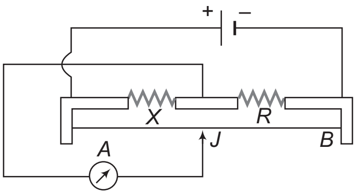


(a) $40 \log_e 4$ (b) $30 \log_e 3$
(c) $20 \log_e 2$ (d) $10 \log_e 2$

11. In order to increase the resistance of a given wire of uniform cross-section to four times its value, a fraction of its length is stretched uniformly till the full length of the wire becomes $\frac{3}{2}$ times the original length. What is the value of this fraction?

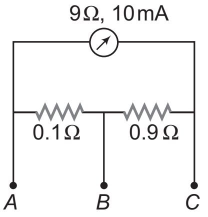
(a) $\frac{1}{4}$ (b) $\frac{1}{8}$
(c) $\frac{1}{16}$ (d) $\frac{1}{6}$

12. The figure shows a meter bridge circuit with $AB = 100$ cm, $X = 12\ \Omega$ and $R = 18\ \Omega$ and the jockey J in the position of balance. If R is now made $8\ \Omega$, through what distance will J have to be moved to obtain balance?

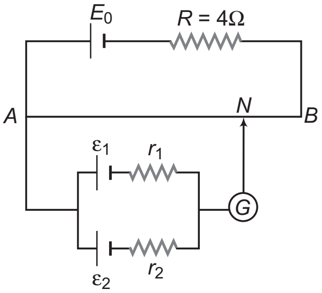


(a) 10 cm (b) 20 cm
(c) 30 cm (d) 40 cm

13. A milliammeter of range 10 mA and resistance 9 Ω is joined in a circuit as shown. The meter gives full scale deflection for current I when A and B are used as its terminals, i.e. current enters at A and leaves at B (C is left isolated). The value of I is



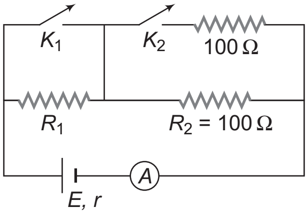
(a) 100 mA (b) 900 mA
(c) 1 A (d) 1.1 A



14. A battery of emf $E_0 = 12$ V is connected across a 4 m long uniform wire having resistance 4 Ω/m. The cell of small emfs $\varepsilon_1 = 2$ V and $\varepsilon_2 = 4$ V having internal resistance 2 Ω and 6 Ω respectively are connected as shown in the figure. If galvanometer shows no deflection at the point N, the distance of point N from the point A is equal to

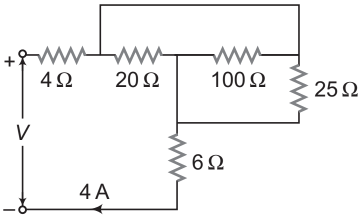
(a) $\frac{5}{3}$ m (b) $\frac{4}{3}$ m
(c) $\frac{3}{2}$ m (d) None of these

15. In the circuit shown, when keys K_1 and K_2 both are closed, the ammeter reads I_0. But when K_1 is open and K_2 is closed, the ammeter reads $I_0/2$. Assuming that ammeter resistance is much less than R_2, the values of r and R_1 in Ω are



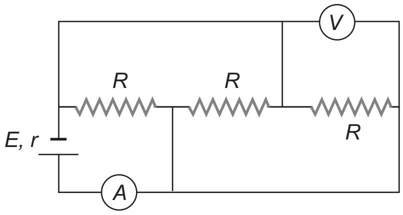
(a) 25, 50 (b) 25, 100
(c) 0, 100 (d) 0, 50

16. In the circuit shown in figure, V must be



(a) 50 V (b) 80 V
(c) 100 V (d) 1290 V

17. In the circuit shown in figure ammeter and voltmeter are ideal. If $E = 4$ V, $R = 9\ \Omega$ and $r = 1\ \Omega$, then readings of ammeter and voltmeter are

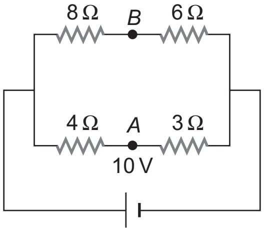


(a) 1 A, 3 V (b) 2 A, 3 V
(c) 3 A, 4 V (d) 4 A, 4 V

18. A moving coil galvanometer is converted into an ammeter reading up to 0.03 A by connecting a shunt of resistance $\frac{r}{4}$. What is the maximum current which can be sent through this galvanometer, if no shunt is used. (Here, r = resistance of galvanometer)

(a) 0.004 A (b) 0.005 A
(c) 0.006 A (d) 0.008 A

19. The potential difference between points A and B is



(a) $\frac{20}{7}$ V (b) $\frac{40}{7}$ V
(c) $\frac{10}{7}$ V (d) zero

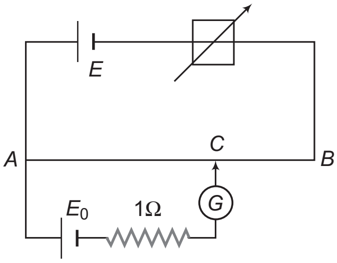
20. Two wires A and B made of same material and having their lengths in the ratio 6 : 1 are connected in series. The potential difference across the wires are 3 V and 2 V respectively. If r_A and r_B are the radii of A and B respectively, then $\frac{r_B}{r_A}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) 1 (d) 2

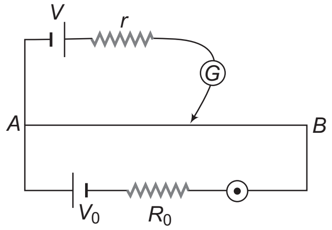
21. A galvanometer of resistance 50 Ω is connected to a battery of 3 V along with resistance of 2950 Ω in series. A full scale deflection of 30 divisions is obtained in the galvanometer. In order to reduce this deflection to 20 divisions, the above series resistance should be

(a) 4450 Ω (b) 5050 Ω
(c) 5550 Ω (d) 6050 Ω

22. Figure shows a potentiometer arrangement with $R_{AB} = 10\,\Omega$ and rheostat of variable resistance x. For $x = 0$ null deflection point is found at 20 cm from A. For unknown value of x null deflection point was at 30 cm from A, then the value of x is



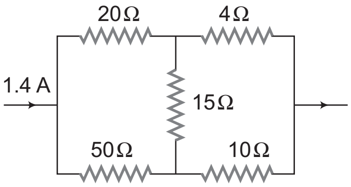
(a) 10 Ω (b) 5 Ω
(c) 2 Ω (d) 1 Ω



23. In the given potentiometer arrangement, the null point

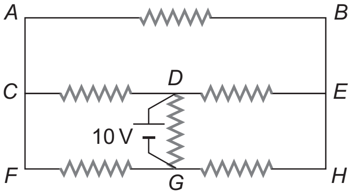
(a) can be obtained for any value of V
(b) can be obtained only if $V < V_0$
(c) can be obtained only if $V > V_0$
(d) can never be obtained

24. In the given figure the current through 4 Ω resistor is



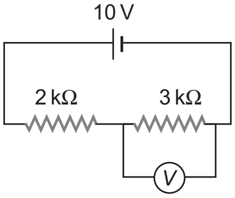
(a) 1.4 A
(b) 0.4 A
(c) 1.0 A
(d) 0.7 A

25. All resistances shown in circuit are 2 Ω each. The current in the resistance between D and E is



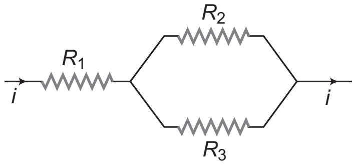
(a) 5 A
(b) 2.5 A
(c) 1 A
(d) 7.5 A

26. In the circuit shown in figure, the resistance of voltmeter is 6 kΩ. The voltmeter reading will be



(a) 6 V
(b) 5 V
(c) 4 V
(d) 3 V

27. For what ratio of R_1, R_2 and R_3 power developed across each resistor is equal?



(a) 1 : 1 : 1
(b) 4 : 4 : 1
(c) 4 : 1 : 1
(d) 1 : 4 : 4

More than One Correct Options

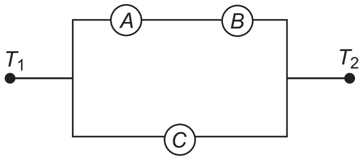
1. Two heaters designed for the same voltage V have different power ratings. When connected individually across a source of voltage V, they produce H amount of heat each in time t_1 and t_2 respectively. When used together across the same source, they produce H amount of heat in time t

   (a) If they are in series, $t = t_1 + t_2$
   (b) If they are in series, $t = 2\,(t_1 + t_2)$
   (c) If they are in parallel, $t = \dfrac{t_1 t_2}{(t_1 + t_2)}$
   (d) If they are in parallel, $t = \dfrac{t_1 t_2}{2(t_1 + t_2)}$

2. Two cells of emf $E_1 = 6$ V and $E_2 = 5$ V are joined in parallel with same polarity on same side, without any external load. If their internal resistances are $r_1 = 2\,\Omega$ and $r_2 = 3\,\Omega$ respectively, then

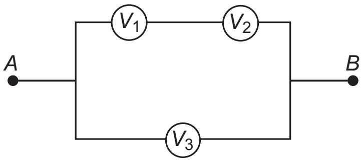
   (a) terminal potential difference across any cell is less than 5 V
   (b) terminal potential difference across any cell is 5.6 V
   (c) current through the cells is 0.2 A
   (d) current through the cells is zero if $E_1 = E_2$

3. Three ammeters A, B and C of resistances R_A, R_B and R_C respectively are joined as shown. When some potential difference is applied across the terminals T_1 and T_2, their readings are I_A, I_B and I_C respectively. Then,



   (a) $I_A = I_B$
   (b) $I_A R_A + I_B R_B = I_C R_C$
   (c) $\dfrac{I_A}{I_C} = \dfrac{R_C}{R_A}$
   (d) $\dfrac{I_B}{I_C} = \dfrac{R_C}{R_A + R_B}$

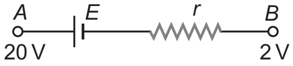
4. Three voltmeters all having different resistances, are joined as shown. When some potential difference is applied across A and B, their readings are V_1, V_2 and V_3. Then,



   (a) $V_1 = V_2$
   (b) $V_1 \neq V_2$
   (c) $V_1 + V_2 = V_3$
   (d) $V_1 + V_2 > V_3$

5. Two conductors made of the same material have lengths L and $2L$ but have equal resistances. The two are connected in series in a circuit in which current is flowing. Which of the following is/are correct?

   (a) The potential difference across the two conductors is the same
   (b) The drift speed is larger in the conductor of length L
   (c) The electric field in the first conductor is twice that in the second
   (d) The electric field in the second conductor is twice that in the first



6. In the figure shown,

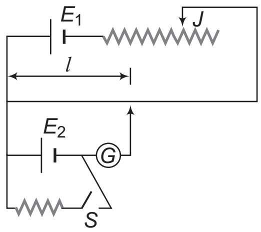
(a) current will flow from A to B

(b) current may flow A to B

(c) current may flow from B to A

(d) the direction of current will depend on E

7. In the potentiometer experiment shown in figure, the null point length is l. Choose the correct options given below.



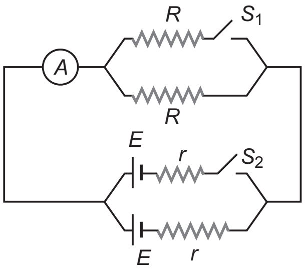
(a) If jockey J is shifted towards right, l will increase

(b) If value of E_1 is increased, l is decreased

(c) If value of E_2 is increased, l is increased

(d) If switch S is closed, l will decrease

8. In the circuit shown in figure, reading of ammeter will

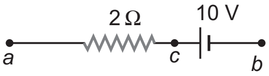


(a) increase if S_1 is closed

(b) decrease if S_1 is closed

(c) increase if S_2 is closed

(d) decrease if S_2 is closed\



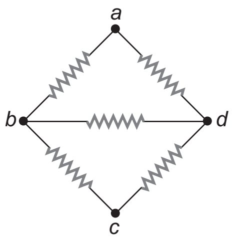
9. In the circuit shown in figure it is given that $V_b - V_a = 2$ volt. Choose the correct options.

(a) Current in the wire is 6 A

(b) Direction of current is from a to b

(c) $V_a - V_c = 12$ volt

(d) $V_c - V_a = 12$ volt



10. Each resistance of the network shown in figure is r. Net resistance between

(a) a and b is $\frac{7}{3}r$

(b) a and c is r

(c) b and d is r

(d) b and d is $\frac{r}{2}$

Comprehension Based Questions

Passage (Q. No. 1 and 2)

The length of a potentiometer wire is 600 *cm and it carries a current of* 40 *mA. For a cell of emf* 2*V and internal resistance* 10 Ω, *the null point is found to be at* 500 *cm. On connecting a voltmeter across the cell, the balancing length is decreased by* 10 *cm.*

1. The voltmeter reading will be

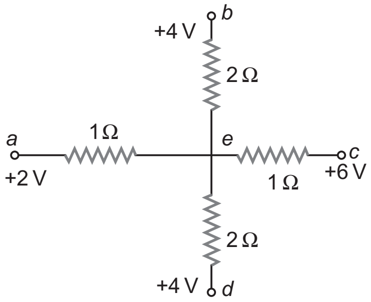
(a) 1.96 V (b) 1.8 V (c) 1.64 V (d) 0.96 V

2. The resistance of the voltmeter is

(a) 500 Ω (b) 290 Ω (c) 490 Ω (d) 20 Ω

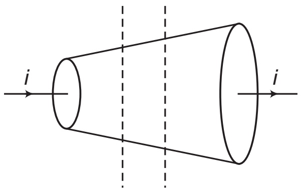
Match the Columns

1. For the circuit shown in figure, match the two columns.



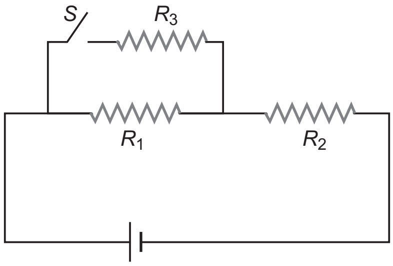
Column I	Column II
(a) current in wire *ae*	(p) 1 A
(b) current is wire *be*	(q) 2 A
(c) current in wire *ce*	(r) 0.5 A
(d) current in wire *de*	(s) None of these

2. Current *i* is flowing through a wire of non-uniform cross-section as shown. Match the following two columns.



Column I	Column II
(a) Current density	(p) is more at 1
(b) Electric field	(q) is more at 2
(c) Resistance per unit length	(r) is same at both sections 1 and 2
(d) Potential difference per unit length	(s) data is insufficient

3. In the circuit shown in figure, after closing the switch S, match the following two columns.

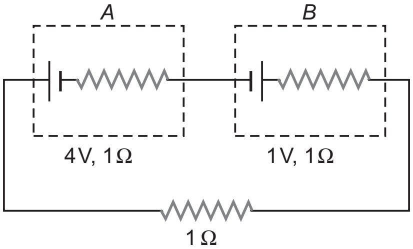


Column I	Column II
(a) current through R_1	(p) will increase
(b) current through R_2	(q) will decrease
(c) potential difference across R_1	(r) will remain same
(d) potential difference across R_2	(s) data insufficient

4. Match the following two columns.

Column I	Column II
(a) Electrical resistance	(p) $[MLT^{-2}A^2]$
(b) Electric potential	(q) $[ML^2T^{-3}A^{-2}]$
(c) Specific resistance	(r) $[ML^2T^{-3}A^{-1}]$
(d) Specific conductance	(s) None of these

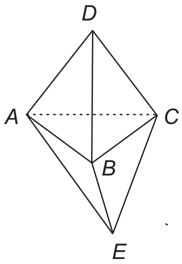
5. In the circuit shown in figure, match the following two columns :



Column I	Column II (In SI units)
(a) potential difference across battery A	(p) zero
(b) potential difference across battery B	(q) 1
(c) net power supplied/ consumed by A	(r) 2
(d) net power supplied/ consumed by B	(s) 3

Subjective Questions

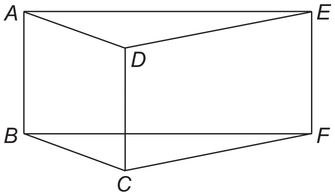
1. Find the equivalent resistance of the triangular bipyramid between the points.



(a) *A* and *C* (b) *D* and *E*

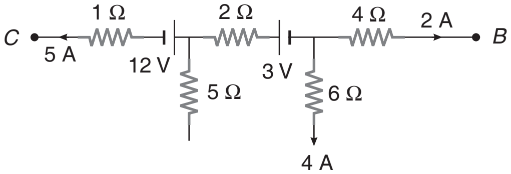
Assume the resistance of each branch to be *R*.

2. Nine wires each of resistance *r* are connected to make a prism as shown in figure. Find the equivalent resistance of the arrangement across

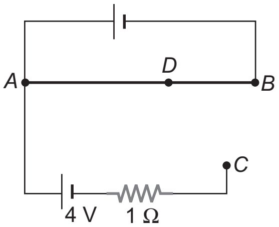


(a) *AD* (b) *AB*

3. The figure shows part of certain circuit, find :

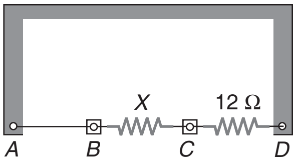


(a) Power dissipated in 5 Ω resistance.
(b) Potential difference $V_C - V_B$.
(c) Which battery is being charged?



4. A 6 V battery of negligible internal resistance is connected across a uniform wire *AB* of length 100 cm. The positive terminal of another battery of emf 4 V and internal resistance 1 Ω is joined to the point *A* as shown in figure. Take the potential at *B* to be zero.

(a) What are the potentials at the points *A* and *C*?
(b) At which point *D* of the wire *AB*, the potential is equal to the potential at *C*?
(c) If the points *C* and *D* are connected by a wire, what will be the current through it?
(d) If the 4V battery is replaced by 7.5 V battery, what would be the answers of parts (a) and (b)?



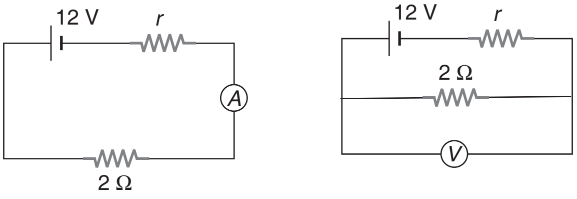
5. A thin uniform wire *AB* of length 1 m, an unknown resistance *X* and a resistance of 12 Ω are connected by thick conducting strips, as shown in the figure. A battery and a galvanometer (with a sliding jockey connected to it) are also available. Connections are to be made to measure the unknown resistance *X*. Using the principle of Wheatstone bridge answer the following questions :

(a) Are there positive and negative terminals on the galvanometer?

(b) Copy the figure in your answer book and show the battery and the galvanometer (with jockey) connected at appropriate points.

(c) After appropriate connections are made, it is found that no deflection takes place in the galvanometer when the sliding jockey touches the wire at a distance of 60 cm from *A*. Obtain the value of the resistance *X*.

6. A galvanometer (coil resistance 99 Ω) is converted into an ammeter using a shunt of 1 Ω and connected as shown in figure (a). The ammeter reads 3 A. The same galvanometer is converted into a voltmeter by connecting a resistance of 101 Ω in series. This voltmeter is connected as shown in figure (b). Its reading is found to be 4/5 of the full scale reading. Find :



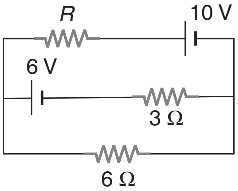
(a) (b)

(a) internal resistance *r* of the cell

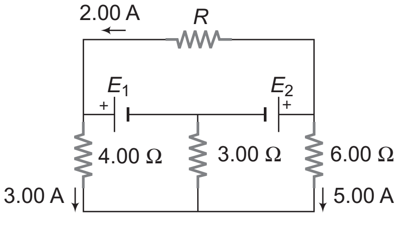
(b) range of the ammeter and voltmeter

(c) full scale deflection current of the galvanometer.

7. In a circuit shown in figure if the internal resistances of the sources are negligible then at what value of resistance *R* will the thermal power generated in it will be the maximum. What is the value of maximum power?

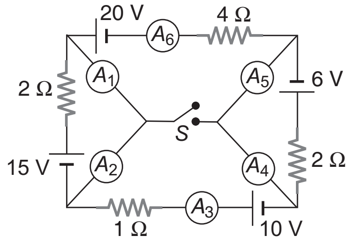


8. In the circuit shown in figure, find :



(a) the current in the 3.00 Ω resistor, (b) the unknown emfs E_1 and E_2 and (c) the resistance *R*.

9. In the circuit shown, all the ammeters are ideal.



(a) If the switch S is open, find the reading of all ammeters and the potential difference across the switch.

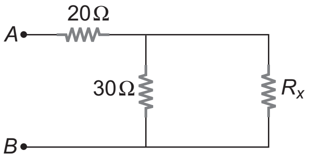
(b) If the switch S is closed, find the current through all ammeters and the switch also.

10. An accumulator of emf 2 V and negligible internal resistance is connected across a uniform wire of length 10 m and resistance 30 Ω. The appropriate terminals of a cell of emf 1.5 V and internal resistance 1 Ω is connected to one end of the wire and the other terminal of the cell is connected through a sensitive galvanometer to a slider on the wire. What is the length of the wire that will be required to produce zero deflection of the galvanometer? How will the balancing length change?

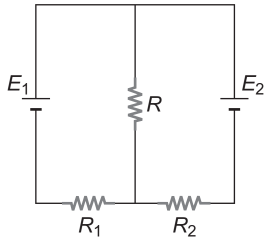
(a) When a coil of resistance 5 Ω is placed in series with the accumulator.

(b) The cell of 1.5 V is shunted with 5 Ω resistor?

11. A circuit shown in the figure has resistances 20 Ω and 30 Ω. At what value of resistance R_x will the thermal power generated in it be practically independent of small variations of that resistance? The voltage between points A and B is supposed to be constant in this case.



12. In the circuit shown in figure, the emfs of batteries are E_1 and E_2 which have internal resistances R_1 and R_2. At what value of the resistance R will the thermal power generated in it be the highest? What it is?



13. A conductor has a temperature independent resistance R and a total heat capacity C. At the moment $t = 0$ it is connected to a DC voltage V. Find the time dependence of the conductor's temperature T assuming the thermal power dissipated into surrounding space to vary as $q = k(T - T_0)$, where k is a constant, T_0 is the surrounding temperature (equal to conductor's temperature at the initial moment).

Answers

Introductory Exercise 23.1

1. 4.375×10^{18} **2.** 38880 C **3.** (a) 337.5 C (b) 2.1×10^{21} **4.** 6.6×10^{15} rps, 1.06 mA

5. 300 C **6.** Yes, from left to right

Introductory Exercise 23.2

1. False

Introductory Exercise 23.3

1. 6.0×10^{-4} m/s **2.** 0.735 μ m/s, 431.4 yr.

Introductory Exercise 23.4

1. 0.18 Ω **2.** True **3.** 15 g **4.** (c)

Introductory Exercise 23.5

1. (d) **2.** 85° C

Introductory Exercise 23.6

1. 5 A, 2.5 A **2.** 0, 2 V, 5 V, 15 V, 3 A from C to B, 7.5 A from D to A.

3. 5 V **4.** $\frac{1}{2}$ A **5.** Zero, 1 A

Introductory Exercise 23.7

1. 3 A, 2 Ω, – 5 V **2.** 36 W, 12 W

Introductory Exercise 23.8

1. $V=\frac{V_1r_2-V_2r_1}{r_1+r_2}$, $r=\frac{r_1r_2}{r_1+r_2}$ **2.** 2 V **3.** 7.5 V, 0.5 Ω

Introductory Exercise 23.9

1. By connecting a resistance of 999 Ω in series with galvanometer

2. By connecting 1 Ω resistance in parallel with it **3.** $(n-1)G$

Introductory Exercise 23.10

1. 1.5 Ω **2.** (a) 320 cm (b) $\frac{3E}{22r}$

Introductory Exercise 23.11

1. (a) **2.** (b) **3.** B is most accurate

Introductory Exercise 23.12

1. 14.2 Ω to 14.3 Ω **2.** See the hints **3.** (c)

Introductory Exercise 23.13

1. $(42\times 10^3 \pm 5\%)\,\Omega$ **2.** Red, Yellow, Blue, Gold

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a,b) **3.** (b) **4.** (d) **5.** (b) **6.** (c) **7.** (d) **8.** (c) **9.** (d) **10.** (a)
11. (d)

Objective Questions

1.(d) **2.**(d) **3.**(a) **4.**(c) **5.**(b) **6.**(b) **7.**(a) **8.**(a) **9.**(b) **10.**(d)
11.(b) **12.**(b) **13.**(b) **14.**(d) **15.**(d) **16.**(d) **17.**(c) **18.**(c) **19.**(c) **20.**(a)
21.(d) **22.**(c) **23.**(b) **24.**(d) **25.**(d) **26.**(c) **27.**(a) **28.**(d) **29.**(a) **30.**(c)
31.(b) **32.**(b) **33.**(b) **34.**(c) **35.**(c) **36.**(d) **37.**(d) **38.**(b) **39.**(a) **40.**(d)
41.(b)

Subjective Questions

1. Yes **2.** False **3.** 1.12 mA **4.** 82 Ω

5. (a) $\frac{1}{2}$ A (b) 1 W, 2 W (c) 6 W (supplied), 3 W (absorbed) **6.** 16 cm from *A*

7. (a) zero (b) 5.0 V (c) 5.0 V **8.** (a) Anti-clockwise (b) E_1 (c) Point *B*

9. −10 V **10.** 1.9×10^{-4} m/s **11.** (a) 9.9 A (b) 5.88 V (c) 0.60 Ω

12. $\frac{\rho l}{\pi ab}$ **14.** 2 W **15.** 954 Ω, 0.002 /°C **16.** 0.569 mm

17. (a) 1.25 V/m (b) $2.84 \times 10^{-8}\ \Omega \cdot \text{m}$ **18.** (a) $2d \times 3d, \frac{V}{\rho d}$ (b) $2d \times 3d, \frac{6vd}{\rho}$

19. (a) $3.65 \times 10^{-8}\ \Omega \cdot \text{m}$ (b) 172.3 A (c) 2.58×10^{-3} m/s

20. $R_1 = 18.18\ \Omega, R_2 = 1.82\ \Omega$ **21.** 5 A **22.** 15 A, $\frac{8}{5}\ \Omega$ **23.** $\frac{8}{3}\ \Omega$, 9 A **24.** $\frac{13}{3}$ A

25. $V_A = 12\text{ V}, V_B = 9\text{ V}, V_C = 3\text{ V}, V_D = -6\text{ V}, V_A' = 12\text{ V}, V_B' = 11\text{ V}, V_C' = 9\text{ V}, V_D' = 6\text{ V}$

26. − 75 V, −50 V, 125 V, 175 V, −25 V, −200 V **27.** (a) 0.1 A, 4.0 V (b) 0.08 A, 4.2 V

28.

Resistance	5 Ω	8 Ω	6 Ω	16 Ω	4 Ω	1 Ω
Current	4 A	0.5 A	3.0 A	0.5 A	1.0 A	4 A
Towards	A	C	C	C	B	E

29. $\frac{20}{3}$ V **30.** (a) (i) 120 V, 80 V (ii) 100 V, 100 V (b) $\frac{1}{60}$ A

31. (a) 5 V (b) 3 V (c) positive terminal on left side **32.** (a) 0.20 Ω (b) 8.7 V

33. 2.5 V **34.** current in all resistors is zero **35.** $\frac{G^2}{G + S}$ **36.** 20.16 Ω **37.** 22.5 V

38. 48 V **39.** 80 Ω **40.** 400 Ω, 3.2 V, 3.238 V **41.** (a) $\frac{ER_v}{R_v + r}$ (b) $4.5 \times 10^{-3}\ \Omega$

42. (a) $I_A \left[1 + \frac{R_A}{R + r}\right]$ (b) 0.0045 Ω **43.** 54 W **44.** 0.6 W, 2 W

45. +14 W, –1 W **46.** (a) 24 W (b) 4 W (c) 20 W

47. (a) 0.80 Ω (b) 1.60 Ω resistor 17.5 A, 2.40 Ω resistor 11.7 A, 4.80 Ω resistor 5.8 A
(c) 35.0 A (d) 28.0 V for each
(e) 1.60 Ω resistor 490 W, 2.40 Ω resistor 327 W, 4.80 Ω resistor 163 W (f) least resistance

48. (a) 273.8 V (b) 1.6 W

49. (a) $\frac{42}{31}\,\Omega$ (b) $\frac{R}{2}$ (c) $\frac{32}{21}\,\Omega$ (d) $\frac{25}{6}\,\Omega$ (e) 6.194 Ω (f) $\frac{5R}{4}$ (g) $\frac{5}{7}\,\Omega$

50. The new equivalent resistance will become 0.6 times **51.** 23.32 Ω

52. (a) $\frac{5}{8}r$ (b) $\frac{4}{3}r$ (c) r (d) $\frac{r}{4}$ (e) r **53.** (a) $r/2$ (b) $4r/5$

LEVEL 2

Single Correct Option

1.(b) **2.**(c) **3.**(b) **4.**(c) **5.**(a) **6.**(b) **7.**(a) **8.**(d) **9.**(b) **10.**(d)
11.(b) **12.**(b) **13.**(c) **14.**(d) **15.**(d) **16.**(b) **17.**(a) **18.**(c) **19.**(d) **20.**(b)
21.(a) **22.**(b) **23.**(d) **24.**(c) **25.**(b) **26.**(b) **27.**(d)

More than One Correct Options

1. (a,c) **2.** (b,c,d) **3.** (a,b,d) **4.** (b,c) **5.** (a,b,c) **6.** (b,c,d) **7.** (a,b,c,d) **8.** (a,c) **9.** (a,d) **10.** (b,d)

Comprehension Based Questions

1. (a) **2.** (c)

Match the Columns

1. (a) → q (b) → s (c) → q (d) → s
2. (a) → p (b) → p (c) → p (d) → p
3. (a) → q (b) → p (c) → q (d) → p
4. (a) → q (b) → r (c) → s (d) → s
5. (a) → s (b) → r (c) → s (d) → r

Subjective Questions

1. (a) $\frac{2}{5}R$ (b) $\frac{2}{3}R$ **2.** (a) $\frac{8}{15}r$ (b) $\frac{3}{5}r$ **3.** (a) 605 W (b) 6 V (c) both

4. (a) 6 V, 2V (b) $AD = 66.7$ cm (c) zero (d) 6 V, –1.5 V, no such point D exists.

5. (a) No (c) 8 Ω **6.** (a) 1.01 Ω (b) 5 A, 9.95 V (c) 0.05 A

7. 2 Ω, 4.5 W **8.** (a) 8 A (b) 36 V, 54 V (c) 9 Ω

9. (a) 9.5 A, 9.5 A, 2 A, 5 A, 5 A, 2 A, 12 V (b) 12.5 A, 2.5 A, 10 A, 7 A, 8 A, 5 A, 15 A

10. 7.5 m (a) 8.75 m (b) 6.25 m **11.** 12 Ω **12.** $R = \frac{R_1R_2}{R_1 + R_2}$, $P_{\max} = \frac{(E_1R_2 + E_2R_1)^2}{4R_1R_2\,(R_1 + R_2)}$

13. $T = T_0 + (1 - e^{-kt/C})\frac{V^2}{kR}$

24

Electrostatics

Chapter Contents

24.1 Introduction
24.2 Electric charge
24.3 Conductor and Insulators
24.4 Charging of a body
24.5 Coulomb's law
24.6 Electric field
24.7 Electric potential energy
24.8 Electric potential
24.9 Relation between electric field and potential
24.10 Equipotential surfaces
24.11 Electric dipole
24.12 Gauss's law
24.13 Properties of a conductor
24.14 Electric field and potential due to charged spherical shell or solid conducting sphere
24.15 Electric field and potential due to a solid sphere of charge

24.1 Introduction

When we comb our hair on a dry day and bring the comb close to tiny pieces of paper, we note that they are swiftly attracted by the comb. Similar phenomena occur if we rub a glass rod or an amber rod with a cloth or with a piece of fur. Why does this happens? What really happens in an electric circuit? How do electric motors and generators work?

The answers to all these questions come from a branch of physics known as **electromagnetism**, the study of electric and magnetic interactions. These interactions involve particles that have a property called electric charge, an inherent property of matter that is as fundamental as mass.

We begin our study of electromagnetism in this chapter by the electric charge. We will see that it is quantized and obeys a conservation principle. Then we will study the interactions of electric charges that are at rest, called **electrostatic** interactions. These interactions are governed by a simple relationship known as **Coulomb's law.** This law is more conveniently described by using the concept of **electric field.**

24.2 Electric Charge

The electrical nature of matter is inherent in atomic structure. An atom consists of a small, relatively massive nucleus that contains particles called protons and neutrons. A proton has a mass 1.673×10^{-27} kg, while a neutron has a slightly greater mass 1.675×10^{-27} kg. Surrounding the nucleus is a diffuse cloud of orbiting particles called electrons. An electron has a mass of 9.11×10^{-31} kg.

Like mass, **electric charge** is an intrinsic property of protons and electrons, and only two types of charge have been discovered positive and negative. A proton has a positive charge, and an electron has a negative charge. A neutron has no net electric charge.

The magnitude of the charge on the proton exactly equals the magnitude of the charge on the electron. The proton carries a charge $+e$ and the electron carries a charge $-e$. The SI unit of charge is coulomb (C) and e has the value

$$\boxed{e = 1.6\times 10^{-19}\,\text{C}}$$

Regarding charge the following points are worth noting:

1. Like charges repel each other and unlike charges attract each other.
2. Charge is a scalar and can be of two types positive or negative.
3. Charge is quantized. The quantum of charge is e. The charge on any body will be some integral multiple of e, i.e.

   $$\boxed{q = \pm ne} \qquad \text{where,}\quad n = 1, 2, 3\ldots$$

   Charge on any body can never be $\left(\frac{1}{3}e\right)$, 1.5$e$, etc.

Note *(i) Apart from charge, energy, angular momentum and mass are also quantized. The quantum of energy is $h\nu$ and that of angular momentum is $\frac{h}{2\pi}$. Quantum of mass is not known till date.*

*(ii) The protons and neutrons are combination of other entities called **quarks**, which have charges $\pm\frac{1}{3}e$ and $\pm\frac{2}{3}e$. However, isolated quarks have not been observed. So, quantum of charge is still e.*

4. During any process, the net electric charge of an isolated system remains constant or we can say that charge is conserved. Pair production and pair annihilation are two examples of conservation of charge.
5. A charged particle at rest produces electric field. A charged particle in an unaccelerated motion produces both electric and magnetic fields but does not radiate energy. But an accelerated charged particle not only produces an electric and magnetic fields but also radiates energy in the form of electromagnetic waves.

Example 24.1 *How many electrons are there in one coulomb of negative charge?*

Solution The negative charge is due to the presence of excess electrons, since they carry negative charge. Because an electron has a charge whose magnitude is $e = 1.6 \times 10^{-19}$ C, the number of electrons is equal to the charge q divided by the charge e on each electron. Therefore, the number n of electrons is

$$n = \frac{q}{e} = \frac{1.0}{1.6 \times 10^{-19}} = 6.25 \times 10^{18}$$ **Ans.**

24.3 Conductors and Insulators

For the purpose of electrostatic theory, all substances can be divided into two main groups, conductors and insulators. In conductors, electric charges are free to move from one place to another, whereas in insulators they are tightly bound to their respective atoms. In an uncharged body, there are equal number of positive and negative charges.

The examples of conductors of electricity are the metals, human body and the earth and that of insulators are glass, hard rubber and plastics. In metals, the free charges are free electrons known as conduction electrons.

Semiconductors are a third class of materials and their electrical properties are somewhere between those of insulators and conductors. Silicon and germanium are well known examples of semiconductors.

24.4 Charging of a Body

Mainly there are the following three methods of charging a body :

Charging by Rubbing

The simplest way to experience electric charges is to rub certain bodies against each other. When a glass rod is rubbed with a silk cloth, the glass rod acquires some positive charge and the silk cloth acquires negative charge by the same amount. The explanation of appearance of electric charge on rubbing is simple. All material bodies contain large number of electrons and equal number of protons in their normal state. When rubbed against each other, some electrons from one body pass onto the other body. The body that donates the electrons becomes positively charged while that which receives the electrons becomes negatively charged. For example, when glass rod is rubbed with silk cloth, glass rod becomes positively charged because it donates the electrons while the silk cloth becomes negatively charged because it receives electrons. Electricity so obtained by rubbing two objects is also known as **frictional electricity.** The other places where the frictional electricity can be observed are when amber is rubbed with wool or a comb is passed through a dry hair. Clouds also become charged by friction.

Charging by Contact

When a negatively charged ebonite rod is rubbed on a metal object, such as a sphere, some of the excess electrons from the rod are transferred to the sphere. Once the electrons are on the metal sphere, where they can move readily, they repel one another and spread out over the sphere's surface. The insulated stand prevents them from flowing to the earth. When the rod is removed, the sphere is left with a negative charge distributed over its surface. In a similar manner, the sphere will be left with a positive charge after being rubbed with a positively charged rod. In this case, electrons from the sphere would be transferred to the rod. The process of giving one object a net electric charge by placing it in contact with another object that is already charged is known as **charging by contact.**

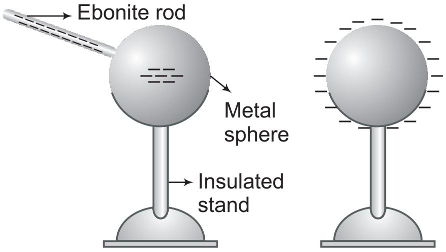


Fig. 24.1

Charging by Induction

It is also possible to charge a conductor in a way that does not involve contact.

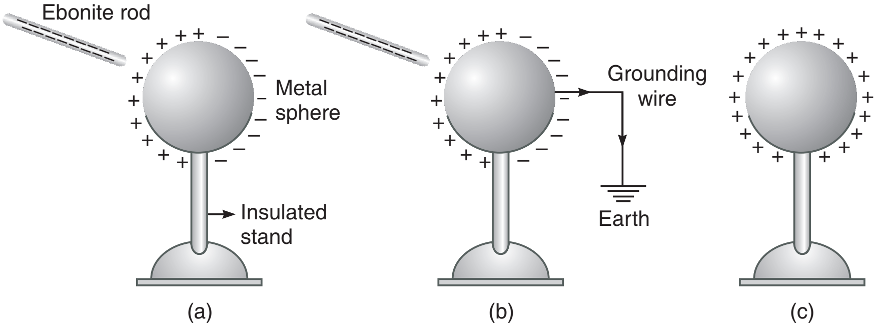


Fig. 24.2

In Fig. (a), a negatively charged rod brought close to (but does not touch) a metal sphere. In the sphere, the free electrons close to the rod move to the other side (by repulsion). As a result, the part of the sphere nearer to the rod becomes positively charged and the part farthest from the rod negatively charged. This phenomenon is called induction. Now, if the rod is removed, the free electrons return to their original places and the charged regions disappear. Under most conditions the earth is a good electric conductor. So, when a metal wire is attached between the sphere and the ground as in figure (b) some of the free electrons leave the sphere and distribute themselves on the much larger earth. If the grounding wire is then removed, followed by the ebonite rod, the sphere is left with a net positive charge.

The process of giving one object a net electric charge without touching the object to a second charged object is called **charging by induction.** The process could also be used to give the sphere a net negative charge, if a positively charged rod were used. Then, electrons would be drawn up from the ground through the grounding wire and onto the sphere.

If the sphere were made from an insulating material like plastic, instead of metal, the method of producing a net charge by induction would not work, because very little charge would flow through the insulating material and down the grounding wire. However, the electric force of the charged rod would have some effect as shown in figure. The electric force would cause the positive and negative charges in the molecules of the insulating material to separate slightly, with the negative charges being pushed away from the negative rod. The surface of the plastic sphere does acquire a slight induced positive charge, although no net charge is created.

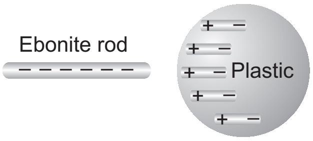


Fig. 24.3

Example 24.2 *If we comb our hair on a dry day and bring the comb near small pieces of paper, the comb attracts the pieces, why?*

Solution This is an example of frictional electricity and induction. When we comb our hair, it gets positively charged by rubbing. When the comb is brought near the pieces of paper some of the electrons accumulate at the edge of the paper piece which is closer to the comb. At the farther end of the piece there is deficiency of electrons and hence, positive charge appears there. Such a redistribution of charge in a material, due to presence of a nearby charged body is called inducion. The comb exerts larger attraction on the negative charges of the paper piece as compared to the repulsion on the positive charge. This is because the negative charges are closer to the comb. Hence, there is a net attraction between the comb and the paper piece.

Example 24.3 *Does the attraction between the comb and the piece of papers last for longer period of time?*

Solution No, because the comb loses its net charge after some time. The excess charge of the comb transfers to earth through our body after some time.

Example 24.4 *Can two similarly charged bodies attract each other?*

Solution Yes, when the charge on one body (q_1) is much greater than that on the other (q_2) and they are close enough to each other so that force of attraction between q_1 and induced charge on the other exceeds the force of repulsion between q_1 and q_2. However, two similar point charges can never attract each other because no induction will take place here.

Example 24.5 *Does in charging the mass of a body change?*

Solution Yes, as charging a body means addition or removal of electrons and electron has a mass.

Example 24.6 *Why a third hole in a socket provided for grounding?*

Solution All electric appliances may end with some charge due to faulty connections. In such a situation charge will be accumulated on the appliance. When the user touches the appliance, he may get a shock. By providing the third hole for grounding all accumulated charge is discharged to the ground and the appliance is safe.

INTRODUCTORY EXERCISE 24.1

1. Is attraction a true test of electrification?
2. Is repulsion a true test of electrification?
3. Why does a phonograph record attract dust particles just after it is cleaned?
4. What is the total charge, in coulombs, of all the electrons in three gram mole of hydrogen atom?

24.5 Coulomb's Law

The law that describes how charges interact with one another was discovered by Charles Augustin de Coulomb in 1785. With a sensitive torsion balance, Coulomb measured the electric force between charged spheres. In Coulomb's experiment, the charged spheres were much smaller than the distance between them so that the charges could be treated as point charges. The results of the experiments of Coulomb and others are summarized in **Coulomb's law.**

The electric force $\mathbf{F}_e$ *exerted by one point charge on another acts along the line between the charges. It varies inversely as the square of the distance separating the charges and is proportional to the product of charges. The force is repulsive if the charges have the same sign and attractive if the charges have opposite signs.*

The magnitude of the electric force exerted by a charge q_1 on another charge q_2 a distance r away is thus, given by

$$F_e = \frac{k\,|q_1 q_2|}{r^2} \qquad \ldots\text{(i)}$$

The value of the proportionality constant k in Coulomb's law depends on the system of units used. In SI units the constant k is

$$k = 8.987551787 \times 10^9 \ \frac{\text{N-m}^2}{\text{C}^2}$$

$$\approx 8.988 \times 10^9 \ \frac{\text{N-m}^2}{\text{C}^2}$$

The value of k is known to such a large number of significant digits because this value is closely related to the speed of light in vacuum. This speed is defined to be exactly $c = 2.99792458 \times 10^8$ m/s. The numerical value of k is defined in terms of c to be precisely.

$$k = \left(10^{-7} \ \frac{\text{N-s}^2}{\text{C}^2}\right) c^2$$

This constant k is often written as $\frac{1}{4\pi\varepsilon_0}$, where ε_0 ("epsilon-nought") is another constant. This appears to complicate matters, but it actually simplifies many formulae that we will encounter in later chapters. Thus, Eq. (i) can be written as

$$\boxed{F_e = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r^2}} \qquad \text{...(ii)}$$

Here, $$\frac{1}{4\pi\varepsilon_0} = \left(10^{-7} \frac{\text{N-s}^2}{\text{C}^2}\right) c^2$$

Substituting value of $c = 2.99792458 \times 10^8$ m/s, we get

$$\boxed{\frac{1}{4\pi\,\varepsilon_0} = 8.99 \times 10^9 \,\text{N-m/C}^2}$$

In examples and problems, we will often use the approximate value

$$\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \ \text{N-m}^2/\text{C}^2$$

Here, the quantity ε_0 is called the **permittivity of free space.** It has the value,

$$\boxed{\varepsilon_0 = 8.854 \times 10^{-12} \ \text{C}^2/\text{N-m}^2}$$

Regarding Coulomb's law, the following points are worth noting:

1. Coulomb's law stated above describes the interaction of two point charges. When two charges exert forces simultaneously on a third charge, the total force acting on that charge is the vector sum of the forces that the two charges would exert individually. This important property, called the **principle of superposition of forces,** holds for any number of charges. Thus,
$$\mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$$
2. The electric force is an action reaction pair, i.e. the two charges exert equal and opposite forces on each other.
3. The electric force is conservative in nature.
4. Coulomb's law as we have stated above can be used for point charges in **vacuum.** If some dielectric is present in the space between the charges, the net force acting on each charge is altered because charges are induced in the molecules of the intervening medium. We will describe this effect later. Here at this moment it is enough to say that the force decreases K times if the medium extends till infinity. Here, K is a dimensionless constant which depends on the medium and called dielectric constant of the medium. Thus,

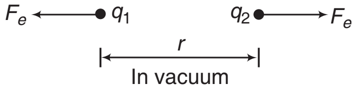


Fig. 24.4

$$F_e = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \qquad \text{(in vacuum)}$$

$$F_e' = \frac{F_e}{K} = \frac{1}{4\pi\varepsilon_0 K} \cdot \frac{q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon} \cdot \frac{q_1 q_2}{r^2} \qquad \text{(in medium)}$$

Here, $\varepsilon = \varepsilon_0 K$ is called permittivity of the medium.

Extra Points to Remember

- In few problems of electrostatics **Lami's theorem** is very useful.

  According to this theorem, "if three concurrent forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ as shown in Fig. 24.5 are in equilibrium or if $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$, then

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

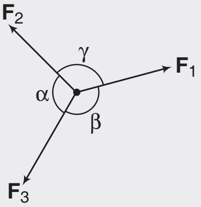


Fig. 24.5

- Suppose the position vectors of two charges q_1 and q_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$, then electric force on charge q_1 due to charge q_2 is,

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2)$$

  Similarly, electric force on q_2 due to charge q_1 is

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3} (\mathbf{r}_2 - \mathbf{r}_1)$$

  Here, q_1 and q_2 are to be substituted with sign. $\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and $\mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$ where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of charges q_1 and q_2.

Example 24.7 *What is the smallest electric force between two charges placed at a distance of 1.0 m?*

Solution $F_e = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{q_1 q_2}{r^2}$ …(i)

For F_e to be minimum $q_1 q_2$ should be minimum. We know that

$$(q_1)_{\min} = (q_2)_{\min} = e = 1.6 \times 10^{-19} \text{ C}$$

Substituting in Eq. (i), we have

$$(F_e)_{\min} = \frac{(9.0 \times 10^9)(1.6 \times 10^{-19})(1.6 \times 10^{-19})}{(1.0)^2}$$

$$= 2.304 \times 10^{-28} \text{ N}$$ **Ans.**

Example 24.8 *Three charges $q_1 = 1\,\mu C$, $q_2 = -2\,\mu C$ and $q_3 = 3\,\mu C$ are placed on the vertices of an equilateral triangle of side 1.0 m. Find the net electric force acting on charge q_1.*

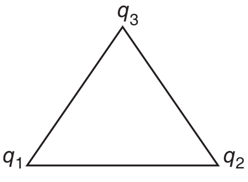


Fig. 24.6

HOW TO PROCEED *Charge q_2 will attract charge q_1 (along the line joining them) and charge q_3 will repel charge q_1. Therefore, two forces will act on q_1, one due to q_2 and another due to q_3. Since, the force is a vector quantity both of these forces (say $\mathbf{F}_1$ and $\mathbf{F}_2$) will be added by vector method. The following are two methods of their addition.*

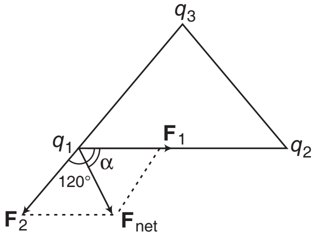


Fig. 24.7

Solution **Method 1.** In the figure,

$$|\mathbf{F}_1| = F_1 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2}$$

$$= \text{magnitude of force between } q_1 \text{ and } q_2$$

$$= \frac{(9.0\times10^9)(1.0\times10^{-6})(2.0\times10^{-6})}{(1.0)^2}$$

$$= 1.8\times10^{-2}\ \text{N}$$

Similarly,

$$|\mathbf{F}_2| = F_2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_3}{r^2}$$

$$= \text{magnitude of force between } q_1 \text{ and } q_3$$

$$= \frac{(9.0\times10^9)(1.0\times10^{-6})(3.0\times10^{-6})}{(1.0)^2}$$

$$= 2.7\times10^{-2}\ \text{N}$$

Now,

$$|\mathbf{F}_{\text{net}}| = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 120^\circ}$$

$$= \left(\sqrt{(1.8)^2 + (2.7)^2 + 2\,(1.8)\,(2.7)\left(-\frac{1}{2}\right)}\right)\times 10^{-2}\ \text{N}$$

$$= 2.38\times10^{-2}\ \text{N}$$

and

$$\tan\alpha = \frac{F_2\sin 120^\circ}{F_1 + F_2\cos 120^\circ}$$

$$= \frac{(2.7\times10^{-2})(0.87)}{(1.8\times10^{-2}) + (2.7\times10^{-2})\left(-\frac{1}{2}\right)}$$

or

$$\alpha = 79.2^\circ$$

Thus, the net force on charge q_1 is 2.38×10^{-2} N at an angle $\alpha = 79.2^\circ$ with a line joining q_1 and q_2 as shown in the figure. **Ans.**

Method 2. In this method let us assume a coordinate axes with q_1 at origin as shown in figure. The coordinates of q_1, q_2 and q_3 in this coordinate system are (0, 0, 0), (1 m, 0, 0) and (0.5 m, 0.87 m, 0) respectively. Now,

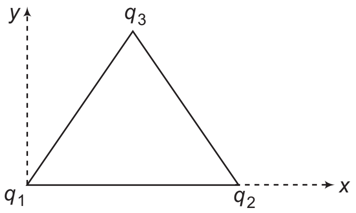


Fig. 24.8

$$\begin{aligned}\mathbf{F}_1 &= \text{force on } q_1 \text{ due to charge } q_2\\ &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1q_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)\\ &= \frac{(9.0\times10^9)(1.0\times10^{-6})(-2.0\times10^{-6})}{(1.0)^3}[(0-1)\,\hat{\mathbf{i}}+(0-0)\,\hat{\mathbf{j}}+(0-0)\,\hat{\mathbf{k}}\,]\\ &= (1.8\times10^{-2}\,\hat{\mathbf{i}})\,\text{N}\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_2 &= \text{force on } q_1 \text{ due to charge } q_3\\ &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1q_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}(\mathbf{r}_1-\mathbf{r}_3)\\ &= \frac{(9.0\times10^9)(1.0\times10^{-6})(3.0\times10^{-6})}{(1.0)^3}[(0-0.5)\,\hat{\mathbf{i}}+(0-0.87)\,\hat{\mathbf{j}}+(0-0)\,\hat{\mathbf{k}}]\\ &= (-1.35\,\hat{\mathbf{i}}-2.349\,\hat{\mathbf{j}})\times10^{-2}\ \text{N}\end{aligned}$$

Therefore, net force on q_1 is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1+\mathbf{F}_2\\ &= (0.45\,\hat{\mathbf{i}}-2.349\,\hat{\mathbf{j}})\times10^{-2}\ \text{N}\end{aligned}$$

Ans.

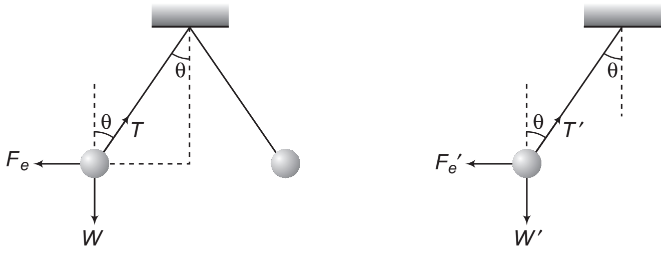
Note *Once you write a vector in terms of* $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$, *there is no need of writing the magnitude and direction of vector separately.*

Example 24.9 *Two identical balls each having a density ρ are suspended from a common point by two insulating strings of equal length. Both the balls have equal mass and charge. In equilibrium each string makes an angle θ with vertical. Now, both the balls are immersed in a liquid. As a result the angle θ does not change. The density of the liquid is σ. Find the dielectric constant of the liquid.*

Solution Each ball is in equilibrium under the following three forces :

(i) tension, (ii) electric force and (iii) weight

So, Lami's theorem can be applied.



In vacuum In liquid

Fig. 24.9

In the liquid,

$$F_e' = \frac{F_e}{K}$$

where, K = dielectric constant of liquid and $W' = W -$ upthrust

Applying Lami's theorem in vacuum

$$\frac{W}{\sin(90° + \theta)} = \frac{F_e}{\sin(180° - \theta)}$$

or
$$\frac{W}{\cos\theta} = \frac{F_e}{\sin\theta} \qquad \text{...(i)}$$

Similarly in liquid,
$$\frac{W'}{\cos\theta} = \frac{F_e'}{\sin\theta} \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{W}{W'} = \frac{F_e}{F_e'}$$

or
$$K = \frac{W}{W - \text{upthrust}} \qquad \left(\text{as } \frac{F_e}{F_e'} = K\right)$$

$$= \frac{V\rho g}{V\rho g - V\sigma g} \qquad (V = \text{volume of ball})$$

or
$$K = \frac{\rho}{\rho - \sigma}$$ **Ans.**

Note *In the liquid F_e and W have changed. Therefore, T will also change.*

INTRODUCTORY EXERCISE 24.2

1. The mass of an electron is 9.11×10^{-31} kg, that of a proton is 1.67×10^{-27} kg. Find the ratio F_e / F_g of the electric force and the gravitational force exerted by the proton on the electron.
2. Find the dimensions and units of ε_0.
3. Three point charges q are placed at three vertices of an equilateral triangle of side a. Find magnitude of electric force on any charge due to the other two.
4. Three point charges each of value $+q$ are placed on three vertices of a square of side a metre. What is the magnitude of the force on a point charge of value $-q$ coulomb placed at the centre of the square?
5. Coulomb's law states that the electric force becomes weaker with increasing distance. Suppose that instead, the electric force between two charged particles were independent of distance. In this case, would a neutral insulator still be attracted towards the comb.
6. A metal sphere is suspended from a nylon thread. Initially, the metal sphere is uncharged. When a positively charged glass rod is brought close to the metal sphere, the sphere is drawn towards the rod. But if the sphere touches the rod, it suddenly flies away from the rod. Explain, why the sphere is first attracted then repelled?
7. Is there any lower limit to the electric force between two particles placed at a certain distance?
8. Does the force on a charge due to another charge depend on the charges present nearby?
9. The electric force on a charge q_1 due to q_2 is $(4\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ N. What is the force on q_2 due to q_1?

24.6 Electric Field

A charged particle cannot directly interact with another particle kept at a distance. A charge produces something called an electric field in the space around it and this electric field exerts a force on any other charge (except the source charge itself) placed in it.

Thus, the region surrounding a charge or distribution of charge in which its electrical effects can be observed is called the **electric field** of the charge or distribution of charge. Electric field at a point can be defined in terms of either a vector function **E** called **'electric field strength'** or a scalar function V called **'electric potential'.** The electric field can also be visualised graphically in terms of **'lines of force'.** Note that all these are functions of position **r** (x, y, z). The field propagates through space with the speed of light, c. Thus, if a charge is suddenly moved, the force it exerts on another charge a distance r away does not change until a time r/c later. In our forgoing discussion, we will see that electric field strength **E** and electric potential V are interrelated. It is similar to a case where the acceleration, velocity and displacement of a particle are related to each other.

Electric Field Strength (E)

Like its gravitational counterpart, the electric field strength (often called electric field) at a point in an electric field is defined as the electrostatic force $\mathbf{F}_e$ per unit positive charge. Thus, if the electrostatic force experienced by a small test charge q_0 is $\mathbf{F}_e$, then field strength at that point is defined as

$$\boxed{\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}_e}{q_0}}$$

The electric field is a vector quantity and its direction is the same as the direction of the force $\mathbf{F}_e$ on a positive test charge. The SI unit of electric field is N/C. Here, it should be noted that the test charge q_0 should be infinitesimally small so that it does not disturb other charges which produces **E**. With the concept of electric field, our description of electric interactions has two parts. First, a given charge distribution acts as a source of electric field. Second, the electric field exerts a force on any charge that is present in this field.

An Electric Field Leads to a Force

Suppose there is an electric field strength **E** at some point in an electric field, then the electrostatic force acting on a charge $+q$ is qE in the direction of **E**, while on the charge $-q$ it is qE in the opposite direction of **E**.

Example 24.10 *An electric field of 10^5 N/C points due west at a certain spot. What are the magnitude and direction of the force that acts on a charge of $+2\,\mu C$ and $-5\,\mu C$ at this spot?*

Solution Force on $+2\,\mu\text{C} = qE = (2 \times 10^{-6})(10^5)$

$= 0.2\ \text{N}$ (due west) **Ans.**

Force on $-5\,\mu\text{C} = (5 \times 10^{-6})(10^5)$

$= 0.5\ \text{N}$ (due east) **Ans.**

Electric Field Due to a Point Charge

The electric field produced by a point charge q can be obtained in general terms from Coulomb's law. First note that the magnitude of the force exerted by the charge q on a test charge q_0 is

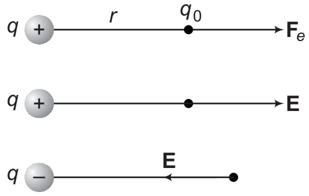


Fig. 24.10

$$F_e = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qq_0}{r^2}$$

then divide this value by q_0 to obtain the magnitude of the field.

$$\boxed{E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}}$$

If q is positive, **E** is directed away from q. On the other hand, if q is negative, then **E** is directed towards q.

The electric field at a point is a vector quantity. Suppose $\mathbf{E}_1$ is the field at a point due to a charge q_1 and $\mathbf{E}_2$ in the field at the same point due to a charge q_2. The resultant field when both the charges are present is

$$\boxed{\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2}$$

If the given charge distribution is continuous, we can use the technique of integration to find the resultant electric field at a point.

Example 24.11 *Two positive point charges $q_1 = 16\ \mu C$ and $q_2 = 4\ \mu C$, are separated in vacuum by a distance of 3.0 m. Find the point on the line between the charges where the net electric field is zero.*

Solution Between the charges the two field contributions have opposite directions, and the net electric field is zero at a point (say P) where the magnitudes of $\mathbf{E}_1$ and $\mathbf{E}_2$ are equal. However, since $q_2 < q_1$, point P must be closer to q_2, in order that the field of the smaller charge can balance the field of the larger charge.

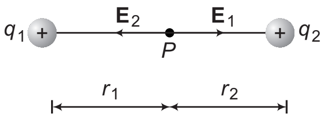


Fig. 24.11

At P, $E_1 = E_2$

or $$\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_1^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_2}{r_2^2}$$

$$\therefore \qquad \frac{r_1}{r_2} = \sqrt{\frac{q_1}{q_2}} = \sqrt{\frac{16}{4}} = 2 \qquad \ldots\text{(i)}$$

Also, $$r_1 + r_2 = 3.0\ \text{m} \qquad \ldots\text{(ii)}$$

Solving these equations, we get

$$r_1 = 2\ \text{m} \quad \text{and} \quad r_2 = 1\ \text{m}$$

Thus, the point P is at a distance of 2 m from q_1 and 1 m from q_2. **Ans.**

Electric Field of a Ring of Charge

A conducting ring of radius R has a total charge q uniformly distributed over its circumference. We are interested in finding the electric field at point P that lies on the axis of the ring at a distance x from its centre.

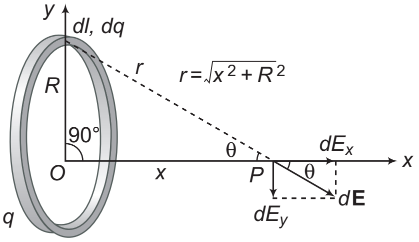


Fig. 24.12

We divide the ring into infinitesimal segments of length *dl*. Each segment has a charge *dq* and acts as a point charge source of electric field.

Let $d\mathbf{E}$ be the electric field from one such segment; the net electric field at P is then the sum of all contributions $d\mathbf{E}$ from all the segments that make up the ring. If we consider two ring segments at the top and bottom of the ring, we see that the contributions $d\mathbf{E}$ to the field at P from these segments have the same x-component but opposite y-components. Hence, the total y-component of field due to this pair of segments is zero. When we add up the contributions from all such pairs of segments, the total field **E** will have only a component along the ring's symmetry axis (the x-axis) with no component perpendicular to that axis (i.e. no y or z-component). So, the field at P is described completely by its x-component E_x.

Calculation of E_x

$$dq = \left(\frac{q}{2\pi R}\right) \cdot dl$$

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dq}{r^2}$$

$$\therefore \qquad dE_x = dE \cos\theta = \left(\frac{1}{4\pi\,\varepsilon_0}\right)\left(\frac{dq}{x^2 + R^2}\right)\left(\frac{x}{\sqrt{x^2 + R^2}}\right)$$

$$= \frac{1}{4\pi\,\varepsilon_0} \cdot \frac{(dq)\,x}{(x^2 + R^2)^{3/2}}$$

$$\therefore \qquad E_x = \int dE_x = \frac{x}{4\pi\varepsilon_0 (x^2 + R^2)^{3/2}} \int dq$$

or

$$\boxed{E_x = \left(\frac{1}{4\pi\varepsilon_0}\right) \frac{qx}{(x^2 + R^2)^{3/2}}}$$

From the above expression, we can see that

(i) $E_x = 0$ at $x = 0$, i.e. field is zero at the centre of the ring. We should expect this, charges on opposite sides of the ring would push in opposite directions on a test charge at the centre, and the forces would add to zero.

(ii) $E_x = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{x^2}$ for $x \gg R$, i.e. when the point P is much farther from the ring, its field is the same as that of a point charge. To an observer far from the ring, the ring would appear like a point, and the electric field reflects this.

(iii) E_x will be maximum where $\frac{dE_x}{dx} = 0$. Differentiating E_x w.r.t. x and putting it equal to zero we get $x = \frac{R}{\sqrt{2}}$ and $E_{\max}$ comes out to be, $\frac{2}{\sqrt[3]{3}} \left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}\right)$.

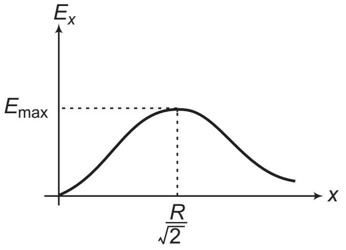


Fig. 24.13

Electric Field of a Line Charge

Positive charge q is distributed uniformly along a line with length $2a$, lying along the y-axis between $y = -a$ and $y = +a$. We are here interested in finding the electric field at point P on the x-axis.

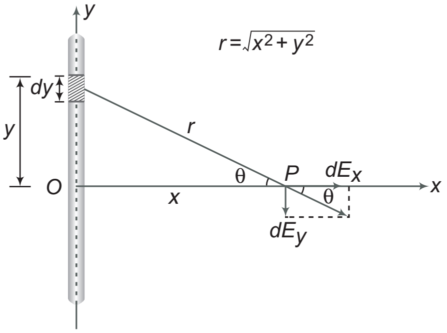


Fig. 24.14

$$\lambda = \text{charge per unit length} = \frac{q}{2a}$$

$$dq = \lambda\, dy = \frac{q}{2a}\, dy$$

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dq}{r^2} = \frac{q}{4\pi\varepsilon_0} \frac{dy}{2a\,(x^2 + y^2)}$$

$$dE_x = dE \cos\theta = \frac{q}{4\pi\varepsilon_0} \cdot \frac{x\,dy}{2a\,(x^2 + y^2)^{3/2}}$$

$$dE_y = -\,dE \sin\theta = -\frac{q}{4\pi\varepsilon_0} \cdot \frac{y\,dy}{2a\,(x^2 + y^2)^{3/2}}$$

$\therefore$
$$E_x = \frac{1}{4\pi\,\varepsilon_0} \cdot \frac{qx}{2a} \int_{-a}^{a} \frac{dy}{(x^2 + y^2)^{3/2}} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{x\sqrt{x^2 + a^2}}$$

and
$$E_y = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2a} \int_{-a}^{a} \frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$$

Thus, electric field is along x-axis only and which has a magnitude,

$$\boxed{E_x = \frac{q}{4\pi\varepsilon_0 x\sqrt{x^2 + a^2}}} \qquad \ldots\text{(i)}$$

From the above expression, we can see that

(i) if $x >> a$, $E_x = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{x^2}$, i.e. if point P is very far from the line charge, the field at P is the same as that of a point charge.

(ii) if we make the line of charge longer and longer, adding charge in proportion to the total length so that λ, the charge per unit length remains constant. In this case, Eq. (i) can be written as

$$E_x = \frac{1}{2\pi\varepsilon_0} \cdot \left(\frac{q}{2a}\right) \cdot \frac{1}{x\sqrt{x^2/a^2 + 1}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0\, x\sqrt{x^2/a^2 + 1}}$$

Now, $x^2/a^2 \to 0$ as $a >> x$, $E_x = \frac{\lambda}{2\pi\varepsilon_0 x}$

Thus, the magnitude of electric field depends only on the distance of point P from the line of charge, so we can say that at any point P at a perpendicular distance r from the line in any direction, the field has magnitude

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}} \qquad \text{(due to infinite line of charge)}$$

or $$E \propto \frac{1}{r}$$

Thus, E-r graph is as shown in Fig. 24.15.

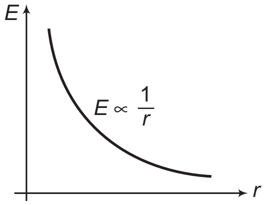


Fig. 24.15

The direction of **E** is radially outward from the line.

Note *Suppose a charge q is placed at a point whose position vector is* $\mathbf{r}_q$ *and we want to find the electric field at a point P whose position vector is* $\mathbf{r}_P$*. Then, in vector form the electric field is given by*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\mathbf{r}_P - \mathbf{r}_q|^3} (\mathbf{r}_P - \mathbf{r}_q)$$

Here, $$\mathbf{r}_P = x_P\hat{\mathbf{i}} + y_P\,\hat{\mathbf{j}} + z_P\,\hat{\mathbf{k}}$$

and $$\mathbf{r}_q = x_q\,\hat{\mathbf{i}} + y_q\hat{\mathbf{j}} + z_q\,\hat{\mathbf{k}}$$

In this equation, q is to be substituted with sign.

Example 24.12 *A charge* $q = 1\,\mu C$ *is placed at point* (1 *m*, 2 *m*, 4 *m*). *Find the electric field at point P* (0, – 4 *m*, 3 *m*).

Solution Here, $$\mathbf{r}_q = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

and $$\mathbf{r}_P = -\,4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

∴ $$\mathbf{r}_P - \mathbf{r}_q = -\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

or $$|\mathbf{r}_P - \mathbf{r}_q| = \sqrt{(-1)^2 + (-6)^2 + (-1)^2} = \sqrt{38}\text{ m}$$

Now, $$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\mathbf{r}_P - \mathbf{r}_q|^3} (\mathbf{r}_P - \mathbf{r}_q)$$

Substituting the values, we have

$$\mathbf{E} = \frac{(9.0 \times 10^9)(1.0 \times 10^{-6})}{(38)^{3/2}} (-\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$$= (-38.42\,\hat{\mathbf{i}} - 230.52\,\hat{\mathbf{j}} - 38.42\,\hat{\mathbf{k}})\text{N/C}$$ **Ans.**

Electric Field Lines

As we have seen, electric charges create an electric field in the space surrounding them. It is useful to have a kind of "map" that gives the direction and indicates the strength of the field at various places. **Field lines,** a concept introduced by Michael Faraday, provide us with an easy way to visualize the electric field.

"An **electric field line** is an imaginary line or curve drawn through a region of space so that its tangent at any point is in the direction of the electric field vector at that point. The relative closeness of the lines at some place give an idea about the intensity of electric field at that point."

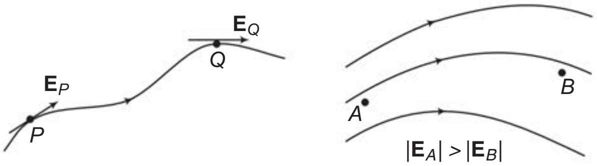


Fig. 24.16

The electric field lines have the following properties :

1. The tangent to a line at any point gives the direction of **E** at that point. This is also the path on which a positive test charge will tend to move if free to do so.
2. Electric field lines always begin on a positive charge and end on a negative charge and do not start or stop in mid-space.
3. The number of lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge. This means, for example that if 100 lines are drawn leaving a $+4\,\mu\text{C}$ charge then 75 lines would have to end on a $-3\,\mu\text{C}$ charge.
4. Two lines can never intersect. If it happens then two tangents can be drawn at their point of intersection, i.e. intensity at that point will have two directions which is absurd.
5. In a uniform field, the field lines are straight parallel and uniformly spaced.

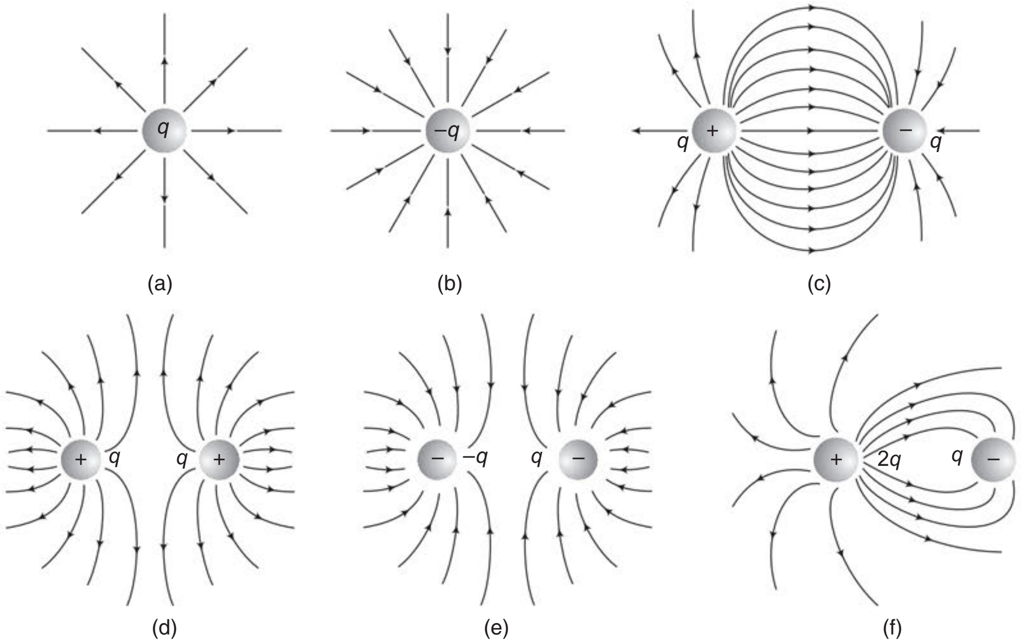


Fig. 24.17

6. The electric field lines can never form closed loops as a line can never start and end on the same charge.

7. Electric field lines also give us an indication of the equipotential surface (surface which has the same potential)
8. Electric field lines always flow from higher potential to lower potential.
9. In a region where there is no electric field, lines are absent. This is why inside a conductor (where electric field is zero) there, cannot be any electric field line.
10. Electric lines of force ends or starts normally from the surface of a conductor.

INTRODUCTORY EXERCISE 24.3

1. The electric field of a point charge is uniform. Is it true or false?
2. Electric field lines are shown in Fig. 24.18. State whether the electric potential is greater at *A* or *B*.

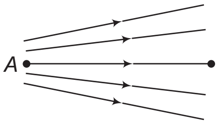


Fig. 24.18

3. A charged particle always move in the direction of electric field. Is this statement true or false?
4. The trajectory of a charged particle is the same as a field line. Is this statement true or false?
5. Figure shows some of the electric field lines due to three point charges q_1, q_2 and q_3 of equal magnitude. What are the signs of each of the three charges?

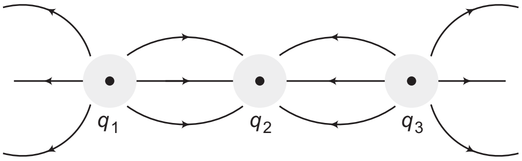


Fig. 24.19

6. Four particles each having a charge *q*, are placed on the four vertices of a regular pentagon. The distance of each corner from the centre is *a*. Find the electric field at the centre of the pentagon.
7. A charge $q = -2.0\ \mu C$ is placed at origin. Find the electric field at (3 *m*, 4 *m*, 0).

24.7 Electric Potential Energy

The electric force between two charges is directed along the line of the charges and depends on the inverse square of their separation, the same as the gravitational force between two masses. Like the gravitational force, the electric force is conservative, so there is a potential energy function *U* associated with it.

When a charged particle moves in an electric field, the field exerts a force that can do work on the particle. This work can always be expressed in terms of electric potential energy. Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field, when a force **F** acts on a particle that moves from point *a* to point *b*, the work $W_{a \to b}$ done by the force is given by

$$W_{a\to b}=\int_a^b \mathbf{F}\cdot d\mathbf{s}=\int_a^b F\cos\theta\, ds$$

where, $d\mathbf{s}$ is an infinitesimal displacement along the particle's path and θ is the angle between **F** and **ds** at each point along the path.

Second, if the force **F** is conservative, the work done by **F** can always be expressed in terms of a **potential energy *U***. When the particle moves from a point where the potential energy is U_a to a point where it is U_b, the change in potential energy is, $\Delta U=U_b-U_a$. This is related by the work $W_{a\to b}$ as

$$W_{a\to b}=U_a-U_b=-(U_b-U_a)=-\Delta U \qquad \text{...(i)}$$

Here, $W_{a\to b}$ is the work done in displacing the particle from a to b by the **conservative force** (here electrostatic) not by us. Moreover we can see from Eq. (i) that if $W_{a\to b}$ is positive, ΔU is negative and the potential energy decreases. So, **whenever the work done by a conservative force is positive, the potential energy of the system decreases and *vice-versa*.** That's what happens when a particle is thrown upwards, the work done by gravity is negative, and the potential energy increases.

Example 24.13 *A uniform electric field E_0 is directed along positive y-direction. Find the change in electric potential energy of a positive test charge q_0 when it is displaced in this field from $y_i=a$ to $y_f=2a$ along the y-axis.*

Solution Electrostatic force on the test charge,

$$F_e=q_0E_0 \qquad \text{(along positive } y\text{-direction)}$$

∴ $W_{i-f}=-\Delta U$

or $\Delta U=-W_{i-f}=-[q_0E_0\,(2a-a)]$

$=-q_0E_0a$ **Ans.**

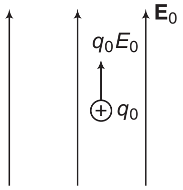


Fig. 24.20

Note *Here, work done by electrostatic force is positive. Hence, the potential energy is decreasing.*

Electric Potential Energy of Two Charges

The idea of electric potential energy is not restricted to the special case of a uniform electric field as in example 24.13. Let us now calculate the work done on a test charge q_0 moving in a non-uniform electric field caused by a single, stationary point charge q.

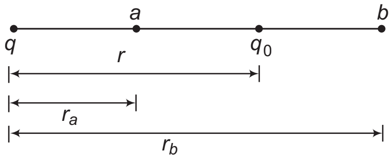


Fig. 24.21

The Coulomb's force on q_0 at a distance r from a fixed charge q is

$$F=\frac{1}{4\pi\varepsilon_0}\cdot\frac{qq_0}{r^2}$$

If the two charges have same signs, the force is repulsive and if the two charges have opposite signs, the force is attractive. The force is not constant during the displacement, so we have to integrate to calculate the work $W_{a\to b}$ done on q_0 by this force as q_0 moves from a to b.

$$\therefore \qquad W_{a\to b} = \int_{r_a}^{r_b} F\, dr = \int_{r_a}^{r_b} \frac{1}{4\pi\varepsilon_0}\cdot\frac{qq_0}{r^2}\, dr = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

Being a conservative force this work is path independent. From the definition of potential energy,

$$U_b - U_a = -W_{a-b} = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$$

We choose the potential energy of the two charge system to be zero when they have infinite separation. This means $U_\infty = 0$. The potential energy when the separation is r is U_r

$$\therefore \qquad U_r - U_\infty = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{\infty}\right)$$

or

$$\boxed{U_r = \frac{qq_0}{4\pi\varepsilon_0}\frac{1}{r}}$$

This is the expression for electric potential energy of two point charges kept at a separation r. In this expression both the charges q and q_0 are to be substituted with sign. The potential energy is positive if the charges q and q_0 have the same sign and negative if they have opposite signs. Note that the above equation is derived by assuming that one of the charges is fixed and the other is displaced. However, the potential energy depends essentially on the separation between the charges and is independent of the spatial location of the charged particles. We emphasize that the potential energy U given by the above equation is a shared property of two charges q and q_0, it is a consequence of the interaction between these two charges. If the distance between the two charges is changed from r_a to r_b, the change in the potential energy is the same whether q is held fixed and q_0 is moved or q_0 is held fixed and q is moved. For this reason we will never use the phrase 'the electric potential energy of a point charge'.

Electric Potential Energy of a System of Charges

The electric potential energy of a system of charges is given by

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

This sum extends over all pairs of charges. We don't let $i = j$, because that would be an interaction of a charge with itself, and we include only terms with $i < j$ to make sure that we count each pair only once.

Thus, to account for the interaction between q_5 and q_4, we include a term with $i = 4$ and $j = 5$ but not a term with $i = 5$ and $j = 4$.

For example, electric potential energy of four point charges q_1, q_2, q_3 and q_4 would be given by

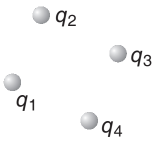


Fig. 24.22

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_4q_3}{r_{43}}+\frac{q_4q_2}{r_{42}}+\frac{q_4q_1}{r_{41}}+\frac{q_3q_2}{r_{32}}+\frac{q_3q_1}{r_{31}}+\frac{q_2q_1}{r_{21}}\right] \qquad \ldots\text{(ii)}$$

Here, all the charges are to be substituted with sign.

Note *Total number of pairs formed by n point charges are* $\frac{n(n-1)}{2}$.

Example 24.14 *Four charges $q_1 = 1\,\mu C$, $q_2 = 2\,\mu C$, $q_3 = -3\,\mu C$ and $q_4 = 4\,\mu C$ are kept on the vertices of a square of side 1m. Find the electric potential energy of this system of charges.*

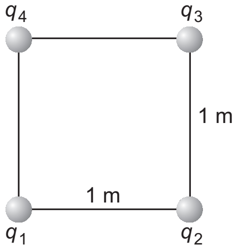


Fig. 24.23

Solution In this problem,

$$r_{41} = r_{43} = r_{32} = r_{21} = 1\,\text{m}$$

and $$r_{42} = r_{31} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\,\text{m}$$

Substituting the proper values with sign in Eq. (ii), we get

$$U = (9.0 \times 10^9)(10^{-6})(10^{-6})\left[\frac{(4)(-3)}{1} + \frac{(4)(2)}{\sqrt{2}} + \frac{(4)(1)}{1} + \frac{(-3)(2)}{1} + \frac{(-3)(1)}{\sqrt{2}} + \frac{(2)(1)}{1}\right]$$

$$= (9.0 \times 10^{-3})\left[-12 + \frac{5}{\sqrt{2}}\right]$$

$$= -7.62 \times 10^{-2}\,\text{J}$$ **Ans.**

Note *Here, negative sign of U implies that positive work has been done by electrostatic forces in assembling these charges at respective distances from infinity.*

Example 24.15 *Two point charges are located on the x-axis, $q_1 = -1\,\mu C$ at $x = 0$ and $q_2 = +1\,\mu C$ at $x = 1$ m.*

(a) Find the work that must be done by an external force to bring a third point charge $q_3 = +1\,\mu C$ from infinity to $x = 2$ m.

(b) Find the total potential energy of the system of three charges.

Solution (a) The work that must be done on q_3 by an external force is equal to the difference of potential energy U when the charge is at $x = 2\,\text{m}$ and the potential energy when it is at infinity.

$$\therefore \quad W = U_f - U_i$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3q_2}{(r_{32})_f} + \frac{q_3q_1}{(r_{31})_f} + \frac{q_2q_1}{(r_{21})_f}\right] - \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3q_2}{(r_{32})_i} + \frac{q_3q_1}{(r_{31})_i} + \frac{q_2q_1}{(r_{21})_i}\right]$$

Here, $$(r_{21})_i = (r_{21})_f$$

and $$(r_{32})_i = (r_{31})_i = \infty$$

$$\therefore \quad W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3q_2}{(r_{32})_f} + \frac{q_3q_1}{(r_{31})_f}\right]$$

Substituting the values, we have

$$W = (9.0 \times 10^9)(10^{-12})\left[\frac{(1)(1)}{(1.0)} + \frac{(1)(-1)}{(2.0)}\right]$$

$$= 4.5 \times 10^{-3}\ \text{J}$$ **Ans.**

(b) The total potential energy of the three charges is given by,

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_3 q_2}{r_{32}} + \frac{q_3 q_1}{r_{31}} + \frac{q_2 q_1}{r_{21}}\right)$$

$$= (9.0 \times 10^9)\left[\frac{(1)(1)}{(1.0)} + \frac{(1)(-1)}{(2.0)} + \frac{(1)(-1)}{(1.0)}\right](10^{-12})$$

$$= -4.5 \times 10^{-3}\ \text{J}$$ **Ans.**

Example 24.16 *Two point charges $q_1 = q_2 = 2\,\mu C$ are fixed at $x_1 = +3$ m and $x_2 = -3$ m as shown in figure. A third particle of mass 1 g and charge $q_3 = -4\,\mu C$ are released from rest at $y = 4.0$ m. Find the speed of the particle as it reaches the origin.*

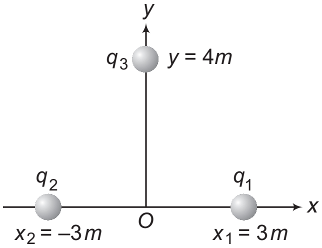


Fig. 24.24

HOW TO PROCEED *Here, the charge q_3 is attracted towards q_1 and q_2 both. So, the net force on q_3 is towards origin.*

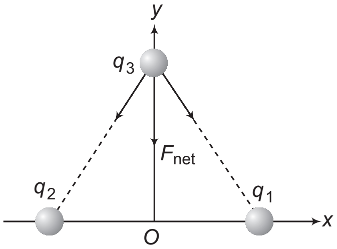


Fig. 24.25

By this force, charge is accelerated towards origin, but this acceleration is not constant. So, to obtain the speed of particle at origin by kinematics we will have to first find the acceleration at some intermediate position and then will have to integrate it with proper limits. On the other hand, it is easy to use energy conservation principle, as the only forces are conservative.

Solution Let v be the speed of particle at origin. From conservation of mechanical energy,

$$U_i + K_i = U_f + K_f$$

or
$$\frac{1}{4\pi\varepsilon_0}\left[\frac{q_3q_2}{(r_{32})_i} + \frac{q_3q_1}{(r_{31})_i} + \frac{q_2q_1}{(r_{21})_i}\right] + 0 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3q_2}{(r_{32})_f} + \frac{q_3q_1}{(r_{31})_f} + \frac{q_2q_1}{(r_{21})_f}\right] + \frac{1}{2}mv^2$$

Here,
$$(r_{21})_i = (r_{21})_f$$

Substituting the proper values, we have

$$(9.0\times 10^9)\left[\frac{(-4)(2)}{(5.0)} + \frac{(-4)(2)}{(5.0)}\right]\times 10^{-12} = (9.0\times 10^9)\left[\frac{(-4)(2)}{(3.0)} + \frac{(-4)(2)}{(3.0)}\right]\times 10^{-12} + \frac{1}{2}\times 10^{-3}\times v^2$$

$\therefore$
$$(9\times 10^{-3})\left(-\frac{16}{5}\right) = (9\times 10^{-3})\left(-\frac{16}{3}\right) + \frac{1}{2}\times 10^{-3}\times v^2$$

$$(9\times 10^{-3})(16)\left(\frac{2}{15}\right) = \frac{1}{2}\times 10^{-3}\times v^2$$

$\therefore$
$$v = 6.2 \text{ m/s}$$
Ans.

INTRODUCTORY EXERCISE 24.4

1. A point charge $q_1 = 1.0\,\mu\text{C}$ is held fixed at origin. A second point charge $q_2 = -2.0\,\mu\text{C}$ and a mass 10^{-4} kg is placed on the *x*-axis, 1.0 m from the origin. The second point charge is released from rest. What is its speed when it is 0.5 m from the origin?
2. A point charge $q_1 = -1.0\,\mu\text{C}$ is held stationary at the origin. A second point charge $q_2 = +2.0\,\mu\text{C}$ moves from the point (1.0 m, 0, 0) to (2.0 m, 0, 0). How much work is done by the electric force on q_2?
3. A point charge q_1 is held stationary at the origin. A second charge q_2 is placed at a point *a*, and the electric potential energy of the pair of charges is -6.4×10^{-8} J. When the second charge is moved to point *b*, the electric force on the charge does 4.2×10^{-8} J of work. What is the electric potential energy of the pair of charges when the second charge is at point *b*?
4. Is it possible to have an arrangement of two point charges separated by finite distances such that the electric potential energy of the arrangement is the same as if the two charges were infinitely far apart? What if there are three charges?

24.8 Electric Potential

As we have discussed in Article 24.6 that an electric field at any point can be defined in two different ways:

(i) by the field strength **E**, and

(ii) by the electric potential V at the point under consideration.

Both **E** and V are functions of position and there is a fixed relationship between these two. Of these, the field strength **E** is a vector quantity while the electric potential V is a scalar quantity. In this article, we will discuss about the electric potential and in the next, the relationship between **E** and V. "**Potential** is the potential energy per unit charge." Electric potential at any point in an electric field is defined as the potential energy per unit charge, same as the field strength is defined as the force per unit charge. Thus,

$$\boxed{V = \frac{U}{q_0} \quad \text{or} \quad U = q_0 V}$$

The SI unit of potential is **volt** (V) which is equal to joule per coulomb. So,

$$1\text{ V} = 1\text{ J/C}$$

The work done by the electrostatic force in displacing a test charge q_0 from a to b in an electric field is defined as the negative of change in potential energy between them, or

$$\Delta U = -W_{a-b}$$

$$\therefore \qquad U_b - U_a = -W_{a-b}$$

We divide this equation by q_0

$$\frac{U_b}{q_0} - \frac{U_a}{q_0} = -\frac{W_{a-b}}{q_0}$$

or

$$V_a - V_b = \frac{W_{a-b}}{q_0}$$

as

$$V = \frac{U}{q_0}$$

Thus, the work done per unit charge by the **electric force** when a charged body moves from a to b is equal to the potential at a minus the potential at b. We sometimes abbreviate this difference as $V_{ab} = V_a - V_b$.

Another way to interpret the potential difference V_{ab} is that the potential at a minus potential at b, equals the work that must be done to move a unit positive charge slowly from b to a against the electric force.

$$\boxed{V_a - V_b = \frac{(W_{b-a})_{\text{external force}}}{q_0}}$$

Absolute Potential at Some Point

Suppose we take the point b at infinity and as a reference point assign the value $V_b = 0$, the above equations can be written as

$$V_a - V_b = \frac{(W_{a-b})_{\text{electric force}}}{q_0} = \frac{(W_{b-a})_{\text{external force}}}{q_0}$$

or

$$V_a = \frac{(W_{a-\infty})_{\text{electric force}}}{q_0} = \frac{(W_{\infty-a})_{\text{external force}}}{q_0}$$

Thus, the absolute electric potential at point a in an electric field can be defined as the work done in displacing a unit positive test charge from infinity to a by the external force or the work done per unit positive charge in displacing it from a to infinity.

Note *The following three formulae are very useful in the problems related to work done in electric field.*

$$(W_{a-b})_{\text{electric force}} = q_0 (V_a - V_b)$$
$$(W_{a-b})_{\text{external force}} = q_0 (V_b - V_a) = -(W_{a-b})_{\text{electric force}}$$
$$(W_{\infty-a})_{\text{external force}} = q_0 V_a$$

Here, q_0, V_a and V_b are to be substituted with sign.

Example 24.17 *The electric potential at point A is 20 V and at B is – 40 V. Find the work done by an external force and electrostatic force in moving an electron slowly from B to A.*

Solution Here, the test charge is an electron, i.e.

$$q_0 = -1.6 \times 10^{-19} \text{ C}$$
$$V_A = 20 \text{ V}$$

and $$V_B = -40 \text{ V}$$

Work done by external force

$$\begin{aligned}(W_{B-A})_{\text{external force}} &= q_0 (V_A - V_B)\\ &= (-1.6 \times 10^{-19})[(20) - (-40)]\\ &= -9.6 \times 10^{-18} \text{ J}\end{aligned}$$ **Ans.**

Work done by electric force

$$\begin{aligned}(W_{B-A})_{\text{electric force}} &= -(W_{B-A})_{\text{external force}}\\ &= -(-9.6 \times 10^{-18} \text{ J})\\ &= 9.6 \times 10^{-18} \text{ J}\end{aligned}$$ **Ans.**

Note *Here, we can see that the electron (a negative charge) moves from B (lower potential) to A (higher potential) and the work done by electric force is positive. Therefore, we may conclude that whenever a negative charge moves from a lower potential to higher potential work done by the electric force is positive or when a positive charge moves from lower potential to higher potential the work done by the electric force is negative.*

Example 24.18 *Find the work done by some external force in moving a charge $q = 2\ \mu C$ from infinity to a point where electric potential is 10^4 V.*

Solution Using the relation,

$$(W_{\infty-a})_{\text{external force}} = qV_a$$

We have, $$\begin{aligned}(W_{\infty-a})_{\text{external force}} &= (2 \times 10^{-6})(10^4)\\ &= 2 \times 10^{-2} \text{ J}\end{aligned}$$ **Ans.**

Electric Potential Due to a Point Charge *q*

From the definition of potential, $$V = \frac{U}{q_0} = \frac{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q\,q_0}{r}}{q_0}$$

or
$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}}$$

Here, r is the distance from the point charge q to the point at which the potential is evaluated.

If q is positive, the potential that it produces is positive at all points; if q is negative, it produces a potential that is negative everywhere. In either case, V is equal to zero at $r = \infty$.

Electric Potential Due to a System of Charges

Just as the electric field due to a collection of point charges is the vector sum of the fields produced by each charge, the electric potential due to a collection of point charges is the scalar sum of the potentials due to each charge.

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}}$$

In this expression, r_i is the distance from the i^{th} charge, q_i, to the point at which V is evaluated. For a continuous distribution of charge along a line, over a surface or through a volume, we divide the charge into elements dq and the sum in the above equation becomes an integral,

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r}}$$

Note *In the equation* $V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}$ *or* $V = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r}$, *if the whole charge is at equal distance* r_0 *from the point where V is to be evaluated, then we can write,*

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_{net}}{r_0}$$

where, q_{net} *is the algebraic sum of all the charges of which the system is made.*

Here there are few examples :

Example (i) *Four charges are placed on the vertices of a square as shown in figure. The electric potential at centre of the square is zero as all the charges are at same distance from the centre and*

$$q_{net} = 4\,\mu C - 2\,\mu C + 2\,\mu C - 4\,\mu C = 0$$

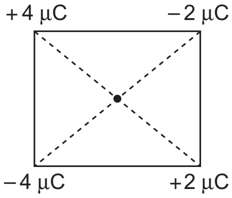
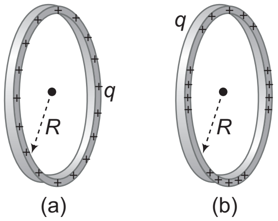


Fig. 24.26

Example (ii) *A charge q is uniformly distributed over the circumference of a ring in Fig. (a) and is non-uniformly distributed in Fig. (b).*



(a) (b)

Fig. 24.27

The electric potential at the centre of the ring in both the cases is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R} \quad \textit{(where, } R = \textit{radius of ring)}$$

and at a distance r from the centre of ring on its axis would be

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\sqrt{R^2 + r^2}}$$

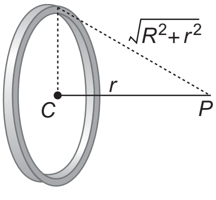


Fig. 24.28

Example 24.19 *Three point charges $q_1 = 1\,\mu C$, $q_2 = -2\,\mu C$ and $q_3 = 3\,\mu C$ are placed at (1 m, 0, 0), (0, 2 m, 0) and (0, 0, 3 m) respectively. Find the electric potential at origin.*

Solution The net electric potential at origin is

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3}\right]$$

Substituting the values, we have

$$V = (9.0 \times 10^9)\left(\frac{1}{1.0} - \frac{2}{2.0} + \frac{3}{3.0}\right) \times 10^{-6}$$

$$= 9.0 \times 10^3 \text{ V}$$ **Ans.**

Example 24.20 *A charge $q = 10\,\mu C$ is distributed uniformly over the circumference of a ring of radius 3 m placed on x-y plane with its centre at origin. Find the electric potential at a point P (0, 0, 4 m).*

Solution The electric potential at point *P* would be

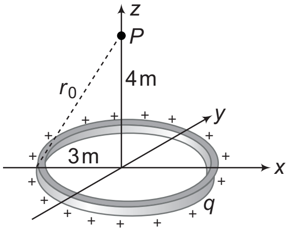


Fig. 24.29

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r_0}$$

Here, r_0 = distance of point *P* from the circumference of ring

$$= \sqrt{(3)^2 + (4)^2} = 5 \text{ m}$$

and $q = 10\,\mu\text{C} = 10^{-5}$ C

Substituting the values, we have

$$V = \frac{(9.0 \times 10^9)(10^{-5})}{(5.0)} = 1.8 \times 10^4 \text{ V}$$ **Ans.**

Variation of Electric Potential on the Axis of a Charged Ring

We have discussed earlier that the electric potential at the centre of a charged ring (whether charged uniformly or non-uniformly) is $\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$ and at a distance r from the centre on the axis of the ring is $\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\sqrt{R^2+r^2}}$. From these expressions, we can see that electric potential is maximum at the centre and decreases as we move away from the centre on the axis. Thus, potential varies with distance r as shown in figure.

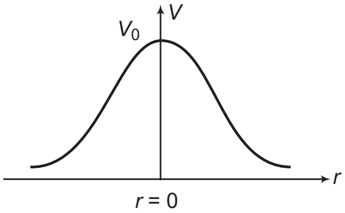


Fig. 24.30

In the figure,
$$V_0 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$$

Electric Potential on the Axis of a Uniformly Charged Disc

Let us find the electric potential at any point P, a distance x on the axis of a uniformly charged circular disc, having surface charge density σ. Let us divide the disc into a large number of thin circular strips and consider a strip of radius r and width dr. Each point of this strip can be assumed to be at equal distance $\sqrt{r^2+x^2}$ from point P. Potential at P due to this circular strip is

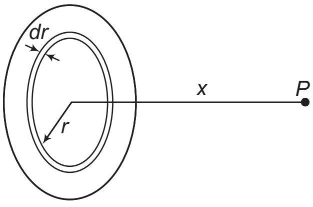


Fig. 24.31

$$dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{dq}{\sqrt{r^2+x^2}}$$

Here,
$$dq = \sigma\,(\text{area of strip}) \quad \text{or} \quad dq = \sigma\,(2\pi r dr)$$

$\therefore$
$$dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\sigma\,(2\pi r dr)}{\sqrt{r^2+x^2}}$$

Thus, the potential due to the whole disc is

$$V = \int_0^R dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r dr}{\sqrt{r^2+x^2}} \quad \text{or} \quad \boxed{V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+x^2} - x\right]}$$

(i) At the centre of the disc, $x = 0$

$$\therefore \qquad \boxed{V\ (\text{centre}) = \frac{\sigma R}{2\varepsilon_0}} \qquad \text{...(i)}$$

(ii) For $x >> R$, using the Binomial expansion for

$$\sqrt{R^2 + x^2} = x\left(1 + \frac{R^2}{x^2}\right)^{1/2} \approx x + \frac{R^2}{2x}$$

$$\therefore \qquad V = \frac{\sigma}{2\varepsilon_0}\left(x + \frac{R^2}{2x} - x\right) = \frac{\sigma R^2}{4\varepsilon_0 x} = \frac{\pi R^2 \sigma}{4\pi\varepsilon_0 x}$$

or
$$\boxed{V = \frac{q}{4\pi\varepsilon_0 x}}$$

as $\pi R^2 \sigma = q$, the total charge on the disc.

This is the relation as obtained due to a point charge. Thus, at far away points, the distribution of charge becomes insignificant. It is difficult to calculate the potential at the points other than on the axis. However, potential on the edge of the disc can be calculated as under.

Potential on the Edge of the Disc

To calculate the potential at point P, let us divide the disc in large number of rings with P as centre. The potential due to one segment between r and $r + dr$ is given as

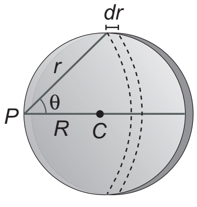


Fig. 24.32

$$dV = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dq}{r}$$

Here,
$$dq = \sigma\ (\text{Area of ring})$$
$$= \sigma\ (2r\theta)\ dr$$

$$\therefore \qquad dV = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma\ (2r\theta)\ dr}{r}$$

$$= \frac{\sigma}{2\pi\varepsilon_0} \cdot \theta\ dr$$

Further,
$$r = 2R \cos\theta$$

$$\therefore \qquad dr = -2R \sin\theta\ d\theta$$

Hence,
$$dV = -\frac{\sigma}{2\pi\varepsilon_0} 2R\theta \sin\theta\ d\theta$$

$$\therefore \qquad V = \int_{\pi/2}^{0} dV = \frac{\sigma R}{\pi\varepsilon_0} \int_0^{\pi/2} \theta \sin\theta\ d\theta$$

Solving, we get
$$\boxed{V = \frac{\sigma R}{\pi\varepsilon_0}} \qquad \text{...(ii)}$$

Comparing Eqs. (i) and (ii), we see that potential at the centre of the disc is greater than the potential at the edge.

Example 24.21 *Find out the points on the line joining two charges + q and – 3q (kept at a distance of 1.0 m) where electric potential is zero.*

Solution Let P be the point on the axis either to the left or to the right of charge $+q$ at a distance r where potential is zero. Hence,

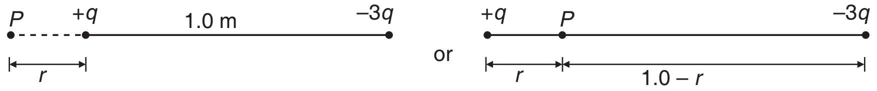


Fig. 24.33

$$V_P = \frac{q}{4\pi\varepsilon_0 r} - \frac{3q}{4\pi\varepsilon_0 (1+r)} = 0$$

Solving this, we get $r = 0.5$ m

Further,
$$V_P = \frac{q}{4\pi\varepsilon_0 r} - \frac{3q}{4\pi\varepsilon_0 (1-r)} = 0$$

which gives $r = 0.25$ m

Thus, the potential will be zero at point P on the axis which is either 0.5 m to the left or 0.25 m to the right of charge $+q$. **Ans.**

INTRODUCTORY EXERCISE 24.5

1. Find V_{ba} if 12 J of work has to be done against an electric field to take a charge of 10^{-2} C from a to b.
2. A rod of length L lies along the x-axis with its left end at the origin. It has a non-uniform charge density $\lambda = \alpha x$, where α is a positive constant.
   (a) What are the units of α?
   (b) Calculate the electric potential at point A where $x = -d$.
3. A charge q is uniformly distributed along an insulating straight wire of length $2l$ as shown in Fig. 24.34. Find an expression for the electric potential at a point located a distance d from the distribution along its perpendicular bisector.

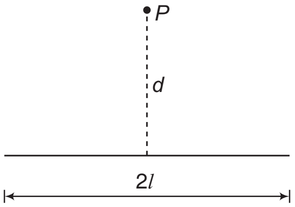


Fig. 24.34

4. A cone made of insulating material has a total charge Q spread uniformly over its sloping surface. Calculate the work done in bringing a small test charge q from infinity to the apex of the cone. The cone has a slope length L.

24.9 Relation Between Electric Field and Potential

As we have discussed above, an invisible space is produced across a charge or system of charges in which any other test charge experiences an electrical force. The vector quantity related to this force is known as electric field. Further, a work is done by this electrostatic force when this test charge is moved from one point to another point. The scalar quantity related to this work done is called potential. Electric field (**E**) and potential (V) are different at different positions. So, they are functions of position.

In a cartesian coordinate system, position of a particle can be represented by three variable coordinates x, y and z. Therefore, **E** and V are functions of three variables x, y and z. In physics, we normally keep least number of variables. So, sometimes E and V are the functions of a single variable x or r. Here, x is the x-coordinate along x-axis and r normally a distance from a point charge or from the centre of a charged sphere or charged spherical shell. From the x-coordinate, we can cover only x-axis. But, from the variable r, we can cover the whole space.

Now, **E** and V functions are related to each other either by differentiation or integration. As far as differentiation is concerned, if there are more than one variables then partial differentiation is done and in case of single variable direct differentiation is required. In case of integration, some limit is required. Limit means value of the function which we get after integration should be known to us at some position. For example, after integrating **E**, we get V. So, value of V should be known at some given position. Without knowing some limit, an unknown in the form of constant of integration remains in the equation. One known limit of V is : potential is zero at infinity.

Conversion of *V* function into E function

This requires differentiation.

Case 1 **When variables are more than one**

In this case,

$$\mathbf{E} = E_x\,\hat{\mathbf{i}} + E_y\,\hat{\mathbf{j}} + E_z\,\hat{\mathbf{k}}$$

Here,

$$E_x = -\frac{\partial V}{\partial x} = -\text{ (partial derivative of } V \text{ w.r.t. } x)$$

$$E_y = -\frac{\partial V}{\partial y} = -\text{ (partial derivative of } V \text{ w.r.t. } y)$$

$$E_z = -\frac{\partial V}{\partial z} = -\text{ (partial derivative of } V \text{ w.r.t. } z)$$

$$\therefore \qquad \boxed{\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]}$$

This is also sometimes written as

$$\boxed{\mathbf{E} = -\text{ gradient } V = -\text{ grad } V = -\nabla V}$$

Example 24.22 *The electric potential in a region is represented as*

$$V = 2x + 3y - z$$

obtain expression for electric field strength.

Solution
$$\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$$

Here,
$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(2x + 3y - z) = 2$$
$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(2x + 3y - z) = 3$$
$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(2x + 3y - z) = -1$$

$\therefore$
$$\mathbf{E} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$$
Ans.

Case 2 **When variable is only one** In this case, electric potential is function of only one variable (say r) and we can write the expression like :

$$\boxed{E = -\frac{dV}{dr}}$$

or
$$E = -\text{ slope of } V\text{-}r \text{ graph}$$

Example Electric potential due to a point charge q at distance r is given as

$$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r} \Rightarrow \frac{dV}{dr} = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}$$

$\therefore$
$$E = -\frac{dV}{dr} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}$$

and we know that this is the expression of electric field due to a point charge.

Note *E is a vector quantity. In the above method, if single variable is x and E comes out to be positive, then direction of E is towards positive x-axis. Negative value of E means direction is towards negative x-axis. If variable is r, then positive value of E means away from the point charge or away from the centre of charged spherical body and negative value of E means towards the charge or towards the centre of charged spherical body.*

Let us take an another example : We wish to find E-r graph corresponding to V-r graph shown in Fig. 24.35.

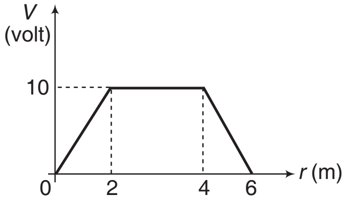


Fig. 24.35

Electric field $E = -5$ V/m for $0 \le r \le 2$ m as slope of V-r graph is 5 V/m. $E = 0$ for 2 m $\le r \le 4$ m as slope of V-r graph in this region is zero. Similarly, $E = 5$ V/m for 4 m $\le r \le 6$ m as slope in this region is – 5 V/m.

So, the corresponding E-r graph is as shown in Fig. 24.36.

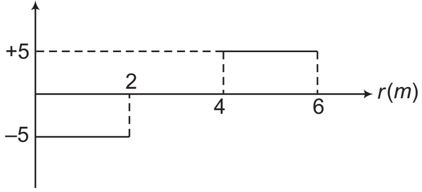


Fig. 24.36

Example 24.23 *The electric potential V at any point x, y, z (all in metre) in space is given by $V = 4x^2$ volt. The electric field at the point (1m, 0, 2 m) is ………V/m.* **(JEE 1992)**

Solution $\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right] \Rightarrow V = 4x^2$

Therefore, $$\frac{\partial V}{\partial x} = 8x \text{ and } \frac{\partial V}{\partial y} = 0 = \frac{\partial V}{\partial z}$$

$$\mathbf{E} = -8x\hat{\mathbf{i}}$$

or **E** at (1 m, 0, 2 m) is $-8\hat{\mathbf{i}}$ V/m.

Conversion of E into *V*

We have learnt, how to find electric field **E** from the electrostatic potential *V*. Let us now discuss how to calculate potential difference or absolute potential if electric field **E** is known. For this, use the relation

$$dV = -\mathbf{E}\cdot d\mathbf{r}$$

or $$\int_A^B dV = -\int_A^B \mathbf{E}\cdot d\mathbf{r}$$

or $$\boxed{V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{r}}$$

Here, $$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

When E is Uniform

Let us take this case with the help of an example.

Example 24.24 *Find V_{ab} in an electric field $\mathbf{E} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\frac{N}{C}$,*

where $\mathbf{r}_a = (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})\ m$ *and* $\mathbf{r}_b = (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}})\ m$

Solution Here, the given field is uniform (constant). So using,

$$dV = -\mathbf{E}\cdot d\mathbf{r}$$

or $$V_{ab} = V_a - V_b = -\int_b^a \mathbf{E}\cdot d\mathbf{r}$$

$$= -\int_{(2,1,-2)}^{(1,-2,1)} (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}})$$

$$= -\int_{(2,1,-2)}^{(1,-2,1)} (2\,dx + 3\,dy + 4\,dz)$$

$$= -[2x + 3y + 4z]_{(2,1,-2)}^{(1,-2,1)}$$

$$= -1\text{V}$$ **Ans.**

Note *In uniform electric field, we can also apply* $\boxed{V = Ed}$

Here, V is the potential difference between any two points, E is the magnitude of uniform electric field and d is the projection of the distance between two points along the electric field.

For example, in the figure for finding the potential difference between points A and B we will have to keep two points in mind,

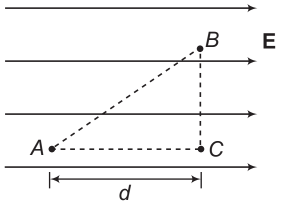


Fig. 24.37

(i) $V_A > V_B$ as electric lines always flow from higher potential to lower potential.

(ii) $d \neq AB$ but $d = AC$

Hence, in the above figure, $V_A - V_B = Ed$

Example 24.25 *In uniform electric field* $\mathbf{E} = 10\ N/C$, *find*

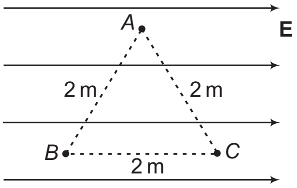


Fig. 24.38

(a) $V_A - V_B$ *(b)* $V_B - V_C$

Solution (a) $V_B > V_A$, So, $V_A - V_B$ will be negative.

Further $d_{AB} = 2 \cos 60° = 1\,\text{m}$

$$\therefore \quad V_A - V_B = -Ed_{AB} = (-10)(1) = -10\ \text{volt}$$ **Ans.**

(b) $V_B > V_C$, so $V_B - V_C$ will be positive.

Further, $d_{BC} = 2.0\,\text{m}$

$$\therefore \quad V_B - V_C = (10)(2) = 20\ \text{volt}$$ **Ans.**

Example 24.26 *A uniform electric field of* $100\ V/m$ *is directed at* $30°$ *with the positive x-axis as shown in figure. Find the potential difference* V_{BA} *if* $OA = 2\ m$ *and* $OB = 4\ m$.

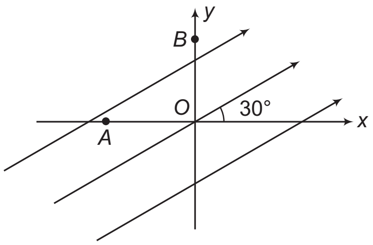


Fig. 24.39

Solution This problem can be solved by both the methods discussed above.

Method 1. Electric field in vector form can be written as

$$\mathbf{E} = (100\cos 30^\circ\,\hat{\mathbf{i}} + 100\sin 30^\circ\,\hat{\mathbf{j}})\ \text{V/m}$$

$$= (50\sqrt{3}\,\hat{\mathbf{i}} + 50\hat{\mathbf{j}})\ \text{V/m}$$

$$A \equiv (-2m, 0, 0)$$

and $$B \equiv (0, 4m, 0)$$

$$\therefore \quad V_{BA} = V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{r}$$

$$= -\int_{(-2m,\,0,\,0)}^{(0,\,4m,\,0)} (50\sqrt{3}\,\hat{\mathbf{i}} + 50\,\hat{\mathbf{j}})\cdot(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}})$$

$$= -[50\sqrt{3}\,x + 50\,y]_{(-2m,\,0,\,0)}^{(0,\,4m,\,0)}$$

$$= -100\,(2+\sqrt{3})\ \text{V}$$ **Ans.**

Method 2. We can also use, $$V = Ed$$

With the view that $V_A > V_B$ or $V_B - V_A$ will be negative.

Here, $$d_{AB} = OA\cos 30^\circ + OB\sin 30^\circ$$

$$= 2\times\frac{\sqrt{3}}{2} + 4\times\frac{1}{2} = (\sqrt{3}+2)$$

$$\therefore \quad V_B - V_A = -Ed_{AB} = -100\,(2+\sqrt{3})$$ **Ans.**

Example 24.27 *A uniform electric field pointing in positive x-direction exists in a region. Let A be the origin, B be the point on the x-axis at $x = +1$ cm and C be the point on the y-axis at $y = +1$ cm. Then, the potentials at the points A, B and C satisfy* **(JEE 2001)**

(a) $V_A < V_B$ (b) $V_A > V_B$

(c) $V_A < V_C$ (d) $V_A > V_C$

Solution Potential decreases in the direction of electric field. Dotted lines are equipotential lines.

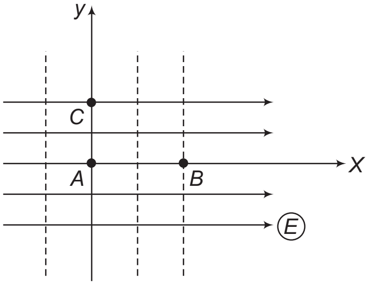


Fig. 24.40

$$\therefore \quad V_A = V_C \quad \text{and} \quad V_A > V_B$$

Hence, the correct option is (b).

Example 24.28 *A non-conducting ring of radius* $0.5\ m$ *carries a total charge of* $1.11\times10^{-10}\,C$ *distributed non-uniformly on its circumference producing an electric field E everywhere in space. The value of the integral* $\int_{l=\infty}^{l=0} -\mathbf{E}\cdot d\mathbf{l}$ *(*$l=0$ *being centre of the ring) in volt is* **(JEE 1997)**

(a) $+2$ *(b)* -1 *(c)* -2 *(d) zero*

Solution $-\int_{l=\infty}^{l=0}\mathbf{E}\cdot d\mathbf{l}=\int_{l=\infty}^{l=0} dV = V\,(\text{centre})-V\,(\text{infinity})$

but $V\,(\text{infinity})=0$

$\therefore\quad -\int_{l=\infty}^{l=0}\mathbf{E}\cdot d\mathbf{l}$ corresponds to potential at centre of ring.

and $$V\,(\text{centre})=\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}=\frac{(9\times10^9)(1.11\times10^{-10})}{0.5}\approx 2\,\text{V}$$

Therefore, the correct answer is (a).

INTRODUCTORY EXERCISE 24.6

1. Determine the electric field strength vector if the potential of this field depends on *x, y* coordinates as

   (a) $V=a\,(x^2-y^2)$ (b) $V=axy$

   where, *a* is a constant.

2. The electrical potential function for an electrical field directed parallel to the *x*-axis is shown in the given graph.

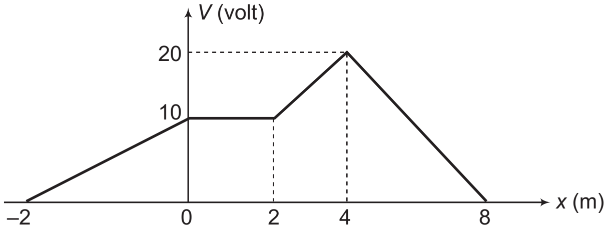


Fig. 24.41

   Draw the graph of electric field strength.

3. The electric potential decreases uniformly from 100 V to 50 V as one moves along the *x*-axis from $x=0$ to $x=5$ m. The electric field at $x=2$ m must be equal to 10 V/m. Is this statement true or false.

4. In the uniform electric field shown in figure, find :

   (a) V_A-V_D
   (b) V_A-V_C
   (c) V_B-V_D
   (d) V_C-V_D

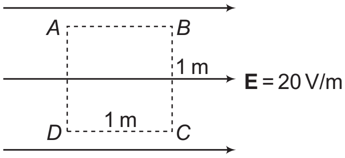


Fig. 24.42

24.10 Equipotential Surfaces

The equipotential surfaces in an electric field have the same basic idea as topographic maps used by civil engineers or mountain climbers. On a topographic map, contour lines are drawn passing through the points having the same elevation. The potential energy of a mass m does not change along a contour line as the elevation is same everywhere.

By analogy to contour lines on a topographic map, an **equipotential surface** is a three-dimensional surface on which the electric potential V is the same at every point on it. An equipotential surface has the following characteristics.

1. Potential difference between any two points in an equipotential surface is zero.
2. If a test charge q_0 is moved from one point to the other on such a surface, the electric potential energy q_0V remains constant.
3. No work is done by the electric force when the test charge is moved along this surface.
4. Two equipotential surfaces can never intersect each other because otherwise the point of intersection will have two potentials which is of course not possible.
5. As the work done by electric force is zero when a test charge is moved along the equipotential surface, it follows that $\mathbf{E}$ must be perpendicular to the surface at every point so that the electric force $q_0\,\mathbf{E}$ will always be perpendicular to the displacement of a charge moving on the surface. Thus, **field lines and equipotential surfaces are always mutually perpendicular.** Some equipotential surfaces are shown in Fig. 24.43.

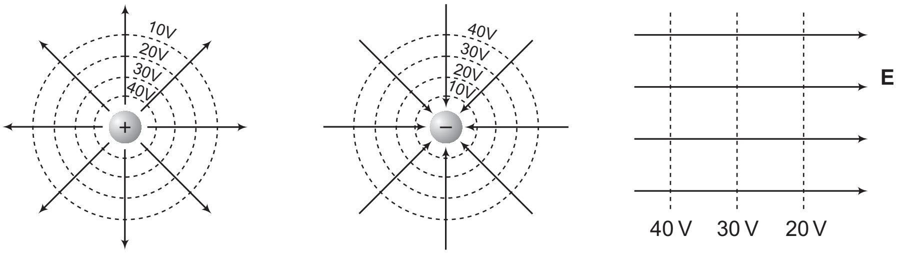


Fig. 24.43

The equipotential surfaces are a family of concentric spheres for a point charge or a sphere of charge and are a family of concentric cylinders for a line of charge or cylinder of charge. For a special case of a uniform field, where the field lines are straight, parallel and equally spaced the equipotential surfaces are parallel planes perpendicular to the field lines.

Note *While drawing the equipotential surfaces we should keep in mind the two main points.*

(i) These are perpendicular to field lines at all places.

(ii) Field lines always flow from higher potential to lower potential.

Example 24.29 *Equipotential spheres are drawn round a point charge. As we move away from the charge, will the spacing between two spheres having a constant potential difference decrease, increase or remain constant.*

Solution $V_1 > V_2$

$$V_1 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r_1} \quad \text{and} \quad V_2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r_2}$$

Now, $$V_1 - V_2 = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

$\therefore$ $$(r_2 - r_1) = \frac{(4\pi\varepsilon_0)(V_1 - V_2)}{q}(r_1 r_2)$$

For a constant potential difference $(V_1 - V_2)$,

$$r_2 - r_1 \propto r_1 r_2$$

i.e. the spacing between two spheres $(r_2 - r_1)$ increases as we move away from the charge, because the product $r_1 r_2$ will increase.

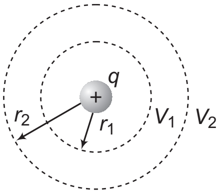


Fig. 24.44

24.11 Electric Dipole

A pair of equal and opposite point charges ±q, that are separated by a fixed distance is known as **electric dipole.** Electric dipole occurs in nature in a variety of situations. The hydrogen fluoride molecule (HF) is typical. When a hydrogen atom combines with a fluorine atom, the single electron of the former is strongly attracted to the later and spends most of its time near the fluorine atom. As a result, the molecule consists of a strongly negative fluorine ion some (small) distance away from a strongly positive ion, though the molecule is electrically neutral overall.

Every electric dipole is characterized by its **electric dipole moment** which is a vector **p** directed from the negative to the positive charge.

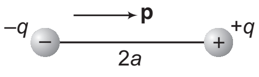


Fig. 24.45

The magnitude of dipole moment is

$$\boxed{p = (2a)\,q}$$

Here, $2a$ is the distance between the two charges.

Electric Potential and Field Due to an Electric Dipole

Consider an electric dipole lying along positive y-direction with its centre at origin.

$$\mathbf{p} = 2aq\,\hat{\mathbf{j}}$$

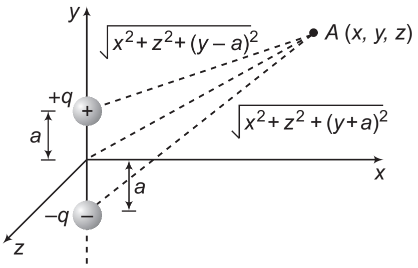


Fig. 24.46

The electric potential due to this dipole at point A (x, y, z) as shown is simply the sum of the potentials due to the two charges. Thus,

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + (y-a)^2 + z^2}} - \frac{q}{\sqrt{x^2 + (y+a)^2 + z^2}}\right]}$$

By differentiating this function, we obtain the electric field of the dipole.

$$E_x = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x}{[x^2 + (y-a)^2 + z^2]^{3/2}} - \frac{x}{[x^2 + (y+a)^2 + z^2]^{3/2}}\right\}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{y-a}{[x^2 + (y-a)^2 + z^2]^{3/2}} - \frac{y+a}{[x^2 + (y+a)^2 + z^2]^{3/2}}\right\}$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{z}{[x^2 + (y-a)^2 + z^2]^{3/2}} - \frac{z}{[x^2 + (y+a)^2 + z^2]^{3/2}}\right\}$$

Special Cases

1. **On the axis of the dipole** (say, along y-axis)

$$x = 0, \quad z = 0$$

$$\therefore \qquad V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{y-a} - \frac{1}{y+a}\right] = \frac{2aq}{4\pi\varepsilon_0\,(y^2 - a^2)}$$

or
$$V = \frac{p}{4\pi\varepsilon_0\,(y^2 - a^2)} \qquad (\text{as } 2aq = p)$$

i.e. at a distance r from the centre of the dipole ($y = r$)

$$V = \frac{p}{4\pi\varepsilon_0\,(r^2 - a^2)} \quad \text{or} \quad \boxed{V_{\text{axis}} \approx \frac{p}{4\pi\varepsilon_0 r^2}} \qquad (\text{for } r >> a)$$

V is positive when the point under consideration is towards positive charge and negative if it is towards negative charge.

Moreover the components of electric field are as under

$$E_x = 0, \quad E_z = 0 \qquad (\text{as } x = 0, \ z = 0)$$

and
$$E_y = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(y-a)^2} - \frac{1}{(y+a)^2}\right]$$

$$= \frac{4ayq}{4\pi\varepsilon_0\,(y^2 - a^2)^2} \quad \text{or} \quad E_y = \frac{1}{4\pi\varepsilon_0}\frac{2py}{(y^2 - a^2)^2}$$

Note that E_y is along positive y-direction or parallel to **p**.

Further, at a distance r from the centre of the dipole ($y = r$).

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{2pr}{(r^2 - a^2)^2} \quad \text{or} \quad \boxed{E_{\text{axis}} \approx \frac{1}{4\pi\varepsilon_0}\cdot\frac{2p}{r^3}} \qquad (\text{for } r >> a)$$

2. **On the perpendicular bisector of dipole**

Say along x-axis (it may be along z-axis also).

$$y = 0, \quad z = 0$$

$$\therefore \qquad V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + a^2}} - \frac{q}{\sqrt{x^2 + a^2}}\right] = 0$$

or $$V_{\perp \text{ bisector}} = 0$$

Moreover the components of electric field are as under,

$$E_x = 0, \quad E_z = 0$$

and $$E_y = \frac{q}{4\pi\varepsilon_0}\left\{\frac{-a}{(x^2 + a^2)^{3/2}} - \frac{a}{(x^2 + a^2)^{3/2}}\right\}$$

$$= \frac{-2aq}{4\pi\varepsilon_0 (x^2 + a^2)^{3/2}}$$

or $$E_y = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{(x^2 + a^2)^{3/2}}$$

Here, negative sign implies that the electric field is along negative y-direction or antiparallel to **p**.

Further, at a distance r from the centre of dipole ($x = r$), the magnitude of electric field is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{(r^2 + a^2)^{3/2}} \quad \text{or} \quad \boxed{E_{\perp \text{ bisector}} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{r^3}} \qquad (\text{for } r >> a)$$

Electric Dipole in Uniform Electric Field

As we have said earlier also, uniform electric field means, at every point the direction and magnitude of electric field is constant. A uniform electric field is shown by parallel equidistant lines. The field due to a point charge or due to an electric dipole is non-uniform in nature. Uniform electric field is found between the plates of a parallel plate capacitor. Now, let us discuss the behaviour of a dipole in uniform electric field.

Force on Dipole

Suppose an electric dipole of dipole moment $|\mathbf{p}| = 2aq$ is placed in a uniform electric field **E** at an angle θ. Here, θ is the angle between **p** and **E**. A force $\mathbf{F}_1 = q\mathbf{E}$ will act on positive charge and $\mathbf{F}_2 = -q\mathbf{E}$ on negative charge. Since, $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude but opposite in direction.

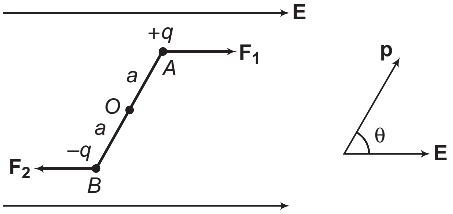


Fig. 24.47

Hence, $\mathbf{F}_1 + \mathbf{F}_2 = 0 \quad \text{or} \quad \mathbf{F}_{\text{net}} = 0$

Thus, net force on a dipole in uniform electric field is zero. While in a non-uniform electric field it may or may not be zero.

Torque on Dipole

The torque of $\mathbf{F}_1$ about O, $\tau_1 = \mathbf{OA} \times \mathbf{F}_1 = q\,(\mathbf{OA} \times \mathbf{E})$

and torque of $\mathbf{F}_2$ about O is, $\tau_2 = \mathbf{OB} \times \mathbf{F}_2 = -\,q(\mathbf{OB} \times \mathbf{E})$

$$= q(\mathbf{BO} \times \mathbf{E})$$

The net torque acting on the dipole is

$$\tau = \tau_1 + \tau_2 = q\,(\mathbf{OA} \times \mathbf{E}) + q\,(\mathbf{BO} \times \mathbf{E})$$
$$= q\,(\mathbf{OA} + \mathbf{BO}) \times \mathbf{E}$$
$$= q\,(\mathbf{BA} \times \mathbf{E})$$

or $$\boxed{\tau = \mathbf{p} \times \mathbf{E}}$$

Thus, the magnitude of torque is $\tau = pE \sin\theta$. The direction of torque is perpendicular to the plane of paper inwards. Further this torque is zero at $\theta = 0°$ or $\theta = 180°$, i.e. when the dipole is parallel or antiparallel to $\mathbf{E}$ and maximum at $\theta = 90°$.

Potential Energy of Dipole

When an electric dipole is placed in an electric field $\mathbf{E}$, a torque $\tau = \mathbf{p} \times \mathbf{E}$ acts on it. If we rotate the dipole through a small angle $d\theta$, the work done by the torque is

$$dW = \tau\, d\theta$$
$$dW = -\,pE \sin\theta\, d\theta$$

The work is negative as the rotation $d\theta$ is opposite to the torque. The change in electric potential energy of the dipole is therefore

$$dU = -\,dW = pE \sin\theta\, d\theta$$

Now, at angle $\theta = 90°$, the electric potential energy of the dipole may be assumed to be zero as net work done by the electric forces in bringing the dipole from infinity to this position will be zero.

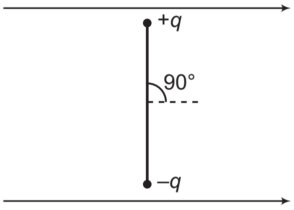


Fig. 24.48

Integrating, $dU = pE \sin\theta\, d\theta$

From 90° to θ, we have $\int_{90°}^{\theta} dU = \int_{90°}^{\theta} pE \sin\theta\, d\theta$

or $$U(\theta) - U(90°) = pE\,[-\cos\theta]_{90°}^{\theta}$$

∴ $$\boxed{U(\theta) = -\,pE\cos\theta = -\,\mathbf{p} \cdot \mathbf{E}}$$

If the dipole is rotated from an angle θ_1 to θ_2, then

$$\text{Work done by external forces} = U(\theta_2) - U(\theta_1)$$

or
$$W_{\text{ext. forces}} = -pE\cos\theta_2 - (-pE\cos\theta_1)$$

or
$$\boxed{W_{\text{ext. forces}} = pE(\cos\theta_1 - \cos\theta_2)}$$

and work done by electric forces,

$$\boxed{W_{\text{electric force}} = -W_{\text{ext. force}} = pE(\cos\theta_2 - \cos\theta_1)}$$

Equilibrium of Dipole

When an electric dipole is placed in a uniform electric field net force on it is zero for any position of the dipole in the electric field. But torque acting on it is zero only at $\theta = 0°$ and 180°. Thus, we can say that at these two positions of the dipole, net force and torque on it is zero or the dipole is in equilibrium

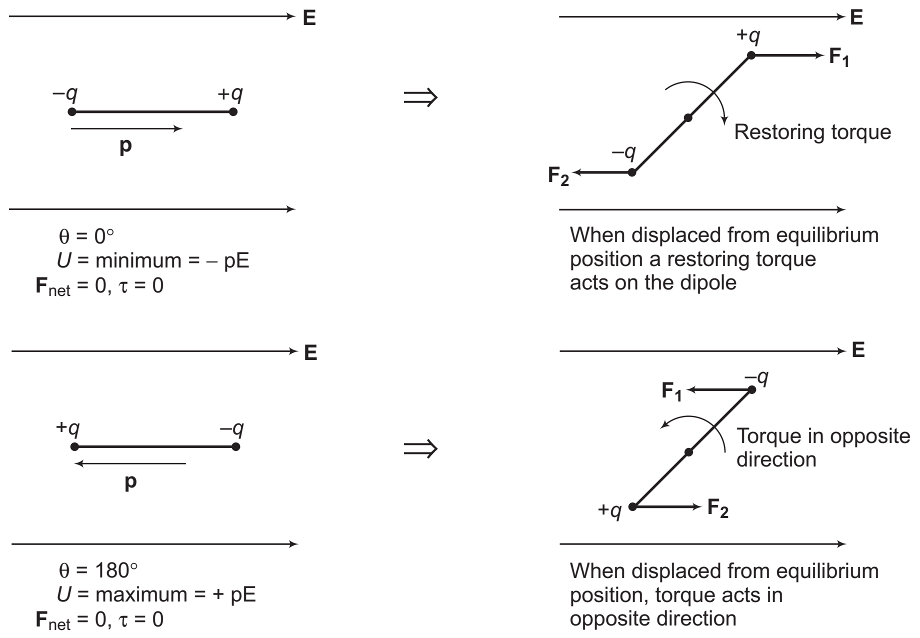

E
$-q$
$+q$
p
$\Rightarrow$
E
$+q$
$\mathbf{F_1}$
Restoring torque
$-q$
$\mathbf{F_2}$
$\theta = 0°$
U = minimum = – pE
$\mathbf{F}_{\text{net}} = 0, \tau = 0$
When displaced from equilibrium position a restoring torque acts on the dipole
E
$+q$
$-q$
p
$\Rightarrow$
E
$\mathbf{F_1}$
$-q$
Torque in opposite direction
$+q$
$\mathbf{F_2}$
$\theta = 180°$
U = maximum = + pE
$\mathbf{F}_{\text{net}} = 0, \tau = 0$
When displaced from equilibrium position, torque acts in opposite direction

Fig. 24.49

Of this, $\theta = 0°$ is the **stable equilibrium position** of the dipole because potential energy in this position is minimum ($U = -pE\cos 0° = -pE$) and when displaced from this position a torque starts acting on it which is restoring in nature and which has a tendency to bring the dipole back in its equilibrium position. On the other hand, at $\theta = 180°$, the potential energy of the dipole is maximum ($U = -pE\cos 180° = +pE$) and when it is displaced from this position, the torque has a tendency to rotate it in other direction. This torque is not restoring in nature. So, this equilibrium is known as **unstable equilibrium position.**

Important Formulae

1. As there are too many formulae in electric dipole, we have summarised them as under :

$$|\mathbf{p}| = (2a)q$$

Direction of $\mathbf{p}$ is from $-q$ to $+q$.

2. If a dipole is placed along y-axis with its centre at origin, then

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+(y-a)^2+z^2}} - \frac{q}{\sqrt{x^2+(y+a)^2+z^2}}\right]$$

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}$$

and $$E_z = -\frac{\partial V}{\partial z}$$

3. On the axis of dipole $x = 0, \; z = 0$

(i)
$$V = \frac{1}{4\pi\varepsilon_0}\frac{p}{(y^2-a^2)}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{p}{r^2-a^2} \qquad \text{if } y = r$$

or $$V_{\text{axis}} \approx \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \qquad \text{if } r >> a$$

(ii)
$$E_x = 0 = E_z \text{ and}$$

$$E = E_y = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2py}{(y^2-a^2)^2} \qquad (\text{along } \mathbf{p})$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2pr}{(r^2-a^2)^2} \qquad \text{if } y = r$$

or $$E_{\text{axis}} \approx \frac{1}{4\pi\varepsilon_0}\cdot\frac{2p}{r^3} \qquad \text{for } r >> a$$

4. On the perpendicular bisector of dipole Along x-axis, $y = 0, \; z = 0$

(i) $$V_{\perp \text{ bisector}} = 0$$

(ii) $$E_x = 0, \quad E_z = 0 \text{ and}$$

$$E_y = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{p}{(x^2+a^2)^{3/2}}$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{(r^2+a^2)^{3/2}} \qquad (\text{opposite to } \mathbf{p})$$

$$\approx \frac{1}{4\pi\varepsilon_0}\cdot\frac{p}{r^3} \qquad \text{for } r >> a$$

5. Dipole in uniform electric field

(i) $\mathbf{F}_{\text{net}} = 0$

(ii) $\tau = \mathbf{p} \times \mathbf{E}$ and $|\tau| = pE\sin\theta$

(iii) $U(\theta) = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta$ with $U(90°) = 0$

(iv) $(W_{\theta_1 \to \theta_2})_{\text{ext. force}} = pE(\cos\theta_1 - \cos\theta_2)$

(v) $(W_{\theta_1 \to \theta_2})_{\text{electric force}} = pE(\cos\theta_2 - \cos\theta_1) = -(W_{\theta_1 \to \theta_2})_{\text{ext. force}}$

(vi) At $\theta = 0°$, $\mathbf{F}_{\text{net}} = 0$, $\tau_{\text{net}} = 0$, U = minimum (stable equilibrium position)

(vii) At $\theta = 180°$, $\mathbf{F}_{\text{net}} = 0$, $\tau_{\text{net}} = 0$, U = maximum (unstable equilibrium position)

Example 24.30 *Draw electric lines of forces due to an electric dipole.*

Solution Electric lines of forces due to an electric dipole are as shown in figure.

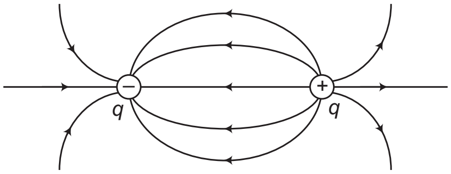


Fig. 24.50

Example 24.31 *Along the axis of a dipole, direction of electric field is always in the direction of electric dipole moment* **p**. *Is this statement true or false?*

Solution False. In the above figure, we can see that direction of electric field is in the opposite direction of **p** between the two charges.

Example 24.32 *At a far away distance r along the axis from an electric dipole electric field is* **E**. *Find the electric field at distance 2r along the perpendicular bisector.*

Solution Along the axis of dipole,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3} \qquad \text{...(i)}$$

This electric field is in the direction of **p**. Along the perpendicular bisector at a distance $2r$,

$$E' = \frac{1}{4\pi\varepsilon_0}\frac{p}{(2r)^3} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$E' = \frac{E}{16}$$

Moreover, E' is in the opposite direction of **p**. Hence,

$$\mathbf{E}' = -\frac{\mathbf{E}}{16}$$ **Ans.**

24.12 Gauss's Law

Gauss's law is a tool of simplifying electric field calculations where there is symmetrical distribution of charge. Many physical systems have symmetry, for example a cylindrical body doesn't look any different if we rotate it around its axis.

Before studying the detailed discussion of Gauss's law let us understand electric flux.

Electric Flux (ϕ)

(i) Electric flux is a measure of the field lines crossing a surface.

(ii) It is a scalar quantity with SI units $\frac{\text{N}}{\text{C}}\text{-}\,\text{m}^2$ or V- m.

(iii) Electric flux passing through a small surface dS is given by

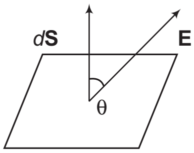


Fig. 24.51

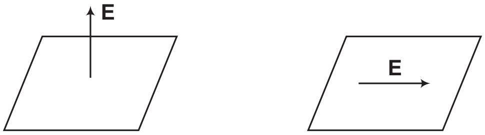
$$\boxed{d\phi = \mathbf{E} \cdot d\mathbf{S} = E\,dS \cos\theta} \qquad \ldots\text{(i)}$$

Here, $d\mathbf{S}$ is an area vector, whose magnitude is equal to dS and whose direction is perpendicular to the surface.

Note *If the surface is open, then d**S** can be taken in either of the two directions perpendicular to the surface, but it should not change even if we rotate the surface.*

*If the surface is closed then by convention, d**S** is normally taken in outward direction.*

(iv) From Eq. (i), we can see that maximum value of $d\phi$ is $E\,dS$, if $\theta = 90°$ or electric lines are perpendicular to the surface. Electric flux is zero, if $\theta = 90°$ or electric lines are tangential to the surface.



$d\phi = E\,dS$ $\qquad$ $d\phi = 0$

Fig. 24.52

(v) Electric flux passing through a large surface is given by

$$\boxed{\phi = \int d\phi = \int \mathbf{E} \cdot d\mathbf{S} = \int E\,dS \cos\theta} \qquad \ldots\text{(ii)}$$

This is basically surface integral of electric flux over the given surface. But normally we do not study surface integral in detail in physics.

Here, are two special cases for calculating the electric flux passing through a surface S of finite size (whether closed or open)

Case 1 $\boxed{\phi = ES}$

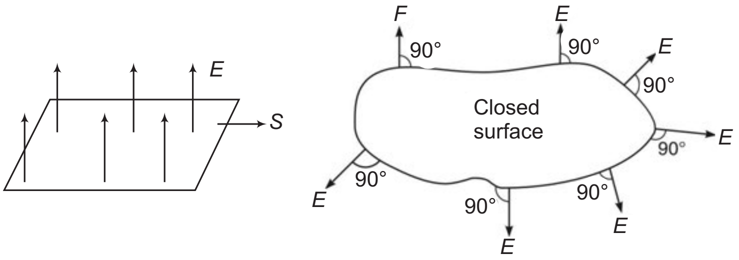


Fig. 24.53

If at every point on the surface, the magnitude of electric field is constant and perpendicular (to the surface).

Case 2

$$\boxed{\phi = 0}$$

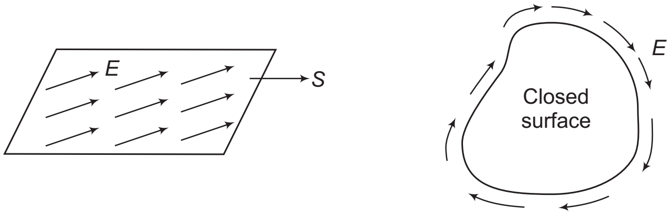


Fig. 24.54

If at all points on the surface the electric field is tangential to the surface.

Gauss's Law

This law gives a relation between the net electric flux through a closed surface and the charge enclosed by the surface. According to this law,

"the net electric flux through any closed surface is equal to the net charge inside the surface divided by ε_0." In symbols, it can be written as

$$\boxed{\phi_e = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0}} \qquad \text{...(i)}$$

where, q_{in} represents the net charge inside the closed surface and **E** represents the electric field at any point on the surface.

In principle, Gauss's law is valid for the electric field of any system of charges or continuous distribution of charge. In practice however, the technique is useful for calculating the electric field only in situations where the degree of symmetry is high. Gauss's law can be used to evaluate the electric field for charge distributions that have spherical, cylindrical or plane symmetry.

Simplified Form of Gauss's Theorem

Gauss's law in simplified form can be written as under

$$\boxed{ES = \frac{q_{\text{in}}}{\varepsilon_0} \quad \text{or} \quad E = \frac{q_{\text{in}}}{S\varepsilon_0}} \qquad \text{...(ii)}$$

but this form of Gauss's law is applicable only under the following two conditions :

(i) The electric field at every point on the surface is either perpendicular or tangential.

(ii) Magnitude of electric field at every point where it is perpendicular to the surface has a constant value (say E).

Here, S is the area where electric field is perpendicular to the surface.

Applications of Gauss's Law

As Gauss's law does not provide expression for electric field but provides only for its flux through a closed surface. To calculate E we choose an imaginary closed surface (called Gaussian surface) in which Eq. (ii) can be applied easily. Let us discuss few simple cases.

Electric field due to a point charge

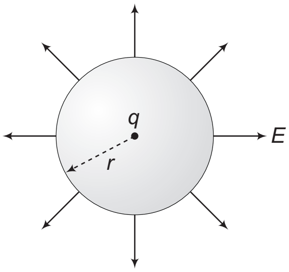


Fig. 24.55

The electric field due to a point charge is everywhere radial. We wish to find the electric field at a distance r from the charge q. We select Gaussian surface, a sphere at distance r from the charge. At every point of this sphere the electric field has the same magnitude E and it is perpendicular to the surface itself. Hence, we can apply the simplified form of Gauss's law,

$$ES = \frac{q_{\text{in}}}{\varepsilon_0}$$

Here, S = area of sphere = $4\pi r^2$ and

q_{in} = net charge enclosing the Gaussian surface = q

$$\therefore \qquad E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$\therefore \qquad \boxed{E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}}$$

It is nothing but Coulomb's law.

Electric field due to a linear charge distribution

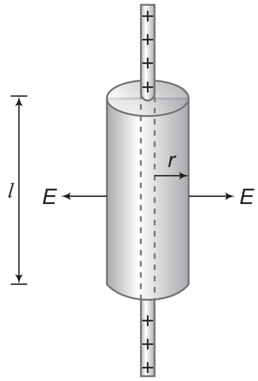


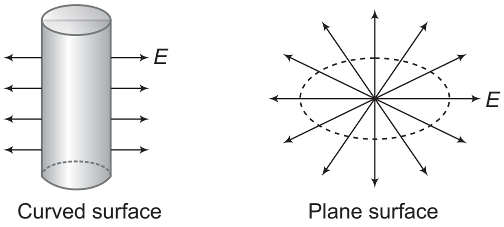
Fig. 24.56

Consider a long line charge with a linear charge density (charge per unit length) λ. We have to calculate the electric field at a point, a distance r from the line charge. We construct a Gaussian surface, a cylinder of any arbitrary length l of radius r and its axis coinciding with the axis of the line charge. This cylinder have three surfaces. One is curved surface and the two plane parallel surfaces. Field lines at plane parallel surfaces are tangential (so flux passing through these surfaces is zero). The magnitude of electric field is having the same magnitude (say E) at curved surface and simultaneously the electric field is perpendicular at every point of this surface.

Hence, we can apply the Gauss's law as

$$ES = \frac{q_{\text{in}}}{\varepsilon_0}$$

Here, S = area of curved surface = $(2\pi rl)$



Curved surface Plane surface

Fig. 24.57

and q_{in} = net charge enclosing this cylinder = λl

$$\therefore \qquad E\,(2\pi r l) = \frac{\lambda l}{\varepsilon_0}$$

$$\therefore \qquad \boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

i.e. $$E \propto \frac{1}{r}$$

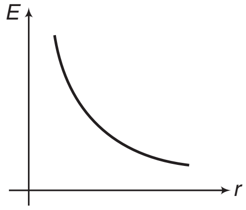


Fig. 24.58

or E-r graph is a rectangular hyperbola as shown in Fig. 24.58.

Electric field due to a plane sheet of charge

Figure shows a portion of a flat thin sheet, infinite in size with constant surface charge density σ (charge per unit area). By symmetry, since the sheet is infinite, the field must have the same magnitude and the opposite directions at two points equidistant from the sheet on opposite sides. Let us draw a Gaussian surface (a cylinder) with one end on one side and other end on the other side and of cross-sectional area S_0. Field lines will be tangential to the curved surface, so flux passing through this surface is zero. At plane surfaces electric field has same magnitude and perpendicular to surface.

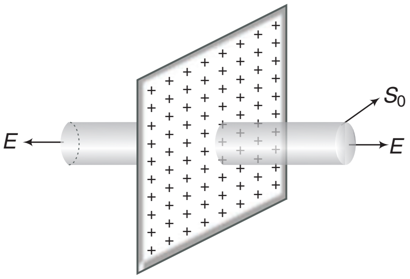


Fig. 24.59

Hence, using $$ES = \frac{q_{\text{in}}}{\varepsilon_0}$$

$$\therefore \qquad E\,(2S_0) = \frac{(\sigma)\,(S_0)}{\varepsilon_0}$$

$$\therefore \qquad \boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

Thus, we see that the magnitude of the field is independent of the distance from the sheet. Practically, an infinite sheet of charge does not exist. This result is correct for real charge sheets if points under consideration are not near the edges and the distances from the sheet are small compared to the dimensions of sheet.

Electric field near a charged conducting surface

When a charge is given to a conducting plate, it distributes itself over the entire outer surface of the plate. The surface density σ is uniform and is the same on both surfaces if plate is of uniform thickness and of infinite size.

This is similar to the previous one the only difference is that this time charges are on both sides.

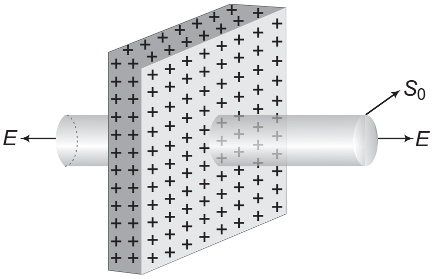


Fig. 24.60

Hence, applying $$ES = \frac{q_{\text{in}}}{\varepsilon_0}$$

Here, $$S = 2S_0 \quad \text{and} \quad q_{\text{in}} = (\sigma)(2S_0)$$

$$\therefore \qquad E(2S_0) = \frac{(\sigma)(2S_0)}{\varepsilon_0}$$

$$\therefore \qquad \boxed{E = \frac{\sigma}{\varepsilon_0}}$$

Thus, field due to a charged conducting plate is twice the field due to plane sheet of charge. It also has same limitations.

Later, we will see that the electric field near a charged conducting surface of any shape is σ/ε_0 and it is normal to the surface.

Note *In case of closed symmetrical body with charge q at its centre, the electric flux linked with each half will be* $\frac{\phi}{2} = \frac{q}{2\varepsilon_0}$. *If the symmetrical closed body has n identical faces with point charge at its centre, flux linked with each face will be* $\frac{\phi}{n} = \frac{q}{n\,\varepsilon_0}$.

✓ *Extra Points to Remember*

- Net electric flux passing through a closed surface in uniform electric field is zero.

Example 24.33 *An electric dipole is placed at the centre of a sphere. Find the electric flux passing through the sphere.*

Solution Net charge inside the sphere $q_{\text{in}} = 0$. Therefore, according to Gauss's law net flux passing through the sphere is zero. **Ans.**

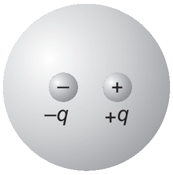


Fig. 24.61

Example 24.34 *A point charge q is placed at the centre of a cube. What is the flux linked*

(a) with all the faces of the cube?

(b) with each face of the cube?

(c) if charge is not at the centre, then what will be the answers of parts (a) and (b)?

Solution (a) According to Gauss's law,

$$\phi_{\text{total}} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \qquad \textbf{Ans.}$$

(b) The cube is a symmetrical body with 6 faces and the point charge is at its centre, so electric flux linked with each face will be

$$\phi_{\text{each face}} = \frac{\phi_{\text{total}}}{6} = \frac{q}{6\varepsilon_0} \qquad \textbf{Ans.}$$

(c) If charge is not at the centre, the answer of part (a) will remain same while that of part (b) will change.

INTRODUCTORY EXERCISE 24.7

1. In figure (a), a charge q is placed just outside the centre of a closed hemisphere. In figure (b), the same charge q is placed just inside the centre of the closed hemisphere and in figure (c), the charge is placed at the centre of hemisphere open from the base. Find the electric flux passing through the hemisphere in all the three cases.

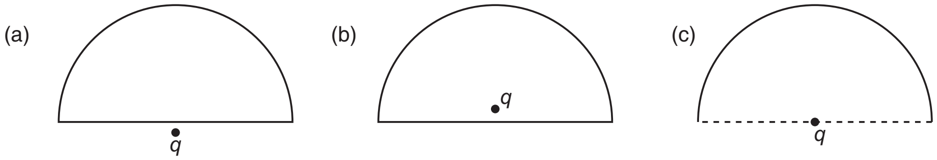


Fig. 24.62

2. Net charge within an imaginary cube drawn in a uniform electric field is always zero. Is this statement true or false?
3. A hemispherical body of radius R is placed in a uniform electric field E. What is the flux linked with the curved surface if, the field is (a) parallel to the base, (b) perpendicular to the base.
4. A cube has sides of length $L = 0.2$ m. It is placed with one corner at the origin as shown in figure. The electric field is uniform and given by $\mathbf{E} = (2.5\text{ N/C})\,\hat{\mathbf{i}} - (4.2\text{ N/C})\,\hat{\mathbf{j}}$. Find the electric flux through the entire cube.

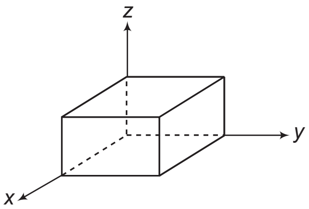


Fig. 24.63

24.13 Properties of a Conductor

Conductors (such as metals) possess free electrons. If a resultant electric field exists in the conductor these free charges will experience a force which will set a current flow. When no current flows, the resultant force and the electric field must be zero. Thus, under **electrostatic conditions** the value of **E** at all points within a conductor is zero. This idea, together with the Gauss's law can be used to prove several interesting facts regarding a conductor.

Excess Charge on a Conductor Resides on its Outer Surface

Consider a charged conductor carrying a charge q and no currents are flowing in it. Now, consider a Gaussian surface inside the conductor everywhere on which $\mathbf{E} = 0$.

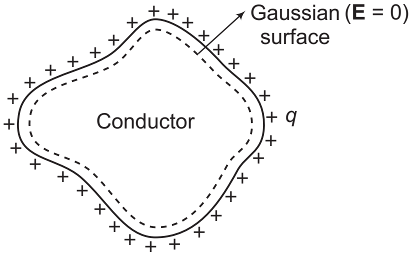


Fig. 24.64

Thus, from Gauss's law,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0}$$

We get, $$q_{\text{in}} = 0, \text{ as } \mathbf{E} = 0$$

Thus, the sum of all charges inside the Gaussian surface is zero. This surface can be taken just inside the surface of the conductor, hence, any charge on the conductor must be on the surface of the conductor. In other words,

"Under electrostatic conditions, the excess charge on a conductor resides on its outer surface."

Electric Field at Any Point Close to the Charged Conductor is $\frac{\sigma}{\varepsilon_0}$

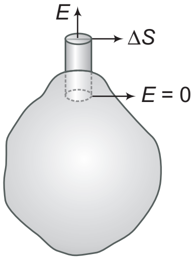


Fig. 24.65

Consider a charged conductor of irregular shape. In general, surface charge density will vary from point to point. At a small surface ΔS, let us assume it to be a constant σ. Let us construct a Gaussian surface in the form of a cylinder of cross-section ΔS. One plane face of the cylinder is inside the conductor and other outside the conductor close to it. The surface inside the conductor does not contribute to the flux as **E** is zero everywhere inside the conductor. The curved surface outside the conductor also does not contribute to flux as **E** is always normal to the charged conductor and hence parallel to the curved surface. Thus, the only contribution to the flux is through the plane face outside the conductor. Thus, from Gauss's law,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0}$$

or
$$E\Delta S = \frac{(\sigma)(\Delta S)}{\varepsilon_0}$$

or
$$\boxed{E = \frac{\sigma}{\varepsilon_0}}$$

Note *(i) Electric field changes discontinuously at the surface of a conductor. Just inside the conductor it is zero and just outside the conductor it is* $\frac{\sigma}{\varepsilon_0}$*. In fact, the field gradually decreases from* $\frac{\sigma}{\varepsilon_0}$ *to zero in a small thickness of about 4 to 5 atomic layers at the surface.*

(ii) For a non-uniform conductor the surface charge density (σ) *varies inversely as the radius of curvature* (ρ) *of that part of the conductor, i.e.*

$$\boxed{\sigma \propto \frac{1}{\textit{Radius of curvature}\ (\rho)}}$$

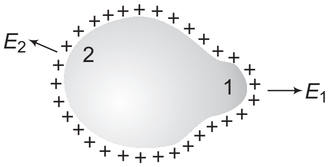


Fig. 24.66

For example in the figure, $\rho_1 < \rho_2 \quad \therefore \quad \sigma_1 > \sigma_2$

or $E_1 > E_2 \quad$ *as* $\quad E = \frac{\sigma}{\varepsilon_0}$

Electric Field and Field Lines are Normal to the Surface of a Conductor

Net field inside a conductor is zero. It implies that no field lines enter a conductor. On the surface of a conductor, electric field and hence field lines are normal to the surface of the conductor.

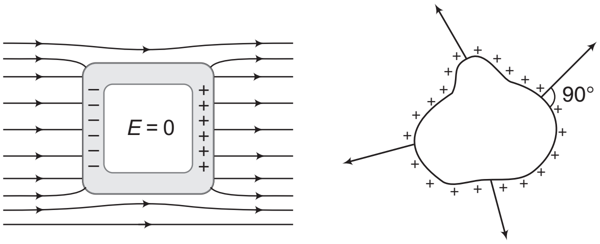


Fig. 24.67

If a conducting box is immersed in a uniform electric field, the field lines near the box are somewhat distorted. Similarly, if a conductor is positively charged, the field lines originate from the surface and are normal at every point and if it is negatively charged the field lines terminate on the surface normally at every point.

Cavity Inside a Conductor

Consider a charge $+q$ suspended in a cavity in a conductor. Consider a Gaussian surface just outside the cavity and inside the conductor. $\mathbf{E} = 0$ on this Gaussian surface as it is inside the conductor. Hence, from Gauss's law,

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0} \quad \text{gives } q_{\text{in}} = 0$$

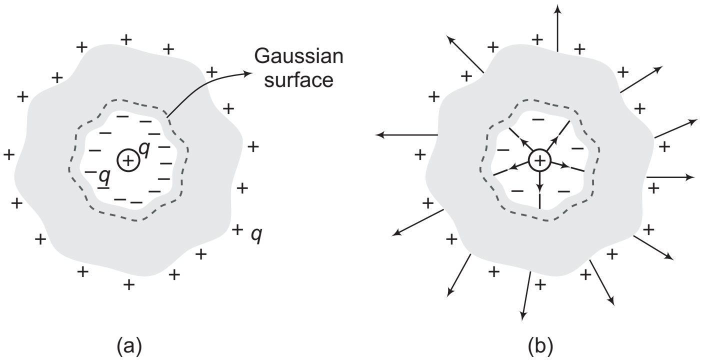


Fig. 24.68

This concludes that a charge of $-q$ must reside on the metal surface of the cavity so that the sum of this induced charge $-q$ and the original charge $+q$ within the Gaussian surface is zero. In other words, a charge q suspended inside a cavity in a conductor induces an equal and opposite charge $-q$ on the surface of the cavity. Further as the conductor is electrically neutral a charge $+q$ is induced on the outer surface of the conductor. As field inside the conductor is zero, the field lines coming from q cannot penetrate into the conductor. The field lines will be as shown in Fig. (b).

The same line of approach can be used to show that the field inside the cavity of a conductor is zero when no charge is suspended in it.

Electrostatic shielding

Suppose we have a very sensitive electronic instrument that we want to protect from external electric fields that might cause wrong measurements. We surround the instrument with a conducting box or we keep the instrument inside the cavity of a conductor. By doing this charge in the conductor is so distributed that the net electric field inside the cavity becomes zero and the instrument is protected from the external fields. This is called electronic shielding.

The Potential of a Charged Conductor Throughout its Volume is Same

In any region in which $\mathbf{E} = 0$ at all points, such as the region very far from all charges or the interior of a charged conductor, the line integral of $\mathbf{E}$ is zero along any path. It means that the potential difference between any two points in the conductor are at the same potential or the interior of a charged conductor is an equipotential region.

24.14 Electric Field and Potential Due to Charged Spherical Shell or Solid Conducting Sphere

Electric Field

At all points inside the charged spherical conductor or hollow spherical shell, electric field $\mathbf{E} = 0$, as there is no charge inside such a sphere. In an isolated charged spherical conductor any excess charge on it is distributed uniformly over its outer surface same as that of charged spherical shell or hollow sphere. The field at external points has the same symmetry as that of a point charge. We can construct a Gaussian surface (a sphere) of radius $r > R$. At all points of this sphere the magnitude of electric field is the same and its direction is perpendicular to the surface.

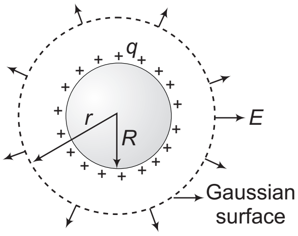


Fig. 24.69

Thus, we can apply

$$ES = \frac{q_{\text{in}}}{\varepsilon_0} \quad \text{or} \quad E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$\therefore \qquad E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

Hence, the electric field at any external point is the same as if the total charge is concentrated at centre.

At the surface of sphere $r = R$,

$$\therefore \qquad E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}$$

Thus, we can write

$$\boxed{\begin{aligned} E_{\text{inside}} &= 0 \\ E_{\text{surface}} &= \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \\ E_{\text{outside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \end{aligned}}$$

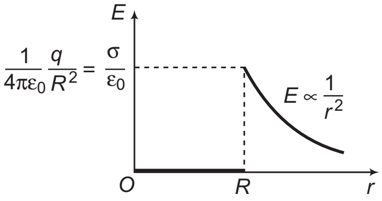


Fig. 24.70

The variation of electric field (E) with the distance from the centre (r) is as shown in Fig. 24.70.

Note *(i) At the surface graph is discontinuous*

(ii) $E_{\text{surface}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2} = \frac{q/4\pi R^2}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

Potential

As we have seen,
$$E_{\text{outside}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

$$\therefore \quad \left(\frac{-dV_{\text{outside}}}{dr}\right) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \qquad \left(E = -\frac{dV}{dr}\right)$$

$$\therefore \quad \int_0^V dV_{\text{outside}} = \frac{-q}{4\pi\varepsilon_0} \int_\infty^r \frac{dr}{r^2} \qquad (V_\infty = 0)$$

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \quad \text{or} \quad V \propto \frac{1}{r}$$

Thus, at external points, the potential at any point is the same when the whole charge is assumed to be concentrated at the centre. At the surface of the sphere, $r = R$

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$$

At some internal point electric field is zero everywhere, therefore, the potential is same at all points which is equal to the potential at surface. Thus, we can write

$$\boxed{\begin{aligned} V_{\text{inside}} = V_{\text{surface}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R} \\ \text{and} \quad V_{\text{outside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \end{aligned}}$$

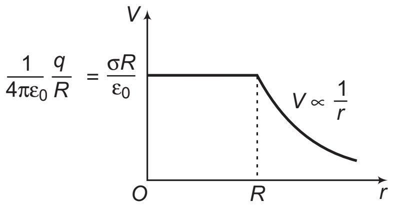


Fig. 24.71

The potential (V) varies with the distance from the centre (r) as shown in Fig. 24.71.

24.15 Electric Field and Potential Due to a Solid Sphere of Charge

Electric Field

Positive charge q is uniformly distributed throughout the volume of a solid sphere of radius R. For finding the electric field at a distance r ($< R$) from the centre let us choose as our Gaussian surface a sphere of radius r, concentric with the charge distribution. From symmetry, the magnitude E of electric field has the same value at every point on the Gaussian surface and the direction of **E** is radial at every point on the surface. So, applying Gauss's law

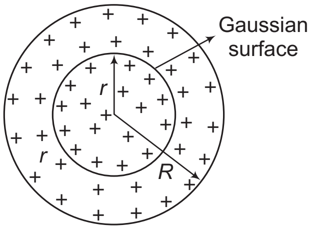


Fig. 24.72

$$ES = \frac{q_{\text{in}}}{\varepsilon_0} \qquad \ldots\text{(i)}$$

Here, $\quad S = 4\pi r^2 \quad \text{and} \quad q_{\text{in}} = (\rho)\left(\frac{4}{3}\pi r^3\right)$

Here, $\quad \rho = \text{charge per unit volume} = \dfrac{q}{\frac{4}{3}\pi R^3}$

Substituting these values in Eq. (i)

We have, $$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^3} \cdot r \quad \text{or} \quad E \propto r$$

At the centre $r = 0,$ so $E = 0$

At surface $r = R,$ so $E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}$

To find the electric field outside the charged sphere, we use a spherical Gaussian surface of radius $r\ (> R)$. This surface encloses the entire charged sphere, so $q_{\text{in}} = q$, and Gauss's law gives

$$E\ (4\pi r^2) = \frac{q}{\varepsilon_0} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

or $$E \propto \frac{1}{r^2}$$

Notice that if we set $r = R$ in either of the two expressions for E (outside and inside the sphere), we get the same result, $$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}$$

this is because E is **continuous function** of r in this case. By contrast, for the charged conducting sphere the magnitude of electric field is discontinuous at $r = R$ (it jumps from $E = 0$ to $E = \sigma/\varepsilon_0$).

Thus, for a uniformly charged solid sphere we have the following formulae for magnitude of electric field :

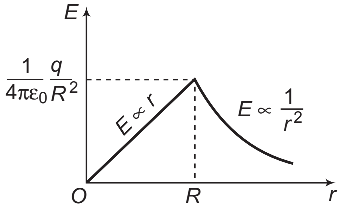


Fig. 24.73

$$\boxed{\begin{aligned} E_{\text{inside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^3} \cdot r \\ E_{\text{surface}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2} \\ E_{\text{outside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \end{aligned}}$$

The variation of electric field (E) with the distance from the centre of the sphere (r) is shown in Fig. 24.73.

Potential

The field intensity outside the sphere is

$$E_{\text{outside}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

$$\frac{dV_{\text{outside}}}{dr} = -E_{\text{outside}}$$

$\therefore$ $$dV_{\text{outside}} = -E_{\text{outside}}\ dr$$

or
$$\int_{\infty}^{V} dV_{\text{outside}} = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\, dr$$

$\therefore$
$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \quad \text{as} \quad V_{\infty} = 0 \quad \text{or} \quad V \propto \frac{1}{r}$$

At $r = R$,
$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$$

i.e. at the surface of the sphere potential is $V_S = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$

The electric intensity inside the sphere,
$$E_{\text{inside}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^3} \cdot r$$

$$\frac{dV_{\text{inside}}}{dr} = -E_{\text{inside}}$$

$\therefore$
$$dV_{\text{inside}} = -E_{\text{inside}}\, dr$$

$\therefore$
$$\int_{V_S}^{V} dV_{\text{inside}} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^3} \int_{R}^{r} r\, dr$$

$\therefore$
$$V - V_S = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^3} \left[\frac{r^2}{2}\right]_R^r$$

Substituting $V_S = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$, we get

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} (1.5\, R^2 - 0.5\, r^2)$$

At the centre $r = 0$ and $V_c = \frac{3}{2}\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}\right) = \frac{3}{2} V_s$, i.e. potential at the centre is 1.5 times the potential at surface.

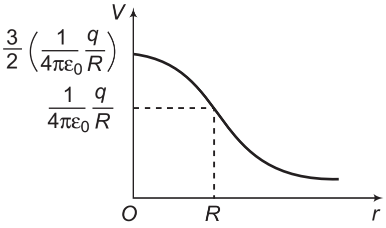


Fig. 24.74

Thus, for a uniformly charged solid sphere we have the following formulae for potential :

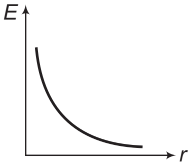
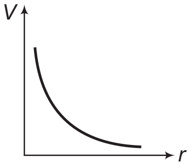
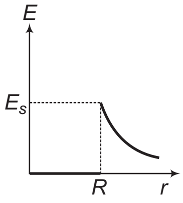
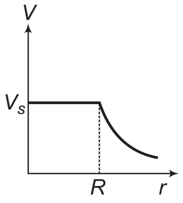
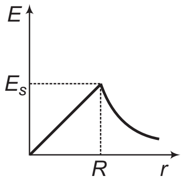
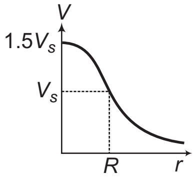
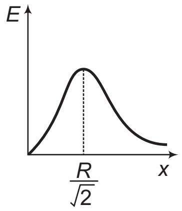
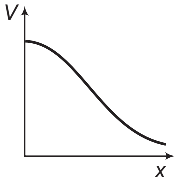
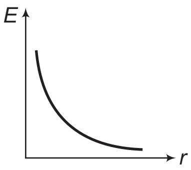
$$\boxed{\begin{aligned} V_{\text{outside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \\ V_{\text{surface}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R} \\ V_{\text{inside}} &= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}\left[\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right] \end{aligned}}$$

and

The variation of potential (V) with distance from the centre (r) is as shown in Fig. 24.74. For inside points variation is parabolic.

List of formulae for field strength *E* and potential *V* $\left(k = \dfrac{1}{4\pi\varepsilon_0}\right)$

Table 24.1

S.No.	Charge Distribution	*E*		*V*	
		Formula	Graph	Formula	Graph
1.	Point charge	$E = \dfrac{kq}{r^2}$		$V = \dfrac{kq}{r}$	
2.	Uniformly charged spherical shell	$E_i = 0$ $E_s = k \cdot \dfrac{q}{R^2} = \dfrac{\sigma}{\varepsilon_0}$ $E_o = \dfrac{kq}{r^2}$		$V_i = V_s = \dfrac{kq}{R}$ $= \dfrac{\sigma R}{\varepsilon_0}$ $V_o = \dfrac{kq}{r}$	
3.	Solid sphere of charge	$E_i = \dfrac{kqr}{R^3}$ $E_s = \dfrac{kq}{R^2}$ $E_o = \dfrac{Kq}{r^2}$		$V_i = \dfrac{kq}{R^3}(1.5\,R^2 - 0.5\,r^2)$ $V_s = \dfrac{kq}{R}$ $V_o = \dfrac{kq}{r}$	
4.	On the axis of uniformly charged ring	$E = \dfrac{kqx}{(R^2 + x^2)^{3/2}}$ At centre $x = 0$ $\therefore \quad E = 0$		$V = \dfrac{kq}{\sqrt{R^2 + x^2}}$ At centre $x = 0$ $\therefore \quad V = \dfrac{kq}{R}$	
5.	Infinitely long line charge	$E = \dfrac{\lambda}{2\pi\,\varepsilon_0 r}$		$\text{PD} = \dfrac{\lambda}{2\pi\,\varepsilon_0} \ln\left(\dfrac{r_2}{r_1}\right)$	Not required

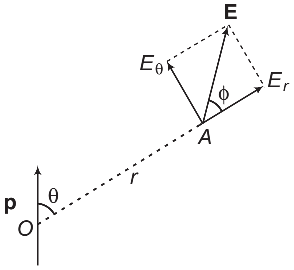
Final Touch Points

1. **Permittivity** Permittivity or absolute permittivity is a measure of resistance that is encountered when forming an electric field in a medium. Thus, permittivity relates to a material's ability to resist an electric field (while unfortunately, the word "permit" suggests the inverse quantity).

   The permittivity of a medium describes how much electric field (more correctly, flux) is generated per unit charge in that medium. More electric flux (per unit charge) exists in a medium with a low permittivity. Vacuum has the lowest permittivity (therefore maximum electric flux per unit charge). Any other dielectric medium has K -times (K = dielectric constant) the permittivity of vacuum. This is because, due to polarization effects electric flux per unit charge deceases K - times ($K > 1$).

2. **Dielectric constant (K)** Also known as relative permittivity of a given material is the ratio of permittivity of the material to the permittivity of vacuum. This is the factor by which the electric force between the two charges is decreased relative to vacuum. Similarly, in the chapter of capacitors we will see that it is the ratio of capacitance of a capacitor using that material as a dielectric compared to a similar capacitor that has vacuum as its dielectric.

3. **Electric field and potential due to a dipole at polar coordinates (r, θ)**



or
$$\boxed{V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

The electric field **E** can be resolved into two components E_r and E_θ, where

or
$$\boxed{E_r = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2p\cos\theta}{r^3}}$$

and
$$\boxed{E_\theta = \frac{1}{4\pi\varepsilon_0}\,\frac{p\sin\theta}{r^3}}$$

The magnitude of resultant electric field $E = \sqrt{E_r^2 + E_\theta^2}$

or
$$\boxed{E = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{1 + 3\cos^2\theta}}$$

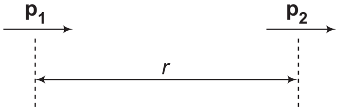
Its inclination ϕ to OA is given by

$$\tan\phi = \frac{E_\theta}{E_r} = \frac{p\sin\theta/4\pi\varepsilon_0 r^3}{2p\cos\theta/4\pi\varepsilon_0 r^3}$$

or
$$\boxed{\tan\phi = \frac{\tan\theta}{2}}$$

4. **Force between two dipoles** The force between two dipoles varies inversely with the fourth power of the distance between their centres or

$$F \propto \frac{1}{r^4}$$



In the, figure, a dipole on left with dipole moment $\mathbf{p}_1$ interacts with the dipole on the right with dipole moment $\mathbf{p}_2$. We assume that the distance between them is quite large. The electric field of the dipole on the left hand side exerts a net force on the dipole on the right hand side. Let us now calculate the net force on the dipole on right hand side.

The electric field at the centre of this dipole

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p_1}{r^3}$$

$$\therefore \qquad dE = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{6p_1}{r^4}\, dr$$

Now, the electric field at the point where $-q$ charge of the dipole lies is given by

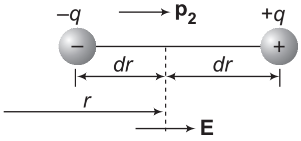
$$E_1 = E + |dE|$$

and force on $-q$ is qE_1 (towards left)

Similarly, electric field at the point where $+q$ charge of the dipole lies is

$$E_2 = E - |dE|$$

and force on $+q$ is qE_2 (towards right)



$\therefore$ Net force on the dipole is

$$F = q\,E_1 - q\,E_2 \qquad \text{(towards left)}$$

$$= 2q\,|dE|$$

$$= \frac{6\,(2qdr)\,p_1}{4\pi\varepsilon_0\, r^4}$$

or

$$F = \frac{6p_1p_2}{4\pi\varepsilon_0\, r^4} \qquad [\text{as } 2q\,(dr) = p_2]$$

Thus, if $\mathbf{p}_1 \,||\, \mathbf{p}_2$, the two dipoles attract each other with a force given by the above relation.

5. **Earthing a conductor** Potential of earth is often taken to be zero. If a conductor is connected to the earth, the potential of the conductor becomes equal to that of the earth, i.e. zero. If the conductor was at some other potential, charges will flow from it to the earth or from the earth to it to bring its potential to zero.

Solved Examples

TYPED PROBLEMS

Type 1. *To find electric potential due to charged spherical shells*

Concept

To find the electric potential due to a conducting sphere (or shell) we should keep in mind the following two points

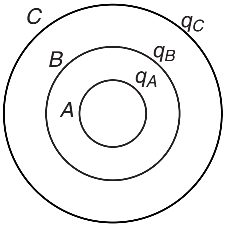
(i) Electric potential on the surface and at any point inside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R} \qquad (R = \text{radius of sphere})$$

(ii) Electric potential at any point outside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \qquad (r = \text{distance of the point from the centre})$$

For example, in the figure shown, potential at A is

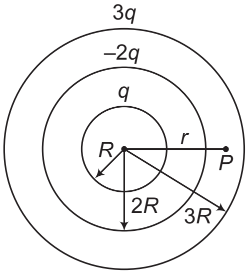


$$V_A = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_A} + \frac{q_B}{r_B} + \frac{q_C}{r_C}\right]$$

Similarly, potential at B is $$V_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_B} + \frac{q_B}{r_B} + \frac{q_C}{r_C}\right]$$

and potential at C is, $$V_C = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_C} + \frac{q_B}{r_C} + \frac{q_C}{r_C}\right]$$

Example 1 *Three conducting spherical shells have charges q, $-2q$ and $3q$ as shown in figure. Find electric potential at point P as shown in figure.*



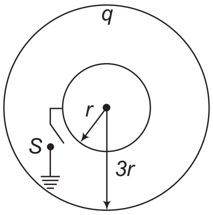
Solution Potential at P,

$$V_P = V_q + V_{-2q} + V_{3q}$$
$$= \frac{kq}{r} - \frac{k(2q)}{r} + \frac{k(3q)}{3R}$$

$$= kq\left(\frac{1}{R} - \frac{1}{r}\right)$$ **Ans.**

Here, $$k = \frac{1}{4\pi\varepsilon_0}$$

Example 2 *Figure shows two conducting thin concentric shells of radii r and 3r. The outer shell carries a charge q. Inner shell is neutral. Find the charge that will flow from inner shell to earth after the switch S is closed.*



Solution Let q' be the charge on inner shell when it is earthed.

Potential of inner shell is zero.

$\therefore$ $$\frac{1}{4\pi\varepsilon_0}\left[\frac{q'}{r} + \frac{q}{3r}\right] = 0$$

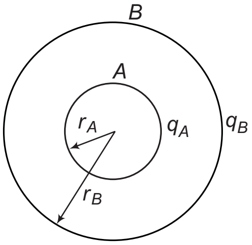
$\therefore$ $$q' = -\frac{q}{3}$$

i.e. $+\frac{q}{3}$ charge will flow from inner shell to earth. **Ans.**

Type 2. *Based on the principle of generator*

Concept

A generator is an instrument for producing high voltages in the million volt region. Its design is based on the principle that if a charged conductor (say A) is brought into contact with a hollow conductor (say B), all of its charge transfers to the hollow conductor no matter how high the potential of the later may be. This can be shown as under:

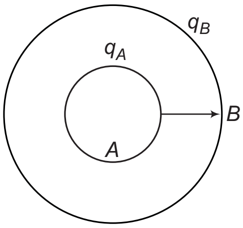


In the figure, $$V_A = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_A} - \frac{q_B}{r_B}\right]$$

and $$V_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_B} - \frac{q_B}{r_B}\right]$$

$$\therefore \qquad V_A - V_B = \frac{q_A}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

From this expression the following conclusions can be drawn :

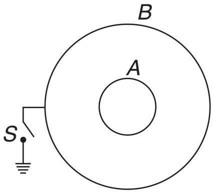


(i) The potential difference (PD) depends on q_A only. It does not depend on q_B.

(ii) If q_A is positive, then $V_A - V_B$ is positive (as $r_A < r_B$), i.e. $V_A > V_B$. So if the two spheres are connected by a conducting wire charge flows from inner sphere to outer sphere (positive charge flows from higher potential to lower potential) till $V_A = V_B$ or $V_A - V_B = 0$. But potential difference will become zero only when $q_A = 0$, i.e. all charge q_A flows from inner sphere to outer sphere.

(iii) If q_A is negative, $V_A - V_B$ is negative, i.e. $V_A < V_B$. Hence, when the two spheres are connected by a thin wire all charge q_A will flow from inner sphere to the outer sphere. Because negative charge flows from lower potential to higher potential. Thus, we see that the whole charge q_A flows from inner sphere to the outer sphere, no matter how high q_B is. Charge always flows from A to B, whether $q_A > q_B$ or $q_B > q_A$, $V_A > V_B$ or $V_B > V_A$.

Example 3 *Initially the spheres A and B are at potentials V_A and V_B. Find the potential of A when sphere B is earthed.*



Solution As we have studied above that the potential difference between these two spheres depends on the charge on the inner sphere only. Hence, the PD will remain unchanged because by earthing the sphere B charge on A remains constant. Let V'_A be the new potential at A. Then,

$$V_A - V_B = V'_A - V'_B$$

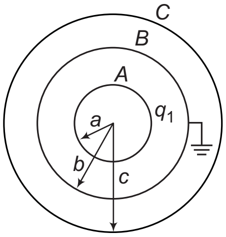
but $V'_B = 0$ as it is earthed. Hence,

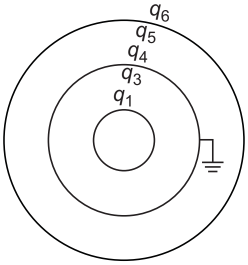
$$V'_A = V_A - V_B \qquad \textbf{Ans.}$$

Type 3. *Based on the charges appearing on different surfaces of concentric spherical shells*

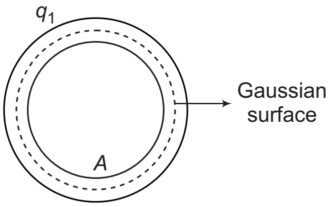
Concept

Figure shows three concentric thin spherical shells A, B and C of radii a, b and c. The shells A and C are given charges q_1 and q_2 and the shell B is earthed. We are interested in finding the charges on inner and outer surfaces of A, B and C. To solve such type of problems we should keep the following points in mind :





(i) The whole charge q_1 will come on the outer surface of A unless some charge is kept inside A. To understand it let us consider a Gaussian surface (a sphere) through the material of A. As the electric field in a conducting material is zero. The flux through this Gaussian surface is zero. Using Gauss's law, the total charge enclosed must be zero.



(ii) Similarly, if we draw a Gaussian surface through the material of B we can see that

$$q_3 + q_1 = 0 \quad \text{or} \quad q_3 = -q_1$$

and if we draw a Gaussian surface through the material of C, then

$$q_5 + q_4 + q_3 + q_1 = 0 \quad \text{or} \quad q_5 = -q_4$$

(iii) $q_5 + q_6 = q_2$. As q_2 charge was given to shell C.

(iv) Potential of B should be zero, as it is earthed. Thus,

$$V_B = 0$$

or

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{b} + \frac{q_3 + q_4}{b} + \frac{q_5 + q_6}{c}\right] = 0$$

So, using the above conditions we can find charges on different surfaces.

We can summarise the above points as under

1. Net charge inside a closed Gaussian surface drawn in any shell is zero. (provided the shell is conducting).

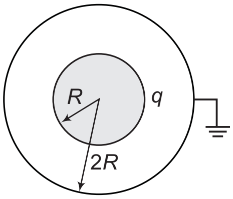
2. Potential of the conductor which is earthed is zero.

3. If two conductors are connected, they are at same potential.

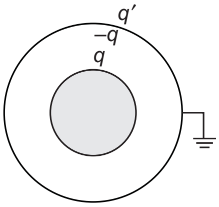
4. Charge remains constant in all conductors except those which are earthed.

5. Charge on the inner surface of the innermost shell is zero provided no charge is kept inside it. In all other shells charge resides on both the surfaces.

6. Equal and opposite charges appear on opposite faces.



Example 4 *A charge q is distributed uniformly on the surface of a solid sphere of radius R. It is covered by a concentric hollow conducting sphere of radius 2R. Find the charges on inner and outer surfaces of hollow sphere if it is earthed.*



Solution The charge on the inner surface of the hollow sphere should be $-q$, because if we draw a closed Gaussian surface through the material of the hollow sphere the total charge enclosed by this Gaussian surface should be zero. Let q' be the charge on the outer surface of the hollow sphere.

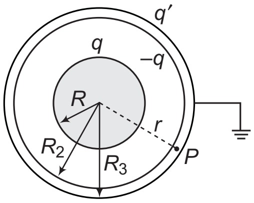
Since, the hollow sphere is earthed, its potential should be zero. The potential on it is due to the charges $q, -q$ and q', Hence,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{2R} - \frac{q}{2R} + \frac{q'}{2R}\right] = 0$$

$\therefore$ $$q' = 0$$ **Ans.**

Therefore, there will be no charge on the outer surface of the hollow sphere.

Example 5 *Solve the above problem if thickness of the hollow sphere is considerable.*



Solution In this case, we can set $V = 0$ at any point on the hollow sphere. Let us select a point P a distance r from the centre, were $R_2 < r < R_3$. So,

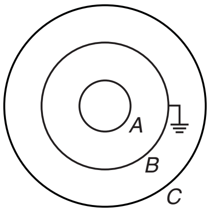
$$V_P = 0$$

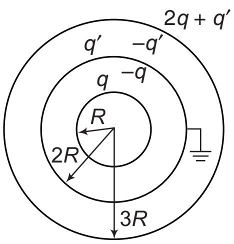
$\therefore$ $$\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} - \frac{q}{r} + \frac{q'}{R_3}\right] = 0$$

$\therefore$ $$q' = 0$$ **Ans.**

i.e. in this case also there will be no charge on the outer surface of the hollow sphere.

Example 6 *Figure shows three concentric thin spherical shells A, B and C of radii R, 2R and 3R. The shell B is earthed and A and C are given charges q and 2q, respectively. Find the charges appearing on all the surfaces of A, B and C.*





Solution Since, there is no charge inside A. The whole charge q given to the shell A will appear on its outer surface. Charge on its inner surface will be zero. Moreover if a Gaussian surface is drawn on the material of shell B, net charge enclosed by it should be zero. Therefore, charge on its inner surface will be $-q$. Now let q' be the charge on its outer surface, then charge on the inner surface of C will be $-q'$ and on its outer surface will be, $2q-(-q')=2q+q'$ as total charge on C is $2q$.

Shell B is earthed. Hence, its potential should be zero.

$$V_B = 0$$

$$\therefore \qquad \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{2R}-\frac{q}{2R}+\frac{q'}{2R}-\frac{q'}{3R}+\frac{2q+q'}{3R}\right]=0$$

Solving this equation, we get

$$q' = -\frac{4}{3}q$$

$$\therefore \qquad 2q+q' = 2q-\frac{4}{3}q=\frac{2}{3}q$$

Therefore, charges on different surfaces in tabular form are given below :

Table 24.2

	A	B	C
Inner surface	0	$-q$	$\frac{4}{3}q$
Outer surface	q	$-\frac{4}{3}q$	$\frac{2}{3}q$

Type 4. *Based on finding electric field due to spherical charge distribution*

Concept

According to Gauss's theorem, at a distance r from centre of sphere,

$$E=\frac{kq_{\text{in}}}{r^2} \qquad \left(k=\frac{1}{4\pi\varepsilon_0}\right)$$

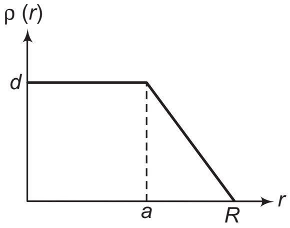
Here, q_{in} is the net charge inside the sphere of radius r . If volume charge density (say ρ) is constant, then

$$q_{\text{in}} = (\text{volume of sphere of radius } r)(\rho)=\frac{4}{3}\pi r^3\rho$$

If ρ is variable, then q_{in} can be obtained by integration.

Passage (Ex. 7 to Ex. 9)

The nuclear charge (Ze) is non-uniformly distributed within a nucleus of radius R. The charge density $\rho(r)$ (charge per unit volume) is dependent only on the radial distance r from the centre of the nucleus as shown in figure. The electric field is only along the radial direction.



Example 7 *The electric field at $r = R$ is* **(JEE 2008)**

(a) independent of a

(b) directly proportional to a

(c) directly proportional to a^2

(d) inversely proportional to a

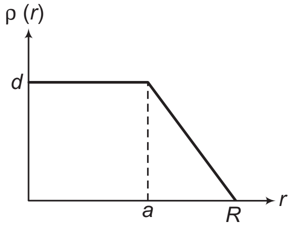
Solution At $r = R$, from Gauss's law

$$E\,(4\pi R^2) = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{Ze}{\varepsilon_0} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze}{R^2}$$

E is independent of a.

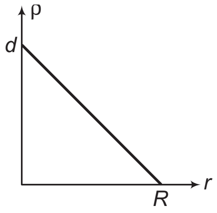
∴ The correct option is (a).

Example 8 *For $a = 0$, the value of d (maximum value of ρ as shown in the figure) is* **(JEE 2008)**



(a) $\frac{3Ze}{4\pi R^3}$ *(b)* $\frac{3Ze}{\pi R^3}$ *(c)* $\frac{4Ze}{3\pi R^3}$ *(d)* $\frac{Ze}{3\pi R^3}$

Solution For $a = 0$,



$$\rho(r) = \left(-\frac{d}{R}\cdot r + d\right)$$

Now, $$\int_0^R (4\pi r^2)\left(d - \frac{d}{R}r\right)dr = \text{net charge} = Ze$$

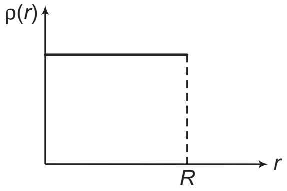
Solving this equation, we get $$d = \frac{3Ze}{\pi R^3}$$

∴ The correct option is (b).

Example 9 *The electric field within the nucleus is generally observed to be linearly dependent on r. This implies* **(JEE 2008)**

(a) $a = 0$ *(b)* $a = \frac{R}{2}$ *(c)* $a = R$ *(d)* $a = \frac{2R}{3}$

Solution In case of solid sphere of charge of uniform volume density



$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R^3}\cdot r \quad \text{or} \quad E \propto r$$

Thus, for E to be linearly dependent on r, volume charge density should be constant.

or $$a = R$$

∴ The correct option is (c).

Type 5. *Based on calculation of electric flux*

Concept

(i) To find electric flux from any closed surface, direct result of Gauss's theorem can be used,

$$\phi = \frac{q_{\text{in}}}{\varepsilon_0}$$

(ii) To find electric flux from an open surface, result of Gauss's theorem and concept of symmetry can be used.

(iii) To find electric flux from a plane surface in uniform electric field,

$$\phi = \mathbf{E} \cdot \mathbf{S} \quad \text{or} \quad ES \cos \theta$$

can be used.

(iv) Net electric flux from a closed surface in uniform electric field is always zero.

Example 10 *The electric field in a region is given by* $\mathbf{E} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. *Here, a and b are constants. Find the net flux passing through a square area of side l parallel to y-z plane.*

Solution A square area of side l parallel to y-z plane in vector form can be written as,

$$\mathbf{S} = l^2\hat{\mathbf{i}}$$

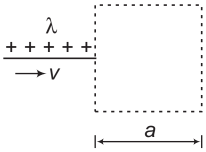
Given, $$\mathbf{E} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

∴ Electric flux passing through the given area will be,

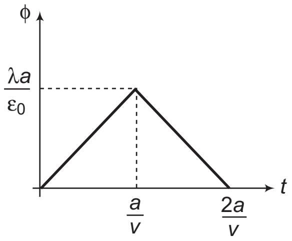
$$\phi = \mathbf{E} \cdot \mathbf{S}$$
$$= (a\hat{\mathbf{i}} + b\hat{\mathbf{j}}) \cdot (l^2\hat{\mathbf{i}})$$
$$= al^2$$

Ans.

Example 11 *Figure shows an imaginary cube of side a. A uniformly charged rod of length a moves towards right at a constant speed v. At t = 0, the right end of the rod just touches the left face of the cube. Plot a graph between electric flux passing through the cube versus time.*



Solution The electric flux passing through a closed surface depends on the net charge inside the surface. Net charge in this case first increases, reaches a maximum value and finally decreases to zero. The same is the case with the electric flux. The electric flux ϕ *versus* time graph is as shown in figure below.

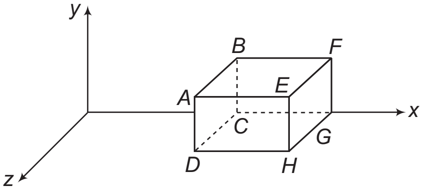


Example 12 *The electric field in a region is given by* $\mathbf{E} = \alpha x \hat{\mathbf{i}}$. *Here,* α *is a constant of proper dimensions. Find*

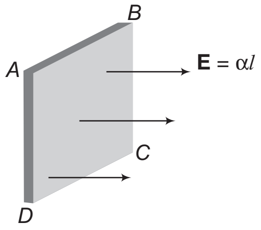
(a) the total flux passing through a cube bounded by the surfaces, $x = l,\ x = 2l,\ y = 0, y = l, z = 0\ , z = l.$

(b) the charge contained inside the above cube.

Solution (a) Electric field is along positive x-direction. Therefore, field lines are perpendicular to faces *ABCD* and *EFGH*. At all other four faces field lines are tangential. So, net flux passing through these four faces will be zero.



Flux entering at face *ABCD* At this face $x = l$



$$\therefore \qquad \mathbf{E} = \alpha l \hat{\mathbf{i}}$$

$\therefore$ Flux entering the cube from this face,

$$\phi_1 = \mathbf{ES} = (\alpha l)\,(l^2) = \alpha l^3$$

Flux leaving the face *EFGH* At this face $x = 2l$

$$\therefore \qquad \mathbf{E} = 2\alpha l \hat{\mathbf{i}}$$

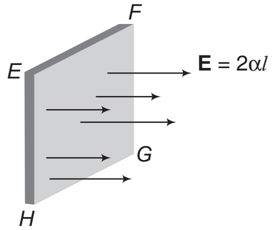
$\therefore$ Flux coming out of this face

$$\begin{aligned}\phi_2 &= \mathbf{ES} = (2\alpha l)\,(l^2)\\ &= 2\alpha l^3\end{aligned}$$

$\therefore$ Net flux passing through the cube,

$$\begin{aligned}\phi_{\text{net}} &= \phi_2 - \phi_1 = 2\alpha l^3 - \alpha l^3\\ &= \alpha l^3\end{aligned}$$

Ans.

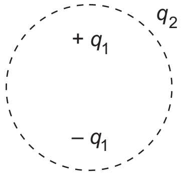


(b) From Gauss's law,

$$\begin{aligned}\phi_{\text{net}} &= \frac{q_{\text{in}}}{\varepsilon_0}\\ q_{\text{in}} &= (\phi_{\text{net}})\,(\varepsilon_0)\\ &= \alpha \varepsilon_0 l^3\end{aligned}$$

Ans.

Example 13 *Consider the charge configuration and a spherical Gaussian surface as shown in the figure. When calculating the flux of the electric field over the spherical surface, the electric field will be due to* **(JEE 2004)**



(a) q_2

(b) only the positive charges

(c) all the charges

(d) $+ q_1$ *and* $- q_1$

Solution At any point over the spherical Gaussian surface, net electric field is the vector sum of electric fields due to $+q_1, -q_1$ and q_2

$\therefore$ The correct option is (c)..

Note *Don't confuse with the electric flux which is zero (net) passing over the Gaussian surface as the net charge enclosing the surface is zero.*

Example 14 *A point charge q is placed on the top of a cone of semi vertex angle θ. Show that the electric flux through the base of the cone is* $\dfrac{q(1-\cos\theta)}{2\varepsilon_0}$.

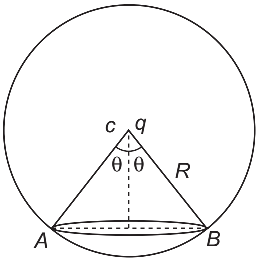
HOW TO PROCEED *This problem can be solved by the method of symmetry. Consider a Gaussian surface, a sphere with its centre at the top and radius the slant length of the cone. The flux through the whole sphere is* q/ε_0. *Therefore, the flux through the base of the cone can be calculated by using the following formula,*

$$\phi_e = \left(\frac{S}{S_0}\right)\cdot\frac{q}{\varepsilon_0}$$

Here, S_0 = *area of whole sphere*

and S = *area of sphere below the base of the cone.*

Solution Let R = slant length of cone = radius of Gaussian sphere



$\therefore$ S_0 = area of whole sphere = $(4\pi R^2)$

S = area of sphere below the base of the cone

$= 2\pi R^2 (1-\cos\theta)$

$\therefore$ The desired flux is,

$$\phi = \left(\frac{S}{S_0}\right)\cdot\frac{q}{\varepsilon_0}$$

$$= \frac{(2\pi R^2)(1-\cos\theta)}{(4\pi R^2)}\cdot\frac{q}{\varepsilon_0}$$

$$= \frac{q(1-\cos\theta)}{2\varepsilon_0}$$

Proved

Note *$S = 2\pi R^2 (1-\cos\theta)$ can be calculated by integration.*

At $\theta = 0°$, $S = 2\pi R^2 (1-\cos 0°) = 0$

$\theta = 90°$, $S = 2\pi R^2 (1-\cos 90°) = 2\pi R^2$

and $\theta = 180°$, $S = 2\pi R^2 (1-\cos 180°) = 4\pi R^2$

Proof

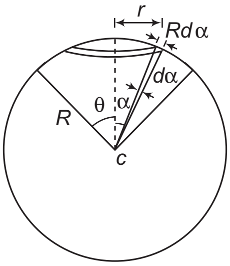
$$dS = (2\pi r)\, R d\alpha$$

$$= (2\pi R \sin\alpha)\, R d\alpha \qquad \text{as } r = R\sin\alpha$$

$$= (2\pi R^2) \sin\alpha \; d\alpha$$

$$\therefore \qquad S = \int_0^{\theta} (2\pi R^2) \sin\alpha \; d\alpha$$

$$\therefore \qquad S = 2\pi R^2 (1 - \cos\theta)$$



Students are advised to remember this result.

Type 6. *Based on E-r and V-r graphs due to two point charges*

Concept

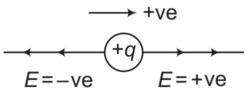
(i)
$$E = \frac{kq}{r^2} \qquad \left(k = \frac{1}{4\pi\varepsilon_0}\right)$$

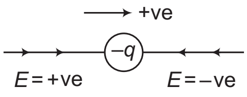
and
$$V = \pm \frac{kq}{r} \qquad \text{(due to a point charge)}$$

(ii) As $r \to 0$, $E \to \infty$ and $V \to \pm \infty$

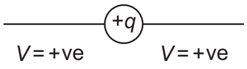
As $r \to \infty$, $E \to 0$ and $V \to 0$

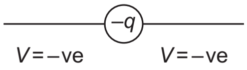
(iii) E is a vector quantity. Due to a point charge, its direction is away from the charge and due to negative charge it is towards the charge. Along one dimension if one direction is taken as positive direction then the other direction is taken as the negative direction.





(iv) V is a scalar quantity. On both sides of a positive charge it is positive and it is negative due to negative charge.

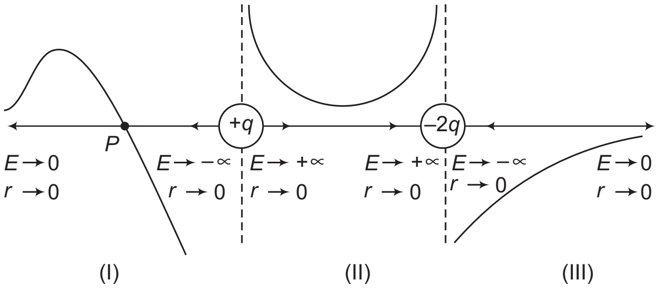




(v) Between zero and zero value, normally we get either a maximum or minimum value.

Example 15 *Draw E - r and V - r graphs due to two point charges $+q$ and $-2q$ kept at some distance along the line joining these two charges.*

Solution **E - r graph**

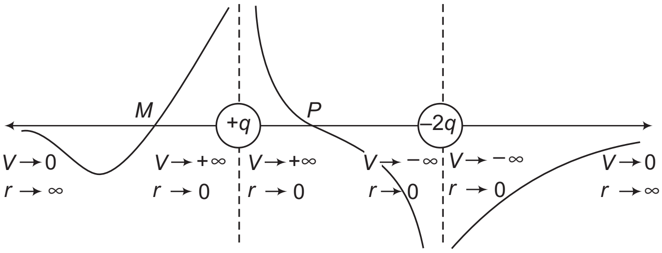


In region I E due to $+q$ is towards left (so negative) and E due to $-2q$ is towards right (so positive). Near $+q$, electric field of $+q$ will dominate. So, net value will be negative. At some point say P both positive and negative values are equal. So, $E_p = 0$. Beyond this point, electric field due to $-2q$ will dominate due to its higher magnitude. So, net value will be positive. $E_p = 0$ and E_∞ (towards left) is also zero. Between zero and zero we will get a maximum positive value.

In region II E due to $+q$ and due to $-2q$ is towards right (so positive). Between the value $+\infty$ and $+\infty$ the graph is as shown in figure.

In region III E due to $+q$ is towards right (so positive) and E due to $-2q$ is towards left (so negative). But electric field of $-2q$ will dominate due to its higher magnitude and lesser distance. Hence, net electric field is always negative.

V - r graph



The logics developed in E - r graph can also be applied here with V - r graph. At point P, positive potential due to $+q$ is equal to negative potential due to $-2q$. Hence, $V_p = 0$, so this point is near $2q$. Same is the case at M.

Type 7. *E - r and V - r graphs due to charged spherical shells of negligible thickness*

Concept

According to Gauss's theorem,

$$E = \frac{kq_{\text{in}}}{r^2} \qquad \left(k = \frac{1}{4\pi\varepsilon_0}\right)$$

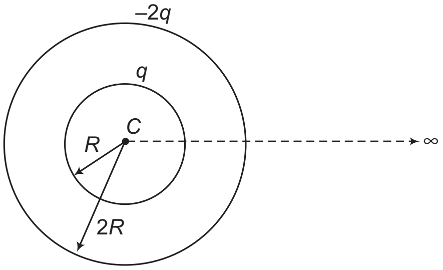
So, only inside charges contribute in the electric field.

$$V = \frac{kq}{R} = \text{constant} \qquad \text{(inside the shell)}$$

$$V = \frac{kq}{r} \neq \text{constant} \qquad \text{(outside the shell)}$$

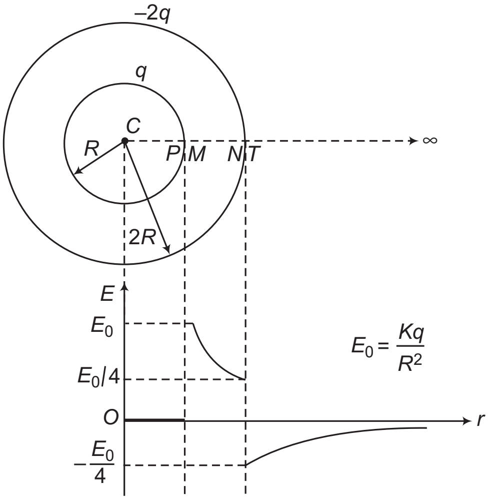
Here, q is the charge on shell.

Example 16 *Draw E - r and V - r graphs due to two charged spherical shells as shown in figure (along the line between C and ∞).*



Solution

E - r graph



C to P $\quad q_{\text{in}} = 0 \quad \Rightarrow \quad E = 0$

At M $\quad E = \frac{kq}{R^2}$ (radially outwards, say positive) $= E_0$ (say)

At N $\quad E = \frac{kq}{(2R)^2} = \frac{kq}{4R^2} = \frac{E_0}{4}$ (radially outwards)

From M to N Value will decrease from E_0 to $\frac{E_0}{4}$

At T $\quad E = \frac{k(-2q+q)}{(2R)^2}$ (radially inwards)

$$= -\frac{E_0}{4}$$

From T to ∞ Value changes from $-\frac{E_0}{4}$ to zero.

V - r graph

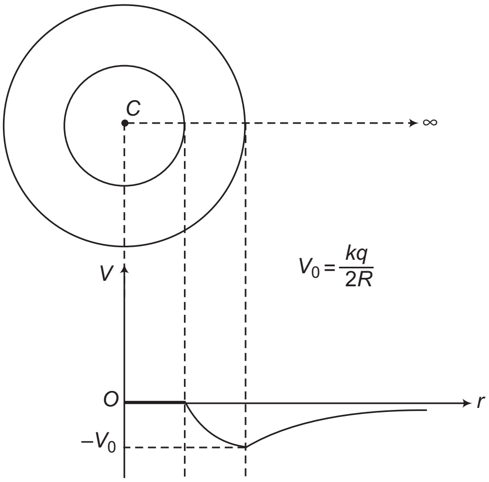
From C to P Points are lying inside both the shells. Hence, potential due to both shells is constant.

$$\therefore \qquad V = \frac{kq}{R} - \frac{k(2q)}{2R} = 0$$

From M to N Potential of $-2q$ will remain constant but potential of q will decrease. So, net value comes out to be negative. At N or T

$$V = \frac{kq}{2R} - \frac{k(2q)}{2R}$$
$$= -\frac{kq}{2R} = -V_0 \qquad \text{(say)}$$

From T to ∞ Value will change from $-V_0$ to zero. The correct graph is as shown below.



Type 8. *Based on motion of a charged particle in uniform electric field*

Concept

(i) In uniform electric field, force on the charged particle is

$$\mathbf{F} = q\mathbf{E}$$

or qE force acts in the direction of electric field if q is positive and in the opposite direction of electric field if q is negative.

(ii) Acceleration of the particle is therefore,

$$a = \frac{F}{m} = \frac{qE}{m}$$

This acceleration is constant. So, path is therefore either a straight line or parabola. If initial velocity is zero or parallel to acceleration or antiparallel to acceleration, then path is straight line. Otherwise in all other cases, path is a parabola.

Example 17 *An electron with a speed of 5.00×10^6 m/s enters an electric field of magnitude 10^3 N/C, travelling along the field lines in the direction that retards its motion.*

(a) How far will the electron travel in the field before stopping momentarily?

(b) *How much time will have elapsed?*

(c) *If the region with the electric field is only 8.00 mm long (too short from the electron to stop with in it), what fraction of the electron's initial kinetic energy will be lost in that region?*

Solution (a) $s = \frac{u^2}{2a} = \frac{u^2}{2\,(qE/m)} = \frac{mu^2}{2qE}$

$$= \frac{(9.1 \times 10^{-31})\,(5 \times 10^6)^2}{2 \times 1.6 \times 10^{-19} \times 10^3}$$

$= 7.1 \times 10^{-2}$ m $= 7.1$ cm **Ans.**

(b) $t = \frac{u}{a} = \frac{u}{qE/m} = \frac{mu}{qE}$

$$= \frac{(9.1 \times 10^{-31})\,(5 \times 10^6)}{(1.6 \times 10^{-19})\,(10^3)}$$

$= 2.84 \times 10^{-8}$s **Ans.**

(c) Loss of energy (in fraction)

$$= \frac{\frac{1}{2}mu^2 - \frac{1}{2}mv^2}{\frac{1}{2}mu^2} = 1 - \frac{u^2}{u^2}$$

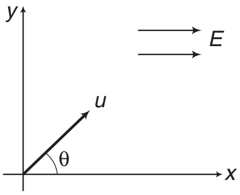
$$= 1 - \frac{u^2 - 2as}{u^2} = \frac{2as}{u^2} = \frac{2qEs}{mu^2}$$

$$= \frac{2 \times 1.6 \times 10^{-19} \times 10^3 \times 8 \times 10^{-3}}{9.1 \times 10^{-31} \times (5 \times 10^6)^2}$$

$= 0.11$ **Ans.**

Example 18 *A charged particle of mass $m = 1$ kg and charge $q = 2\,\mu C$ is thrown from a horizontal ground at an angle $\theta = 45°$ with speed 20 m/s. In space a horizontal electric field $E = 2 \times 10^7$ V/m exist. Find the range on horizontal ground of the projectile thrown.*

Solution The path of the particle will be a parabola, but along x-axis also motion of the particle will be accelerated. Time of flight of the projectile is



$$T = \frac{2u_y}{a_y} = \frac{2u_y}{g} = \frac{2 \times 20 \cos 45°}{10} = 2\sqrt{2} \text{ s}$$

Horizontal range of the particle will be

$$R = u_x T + \frac{1}{2} a_x T^2$$

Here, $a_x = \frac{qE}{m} = \frac{(2 \times 10^{-6})\,(2 \times 10^7)}{1} = 40 \text{ m/s}^2$

$\therefore$ $R = (20 \cos 45°)\,(2\sqrt{2}) + \frac{1}{2}(40)\,(2\sqrt{2})^2$

$= 40 + 160$

$= 200$ m **Ans.**

Type 9. *To find potential difference between two points when electric field is known*

Concept

In Article 24.9, we have already read the relation between **E** and V. There we have taken a simple case when electric field was uniform. Here, two more cases are possible depending on the nature of **E**.

When E has a Function Like $f_1(x)\hat{\mathbf{i}} + f_2(y)\hat{\mathbf{j}} + f_3(z)\hat{\mathbf{k}}$

In this case also, we will use the same approach. Let us take an example.

Example 19 *Find the potential difference V_{AB} between $A(2\text{m}, 1\text{m}, 0)$ and $B(0, 2\text{m}, 4\text{m})$ in an electric field,*

$$\mathbf{E} = (x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\ V/m$$

Solution

$$dV = -\mathbf{E}\cdot d\mathbf{r}$$

$$\int_B^A dV = -\int_{(0,\,2,\,4)}^{(2,\,1,\,0)} (x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$\therefore \quad V_A - V_B = -\int_{(0,\,2,\,4)}^{(2,\,1,\,0)} (xdx - 2ydy + zdz)$$

$$\text{or} \quad V_{AB} = -\left[\frac{x^2}{2} - y^2 + \frac{z^2}{2}\right]_{(0,\,2,\,4)}^{(2,\,1,\,0)}$$

$$= 3 \text{ volt}$$ **Ans.**

When E · *d*r becomes a Perfect Differential.

Same method is used when $\mathbf{E}\cdot d\mathbf{r}$ becomes a perfect differential. The following example will illustrate the theory.

Example 20 *Find potential difference V_{AB} between $A(0, 0, 0)$ and $B(1m, 1m, 1m)$ in an electric field (a) $\mathbf{E} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ (b) $\mathbf{E} = 3x^2y\hat{\mathbf{i}} + x^3\hat{\mathbf{j}}$*

Solution (a)

$$dV = -\mathbf{E}\cdot d\mathbf{r}$$

$$\therefore \quad \int_B^A dV = -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} (y\hat{\mathbf{i}} + x\hat{\mathbf{j}})\cdot(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$\text{or} \quad V_A - V_B = -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} (y\,dx + x\,dy)$$

$$\text{or} \quad V_{AB} = -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} d\,(xy) \qquad [\text{as } y\,dx + x\,dy = d\,(xy)]$$

$$\therefore \quad V_{AB} = -[xy]_{(1,1,1)}^{(0,0,0)} = 1\text{ V}$$ **Ans.**

(b)

$$dV = -\mathbf{E}\cdot d\mathbf{r}$$

$$\therefore \quad \int_B^A dV = -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} (3x^2y\hat{\mathbf{i}} + x^3\hat{\mathbf{j}})\bullet(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$\text{or} \quad V_A - V_B = -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} (3x^2ydx + x^3dy)$$

$$= -\int_{(1,\,1,\,1)}^{(0,\,0,\,0)} d\,(x^3y)$$

$$\therefore \quad V_{AB} = -[x^3y]_{(1,1,1)}^{(0,0,0)} = 1\text{ V}$$ **Ans.**

Type 10. *Based on oscillations of a dipole*

Concept

In uniform electric field, net force on a dipole is zero at all angles. But net torque is zero for $\theta = 0°$ or $180°$. Here, $\theta = 0°$ is the stable equilibrium position and $\theta = 180°$ is unstable equilibrium position. If the dipole is released from any angle other than 0° or 180°, it rotates towards 0°. In this process electrostatic potential energy of the dipole decreases. But rotational kinetic energy increases. At two angles θ_1 and θ_2, we can apply the equation

$$U_{\theta_1} + K_{\theta_1} = U_{\theta_2} + K_{\theta_2}$$

or
$$-pE\cos\theta_1 + \frac{1}{2}I\omega_1^2 = -pE\cos\theta_2 + \frac{1}{2}I\omega_2^2$$

Moreover, if the dipole is displaced from stable equilibrium position ($\theta = 0°$), then it starts rotational oscillations. For small value of θ, these oscillations are simple harmonic in nature.

Example 21 *An electric dipole of dipole moment p is placed in a uniform electric field E in stable equilibrium position. Its moment of inertia about the centroidal axis is I. If it is displaced slightly from its mean position, find the period of small oscillations.*

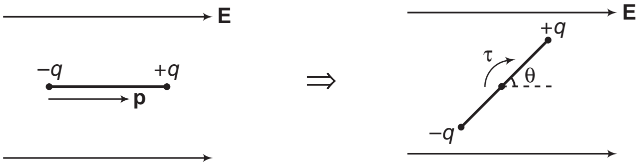
Solution When displaced at an angle θ from its mean position, the magnitude of restoring torque is

$$\tau = -pE\sin\theta$$

For small angular displacement $\sin\theta \approx \theta$

$$\tau = -pE\theta$$

The angular acceleration is $\alpha = \frac{\tau}{I} = -\left(\frac{pE}{I}\right)\theta = -\omega^2\theta$



where, $\omega^2 = \frac{pE}{I} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{pE}}$ **Ans.**

Type 11. *Based on the work done (by external forces) in moving a charge from one point to another point*

Concept

If kinetic energy of the particle is not changed, then

$$W = \Delta U = U_f - U_i = q(V_f - V_i) \quad \text{or} \quad q(\Delta U)$$

Here, q is the charge to be displaced and V_i and V_f are the initial and final potentials.

Example 22 *Two identical thin rings, each of radius R, are coaxially placed a distance R apart. If Q_1 and Q_2 are respectively the charges uniformly spread on the two rings, the work done in moving a charge q from the centre of one ring to that of the other is* **(JEE 1992)**

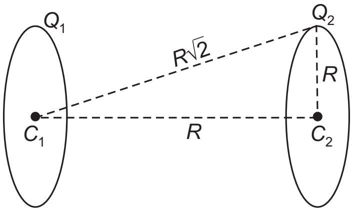
(a) zero

(b) $\dfrac{q(Q_1 - Q_2)(\sqrt{2} - 1)}{\sqrt{2}\,(4\pi\varepsilon_0\, R)}$

(c) $\dfrac{q\sqrt{2}(Q_1 + Q_2)}{(4\pi\varepsilon_0 R)}$

(d) $q(Q_1 / Q_2)(\sqrt{2} + 1)\sqrt{2}\,(4\pi\varepsilon_0 R)$

Solution $V_{C_1} = V_{Q_1} + V_{Q_2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1}{R} + \dfrac{1}{4\pi\,\varepsilon_0}\dfrac{Q_2}{R\sqrt{2}} = \dfrac{1}{4\pi\,\varepsilon_0 R}\left(Q_1 + \dfrac{Q_2}{\sqrt{2}}\right)$



Similarly,
$$V_{C_2} = \frac{1}{4\pi\,\varepsilon_0 R}\left(Q_2 + \frac{Q_1}{\sqrt{2}}\right)$$

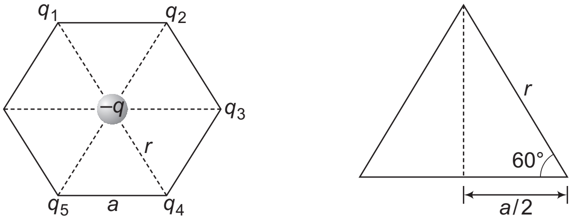
∴
$$\begin{aligned}\Delta V &= V_{C_1} - V_{C_2}\\ &= \frac{1}{4\pi\varepsilon_0 R}\left[(Q_1 - Q_2) - \frac{1}{\sqrt{2}}(Q_1 - Q_2)\right]\\ &= \frac{Q_1 - Q_2}{\sqrt{2}(4\pi\,\varepsilon_0 R)}(\sqrt{2} - 1)\end{aligned}$$
$$W = q\Delta V = q\,(Q_1 - Q_2)\,(\sqrt{2} - 1)/\sqrt{2}(4\pi\,\varepsilon_0 R)$$

∴ The correct option is (b).

Miscellaneous Examples

Example 23 *Five point charges each of value + q are placed on five vertices of a regular hexagon of side 'a' metre. What is the magnitude of the force on a point charge of value – q coulomb placed at the centre of the hexagon?*

Solution



$$\frac{a/2}{r} = \cos 60° = \frac{1}{2}$$

$$\therefore \qquad a = r$$

$$q_1 = q_2 = \ldots = q_5 = q$$

Net force on $-q$ is only due to q_3 because forces due to q_1 and due to q_4 are equal and opposite so cancel each other. Similarly, forces due to q_2 and q_5 also cancel each other. Hence, the net force on $-q$ is

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(q)(q)}{r^2} \qquad \text{(towards } q_3\text{)}$$

or

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2}{r^2}$$

Ans.

Example 24 *A point charge* $q_1 = 9.1\,\mu C$ *is held fixed at origin. A second point charge* $q_2 = -0.42\,\mu C$ *and a mass* 3.2×10^{-4} *kg is placed on the x-axis,* 0.96 *m from the origin. The second point charge is released at rest. What is its speed when it is* 0.24 *m from the origin?*

Solution From conservation of mechanical energy, we have

Decrease in electrostatic potential energy = Increase in kinetic energy

or

$$\frac{1}{2}mv^2 = U_i - U_f = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{r_f - r_i}{r_i r_f}\right)$$

$$\therefore \qquad v = \sqrt{\frac{q_1 q_2}{2\pi\varepsilon_0 m}\left(\frac{r_f - r_i}{r_i r_f}\right)}$$

$$= \sqrt{\frac{(9.1 \times 10^{-6})(-0.42 \times 10^{-6}) \times 2 \times 9 \times 10^9}{3.2 \times 10^{-4}}\left(\frac{0.24 - 0.96}{(0.24)(0.96)}\right)}$$

$$= 26 \text{ m/s}$$

Ans.

Example 25 *A point charge* $q_1 = -5.8\,\mu C$ *is held stationary at the origin. A second point charge* $q_2 = +4.3\,\mu C$ *moves from the point* (0.26 *m*, 0, 0) *to* (0.38 *m*, 0, 0)*. How much work is done by the electric force on* q_2*?*

Solution Work done by the electrostatic forces $= U_i - U_f$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{r_f - r_i}{r_i\, r_f}\right)$$

$$= \frac{(-5.8 \times 10^{-6})(4.3 \times 10^{-6})(9 \times 10^9)(0.38 - 0.26)}{(0.38)(0.26)}$$

$$= -0.272 \text{ J}$$

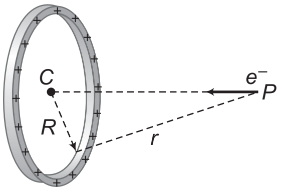
Ans.

Example 26 *A uniformly charged thin ring has radius* 10.0 *cm and total charge* + 12.0 μ*C. An electron is placed on the ring's axis a distance* 25.0 *cm from the centre of the ring and is constrained to stay on the axis of the ring. The electron is then released from rest.*

(a) describe the subsequent motion of the electron.

(b) find the speed of the electron when it reaches the centre of the ring.

Solution (a) The electron will be attracted towards the centre C of the ring. At C net force is zero, but on reaching C, electron has some kinetic energy and due to inertia it crosses C, but on the other side it is further attracted towards C. Hence, motion of electron is oscillatory about point C.



(b) As the electron approaches C, its speed (hence, kinetic energy) increases due to force of attraction towards the centre C. This increase in kinetic energy is at the cost of electrostatic potential energy. Thus,

$$\frac{1}{2}mv^2 = U_i - U_f$$

$$= U_P - U_C = (-e)\,[V_P - V_C] \qquad \text{...(i)}$$

Here, V is the potential due to ring.

$$V_P = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r} \qquad (q = \text{charge on ring})$$

$$= \frac{(9\times10^9)\,(12\times10^{-9})}{(\sqrt{(10)^2+(25)^2})\times10^{-2}} = 401\text{ V}$$

$$V_C = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$$

$$= \frac{(9\times10^9)\,(12\times10^{-9})}{10\times10^{-2}} = 1080\text{ V}$$

Substituting the proper values in Eq. (i), we have

$$\frac{1}{2}\times9.1\times10^{-31}\times v^2 = (-1.6\times10^{-19})\,(401-1080)$$

∴ $v = 15.45\times10^6$ m/s **Ans.**

Example 27 *Two points A and B are* 2 *cm apart and a uniform electric field E acts along the straight line AB directed from A to B with E* = 200 *N/C. A particle of charge* + 10^{-6} *C is taken from A to B along AB. Calculate*

(a) the force on the charge

(b) the potential difference $V_A - V_B$ *and*

(c) the work done on the charge by **E**

Solution (a) Electrostatic force on the charge,

$$F = qE = (10^{-6})(200)$$
$$= 2 \times 10^{-4}\ \text{N} \qquad \textbf{Ans.}$$

(b) In uniform electric field,

PD, $V = E \cdot d$

or $V_A - V_B = 200 \times 2 \times 10^{-2}$

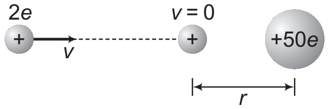
$= 4\ \text{V}$ **Ans.**

(c) $W = (2 \times 10^{-4})(2 \times 10^{-2}) \cos 0°$

$= 4 \times 10^{-6}\ \text{J}$ **Ans.**

Example 28 *An alpha particle with kinetic energy* 10 *MeV is heading towards a stationary tin nucleus of atomic number* 50. *Calculate the distance of closest approach. Initially they were far apart.*

Solution Due to repulsion by the tin nucleus, the kinetic energy of the α-particle gradually decreases at the expense of electrostatic potential energy.



∴ Decrease in kinetic energy = increase in potential energy

or $$\frac{1}{2}mv^2 = U_f - U_i$$

or $$\frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r} - 0$$

∴ $$r = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(2e)(50e)}{(\text{KE})}$$

Substituting the values,

$$r = \frac{(9 \times 10^9)(2 \times 1.6 \times 10^{-19})(1.6 \times 10^{-19} \times 50)}{10 \times 10^6 \times 1.6 \times 10^{-19}}$$

$$= 14.4 \times 10^{-15}\ \text{m} \qquad \textbf{Ans.}$$

Example 29 *Three point charges of* 1 *C*, 2 *C and* 3 *C are placed at the corners of an equilateral triangle of side* 1 *m. Calculate the work required to move these charges to the corners of a smaller equilateral triangle of side* 0.5 *m.*

Solution Work done $= U_f - U_i$

$$= \frac{1}{4\pi\ \varepsilon_0}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)[q_3 q_2 + q_3 q_1 + q_2 q_1]$$

$$= 9 \times 10^9 \left(\frac{1}{0.5} - \frac{1}{1}\right)[(3)(2) + (3)(1) + (2)(1)]$$

$$= 99 \times 10^9\ \text{J} \qquad \textbf{Ans.}$$

Note *Work done by electrostatic forces is $U_i - U_f$ but work done by external forces is $U_f - U_i$. Sometimes in a simple way it is asked, find the work done. It means $U_f - U_i$.*

Example 30 *Consider a spherical surface of radius 4 m centred at the origin. Point charges + q and – 2q are fixed at points A (2 m, 0, 0) and B (8 m, 0, 0) respectively. Show that every point on the spherical surface is at zero potential.*

Solution Let $P(x, y, z)$ be any point on the sphere. From the property of the sphere,

$$x^2 + y^2 + z^2 = (4^2) = 16 \quad \text{...(i)}$$

Further, $$PA = \sqrt{(x-2)^2 + y^2 + z^2} \quad \text{...(ii)}$$

and $$PB = \sqrt{(x-8)^2 + y^2 + z^2} \quad \text{...(iii)}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{PA} - \frac{2q}{PB}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(x-2)^2 + y^2 + z^2}} - \frac{2q}{\sqrt{(x-8)^2 + y^2 + z^2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + z^2 + 4 - 4x}} - \frac{2q}{\sqrt{x^2 + y^2 + z^2 + 64 - 16x}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{16 + 4 - 4x}} - \frac{2q}{\sqrt{16 + 64 - 16x}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{20 - 4x}} - \frac{q}{\sqrt{20 - 4x}}\right]$$

$$= 0$$

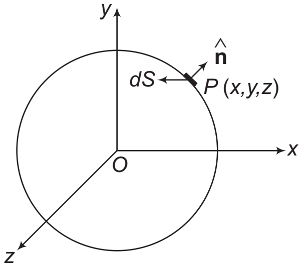
Proved

Example 31 *The intensity of an electric field depends only on the coordinates x and y as follows*

$$\mathbf{E} = \frac{a\,(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})}{x^2 + y^2}$$

where, a is a constant and $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *are the unit vectors of the x and y-axes. Find the charge within a sphere of radius R with the centre at the origin.*

Solution At any point $P(x, y, z)$ on the sphere a unit vector perpendicular to the sphere radially outwards is



$$\hat{\mathbf{n}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\hat{\mathbf{i}} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\hat{\mathbf{j}} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\hat{\mathbf{k}}$$

$$= \frac{x}{R}\hat{\mathbf{i}} + \frac{y}{R}\hat{\mathbf{j}} + \frac{z}{R}\hat{\mathbf{k}} \quad \text{as} \quad x^2 + y^2 + z^2 = R^2$$

Let us find the electric flux passing through a small area dS at point P on the sphere,

$$d\phi = \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \left\{\frac{ax^2}{R\,(x^2 + y^2)} + \frac{ay^2}{R\,(x^2 + y^2)}\right\} dS$$

$$= \left(\frac{a}{R}\right) dS$$

Here, we note that $d\phi$ is independent of the coordinates x, y and z. Therefore, total flux passing through the sphere

$$\phi = \int d\phi = \frac{a}{R}\int dS = \left(\frac{a}{R}\right)(4\pi R^2)$$

$$= 4\pi a R$$

From Gauss's law,

$$\phi = \frac{q_{\text{in}}}{\varepsilon_0} \quad \text{or} \quad (4\pi a R) = \frac{q_{\text{in}}}{\varepsilon_0}$$

$\therefore$ $$q_{\text{in}} = 4\pi\varepsilon_0 a R$$ **Ans.**

Example 32 *Find the electric field caused by a disc of radius a with a uniform surface charge density σ (charge per unit area), at a point along the axis of the disc a distance x from its centre.*

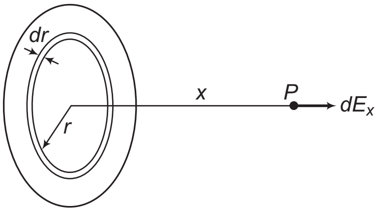
Solution We can assume this charge distribution as a collection of concentric rings of charge.

$$dA = (2\pi r)\, dr$$

$$dq = \sigma\, dA = (2\pi\sigma r)\, dr$$

$$dE_x = \frac{1}{4\pi\varepsilon_0}\cdot\frac{(dq)x}{(x^2+r^2)^{3/2}}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{(2\pi\sigma r\, dr)\, x}{(x^2+r^2)^{3/2}}$$



$\therefore$ $$E_x = \int_0^a dE_x$$

$$= \int_0^a \frac{(2\pi\sigma r\, dr)\, x}{4\pi\, \varepsilon_0\, (x^2+r^2)^{3/2}}$$

$$= \frac{\sigma x}{2\varepsilon_0}\int_0^a \frac{r\, dr}{(x^2+r^2)^{3/2}}$$

or $$E_x = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{a^2/x^2+1}}\right]$$

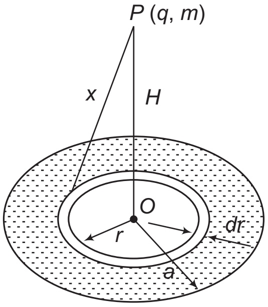
If the charge distribution gets very large, i.e. $a >> x$, the term $\frac{1}{\sqrt{a^2/x^2+1}}$ becomes negligibly small, and we get $E = \frac{\sigma}{2\varepsilon_0}$.

Thus, we can say that electric field produced by an infinite plane sheet of charge is independent of the distance from the sheet. Thus, the field is uniform, its direction is everywhere perpendicular to the sheet.

Example 33 *A non-conducting disc of radius a and uniform positive surface charge density σ is placed on the ground with its axis vertical. A particle of mass m and positive charge q is dropped, along the axis of the disc from a height H with zero initial velocity. The particle has* $q/m = 4\varepsilon_0\, g/\sigma$.

(a) Find the value of H if the particle just reaches the disc.

(b) Sketch the potential energy of the particle as a function of its height and find its equilibrium position.



Solution Potential at a height ***H* on the axis of the disc *V*(*P*)**. The charge dq contained in the ring shown in figure, $dq = (2\pi r dr)\sigma$

Potential of P due to this ring

$$dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{dq}{x}, \quad \text{where } x = \sqrt{H^2 + r^2}$$

$$dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{(2\pi r dr)\,\sigma}{\sqrt{H^2 + r^2}} = \frac{\sigma}{2\varepsilon_0}\,\frac{r dr}{\sqrt{H^2 + r^2}}$$

$\therefore$ Potential due to the complete disc,

$$V_P = \int_{r=0}^{r=a} dV = \frac{\sigma}{2\varepsilon_0}\int_{r=0}^{r=a}\frac{r dr}{\sqrt{H^2 + r^2}}$$

$$V_P = \frac{\sigma}{2\varepsilon_0}\,[\sqrt{a^2 + H^2} - H]$$

Potential at centre, O will be

$$V_O = \frac{\sigma a}{2\varepsilon_0} \qquad (H = 0)$$

(a) Particle is released from P and it just reaches point O. Therefore, from conservation of mechanical energy

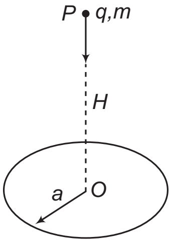
decrease in gravitational potential energy = increase in electrostatic potential energy

(Δ KE = 0 because $K_i = K_f = 0$)

$$\therefore \quad mgH = q\,[V_O - V_P]$$

or
$$gH = \left(\frac{q}{m}\right)\left(\frac{\sigma}{2\varepsilon_0}\right)[a - \sqrt{a^2 + H^2} + H] \qquad \ldots\text{(i)}$$

$$\frac{q}{m} = \frac{4\varepsilon_0 g}{\sigma} \Rightarrow \frac{q\sigma}{2\varepsilon_0 m} = 2g$$



Substituting in Eq. (i), we get

$$gH = 2g\,[a + H - \sqrt{a^2 + H^2}]$$

or
$$\frac{H}{2} = (a + H) - \sqrt{a^2 + H^2}$$

or
$$\sqrt{a^2 + H^2} = a + \frac{H}{2}$$

or
$$a^2 + H^2 = a^2 + \frac{H^2}{4} + aH$$

or
$$\frac{3}{4}H^2 = aH \quad \text{or} \quad H = \frac{4}{3}a \quad \text{and} \quad H = 0$$

$$\therefore \quad H = (4/3)\,a$$

Ans.

(b) Potential energy of the particle at height H = Electrostatic potential energy + gravitational potential energy

$$\therefore \quad U = qV + mgH$$

Here, V = Potential at height H

$$\therefore \quad U = \frac{\sigma q}{2\varepsilon_0}\,[\sqrt{a^2 + H^2} - H] + mgH \qquad \ldots\text{(ii)}$$

At equilibrium position, $F = \frac{-dU}{dH} = 0$

Differentiating Eq. (ii) w.r.t. H,

or $$mg + \frac{\sigma q}{2\varepsilon_0}\left[\left(\frac{1}{2}\right)(2H)\frac{1}{\sqrt{a^2+H^2}} - 1\right] = 0 \qquad \left[\frac{\sigma q}{2\varepsilon_0} = 2mg\right]$$

∴ $$mg + 2mg\left[\frac{H}{\sqrt{a^2+H^2}} - 1\right] = 0$$

or $$1 + \frac{2H}{\sqrt{a^2+H^2}} - 2 = 0 \Rightarrow \frac{2H}{\sqrt{a^2+H^2}} = 1$$

or $$\frac{H^2}{a^2+H^2} = \frac{1}{4} \quad \text{or} \quad 3H^2 = a^2$$

or $$H = \frac{a}{\sqrt{3}}$$ **Ans.**

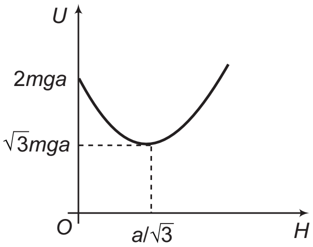
From Eq. (ii), we can see that

$$U = 2\ mga \text{ at } H = 0 \text{ and}$$
$$U = U_{\min} = \sqrt{3}\ mga \text{ at } H = \frac{a}{\sqrt{3}}$$

Therefore, U-H graph will be as shown.

Note that at $H = \frac{a}{\sqrt{3}}$, U is minimum.

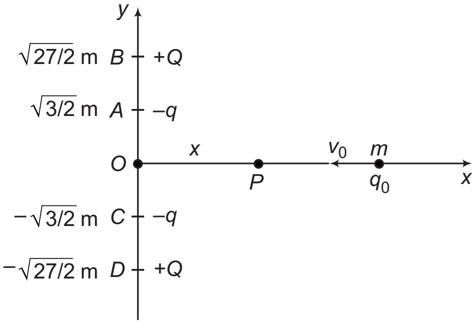
Therefore, $H = \frac{a}{\sqrt{3}}$ is stable equilibrium position.



Example 34 *Four point charges* $+8\,\mu C$, $-1\,\mu C$, $-1\,\mu C$ *and* $+8\,\mu C$ *are fixed at the points* $-\sqrt{27/2}$ *m,* $-\sqrt{3/2}$ *m,* $+\sqrt{3/2}$ *m and* $+\sqrt{27/2}$ *m respectively on the Y-axis. A particle of mass* 6×10^{-4} *kg and charge* $+0.1\,\mu C$ *moves along the –X direction. Its speed at* $x = +\infty$ *is* v_0*. Find the least value of* v_0 *for which the particle will cross the origin. Find also the kinetic energy of the particle at the origin. Assume that space is gravity free.*

Solution In the figure,

$$q = 1\,\mu\text{C} = 10^{-6}\text{ C}$$
$$q_0 = +0.1\,\mu\text{C} = 10^{-7}\text{ C}$$
$$m = 6\times10^{-4}\text{ kg}$$
and $$Q = 8\mu\text{C} = 8\times10^{-6}\text{ C}$$



Let P be any point at a distance x from origin O. Then,

$$AP = CP = \sqrt{\frac{3}{2} + x^2}$$
$$BP = DP = \sqrt{\frac{27}{2} + x^2}$$

Electric potential at point P will be

$$V = \frac{2kQ}{BP} - \frac{2kq}{AP}$$

where, $$k = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2$$

$\therefore$ $$V = 2 \times 9 \times 10^9 \left[\frac{8 \times 10^{-6}}{\sqrt{\frac{27}{2} + x^2}} - \frac{10^{-6}}{\sqrt{\frac{3}{2} + x^2}} \right]$$

$$V = 1.8 \times 10^4 \left[\frac{8}{\sqrt{\frac{27}{2} + x^2}} - \frac{1}{\sqrt{\frac{3}{2} + x^2}} \right] \qquad \text{...(i)}$$

$\therefore$ Electric field at P is

$$E = -\frac{dV}{dx} = -1.8 \times 10^4 \left[(8)\left(-\frac{1}{2}\right)\left(\frac{27}{2} + x^2\right)^{-3/2} - (1)\left(-\frac{1}{2}\right)\left(\frac{3}{2} + x^2\right)^{-3/2} \right](2x)$$

$E = 0$ on x-axis where

$$\frac{8}{\left(\frac{27}{2} + x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2} + x^2\right)^{3/2}}$$

$\Rightarrow$ $$\frac{(4)^{3/2}}{\left(\frac{27}{2} + x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2} + x^2\right)^{3/2}}$$

$\Rightarrow$ $$\left(\frac{27}{2} + x^2\right) = 4\left(\frac{3}{2} + x^2\right)$$

This equation gives $$x = \pm\sqrt{\frac{5}{2}} \text{ m}$$

The least value of kinetic energy of the particle at infinity should be enough to take the particle upto $x = +\sqrt{\frac{5}{2}}$ m because

at $x = +\sqrt{\frac{5}{2}}$ m, $E = 0 \Rightarrow$ Electrostatic force on charge q_0 is zero or $F_e = 0$

for $x > \sqrt{\frac{5}{2}}$ m, E is repulsive (towards positive x-axis)

and for $x < \sqrt{\frac{5}{2}}$ m, E is attractive (towards negative x-axis)

Now, from Eq. (i), potential at $x = \sqrt{\frac{5}{2}}$ m

$$V = 1.8. \times 10^4 \left[\frac{8}{\sqrt{\frac{27}{2} + \frac{5}{2}}} - \frac{1}{\sqrt{\frac{3}{2} + \frac{5}{2}}} \right]$$

$$V = 2.7 \times 10^4 \text{ V}$$

Applying energy conservation at $x = \infty$ and $x = \sqrt{\frac{5}{2}}$ m

$$\frac{1}{2}mv_0^2 = q_0V \qquad \text{...(ii)}$$

$$\therefore \qquad v_0 = \sqrt{\frac{2q_0V}{m}}$$

Substituting the values,

$$v_0 = \sqrt{\frac{2\times10^{-7}\times2.7\times10^4}{6\times10^{-4}}}$$

$$v_0 = 3 \text{ m/s}$$ **Ans.**

∴ Minimum value of v_0 is 3 m/s.

From Eq. (i), potential at origin ($x = 0$) is

$$V_0 = 1.8\times10^4\left[\frac{8}{\sqrt{\frac{27}{2}}} - \frac{1}{\sqrt{\frac{3}{2}}}\right] = 2.4\times10^4 \text{ V}$$

Let T be the kinetic energy of the particle at origin.

Applying energy conservation at $x = 0$ and at $x = \infty$

$$T + q_0V_0 = \frac{1}{2}mv_0^2$$

But,

$$\frac{1}{2}mv_0^2 = q_0V \qquad \text{[from Eq. (ii)]}$$

$$\therefore \qquad T = q_0\,(V - V_0)$$

$$T = (10^{-7})\,(2.7\times10^4 - 2.4\times10^4)$$

$$T = 3\times10^{-4} \text{ J}$$ **Ans.**

Note *$E = 0$ or F_e on q_0 is zero at $x = 0$ and $x = \pm\sqrt{\frac{5}{2}}$ m. Of these $x = 0$ is stable equilibrium position and $x = \pm\sqrt{\frac{5}{2}}$ m is unstable equilibrium position.*

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : An independent negative charge moves itself from point A to point B. Then, potential at A should be less than potential at B.

Reason : While moving from A to B kinetic energy of electron will increase.

2. Assertion : When two unlike charges are brought nearer, their electrostatic potential energy decreases.

Reason : All conservative forces act in the direction of decreasing potential energy.

3. Assertion : At a point electric potential is decreasing along x-axis at a rate of 10 V/m. Therefore, x-component of electric field at this point should be 10 V/m along x-axis.

Reason : Magnitude of $E_x = \dfrac{\partial V}{\partial x}$

4. Assertion : Electric potential on the surface of a charged sphere of radius R is V. Then electric field at a distance $r = \dfrac{R}{2}$ from centre is $\dfrac{V}{2R}$. Charge is distributed uniformly over the volume.

Reason : From centre to surface, electric field varies linearly with r. Here, r is distance from centre.

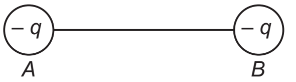
5. Assertion : Gauss's theorem can be applied only for a closed surface.

Reason : Electric flux can be obtained passing from an open surface also.

6. Assertion : In the electric field $\mathbf{E} = (4\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ N/C, electric potential at $A(4\,\text{m}, 0)$ is more than the electric potential at $B(0, 4\,\text{m})$.

Reason : Electric lines of forces always travel from higher potential to lower potential.

7. Assertion : Two charges $-q$ each are fixed at points A and B. When a third charge $-q$ is moved from A to B, electrical potential energy first decreases than increases.



Reason : Along the line joining A and B, the third charge is in stable equilibrium position at centre.

8. **Assertion :** A small electric dipole is moved translationally from higher potential to lower potential in uniform electric field. Work done by electric field is positive.

   Reason : When a positive charge is moved from higher potential to lower potential, work done by electric field is positive.

9. **Assertion :** In case of charged spherical shells, E-r graph is discontinuous while V-r graph is continuous.

   Reason : According to Gauss's theorem only the charge inside a closed surface can produce electric field at some point.

10. **Assertion :** If we see along the axis of a charged ring, the magnitude of electric field is minimum at centre and magnitude of electric potential is maximum.

    Reason : Electric field is a vector quantity while electric potential is scalar.

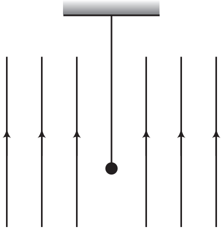
Objective Questions

1. Units of electric flux are

(a) $\frac{\text{N-m}^2}{\text{C}^2}$ (b) $\frac{\text{N}}{\text{C}^2\text{-m}^2}$

(c) volt-m (d) volt- m^3

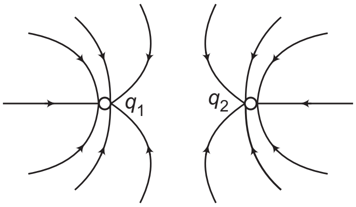
2. A neutral pendulum oscillates in a uniform electric field as shown in figure. If a positive charge is given to the pendulum, then its time period



(a) will increase (b) will decrease

(c) will remain constant (d) will first increase then decrease

3. Identify the correct statement about the charges q_1 and q_2, then



(a) q_1 and q_2 both are positive (b) q_1 and q_2 both are negative

(c) q_1 is positive q_2 is negative (d) q_2 is positive and q_1 is negative

4. Three identical charges are placed at corners of an equilateral triangle of side l. If force between any two charges is F, the work required to double the dimensions of triangle is

(a) $-3\,Fl$ (b) $3\,Fl$

(c) $(-3/2)\,Fl$ (d) $(3/2)\,Fl$

5. A proton, a deuteron and an alpha particle are accelerated through potentials of V, 2 V and 4 V respectively. Their velocity will bear a ratio

(a) $1:1:1$ (b) $1:\sqrt{2}:1$
(c) $\sqrt{2}:1:1$ (d) $1:1:\sqrt{2}$

6. Electric potential at a point P, r distance away due to a point charge q kept at point A is V. If twice of this charge is distributed uniformly on the surface of a hollow sphere of radius $4r$ with centre at point A, the potential at P now is

(a) V (b) $V/2$
(c) $V/4$ (d) $V/8$

7. Four charges $+q, -q, +q$ and $-q$ are placed in order on the four consecutive corners of a square of side a. The work done in interchanging the positions of any two neighbouring charges of the opposite sign is

(a) $\frac{q^2}{4\pi\varepsilon_0 a}(-4+\sqrt{2})$ (b) $\frac{q^2}{4\pi\varepsilon_0 a}(4+2\sqrt{2})$
(c) $\frac{q^2}{4\pi\varepsilon_0 a}(4-2\sqrt{2})$ (d) $\frac{q^2}{4\pi\varepsilon_0 a}(4+\sqrt{2})$

8. Two concentric spheres of radii R and $2R$ are charged. The inner sphere has a charge of $1\,\mu\text{C}$ and the outer sphere has a charge of $2\,\mu\text{C}$ of the same sign. The potential is 9000 V at a distance $3R$ from the common centre. The value of R is

(a) 1 m (b) 2 m
(c) 3 m (d) 4 m

9. A ring of radius R is having two charges q and $2q$ distributed on its two half parts. The electric potential at a point on its axis at a distance of $2\sqrt{2}\,R$ from its centre is $\left(k = \frac{1}{4\pi\varepsilon_0}\right)$

(a) $\frac{3kq}{R}$ (b) $\frac{kq}{3R}$
(c) $\frac{kq}{R}$ (d) $\frac{kq}{\sqrt{3}R}$

10. A particle A having a charge of 2.0×10^{-6} C and a mass of 100 g is fixed at the bottom of a smooth inclined plane of inclination 30°. Where should another particle B having same charge and mass, be placed on the inclined plane so that B may remain in equilibrium?

(a) 8 cm from the bottom (b) 13 cm from the bottom
(c) 21 cm from the bottom (d) 27 cm from the bottom

11. Four positive charges $(2\sqrt{2}-1)Q$ are arranged at the four corners of a square. Another charge q is placed at the centre of the square. Resulting force acting on each corner charge is zero if q is

(a) $-\frac{7Q}{4}$ (b) $-\frac{4Q}{7}$
(c) $-Q$ (d) $-(\sqrt{2}+1)Q$

12. A proton is released from rest, 10 cm from a charged sheet carrying charged density of $-2.21 \times 10^{-9}\,\text{C/m}^2$. It will strike the sheet after the time (approximately)

(a) $4\,\mu\text{s}$ (b) $2\,\mu\text{s}$
(c) $2\sqrt{2}\,\mu\text{s}$ (d) $4\sqrt{2}\,\mu\text{s}$

13. Two point charges $+q$ and $-q$ are placed a distance x apart. A third charge is so placed that all the three charges are in equilibrium. Then,

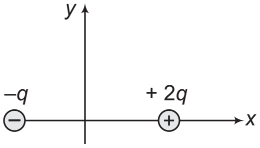
(a) unknown charge is $-4q/9$

(b) unknown charge is $-9q/4$

(c) it should be at $(x/3)$ from smaller charge between them

(d) None of the above

14. Charges $2q$ and $-q$ are placed at $(a, 0)$ and $(-a, 0)$ as shown in the figure. The coordinates of the point at which electric field intensity is zero will be $(x, 0)$, where



(a) $-a < x < a$ (b) $x < -a$

(c) $x > -a$ (d) $0 < x < a$

15. Five point charges ($+q$ each) are placed at the five vertices of a regular hexagon of side $2a$. What is the magnitude of the net electric field at the centre of the hexagon?

(a) $\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}$ (b) $\frac{q}{16\pi\varepsilon_0 a^2}$

(c) $\frac{\sqrt{2}q}{4\pi\varepsilon_0 a^2}$ (d) $\frac{5q}{16\pi\varepsilon_0 a^2}$

16. Two identical small conducting spheres having unequal positive charges q_1 and q_2 are separated by a distance r. If they are now made to touch each other and then separated again to the same distance, the electrostatic force between them in this case will be

(a) less than before (b) same as before

(c) more than before (d) zero

17. Three concentric conducting spherical shells carry charges $+4Q$ on the inner shell $-2Q$ on the middle shell and $+6Q$ on the outer shell. The charge on the inner surface of the outer shell is

(a) 0 (b) $4Q$

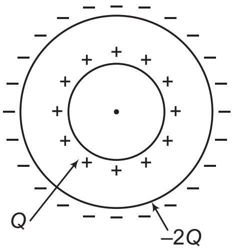
(c) $-Q$ (d) $-2Q$

18. 1000 drops of same size are charged to a potential of 1 V each. If they coalesce to form a single drop, its potential would be

(a) V (b) 10 V

(c) 100 V (d) 1000 V

19. Two concentric conducting spheres of radii R and $2R$ are carrying charges Q and $-2Q$, respectively. If the charge on inner sphere is doubled, the potential difference between the two spheres will



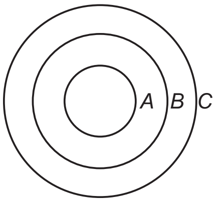
(a) become two times

(b) become four times

(c) be halved

(d) remain same

20. Charges Q, $2Q$ and $-Q$ are given to three concentric conducting spherical shells A, B and C respectively as shown in figure. The ratio of charges on the inner and outer surfaces of shell C will be



(a) $+\frac{3}{4}$ (b) $\frac{-3}{4}$

(c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

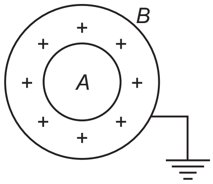
21. The electric field in a region of space is given by $\mathbf{E} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ N/C. The flux of $\mathbf{E}$ due to this field through an area $1\,\text{m}^2$ lying in the y-z plane, in SI units, is

(a) 5
(b) 10
(c) 2
(d) $5\sqrt{29}$

22. A charge Q is placed at each of the two opposite corners of a square. A charge q is placed at each of the other two corners. If the resultant force on each charge q is zero, then

(a) $q = \sqrt{2}\,Q$
(b) $q = -\sqrt{2}\,Q$
(c) $q = 2\sqrt{2}\,Q$
(d) $q = -2\sqrt{2}\,Q$

23. A and B are two concentric spherical shells. If A is given a charge $+q$ while B is earthed as shown in figure, then



(a) charge on the outer surface of shell B is zero
(b) the charge on B is equal and opposite to that of A
(c) the field inside A and outside B is zero
(d) All of the above

24. A solid sphere of radius R has charge 'q' uniformly distributed over its volume. The distance from its surface at which the electrostatic potential is equal to half of the potential at the centre is

(a) R
(b) $2R$
(c) $\frac{R}{3}$
(d) $\frac{R}{2}$

25. Four dipoles each of magnitudes of charges $\pm e$ are placed inside a sphere. The total flux of $\mathbf{E}$ coming out of the sphere is

(a) zero
(b) $\frac{4e}{\varepsilon_0}$
(c) $\frac{8e}{\varepsilon_0}$
(d) None of these

26. A pendulum bob of mass m carrying a charge q is at rest with its string making an angle θ with the vertical in a uniform horizontal electric field E. The tension in the string is

(a) $\frac{mg}{\sin\theta}$
(b) mg
(c) $\frac{qE}{\sin\theta}$
(d) $\frac{qE}{\cos\theta}$

27. Two isolated charged conducting spheres of radii a and b produce the same electric field near their surfaces. The ratio of electric potentials on their surfaces is

(a) $\frac{a}{b}$
(b) $\frac{b}{a}$
(c) $\frac{a^2}{b^2}$
(d) $\frac{b^2}{a^2}$

28. Two point charges $+q$ and $-q$ are held fixed at $(-a, 0)$ and $(a, 0)$ respectively of a x-y coordinate system, then

(a) the electric field $\mathbf{E}$ at all points on the x-axis has the same direction
(b) $\mathbf{E}$ at all points on the y-axis is along $\hat{\mathbf{i}}$
(c) positive work is done in bringing a test charge from infinity to the origin
(d) All of the above

29. A conducting shell S_1 having a charge Q is surrounded by an uncharged concentric conducting spherical shell S_2. Let the potential difference between S_1 and that S_2 be V. If the shell S_2 is now given a charge $-3Q$, the new potential difference between the same two shells is

(a) V (b) $2V$

(c) $4V$ (d) $-2V$

30. At a certain distance from a point charge, the field intensity is 500 V/m and the potential is – 3000 V. The distance to the charge and the magnitude of the charge respectively are

(a) 6 m and 6 μC (b) 4 m and 2 μC

(c) 6 m and 4 μC (d) 6 m and 2 μC

31. Two point charges q_1 and q_2 are placed at a distance of 50 m from each other in air, and interact with a certain force. The same charges are now put in oil whose relative permittivity is 5. If the interacting force between them is still the same, their separation now is

(a) 16.6 m (b) 22.3 m

(c) 28.4 m (d) 25.0 cm

32. An infinite line of charge λ per unit length is placed along the y-axis. The work done in moving a charge q from $A(a, 0)$ to $B(2a, 0)$ is

(a) $\dfrac{q\lambda}{2\pi\varepsilon_0}\ln 2$ (b) $\dfrac{q\lambda}{2\pi\varepsilon_0}\ln\left(\dfrac{1}{2}\right)$

(c) $\dfrac{q\lambda}{4\pi\varepsilon_0}\ln\sqrt{2}$ (d) $\dfrac{q\lambda}{4\pi\varepsilon_0}\ln 2$

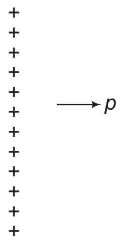
33. An electric dipole is placed perpendicular to an infinite line of charge at some distance as shown in figure. Identify the correct statement.

(a) The dipole is attracted towards the line charge

(b) The dipole is repelled away from the line charge

(c) The dipole does not experience a force

(d) The dipole experiences a force as well as a torque



34. An electrical charge 2×10^{-8} C is placed at the point (1, 2, 4) m. At the point (4, 2, 0) m, the electric

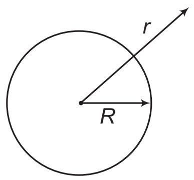
(a) potential will be 36 V

(b) field will be along y-axis

(c) field will increase if the space between the points is filled with a dielectric

(d) All of the above

35. If the potential at the centre of a uniformly charged hollow sphere of radius R is V, then electric field at a distance r from the centre of sphere will be $(r > R)$



(a) $\dfrac{VR}{r^2}$ (b) $\dfrac{Vr}{R^2}$ (c) $\dfrac{VR}{r}$ (d) $\dfrac{VR}{R^2 + r^2}$

36. There is an electric field E in x-direction. If the work done on moving a charge of 0.2 C through a distance of 2 m along a line making an angle 60° with x-axis is 4 J, then what is the value of E?

(a) $\sqrt{3}$ N/C
(b) 4 N/C
(c) 5 N/C
(d) 20 N/C

37. Two thin wire rings each having radius R are placed at a distance d apart with their axes coinciding. The charges on the two rings are $+Q$ and $-Q$. The potential difference between the centres of the two rings is

(a) zero
(b) $\dfrac{Q}{4\pi\varepsilon_0}\left[\dfrac{1}{R}-\dfrac{1}{\sqrt{R^2+d^2}}\right]$
(c) $\dfrac{Q}{4\pi\varepsilon_0 d^2}$
(d) $\dfrac{Q}{2\pi\varepsilon_0}\left[\dfrac{1}{R}-\dfrac{1}{\sqrt{R^2+d^2}}\right]$

38. The electric field at a distance 2 cm from the centre of a hollow spherical conducting shell of radius 4 cm having a charge of 2×10^{-3} C on its surface is

(a) 1.1×10^{10} V/m
(b) 4.5×10^{-10} V/m
(c) 4.5×10^{10} V/m
(d) zero

39. Charge Q is given a displacement $\mathbf{r}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}$ in an electric field $\mathbf{E}=E_1\hat{\mathbf{i}}+E_2\hat{\mathbf{j}}$. The work done is

(a) $Q(E_1a+E_2b)$
(b) $Q\sqrt{(E_1a)^2+(E_2b)^2}$
(c) $Q(E_1+E_2)\sqrt{a^2+b^2}$
(d) $Q\sqrt{E_1^2+E_2^2}\sqrt{a^2+b^2}$

Subjective Questions

Note *You can take approximations in the answers.*

1. A certain charge Q is divided into two parts q and $Q-q$, which are then separated by a certain distance. What must q be in terms of Q to maximize the electrostatic repulsion between the two charges?

2. An α-particle is the nucleus of a helium atom. It has a mass $m=6.64\times10^{-27}$ kg and a charge $q=+2e=3.2\times10^{-19}$ C. Compare the force of the electric repulsion between two α-particles with the force of gravitational attraction between them.

3. What is the charge per unit area in C/m^2 of an infinite plane sheet of charge if the electric field produced by the sheet of charge has magnitude 3.0 N/C?

4. A circular wire loop of radius R carries a total charge q distributed uniformly over its length. A small length x ($<<R$) of the wire is cut off. Find the electric field at the centre due to the remaining wire.

5. Two identical conducting spheres, fixed in space, attract each other with an electrostatic force of 0.108 N when separated by 50.0 cm, centre-to-centre. A thin conducting wire then connects the spheres. When the wire is removed, the spheres repel each other with an electrostatic force of 0.0360 N. What were the initial charges on the spheres?

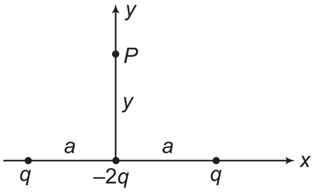
6. Show that the torque on an electric dipole placed in a uniform electric field is

$$\tau=\mathbf{p}\times\mathbf{E}$$

independent of the origin about which torque is calculated.

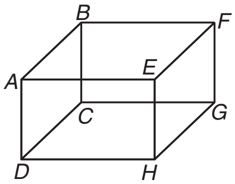
7. Three point charges $q, -2q$ and q are located along the x-axis as shown in figure. Show that the electric field at P $(y >> a)$ along the y-axis is,

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{3qa^2}{y^4}\hat{\mathbf{j}}$$



Note *This charge distribution which is essentially that of two electric dipoles is called an electric quadrupole. Note that* **E** *varies as* r^{-4} *for a quadrupole compared with variations of* r^{-3} *for the dipole and* r^{-2} *for a monopole (a single charge).*

8. A charge q is placed at point D of the cube. Find the electric flux passing through the face $EFGH$ and face $AEHD$.



9. Point charges q_1 and q_2 lie on the x-axis at points $x = -a$ and $x = +a$ respectively.

(a) How must q_1 and q_2 be related for the net electrostatic force on point charge $+Q$, placed at $x = +a/2$, to be zero?

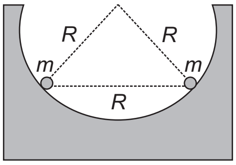
(b) With the same point charge $+Q$ now placed at $x = +3a/2$.

10. Two particles (free to move) with charges $+q$ and $+4q$ are a distance L apart. A third charge is placed so that the entire system is in equilibrium.

(a) Find the location, magnitude and sign of the third charge.

(b) Show that the equilibrium is unstable.

11. Two identical beads each have a mass m and charge q. When placed in a hemispherical bowl of radius R with frictionless, non-conducting walls, the beads move, and at equilibrium they are a distance R apart (figure). Determine the charge on each bead.



12. Three identical small balls, each of mass 0.1 g, are suspended at one point on silk thread having a length of $l = 20\,\text{cm}$. What charges should be imparted to the balls for each thread to form an angle of $\alpha = 30°$ with the vertical?

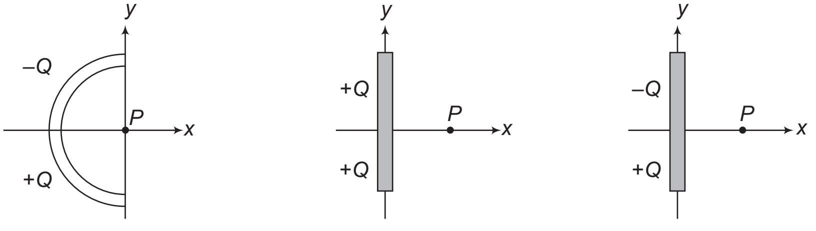
13. Three charges, each equal to q, are placed at the three corners of a square of side a. Find the electric field at fourth corner.

14. A point charge $q = -8.0$ nC is located at the origin. Find the electric field vector at the point $x = 1.2$ m, $y = -1.6$ m.

15. Find the electric field at the centre of a uniformly charged semicircular ring of radius R. Linear charge density is λ.

16. Find the electric field at a point P on the perpendicular bisector of a uniformly charged rod. The length of the rod is L, the charge on it is Q and the distance of P from the centre of the rod is a.

17. Find the direction of electric field at point P for the charge distribution as shown in figure.

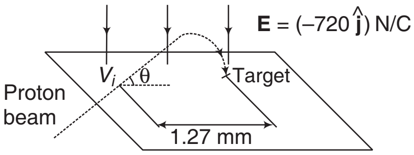


(a) (b) (c)

18. A clock face has charges $-q, -2q, -3q, \ldots -12q$ fixed at the position of the corresponding numerals on the dial. The clock hands do not disturb the net field due to point charges. At what time does the hour hand point in the direction of the electric field at the centre of the dial.

19. A charged particle of mass $m = 1$ kg and charge $q = 2\,\mu\text{C}$ is thrown from a horizontal ground at an angle $\theta = 45°$ with the speed 25 m/s. In space, a horizontal electric field $E = 2 \times 10^7$ V/m exists in the direction of motion. Find the range on horizontal ground of the projectile thrown. Take $g = 10\ \text{m/s}^2$.

20. Protons are projected with an initial speed $v_i = 9.55 \times 10^3$ m/s into a region where a uniform electric field $\mathbf{E} = (-720\,\hat{\mathbf{j}})$ N/C is present, as shown in figure. The protons are to hit a target that lies at a horizontal distance of 1.27 mm from the point where the protons are launched. Find

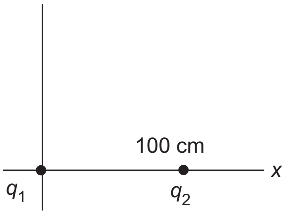


(a) the two projection angles θ that result in a hit and
(b) the total time of flight for each trajectory.

21. At some instant the velocity components of an electron moving between two charged parallel plates are $v_x = 1.5 \times 10^5$ m/s and $v_y = 3.0 \times 10^6$ m/s. Suppose that the electric field between the plates is given by $\mathbf{E} = (120\ \text{N/C})\,\hat{\mathbf{j}}$.

(a) What is the acceleration of the electron?
(b) What will be the velocity of the electron after its x-coordinate has changed by 2.0 cm?

22. A point charge $q_1 = +2\,\mu\text{C}$ is placed at the origin of coordinates. A second charge, $q_2 = -3\,\mu\text{C}$, is placed on the x-axis at $x = 100$ cm. At what point (or points) on the x-axis will the absolute potential be zero?



23. A charge Q is spread uniformly in the form of a line charge density $\lambda = \dfrac{Q}{3a}$ on the sides of an equilateral triangle of perimeter $3a$. Calculate the potential at the centroid C of the triangle.

24. A uniform electric field of magnitude 250 V/m is directed in the positive x-direction. A $+12\,\mu\text{C}$ charge moves from the origin to the point $(x, y) = (20.0\text{ cm}, 5.0\text{ cm})$.

(a) What was the change in the potential energy of this charge?

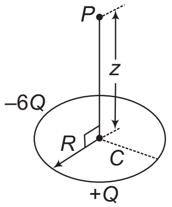
(b) Through what potential difference did the charge move?

25. A small particle has charge $-5.00\,\mu\text{C}$ and mass 2.00×10^{-4} kg. It moves from point A, where the electric potential is $V_A = +200$ V, to point B, where the electric potential is $V_B = +800$ V. The electric force is the only force acting on the particle. The particle has speed 5.00 m/s at point A. What is its speed at point B? Is it moving faster or slower at B than at A? Explain.

26. A plastic rod has been formed into a circle of radius R. It has a positive charge $+Q$ uniformly distributed along one-quarter of its circumference and a negative charge of $-6Q$ uniformly distributed along the rest of the circumference (figure). With $V = 0$ at infinity, what is the electric potential

(a) at the centre C of the circle and

(b) at point P, which is on the central axis of the circle at distance z from the centre?



27. A point charge $q_1 = +2.40\,\mu\text{C}$ is held stationary at the origin. A second point charge $q_2 = -4.30\,\mu\text{C}$ moves from the point $x = 0.150$ m, $y = 0$ to the point $x = 0.250$ m, $y = 0.250$ m. How much work is done by the electric force on q_2?

28. A point charge $q_1 = 4.00$ nC is placed at the origin, and a second point charge $q_2 = -3.00$ nC is placed on the x-axis at $x = +20.0$ cm. A third point charge $q_3 = 2.00$ nC is placed on the x-axis between q_1 and q_2. (Take as zero the potential energy of the three charges when they are infinitely far apart).

(a) What is the potential energy of the system of the three charges if q_3 is placed at $x = +10.0$ cm?

(b) Where should q_3 be placed to make the potential energy of the system equal to zero?

29. Three point charges, which initially are infinitely far apart, are placed at the corners of an equilateral triangle with sides d. Two of the point charges are identical and have charge q. If zero net work is required to place the three charges at the corners of the triangles, what must the value of the third charge be?

30. The electric field in a certain region is given by $\mathbf{E} = (5\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}})$ kV/m. Find the difference in potential $V_B - V_A$. If A is at the origin and point B is at (a) (0, 0, 5) m, (b) (4, 0, 3) m.

31. In a certain region of space, the electric field is along $+y$-direction and has a magnitude of 400 V/m. What is the potential difference from the coordinate origin to the following points?

(a) $x = 0$, $y = 20$ cm, $z = 0$

(b) $x = 0$, $y = -30$ cm, $z = 0$

(c) $x = 0$, $y = 0$, $z = 15$ cm

32. An electric field of 20 N/C exists along the x-axis in space. Calculate the potential difference $V_B - V_A$ where the points A and B are given by

(a) $A = (0, 0)$, $B = (4\text{ m}, 2\text{ m})$

(b) $A = (4\text{ m}, 2\text{ m})$, $B = (6\text{ m}, 5\text{ m})$

33. The electric potential existing in space is $V(x, y, z) = A(xy + yz + zx)$.

(a) Write the dimensional formula of A.

(b) Find the expression for the electric field.

(c) If A is 10 SI units, find the magnitude of the electric field at (1 m, 1 m, 1 m)

34. An electric field $\mathbf{E} = (20\,\hat{\mathbf{i}} + 30\,\hat{\mathbf{j}})$ N/C exists in the space. If the potential at the origin is taken to be zero, find the potential at $(2\,m, 2\,m)$.

35. In a certain region of space, the electric potential is $V(x, y, z) = Axy - Bx^2 + Cy$, where A, B and C are positive constants.

(a) Calculate the x, y and z-components of the electric field.

(b) At which points is the electric field equal to zero?

36. A sphere centered at the origin has radius 0.200 m. A $-500\,\mu\text{C}$ point charge is on the x-axis at $x = 0.300$ m. The net flux through the sphere is 360 N-m^2/ C. What is the total charge inside the sphere?

37. (a) A closed surface encloses a net charge of $-3.60\,\mu\text{C}$. What is the net electric flux through the surface?

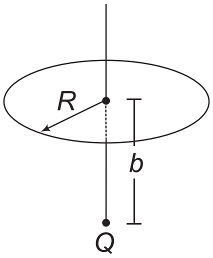
(b) The electric flux through a closed surface is found to be 780 N-m^2/C. What quantity of charge is enclosed by the surface?

(c) The closed surface in part (b) is a cube with sides of length 2.50 cm. From the information given in part (b), is it possible to tell where within the cube the charge is located? Explain.

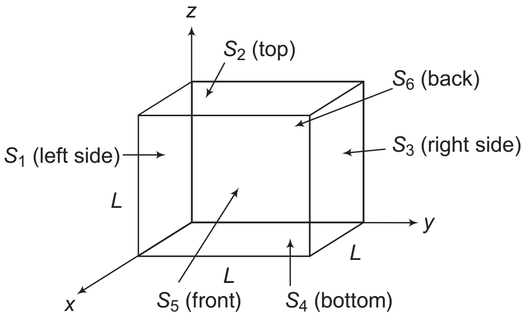
38. The electric field in a region is given by $\mathbf{E} = \frac{3}{5}E_0\hat{\mathbf{i}} + \frac{4}{5}E_0\hat{\mathbf{j}}$ with $E_0 = 2.0 \times 10^3$ N/C. Find the flux of this field through a rectangular surface of area 0.2 m^2 parallel to the y-z plane.

39. The electric field in a region is given by $\mathbf{E} = \frac{E_0 x}{l}\hat{\mathbf{i}}$. Find the charge contained inside a cubical volume bounded by the surfaces $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$. Take $E_0 = 5 \times 10^3$ N/C, $l = 2$ cm and $a = 1$ cm .

40. A point charge Q is located on the axis of a disc of radius R at a distance b from the plane of the disc (figure). Show that if one-fourth of the electric flux from the charge passes through the disc, then $R = \sqrt{3}\,b$.



41. A cube has sides of length L. It is placed with one corner at the origin as shown in figure. The electric field is uniform and given by $\mathbf{E} = -B\hat{\mathbf{i}} + C\hat{\mathbf{j}} - D\hat{\mathbf{k}}$, where B, C and D are positive constants.



(a) Find the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5 and S_6.

(b) Find the electric flux through the entire cube.

42. Two point charges q and $-q$ are separated by a distance $2l$. Find the flux of electric field strength vector across the circle of radius R placed with its centre coinciding with the mid-point of line joining the two charges in the perpendicular plane.

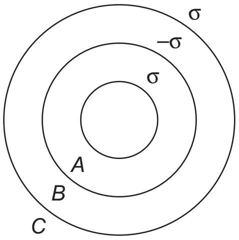
43. A point charge q is placed at the origin. Calculate the electric flux through the open hemispherical surface : $(x-a)^2 + y^2 + z^2 = a^2, x \geq a$

44. A charge Q is distributed over two concentric hollow spheres of radii r and $R (> r)$ such that the surface charge densities are equal. Find the potential at the common centre.

45. A charge q_0 is distributed uniformly on a ring of radius R. A sphere of equal radius R is constructed with its centre on the circumference of the ring. Find the electric flux through the surface of the sphere.

46. Two concentric conducting shells A and B are of radii R and $2R$. A charge $+q$ is placed at the centre of the shells. Shell B is earthed and a charge q is given to shell A. Find the charge on outer surface of A and B.

47. Three concentric metallic shells A, B and C of radii a, b and c $(a < b < c)$ have surface charge densities, $\sigma, -\sigma$ and σ respectively.



(a) Find the potentials of three shells A, B and C.

(b) It is found that no work is required to bring a charge q from shell A to shell C, then obtain the relation between the radii a, b and c.

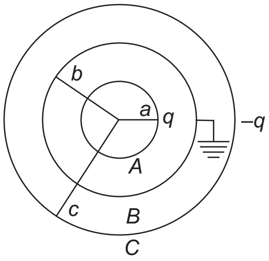
48. A charge Q is placed at the centre of an uncharged, hollow metallic sphere of radius a,

(a) Find the surface charge density on the inner surface and on the outer surface.

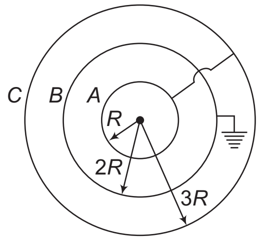
(b) If a charge q is put on the sphere, what would be the surface charge densities on the inner and the outer surfaces?

(c) Find the electric field inside the sphere at a distance x from the centre in the situations (a) and (b).

49. Figure shows three concentric thin spherical shells A, B and C of radii a, b and c respectively. The shells A and C are given charges q and $-q$ respectively and the shell B is earthed. Find the charges appearing on the surfaces of B and C.

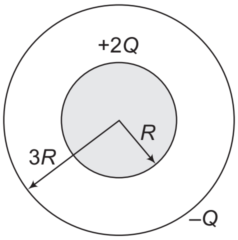


50. Three spherical shells have radii R, $2R$ and $3R$ respectively. Total charge on A and C is $3q$. Find the charges on different surfaces of A, B and C. The connecting wire does not touch the shell B.



51. In the above problem, the charges on different surfaces if a charge q is placed at the centre of the shell with all other conditions remaining the same.

52. A solid sphere of radius R has a charge $+2Q$. A hollow spherical shell of radius $3R$ placed concentric with the first sphere that has net charge $-Q$.



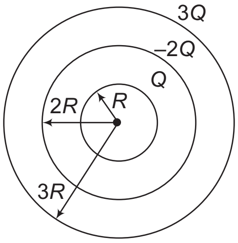
(a) Find the electric field between the spheres at a distance r from the centre of the inner sphere. $[R < r < 3R]$

(b) Calculate the potential difference between the spheres.

(c) What would be the final distribution of charges, if a conducting wire joins the spheres?

(d) Instead of (c), if the inner sphere is earthed, what is the charge on it?

53. Three concentric conducting spherical shells of radii R, $2R$ and $3R$ carry charges Q, $-2Q$ and $3Q$, respectively.



(a) Find the electric potential at $r = R$ and $r = 3R$, where r is the radial distance from the centre.

(b) Compute the electric field at $r = \frac{5}{2}R$

(c) Compute the total electrostatic energy stored in the system.

The inner shell is now connected to the external one by a conducting wire, passing through a very small hole in the middle shell.

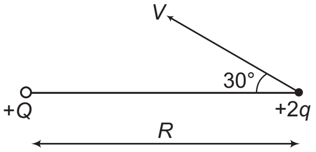
(d) Compute the charges on the spheres of radii R and $3R$.

(e) Compute the electric field at $r = \frac{5}{2}R$.

LEVEL 2

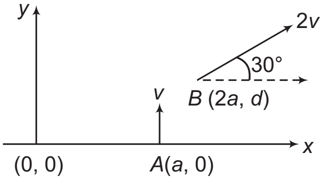
Single Correct Option

1. In the diagram shown, the charge $+Q$ is fixed. Another charge $+2q$ and mass M is projected from a distance R from the fixed charge. Minimum separation between the two charges if the velocity becomes $\frac{1}{\sqrt{3}}$ times of the projected velocity, at this moment is (Assume gravity to be absent)



(a) $\frac{\sqrt{3}}{2}R$ (b) $\frac{1}{\sqrt{3}}R$ (c) $\frac{1}{2}R$ (d) None of these

2. A uniform electric field of strength $\mathbf{E}$ exists in a region. An electron enters a point A with velocity v as shown. It moves through the electric field and reaches at point B. Velocity of particle at B is $2v$ at 30° with x-axis. Then,



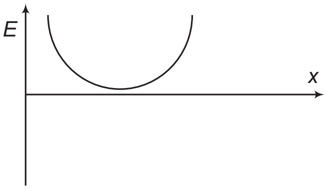
(a) electric field $\mathbf{E} = -\frac{3mv^2}{2ea}\hat{\mathbf{i}}$

(b) rate of doing work done by electric field at B is $\frac{3mv^3}{2ea}$

(c) Both (a) and (b) are correct

(d) Both (a) and (b) are wrong

3. Two point charges a and b whose magnitudes are same, positioned at a certain distance along the positive x-axis from each other. a is at origin. Graph is drawn between electrical field strength and distance x from a. E is taken positive if it is along the line joining from a to b. From the graph it can be decided that

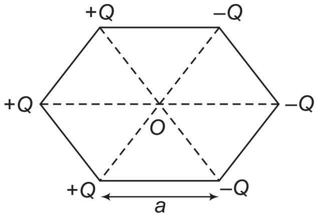


(a) a is positive, b is negative (b) a and b both are positive

(c) a and b both are negative (d) a is negative, b is positive

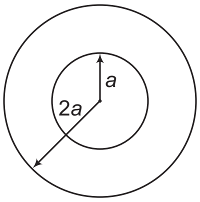
Note *Graph is drawn only between a and b.*

4. Six charges are placed at the vertices of a rectangular hexagon as shown in the figure. The electric field on the line passing through point O and perpendicular to the plane of the figure as a function of distance x from point O is ($x >> a$)



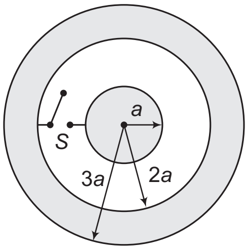
(a) 0 (b) $\dfrac{Qa}{\pi \varepsilon_0 x^3}$ (c) $\dfrac{2Qa}{\pi \varepsilon_0 x^3}$ (d) $\dfrac{\sqrt{3}Qa}{\pi \varepsilon_0 x^3}$

5. If the electric potential of the inner shell is 10 V and that of the outer shell is 5 V, then the potential at the centre will be



(a) 10 V (b) 5 V (c) 15 V (d) zero

6. A solid conducting sphere of radius a having a charge q is surrounded by a concentric conducting spherical shell of inner radius $2a$ and outer radius $3a$ as shown in figure. Find the amount of heat produced when switch is closed $\left(k = \dfrac{1}{4\pi\varepsilon_0}\right)$



(a) $\dfrac{kq^2}{2a}$ (b) $\dfrac{kq^2}{3a}$ (c) $\dfrac{kq^2}{4a}$ (d) $\dfrac{kq^2}{6a}$

7. There are four concentric shells A, B, C and D of radii $a, 2a, 3a$ and $4a$ respectively. Shells B and D are given charges $+q$ and $-q$ respectively. Shell C is now earthed. The potential difference $V_A - V_C$ is $\left(k = \dfrac{1}{4\pi\varepsilon_0}\right)$

(a) $\dfrac{kq}{2a}$ (b) $\dfrac{kq}{3a}$ (c) $\dfrac{kq}{4a}$ (d) $\dfrac{kq}{6a}$

8. Potential difference between centre and surface of the sphere of radius R and uniform volume charge density ρ within it will be

(a) $\dfrac{\rho R^2}{6\,\varepsilon_0}$ (b) $\dfrac{\rho R^2}{4\,\varepsilon_0}$

(c) $\dfrac{\rho R^2}{3\,\varepsilon_0}$ (d) $\dfrac{\rho R^2}{2\,\varepsilon_0}$

9. A positively charged disc is placed on a horizontal plane. A charged particle is released from a certain height on its axis. The particle just reaches the centre of the disc. Select the correct alternative.

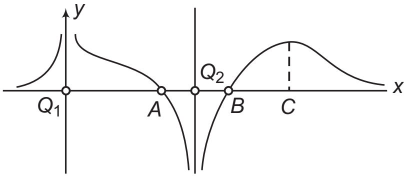
(a) Particle has negative charge on it

(b) Total potential energy (gravitational + electrostatic) of the particle first increases, then decreases

(c) Total potential energy of the particle first decreases, then increases

(d) Total potential energy of the particle continuously decreases

10. The curve represents the distribution of potential along the straight line joining the two charges Q_1 and Q_2 (separated by a distance r) then which of the following statements are correct?



1. $|Q_1| > |Q_2|$
2. Q_1 is positive in nature
3. A and B are equilibrium points
4. C is a point of unstable equilibrium

(a) 1 and 2 (b) 1, 2 and 3

(c) 1, 2 and 4 (d) 1, 2, 3 and 4

11. A point charge $q_1 = q$ is placed at point P. Another point charge $q_2 = -q$ is placed at point Q. At some point $R(R \neq P, R \neq Q)$, electric potential due to q_1 is V_1 and electric potential due to q_2 is V_2. Which of the following is correct?

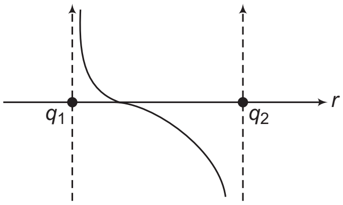
(a) Only for some points $V_1 > V_2$

(b) Only for some points $V_2 > V_1$

(c) For all points $V_1 > V_2$

(d) For all points $V_2 > V_1$

12. The variation of electric field between two charges q_1 and q_2 along the line joining the charges is plotted against distance from q_1 (taking rightward direction of electric field as positive) as shown in the figure. Then, the correct statement is



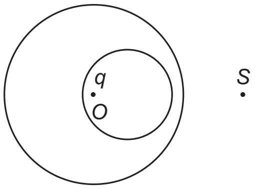
(a) q_1 and q_2 are positive and $q_1 < q_2$

(b) q_1 and q_2 are positive and $q_1 > q_2$

(c) q_1 is positive and q_2 is negative $q_1 < |q_2|$

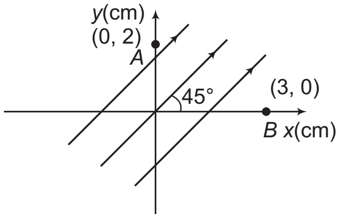
(d) q_1 and q_2 are negative and $|q_1| < |q_2|$

13. A charge q is placed at O in the cavity in a spherical uncharged conductor. Point S is outside the conductor. If q is displaced from O towards S (still remaining within the cavity)



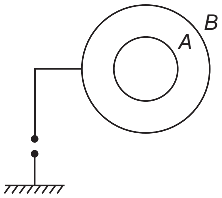
(a) electric field at S will increase
(b) electric field at S will decrease
(c) electric field at S will first increase and then decrease
(d) electric field at S will not change

14. A uniform electric field of 400 V/m is directed at 45° above the x-axis as shown in the figure. The potential difference $V_A - V_B$ is given by



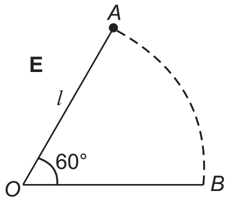
(a) 0
(b) 4 V
(c) 6.4 V
(d) 2.8 V

15. Initially the spheres A and B are at potentials V_A and V_B respectively. Now, sphere B is earthed by closing the switch. The potential of A will now become



(a) 0
(b) V_A
(c) $V_A - V_B$
(d) V_B

16. A particle of mass m and charge q is fastened to one end of a string of length l. The other end of the string is fixed to the point O. The whole system lies on a frictionless horizontal plane. Initially, the mass is at rest at A. A uniform electric field in the direction shown is then switched on. Then,



(a) the speed of the particle when it reaches B is $\sqrt{\frac{2qEl}{m}}$
(b) the speed of the particle when it reaches B is $\sqrt{\frac{qEl}{m}}$
(c) the tension in the string when the particle reaches at B is qE
(d) the tension in the string when the particle reaches at B is zero

17. A charged particle of mass m and charge q is released from rest from the position $(x_0, 0)$ in a uniform electric field $E_0\hat{\mathbf{j}}$. The angular momentum of the particle about origin

(a) is zero
(b) is constant
(c) increases with time
(d) decreases with time

18. A charge $+Q$ is uniformly distributed in a spherical volume of radius R. A particle of charge $+q$ and mass m projected with velocity v_0 from the surface of the spherical volume to its centre inside a smooth tunnel dug across the sphere. The minimum value of v_0 such that it just reaches the centre (assume that there is no resistance on the particle except electrostatic force) of the spherical volume is

(a) $\sqrt{\dfrac{Qq}{2\pi\varepsilon_0 mR}}$ (b) $\sqrt{\dfrac{Qq}{\pi\varepsilon_0 mR}}$

(c) $\sqrt{\dfrac{2Qq}{\pi\varepsilon_0 mR}}$ (d) $\sqrt{\dfrac{Qq}{4\pi\varepsilon_0 mR}}$

19. Two identical coaxial rings each of radius R are separated by a distance of $\sqrt{3}R$. They are uniformly charged with charges $+Q$ and $-Q$ respectively. The minimum kinetic energy with which a charged particle (charge $+q$) should be projected from the centre of the negatively charged ring along the axis of the rings such that it reaches the centre of the positively charged ring is

(a) $\dfrac{Qq}{4\pi\varepsilon_0 R}$ (b) $\dfrac{Qq}{2\pi\varepsilon_0 R}$ (c) $\dfrac{Qq}{8\pi\varepsilon_0 R}$ (d) $\dfrac{3Qq}{4\pi\varepsilon_0 R}$

20. A uniform electric field exists in x-y plane. The potential of points $A(2\,\text{m}, 2\,\text{m})$, $B(-2\,\text{m}, 2\,\text{m})$ and $C(2\,\text{m}, 4\,\text{m})$ are 4 V, 16 V and 12 V respectively. The electric field is

(a) $(4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ V/m (b) $(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ V/m

(c) $-(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ V/m (d) $(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$ V/m

21. Two fixed charges $-2Q$ and $+Q$ are located at points $(-3a, 0)$ and $(+3a, 0)$ respectively. Then, which of the following statement is correct?

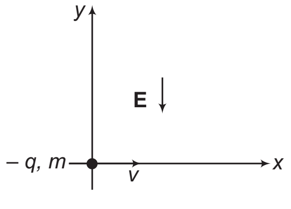
(a) Points where the electric potential due to the two charges is zero in x-y plane, lie on a circle of radius $4a$ and centre $(5a, 0)$

(b) Potential is zero at $x = a$ and $x = 9a$

(c) Both (a) and (b) are wrong

(d) Both (a) and (b) are correct

22. A particle of mass m and charge $-q$ is projected from the origin with a horizontal speed v into an electric field of intensity E directed downward. Choose the **wrong** statement. Neglect gravity



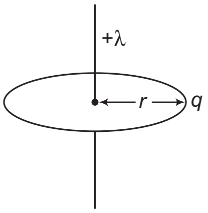
(a) The kinetic energy after a displacement y is qEy

(b) The horizontal and vertical components of acceleration are $a_x = 0, a_y = \dfrac{qE}{m}$

(c) The equation of trajectory is $y = \dfrac{1}{2}\left(\dfrac{qEx^2}{mv^2}\right)$

(d) The horizontal and vertical displacements x and y after a time t are $x = vt$ and $y = \dfrac{1}{2}a_y t^2$

23. A particle of charge $-q$ and mass m moves in a circle of radius r around an infinitely long line charge of linear charge density $+\lambda$. Then, time period will be

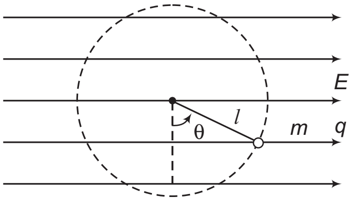


(a) $T = 2\pi r\sqrt{\dfrac{m}{2k\lambda q}}$ (b) $T^2 = \dfrac{4\pi^2 m}{2k\lambda q} r^3$

(c) $T = \dfrac{1}{2\pi r}\sqrt{\dfrac{2k\lambda q}{m}}$ (d) $T = \dfrac{1}{2\pi r}\sqrt{\dfrac{m}{2k\lambda q}}$

where, $k = \dfrac{1}{4\pi\varepsilon_0}$

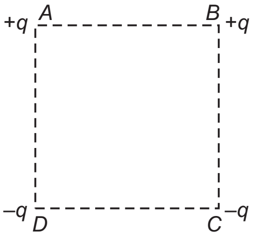
24. A small ball of mass m and charge $+q$ tied with a string of length l, rotating in a vertical circle under gravity and a uniform horizontal electric field E as shown. The tension in the string will be minimum for



(a) $\theta = \tan^{-1}\left(\dfrac{qE}{mg}\right)$ (b) $\theta = \pi$

(c) $\theta = 0°$ (d) $\theta = \pi + \tan^{-1}\left(\dfrac{qE}{mg}\right)$

25. Four point charges A, B, C and D are placed at the four corners of a square of side a. The energy required to take the charges C and D to infinity (they are also infinitely separated from each other) is

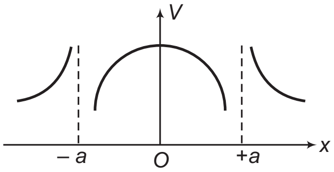


(a) $\dfrac{q^2}{4\pi\varepsilon_0 a}$ (b) $\dfrac{2q^2}{\pi\varepsilon_0 a}$

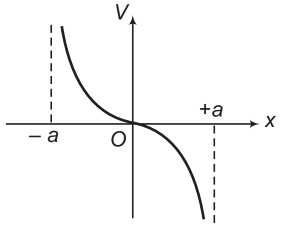
(c) $\dfrac{q^2}{4\pi\varepsilon_0 a}(\sqrt{2}+1)$ (d) $\dfrac{q^2}{4\pi\varepsilon_0 a}(\sqrt{2}-1)$

26. Two identical positive charges are placed at $x = -a$ and $x = a$. The correct variation of potential V along the x-axis is given by

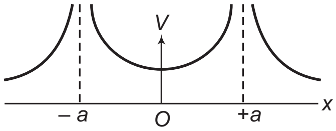
(a)



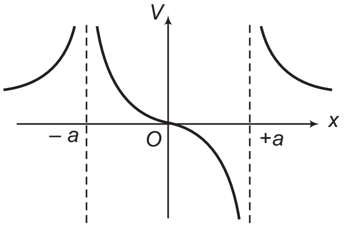
(b)



(c)



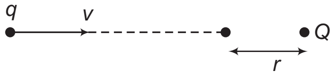
(d)



27. Two identical charges are placed at the two corners of an equilateral triangle. The potential energy of the system is U. The work done in bringing an identical charge from infinity to the third vertex is

(a) U
(b) $2U$
(c) $3U$
(d) $4U$

28. A charged particle q is shot from a large distance towards another charged particle Q which is fixed, with a speed v. It approaches Q up to a closest distance r and then returns. If q were given a speed $2v$, the distance of approach would be

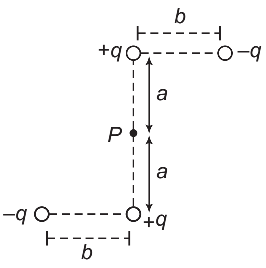


(a) r
(b) $2r$
(c) $r/2$
(d) $r/4$

29. Two identical charged spheres are suspended by strings of equal length. The strings make an angle of 30° with each other. When suspended in a liquid of density 0.8 g/cc, the angle remains the same. The dielectric constant of the liquid is [density of the material of sphere is 1.6 g/cc]

(a) 2
(b) 4
(c) 2.5
(d) 3.5

30. The electrostatic potential due to the charge configuration at point P as shown in figure for $b << a$ is



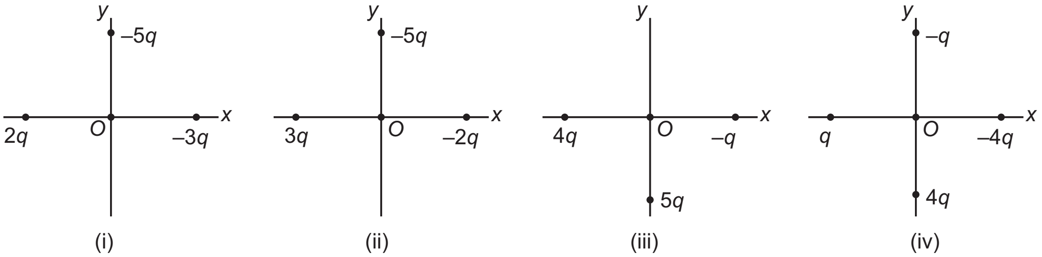
(a) $\dfrac{2q}{4\pi\varepsilon_0 a}$

(b) $\dfrac{2qb^2}{4\pi\varepsilon_0 a^3}$

(c) $\dfrac{qb^2}{4\pi\varepsilon_0 a^3}$

(d) zero

31. The figure shows four situations in which charged particles are at equal distances from the origin. If E_1, E_2, E_3 and E_4 be the magnitude of the net electric fields at the origin in four situations (i), (ii), (iii) and (iv) respectively, then



(a) $E_1 = E_2 = E_3 = E_4$
(b) $E_1 = E_2 > E_3 > E_4$
(c) $E_1 < E_2 < E_3 = E_4$
(d) $E_1 > E_2 = E_3 < E_4$

32. An isolated conducting sphere whose radius $R = 1$ m has a charge $q = \frac{1}{9}$ nC. The energy density at the surface of the sphere is

(a) $\frac{\varepsilon_0}{2}$ J/m^3
(b) ε_0 J/m^3
(c) $2\varepsilon_0$ J/m^3
(d) $\frac{\varepsilon_0}{3}$ J/m^3

33. Two conducting concentric, hollow spheres A and B have radii a and b respectively, with A inside B. Their common potentials is V. A is now given some charge such that its potential becomes zero. The potential of B will now be

(a) 0
(b) $V(1 - a/b)$
(c) Va/b
(d) Vb/a

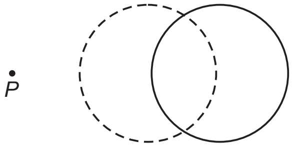
34. In a uniform electric field, the potential is 10 V at the origin of coordinates and 8 V at each of the points (1, 0, 0), (0, 1, 0) and (0, 0, 1). The potential at the point (1, 1, 1) will be

(a) 0 (b) 4 V (c) 8 V (d) 10 V

35. There are two uncharged identical metallic spheres 1 and 2 of radius r separated by a distance d ($d >> r$). A charged metallic sphere of same radius having charge q is touched with one of the sphere. After some time it is moved away from the system. Now, the uncharged sphere is earthed. Charge on earthed sphere is

(a) $+\frac{q}{2}$
(b) $-\frac{q}{2}$
(c) $-\frac{qr}{2d}$
(d) $-\frac{qd}{2r}$

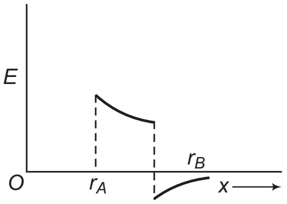
36. Figure shows a closed dotted surface which intersects a conducting uncharged sphere. If a positive charge is placed at the point P, the flux of the electric field through the closed surface



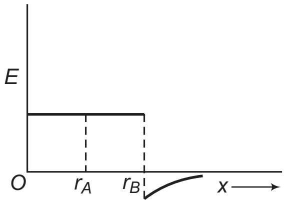
(a) will remain zero
(b) will become positive
(c) will become negative
(d) data insufficient

37. Two concentric conducting thin spherical shells A and B having radii r_A and $r_B (r_B > r_A)$ are charged to Q_A and $-Q_B (|Q_B| > |Q_A|)$. The electrical field along a line passing through the centre is

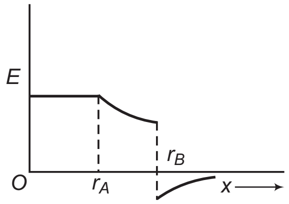
(a)



(b)



(c)



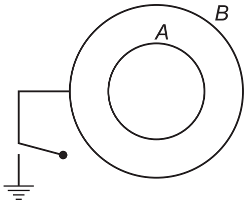
(d) None of these

38. The electric potential at a point (x, y) in the x-y plane is given by $V = -kxy$. The field intensity at a distance r in this plane, from the origin is proportional to

(a) r^2
(b) r
(c) $1/r$
(d) $1/r^2$

More than One Correct Options

1. Two concentric shells have radii R and $2R$ charges q_A and q_B and potentials 2 V and (3/2) V respectively. Now, shell B is earthed and let charges on them become q_A' and q_B'. Then,



(a) $q_A/q_B = 1/2$
(b) $q_A'/q_B' = 1$
(c) potential of A after earthing becomes (3/2) V
(d) potential difference between A and B after earthing becomes $V/2$

2. A particle of mass 2 kg and charge 1 mC is projected vertically with a velocity 10 ms^{-1}. There is a uniform horizontal electric field of 10^4 N/C, then

(a) the horizontal range of the particle is 10 m
(b) the time of flight of the particle is 2 s
(c) the maximum height reached is 5 m
(d) the horizontal range of the particle is 5 m

3. At a distance of 5 cm and 10 cm from surface of a uniformly charged solid sphere, the potentials are 100 V and 75 V respectively. Then,

(a) potential at its surface is 150 V
(b) the charge on the sphere is $\frac{50}{3} \times 10^{-10}$ C
(c) the electric field on the surface is 1500 V/m
(d) the electric potential at its centre is 25 V

4. Three charged particles are in equilibrium under their electrostatic forces only. Then,

(a) the particles must be collinear

(b) all the charges cannot have the same magnitude

(c) all the charges cannot have the same sign

(d) the equilibrium is unstable

5. Charges Q_1 and Q_2 lie inside and outside respectively of a closed surface S. Let E be the field at any point on S and ϕ be the flux of E over S.

(a) If Q_1 changes, both E and ϕ will change

(b) If Q_2 changes, E will change but ϕ will not change

(c) If $Q_1 = 0$ and $Q_2 \neq 0$, then $E \neq 0$ but $\phi = 0$

(d) If $Q_1 \neq 0$ and $Q_2 = 0$, then $E = 0$ but $\phi \neq 0$

6. An electric dipole is placed at the centre of a sphere. Mark the correct options.

(a) The flux of the electric field through the sphere is zero

(b) The electric field is zero at every point of the sphere

(c) The electric field is not zero at anywhere on the sphere

(d) The electric field is zero on a circle on the sphere

7. Mark the correct options.

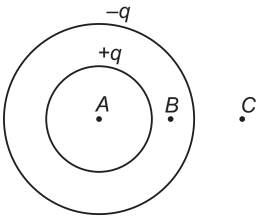
(a) Gauss's law is valid only for uniform charge distributions

(b) Gauss's law is valid only for charges placed in vacuum

(c) The electric field calculated by Gauss's law is the field due to all the charges

(d) The flux of the electric field through a closed surface due to all the charges is equal to the flux due to the charges enclosed by the surface

8. Two concentric spherical shells have charges $+q$ and $-q$ as shown in figure. Choose the correct options.



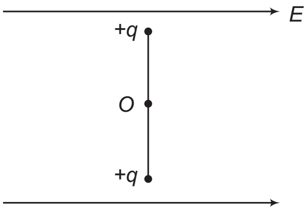
(a) At A electric field is zero, but electric potential is non-zero

(b) At B electric field and electric potential both are non-zero

(c) At C electric field is zero but electric potential is non-zero

(d) At C electric field and electric potential both are zero

9. A rod is hinged (free to rotate) at its centre O as shown in figure. Two point charges $+q$ and $+q$ are kept at its two ends. Rod is placed in uniform electric field E as shown. Space is gravity free. Choose the correct options.

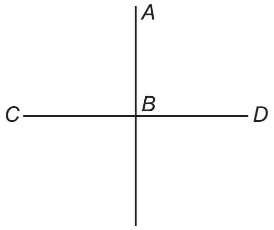


(a) Net force from the hinge on the rod is zero

(b) Net force from the hinge on the rod is leftwards

(c) Equilibrium of rod is neutral

(d) Equilibrium of rod is stable



10. Two charges $+Q$ each are fixed at points C and D. Line AB is the bisector line of CD. A third charge $+q$ is moved from A to B, then from B to C.

(a) From A to B electrostatic potential energy will decrease

(b) From A to B electrostatic potential energy will increase

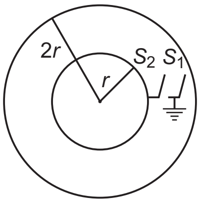
(c) From B to C electrostatic potential energy will increase

(d) From B to C electrostatic potential energy will decrease

Comprehension Based Questions

Passage I (Q. No. 1 to 3)

There are two concentric spherical shell of radii r and $2r$. Initially, a charge Q is given to the inner shell and both the switches are open.



1. If switch S_1 is closed and then opened, charge on the outer shell will be

(a) Q (b) $Q/2$

(c) $-Q$ (d) $-Q/2$

2. Now, S_2 is closed and opened. The charge flowing through the switch S_2 in the process is

(a) Q (b) $Q/4$

(c) $Q/2$ (d) $2Q/3$

3. The two steps of the above two problems are repeated n times, the potential difference between the shells will be

(a) $\frac{1}{2^{n+1}}\left[\frac{Q}{4\pi\varepsilon_0 r}\right]$ (b) $\frac{1}{2^n}\left[\frac{Q}{4\pi\varepsilon_0 r}\right]$

(c) $\frac{1}{2^n}\left[\frac{Q}{2\pi\varepsilon_0 r}\right]$ (d) $\frac{1}{2^{n-1}}\left[\frac{Q}{2\pi\varepsilon_0 r}\right]$

Passage II (Q. No. 4 to 7)

A sphere of charge of radius R carries a positive charge whose volume charge density depends only on the distance r from the ball's centre as $\rho = \rho_0\left(1 - \frac{r}{R}\right)$, where ρ_0 is a constant. Assume ε as the permittivity of space.

4. The magnitude of electric field as a function of the distance r inside the sphere is given by

(a) $E = \frac{\rho_0}{\varepsilon}\left[\frac{r}{3} - \frac{r^2}{4R}\right]$ (b) $E = \frac{\rho_0}{\varepsilon}\left[\frac{r}{4} - \frac{r^2}{3R}\right]$

(c) $E = \frac{\rho_0}{\varepsilon}\left[\frac{r}{3} + \frac{r^2}{4R}\right]$ (d) $E = \frac{\rho_0}{\varepsilon}\left[\frac{r}{4} + \frac{r^2}{3R}\right]$

5. The magnitude of the electric field as a function of the distance r outside the ball is given by

(a) $E = \frac{\rho_0 R^3}{8\varepsilon r^2}$ (b) $E = \frac{\rho_0 R^3}{12\varepsilon r^2}$

(c) $E = \frac{\rho_0 R^2}{8\varepsilon r^3}$ (d) $E = \frac{\rho_0 R^2}{12\varepsilon r^3}$

6. The value of distance r_m at which electric field intensity is maximum is given by

(a) $r_m = \frac{R}{3}$ (b) $r_m = \frac{3R}{2}$

(c) $r_m = \frac{2R}{3}$ (d) $r_m = \frac{4R}{3}$

7. The maximum electric field intensity is

(a) $E_m = \frac{\rho_0 R}{9\,\varepsilon}$ (b) $E_m = \frac{\rho_0 \varepsilon}{9R}$

(c) $E_m = \frac{\rho_0 R}{3\varepsilon}$ (d) $E_m = \frac{\rho_0 R}{6\,\varepsilon}$

Passage III (Q. No. 8 to 10)

A solid metallic sphere of radius a is surrounded by a conducting spherical shell of radius b ($b > a$). The solid sphere is given a charge Q. A student measures the potential at the surface of the solid sphere as V_a and the potential at the surface of spherical shell as V_b. After taking these readings, he decides to put charge of $-4Q$ on the shell. He then noted the readings of the potential of solid sphere and the shell and found that the potential difference is ΔV. He then connected the outer spherical shell to the earth by a conducting wire and found that the charge on the outer surface of the shell as q_1.

He then decides to remove the earthing connection from the shell and earthed the inner solid sphere. Connecting the inner sphere with the earth he observes the charge on the solid sphere as q_2. He then wanted to check what happens if the two are connected by the conducting wire. So he removed the earthing connection and connected a conducting wire between the solid sphere and the spherical shell. After the connections were made he found the charge on the outer shell as q_3.

Answer the following questions based on the readings taken by the student at various stages.

8. Potential difference (ΔV) measured by the student between the inner solid sphere and outer shell after putting a charge $-4Q$ is

(a) $V_a - 3V_b$ (b) $3(V_a - V_b)$

(c) V_a (d) $V_a - V_b$

9. q_2 is

(a) Q (b) $Q\left(\frac{a}{b}\right)$

(c) $-4Q$ (d) zero

10. q_3 is

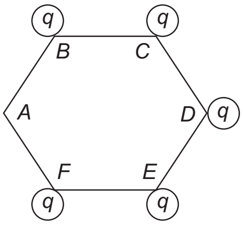
(a) $\frac{Q(a+b)}{a-b}$ (b) $\frac{Qa^2}{b}$

(c) $\frac{Q(a-b)}{b}$ (d) $-\frac{Qb}{a}$

Match the Columns

1. Five identical charges are kept at five vertices of a regular hexagon. Match the following two columns at centre of the hexagon. If in the given situation electric field at centre is E. Then,

Column I	Column II
(a) If charge at B is removed, then electric field will become	(p) $2E$
(b) If charge at C is removed, then electric field will become	(q) E
(c) If charge at D is removed then electric field will become	(r) zero
(d) If charges at B and C both are removed, then electric field will become	(s) $\sqrt{3}\,E$



Note *Only magnitudes of electric field are given.*

2. In an electric field $\mathbf{E} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ N/C, electric potential at origin is 0 V. Match the following two columns.

Column I	Column II
(a) Potential at (4 m, 0)	(p) 8 V
(b) Potential at (–4 m, 0)	(q) – 8 V
(c) Potential at (0, 4 m)	(r) 16 V
(d) Potential at (0, – 4 m)	(s) – 16 V

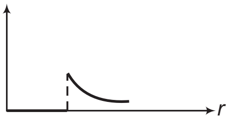
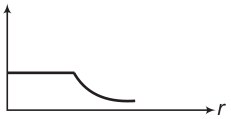
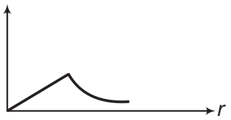
3. Electric potential on the surface of a solid sphere of charge is V. Radius of the sphere is 1m. Match the following two columns.

Column I	Column II
(a) Electric potential at $r = \frac{R}{2}$	(p) $\frac{V}{4}$
(b) Electric potential at $r = 2R$	(q) $\frac{V}{2}$
(c) Electric field at $r = \frac{R}{2}$	(r) $\frac{3V}{4}$
(d) Electric field at $r = 2R$	(s) None of these

4. Match the following two columns.

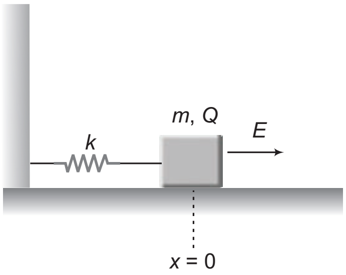
Column I	Column II
(a) Electric potential	(p) $[\mathrm{MLT^{-3}A^{-1}}]$
(b) Electric field	(q) $[\mathrm{ML^3T^{-3}A^{-1}}]$
(c) Electric flux	(r) $[\mathrm{ML^2T^{-3}A^{-1}}]$
(d) Permittivity of free space	(s) None of these

5. Match the following two columns.

Column I	Column II
(a) Electric field due to charged spherical shell	(p) 
(b) Electric potential due to charged spherical shell	(q) 
(c) Electric field due to charged solid sphere	(r) 
(d) Electric potential due to charged solid sphere	(s) None of these

Subjective Questions

1. A 4.00 kg block carrying a charge $Q = 50.0\,\mu\text{C}$ is connected to a spring for which $k = 100\,\text{N/m}$. The block lies on a frictionless horizontal track, and the system is immersed in a uniform electric field of magnitude $E = 5.00 \times 10^5\ \text{V/m}$, directed as shown in figure. If the block is released from rest when the spring is unstretched (at $x = 0$).



(a) By what maximum amount does the spring expand?
(b) What is the equilibrium position of the block?
(c) Show that the block's motion is simple harmonic and determine its period.
(d) Repeat part (a) if the coefficient of kinetic friction between block and surface is 0.2.

2. A particle of mass m and charge $-Q$ is constrained to move along the axis of a ring of radius a. The ring carries a uniform charge density $+\lambda$ along its length. Initially, the particle is in the centre of the ring where the force on it is zero. Show that the period of oscillation of the particle when it is displaced slightly from its equilibrium position is given by

$$T = 2\pi\sqrt{\frac{2\varepsilon_0 m a^2}{\lambda Q}}$$

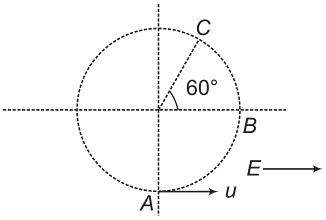
3. Three identical conducting plane parallel plates, each of area A are held with equal separation d between successive surfaces. Charges Q, $2Q$, and $3Q$ are placed on them. Neglecting edge effects, find the distribution of charges on the six surfaces.

4. A long non-conducting, massless rod of length L pivoted at its centre and balanced with a weight w at a distance x from the left end. At the left and right ends of the rod are attached small conducting spheres with positive charges q and $2q$ respectively. A distance h directly beneath each of these spheres is a fixed sphere with positive charge Q.
   (a) Find the distance x where the rod is horizontal and balanced.
   (b) What value should h have so that the rod exerts no vertical force on the bearing when the rod is horizontal and balanced?

Note *Ignore the force between Q (beneath q) and 2q and the force between Q (beneath 2q) and q. Also the force between Q and Q.*

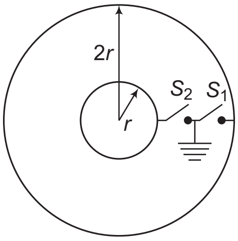
5. The electric potential varies in space according to the relation $V = 3x + 4y$. A particle of mass 10 kg starts from rest from point (2, 3.2) m under the influence of this field. Find the velocity of the particle when it crosses the x-axis. The charge on the particle is $+1\,\mu\text{C}$. Assume $V\,(x, y)$ are in SI units.

6. A simple pendulum with a bob of mass $m = 1$ kg, charge $q = 5\,\mu\text{C}$ and string length $l = 1$ m is given a horizontal velocity u in a uniform electric field $E = 2 \times 10^6$ V/m at its bottommost point A, as shown in figure. It is given that the speed u is such that the particle leaves the circle at point C. Find the speed u (Take $g = 10\text{ m/s}^2$)

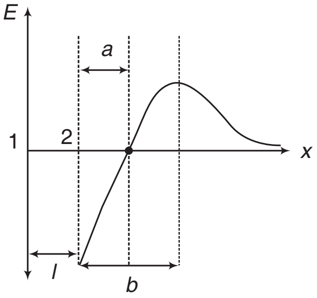


7. Eight point charges of magnitude Q are arranged to form the corners of a cube of side L. The arrangement is made in manner such that the nearest neighbour of any charge has the opposite sign. Initially, the charges are held at rest. If the system is let free to move, what happens to the arrangement? Does the cube-shape shrink or expand? Calculate the velocity of each charge when the side-length of the cube formation changes from L to nL. Assume that the mass of each point charge is m.

8. There are two concentric spherical shells of radii r and $2r$. Initially, a charge Q is given to the inner shell. Now, switch S_1 is closed and opened then S_2 is closed and opened and the process is repeated n times for both the keys alternatively. Find the final potential difference between the shells.



9. Two point charges Q_1 and Q_2 are positioned at points 1 and 2. The field intensity to the right of the charge Q_2 on the line that passes through the two charges varies according to a law that is represented schematically in the figure. The field intensity is assumed to be positive if its direction coincides with the positive direction on the x-axis. The distance between the charges is l.



(a) Find the sign of each charge.

(b) Find the ratio of the absolute values of the charges $\left|\dfrac{Q_1}{Q_2}\right|$

(c) Find the value of b where the field intensity is maximum.

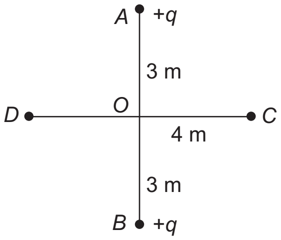
10. A conducting sphere S_1 of radius r is attached to an insulating handle. Another conducting sphere S_2 of radius $R\,(> r)$ is mounted on an insulating stand, S_2 is initially uncharged. S_1 is given a charge Q. Brought into contact with S_2 and removed. S_1 is recharged such that the charge on it is again Q and it is again brought into contact with S_2 and removed. This procedure is repeated n times.

(a) Find the electrostatic energy of S_2 after n such contacts with S_1.

(b) What is the limiting value of this energy as $n \to \infty$?

11. A proton of mass m and accelerated by a potential difference V gets into a uniform electric field of a parallel plate capacitor parallel to plates of length l at mid-point of its separation between plates. The field strength in it varies with time as $E = at$, where a is a positive constant. Find the angle of deviation of the proton as it comes out of the capacitor. (Assume that it does not collide with any of the plates.)

12. Two fixed, equal, positive charges, each of magnitude 5×10^{-5} C are located at points A and B separated by a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of the line AB. The moving charge when it reaches the point C at a distance of 4 m from O, has a kinetic energy of 4 J. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C.

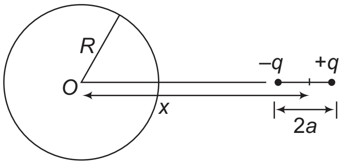


13. Positive charge Q is uniformly distributed throughout the volume of a sphere of radius R. A point mass having charge $+q$ and mass m is fired towards the centre of the sphere with velocity

v from a point A at distance $r(r > R)$ from the centre of the sphere. Find the minimum velocity v so that it can penetrate $R/2$ distance of the sphere. Neglect any resistance other than electric interaction. Charge on the small mass remains constant throughout the motion.

14. Two concentric rings placed in a gravity free region in yz-plane one of radius R carries a charge $+Q$ and second of radius $4R$ and charge $-8Q$ distributed uniformly over it. Find the minimum velocity with which a point charge of mass m and charge $+q$ should be projected from a point at a distance $3R$ from the centre of rings on its axis so that it will reach to the centre of the rings.

15. An electric dipole is placed at a distance x from centre O on the axis of a charged ring of radius R and charge Q uniformly distributed over it.



(a) Find the net force acting on the dipole.

(b) What is the work done in rotating the dipole through 180°?

16. A point charge $-q$ revolves around a fixed charge $+Q$ in elliptical orbit. The minimum and maximum distance of q from Q are r_1 and r_2, respectively. The mass of revolving particle is m. $Q > q$ and assume no gravitational effects. Find the velocity of q at positions when it is at r_1 and r_2 distance from Q.

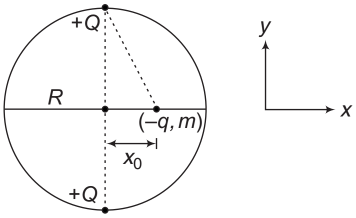
17. Three concentric, thin, spherical, metallic shells have radii 1, 2, and 4 cm and they are held at potentials 10, 0 and 40 V respectively. Taking the origin at the common centre, calculate the following:

(a) Potential at $r = 1.25$ cm

(b) Potential at $r = 2.5$ cm

(c) Electric field at $r = 1.25$ cm

18. A thin insulating wire is stretched along the diameter of an insulated circular hoop of radius R. A small bead of mass m and charge $-q$ is threaded onto the wire. Two small identical charges are tied to the hoop at points opposite to each other, so that the diameter passing through them is perpendicular to the thread (see figure). The bead is released at a point which is a distance x_0 from the centre of the hoop. Assume that $x_0 << R$.



(a) What is the resultant force acting on the charged bead?

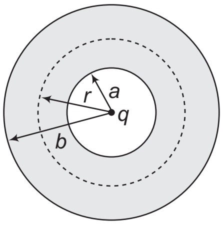
(b) Describe (qualitatively) the motion of the bead after it is released.

(c) Use the assumption that $\frac{x}{R} << 1$ to obtain an approximate equation of motion, and find the displacement and velocity of the bead as functions of time.

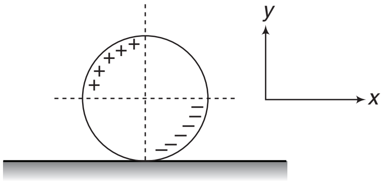
(d) When will the velocity of the bead will become zero for the first time?

19. The region between two concentric spheres of radii a and $b\,(> a)$ contains volume charge density $\rho(r) = \dfrac{C}{r}$, where C is a constant and r is the radial distance as shown in figure. A point charge q is placed at the origin, $r = 0$.

Find the value of C for which the electric field in the region between the spheres is constant (i.e. r independent).

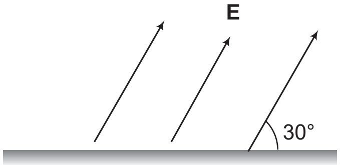


20. A non-conducting ring of mass m and radius R is charged as shown. The charge density, i.e. charge per unit length is λ. It is then placed on a rough non-conducting horizontal plane. At time $t = 0$, a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{i}}$ is switched on and the ring starts rolling without sliding. Determine the friction force (magnitude and direction) acting on the ring when it starts moving.



21. A rectangular tank of mass m_0 and charge Q over it is placed on a smooth horizontal floor. A horizontal electric field E exists in the region. Rain drops are falling vertically in the tank at the constant rate of n drops per second. Mass of each drop is m. Find velocity of tank as function of time.

22. In a region, an electric field $E = 15$ N/C making an angle of 30° with the horizontal plane is present. A ball having charge $2C$, mass 3 kg and coefficient of restitution with ground 1/2 is projected at an angle of 30° with the horizontal in the direction of electric field with speed 20 m/s. Find the horizontal distance travelled by ball from first drop to the second drop.



Answers

Introductory Exercise 24.1

1. No, as attraction can take place between a charged and an uncharged body too.
2. Yes
3. Record gets charged when cleaned and then by induction, it attracts dust particles.
4. -2.89×10^5 C

Introductory Exercise 24.2

1. 2.27×10^{39} **2.** $[\mathrm{M^{-1}\,L^{-3}\,T^{4}\,A^{2}}]$, $\dfrac{\mathrm{C}}{\mathrm{V\text{-}m}}$ **3.** $\dfrac{\sqrt{3}}{4\pi\varepsilon_0}\left(\dfrac{q}{a}\right)^2$ **4.** $\dfrac{1}{2\pi\varepsilon_0}\cdot\left(\dfrac{q}{a}\right)^2$
5. No **6.** Induction → Conduction → Repulsion **7.** Yes
8. No **9.** $(-4\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ N

Introductory Exercise 24.3

1. False **2.** At A **3.** False **4.** False
5. q_1 and q_3 are positive and q_2 is negative **6.** $\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{q}{a^2}$
7. $-(4.32\,\hat{\mathbf{i}} + 5.76\,\hat{\mathbf{j}}) \times 10^2$ N/C

Introductory Exercise 24.4

1. 18.97 m/s **2.** -9 mJ **3.** -10.6×10^{-8} J **4.** No, Yes

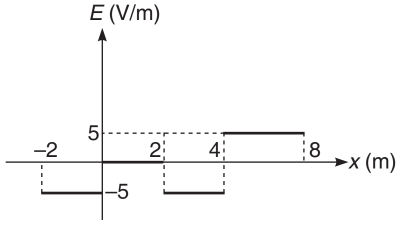
Introductory Exercise 24.5

1. 1.2×10^3 V **2.** (a) $\dfrac{\mathrm{C}}{\mathrm{m}^2}$ (b) $\dfrac{1}{4\pi\varepsilon_0} \times \alpha \left[L - d\ln\left(1 + \dfrac{L}{d}\right)\right]$
3. $V = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{q}{2l}\ln\left(\dfrac{\sqrt{l^2+d^2}+l}{\sqrt{l^2+d^2}-l}\right)$ **4.** $\dfrac{Qq}{2\pi\varepsilon_0 L}$

Introductory Exercise 24.6

1. (a) $\mathbf{E} = -2a(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$ (b) $\mathbf{E} = -a(y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$

2.



3. False **4.** (a) Zero (b) 20 V (c) -20 V (d) -20 V

Introductory Exercise 24.7

1. (a) Zero (b) $\dfrac{q}{\varepsilon_0}$ (c) $\dfrac{q}{2\varepsilon_0}$ **2.** True **3.** (a) Zero (b) $\pi R^2 E$ **4.** Zero

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (a,b) **3.** (d) **4.** (b) **5.** (b) **6.** (d) **7.** (a,b) **8.** (d) **9.** (c) **10.** (b)

Objective Questions

1. (c) **2.** (a) **3.** (b) **4.** (c) **5.** (d) **6.** (b) **7.** (c) **8.** (a) **9.** (c) **10.** (d)
11. (a) **12.** (a) **13.** (d) **14.** (b) **15.** (b) **16.** (c) **17.** (d) **18.** (c) **19.** (a) **20.** (d)
21. (a) **22.** (d) **23.** (d) **24.** (c) **25.** (a) **26.** (c) **27.** (a) **28.** (b) **29.** (a) **30.** (d)
31. (b) **32.** (b) **33.** (a) **34.** (a) **35.** (a) **36.** (d) **37.** (d) **38.** (d) **39.** (a)

Subjective Questions

1. $q = \frac{Q}{2}$ **2.** $\frac{F_e}{F_g} = 3.1 \times 10^{35}$ **3.** 5.31×10^{-11} C/m^2 **4.** $\frac{qx}{8\pi^2 \varepsilon_0 R^3}$

5. $\pm 3\ \mu$C, $\mp 1\ \mu$C **8.** $\frac{q}{24\varepsilon_0}$, zero **9.** (a) $q_1 = 9q_2$ (b) $q_1 = -25q_2$

10. (a) Third charge is $-\frac{4q}{9}$ at a distance of $\frac{L}{3}$ from q between the two charges.

11. $\left(\frac{4\pi\,\varepsilon_0 mgR^2}{\sqrt{3}}\right)^{1/2}$ **12.** 3.3×10^{-8} C **13.** $(2\sqrt{2} + 1)\frac{q}{8\pi\varepsilon_0 a^2}$

14. $(14.4\hat{\mathbf{j}} - 10.8\,\hat{\mathbf{i}})$ N/C **15.** $\frac{\lambda}{2\pi\,\varepsilon_0 R}$ **16.** $\frac{Q}{2\pi\,\varepsilon_0 a\sqrt{L^2 + 4a^2}}$

17. (a) Along positive y-axis (b) Along positive x-axis (c) Along positive y-axis

18. 9.30 **19.** 312.5 m **20.** (a) 37° and 53° (b) 1.66×10^{-7} s, 2.2×10^{-7} s

21. (a) $(-2.1 \times 10^{13}\ \hat{\mathbf{j}})$ m/s^2 (b) $(1.5\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \times 10^5$ m/s

22. At $x = 40$ cm and $x = -200$ cm **23.** $V = 2.634\frac{Q}{4\pi\,\varepsilon_0 a}$

24. (a) -6×10^{-4} J (b) 50 V **25.** 7.42 m/s, faster

26. (a) $\frac{-5Q}{4\pi\,\varepsilon_0 R}$ (b) $\frac{-5Q}{4\pi\,\varepsilon_0\sqrt{R^2 + z^2}}$ **27.** -0.356 J

28. (a) -3.6×10^{-7} J (b) x = 0.0743 m **29.** $-q/2$

30. (a) zero (b) -20 kV **31.** (a) -80 V (b) 120 V (c) 0 V

32. (a) -80 V (b) -40 V

33. (a) [MT^{-3} A^{-1}] (b) $-A\{(y + z)\hat{\mathbf{i}} + (x + z)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}}\}$ (c) $20\sqrt{3}$ N/C **34.** -100 V

35. (a) $E_x = -Ay + 2Bx, E_y = -Ax - C, E_z = 0,$ (b) $x = -C/A,\ y = -2BC/A^2$, any value of z

36. 3.19 nC **37.** (a) $-4.07 \times 10^5\frac{\text{N-m}^2}{\text{C}}$ (b) 6.91 nC (c) No **38.** $240\frac{\text{N-m}^2}{\text{C}}$ **39.** 2.2×10^{-12} C

41. (a) $\phi_{s_1} = -CL^2, \phi_{s_2} = -DL^2, \phi_{s_3} = CL^2, \phi_{s_4} = DL^2, \phi_{s_5} = -BL^2, \phi_{s_6} = BL^2$ (b) zero

42. $\frac{q}{\varepsilon_0}\left[1-\frac{1}{\sqrt{1+(R/l)^2}}\right]$ **43.** $\frac{q}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{2}}\right)$

44. $\frac{Q}{4\pi\varepsilon_0}\left(\frac{R+r}{R^2+r^2}\right)$ **45.** $\frac{q_0}{3\varepsilon_0}$ **46.** 2q, Zero

47. (a) $V_A=\frac{\sigma}{\varepsilon_0}(a-b+c), V_B=\frac{\sigma}{\varepsilon_0}\left(\frac{a^2}{b}-b+c\right), V_C=\frac{\sigma}{\varepsilon_0}\left(\frac{a^2}{c}-\frac{b^2}{c}+c\right)$ (b) $c=a+b$

48. (a) $\frac{-Q}{4\pi a^2},\frac{Q}{4\pi a^2}$ (b) $\frac{-Q}{4\pi a^2},\frac{Q+q}{4\pi a^2}$ (c) $\frac{Q}{4\pi\varepsilon_0 x^2}$ in both situations

49. Inner surface of $B\rightarrow -q$, outer surface of $B\rightarrow\frac{b}{c}q$, inner surface of $C\rightarrow\left(\frac{-bq}{c}\right)$, outer surface of $C\rightarrow\left(\frac{b}{c}-1\right)q$

50.

	A	B	C
Inner Surface	0	$-\frac{6}{11}q$	$\frac{18}{11}q$
Outer Surface	$\frac{6}{11}q$	$-\frac{18}{11}q$	$\frac{9}{11}q$

51.

	A	B	C
Inner Surface	$-q$	$-2q$	$\frac{4}{3}q$
Outer Surface	$2q$	$-\frac{4}{3}q$	$\frac{2}{3}q$

52. (a) $\frac{Q}{2\pi\varepsilon_0 r^2}$ (b) $\frac{Q}{3\pi\varepsilon_0 R}$ (c) zero on inner and Q on outer (d) $\frac{Q}{3}$

53. (a) $\frac{Q}{4\pi\varepsilon_0 R},\frac{Q}{6\pi\varepsilon_0 R}$ (b) $\frac{-Q}{25\pi\,\varepsilon_0 R^2}\hat{\mathbf{r}}$ (c) $\frac{Q^2}{4\pi\,\varepsilon_0 R}$ (d) $Q_1=\frac{Q}{2}, Q_2=\frac{7Q}{2}$ (e) $\frac{-3Q}{50\pi\,\varepsilon_0 R^2}\hat{\mathbf{r}}$

LEVEL 2

Single Correct Option

1.(a) **2.**(a) **3.**(a) **4.**(b) **5.**(a) **6.**(c) **7.**(d) **8.**(a) **9.**(c) **10.**(a)
11.(c) **12.**(a) **13.**(d) **14.**(d) **15.**(c) **16.**(b) **17.**(c) **18.**(d) **19.**(a) **20.**(d)
21.(d) **22.**(a) **23.**(a) **24.**(d) **25.**(c) **26.**(c) **27.**(b) **28.**(d) **29.**(a) **30.**(c)
31.(a) **32.**(a) **33.**(b) **34.**(b) **35.**(c) **36.**(c) **37.**(a) **38.**(b)

More than One Correct Options

1.(a,d) **2.**(a,b,c) **3.**(a,b,c) **4.**(all) **5.**(a,b,c) **6.**(a,c) **7.**(c,d) **8.**(a,b,d) **9.**(b,c) **10.**(b,c)

Comprehension Based Questions

1.(c) **2.**(c) **3.**(a) **4.**(a) **5.**(b) **6.**(c) **7.**(a) **8.**(d) **9.**(b) **10.**(c)

Match the Columns

1. (a) → s (b) → q (c) → r (d) → p
2. (a) → q (b) → p (c) → s (d) → r
3. (a) → s (b) → q (c) → q (d) → p
4. (a) → r (b) → p (c) → q (d) → s
5. (a) → p (b) → q (c) → r (d) → s

Subjective Questions

1. (a) 0.5 m (b) 0.25 m (c) 1.26 s (d) 0.34 m

3. $3Q, -2Q, 2Q, 0, 0, 3Q$

4. (a) $\frac{L}{2}\left[1+\frac{Qq}{(4\pi\varepsilon_0)Wh^2}\right]$ (b) $\sqrt{\frac{3Qq}{(4\pi\varepsilon_0)W}}$

5. 2.0×10^{-3} m/s

6. 6 m/s **7.** shrink, $\sqrt{\frac{Q^2(1-n)(3\sqrt{6}+\sqrt{2}-3\sqrt{3})}{4nm\pi L\varepsilon_0\sqrt{6}}}$

8. $\frac{1}{2^{n+1}}\left[\frac{Q}{4\pi\varepsilon_0 r}\right]$

9. (a) Q_2 is negative and Q_1 is positive (b) $\left(\frac{l+a}{a}\right)^2$ (c) $\frac{l}{\left(\frac{l+a}{a}\right)^{2/3}-1}$

10. (a) $\frac{q_n^2}{8\pi\varepsilon_0 R}$, where $q_n=\frac{QR}{r}\left[1-\left(\frac{R}{R+r}\right)^n\right]$ (b) $\frac{Q^2R}{8\pi\varepsilon_0 r^2}$

11. $\theta=\tan^{-1}\left\{\frac{al^2}{4V}\sqrt{\frac{m}{2eV}}\right\}$

12. $\sqrt{72}$ m

13. $\left[\frac{1}{2\pi\varepsilon_0}\frac{Qq}{Rm}\left(\frac{r-R}{r}+\frac{3}{8}\right)\right]^{\frac{1}{2}}$

14. $\sqrt{\frac{Qq}{2\pi\varepsilon_0 mR}\left(\frac{3\sqrt{10}-5}{5\sqrt{10}}\right)}$

15. (a) $\frac{aqQ}{2\pi\varepsilon_0}\frac{R^2-2x^2}{(R^2+x^2)^{5/2}}$ (b) $\frac{aqQx}{\pi\varepsilon_0(R^2+x^2)^{3/2}}$

16. $\sqrt{\frac{Q\ q\ r_2}{2\pi\varepsilon_0 mr_1(r_1+r_2)}}$, $\sqrt{\frac{Qq\,r_1}{2\pi\varepsilon_0 mr_2(r_1+r_2)}}$

17. (a) 6 V (b) 16 V (c) 1280 V/m

18. (a) $F=-\frac{2kQqx}{(R^2+x^2)^{3/2}}$ (b) Periodic between $\pm\, x_0$

(c) $x=x_0\cos\omega t$, $v=-\omega\, x_0\sin\omega t$, where $\omega=\sqrt{\frac{2Qqk}{mR^3}}$ (d) $t=\sqrt{\frac{\pi^2mR^3}{2Qqk}}$, Here $k=\frac{1}{4\pi\varepsilon_0}$

19. $C=\frac{q}{2\pi a^2}$

20. $f=\lambda RE_0$ along positive x-axis

21. $v=QE\left(\frac{t}{m_0+mnt}\right)$

22. $70\sqrt{3}$ m

25

Capacitors

Chapter Contents

25.1 Capacitance
25.2 Energy stored in a charged capacitor
25.3 Capacitors
25.4 Mechanical force on a charged conductor
25.5 Capacitors in series and parallel
25.6 Two laws in capacitors
25.7 Energy density
25.8 C-R circuits
25.9 Methods of finding equivalent resistance and capacitance

25.1 Capacitance

In practice, we cannot handle free point charges or hold them fixed at desired positions. A practical way to handle a charge would be to put it on a conductor. Thus, one use of a conductor is to store electric charge (or electric potential energy). But, every conductor has its maximum limit of storing the electric charge or potential energy. Beyond that limit, the dielectric in which the conductor is placed, becomes ionized. A capacitor is a device which can store more electric charge or potential energy compared to an isolated conductor.

Capacitors have a tremendous number of applications. In the flash light used by photographers, the energy and charge stored in a capacitor are recovered quickly. In other applications, the energy is released more slowly.

Capacitance of an Isolated Conductor

When a charge q is given to a conductor, it spreads over the outer surface of the conductor. The whole conductor comes to the same potential (say V). This potential V is directly proportional to the charge q, i.e.

$$V \propto q$$

When the proportionality sign is removed, a constant of proportionality $\frac{1}{C}$ comes in picture.

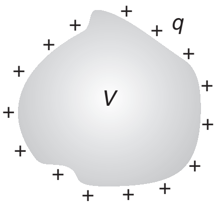


Fig. 25.1

Hence, $$V = \frac{q}{C}$$

or $$C = \frac{q}{V}$$

Here, C is called the **capacitance** of the conductor. The SI unit of capacitance is called **one farad** (1 F). One farad is equal to one coulomb per volt (1 C / V)

$\therefore$ $$1\text{ F} = 1\text{ farad} = 1\text{ C/V} = 1\text{ coulomb/volt}$$

Note *(i) An obvious question arises in mind that when a conductor stores electric charge and energy then why not the unit of capacitance is coulomb or joule. For example, the capacity of a storage tank is given in litres (the unit of volume) or gallons not in the name of some scientist. The reason is simple the capacity of tank does not depend on medium in which it is kept. While the capacity of a conductor to store charge (or energy) depends on the medium in which it is kept. It varies from medium to medium. So, it is difficult to express the capacity in terms of coulomb or joule. Because in that case we will have to mention the medium also.*

For example, we will say like this, capacity of this conductor in water is 1 C in oil it is 5 C, etc. On the other hand, the C discussed above gives us an idea about the dimensions of the conductor nothing about the charge which it can store because as we said earlier also it will vary from medium to medium. By knowing the C (or the dimensions of conductor) a physics student can easily find the maximum charge which it can store, provided the medium is also given.

(ii) Farad in itself is a large unit. Microfarad (μF) is used more frequently.

Method of Finding Capacitance of a Conductor

Give a charge q to the conductor. Find potential on it due to charge q. This potential V will be a function of q and finally find q/V, which is the desired capacitance C.

Capacitance of a Spherical Conductor

When a charge q is given to a spherical conductor of radius R, the potential on it is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$$

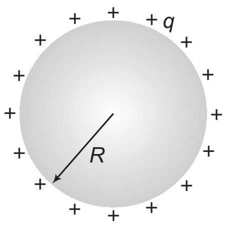


Fig. 25.2

From this expression, we find that $\quad \frac{q}{V} = 4\pi\varepsilon_0 R = C$

Thus, capacitance of the spherical conductor is

$$\boxed{C = 4\pi\varepsilon_0 R}$$

From this expression, we can draw the following conclusions :

(i) $C \propto R$ or C depends on R only. Which we have already stated that C depends on the dimensions of the conductor. Moreover if two conductors have radii R_1 and R_2, then

$$\frac{C_1}{C_2} = \frac{R_1}{R_2}$$

(ii) Earth is also a spherical conductor of radius $R = 6.4 \times 10^6$ m. The capacity of earth is therefore,

$$C = \left(\frac{1}{9 \times 10^9}\right)(6.4 \times 10^6)$$

$$\approx 711 \times 10^{-6} \text{ F}$$

or $\quad C = 711\,\mu\text{F}$

From here, we can see that farad is a large unit. As capacity of such a huge conductor is only $711\,\mu$F.

Extra Points to Remember

- **Dielectric strength of an insulator** In an insulator, most of the electrons are tightly bounded with the nucleus. If an electric field is applied on this insulator, an electrostatic force acts on these electrons in the opposite direction of electric field. As electric field increases, this force also increases. After a certain value of electric field, force becomes so large that these bounded electrons are knocked out or ionized. This maximum electric field is called dielectric strength of insulator. Its unit is N/C or V/m.
- With the help of capacitance (or dimensions of conductor) and dielectric strength of an insulator we can determine the maximum charge or energy which can be stored by this conductor.

Example 25.1 *Capacitance of a conductor is $1\,\mu F$. What charge is required to raise its potential to $100 V$?*

Solution Using the equation

$$q = CV$$

We have, $\quad q = (1\,\mu\text{F})(100\,\text{V})$

$$= 100\,\mu\text{C}$$ **Ans.**

Example 25.2 *Radius of a spherical conductor is 2 m. This is kept in a dielectric medium of dielectric constant 10^6 N/C. Find*

(a) capacitance of the conductor

(b) maximum charge which can be stored on this conductor.

Solution (a)

$$C = 4\pi\varepsilon_0 R$$

$$= \left(\frac{1}{9\times 10^9}\right)(2)$$

$$= 2.2\times 10^{-10}\ \text{F}$$ **Ans.**

(b) Maximum electric field on the surface of spherical conductor is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

This should not exceed 10^6 N/C.

$$\therefore \quad E_{\max} = \frac{1}{4\pi\varepsilon_0}\frac{q_{\max}}{R^2} = 10^6$$

$$\Rightarrow \quad q_{\max} = (4\pi\varepsilon_0)R^2(10^6)$$

$$= \left(\frac{1}{9\times 10^9}\right)(2)^2(10^6)$$

$$= 4.4\times 10^{-4}\ \text{C}$$ **Ans.**

25.2 Energy Stored in a Charged Capacitor

Work has to be done in charging a conductor against the force of repulsion by the already existing charges on it. This work is stored as a potential energy in the electric field of the conductor. Suppose a conductor of capacity C is charged to a potential V_0 and let q_0 be the charge on the conductor at this instant. The potential of the conductor when (during charging) the charge on it was $q\ (< q_0)$ is

$$V = \frac{q}{C}$$

Now, work done in bringing a small charge dq at this potential is

$$dW = Vdq = \left(\frac{q}{C}\right)dq$$

$\therefore$ Total work done in charging it from 0 to q_0 is

$$W = \int_0^{q_0} dW = \int_0^{q_0}\frac{q}{C}\,dq = \frac{1}{2}\frac{q_0^2}{C}$$

This work is stored as the potential energy,

$$\therefore \quad U = \frac{1}{2}\frac{q_0^2}{C}$$

Further by using $q_0 = CV_0$, we can write this expression also as

$$U = \frac{1}{2} CV_0^2 = \frac{1}{2} q_0 V_0$$

In general, if a conductor of capacity C is charged to a potential V by giving it a charge q, then

$$\boxed{U = \frac{1}{2} CV^2 = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2} qV}$$

Redistribution of Charge

Let us take an analogous example. Some liquid is filled in two vessels of different sizes upto different heights. These are joined through a valve which was initially closed. When the valve is opened, the level in both the vessels becomes equal but the volume of liquid in the right side vessel is more than the liquid in the left side vessel. This is because the base area (or capacity) of this vessel is more.

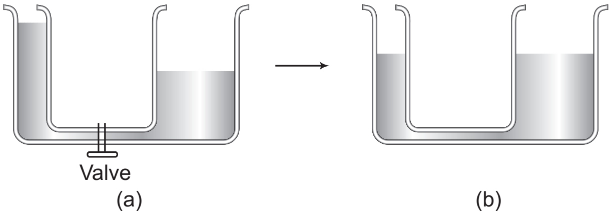


Fig. 25.3

Now, suppose two conductors of capacities C_1 and C_2 have charges q_1 and q_2 respectively when they are joined together by a conducting wire, charge redistributes in these conductors in the ratio of their capacities. Charge redistributes till potential of both the conductors becomes equal. Thus, let q_1' and q_2' be the final charges on them, then

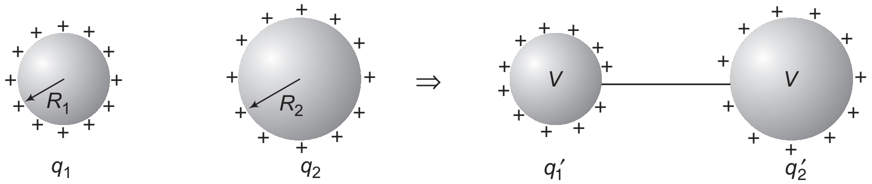


Fig. 25.4

$$q' \propto C \quad \text{or} \quad \frac{q_1'}{q_2'} = \frac{C_1}{C_2}$$

and if they are spherical conductors, then

$$\frac{C_1}{C_2} = \frac{R_1}{R_2}$$

$$\therefore \qquad \frac{q_1'}{q_2'} = \frac{C_1}{C_2} = \frac{R_1}{R_2}$$

Since, the total charge is $(q_1 + q_2)$. Therefore,

$$q_1' = \left(\frac{C_1}{C_1 + C_2}\right)(q_1 + q_2) = \left(\frac{R_1}{R_1 + R_2}\right)(q_1 + q_2)$$

and
$$q_2' = \left(\frac{C_2}{C_1 + C_2}\right)(q_1 + q_2) = \left(\frac{R_2}{R_1 + R_2}\right)(q_1 + q_2)$$

Note *The common potential is given by*

$$V = \frac{\text{Total charge}}{\text{Total capacity}} = \frac{q_1 + q_2}{C_1 + C_2}$$

Loss of Energy During Redistribution of Charge

We can show that in redistribution of charge energy is always lost.

Initial potential energy, $$U_i = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2}$$

Final potential energy, $$U_f = \frac{1}{2}\frac{(q_1 + q_2)^2}{C_1 + C_2}$$

$$\Delta U = U_i - U_f = \frac{1}{2}\left[\frac{q_1^2}{C_1} + \frac{q_2^2}{C_2} - \frac{(q_1 + q_2)^2}{C_1 + C_2}\right]$$

or
$$\Delta U = \frac{1}{2C_1C_2\,(C_1 + C_2)}[q_1^2C_1C_2 + q_1^2C_2^2 + q_2^2C_1^2 + q_2^2C_1C_2 - q_1^2C_1C_2 - q_2^2C_1C_2 - 2q_1q_2C_1C_2]$$

$$= \frac{C_1^2C_2^2}{2C_1C_2\,(C_1 + C_2)}\left[\frac{q_1^2}{C_1^2} + \frac{q_2^2}{C_2^2} - \frac{2q_1q_2}{C_1C_2}\right]$$

$$= \frac{C_1C_2}{2\,(C_1 + C_2)}[V_1^2 + V_2^2 - 2V_1V_2]$$

or
$$\boxed{\Delta U = \frac{C_1C_2}{2\,(C_1 + C_2)}(V_1 - V_2)^2}$$

Now, as C_1, C_2 and $(V_1 - V_2)^2$ are always positive. $U_i > U_f$, i.e. there is a decrease in energy. Hence, energy is always lost in redistribution of charge. Further,

$$\Delta U = 0 \quad \text{if} \quad V_1 = V_2$$

this is because no flow of charge takes place when both the conductors are at same potential.

Example 25.3 *Two isolated spherical conductors have radii* 5 *cm and* 10 *cm, respectively. They have charges of* 12 μ*C and* – 3 μ*C. Find the charges after they are connected by a conducting wire. Also find the common potential after redistribution.*

Solution

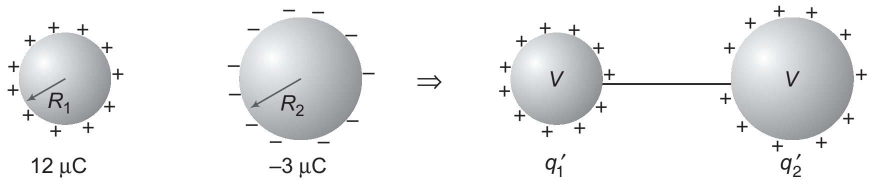


Fig. 25.5

Net charge = $(12-3)\,\mu\text{C} = 9\,\mu\text{C}$

Charge is distributed in the ratio of their capacities (or radii in case of spherical conductors), i.e.

$$\frac{q_1'}{q_2'} = \frac{R_1}{R_2} = \frac{5}{10} = \frac{1}{2}$$

$$\therefore \qquad q_1' = \left(\frac{1}{1+2}\right)(9) = 3\,\mu\text{C}$$

and

$$q_2' = \left(\frac{2}{1+2}\right)(9) = 6\,\mu\text{C}$$

$$\text{Common potential, } V = \frac{q_1 + q_2}{C_1 + C_2} = \frac{(9 \times 10^{-6})}{4\pi\varepsilon_0 (R_1 + R_2)}$$

$$= \frac{(9 \times 10^{-6})(9 \times 10^{9})}{(15 \times 10^{-2})}$$

$$= 5.4 \times 10^{5}\ \text{V}$$

Ans.

Example 25.4 *An insulated conductor initially free from charge is charged by repeated contacts with a plate which after each contact is replenished to a charge Q. If q is the charge on the conductor after first operation prove that the maximum charge which can be given to the conductor in this way is* $\frac{Qq}{Q-q}$.

Solution Let C_1 be the capacity of plate and C_2 that of the conductor. After first contact charge on conductor is q. Therefore, charge on plate will remain $Q - q$. As the charge redistributes in the ratio of capacities.

$$\frac{Q-q}{q} = \frac{C_1}{C_2} \qquad \text{...(i)}$$

Let q_m be the maximum charge which can be given to the conductor. Then, flow of charge from the plate to the conductor will stop when,

$$V_{\text{conductor}} = V_{\text{plate}}$$

$$\therefore \qquad \frac{q_m}{C_2} = \frac{Q}{C_1} \quad \Rightarrow \quad q_m = \left(\frac{C_2}{C_1}\right) Q$$

Substituting $\frac{C_2}{C_1}$ from Eq. (i), we get

$$q_m = \frac{Qq}{Q-q}$$

Hence Proved.

Example 25.5 *A conducting sphere S_1 of radius r is attached to an insulating handle. Another conducting sphere S_2 of radius R is mounted on an insulating stand. S_2 is initially uncharged. S_1 is given a charge Q, brought into contact with S_2 and removed. S_1 is recharged such that the charge on it is again Q and it is again brought into contact with S_2 and removed. This procedure is repeated n times.* **(JEE 1998)**

(a) Find the electrostatic energy of S_2 after n such contacts with S_1.

(b) What is the limiting value of this energy as $n \to \infty$?

Solution Capacities of conducting spheres are in the ratio of their radii. Let C_1 and C_2 be the capacities of S_1 and S_2, then

$$\frac{C_2}{C_1} = \frac{R}{r}$$

(a) Charges are distributed in the ratio of their capacities. Let in the first contact, charge acquired by S_2 is q_1. Therefore, charge on S_1 will be $Q - q_1$. Say it is q'_1

$$\therefore \qquad \frac{q_1}{q_1'} = \frac{q_1}{Q - q_1} = \frac{C_2}{C_1} = \frac{R}{r}$$

$$\therefore \qquad q_1 = Q\left(\frac{R}{R+r}\right) \qquad \dots\text{(i)}$$

In the second contact, S_1 again acquires the same charge Q.

Therefore, total charge in S_1 and S_2 will be

$$Q + q_1 = Q\left(1 + \frac{R}{R+r}\right)$$

This charge is again distributed in the same ratio. Therefore, charge on S_2 in second contact,

$$q_2 = Q\left(1 + \frac{R}{R+r}\right)\left(\frac{R}{R+r}\right)$$

$$= Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2\right]$$

Similarly, $$q_3 = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \left(\frac{R}{R+r}\right)^3\right]$$

and $$q_n = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \dots + \left(\frac{R}{R+r}\right)^n\right]$$

or $$q_n = Q\frac{R}{r}\left[1 - \left(\frac{R}{R+r}\right)^n\right] \qquad \dots\text{(ii)}$$

$$\left[S_n = \frac{a\,(1 - r^n)}{(1 - r)}\right]$$

Therefore, electrostatic energy of S_2 after n such contacts

$$= \frac{q_n^2}{2(4\pi\varepsilon_0 R)} \quad \text{or} \quad U_n = \frac{q_n^2}{8\pi\varepsilon_0 R}$$

where, q_n can be written from Eq. (ii).

(b) As $n \to \infty$

$$q_\infty = Q\frac{R}{r}$$

$$\therefore \qquad U_\infty = \frac{q_\infty^2}{2C} = \frac{Q^2R^2/r^2}{8\pi\varepsilon_0 R}$$

or

$$U_\infty = \frac{Q^2R}{8\pi\varepsilon_0 r^2}$$

INTRODUCTORY EXERCISE 25.1

1. Find the dimensions of capacitance.
2. No charge will flow when two conductors having the same charge are connected to each other. Is this statement true or false?
3. Two conductors of capacitance 1μF and 2 μF are charged to +10 V and –20 V. They are now connected by a conducting wire. Find
   (a) their common potential
   (b) the final charges on them
   (c) the loss of energy during redistribution of charges.

25.3 Capacitors

Any two conductors separated by an insulator (or a vacuum) form a capacitor.

In most practical applications, each conductor initially has zero net charge, and electrons are transferred from one conductor to the other. This is called charging of the conductor. Then, the two conductors have charges with equal magnitude and opposite sign, and the net charge on the capacitor as a whole remains zero. When we say that a capacitor has charge q we mean that the conductor at higher potential has charge $+q$ and the conductor at lower potential has charge $-q$. In circuit diagram, a capacitor is represented by two parallel lines as shown in Fig. 25.7.

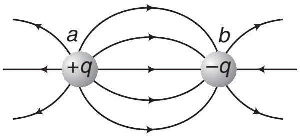


Fig. 25.6

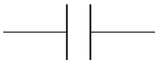


Fig. 25.7

One common way to charge a capacitor is to connect the two conductors to opposite terminals of a battery. This gives a fixed potential difference V_{ab} between the conductors, which is just equal to the voltage of the battery. The ratio $\frac{q}{V_{ab}}$ is called the capacitance of the capacitor. Hence,

$$\boxed{C = \frac{q}{V_{ab}}} \qquad \text{(capacitance of a capacitor)}$$

Calculation of Capacitance

Give a charge $+q$ to one plate and $-q$ to the other plate. Then, find potential difference V between the plates. Now,

$$\boxed{C = \frac{q}{V}}$$

Parallel Plate Capacitor

Two metallic parallel plates of any shape but of same size and separated by a small distance constitute parallel plate capacitor. Suppose the area of each plate is A and the separation between the two plates is d. Also assume that the space between the plates contains vacuum.

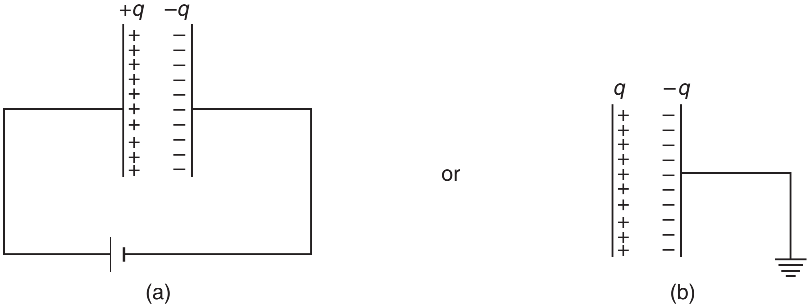


Fig. 25.8

We put a charge q on one plate and a charge $-q$ on the other. This can be done either by connecting one plate with the positive terminal and the other with negative plate of a battery [as shown in Fig. (a)] or by connecting one plate to the earth and by giving a charge $+q$ to the other plate only. This charge will induce a charge $-q$ on the earthed plate. The charges will appear on the facing surfaces. The charge density on each of these surfaces has a magnitude $\sigma = q/A$.

If the plates are large as compared to the separation between them, then the electric field between the plates (at point B) is uniform and perpendicular to the plates except for a small region near the edge. The magnitude of this uniform field E may be calculated by using the fact that both positive and negative plates produce the electric field in the same direction (from positive plate towards negative plate) of magnitude $\sigma/2\varepsilon_0$ and therefore, the net electric field between the plates will be

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

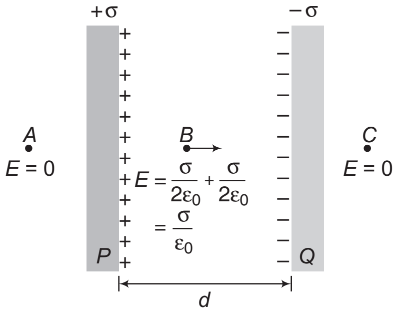

+σ
−σ
A
E = 0
B
$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$
C
E = 0
P
Q
d

Fig. 25.9

Outside the plates (at points A and C) the field due to positive sheet of charge and negative sheet of charge are in opposite directions. Therefore, net field at these points is zero.

The potential difference between the plates is

$$\therefore \qquad V = E \cdot d = \left(\frac{\sigma}{\varepsilon_0}\right) d = \frac{qd}{A\varepsilon_0}$$

$\therefore$ The capacitance of the parallel plate capacitor is

$$C = \frac{q}{V} = \frac{A\varepsilon_0}{d}$$

or

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

Note *(i) Instead of two plates if there are n similar plates at equal distances from each other and the alternate plates are connected together, the capacitance of the arrangement is given by*

$$C = \frac{(n-1)\,\varepsilon_0 A}{d}$$

(ii) From the above relation, it is clear that the capacitance depends only on geometrical factors (A and d).

Effect of Dielectrics

Most capacitors have a dielectric between their conducting plates. Placing a solid dielectric between the plates of a capacitor serves the following three functions :

(i) It solves the problem of maintaining two large metal sheets at a very small separation without actual contact.

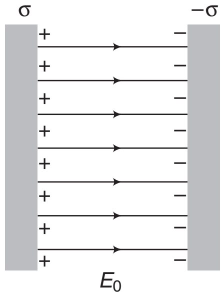


Fig. 25.10

(ii) It increases the maximum possible potential difference which can be applied between the plates of the capacitor without the dielectric breakdown. Many dielectric materials can tolerate stronger electric fields without breakdown than can air.

(iii) It increases the capacitance of the capacitor.

When a dielectric material is inserted between the plates (keeping the charge to be constant) the electric field and hence the potential difference decreases by a factor K (the dielectric constant of the dielectric).

$$\therefore \qquad \boxed{E = \frac{E_0}{K} \quad \text{and} \quad V = \frac{V_0}{K}} \qquad \text{(When } q \text{ is constant)}$$

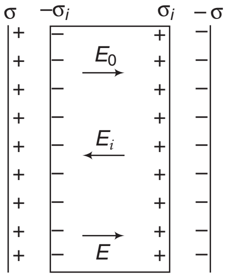


Fig. 25.11

Electric field is decreased because an induced charge of the opposite sign appears on each surface of the dielectric. This induced charge produces an electric field inside the dielectric in opposite directions and as a result net electric field is decreased. The induced charge in the dielectric can be calculated as under

$$E = E_0 - E_i \quad \text{or} \quad \frac{E_0}{K} = E_0 - E_i$$

$$\therefore \qquad E_i = E_0\left(1 - \frac{1}{K}\right)$$

Therefore,

$$\frac{\sigma_i}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}\left(1 - \frac{1}{K}\right)$$

or

$$\boxed{\sigma_i = \sigma\left(1 - \frac{1}{K}\right)}$$

or

$$\boxed{q_i = q\left(1 - \frac{1}{K}\right)}$$

For a conductor $K = \infty$. Hence,

$$q_i = q,\ \sigma_i = \sigma \quad \text{and} \quad E = 0$$

and otherwise

$$q_i < q$$

Thus,

$$\boxed{q_i \le q, \quad \sigma_i \le \sigma}$$

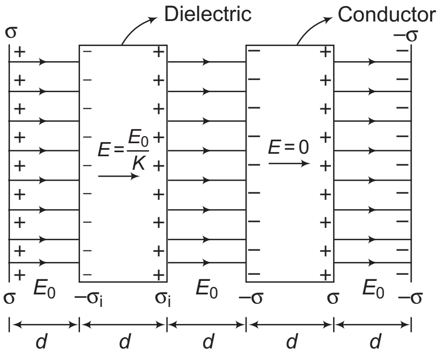


Fig. 25.12

Hence, we can conclude the above discussion as under:

(i) $E_{\text{vacuum}} = E_0 = \dfrac{\sigma}{\varepsilon_0} = \dfrac{q}{A\varepsilon_0}$

(ii) $E_{\text{dielectric}} = \dfrac{E_0}{K}$ (here, K = dielectric constant)

(iii) $E_{\text{conductor}} = 0$ (as $K = \infty$)

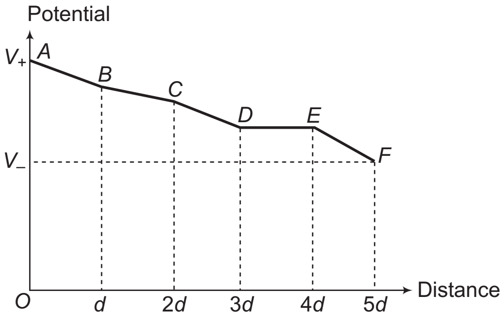


Fig. 25.13

If we plot a graph between potential and distance from positive plate, it will be as shown in Fig. 25.13: Modulus of,

Slope of $AB =$ slope of $CD =$ slope of $EF = E_0$

Slope of $BC = \dfrac{E_0}{K}$

and slope of $DE = 0$

Further, the potential difference between positive and negative plate is

$$V_+ - V_- = E_0 d + \frac{E_0}{K} \cdot d + E_0 d + 0 + E_0 d$$

$$= 3E_0 d + \frac{E_0}{K} \cdot d$$

Here we have used PD $= Ed$. (in uniform electric field)

Capacitance of a Capacitor Partially Filled with Dielectric

Suppose, a dielectric is partially filled with a dielectric (dielectric constant $= K$) as shown in figure. If a charge q is given to the capacitor, an induced charge q_i is developed on the dielectric.

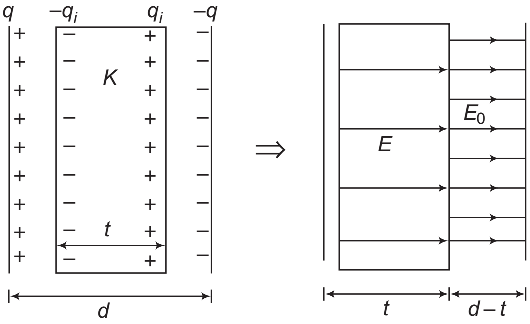


Fig. 25.14

where, $$q_i = q\left(1 - \frac{1}{K}\right)$$

Moreover, if E_0 is the electric field in the region where dielectric is absent, then electric field inside the dielectric will be $E = E_0/K$. The potential difference between the plates of the capacitor is

$$V = V_+ - V_- = Et + E_0\,(d - t)$$

$$= \frac{E_0}{K} t + E_0\,(d - t) = E_0\left(d - t + \frac{t}{K}\right)$$

$$= \frac{\sigma}{\varepsilon_0}\left(d - t + \frac{t}{K}\right) = \frac{q}{A\varepsilon_0}\left(d - t + \frac{t}{K}\right)$$

Now, as per the definition of capacitance,

$$C = \frac{q}{V} = \frac{\varepsilon_0 A}{d - t + \dfrac{t}{K}} \qquad \text{or} \qquad \boxed{C = \frac{\varepsilon_0 A}{d - t + \dfrac{t}{K}}}$$

Different Cases

(i) If more than one dielectric slabs are placed between the capacitor, then

$$C = \frac{\varepsilon_0 A}{(d - t_1 - t_2 - \ldots - t_n) + \left(\dfrac{t_1}{K_1} + \dfrac{t_2}{K_2} + \ldots + \dfrac{t_n}{K_n}\right)}$$

(ii) If the slab completely filles the space between the plates, then $t = d$ and therefore,

$$C = \frac{\varepsilon_0 A}{d/K} = \frac{K\varepsilon_0 A}{d}$$

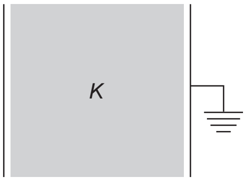


Fig. 25.15

(iii) If a conducting slab ($K = \infty$) is placed between the plates, then

$$C = \frac{\varepsilon_0 A}{d - t + \dfrac{t}{\infty}} = \frac{\varepsilon_0 A}{d - t}$$

This can be explained from the following figure:

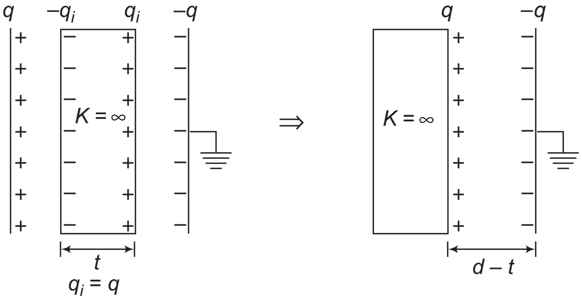


Fig. 25.16

(iv) If the space between the plates is completely filled with a conductor, then $t = d$ and $K = \infty$.

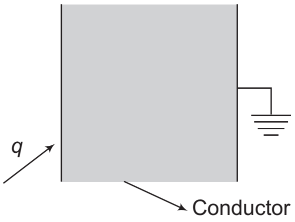


Fig. 25.17

Then,

$$C = \frac{\varepsilon_0 A}{d - d + \dfrac{d}{\infty}} = \infty$$

The significance of infinite capacitance can be explained as under:

If one of the plates of a capacitor is earthed and the second one is given a charge q, then the whole charge transfers to earth and as the capacity of earth is very large compared to the capacitor we can say that the capacitance has become infinite.

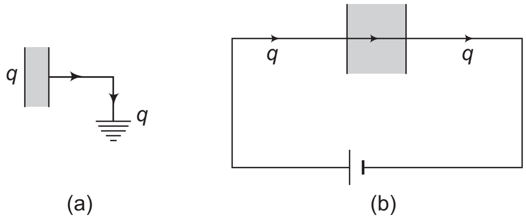


Fig. 25.18

Alternatively, if the plates of the capacitor are connected to a battery, the current starts flowing in the circuit. Thus, is as much charge enters the positive plate of the capacitor, the same charge leaves the negative plate. So, we can say, the positive plate can accept infinite amount of charge or its capacitance has become infinite.

Energy Stored in Charged Capacitor

A charged capacitor stores an electric potential energy in it, which is equal to the work required to charge it. This energy can be recovered if the capacitor is allowed to discharge. If the charging is done by a battery, electrical energy is stored at the expense of chemical energy of battery.

Suppose at time t, a charge q is present on the capacitor and V is the potential of the capacitor. If dq amount of charge is brought against the forces of the field due to the charge already present on the capacitor, the additional work needed will be

$$dW = (dq)V = \left(\frac{q}{C}\right) \cdot dq \qquad (\text{as } V = q/C)$$

$\therefore$ Total work to charge a capacitor to a charge q_0,

$$W = \int dW = \int_0^{q_0} \left(\frac{q}{C}\right) \cdot dq = \frac{q_0^2}{2C}$$

$\therefore$ Energy stored by a charged capacitor,

$$U = W = \frac{q_0^2}{2C} = \frac{1}{2}CV_0^2 = \frac{1}{2}q_0V_0$$

Thus, if a capacitor is given a charge q, the potential energy stored in it is

$$\boxed{U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}qV}$$

The above relation shows that the charged capacitor is the electrical analog of a stretched spring whose elastic potential energy is $\frac{1}{2}Kx^2$. The charge q is analogous to the elongation x and $\frac{1}{C}$, i.e. the reciprocal of capacitance to the force constant k.

Extra Points to Remember

- **Capacitance of a spherical conductor enclosed by an earthed concentric spherical shell**

  If a charge q is given to the inner spherical conductor, it spreads over the outer surface of it and a charge $-q$ appears on the inner surface of the shell. The electric field is produced only between the two. From the principle of generator, the potential difference between the two will depend on the inner charge q only and is given by

  $$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

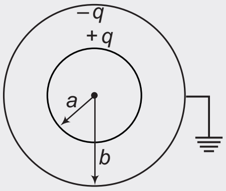
  

  Fig. 25.19

  Hence, the capacitance of this system

  $$C = \frac{q}{V}$$

  or $$\boxed{C = 4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right) = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)}$$

  From this expression we see that if $b = \infty$, $C = 4\pi\varepsilon_0 a$, which corresponds to that of an isolated sphere, i.e. the charged sphere may be regarded as a capacitor in which the outer surface has been removed to infinity.

- **Capacitance of a cylindrical capacitor** When a metallic cylinder of radius a is placed coaxially inside an earthed hollow metallic cylinder of radius b $(> a)$ we get cylindrical capacitor. If a charge q is given to the inner cylinder, induced charge $-q$ will reach to the inner surface of the outer cylinder. Assume that the capacitor is of very large length ($l >> b$) so that the lines of force are radial. Using Gauss's law, we can prove that

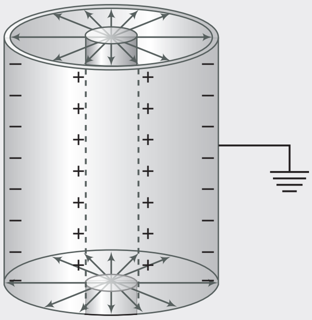
  

  Fig. 25.20

  $$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{for} \quad a \le r \le b$$

  Here, λ = charge per unit length

  Therefore, the potential difference between the cylinders

  $$V = -\int_b^a \mathbf{E} \cdot d\mathbf{r} = -\int_b^a \frac{\lambda}{2\pi\varepsilon_0 r}\,dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

  $$\therefore \quad \frac{\lambda}{V} = \frac{\text{charge/length}}{\text{potential difference}} = \frac{\text{capacitance}}{\text{length}} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

  Hence, $$\boxed{\text{capacitance per unit length} = \frac{2\pi\varepsilon_0}{\ln(b/a)}}$$

Example 25.6 *A parallel-plate capacitor has capacitance of* 1.0 *F. If the plates are* 1.0 *mm apart, what is the area of the plates?*

Solution ∵
$$C = \frac{\varepsilon_0 A}{d}$$

∴
$$A = \frac{Cd}{\varepsilon_0} = \frac{(1)(10^{-3})}{8.86 \times 10^{-12}}$$
$$= 1.1 \times 10^8 \text{ m}^2$$

Example 25.7 *Two parallel plate vacuum capacitors have areas A_1 and A_2 and equal plate spacing d. Show that when the capacitors are connected in parallel, the equivalent capacitance is the same as for a single capacitor with plate area $A_1 + A_2$ and spacing d.* ***Note :*** *In parallel $C = C_1 + C_2$.*

Solution $C = C_1 + C_2$ (in parallel)

∴
$$\frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 A_1}{d} + \frac{\varepsilon_0 A_2}{d}$$

or
$$A = A_1 + A_2$$

Example 25.8 *(a) Two spheres have radii a and b and their centres are at a distance d apart. Show that the capacitance of this system is*
$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} + \frac{1}{b} \pm \frac{2}{d}}$$

provided that d is large compared with a and b.

(b) Show that as d approaches infinity the above result reduces to that of two isolated spheres in series. ***Note :*** *In series,* $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$.

Solution (a) PD, $V = V_1 - V_2$

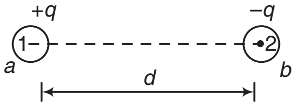


Fig. 25.21

$$= \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q}{a} - \frac{q}{d}\right) - \left(\frac{-q}{b} + \frac{q}{d}\right)\right]$$

∴
$$C = \frac{q}{V} = \frac{4\pi\varepsilon_0}{\frac{1}{a} + \frac{1}{b} - \frac{2}{d}}$$

If − q is given to first sphere and + q to second sphere, then
$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} + \frac{1}{b} + \frac{2}{d}}$$

(b) If $d \to \infty$, then $C = \dfrac{4\pi\varepsilon_0}{\dfrac{1}{a} + \dfrac{1}{b}} = \dfrac{(4\pi\varepsilon_0)(ab)}{a+b}$

In series, $C_{\text{net}} = \dfrac{C_1 C_2}{C_1 + C_2} = \dfrac{(4\pi\,\varepsilon_0\, a)(4\pi\,\varepsilon_0 b)}{(4\pi\,\varepsilon_0 a) + (4\pi\varepsilon_0 b)} = \dfrac{(4\pi\varepsilon_0)\,ab}{a+b}$ **Hence Proved.**

INTRODUCTORY EXERCISE 25.2

1. A capacitor has a capacitance of $7.28\ \mu\text{F}$. What amount of charge must be placed on each of its plates to make the potential difference between its plates equal to $25.0\ \text{V}$?
2. A parallel plate air capacitor of capacitance $245\ \mu\text{F}$ has a charge of magnitude $0.148\ \mu\text{C}$ on each plate. The plates are 0.328 mm apart.
   (a) What is the potential difference between the plates?
   (b) What is the area of each plate?
   (c) What is the surface charge density on each plate?
3. Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electric field is $E_0 = 3.20 \times 10^5$ V/m. When the space is filled with dielectric, the electric field is $E = 2.50 \times 10^5$ V/m.
   (a) What is the dielectric constant?
   (b) What is the charge density on each surface of the dielectric?

25.4 Mechanical Force on a Charged Conductor

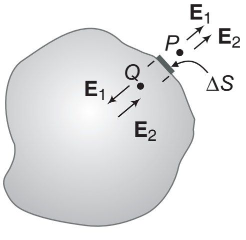


Fig. 25.22

We know that similar charges repel each other, hence the charge on any part of surface of the conductor is repelled by the charge on its remaining part. The surface of the conductor thus experiences a mechanical force.

The electric field at any point P near the conductor's surface can be assumed as due to a small part of the surface of area say ΔS immediately in the neighbourhood of the point under consideration and due to the rest of the surface. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the field intensities due to these parts respectively.

Then, total electric field, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$

$\mathbf{E}$ has a magnitude σ/ε_0 at any point P just outside the conductor and is zero at point Q just inside the conductor. Thus,

$$E_1 + E_2 = \sigma/\varepsilon_0 \text{ at } P$$

and

$$E_1 - E_2 = 0 \text{ at } Q$$

$$\therefore \qquad E_1 = E_2 = \frac{\sigma}{2\varepsilon_0}$$

Hence, the force experienced by small surface of area ΔS due to the charge on the rest of the surface is

$$F = qE_2 = (\sigma\Delta S)(E_2) = \frac{(\sigma^2)(\Delta S)}{2\varepsilon_0}$$

$$\therefore \qquad \frac{\text{Force}}{\text{Area}} = \frac{F}{\Delta S} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2 \qquad \left(\text{as } E = \frac{\sigma}{\varepsilon_0}\right)$$

$$\therefore \qquad \boxed{\frac{\text{Force}}{\text{Area}} = \frac{1}{2}\varepsilon_0 E^2}$$

Force between the Plates of a Capacitor

Consider a parallel plate capacitor with plate area A. Suppose a positive charge q is given to one plate and a negative charge $-q$ to the other plate. The electric field on the negative plate due to positive charge is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{q}{2A\varepsilon_0}$$

The magnitude of force on the charge in negative plate is

$$F = qE = \frac{q^2}{2A\varepsilon_0}$$

This is the force with which both the plates attract each other. Thus,

$$\boxed{F = \frac{q^2}{2A\varepsilon_0}}$$

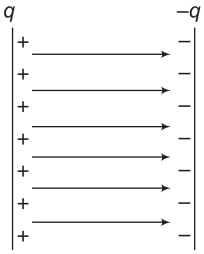


Fig. 25.23

Example 25.9 *A capacitor is given a charge q. The distance between the plates of the capacitor is d. One of the plates is fixed and the other plate is moved away from the other till the distance between them becomes 2d. Find the work done by the external force.*

Solution When one plate is fixed, the other is attracted towards the first with a force

$$F = \frac{q^2}{2A\varepsilon_0} = \text{constant}$$

Hence, an external force of same magnitude will have to be applied in opposite direction to increase the separation between the plates.

$$\therefore \qquad W = F(2d - d) = \frac{q^2 d}{2A\varepsilon_0} \qquad \textbf{Ans.}$$

Alternate solution $$W = \Delta U = U_f - U_i = \frac{q^2}{2C_f} - \frac{q^2}{2C_i} \qquad \ldots\text{(i)}$$

Here, $$C_f = \frac{\varepsilon_0 A}{2d} \quad \text{and} \quad C_i = \frac{\varepsilon_0 A}{d}$$

Substituting in Eq. (i), we have

$$W = \frac{q^2}{2\left(\frac{\varepsilon_0 A}{2d}\right)} - \frac{q^2}{2\left(\frac{\varepsilon_0 A}{d}\right)} = \frac{q^2 d}{2\varepsilon_0 A} \qquad \textbf{Ans.}$$

25.5 Capacitors in Series and Parallel

In Series

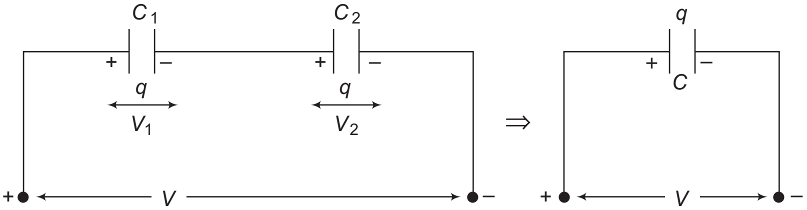


Fig. 25.24

In a series connection, the magnitude of charge on all plates is same. The potential is distributed in the inverse ratio of the capacity (as $V = q/C$ or $V \propto 1/C$). Thus, in the figure, if a potential difference V is applied across the two capacitors C_1 and C_2, then

$$\boxed{\frac{V_1}{V_2} = \frac{C_2}{C_1}}$$

or
$$V_1 = \left(\frac{C_2}{C_1 + C_2}\right)V \quad \text{and} \quad V_2 = \left(\frac{C_1}{C_1 + C_2}\right)V$$

Further, in the figure,
$$V = V_1 + V_2 \quad \text{or} \quad \frac{q}{C} = \frac{q}{C_1} + \frac{q}{C_2}$$

or
$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

Here, C is the **equivalent capacitance**.

The equivalent capacitance of the series combination is defined as the capacitance of a single capacitor for which the charge q is the same as for the combination, when the same potential difference V is applied across it. In other words, the combination can be replaced by an equivalent capacitor of capacitance C. We can extend this analysis to any number of capacitors in series. We find the following result for the equivalent capacitance.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$

Following points are important in case of series combination of capacitors.

(i) In a series connection, the equivalent capacitance is always less than any individual capacitance.

(ii) For the equivalent capacitance of two capacitors it is better to remember the following form

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

For example, equivalent capacitance of two capacitors $C_1 = 6\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$ is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \left(\frac{6 \times 3}{6 + 3}\right)\mu\text{F} = 2\,\mu\text{F}$$

(iii) If n capacitors of equal capacity C are connected in series, then their equivalent capacitance is $\frac{C}{n}$.

Example 25.10 *In the circuit shown in figure, find*

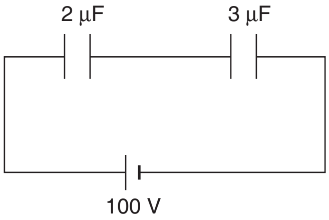


Fig. 25.25

(a) the equivalent capacitance,
(b) the charge stored in each capacitor and
(c) the potential difference across each capacitor.

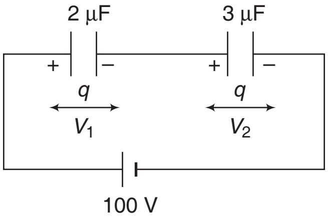


Fig. 25.26

Solution (a) The equivalent capacitance

$$C = \frac{C_1C_2}{C_1 + C_2}$$

or $$C = \frac{(2)(3)}{2+3} = 1.2\mu\text{F}$$ **Ans.**

(b) The charge q stored in each capacitor is

$$q = CV = (1.2 \times 10^{-6})(100)\text{ C}$$
$$= 120\ \mu\text{C}$$ **Ans.**

(c) In series combination, $$V \propto \frac{1}{C} \quad \text{or} \quad \frac{V_1}{V_2} = \frac{C_2}{C_1}$$

∴ $$V_1 = \left(\frac{C_2}{C_1 + C_2}\right)V = \left(\frac{3}{2+3}\right)(100) = 60\text{ V}$$

and $$V_2 = V - V_1 = 100 - 60 = 40\text{ V}$$ **Ans.**

In Parallel

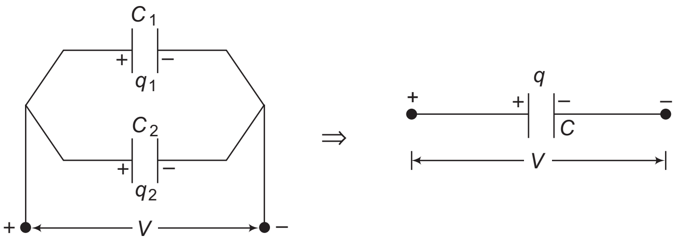


Fig. 25.27

The arrangement shown in figure is called a parallel connection. In a parallel combination, the potential difference for all individual capacitors is the same and the total charge q is distributed in the ratio of their capacities. (as $q = CV$ or $q \propto C$ for same potential difference). Thus,

$$\boxed{\frac{q_1}{q_2} = \frac{C_1}{C_2}}$$

or $$q_1 = \left(\frac{C_1}{C_1 + C_2}\right) q \quad \text{and} \quad q_2 = \left(\frac{C_2}{C_1 + C_2}\right) q$$

The parallel combination is equivalent to a single capacitor with the same total charge $q = q_1 + q_2$ and potential difference V.

Thus, $$q = q_1 + q_2$$

or $$CV = C_1 V + C_2 V$$

or $$\boxed{C = C_1 + C_2}$$

In the same way, we can show that for any number of capacitors in parallel,

$$\boxed{C = C_1 + C_2 + C_3 + \dots}$$

In a parallel combination, the equivalent capacitance is always greater than any individual capacitance.

Example 25.11 *In the circuit shown in figure, find*

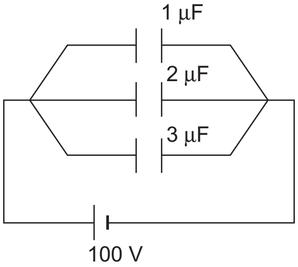


Fig. 25.28

(a) the equivalent capacitance and

(b) the charge stored in each capacitor.

Solution (a) The capacitors are in parallel. Hence, the equivalent capacitance is

$$C = C_1 + C_2 + C_3$$

or $$C = (1 + 2 + 3) = 6\,\mu\text{F}$$ **Ans.**

(b) Total charge drawn from the battery,

$$q = CV = 6 \times 100\,\mu\text{C} = 600\,\mu\text{C}$$

This charge will be distributed in the ratio of their capacities. Hence,

$$q_1 : q_2 : q_3 = C_1 : C_2 : C_3 = 1 : 2 : 3$$

$\therefore$ $$q_1 = \left(\frac{1}{1 + 2 + 3}\right) \times 600 = 100\,\mu\text{C}$$

$$q_2 = \left(\frac{2}{1 + 2 + 3}\right) \times 600 = 200\,\mu\text{C}$$

and $$q_3 = \left(\frac{3}{1 + 2 + 3}\right) \times 600 = 300\,\mu\text{C}.$$ **Ans.**

Alternate solution Since the capacitors are in parallel, the PD across each of them is 100 V. Therefore, from $q = CV$, the charge stored in $1\mu F$ capacitor is $100\mu C$, in $2\mu F$ capacitor is $200\mu C$ and that in $3\mu F$ capacitor is $300\mu C$.

INTRODUCTORY EXERCISE 25.3

1. Find charges on different capacitors.

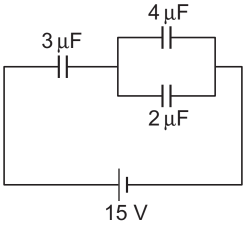


Fig. 25.29

2. Find charges on different capacitors.

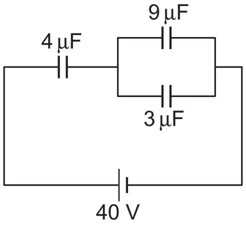


Fig. 25.30

25.6 Two Laws in Capacitors

Like an electric circuit having resistances and batteries in a complex circuit containing capacitors and the batteries charges on different capacitors can be obtained with the help of Kirchhoff 's laws.

First Law

This law is basically law of conservation of charge which is normally applied across a battery or in an isolated system.

(i) In case of a battery, both terminals of the battery supply equal amount of charge.

(ii) In an isolated system (not connected to any source of charge like terminal of a battery or earth) net charge remains constant.

For example, in the Fig. 25.31, the positive terminal of the battery supplies a positive charge $q_1 + q_2$. Similarly, the negative terminal supplies a negative charge of magnitude $q_3 + q_4$. Hence,

$$q_1 + q_2 = q_3 + q_4$$

Further, the plates enclosed by the dotted lines form an isolated system, as they are neither connected to a battery terminal nor to the earth. Initially, no charge was present in these plates. Hence, after charging net charge on these plates should also be zero. Or,

$$q_3 + q_5 - q_1 = 0 \quad \text{and} \quad q_4 - q_2 - q_5 = 0$$

So, these are the three equations which can be obtained from the first law.

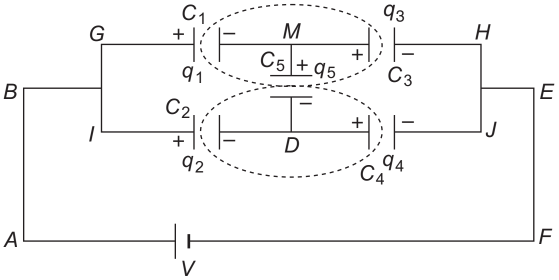


Fig. 25.31

Second Law

In a capacitor, potential drops by q/C when one moves from positive plate to the negative plate and in a battery it drops by an amount equal to the emf of the battery. Applying second law in loop *ABGHEFA*, we have

$$-\frac{q_1}{C_1}-\frac{q_3}{C_3}+V=0$$

Similarly, the second law in loop *GMDIG* gives the equation,

$$-\frac{q_1}{C_1}-\frac{q_5}{C_5}+\frac{q_2}{C_2}=0$$

Example 25.12 *Find the charges on the three capacitors shown in figure.*

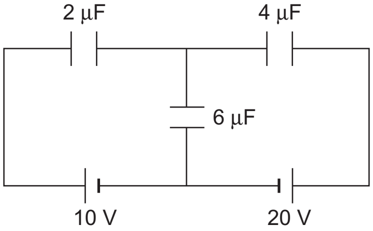


Fig. 25.32

Solution Let the charges in three capacitors be as shown in Fig. 25.33.

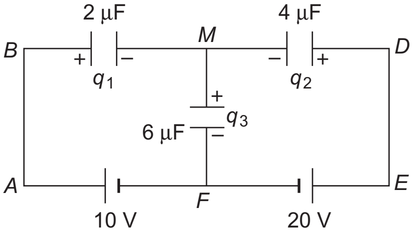


Fig. 25.33

Charge supplied by 10 V battery is q_1 and that from 20 V battery is q_2. Thus,

$$q_1+q_2=q_3 \qquad \ldots\text{(i)}$$

This relation can also be obtained by a different method. The charges on the three plates which are in contact add to zero. Because these plates taken together form an isolated system which can't receive charges from the batteries. Thus,

$$q_3-q_1-q_2=0$$

or

$$q_3=q_1+q_2$$

Applying second law in loops *BMFAB* and *MDEFM*, we have

$$-\frac{q_1}{2}-\frac{q_3}{6}+10=0$$

or $$q_3+3q_1=60 \qquad \text{...(ii)}$$

and $$\frac{q_2}{4}-20+\frac{q_3}{6}=0$$

or $$3q_2+2q_3=240 \qquad \text{...(iii)}$$

Solving the above three equations, we have

$$q_1=\frac{10}{3}\,\mu\text{C}$$

$$q_2=\frac{140}{3}\,\mu\text{C}$$

and $$q_3=50\,\mu\text{C}$$

Thus, charges on different capacitors are as shown in Fig. 25.34.

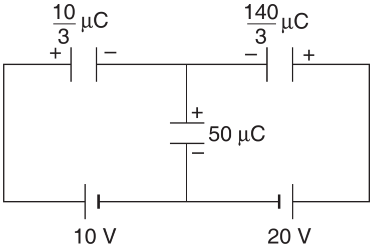


Fig. 25.34

Note *In the problem q_1, q_2 and q_3 are already in microcoulombs.*

25.7 Energy Density (*u*)

The potential energy of a charged conductor or a capacitor is stored in the electric field. The energy per unit volume is called the energy density (u). Energy density in a dielectric medium is given by

$$\boxed{u=\frac{1}{2}\varepsilon_0 KE^2}$$

This relation shows that the energy stored per unit volume depends on E^2. If E is the electric field in a space of volume dV, then the total stored energy in an electrostatic field is given by

$$\boxed{U=\frac{1}{2}\varepsilon_0 K\int E^2\,dV}$$

and if E is uniform throughout the volume (electric field between the plates of a capacitor is almost uniform), then the total stored energy can be given by

$$\boxed{U=u\,(\text{Total volume})=\frac{1}{2}K\varepsilon_0 E^2 V}$$

Example 25.13 *Using the concept of energy density, find the total energy stored in a*

(a) parallel plate capacitor

(b) charged spherical conductor.

Solution (a) Electric field is uniform between the plates of the capacitor. The magnitude of this field is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$$

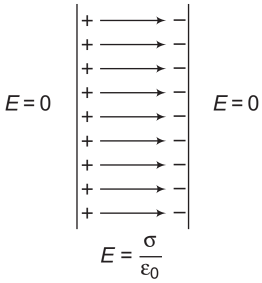


Fig. 25.35

Therefore, the energy density (u) should also be constant.

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{q^2}{2A^2\varepsilon_0}$$

∴ Total stored energy,

$$U = (u)\,(\text{total volume})$$

$$= \left(\frac{q^2}{2A^2\varepsilon_0}\right)(A \cdot d) = \frac{q^2}{2\left(\frac{A\varepsilon_0}{d}\right)}$$

$$= \frac{q^2}{2C} \qquad \left(\text{as } C = \frac{A\varepsilon_0}{d}\right)$$

∴ $$U = \frac{q^2}{2C}$$ **Ans.**

(b) In case of a spherical conductor (of radius R) the excess charge resides on the outer surface of the conductor. The field inside the conductor is zero. It extends from surface to infinity. And since the potential energy is stored in the field only, it will be stored in the region extending from surface to infinity. But as the field is non-uniform, the energy density u is also non-uniform. So, the total energy will be calculated by integration. Electric field at a distance r from the centre is

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

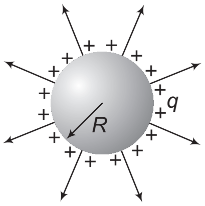


Fig. 25.36

∴ $$u(r) = \frac{1}{2}\varepsilon_0 E^2$$

$$= \frac{1}{2}\varepsilon_0 \left\{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\right\}^2$$

Energy stored in a volum $dV = (4\pi r^2)\,dr$ is

$$dU = u\,dV$$

∴ Total energy stored, $U = \int_{r=R}^{r=\infty} dU$

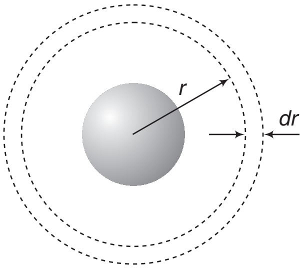


Fig. 25.37

Substituting the values, we get

$$U = \frac{q^2}{2\,(4\pi\varepsilon_0 R)}$$

or $$U = \frac{q^2}{2C}$$ **Ans.**

(as $C = 4\pi\varepsilon_0 R$)

25.8 *C-R* Circuits

Charging of a Capacitor in *C-R* Circuit

To understand the charging of a capacitor in *C-R* circuit, let us first consider the charging of a capacitor without resistance.

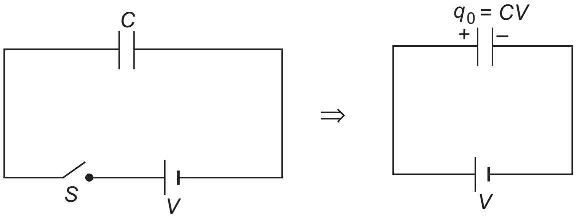


Fig. 25.38

Consider a capacitor connected to a battery of emf V through a switch S. When we close the switch the capacitor gets charged immediately. Charging takes no time. A charge $q_0 = CV$ appears in the capacitor as soon as switch is closed and the q-t graph in this case is a straight line parallel to t-axis as shown in Fig. 25.39

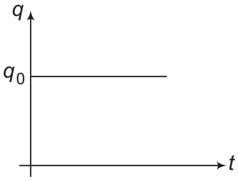


Fig. 25.39

If there is some resistance in the circuit charging takes some time. Because resistance opposes the charging (or current flow in the circuit). Final charge (called steady state charge) is still q_0 but it is acquired after a long period of time. The q-t equation in this case is

$$\boxed{q = q_0\,(1 - e^{-t/\tau_C})}$$

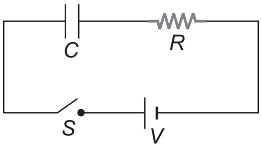


Fig. 25.40

Here, $q_0 = CV$ and $\tau_C = CR =$ time constant.

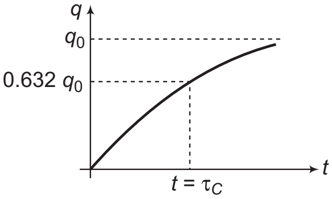


Fig. 25.41

q-t graph is an exponentially increasing graph. The charge q increases exponentially from 0 to q_0.

From the graph and equation, we see that

at $t = 0,\ q = 0$ and at $t = \infty,\ q = q_0$

Definition of τ_C

At $t = \tau_C$, $q = q_0\ (1 - e^{-1}) \approx 0.632\ q_0$

Hence, τ_C can be defined as the time in which 63.2% charging is over. Note that τ_C is the time.

Hence, $$[\tau_C] = [\text{time}]$$

or $$[CR] = [\mathrm{M}^0\mathrm{L}^0\ \mathrm{T}]$$

Proof : Now, let us derive the q-t relation discussed above.

Suppose the switch is closed at time $t = 0$. At some instant of time, let charge in the capacitor is $q\ (< q_0)$ and it is still increasing and hence current is flowing in the circuit.

Applying loop law in $ABEDA$, we get

$$-\frac{q}{C} - iR + V = 0$$

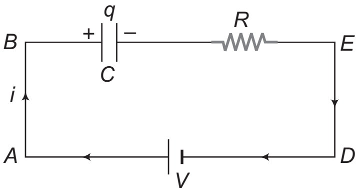


Fig. 25.42

Here, $$i = \frac{dq}{dt}$$

$\therefore$ $$-\frac{q}{C} - \left(\frac{dq}{dt}\right)R + V = 0$$

$\therefore$ $$\frac{dq}{V - \frac{q}{C}} = \frac{dt}{R}$$

or $$\int_0^q \frac{dq}{V - \frac{q}{C}} = \int_0^t \frac{dt}{R}$$

This gives $$q = CV\ (1 - e^{-\frac{t}{CR}})$$

Substituting $CV = q_0$ and $CR = \tau_C$, we have

$$q = q_0\ (1 - e^{-t/\tau_C}).$$

Charging Current

Current flows in a C-R circuit during charging of a capacitor. Once charging is over or the steady state condition is reached the current becomes zero. The current at any time t can be calculated by differentiating q with respect to t. Hence,

$$i = \frac{dq}{dt} = \frac{d}{dt}\{q_0\ (1 - e^{-t/\tau_C})\}$$

or $$i = \frac{q_0}{\tau_C}\ e^{-t/\tau_C}$$

Substituting $q_0 = CV$ and $\tau_C = CR$, we have

$$i = \frac{V}{R}\ e^{-t/\tau_C}$$

By letting, $\frac{V}{R} = i_0$ $\boxed{i = i_0\, e^{-t/\tau_C}}$

i.e. current decreases exponentially with time.

The *i-t* graph is as shown in Fig. 25.43.

Here, $i_0 = \frac{V}{R}$ is the current at time $t = 0$. This is the current which would had been in the absence of capacitor in the circuit.

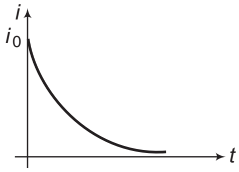


Fig. 25.43

Discharging of a Capacitor in *C-R* Circuit

To understand discharging through a *C-R* circuit again we first consider the discharging without resistance.

Suppose a capacitor has a charge q_0. The positive plate has a charge $+q_0$ and negative plate $-q_0$. It implies that the positive plate has deficiency of electrons and negative plate has excess of electrons. When the switch is closed, the extra electrons on negative plate immediately rush to the positive plate and net charge on both plates becomes zero. So, we can say that discharging takes place immediately.

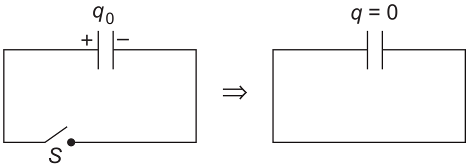


Fig. 25.44

In case of a *C-R* circuit, discharging also takes time. Final charge on the capacitor is still zero but after sufficiently long period of time. The *q-t* equation in this case is

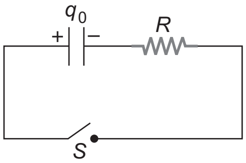


Fig. 25.45

$$\boxed{q = q_0 e^{-t/\tau_C}}$$

Thus, q decreases exponentially from q_0 to zero, as shown in Fig. 25.46.

From the graph and the equation, we see that

At $t = 0,\ q = q_0$

At $t = \infty,\ q = 0.$

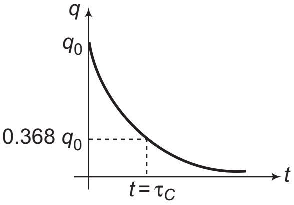


Fig. 25.46

Definition of Time Constant (τ_C)

In case of discharging, definition of τ_C is changed.

At time $t = \tau_C$,

$$q = q_0 e^{-1} = 0.368\, q_0$$

Hence, in this case τ_C can be defined as the time when charge reduces to 36.8% of its maximum value q_0.

Discharging Current

During discharging, current flows in the circuit till q becomes zero. This current can be found by differentiating q with respect to t but with negative sign because charge is decreasing with time. So,

$$i = \left(-\frac{dq}{dt}\right) = -\frac{d}{dt}(q_0 e^{-t/\tau_C})$$

$$= \frac{q_0}{\tau_C} e^{-t/\tau_C}$$

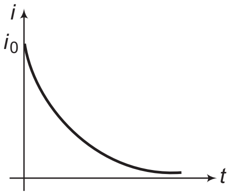


Fig. 25.47

By letting, $$\frac{q_0}{\tau_C} = i_0$$

We have, $$\boxed{i = i_0 e^{-t/\tau_C}}$$

This is an exponentially decreasing equation. Thus, i-t graph decreases exponentially with time from i_0 to 0. The i-t graph is as shown in Fig. 25.47.

25.9 Methods of Finding Equivalent Resistance and Capacitance

We know that in series,

$$R_{eq} = R_1 + R_2 + \ldots + R_n$$

and $$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

and in parallel, $$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

and $$C_{eq} = C_1 + C_2 + \ldots + C_n$$

Sometimes there are circuits in which resistances/capacitors are in mixed grouping. To find R_{eq} or C_{eq} for such circuits few methods are suggested here which will help you in finding R_{eq} or C_{eq}.

Method of Same Potential

Give any arbitrary potentials ($V_1, V_2, \ldots$ etc.) to all terminals of capacitors/resistors. But notice that the points connected directly by a conducting wire will have at the same potential. The capacitors/resistors having the same PD are in parallel. Make a table corresponding to the figure. Now, corresponding to this table a simplified figure can be formed and from this figure C_{eq} and R_{eq} can be calculated.

Example 25.14 *Find equivalent capacitance between points A and B as shown in figure.*

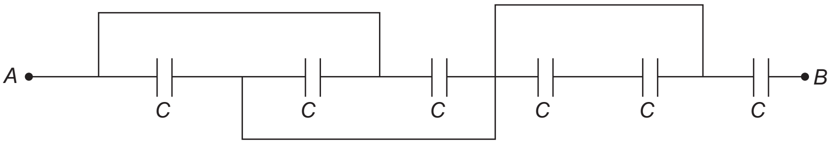


Fig. 25.48

Solution Three capacitors have PD, $V_1 - V_2$. So, they are in parallel. Their equivalent capacitance is $3C$.

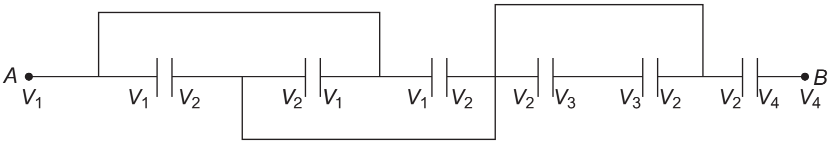


Fig. 25.49

Two capacitors have PD, $V_2 - V_3$. So, their equivalent capacitance is $2C$ and lastly there is one capacitor across which PD is $V_2 - V_4$. So, let us make a table corresponding to this information.

Table 25.1

PD	Capacitance
$V_1 - V_2$	$3C$
$V_2 - V_3$	$2C$
$V_2 - V_4$	C

Now, corresponding to this table, we make a simple figure as shown in Fig. 25.50.

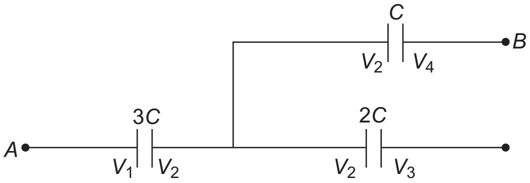


Fig. 25.50

As we have to find the equivalent capacitance between points A and B, across which PD is $V_1 - V_4$. From the simplified figure, we can see that the capacitor of capacitance $2C$ is out of the circuit and points A and B are as shown. Now, $3C$ and C are in series and their equivalent capacitance is

$$C_{eq} = \frac{(3C)(C)}{3C + C} = \frac{3}{4}C$$ **Ans.**

EXERCISE *Find equivalent capacitance between points A and B.*

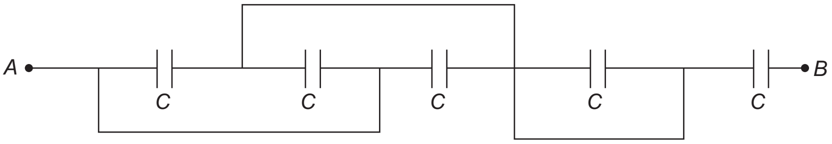


Fig. 25.51

HINT *In case PD across any capacitor comes out to be zero (i.e. the plates are short circuited), then this capacitor will not store charge. So ignore this capacitor.*

Ans. $\frac{3}{4}C$

EXERCISE *Identical metal plates are located in air at equal distance d from one another as shown in figure. The area of each plate is A. Find the capacitance of the system between points P and Q if plates are interconnected as shown.*

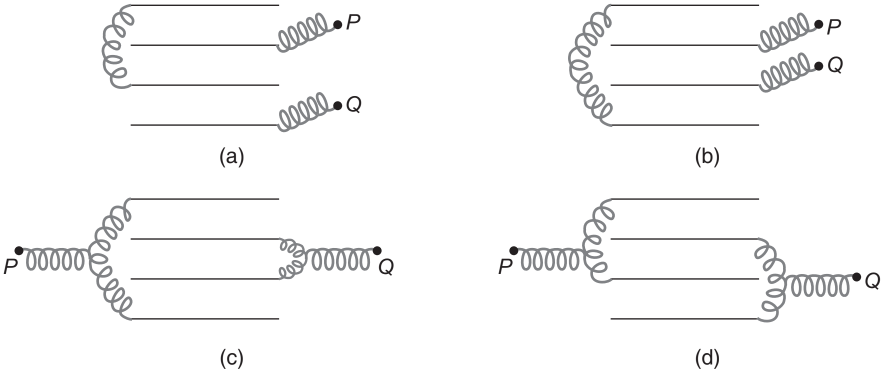


Fig. 25.52

Ans. (a) $\frac{2}{3}\frac{\varepsilon_0 A}{d}$ (b) $\frac{3}{2}\frac{\varepsilon_0 A}{d}$ (c) $\frac{2\varepsilon_0 A}{d}$ (d) $\frac{3\varepsilon_0 A}{d}$

EXERCISE *Find equivalent resistance between A and B.*

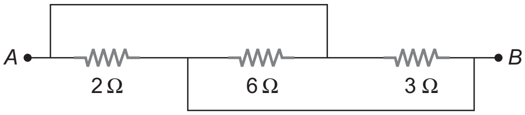


Fig. 25.53

Ans. $1\,\Omega$

EXERCISE *Find equivalent capacitance between points A and B.*

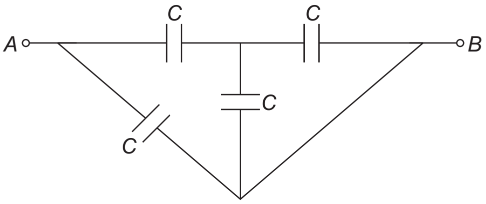


Fig. 25.54

Ans. $\frac{5}{3}C.$

Infinite Series Problems

This circuit consists of an infinite series of identical loops. To find C_{eq} or R_{eq} of such a series first we consider by ourself a value (say x) of C_{eq} or R_{eq}. Then, we break the chain in such a manner that only one loop is left with us and in place of the remaining portion we connect a capacitor or resistor x. Then, we find the C_{eq} or R_{eq} and put it equal to x. With this we get a quadratic equation in x. By solving this equation, we can find the desired value of x.

Example 25.15 *An infinite ladder network is constructed with* $1\,\Omega$ *and* $2\,\Omega$ *resistors as shown. Find the equivalent resistance between points A and B.*

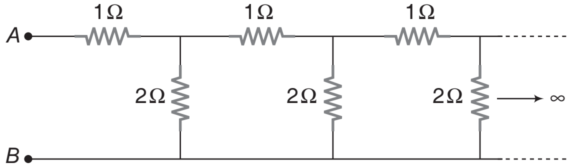


Fig. 25.55

Solution Let the equivalent resistance between A and B is x. We may consider the given circuit as shown in Fig. 25.56.

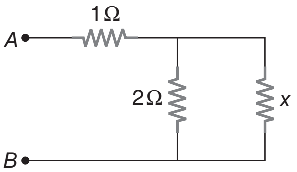


Fig. 25.56

In this diagram,

$$R_{AB} = \frac{2x}{2+x} + 1 \quad \text{or} \quad x = \frac{2x}{2+x} + 1 \qquad (\text{as } R_{AB} = x)$$

$$\text{or} \qquad x(2+x) = 2x + 2 + x \quad \text{or} \quad x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = -1\,\Omega \text{ and } 2\,\Omega$$

Ignoring the negative value, we have $R_{AB} = x = 2\,\Omega$ **Ans.**

Note *Care should be taken while breaking the chain. It should be broken from those points from where the broken chain resembles with the original chain.*

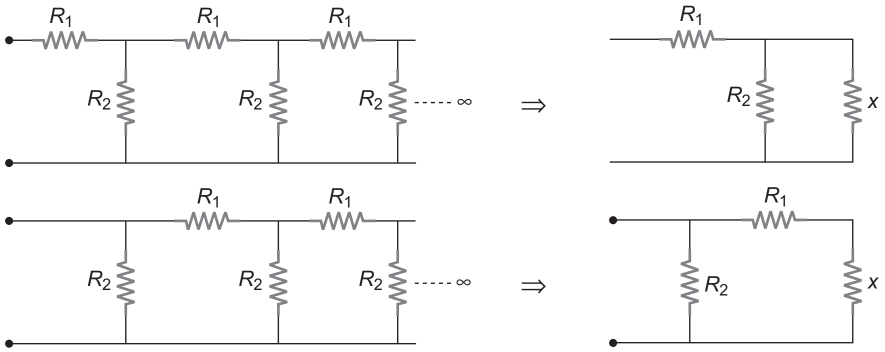


Fig. 25.57

EXERCISE *Find equivalent resistance between A and B.*

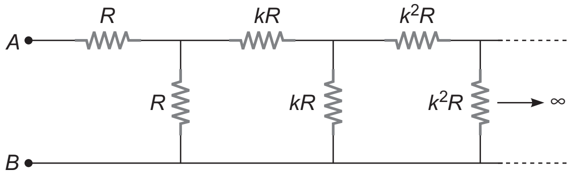


Fig. 25.58

HINT Let $R_{AB} = x$, then the resistance of the broken chain will be kx.

Ans. $R\,[(2k-1) + \sqrt{4k^2 + 1}]/2k$

Method of Symmetry

Symmetry of a circuit can be checked in the following four manners :

1. Points which are symmetrically located about the starting and last points are at same potentials. So, the resistances/capacitors between these points can be ignored. The following example will illustrate the theory.

Example 25.16 *Twelve resistors each of resistance r are connected together so that each lies along the edge of the cube as shown in figure. Find the equivalent resistance between*

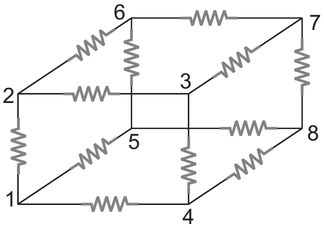


Fig. 25.59

(a) 1 *and* 4 *(b)* 1 *and* 3

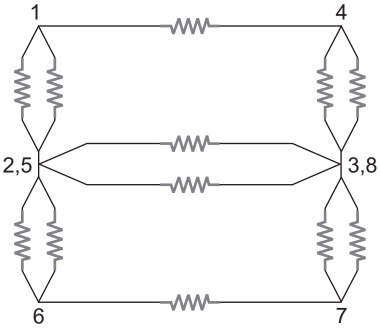


Fig. 25.60

Solution (a) **Between 1 and 4 :** Points 2 and 5 are symmetrically located w.r.t. points 1 and 4. So, they are at same potentials.

Similarly, points 3 and 8 are also symmetrically located w.r.t. points 1 and 4. So, they are again at same potential.

Now, we have 12 resistors each of resistance r connected across 1 and 2, 2 and 3,…, etc. So, redrawing them with the assumption that 2 and 5 are at same potential and 3 and 8 are at same potential. The new figure is as shown in Fig.25.60.

Now, we had to find the equivalent resistance between 1 and 4. We can now simplify the circuit as

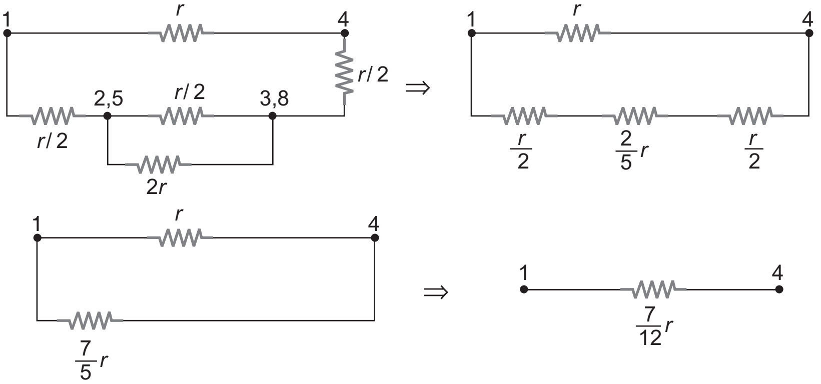


Fig. 25.61

Thus, the equivalent resistance between points 1 and 4 is $\frac{7}{12}r$. **Ans.**

(b) **Between 1 and 3 :** Points 6 and 8 are symmetrically located w.r.t. points 1 and 3. Similarly, points 2 and 4 are located symmetrically w.r.t. points 1 and 3. So, points 6 and 8 are at same potential. Similarly, 2 and 4 are at same potentials. Redrawing the simple circuit, we have Fig. 25.62.

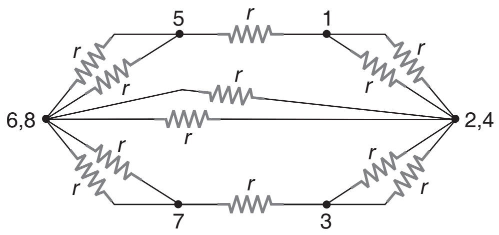


Fig. 25.62

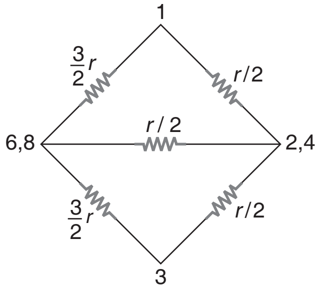


Fig. 25.63

Between 1 and 3, a balanced Wheatstone bridge is formed as shown in Fig. 25.63.

So, the resistance between 2 and 6 and between 4 and 8 can be removed.

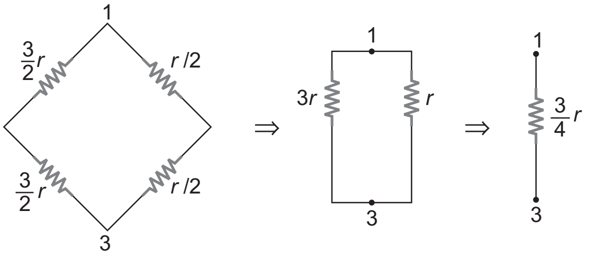


Fig. 25.64

Thus, the equivalent resistance between 1 and 3 is $\frac{3}{4}r$. **Ans.**

EXERCISE *Fourteen identical resistors each of resistance r are connected as shown. Calculate equivalent resistance between A and B.*

Ans. $1.2r$

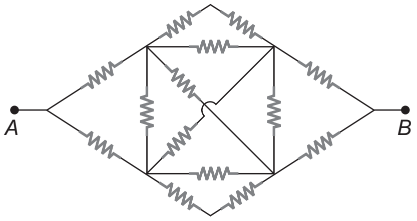


Fig. 25.65

EXERCISE *Eight identical resistances r each are connected along edges of a pyramid having square base ABCD as shown. Calculate equivalent resistance between A and O.*

Ans. $\frac{7r}{15}$

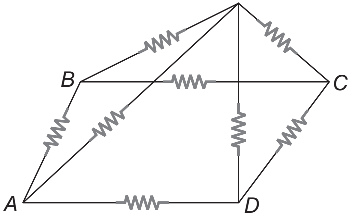


Fig. 25.66

2. If points A and B are connected to a battery and AB is a line of symmetry, then all points lying on perpendiculars drawn to AB are at the same potential. For example,

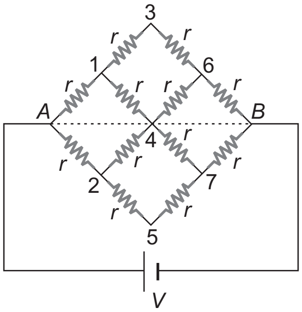


Fig. 25.67

In Fig. 25.67, points (1, 2), (3, 4, 5) and (6, 7) are at same potential. So, we can join these points and draw a simple circuit as shown in Fig. 25.68.

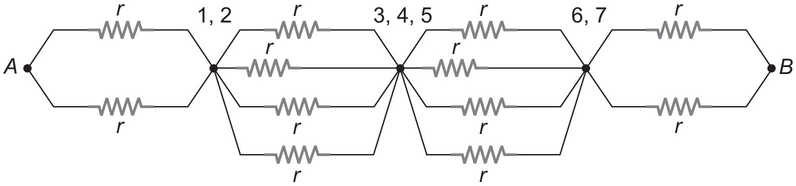


Fig. 25.68

Now, the equivalent of this series combination is

$$R_{\text{eq}} = \frac{r}{2} + \frac{r}{4} + \frac{r}{4} + \frac{r}{2} = \frac{3r}{2}$$

EXERCISE *Solve the same problem by connection removal method (will be discussed later).*

3. Even if AB is not a line of symmetry but its perpendicular bisector is, then all the points on this perpendicular bisector are at the same potential.

For example,

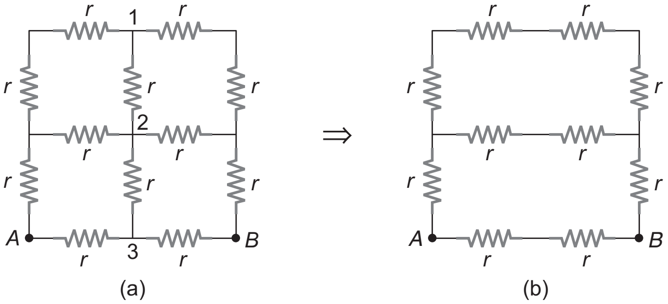


Fig. 25.69

In Fig. (a), AB is not a line of symmetry but, 1, 2 and 3 are line of symmetry. Hence, they are at same potential (if A and B are connected to a battery). This makes the resistors between 1 & 2 and 2 and 3 redundant because no current flows through them. So, the resistance between them can be removed [as shown in Fig. (b)]. The equivalent resistance between A and B can now be easily determined as $\frac{5r}{4}$.

4. Each wire in the cube has a resistance r. We are interested in calculating the equivalent resistance between A and B.

   This is a three-dimensional case and in place of a line of symmetry involving points A and B we locate a plane of symmetry involving A and B.

   Such a plane is the plane $ABce$ and for this plane points d and f and g and h have the same potential.

   The equivalent resistance between A and B can now be easily worked out (Using Wheatstone's bridge principle) as

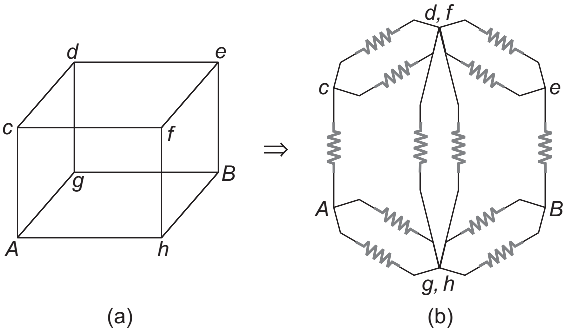


Fig. 25.70

$$R_{\text{eq}} = \frac{3r}{4}$$

Connection Removal Method

This method is useful when the circuit diagram is symmetric except for the fact that the input and output are reversed. That is the flow of current is a mirror image between input and output above a particular axis. In such cases, some junctions are unnecessarily made. Even if we remove that junction there is no difference in the remaining circuit or current distribution. But after removing the junction, the problem becomes very simple. The following example illustrates the theory.

Example 25.17 *Find the equivalent resistance between points A and B.*

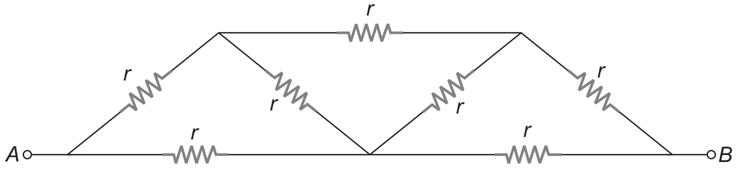


Fig. 25.71

Solution

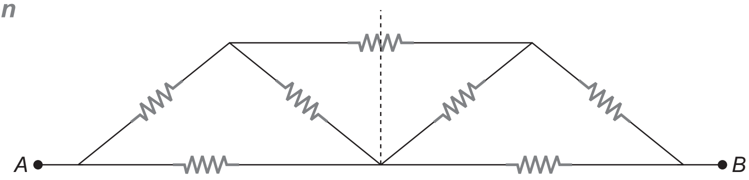


Fig. 25.72

Input and output circuits are mirror images of each other about the dotted line as shown. So, if a current i enters from A and leaves from B, it will distribute as shown below.

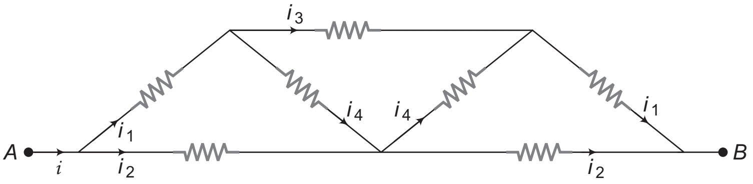


Fig. 25.73

Now, we can see in figure that the junction where i_2 and i_4 are meeting can be removed easily and then the circuit becomes simple.

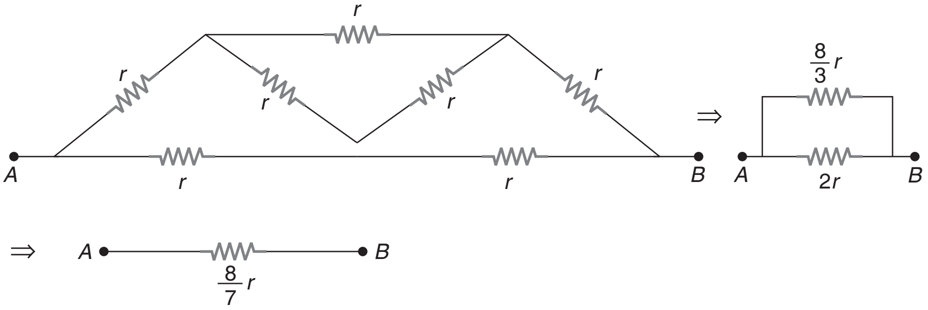


Fig. 25.74

Hence, the equivalent resistance between A and B is $\frac{8}{7}r$. **Ans.**

EXERCISE *Eight identical resistances r each are connected as shown. Find equivalent resistance between A and D.*

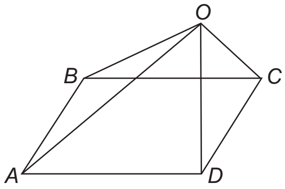


Fig. 25.75

Ans. $\frac{8r}{15}$

EXERCISE *Twelve resistors each of resistance r are connected as shown. Find equivalent resistance between A and B.*

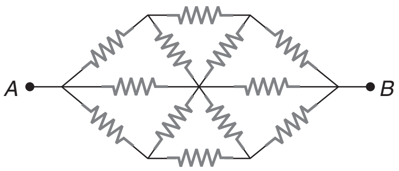


Fig. 25.76

Ans. $(4/5)\,r$

EXERCISE *Find equivalent resistance between A and B.*

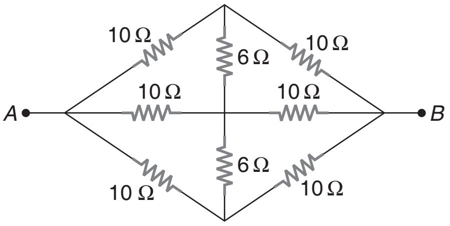


Fig. 25.77

Ans. $\frac{20}{3}\Omega$.

Wheatstone Bridge Circuits

Wheatstone bridge in case of resistors has already been discussed in the chapter of current electricity.

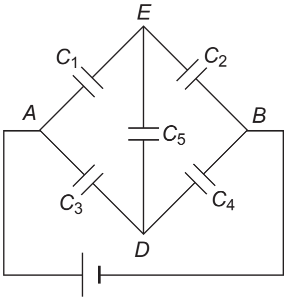


Fig. 25.78

For capacitor, theory is same.

If $\frac{C_1}{C_2}=\frac{C_3}{C_4}$, bridge is said to be balanced and in that case

$$V_E=V_D \quad \text{or} \quad V_E-V_D \quad \text{or} \quad V_{ED}=0$$

i.e. no charge is stored in C_5. Hence, it can be removed from the circuit.

EXERCISE In the circuit shown in figure, prove that $V_{AB}=0$ if $\frac{R_1}{R_2}=\frac{C_2}{C_1}$.

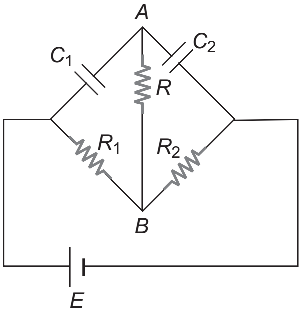


Fig. 25.79

By Distributing Current/Charge

Sometimes none of the above five methods is applicable. So, this one is the last and final method which can be applied everywhere. Of course this method is a little bit lengthy but is applicable everywhere, under all conditions. In this method, we assume a main current/charge, i or q. Distribute it in different resistors/capacitors as i_1, i_2 ... (or $q_1, q_2, \ldots$, etc.). Using Kirchhoff's laws, we find $i_1, i_2, \ldots$ etc., (or $q_1, q_2, \ldots$, etc.) in terms of i (or q). Then, find the potential difference between starting and end points through any path and equate it with iR_{net} or q/C_{net}. By doing so, we can calculate R_{net} or C_{net}.

The following example is in support of the theory.

Example 25.18 *Find the equivalent capacitance between A and B.*

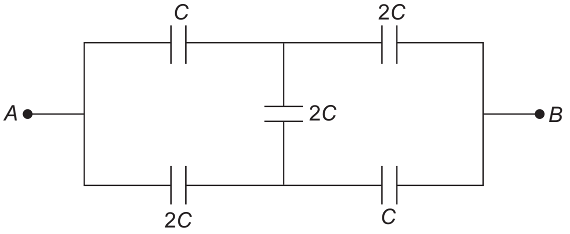


Fig. 25.80

Solution The given circuit forms a Wheatstone bridge. But the bridge is not balanced. Let us suppose point A is connected to the positive terminal of a battery and B to the negative terminal of the same battery; so that a total charge q is stored in the capacitors. Just by seeing input and output symmetry, we can say that charges will be distributed as shown below.

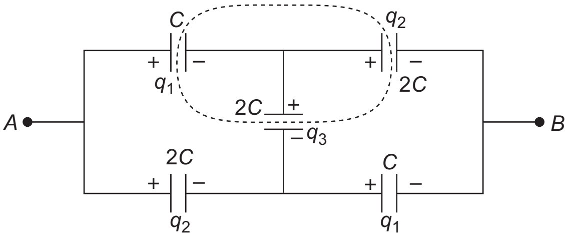


Fig. 25.81

$$q_1 + q_2 = q \qquad \ldots\text{(i)}$$

Applying second law, we have

$$-\frac{q_1}{C} - \frac{q_3}{2C} + \frac{q_2}{2C} = 0$$

or

$$q_2 - q_3 - 2q_1 = 0 \qquad \ldots\text{(ii)}$$

Plates inside the dotted line form an isolated system. Hence,

$$q_2 + q_3 - q_1 = 0 \qquad \ldots\text{(iii)}$$

Solving these three equations, we have

$$q_1 = \frac{2}{5}q, \quad q_2 = \frac{3}{5}q \quad \text{and} \quad q_3 = -\frac{q}{5}$$

Now, let C_{eq} be the equivalent capacitance between A and B. Then,

$$V_A - V_B = \frac{q}{C_{\text{eq}}} = \frac{q_1}{C} + \frac{q_2}{2C}$$

$$\therefore \quad \frac{q}{C_{\text{eq}}} = \frac{2q}{5C} + \frac{3q}{10C} = \frac{7q}{10C}$$

$$\therefore \quad C_{\text{eq}} = \frac{10}{7}C$$

Ans.

Final Touch Points

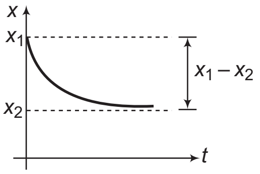
1. Now, onwards we will come across the following integration very frequently. So, remember the result as such.

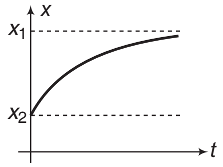
   If $\int_0^x \frac{dx}{a-bx} = \int_0^t c\,dt,$ then $x = \frac{a}{b}(1-e^{-bct})$

   and if $\int_{x_0}^x \frac{dx}{a-bx} = \int_0^t c\,dt,$ then $x = \frac{a}{b} - \left(\frac{a}{b} - x_0\right)e^{-bct}$

   Here, *a, b* and *c* are constants.

2. Sometimes a physical quantity *x* decreases from x_1 to x_2, exponentially, then *x*-*t* equation is like





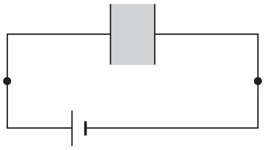
$$x = x_2 + (x_1 - x_2)\,e^{-Kt}$$

   Here, *K* is a constant.

   Similarly, if *x* increases from x_2 to x_1 exponentially, then *x*-*t* equation is

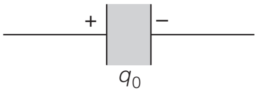
$$x = x_2 + (x_1 - x_2)(1 - e^{-Kt})$$

3. **Leakage Current Through a Capacitor** The space between the capacitor's plates is filled with a dielectric and we assume that no current flows through it when the capacitor is connected to a battery as in figure (a) or if the capacitor is charged, the charge on its plates remains forever. But every insulator has some conductivity. On account of which some current flows through the capacitor if connected to a battery. This small current is known as the leakage current. Similarly, when it is charged, the charge does not remain as it is for a long period of time. But it starts discharging. Or we can say it becomes a case of discharging of a capacitor in *C*-*R* circuit.



(a)

or



(b)

   In both the cases, we will first find the resistance of the dielectric.

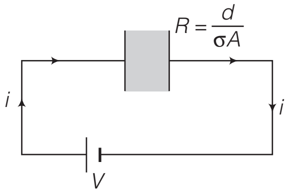
$$R = \frac{l}{\sigma A} \qquad (\sigma = \text{specific conductance})$$

   Here, $l = d$ (the distance between the plates of the capacitor)

$$\therefore \qquad \boxed{R = \frac{d}{\sigma A}}$$

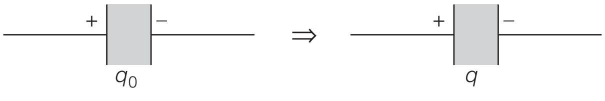
   Thus, the leakage current in the circuit shown in the figure is

$$i = \frac{V}{R}$$



Similarly, if the capacitor is given a charge q_0 at time $t = 0$, then after time t, q charge will remain on it, where

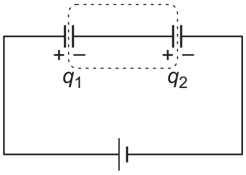
$$\boxed{q = q_0 e^{-t/\tau_C}} \qquad \text{(Discharging of a capacitor)}$$



At $t = 0$ At $t = t$

Here,
$$\tau_C = CR = \left(\frac{K\varepsilon_0 A}{d}\right)\left(\frac{d}{\sigma A}\right) \quad \text{or} \quad \boxed{\tau_C = \frac{K\varepsilon_0}{\sigma}}$$

4. If capacitors are in series, then charges on them are equal, provided they are initially uncharged. This can be proved by the following illustration :



Let us suppose that charges on two capacitors are q_1 and q_2. The two plates encircled by dotted lines form an isolated system. So, net charge on them will remain constant.

$$\Sigma q_f = \Sigma q_i$$

If initially they are uncharged, then

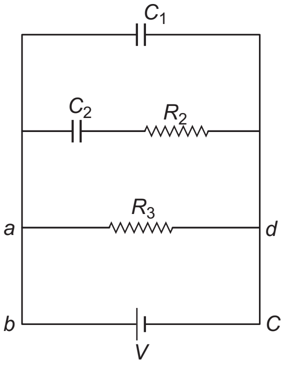
$$\Sigma q_i = 0$$

$$\Sigma q_f \text{ is also zero}$$

or
$$-q_1 + q_2 = 0 \quad \text{or} \quad q_1 = q_2$$

So, this proves that charges are equal if initially they are uncharged.

5. **Independent parallel circuit**



Three circuits shown in figure are independently connected in parallel with the battery. Potential difference across each of the circuit is V. By this potential difference, capacitor C_1 is immediately charged. Capacitor C_2 is exponentially charged and current grows immediately in R_3. Thus,

$$q_{C_1} = C_1 V \qquad \text{(immediately)}$$

$$q_{C_2} = C_2 V \left(1 - e^{-\frac{t}{C_2 R_2}}\right)$$

$$I_{R_3} = \frac{V}{R_3} \qquad \text{(immediately)}$$

Note *If any resistance or capacitance is connected between abcd, then it no longer remains an independent parallel circuit.*

Solved Examples

TYPED PROBLEMS

Type 1. *In a complex capacitor circuit method of finding values of q and V across different capacitors if values across one capacitor are known*

Concept

In series, q is same and V distributes in inverse ratio of capacity.

As, $$V = \frac{q}{C} \quad \Rightarrow \quad V \propto \frac{1}{C} \qquad (q \text{ is same})$$

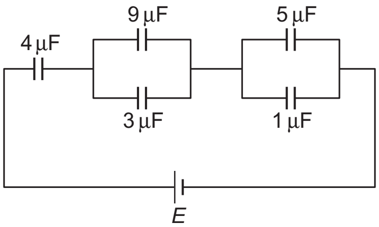
If capacitance is double, then V will be half.

In parallel, V is same and q distributes in direct ratio of capacity.

As, $$q = CV \quad \Rightarrow \quad q \propto C \qquad (V \text{ is same})$$

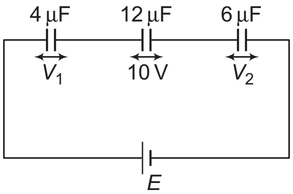
If C is double, then q is also double.

Example 1



In the circuit shown in figure potential difference across $3\,\mu F$ is $10\,V$. Find potential difference and charge stored in different capacitors. Also find emf of the battery E.

Solution The given circuit can be simplified as under



V_1: $4\,\mu\text{F}$ is $\frac{1}{3}$rd of $12\,\mu\text{F}$. Therefore, V_1 is thrice of 10 V or 30 V.

V_2: $6\,\mu\text{F}$ is half of $12\,\mu\text{F}$. Therefore, V_2 is twice of 10 V or 20 V.

$$\begin{aligned} E &= V_1 + 10 + V_2 \\ &= 30 + 10 + 20 \\ &= 60 \text{ V} \end{aligned}$$

Ans.

Potential difference and charge on different capacitors in tabular form are given below.

Table 25.2

Capacitance	Potential difference	Charge $q = CV$
$4\,\mu F$	30 V	$120\,\mu C$
$9\,\mu F$	10 V	$90\,\mu C$
$3\,\mu F$	10 V	$30\,\mu C$
$5\,\mu F$	20 V	$100\,\mu C$
$1\,\mu F$	20 V	$20\,\mu C$

Type 2. *Configuration of capacitor is changed and change in five quantities q, C, V, U and E is asked*

Concept

Some problems are asked when a capacitor is charged through a battery and then the configuration of capacitor is changed :

(i) either by inserting a dielectric slab or removing the slab (if it already exists) or

(ii) by changing the distance between the plates of capacitor or

(iii) by both.

The questions will be based on the change in electric field, potential, etc. In such problems, two cases are possible.

Case 1 **When battery is removed after charging**

If the battery is removed after charging, then the charge stored in the capacitor remains constant.

$$q = \text{constant}$$

First of all, find the change in capacitance and according to the formula find the change in other quantities.

Example 2 *An air capacitor is first charged through a battery. The charging battery is then removed and a dielectric slab of dielectric constant $K = 4$ is inserted between the plates. Simultaneously, the distance between the plates is reduced to half, then find change in C, E, V and U.*

Solution Change in capacitance, $$C = \frac{K\varepsilon_0 A}{d} \propto \frac{K}{d}$$

$\therefore$ Capacitance will become 8 times $\left(K = 4,\ d' = \dfrac{d}{2}\right)$

Change in electric field

$$\therefore \quad E = E_0/K$$

$$\text{or} \quad E \propto \frac{1}{K} \qquad (\text{if } q = \text{constant})$$

or the electric field will become $\frac{1}{4}$ times its initial value.

Change in potential difference,

$$V = \frac{q}{C}$$

or $$V \propto \frac{1}{C} \qquad \text{(if } q = \text{constant)}$$

Therefore, potential difference becomes $\frac{1}{8}$ times of its initial value.

Alternate method

$$V = Ed$$

Electric field has become $\frac{1}{4}$ times its initial value and d is reduced to half. Hence, V becomes $\frac{1}{8}$ times.

Change in stored potential energy,

$$U = \frac{1}{2}\frac{q^2}{C}$$

or $$U \propto \frac{1}{C} \qquad \text{(if } q = \text{constant)}$$

Capacitance has become 8 times. Therefore, the stored potential energy U will become $\frac{1}{8}$ times.

Case 2 When battery remains connected

If the battery remains connected, the potential difference V becomes constant. So, in the above example, capacitance will become 8 times.

The charge stored ($q = CV$ *or* $q \propto C$) will also increase to 8 times. The electric field $\left(E = \frac{V}{d} \text{ or } E \propto \frac{1}{d}\right)$ becomes twice and the stored PE $\left(U = \frac{1}{2}CV^2 \text{ or } U \propto C\right)$ is 8 times.

Type 3. *To find self energy of a system of charges*

Concept

The self energy of a system of charges is

$$\boxed{U_s = \int_0^q V dq}$$

This comes out to be equal to $\frac{q^2}{2C}$ in case of a capacitor or conductor.

A point charge does not have any self energy.

Example 3 *Find the electric potential energy of a uniformly charged sphere.*

Solution Consider a uniformly charged sphere of radius R having a total charge q_0. The volume charge density is

$$\rho = \frac{q_0}{\frac{4}{3}\pi R^3} = \frac{3q_0}{4\pi R^3}$$

When the radius of the sphere is r, the charge contained in it is

$$q = \left(\frac{4}{3}\pi r^3\right)\rho = \left(\frac{q_0}{R^3}\right) r^3$$

The potential at the surface is

$$V = \frac{q}{4\pi\varepsilon_0 r} = \frac{q_0}{4\pi\varepsilon_0 R^3} r^2$$

The charge needed to increase the radius from r to $r + dr$ is

$$\begin{aligned} dq &= (4\pi r^2)\, dr\, \rho \\ &= \frac{3q_0}{R^3} \cdot r^2\, dr \end{aligned}$$

∴ The self energy of the sphere is

$$\begin{aligned} U_s &= \int_0^R V\, dq \\ &= \int_0^R \left(\frac{q_0}{4\pi\varepsilon_0 R^3}\, r^2\right)\left(\frac{3q_0}{R^3} \cdot r^2\, dr\right) \\ &= \frac{3q_0^2}{20\,\pi\varepsilon_0 R} \end{aligned}$$

Ans.

Example 4 *Find the electric potential energy of a uniformly charged, thin spherical shell.*

Solution Consider a uniformly charged thin spherical shell of radius R having a total charge q_0. Suppose at some instant a charge q is placed on the shell. The potential at the surface is

$$V = \frac{q}{4\pi\varepsilon_0 R}$$

∴ The self energy of the shell is

$$\begin{aligned} U_s &= \int_0^{q_0} V\, dq \\ &= \int_0^{q_0} \left(\frac{q}{4\pi\varepsilon_0 R}\right) dq \\ &= \frac{q_0^2}{8\pi\varepsilon_0 R} \end{aligned}$$

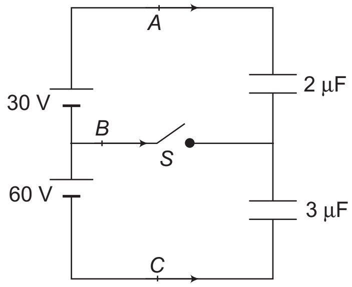
Ans.

Type 4. *Based on flow of charge when position of a switch is changed*

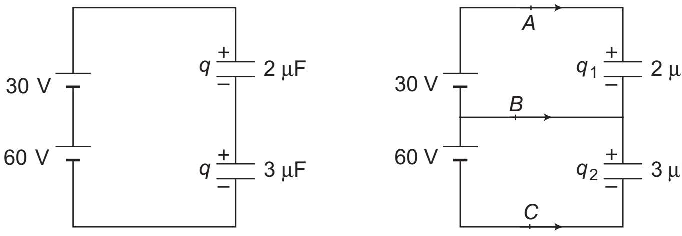
Concept

From the flow of charge we mean that when a switch in a circuit is either closed or opened or it is shifted from one position to the other, then how much charge will flow through certain points of the circuit. Such problems can be solved by finding charges on different capacitors at initial and final positions and then by the difference we can find the charge flowing through a certain point. The following example will illustrate the theory.

Example 5 *What charges will flow through A, B and C in the directions shown in the figure when switch S is closed?*



Solution Let us draw two figures and find the charge on both the capacitors before closing the switch and after closing the switch.



(a) (b)

Refer Fig. (a), when switch is open Both capacitors are in series. Hence, their equivalent capacitance is

$$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2} = \frac{(2)(3)}{2+3} = \frac{6}{5}\,\mu\text{F}$$

Therefore, charge on both capacitors will be same. Hence, using $q = CV$, we get

$$q = (30 + 60)\left(\frac{6}{5}\right)\mu\text{C} = 108\,\mu\text{C}$$

Refer Fig. (b), when switch is closed Let q_1 and q_2 be the charges (in μC) on two capacitors. Then, applying second law in upper and lower loops, we have

$$30 - \frac{q_1}{2} = 0 \quad \text{or} \quad q_1 = 60\,\mu\text{C}$$

$$60 - \frac{q_2}{3} = 0 \quad \text{or} \quad q_2 = 180\,\mu\text{C}$$

Charges q_1 and q_2 can be calculated alternatively by seeing that upper plate of 2 μF capacitor is connected with positive terminal of 30 V battery. Therefore, they are at the same potential. Similarly, the lower plate of this capacitor is at the same potential as that of the negative terminal of 30 V battery. So, we can say that PD across 2 μF capacitor is also 30 V.

$$\therefore \quad q_1 = (C)\,(\text{PD}) = (2)\,(30)\,\mu\text{C} = 60\,\mu\text{C}$$

Similarly, PD across 3 μF capacitor is same as that between 60 V battery. Hence,

$$q_2 = (3)\,(60)\,\mu\text{C} = 180\,\mu\text{C}$$

Now, let q_A charge flows from A in the direction shown. This charge goes to the upper plate of 2 μF capacitor. Initially, it had a charge $+q$ and final charge on it is $+q_1$. Hence,

$$q_1 = q + q_A$$

or $$q_A = q_1 - q = 60 - 108$$

$$= -48\,\mu\text{C}$$ **Ans.**

Similarly, charge q_B goes to the upper plate of 3 μF capacitor and lower plate of 2 μF capacitor. Initially, both the plates had a charge $+q - q$ or zero. And finally they have a charge $(q_2 - q_1)$. Hence,

$$(q_2 - q_1) = q_B + 0$$

∴ $$q_B = q_2 - q_1 = 180 - 60$$

$$= 120\,\mu\text{C}$$ **Ans.**

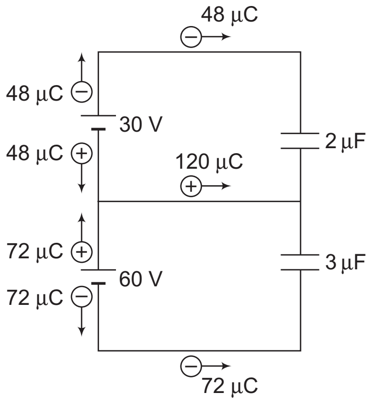
Charge q_C goes to the lower plate of 3 μF capacitor. Initially, it had a charge $-q$ and finally $-q_2$. Hence,

$$-q_2 = (-q) + q_C$$

∴ $$q_C = q - q_2 = 108 - 180$$

$$= -72\,\mu\text{C}$$ **Ans.**

So, the charges will flow as shown below

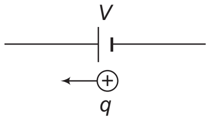


Type 5. *Based on heat generation or loss of energy during shifting of switch*

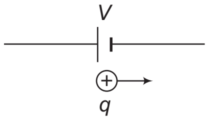
Concept

By heat generation (or loss of energy), we mean that when a switch is shifted from one position to the other, what amount of heat will be generated (or loss will be there) in the circuit. Such problems can be solved by simple energy conservation principle. For this, remember that when a charge $+q$ flows from negative terminal to the positive terminal inside a battery of emf V is supplied an energy,

$$E = qV$$



Energy supplied = qV



Energy consumed = qV

and if opposite is the case, i.e. charge + q flows in opposite direction, then it consumes energy by the same amount.

Now, from energy conservation principle we can find the heat generated (or loss of energy) in the circuit in shifting the switch.

Heat generated or loss of energy = energy supplied by the battery/batteries
– energy consumed by the battery/batteries + $\Sigma U_i - \Sigma U_f$

Here, ΣU_i = energy stored in all the capacitors initially and

ΣU_f = energy stored in all the capacitors finally

Example 6 *Find loss of energy in example* 5.

Solution In the above example, energy is supplied by 60 V battery and consumed by 30 V battery. Using $E = qV$, we have

$$\text{Energy supplied} = (72 \times 10^{-6})\,(60)$$

$$= 4.32 \times 10^{-3}\ \text{J}$$

$$\text{Energy consumed} = (48 \times 10^{-6})\,(30)$$

$$= 1.44 \times 10^{-3}\ \text{J}$$

$$\Sigma U_i = \frac{1}{2} \times \frac{6}{5} \times 10^{-6} \times (90)^2$$

$$= 4.86 \times 10^{-3}\ \text{J}$$

and $$\Sigma U_f = \frac{1}{2} \times 2 \times 10^{-6} \times (30)^2 + \frac{1}{2} \times 3 \times 10^{-6} \times (60)^2$$

$$= 6.3 \times 10^{-3}\ \text{J}$$

$\therefore$ $$\text{Loss of energy} = (4.32 - 1.44 + 4.86 - 6.3) \times 10^{-3}\ \text{J}$$

$$= 1.44 \times 10^{-3}\ \text{J}$$ **Ans.**

Example 7 *Prove that in charging a capacitor half of the energy supplied by the battery is stored in the capacitor and remaining half is lost during charging.*

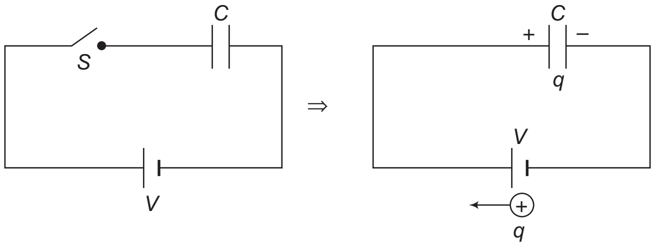
Solution When switch S is closed, $q = CV$ charge is stored in the capacitor.

Charge transferred from the battery is also q.

Hence,

$$\text{energy supplied by the battery} = qV = (CV)\,(V) = CV^2.$$

Half of its energy, i.e. $\frac{1}{2}CV^2$ is stored in the capacitor and the remaining 50 % or $\frac{1}{2}CV^2$ is lost.



Type 6. *Two or more than two capacitors are charged from different batteries and then connected in parallel*

Concept

In parallel, they come to a common potential given by

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

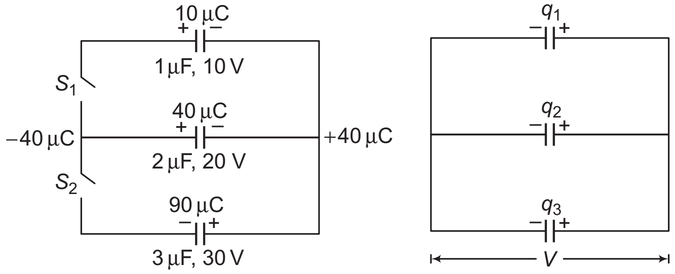
Moreover, total charge on them distributes in direct ratio of their capacity or we can also find the final charges on the capacitors by using the equation, $q = CV$

Example 8 *Three capacitors of capacities* $1\,\mu F$, $2\,\mu F$ *and* $3\,\mu F$ *are charged by* $10\,V$, $20\,V$ *and* $30\,V$ *respectively. Now, positive plates of first two capacitors are connected with the negative plate of third capacitor on one side and negative plates of first two capacitors are connected with positive plate of third capacitor on the other side. Find*

(a) common potential V

(b) final charges on different capacitors

Solution



Total charge on all three capacitors $= 40\,\mu\text{C}$

Total capacity $= (1 + 2 + 3)\,\mu\text{F} = 6\,\mu\text{F}$

(a) Common potential,

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

$$= \frac{40\,\mu\text{C}}{6\,\mu\text{F}} = \frac{20}{3}\ \text{volt}$$

(b)

$$q_1 = C_1V = (1\,\mu\text{F})\left(\frac{20}{3}\right) = \frac{20}{3}\,\mu\text{C}$$

$$q_2 = C_2V = (2\,\mu\text{F})\left(\frac{20}{3}\right) = \frac{40}{3}\,\mu\text{C}$$

$$q_3 = C_3V = (3\,\mu\text{F})\left(\frac{20}{3}\right) = 20\,\mu\text{C}$$

Type 7. *To find final charges on different capacitors when position of switch is changed (opened, closed or shifted from one position to other position)*

Concept

(i) To find current in a C-R circuit at any time t, a capacitor may be assumed a battery of emf E or $V = \frac{q}{C}$.

(ii) Difference between a normal battery and a capacitor battery is, emf of a normal battery remains constant while emf of a capacitor battery keeps on changing with q.

(iii) Before changing the position of switch, every loop of the circuit may not be balanced by Kirchhoff's second equation of potential. So, in a single loop problem rotate a charge q, either clockwise or anti-clockwise. From this charge q, some of the charges on capacitors may increase and others may decrease. To check this, always concentrate on positive plate of each capacitors. If positive charge comes towards this plate, then charge on this capacitor will increase.

(iv) With these charges, apply Kirchhoff's loop equation and find the final charges on them.

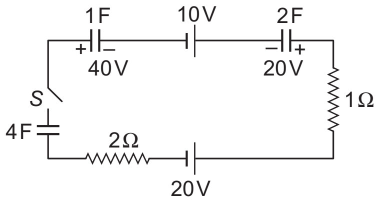
(v) In this redistribution of charges, there is some loss of energy as discussed in type 5.

(vi) If there are only capacitors in the circuit, then redistribution of charges is immediate and if there are resistors in the circuit, then redistribution is exponential.

(vii) If there are only capacitors, then loss is in the form of electromagnetic waves and if there are resistors in the circuit, then loss is in the form of heat.

Further, this loss is proportional to R if resistors are in series ($H = i^2Rt$ or $H \propto R$ as i is same in series) and this loss is inversely proportional to R if resistors are in parallel ($H = \frac{V^2t}{R}$ or $H \propto \frac{1}{R}$ as V is same in parallel).

Example 9 *In the circuit shown in figure, switch S is closed at time $t = 0$. Find*



(a) Initial current at $t = 0$ and final current at $t = \infty$ in the loop.

(b) Total charge q flown from the switch.

(c) Final charges on capacitors in steady state at time $t = \infty$.

(d) Loss of energy during redistribution of charges.

(e) Individual loss across $1\,\Omega$ and $2\,\Omega$ resistance.

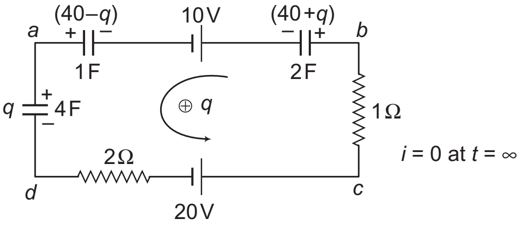
Solution (a) **At $t = 0$** Three capacitors may be assumed like batteries of emf 40 V, 20 V and 0 V.

$$\therefore \quad i = \frac{\text{Net emf}}{\text{Total resistance}} = \frac{40 + 20 - 10 - 20}{1 + 2}$$

$$= 10 \text{ A} \qquad \text{(anti-clockwise)}$$

At $t = \infty$ When charge redistribution is complete and loop is balanced by Kirchhoff's second equation of potential, current in the loop becomes zero, as insulator is filled between the capacitors.

(b) Redistribution current was anti-clockwise. So, we can assume that $+q$ charge rotates anti-clockwise in the loop (between time $t = 0$ and $t = \infty$). After this rotation of charges, final charges on different capacitors are as shown below.



Applying loop equation in loop *abcda*,

$$-\frac{(40-q)}{1}+10+\frac{(40+q)}{2}-20+\frac{q}{4}=0$$

Solving this equation, we get $q=\dfrac{120}{7}\text{ C}$ **Ans.**

(c) **Final charges**

$$q_{1\text{F}}=40-q=40-\frac{120}{7}=\frac{160}{7}\text{ C}$$

$$q_{2\text{F}}=40+q=40+\frac{120}{7}=\frac{400}{7}\text{ C}$$

$$q_{4\text{F}}=q=\frac{120}{7}\text{ C}$$

(d) **Total loss of energy during redistribution**

$$\Sigma U_i=\frac{1}{2}\times(1)\,(40)^2+\frac{1}{2}\times(2)\,(20)^2=1200\text{ J}$$

$$\Sigma U_f=\frac{1}{2}\times\frac{(160/7)^2}{1}+\frac{1}{2}\times\frac{(400/7)^2}{2}+\frac{1}{2}\times\frac{(120/7)^2}{4}$$

$$=1114.3\text{ J}$$

$$\Delta U=\Sigma U_f-\Sigma U_i=-85.7\text{ J}$$

20 V battery will supply energy but 10 V battery will consume energy. So,

$$\text{Total energy supplied}=20\times\frac{120}{7}-10\times\frac{120}{7}=171.4\text{ J}$$

$$\text{Total heat produced}=\text{Energy supplied}-\Delta U$$

$$=171.4-(-85.7)=257.1\text{ J}$$ **Ans.**

(e) Resistors are in series. Hence,

$$H\propto R\quad\text{or}\quad\frac{H_1}{H_2}=\frac{R_1}{R_2}=\frac{1\,\Omega}{2\,\Omega}=\frac{1}{2}$$

$\therefore$ $$H_1=\left(\frac{1}{1+2}\right)(257.1)=85.7\text{ J}$$ **Ans.**

$$H_2=\left(\frac{2}{1+2}\right)(257.1)=171.4\text{ J}$$ **Ans.**

Note *(i) For making the calculations simple, we have taken capacities in Farad, otherwise Farad is a large unit.*
(ii) For two loop problems, we will rotate two charges q_1 and q_2.

Type 8. *Shortcut method of finding time varying functions in a C-R circuit like q or i etc.*

Concept

(i) At time $t = 0$, when capacitor is uncharged it offers maximum current passing through it. So, it may be assumed like a conducting wire of zero resistance. With this concept, find initial values of q or i etc.

(ii) At time $t = \infty$, when capacitor is fully charged it does not allow current through it, as insulator is filled between the plates. So, its resistance may be assumed as infinite. With this concept, find steady state values at time $t = \infty$ of q or i etc.

(iii) **Equivalent time constant** To find the equivalent time constant of a circuit, the following steps are followed :

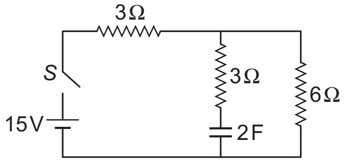
(a) Short-circuit the battery.

(b) Find net resistance across the capacitor (suppose it is R_{net})

(c) $\tau_C = (R_{\text{net}})C$

(iv) In C - R circuit, increase or decrease is always exponential. So, first make exponential graph and then write exponential equation corresponding to this graph with the time constant obtained by the method discussed above.

Example 10 *Switch S is closed at time $t = 0$ in the circuit shown in figure.*



(a) Find the time varying quantities in the circuit.

(b) Find their values at time $t = 0$.

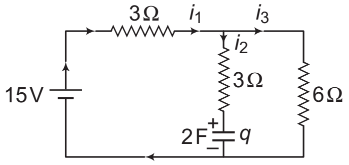
(c) Find their values at time $t = \infty$

(d) Find time constant of all time varying functions.

(e) Make their exponential graphs and write their exponential equations.

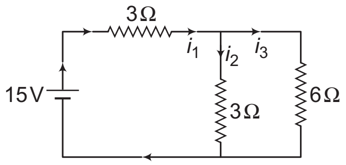
(f) Just write the equations to solve them to find different time varying functions.

Solution (a)



There are four times variable functions i_1, i_2, i_3 and q.

(b) At $t = 0$, equivalent resistance of capacitor is zero. So, the simple circuit is as shown below



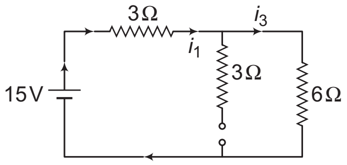
$$R_{\text{net}} = 3 + \frac{3 \times 6}{3 + 6} = 5\,\Omega$$

$$i_1 = \frac{15}{5} = 3\text{ A} \quad \Rightarrow \quad \frac{i_2}{i_3} = \frac{6}{3} = \frac{2}{1}$$

$$\therefore \qquad i_2 = \frac{2}{2+1}\,(3\text{ A}) = 2\text{ A}$$

$$i_3 = \left(\frac{1}{2+1}\right)(3\text{ A}) = 1\text{ A} \quad \text{and} \quad q = 0$$

(c) At $t = \infty$, equivalent resistance of capacitor is infinite. So, equivalent circuit is as shown below



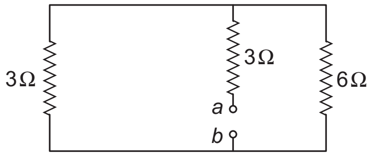
$$i_1 = i_3 = \frac{15}{3+6} = \frac{5}{3}\text{ A}$$

$$i_2 = 0$$

$$V_{2\text{F}} = V_{6\,\Omega} = iR = \left(\frac{5}{3}\right)(6) = 10\text{ volt}$$

$$\therefore \qquad q = CV = (2)\,(10) = 20\text{ C}$$

(d) By short-circuiting the battery, the simplified circuit is as shown below

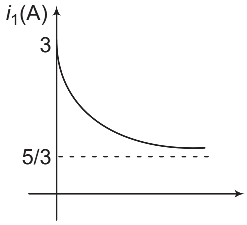


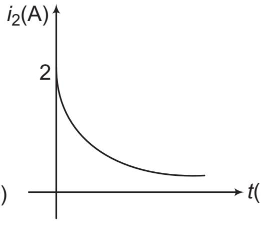
Net resistance across capacitor or ab is

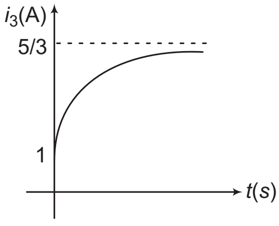
$$R_{\text{net}} = 3 + \frac{3 \times 6}{3 + 6} = 5\,\Omega$$

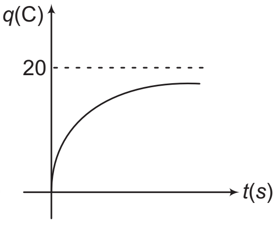
$$\therefore \qquad \tau_C = CR_{\text{net}} = (2)\,(5) = 10\text{ s}$$

(e) Exponential graphs and their exponential equations are as under.









$$i_1 = \frac{5}{3} + \left(3 - \frac{5}{3}\right) e^{\frac{-t}{\tau_C}} = \frac{5}{3} + \frac{4}{3}\, e^{-\frac{t}{10}}$$

$$i_2 = 2e^{-\frac{t}{\tau_C}} = 2e^{-\frac{t}{10}}$$

$$i_3 = 1 + \left(\frac{5}{3} - 1\right)(1 - e^{-\frac{t}{\tau_C}}) = 1 + \frac{2}{3}\,(1 - e^{-\frac{t}{10}})$$

$$q = 20\,(1 - e^{-\frac{t}{\tau_C}}) = 20\,(1 - e^{-\frac{t}{10}})$$

(f) Unknowns are four: i_1, i_2, i_3 and q. So, corresponding to the figure of part (a), four equations are

$$i_1 = i_2 + i_3 \qquad \text{...(i)}$$

$$i_2 = \frac{dq}{dt} \qquad \text{...(ii)}$$

Applying loop equation in left hand side loop,

$$+15 - 3i_1 - 3i_2 - \frac{q}{2} = 0 \qquad \text{...(iii)}$$

Applying loop equation in right hand side loop,

$$+\frac{q}{2} + 3i_2 - 6i_3 = 0 \qquad \text{...(iv)}$$

Solving these equations (with some integration), we can find same time functions as we have obtained in part (e).

Type 9. *To find current and hence potential difference between two points in a wire having a capacitor*

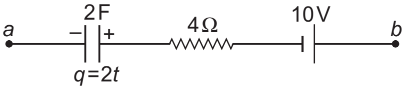
Concept

If charge on capacitor is constant, then current through capacitor wire is zero. If charge is variable, then current is non-zero. Magnitude of this current is

$$i = \left|\frac{dq}{dt}\right|$$

and direction of this current is towards the positive plate if charge is increasing and away from the positive plate if charge is decreasing.

Example 11 *In the circuit shown in figure, find* V_{ab} *at* 1 s.

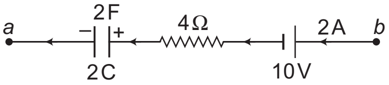


Solution Charge on capacitor is increasing. So, there is a current in the circuit from right to left. This current is given by

$$i = \frac{dq}{dt} = 2 \text{ A}$$

At 1 s, $q = 2$ C.

So, at 1 s, circuit is as shown in figure.



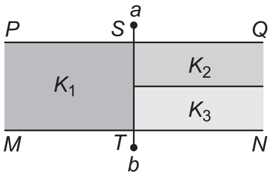
$$V_a + \frac{2}{2} + (2)(4) + 10 = V_b$$

∴ $$V_a - V_b$$

or $$V_{ab} = -19 \text{ volt}$$ **Ans.**

Type 10. *To find capacitance of a capacitor filled with two or more than two dielectrics*

Concept



PQ and *MN* are two metallic plates.

If we wish to find net capacitance between *a* and *b*, then

$$V_{PS} = V_{SQ} = V_1 \text{ (say)}$$

$$V_{MT} = V_{TN} = V_2 \text{ (say)}$$

Hence, $\quad V_{PS} - V_{MT} = V_{SQ} - V_{TN} = V_1 - V_2$

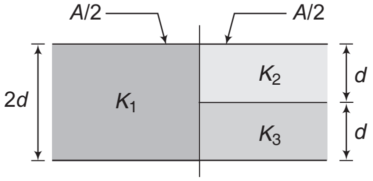
Therefore, there are two capacitors, one on right hand side and other on left hand side which are in parallel.

$\therefore \quad C = C_{\text{RHS}} + C_{\text{LHS}}$

For C_{RHS}, we can use the formula,

$$C = \frac{\varepsilon_0 A}{d - t_1 - t_2 + \dfrac{t_1}{K_1} + \dfrac{t_2}{K_2}}$$

Example 12 *What is capacitance of the capacitor shown in figure?*



Solution $C = C_{\text{LHS}} + C_{\text{RHS}}$

$$= \frac{K_1 \varepsilon_0 (A/2)}{2d} + \frac{\varepsilon_0 (A/2)}{(2d - d - d) + (d/K_2) + (d/K_3)}$$

$$= \frac{\varepsilon_0 A}{2d}\left[\frac{K_1}{2} + \frac{K_2 K_3}{K_2 + K_3}\right]$$

Type 11. *To find charge on different capacitors in a C-R circuit*

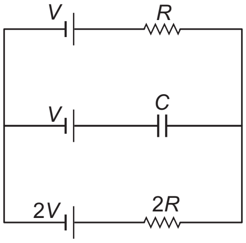
Concept

In a *C - R* circuit, charge on different capacitors is normally asked either at $t = 0$, $t = \infty$ or $t = t$. If nothing is given in the question, then we have to find charges on capacitors at $t = \infty$ or steady state charges.

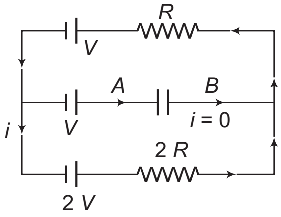
In steady state, no current flows through a wire having capacitor. But, if there is any other closed circuit then current can flow through that circuit. So, first find this current and then steady state potential difference (say V_0) across two plates of capacitor. Now,

$$q_0 = CV_0 = \text{steady state charge}$$

Example 13 *Find potential difference across the capacitor (obviously in steady state)*



Solution In steady state condition, no current will flow through the capacitor C. Current in the outer circuit,



$$i = \frac{2V - V}{2R + R} = \frac{V}{3R}$$

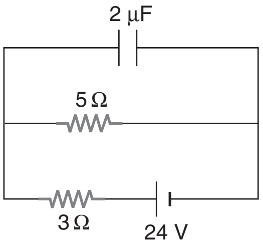
Potential difference between A and B,

$$V_A - V + V + iR = V_B$$

$$\therefore \qquad V_B - V_A = iR = \left(\frac{V}{3R}\right) R = \frac{V}{3}$$

Note *In this problem, charge stored in the capacitor can also be asked, which is equal to* $q = C\frac{V}{3}$ *with positive charge on B side and negative on A side because* $V_B > V_A$.

Example 14 *Find the charge stored in the capacitor.*



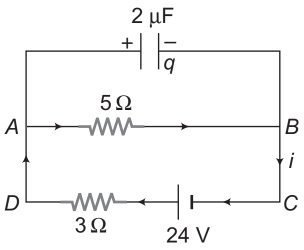
HOW TO PROCEED *Insulator is filled between the plates of the capacitor. Therefore, a capacitor does not allow current flow through it after charging is over. Hence, in the circuit current will flow through 3 Ω and 5 Ω resistances and it will not flow through the capacitor. To find the charge stored in the capacitor, we need the PD across it. So, first we will find PD across the capacitor and then apply,*

$$q = CV$$

where, V = *PD across the capacitor.*

Solution As we said earlier also, current will flow in loop $ABCDA$ when charging is over. And this current is

$$i = \frac{\text{Net emf}}{\text{Total resistance}} = \frac{24}{5+3} = 3\ \text{A}$$



Now, PD across the capacitor is equal to the PD across the 5 Ω resistance.

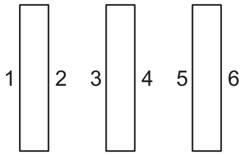
Hence, $$V = V_A - V_B = iR = (3)(5) = 15 \text{ V}$$

∴ $$q = CV = (2 \times 15)\,\mu\text{C} = 30\,\mu\text{C}$$ **Ans.**

Note *$V_A - V_B = 15$ V, therefore $V_A > V_B$, i.e. the positive charge will be collected on the left plate of the capacitor and negative on the right plate.*

Type 12. *To find distribution of charges on different faces on parallel conducting plates*

Concept



(i) $q_1 = q_6 = \dfrac{q_{\text{Total}}}{2}$

(ii) $q_2 = -q_3$ and $q_4 = -q_5$

Note *The above two results are proved in example 15.*

(iii) Electric field between 2 and 3 is due to the charges q_2 and q_3. This electric field is given by

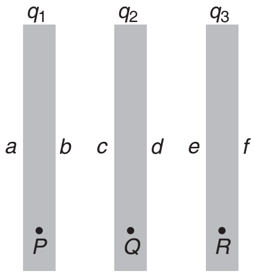
$$E = \frac{\sigma}{\varepsilon_0} = \frac{q/A}{\varepsilon_0} \qquad [|q_2| = |q_3| = q]$$

(iv) This electric field is uniform, so potential difference between any two points is given by

$$V = Ed$$

(v) If two plates are connected to each other, then distance between the plates is required, otherwise there is no requirement of that.

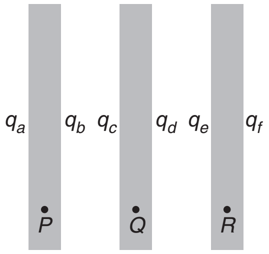
Example 15 *Three parallel metallic plates each of area A are kept as shown in figure and charges q_1, q_2 and q_3 are given to them. Find the resulting charge distribution on the six surfaces, neglecting edge effects as usual.*



Solution The plate separations do not affect the distribution of charge in this problem.

In the figure, $$q_b = q_1 - q_a, \quad q_d = q_2 - q_c$$

and $$q_f = q_3 - q_e.$$



Electric field at point P is zero because this point is lying inside a conductor.

$$E_P = 0$$

At P, charge q_a will give an electric field towards right. All other charges q_b, q_c … , etc., will give the electric field towards left. So,

$$\frac{1}{2A\varepsilon_0}[q_a - (q_1 - q_a) - q_c - (q_2 - q_c) - q_e - (q_3 - q_e)] = 0$$

or $$2q_a - q_1 - q_2 - q_3 = 0$$

or $$q_a = \frac{q_1 + q_2 + q_3}{2}$$

Similarly the condition, $E_R = 0$

will give the result,

$$q_f = \frac{q_1 + q_2 + q_3}{2}$$

From here we may conclude that, **half of the sum of all charges appears on each of the two outermost surfaces of the system of plates.**

Further we have a condition,

$$E_Q = 0$$

$$\frac{1}{2A\varepsilon_0}[q_a + (q_1 - q_a) + q_c - (q_2 - q_c) - q_e - (q_3 - q_e)] = 0$$

or $$q_1 + 2q_c - q_2 - q_3 = 0$$

$\therefore$ $$q_c = \frac{q_2 + q_3 - q_1}{2}$$

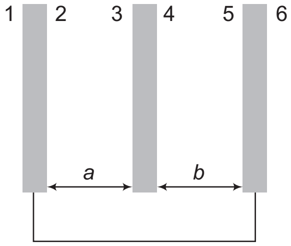
$$q_b = q_1 - q_a = \frac{q_1 - q_2 - q_3}{2} = -q_c$$

Similarly, we can show that

$$q_d = -q_e.$$

From here we can find another important result that **the pairs of opposite surfaces like *b, c* and *d,e* carry equal and opposite charges.**

Example 16 *Three identical metallic plates are kept parallel to one another at a separation of a and b. The outer plates are connected by a thin conducting wire and a charge Q is placed on the central plate. Find final charges on all the six plate's surfaces.*

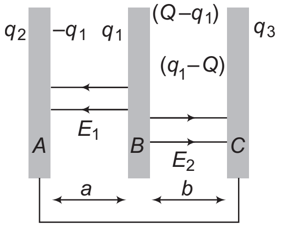


Solution Let the charge distribution in all the six faces be as shown in figure. While distributing the charge on different faces, we have used the fact that two opposite faces have equal and opposite charges on them.

Net charge on plates A and C is zero. Hence,

$$q_2 - q_1 + q_3 + q_1 - Q = 0$$

or $$q_2 + q_3 = Q \qquad \text{...(i)}$$



Further A and C are at same potentials. Hence,

$$V_B - V_A = V_B - V_C$$

or $$E_1 a = E_2 b$$

$\therefore$ $$\frac{q_1}{A\varepsilon_0} \cdot a = \frac{Q - q_1}{A\varepsilon_0} \cdot b \qquad (A = \text{Area of plates})$$

$\therefore$ $$q_1 a = (Q - q_1)\, b$$

$\therefore$ $$q_1 = \frac{Qb}{a + b} \qquad \text{...(ii)}$$

Electric field inside any conducting plate (say inside C) is zero. Therefore,

$$\frac{q_2}{2A\varepsilon_0} - \frac{q_1}{2A\varepsilon_0} + \frac{q_1}{2A\varepsilon_0} + \frac{Q - q_1}{2A\varepsilon_0} + \frac{q_1 - Q}{2A\varepsilon_0} - \frac{q_3}{2A\varepsilon_0} = 0$$

$\therefore$ $$q_2 - q_3 = 0 \qquad \text{...(iii)}$$

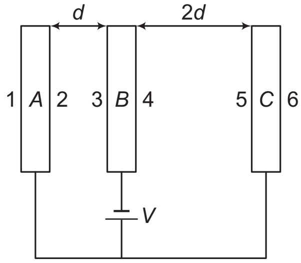
Solving these three equations, we get $q_1 = \dfrac{Qb}{a + b}$, $q_2 = q_3 = \dfrac{Q}{2}$

Hence, charge on different faces are as follows.

Table 25.3

Face	Charge
1	$q_2 = \frac{Q}{2}$
2	$-q_1 = -\frac{Qb}{a+b}$
3	$q_1 = \frac{Qb}{a+b}$
4	$Q - q_1 = \frac{Qa}{a+b}$
5	$q_1 - Q = -\frac{Qa}{a+b}$
6	$q_3 = \frac{Q}{2}$

Example 17 *Area of each plate is A. The conducting plates are connected to a battery of emf V volts. Find charges q_1 to q_6.*



Solution Net charge drawn from the battery is zero or

$$q_{\text{Total}} = 0$$

$\therefore$
$$q_1 = q_6 = \frac{q_{\text{Total}}}{2} = 0$$

$$V_{AB} = V \text{ with } V_A > V_B$$

$\therefore$
$$|q_2| = |q_3| = CV = \left(\frac{\varepsilon_0 A}{d}\right) V$$

$$q_2 = +\frac{\varepsilon_0 AV}{d} \text{ and } q_3 = -\frac{\varepsilon_0 AV}{d}$$

Similarly,

$$V_{BC} = V \text{ with } V_C > V_B$$

$\therefore$
$$|q_4| = |q_5| = CV = \left(\frac{\varepsilon_0 A}{2d}\right) V$$

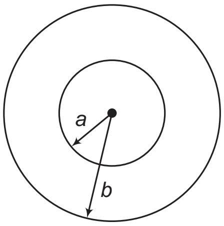
with
$$q_5 = +\frac{\varepsilon_0 AV}{2d} \text{ and } q_4 = -\frac{\varepsilon_0 AV}{2d}$$

Type 13. *To find total electrostatic potential energy due to spherical charged shells*

Concept

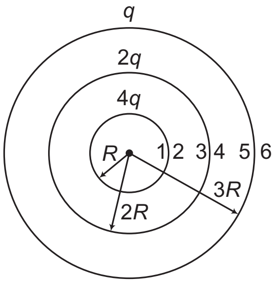
(i) Capacity of a spherical capacitor is given by

$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$



(ii) $U = \frac{1}{2}\frac{q^2}{C}$

Example 18 *In the figure shown,*



(a) Find q_1 to q_6.

(b) Total electrostatic potential energy.

Solution (a)

$$q_1 = 0$$

$$q_2 = 4q$$

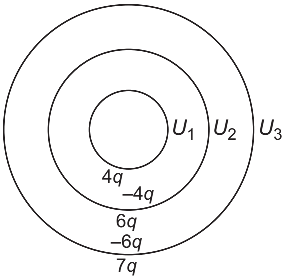
$$q_3 = -q_2 = -4q$$

$$q_4 = 2q - q_3 = 6q$$

$$q_5 = -q_4 = -6q$$

$$q_6 = q - q_5 = 7q$$

(b) $U_{\text{Total}} = U_1 + U_2 + U_3$



Here, $$U_1 = \frac{1}{2}\frac{(4q)^2}{C_1}$$

where, $$C_1 = \frac{4\pi\varepsilon_0}{\frac{1}{R} - \frac{1}{2R}}$$

$$U_2 = \frac{1}{2}\frac{(6q)^2}{C_2}$$

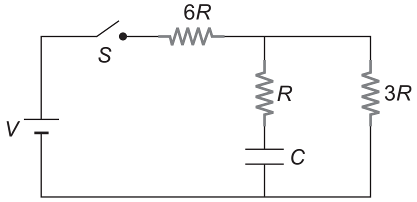
where, $$C_2 = \frac{4\pi\varepsilon_0}{\frac{1}{2R} - \frac{1}{3R}}$$

and $$U_3 = \frac{1}{2}\frac{(7q)^2}{C_3}$$

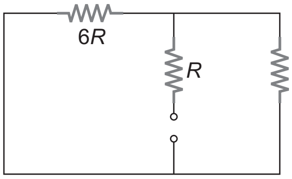
where, $$C_3 = \frac{4\pi\varepsilon_0}{\frac{1}{3R} - \frac{1}{\infty}} = 4\pi\varepsilon_0(3R)$$

Miscellaneous Examples

Example 19 *In the circuit shown in figure switch S is closed at time $t = 0$. Find the current through different wires and charge stored on the capacitor at any time t.*



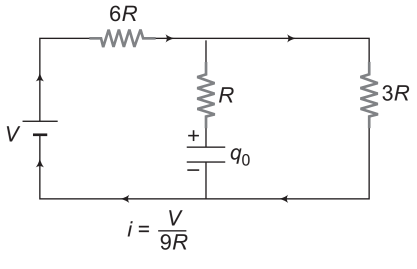
Solution **Calculation of τ_C** Equivalent resistance across capacitor after short-circuiting the battery is



$$R_{\text{net}} = R + \frac{(6R)(3R)}{6R + 3R} = 3R$$

$$\therefore \qquad \tau_C = (C)R_{\text{net}} = 3RC$$

Calculation of steady state charge q_0 At $t = \infty$, capacitor is fully charged and no current flows through it.



PD across capacitor = PD across $3R$

$$= \left(\frac{V}{9R}\right)(3R)$$

$$= \frac{V}{3}$$

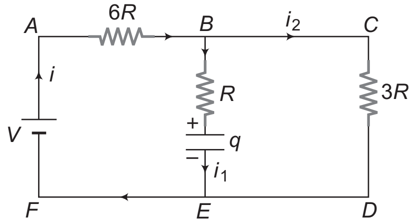
$$\therefore \qquad q_0 = \frac{CV}{3}$$

Now, let charge on the capacitor at any time t be q and current through it is i_1. Then,

$$q = q_0\,(1 - e^{-t/\tau_C})$$

and $$i_1 = \frac{dq}{dt} = \frac{q_0}{\tau_C}\, e^{-t/\tau_C} \qquad \text{...(i)}$$

Applying Kirchhoff's second law in loop $ACDFA$, we have



$$-6iR - 3i_2R + V = 0$$

or $$2i + i_2 = \frac{V}{3R} \qquad \text{...(ii)}$$

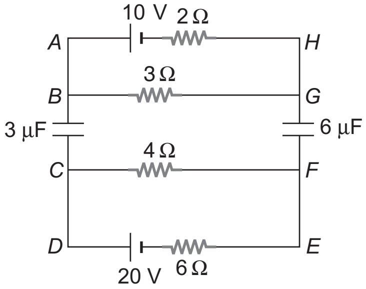
Applying Kirchhoff's junction law at B, we have

$$i = i_1 + i_2 \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we have

$$i_2 = \frac{V}{9R} - \frac{2}{3}i_1 = \frac{V}{9R} - \frac{2q_0}{3\tau_C}\,e^{-t/\tau_C} \quad \text{and} \quad i = \frac{V}{9R} + \frac{q_0}{3\tau_C}\,e^{-t/\tau_C}$$

Example 20 *In the circuit shown in figure, find the steady state charges on both the capacitors.*



HOW TO PROCEED *In steady state a capacitor offers an infinite resistance. Therefore, the two circuits ABGHA and CDEFC have no relation with each other. Hence, the battery of emf* 10 *V is not going to contribute any current in the lower circuit. Similarly, the battery of emf* 20 *V will not contribute to the current in the upper circuit. So, first we will calculate the current in the two circuits, then find the potential difference* V_{BG} *and* V_{CF} *and finally we can connect two batteries of emf* V_{BG} *and* V_{CF} *across the capacitors to find the charges stored in them.*

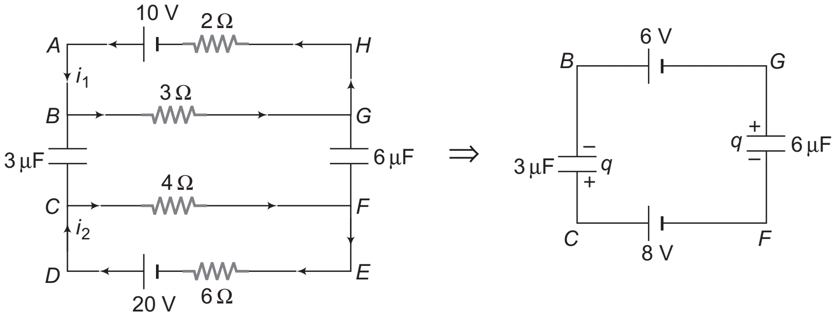
Solution Current in the upper circuit, $i_1 = \dfrac{10}{3+2} = 2$ A

$$\therefore \qquad V_{BG} = V_B - V_G = 3i_1 = 3 \times 2 = 6\text{ V}$$

Current in the lower circuit, $i_2 = \dfrac{20}{4+6} = 2\text{ A}$

$\therefore$ $V_{CF} = V_C - V_F = 4i_2 = 4 \times 2 = 8\text{ V}$

Charge on both the capacitors will be same. Let it be q. Applying Kirchhoff's second law in loop *BGFCB*,



$$-6 - \frac{q}{6 \times 10^{-6}} + 8 - \frac{q}{3 \times 10^{-6}} = 0$$

or $$\frac{(10^6)\,q}{2} = 2$$

or $$q = 4 \times 10^{-6}\text{ C}$$

or $$q = 4\,\mu\text{C}$$ **Ans.**

Example 21 *An isolated parallel plate capacitor has circular plates of radius* 4.0 *cm. If the gap is filled with a partially conducting material of dielectric constant K and conductivity* $5.0 \times 10^{-14}\ \Omega^{-1}m^{-1}$. *When the capacitor is charged to a surface charge density of* $15\,\mu C/cm^2$, *the initial current between the plates is* $1.0\,\mu A$?

(a) Determine the value of dielectric constant K.

(b) If the total joule heating produced is 7500 *J, determine the separation of the capacitor plates.*

Solution (a) This is basically a problem of discharging of a capacitor from inside the capacitor. Charge at any time t is

$$q = q_0 e^{-t/\tau_C}$$

Here, q_0 = (area of plates) (surface charge density)

and discharging current, $$i = \left(\frac{-dq}{dt}\right) = \frac{q_0}{\tau_C} \cdot e^{-t/\tau_C} = i_0 e^{-t/\tau_C}$$

Here, $$i_0 = \frac{q_0}{\tau_C} = \frac{q_0}{CR}$$

$$C = \frac{K\varepsilon_0 A}{d} \quad \text{and} \quad R = \frac{d}{\sigma A}$$

$\therefore$ $$CR = \frac{K\varepsilon_0}{\sigma}$$

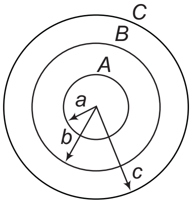
Therefore,
$$i_0 = \frac{\dfrac{q_0}{K\varepsilon_0}}{\sigma} = \frac{\sigma q_0}{K\varepsilon_0} \quad \Rightarrow \quad K = \frac{\sigma q_0}{i_0 \varepsilon_0}$$

Substituting the values, we have
$$K = \frac{(5.0 \times 10^{-14})\,(\pi)\,(4.0)^2\,(15 \times 10^{-6})}{(1.0 \times 10^{-6})\,(8.86 \times 10^{-12})} = 4.25$$ **Ans.**

(b)
$$U = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}\frac{q_0^2}{\dfrac{K\varepsilon_0 A}{d}}$$

$\therefore$
$$d = \frac{2K\varepsilon_0 AU}{q_0^2}$$
$$= \frac{2 \times 4.25 \times 8.86 \times 10^{-12} \times \pi \times (4.0 \times 10^{-2})^2 \times 7500}{(15 \times 10^{-6} \times \pi \times 4.0 \times 4.0)^2}$$
$$= 5.0 \times 10^{-3}\ \text{m} = 5.0\ \text{mm}$$ **Ans.**

Example 22 *Three concentric conducting shells A, B and C of radii a, b and c are as shown in figure. A dielectric of dielectric constant K is filled between A and B. Find the capacitance between A and C.*



HOW TO PROCEED *When the dielectric is filled between A and B, the electric field will change in this region. Therefore, the potential difference and hence the capacitance of the system will change. So, first find the electric field E(r) in the region a ≤ r ≤ c. Then, find the PD (V) between A and C and finally the capacitance of the system will be*

$$C = \frac{q}{V}$$

Here, q = charge on A

Solution
$$E(r) = \frac{q}{4\pi\varepsilon_0 K r^2} \quad \text{for} \quad a \le r \le b$$
$$= \frac{q}{4\pi\varepsilon_0 r^2} \quad \text{for} \quad b \le r \le c$$

Using, $dV = -\int \mathbf{E} \cdot d\mathbf{r}$

the PD between A and C is

$\therefore$
$$V = V_A - V_C = -\int_a^b \frac{q}{4\pi\varepsilon_0 K r^2} \cdot dr - \int_b^c \frac{q}{4\pi\varepsilon_0 r^2}\, dr$$
$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{K}\left(\frac{1}{a} - \frac{1}{b}\right) + \left(\frac{1}{b} - \frac{1}{c}\right)\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{(b-a)}{Kab} + \frac{(c-b)}{bc}\right]$$
$$= \frac{q}{4\pi\varepsilon_0 Kabc}\,[c\,(b-a) + Ka\,(c-b)]$$

$\therefore$ The desired capacitance is
$$C = \frac{q}{V} = \frac{4\pi\varepsilon_0 Kabc}{Ka\,(c-b) + c\,(b-a)}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** From the relation $C = \frac{q}{V}$. We can say that, if more charge q is given to a conductor, its capacitance should increase.

   Reason : Ratio $\frac{q}{V}$ will remain constant for a given conductor.

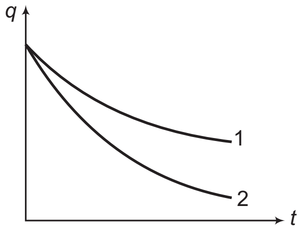
2. **Assertion :** A parallel plate capacitor is first charged and then distance between the plates is increased. In this process, electric field between the plates remains the same, while potential difference gets decreased.

   Reason : $E = \frac{q}{A\,\varepsilon_0}$ and $V = \frac{q}{Ad\,\varepsilon_0}$. Since, q remains same, E will remain same while V will decrease.

3. **Assertion :** When an uncharged capacitor is charged by a battery, only 50% of the energy supplied by a battery is stored in the capacitor.

   Reason : Rest 50% is lost.

4. **Assertion :** Discharging graphs of two C-R circuits having the same value of C is shown in figure. From the graph we can say that $\tau_{C_1} > \tau_{C_2}$.

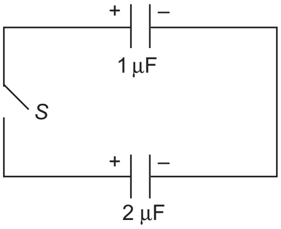


   Reason : $R_1 > R_2$.

5. **Assertion :** In series combination, charges on two capacitors are always equal.

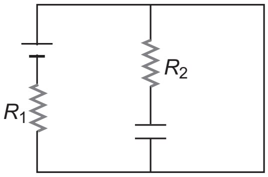
   Reason : If charges are same, the total potential difference applied across two capacitors will be distributed in inverse ratio of capacities.

6. **Assertion :** Two capacitors are charged from the same battery and then connected as shown. A current will flow in anti-clockwise direction as soon as switch is closed.



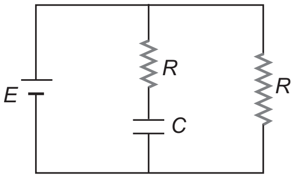
Reason : In steady state charges on two capacitors are in the ratio 1 : 2..

7. **Assertion :** In the circuit shown in figure no charge will be stored in the capacitor.



Reason : Current through R_2 will be zero.

8. **Assertion :** In the circuit shown in figure, time constant of charging of capacitor is $\frac{CR}{2}$.

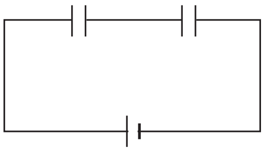


Reason : In the absence of capacitor in the circuit, two resistors are in parallel with the battery.

9. **Assertion :** Two capacitors are connected in series with a battery. Energy stored across them is in inverse ratio of their capacity.

Reason : $U = \frac{1}{2} qV$ or $U \propto qV$.

10. **Assertion :** In the circuit shown in figure, when a dielectric slab is inserted in C_2, the potential difference across C_2 will decrease.



Reason : By inserting the slab a current will flow in the circuit in clockwise direction.

Objective Questions

1. The separation between the plates of a charged parallel- plate capacitor is increased. The force between the plates

(a) increases
(b) decreases
(c) remains same
(d) first increases then decreases

2. If the plates of a capacitor are joined together by a conducting wire, then its capacitance

(a) remains unchanged
(b) decreases
(c) becomes zero
(d) becomes infinite

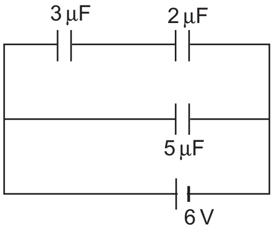
3. Two metal spheres of radii a and b are connected by a thin wire. Their separation is very large compared to their dimensions. The capacitance of this system is

(a) $4\pi\varepsilon_0(ab)$
(b) $2\pi\varepsilon_0(a+b)$
(c) $4\pi\varepsilon_0(a+b)$
(d) $4\pi\varepsilon_0\left(\frac{a^2+b^2}{2}\right)$

4. n identical capacitors are connected in parallel to a potential difference V. These capacitors are then reconnected in series, their charges being left undisturbed. The potential difference obtained is

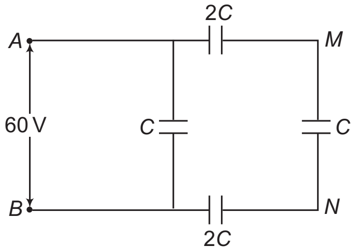
(a) zero
(b) $(n-1)V$
(c) nV
(d) n^2V

5. In the circuit shown in figure, the ratio of charge on 5 μF and 2 μF capacitor is



(a) 5/4
(b) 5/3
(c) 3/8
(d) None of these

6. In the circuit shown, a potential difference of 60 V is applied across AB. The potential difference between the points M and N is

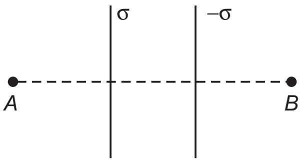


(a) 10 V
(b) 15 V
(c) 20 V
(d) 30 V

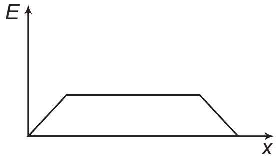
7. In Milikan's oil drop experiment, an oil drop of radius r and charge q is held in equilibrium between the plates of a charged parallel-plate capacitor when the potential difference is V. To keep a drop of radius $2r$ and with a charge $2q$ in equilibrium between the plates the potential difference V required is

(a) V
(b) $2V$
(c) $4V$
(d) $8V$

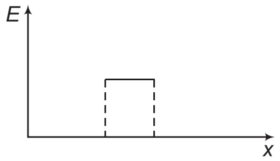
8. Two large parallel sheets charged uniformly with surface charge density σ and $-\sigma$ are located as shown in the figure. Which one of the following graphs shows the variation of electric field along a line perpendicular to the sheets as one moves from A to B?



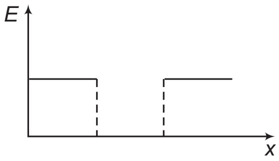
(a)



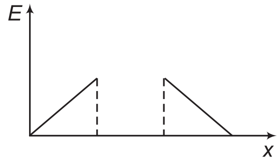
(b)



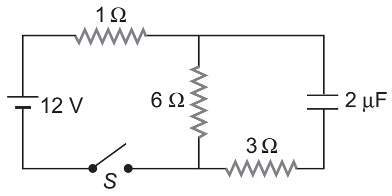
(c)



(d)



9. When the switch is closed, the initial current through the 1 Ω resistor is

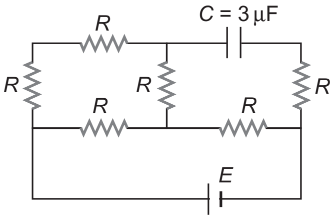


(a) 2 A (b) 4 A (c) 3 A (d) 6 A

10. A capacitor of capacitance C carrying charge Q is connected to a source of emf E. Finally, the charge on capacitor would be

(a) Q (b) $Q + CE$ (c) CE (d) None of these

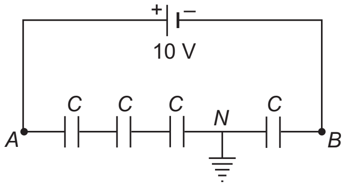
11. In the circuit, the potential difference across the capacitor is 10 V. Each resistance is of 3 Ω. The cell is ideal. The emf of the cell is



(a) 14 V (b) 16 V (c) 18 V (d) 24 V

12. Four identical capacitors are connected in series with a 10 V battery as shown in the figure. The point N is earthed. The potentials of points A and B are

(a) 10 V, 0 V
(b) 7.5 V, – 2.5 V
(c) 5 V, – 5 V
(d) 7.5 V, 2.5 V



13. A capacitor of capacity 2 μF is charged to 100 V. What is the heat generated when this capacitor is connected in parallel to an another capacitor of same capacity?

(a) 2.5 mJ
(b) 5.0 mJ
(c) 10 mJ
(d) 4 mJ

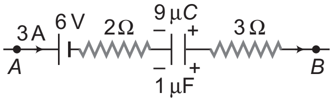
14. A charged capacitor is discharged through a resistance. The time constant of the circuit is η. Then, the value of time constant for the power dissipated through the resistance will be

(a) η
(b) 2η
(c) $\eta/2$
(d) zero

15. A capacitor is charged by a cell of emf E and the charging battery is then removed. If an identical capacitor is now inserted in the circuit in parallel with the previous capacitor, the potential difference across the new capacitor is

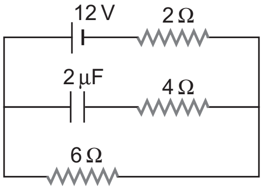
(a) $2E$
(b) E
(c) $E/2$
(d) zero

16. The potential difference $V_A - V_B$ between points A and B for the circuit segment shown in figure at the given instant is



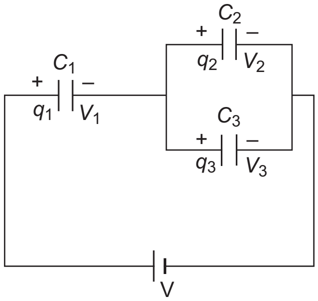
(a) 12 V
(b) – 12 V
(c) 6 V
(d) – 6 V

17. For the circuit arrangement shown in figure, in the steady state condition charge on the capacitor is

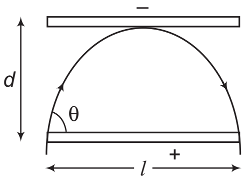


(a) 12 μC
(b) 14 μC
(c) 20 μC
(d) 18 μC

18. In the circuit as shown in figure if all the symbols have their usual meanings, then identify the correct statement,



(a) $q_2 = q_3$; $V_2 = V_3$
(b) $q_1 = q_2 + q_3$; $V_2 = V_3$
(c) $q_1 = q_2 + q_3$; $V = V_1 + V_2 + V_3$
(d) $q_1 + q_2 + q_3 = 0$; $V_2 = V_3 = V - V_1$



19. An electron enters the region between the plates of a parallel-plate capacitor at an angle θ to the plates. The plate width is l. The plate separation is d. The electron follows the path shown, just missing the upper plate. Neglect gravity. Then,

(a) $\tan\theta = 2d/l$

(b) $\tan\theta = 4d/l$

(c) $\tan\theta = 8d/l$

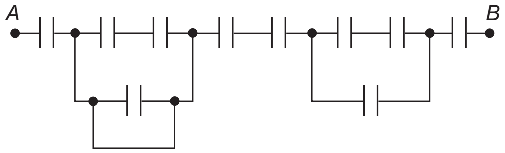
(d) The data given is insufficient to find a relation between d, l and θ

20. An infinite sheet of charge has a surface charge density of $10^{-7}\,\text{C/m}^2$. The separation between two equipotential surfaces whose potentials differ by 5 V is

(a) 0.64 cm (b) 0.88 mm

(c) 0.32 cm (d) 5×10^{-7} m

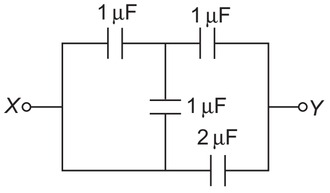
21. Find the equivalent capacitance across A and B for the arrangement shown in figure. All the capacitors are of capacitance C



(a) $\dfrac{3C}{14}$ (b) $\dfrac{C}{8}$

(c) $\dfrac{3C}{16}$ (d) None of these

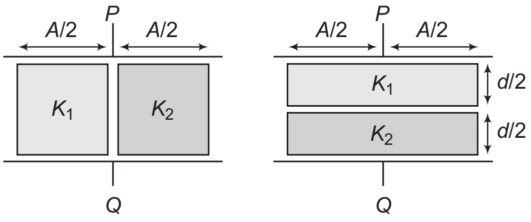
22. The equivalent capacitance between X and Y is



(a) $5/6\,\mu\text{F}$ (b) $7/6\,\mu\text{F}$

(c) $8/3\,\mu\text{F}$ (d) $1\,\mu\text{F}$

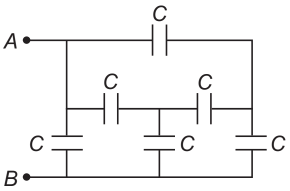
23. In the arrangement shown in figure, dielectric constant $K_1 = 2$ and $K_2 = 3$. If the capacitance across P and Q are C_1 and C_2 respectively, then C_1/C_2 will be (the gaps shown are negligible)



(a) 1 : 1 (b) 2 : 3

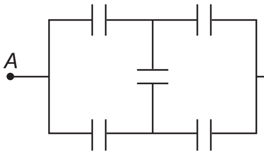
(c) 9 : 5 (d) 25 : 24

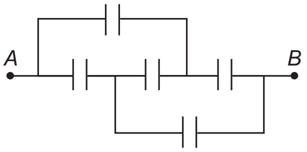
24. Six equal capacitors each of capacitance C are connected as shown in the figure. The equivalent capacitance between points A and B, is

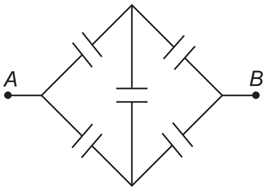


(a) $1.5\,C$
(b) C
(c) $2C$
(d) $0.5\,C$

25. Four ways of making a network of five capacitors of the same value are shown in four choices. Three out of four are identical. The one which is different is

(a)


(b)


(c)


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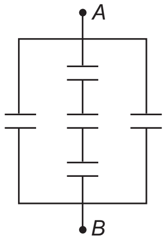
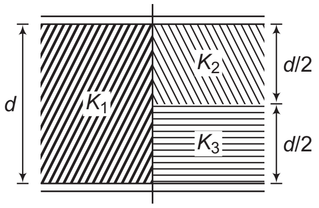
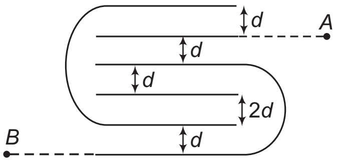


26. The equivalent capacitance of the arrangement shown in figure, if A is the area of each plate, is



(a) $C = \frac{\varepsilon_0 A}{d}\left[\frac{K_1}{2} + \frac{K_2 + K_3}{K_2 K_3}\right]$
(b) $C = \frac{\varepsilon_0 A}{d}\left[\frac{K_1}{2} + \frac{K_2 K_3}{K_2 + K_3}\right]$
(c) $C = \frac{\varepsilon_0 A}{2d}\left[K_1 + \frac{K_2 K_3}{K_2 + K_3}\right]$
(d) $C = \frac{\varepsilon_0 A}{d}\left[K_1 + \frac{K_2 K_3}{K_2 + K_3}\right]$

27. Find equivalent capacitance between points A and B. [Assume each conducting plate is having same dimensions and neglect the thickness of the plate, $\frac{\varepsilon_0 A}{d} = 7\,\mu\text{F}$, where A is area of plates]

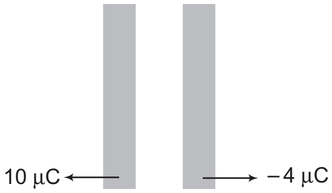


(a) $7\,\mu\text{F}$
(b) $11\,\mu\text{F}$
(c) $12\,\mu\text{F}$
(d) $15\,\mu\text{F}$

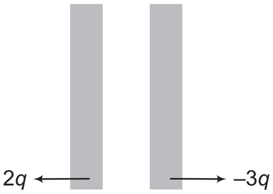
Subjective Questions

Note *You can take approximations in the answers.*

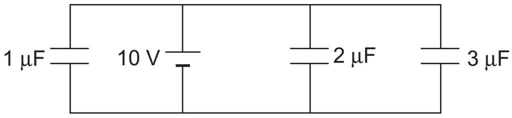
1. Two metallic plates are kept parallel to one another and charges are given to them as shown in figure. Find the charge on all the four faces.



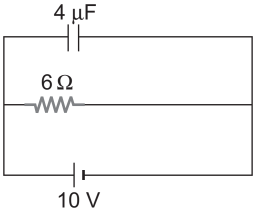
2. Charges $2q$ and $-3q$ are given to two identical metal plates of area of cross-section A. The distance between the plates is d. Find the capacitance and potential difference between the plates.



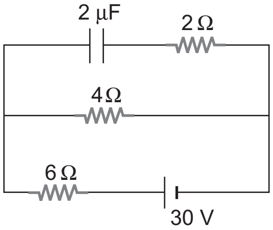
3. Find the charge stored in all the capacitors.



4. Find the charge stored in the capacitor.



5. Find the charge stored in the capacitor.



6. A 1 μF capacitor and a 2 μF capacitor are connected in series across a 1200 V supply line.

(a) Find the charge on each capacitor and the voltage across them.

(b) The charged capacitors are disconnected from the line and from each other and reconnected with terminals of like sign together. Find the final charge on each and the voltage across them.

7. A 100 μF capacitor is charged to 100 V. After the charging, battery is disconnected. The capacitor is then connected in parallel to another capacitor. The final voltage is 20 V. Calculate the capacity of second capacitor.

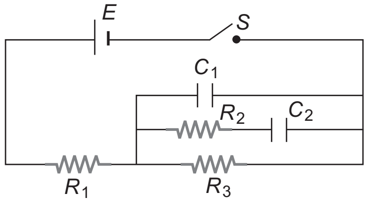
8. An uncharged capacitor C is connected to a battery through a resistance R. Show that by the time the capacitor gets fully charged, the energy dissipated in R is the same as the energy stored in C.

9. How many time constants will elapse before the current in a charging R-C circuit drops to half of its initial value?

10. A capacitor of capacitance C is given a charge q_0. At time $t = 0$ it is connected to an uncharged capacitor of equal capacitance through a resistance R. Find the charge on the first capacitor and the second capacitor as a function of time t. Also plot the corresponding q-t graphs.

11. A capacitor of capacitance C is given a charge q_0. At time $t = 0$, it is connected to a battery of emf E through a resistance R. Find the charge on the capacitor at time t.

12. Determine the current through the battery in the circuit shown in figure.



(a) immediately after the switch S is closed

(b) after a long time.

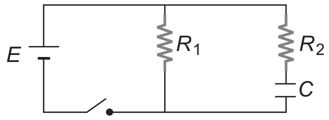
13. For the circuit shown in figure, find

(a) the initial current through each resistor

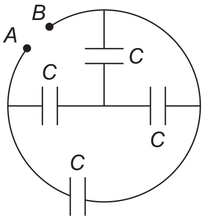
(b) steady state current through each resistor

(c) final energy stored in the capacitor

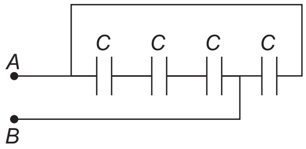
(d) time constant of the circuit when switch is opened.



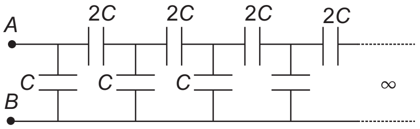
14. Find equivalent capacitance between points A and B,



(a)



(b)



(c)

15. A 4.00 μF capacitor and a 6.00 μF capacitor are connected in parallel across a 660 V supply line.

(a) Find the charge on each capacitor and the voltage across each.

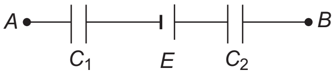
(b) The charged capacitors are disconnected from the line and from each other, and then reconnected to each other with terminals of unlike sign together. Find the final charge on each and the voltage across each.

16. A 5.80 μF parallel-plate air capacitor has a plate separation of 5.00 mm and is charged to a potential difference of 400 V. Calculate the energy density in the region between the plates, in J/m^3.

17. The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of 1.60×10^7 V/m. The capacitor is to have a capacitance of 1.25×10^{-9} F and must be able to withstand a maximum potential difference of 5500 V. What is the minimum area the plates of the capacitor may have?

18. Two condensers are in parallel and the energy of the combination is 0.1 J, when the difference of potential between terminals is 2 V. With the same two condensers in series, the energy is 1.6×10^{-2} J for the same difference of potential across the series combination. What are the capacities?

19. A circuit has section AB as shown in figure. The emf of the source equals $E = 10$ V, the capacitor capacitances are equal to $C_1 = 1.0\,\mu\text{F}$ and $C_2 = 2.0\,\mu\text{F}$, and the potential difference $V_A - V_B = 5.0$ V. Find the voltage across each capacitor.



20. Several 10 pF capacitors are given, each capable of withstanding 100 V. How would you construct :

(a) a unit possessing a capacitance of 2 pF and capable of withstanding 500 V?

(b) a unit possessing a capacitance of 20 pF and capable of withstanding 300 V?

21. Two, capacitors A and B are connected in series across a 100 V supply and it is observed that the potential difference across them are 60 V and 40 V. A capacitor of 2 μF capacitance is now connected in parallel with A and the potential difference across B rises to 90 V. Determine the capacitance of A and B.

22. A 10.0 μF parallel-plate capacitor with circular plates is connected to a 12.0 V battery.

(a) What is the charge on each plate?

(b) How much charge would be on the plates if their separation were doubled while the capacitor remained connected to the battery?

(c) How much charge would be on the plates if the capacitor were connected to the 12.0 V battery after the radius of each plate was doubled without changing their separation?

23. A 450 μF capacitor is charged to 295 V. Then, a wire is connected between the plates. How many joule of thermal energy are produced as the capacitor discharges if all of the energy that was stored goes into heating the wire?

24. The plates of a parallel-plate capacitor in vacuum are 5.00 mm apart and 2.00 m^2 in area. A potential difference of 10,000 V is applied across the capacitor. Compute

(a) the capacitance

(b) the charge on each plate, and

(c) the magnitude of the electric field in the space between them.

25. Three capacitors having capacitances of $8.4\,\mu\text{F}$, $8.2\,\mu\text{F}$ and $4.2\,\mu\text{F}$ are connected in series across a 36 V potential difference.

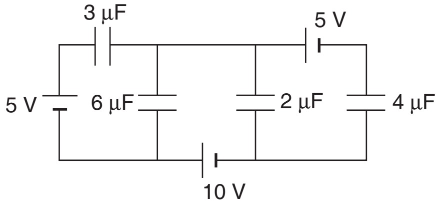
(a) What is the charge on $4.2\,\mu\text{F}$ capacitor?

(b) What is the total energy stored in all three capacitors?

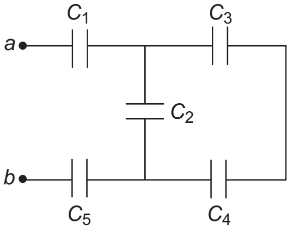
(c) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other, with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination?

(d) What is the total energy now stored in the capacitors?

26. Find the charges on $6\,\mu\text{F}$ and $4\,\mu\text{F}$ capacitors.



27. In figure, $C_1 = C_5 = 8.4\,\mu\text{F}$ and $C_2 = C_3 = C_4 = 4.2\,\mu\text{F}$. The applied potential is $V_{ab} = 220$ V.



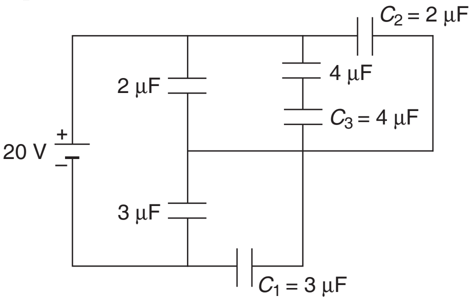
(a) What is the equivalent capacitance of the network between points a and b?

(b) Calculate the charge on each capacitor and the potential difference across each capacitor.

28. Two condensers A and B each having slabs of dielectric constant $K = 2$ are connected in series. When they are connected across 230 V supply, potential difference across A is 130 V and that across B is 100 V. If the dielectric in the condenser of smaller capacitance is replaced by one for which $K = 5$, what will be the values of potential difference across them?

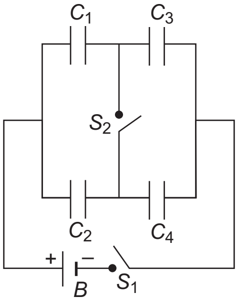
29. A capacitor of capacitance $C_1 = 1.0\,\mu\text{F}$ charged upto a voltage $V = 110$ V is connected in parallel to the terminals of a circuit consisting of two uncharged capacitors connected in series and possessing the capacitance $C_2 = 2.0\,\mu\text{F}$ and $C_3 = 3.0\,\mu\text{F}$. What charge will flow through the connecting wires?

30. In figure, the battery has a potential difference of 20 V. Find



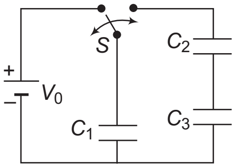
(a) the equivalent capacitance of all the capacitors across the battery and
(b) the charge stored on that equivalent capacitance.
Find the charge on
(c) capacitor 1,
(d) capacitor 2, and
(e) capacitor 3.

31. In figure, battery B supplies 12 V. Find the charge on each capacitor



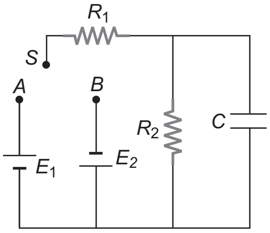
(a) first when only switch S_1 is closed and
(b) later when S_2 is also closed.
(Take $C_1 = 1.0\,\mu\text{F}, C_2 = 2.0\,\mu\text{F}, C_3 = 3.0\,\mu\text{F}$ and $C_4 = 4.0\,\mu\text{F}$)

32. When switch S is thrown to the left in figure, the plates of capacitor 1 acquire a potential difference V_0. Capacitors 2 and 3 are initially uncharged. The switch is now thrown to the right. What are the final charges q_1, q_2 and q_3 on the capacitors?



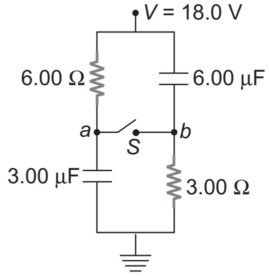
33. A parallel-plate capacitor has plates of area A and separation d and is charged to a potential difference V. The charging battery is then disconnected, and the plates are pulled apart until their separation is $2d$. Derive expression in terms of A, d and V for
(a) the new potential difference
(b) the initial and final stored energies, U_i and U_f and
(c) the work required to increase the separation of plates from d to $2d$.

34. In the circuit shown in figure $E_1 = 2E_2 = 20\text{ V}, R_1 = R_2 = 10\text{ k}\Omega$ and $C = 1\,\mu\text{F}$. Find the current through R_1, R_2 and C when



(a) S has been kept connected to A for a long time.
(b) The switch is suddenly shifted to B.

35. (a) What is the steady state potential of point a with respect to point b in figure when switch S is open?

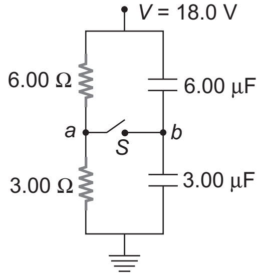


(b) Which point, a or b, is at the higher potential?

(c) What is the final potential of point b with respect to ground when switch S is closed?

(d) How much does the charge on each capacitor change when S is closed?

36. (a) What is the potential of point a with respect to point b in figure, when switch S is open?

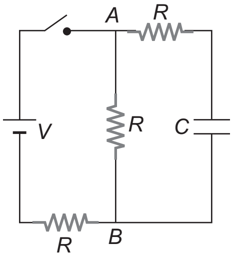


(b) Which point, a or b, is at the higher potential?

(c) What is the final potential of point b with respect to ground when switch S is closed?

(d) How much charge flows through switch S when it is closed?

37. In the circuit shown in figure, the battery is an ideal one with emf V. The capacitor is initially uncharged. The switch S is closed at time $t = 0$.



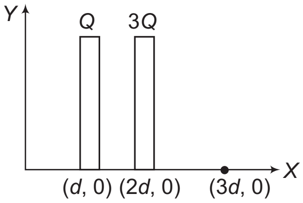
(a) Find the charge Q on the capacitor at time t.

(b) Find the current in AB at time t. What is its limiting value as $t \rightarrow \infty$?

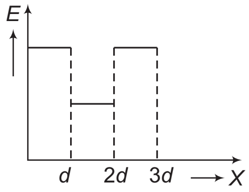
LEVEL 2

Single Correct Option

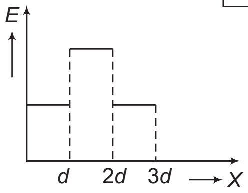
1. Two very large thin conducting plates having same cross-sectional area are placed as shown in figure. They are carrying charges Q and $3Q$, respectively. The variation of electric field as a function at x (for $x = 0$ to $x = 3d$) will be best represented by



(a)



(b)



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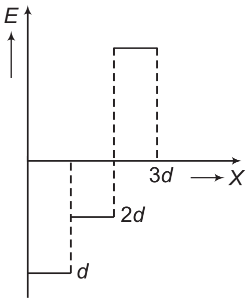
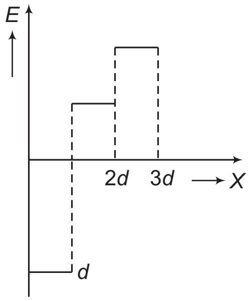
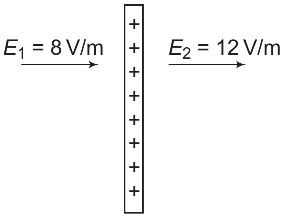


(d)



2. The electric field on two sides of a thin sheet of charge is shown in the figure. The charge density on the sheet is



(a) $2\,\varepsilon_0$

(b) $4\,\varepsilon_0$

(c) $10\,\varepsilon_0$

(d) zero

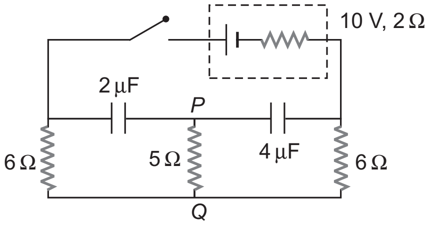
3. In the circuit shown in figure, the capacitors are initially uncharged. The current through resistor PQ just after closing the switch is

(a) 2 A from P to Q

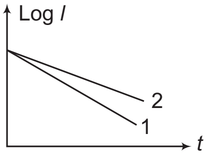
(b) 2 A from Q to P

(c) 6 A from P to Q

(d) zero



4. A graph between current and time during charging of a capacitor by a battery in series with a resistor is shown. The graphs are drawn for two circuits. R_1, R_2, C_1, C_2 and V_1, V_2 are the values of resistance, capacitance and EMF of the cell in the two circuits. If only two parameters (out of resistance, capacitance, EMF) are different in the two circuits. What may be the correct option(s)?

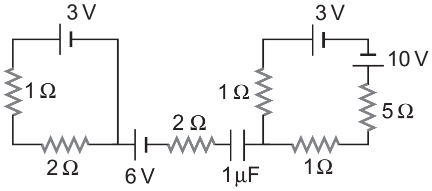


(a) $V_1 = V_2, R_1 > R_2, C_1 > C_2$
(b) $V_1 > V_2, R_1 > R_2, C_1 = C_2$
(c) $V_1 < V_2, R_1 < R_2, C_1 = C_2$
(d) $V_1 < V_2, R_1 = R_2, C_1 < C_2$

5. A capacitor of capacitance C is charged by a battery of emf E and internal resistance r. A resistance $2r$ is also connected in series with the capacitor. The amount of heat liberated inside the battery by the time capacitor is 50% charged is

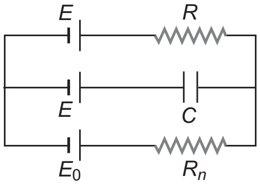
(a) $\frac{3}{8}E^2C$
(b) $\frac{E^2C}{6}$
(c) $\frac{E^2C}{12}$
(d) $\frac{E^2C}{24}$

6. For the circuit shown in the figure, determine the charge on capacitor in steady state.



(a) 4 μC
(b) 6 μC
(c) 1 μC
(d) Zero

7. For the circuit shown in the figure, find the charge stored on capacitor in steady state.

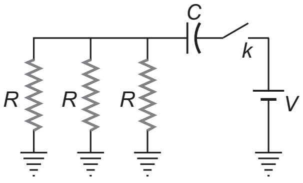


(a) $\frac{RC}{R+R_0}E$
(b) $\frac{RC}{R_0}(E-E_0)$
(c) zero
(d) $\frac{RC}{R+R_0}(E-E_0)$

8. Two similar parallel-plate capacitors each of capacity C_0 are connected in series. The combination is connected with a voltage source of V_0. Now, separation between the plates of one capacitor is increased by a distance d and the separation between the plates of another capacitor is decreased by the distance $d/2$. The distance between the plates of each capacitor was d before the change in separation. Then, select the correct choice.

(a) The new capacity of the system will increase
(b) The new capacity of the system will decrease
(c) The new capacity of the system will remain same
(d) data insufficient

9. The switch shown in the figure is closed at $t = 0$. The charge on the capacitor as a function of time is given by

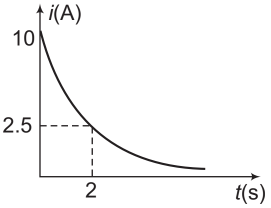


(a) $CV\ (1 - e^{-t/RC})$
(b) $3CV\ (1 - e^{-t/RC})$
(c) $CV\ (1 - e^{-3t/RC})$
(d) $CV\ (1 - e^{-t/3RC})$

10. A 2 μF capacitor C_1 is charged to a voltage 100 V and a 4 μF capacitor C_2 is charged to a voltage 50 V. The capacitors are then connected in parallel. What is the loss of energy due to parallel connection?

(a) 1.7 J
(b) 0.17 J
(c) 1.7×10^{-2} J
(d) 1.7×10^{-3} J

11. The figure shows a graph of the current in a discharging circuit of a capacitor through a resistor of resistance 10 Ω.



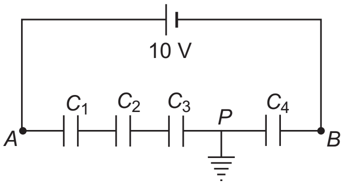
(a) The initial potential difference across the capacitor is 100 V

(b) The capacitance of the capacitor is $\dfrac{1}{10 \ln 2}$ F

(c) The total heat produced in the circuit will be $\dfrac{500}{\ln 2}$ J

(d) All of the above

12. Four capacitors are connected in series with a battery of emf 10 V as shown in the figure. The point *P* is earthed. The potential of point *A* is equal in magnitude to potential of point *B* but opposite in sign if

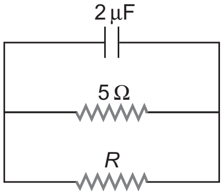


(a) $C_1 + C_2 + C_3 = C_4$
(b) $\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} = \dfrac{1}{C_4}$
(c) $\dfrac{C_1 C_2 C_3}{C_1^2 + C_2^2 + C_3^2} = C_4$
(d) It is never possible

13. A capacitor of capacity *C* is charged to a potential difference *V* and another capacitor of capacity 2*C* is charged to a potential difference 4*V*. The charging batteries are disconnected and the two capacitors are connected with reverse polarity (i.e. positive plate of first capacitor is connected to negative plate of second capacitor). The heat produced during the redistribution of charge between the capacitors will be

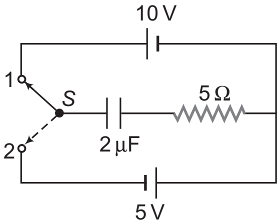
(a) $\dfrac{125\,CV^2}{3}$
(b) $\dfrac{50\,CV^2}{3}$
(c) $2CV^2$
(d) $\dfrac{25\,CV^2}{3}$

14. A capacitor of capacitance 2 μF is charged to a potential difference of 5 V. Now, the charging battery is disconnected and the capacitor is connected in parallel to a resistor of 5 Ω and another unknown resistor of resistance *R* as shown in figure. If the total heat produced in 5 Ω resistance is 10 μJ, then the unknown resistance *R* is equal to



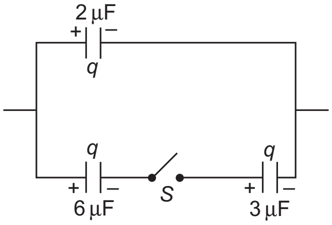
(a) 10 Ω
(b) 15 Ω
(c) $\dfrac{10}{3}$ Ω
(d) 7.5 Ω

15. In the circuit shown in figure switch S is thrown to position 1 at $t = 0$. When the current in the resistor is 1 A, it is shifted to position 2. The total heat generated in the circuit after shifting to position 2 is



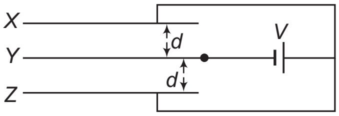
(a) zero (b) 625 μJ (c) 100 μJ (d) None of these

16. The flow of charge through switch S if it is closed is



(a) zero (b) $q/4$ (c) $2q/3$ (d) $q/3$

17. Consider the arrangement of three plates X, Y and Z each of the area A and separation d. The energy stored when the plates are fully charged is

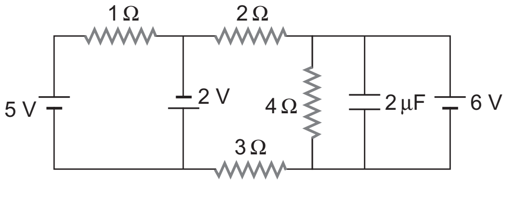


(a) $\varepsilon_0 AV^2/2d$ (b) $\varepsilon_0 AV^2/d$
(c) $2\varepsilon_0 AV^2/d$ (d) $3\varepsilon_0 AV^2/d$

18. Consider a capacitor – charging circuit. Let Q_1 be the charge given to the capacitor in time interval of 20 ms and Q_2 be the charge given in the next time interval of 20 ms. Let 10 μC charge be deposited in a time interval t_1 and the next 10 μC charge is deposited in the next time interval t_2. Then,

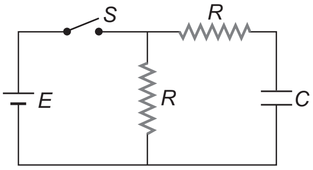
(a) $Q_1 > Q_2$, $t_1 > t_2$ (b) $Q_1 > Q_2$, $t_1 < t_2$
(c) $Q_1 < Q_2$, $t_1 > t_2$ (d) $Q_1 < Q_2$, $t_1 < t_2$

19. The current in 1 Ω resistance and charge stored in the capacitor are



(a) 4 A , 6 μC (b) 7 A, 12 μC (c) 4 A , 12 μC (d) 7 A, 6 μC

20. A capacitor C is connected to two equal resistances as shown in the figure. Consider the following statements.



(i) At the time of charging of capacitor time constant of the circuit is $2CR$

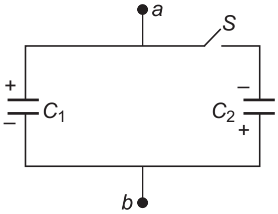
(ii) At the time of discharging of the capacitor the time constant of the circuit is CR

(iii) At the time of discharging of the capacitor the time constant of the circuit is $2CR$

(iv) At the time of charging of capacitor the time constant of the circuit is CR

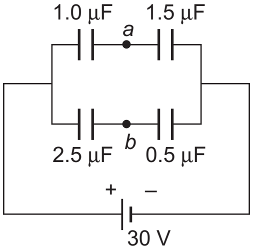
(a) Statements (i) and (ii) only are correct
(b) Statements (ii) and (iii) only are correct
(c) Statements (iii) and (iv) only are correct
(d) Statements (i) and (iii) only are correct

21. Two capacitors $C_1 = 1\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$ each are charged to a potential difference of 100 V but with opposite polarity as shown in the figure. When the switch S is closed, the new potential difference between the points a and b is



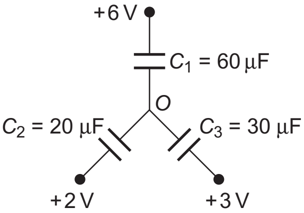
(a) 200 V
(b) 100 V
(c) 50 V
(d) 25 V

22. Four capacitors are connected as shown in figure to a 30 V battery. The potential difference between points a and b is



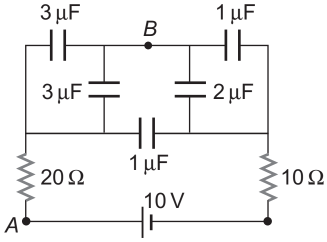
(a) 5 V
(b) 9 V
(c) 10 V
(d) 13 V

23. Three uncharged capacitors of capacitance C_1, C_2 and C_3 are connected to one another as shown in figure. The potential at O will be



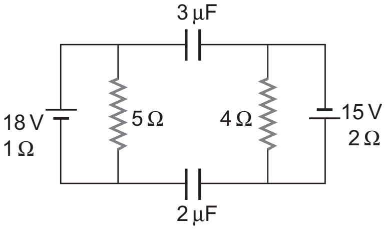
(a) 3 V
(b) $\frac{49}{11}$ V
(c) 4 V
(d) $\frac{3}{11}$ V

24. In the circuit shown in figure, the potential difference between the points A and B in the steady state is



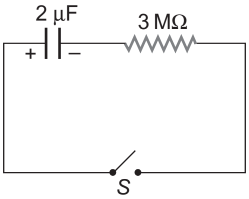
(a) zero (b) 6 V (c) 4 V (d) $\frac{10}{3}$ V

25. Two cells, two resistors and two capacitors are connected as shown in figure. The charge on 2 μF capacitor is



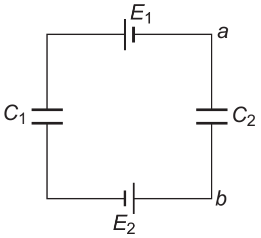
(a) 30 μC (b) 20 μC (c) 25 μC (d) 48 μC

26. In the circuit shown in figure, the capacitor is charged with a cell of 5 V. If the switch is closed at $t = 0$, then at $t = 12$ s, charge on the capacitor is



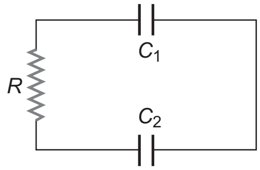
(a) (0.37) 10 μC (b) $(0.37)^2$ 10 μC (c) (0.63) 10 μC (d) $(0.63)^2$ 10 μC

27. The potential difference between points a and b of circuits shown in the figure is



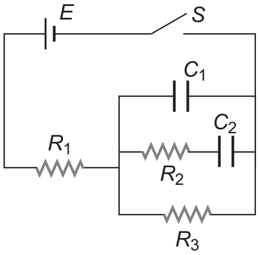
(a) $\left(\frac{E_1 + E_2}{C_1 + C_2}\right) C_2$ (b) $\left(\frac{E_1 - E_2}{C_1 + C_2}\right) C_2$ (c) $\left(\frac{E_1 + E_2}{C_1 + C_2}\right) C_1$ (d) $\left(\frac{E_1 - E_2}{C_1 + C_2}\right) C_1$

28. A capacitor C_1 is charged to a potential V and connected to another capacitor in series with a resistor R as shown. It is observed that heat H_1 is dissipated across resistance R, till the circuit reaches steady state. Same process is repeated using resistance of $2R$. If H_2 is heat dissipated in this case, then



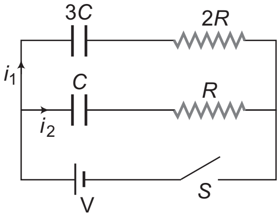
(a) $\dfrac{H_2}{H_1} = 1$
(b) $\dfrac{H_2}{H_1} = 4$
(c) $\dfrac{H_2}{H_1} = \dfrac{1}{4}$
(d) $\dfrac{H_2}{H_1} = 2$

29. In the circuit diagram, the current through the battery immediately after the switch S is closed is



(a) zero
(b) $\dfrac{E}{R_1}$
(c) $\dfrac{E}{R_1 + R_2}$
(d) $\dfrac{E}{R_1 + \dfrac{R_2 R_3}{R_2 + R_3}}$

30. In the circuit shown, switch S is closed at $t = 0$. Let i_1 and i_2 be the current at any finite time t, then the ratio i_1 / i_2

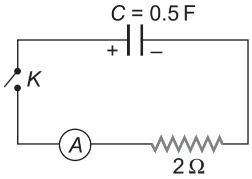


(a) is constant
(b) increases with time
(c) decreases with time
(d) first increases and then decreases

31. A leaky parallel capacitor is filled completely with a material having dielectric constant $K = 5$ and electrical conductivity $\sigma = 7.4 \times 10^{-12} \Omega^{-1}\, m^{-1}$. Charge on the plate at instant $t = 0$ is $q = 8.885 \mu C$. Then, time constant of leaky capacitor is

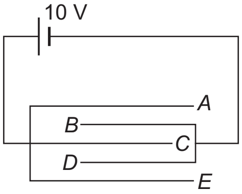
(a) 3 s
(b) 4 s
(c) 5 s
(d) 6 s

32. A charged capacitor is allowed to discharge through a resistor by closing the key at the instant $t = 0$. At the instant $t = (\ln 4)\,\mu s$, the reading of the ammeter falls half the initial value. The resistance of the ammeter is equal to



(a) 0.5 Ω
(b) 1 Ω
(c) 2 Ω
(d) 4 Ω

33. Five identical capacitor plates are arranged such that they make four capacitors each of 2 μF. The plates are connected to a source of emf 10 V. The charge on plate C is



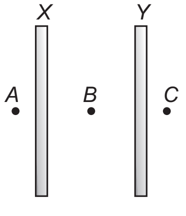
(a) + 20 μC
(b) + 40 μC
(c) + 60 μC
(d) + 80 μC

34. A capacitor of capacitance C is charged to a potential difference V from a cell and then disconnected from it. A charge $+Q$ is now given to its positive plate. The potential difference across the capacitor is now

(a) V
(b) $V + \dfrac{Q}{C}$
(c) $V + \dfrac{Q}{2C}$
(d) $V - \dfrac{Q}{C}$, if $Q < CV$

More than One Correct Options

1. X and Y are large, parallel conducting plates close to each other. Each face has an area A. X is given a charge Q. Y is without any charge. Points A, B and C are as shown in the figure.



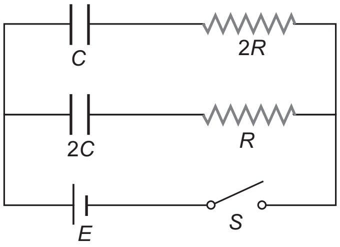
(a) The field at B is $\dfrac{Q}{2\varepsilon_0 A}$

(b) The field at B is $\dfrac{Q}{\varepsilon_0 A}$

(c) The fields at A, B and C are of the same magnitude

(d) The fields at A and C are of the same magnitude, but in opposite directions

2. In the circuit shown in the figure, switch S is closed at time $t = 0$. Select the correct statements.



(a) Rate of increase of charge is same in both the capacitors

(b) Ratio of charge stored in capacitors C and $2C$ at any time t would be 1 : 2

(c) Time constants of both the capacitors are equal

(d) Steady state charges on capacitors C and $2C$ are in the ratio of 1 : 2

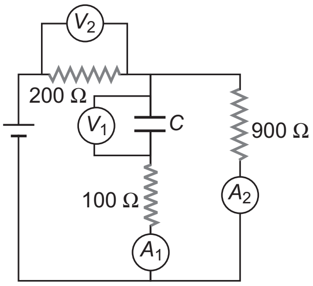
3. An electrical circuit is shown in the given figure. The resistance of each voltmeter is infinite and each ammeter is 100 Ω. The charge on the capacitor of 100 μF in steady state is 4 mC. Choose correct statement(s) regarding the given circuit.

(a) Reading of voltmeter V_2 is 16 V

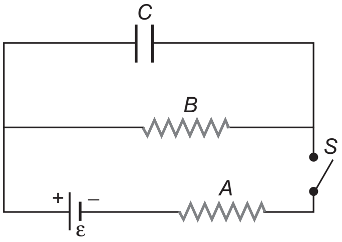
(b) Reading of ammeter A_1 is zero and A_2 is 1/25 A

(c) Reading of voltmeter V_1 is 40 V

(d) Emf of the ideal cell is 66 V



4. In the circuit shown, A and B are equal resistances. When S is closed, the capacitor C charges from the cell of emf ε and reaches a steady state.



(a) During charging, more heat is produced in A than in B

(b) In steady state, heat is produced at the same rate in A and B

(c) In the steady state, energy stored in C is $\frac{1}{4}C\varepsilon^2$

(d) In the steady state energy stored in C is $\frac{1}{8}C\varepsilon^2$

5. A parallel-plate capacitor is charged from a cell and then isolated from it. The separation between the plates is now increased

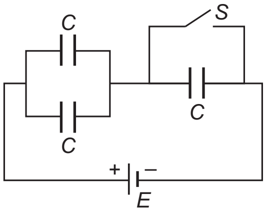
(a) The force of attraction between the plates will decrease

(b) The field in the region between the plates will not change

(c) The energy stored in the capacitor will increase

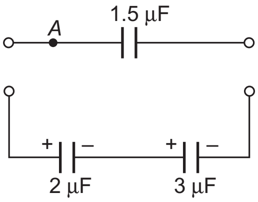
(d) The potential difference between the plates will decrease

6. In the circuit shown, each capacitor has a capacitance C. The emf of the cell is E. If the switch S is closed, then



(a) positive charge will flow out of the positive terminal of the cell
(b) positive charge will enter the positive terminal of the cell
(c) the amount of the charge flowing through the cell will be $\frac{1}{3}CE$
(d) the amount of charge flowing through the cell is $\left(\frac{4}{3}\right)CE$

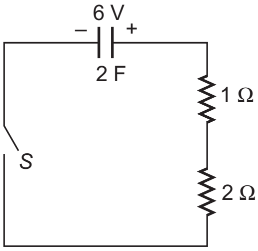
7. Two capacitors of 2 μF and 3 μF are charged to 150 V and 120 V, respectively. The plates of capacitor are connected as shown in the figure. An uncharged capacitor of capacity 1.5 μF falls to the free end of the wire. Then,



(a) charge on 1.5 μF capacitor is 180 μC
(b) charge on 2 μF capacitor is 120 μC
(c) positive charge flows through A from right to left
(d) positive charge flows through A from left to right

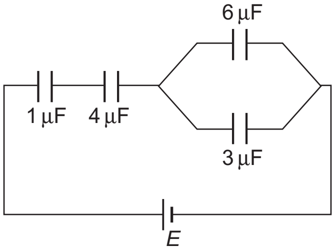
8. A parallel plate capacitor is charged and then the battery is disconnected. When the plates of the capacitor are brought closer, then
(a) energy stored in the capacitor decreases
(b) the potential difference between the plates decreases
(c) the capacitance increases
(d) the electric field between the plates decreases

9. A capacitor of 2 F (practically not possible to have a capacity of 2 F) is charged by a battery of 6 V. The battery is removed and circuit is made as shown. Switch is closed at time $t = 0$. Choose the correct options.



(a) At time $t = 0$ current in the circuit is 2 A
(b) At time $t = (6 \ln 2)$ second, potential difference across capacitor is 3 V
(c) At time $t = (6 \ln 2)$ second, potential difference across 1 Ω resistance is 1 V
(d) At time $t = (6 \ln 2)$ second, potential difference across 2 Ω resistance is 2 V.

10. Given that potential difference across 1 μF capacitor is 10 V. Then,

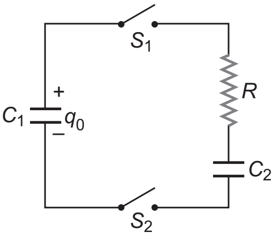


(a) potential difference across 4 μF capacitor is 40 V
(b) potential difference across 4 μF capacitor is 2.5 V
(c) potential difference across 3 μF capacitor is 5 V
(d) value of E is 50 V

Comprehension Based Questions

Passage I (Q. No. 1 and 2)

The capacitor C_1 in the figure shown initially carries a charge q_0. When the switches S_1 and S_2 are closed, capacitor C_1 is connected in series to a resistor R and a second capacitor C_2, which is initially uncharged.



1. The charge flown through wires as a function of time t is

(a) $q_0 e^{-t/RC} + \frac{C}{C_2} q_0$ (b) $\frac{q_0 C}{C_1} \times [1 - e^{-t/RC}]$

(c) $q_0 \frac{C}{C_1} e^{-t/CR}$ (d) $q_0 e^{-t/RC}$

where, $C = \frac{C_1 C_2}{C_1 + C_2}$

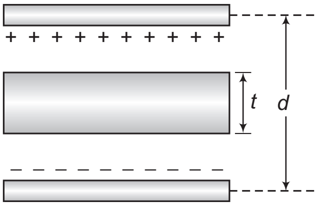
2. The total heat dissipated in the circuit during the discharging process of C_1 is

(a) $\frac{q_0^2}{2C_1^2} \times C$ (b) $\frac{q_0^2}{2C}$

(c) $\frac{q_0^2 C_2}{2C_1^2}$ (d) $\frac{q_0^2}{2C_1 C_2}$

Passage II (Q. No. 3 and 4)

Figure shows a parallel plate capacitor with plate area A and plate separation d. A potential difference is being applied between the plates. The battery is then disconnected and a dielectric slab of dielectric constant K is placed in between the plates of the capacitor as shown.



Now, answer the following questions based on above information.

3. The electric field in the gaps between the plates and the dielectric slab will be

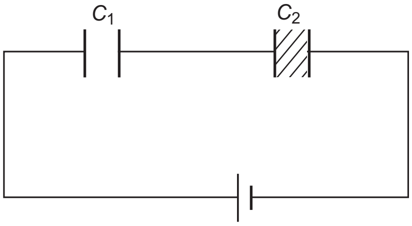
(a) $\frac{\varepsilon_0 AV}{d}$ (b) $\frac{V}{d}$ (c) $\frac{KV}{d}$ (d) $\frac{V}{d-t}$

4. The electric field in the dielectric slab is

(a) $\frac{V}{Kd}$ (b) $\frac{KV}{d}$ (c) $\frac{V}{d}$ (d) $\frac{KV}{t}$

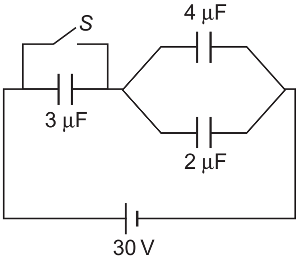
Match the Columns

1. In the figure shown, $C_1 = 4\,\mu\text{F}$ (without dielectric) and $C_2 = 4\,\mu\text{F}$ (with a dielectric slab of dielectric constant $K = 2$) . Now, the same slab after removing from C_2 is filled in C_1. Then, match the following two columns.



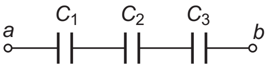
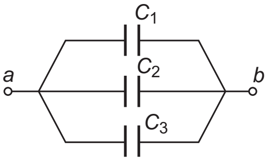
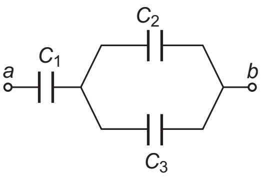
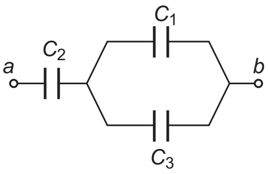
Column I	Column II
(a) Charge on C_2	(p) will increase
(b) Energy stored in C_2	(q) will decrease
(c) Potential difference across C_2	(r) will remain same
(d) Electric field between the plates of C_2	(s) data insufficient

2. In the circuit shown in figure, match the following two columns for the flow of charge when switch is closed.

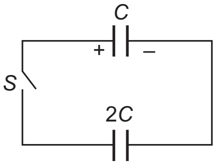


Column I	Column II
(a) From the battery	(p) 40 μC
(b) From 2 μF capacitor	(q) 100 μC
(c) From 3 μF	(r) 60 μC
(d) From 4 μF capacitor	(s) None of these

3. Three identical capacitors are connected in three different configurations as shown in Column II. Points a and b are connected with a battery. Match the two columns.

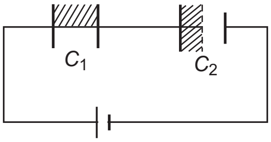
Column I	Column II	
(a) Maximum charge on C_1	(p)	
(b) Minimum charge on C_2	(q)	
(c) Maximum potential difference across C_1	(r)	
(d) Minimum potential difference across C_1	(s)	

4. A capacitor C is charged by a battery of V volts. Then, it is connected to an uncharged capacitor of capacity $2C$ as shown in figure. Now, match the following two columns.



Column I	Column II
(a) After closing the switch energy stored in C.	(p) $\frac{1}{9}CV^2$
(b) After closing the switch energy stored in $2C$.	(q) $\frac{1}{6}CV^2$
(c) After closing the switch loss of energy during redistribution of charge.	(r) $\frac{1}{18}CV^2$
	(s) None of these

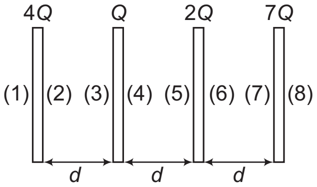
5. Two identical sized capacitors C_1 and C_2 are connected with a battery as shown in figure. Capacitor plates are square plates. A dielectric slab of dielectric constant $K = 2$, is filled in half the region of the two capacitors as shown :



$C \rightarrow$ capacity, $q \rightarrow$ charge stored, $U \rightarrow$ energy stored. Match the following two columns.

Column I	Column II
(a) C_1/C_2	(p) 9/4
(b) q_1/q_2	(q) 4/9
(c) U_1/U_2	(r) 4/3
	(s) None of these

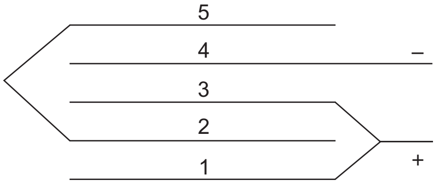
6. Four large parallel identical conducting plates are arranged as shown.



Column I	Column II
(a) Surfaces having charges of same magnitude and sign	(p) 1 and 8
(b) Surfaces having positive charges	(q) 3 and 5
(c) Uncharged surfaces	(r) 2 and 3
(d) Charged surfaces	(s) 6 and 7

Subjective Questions

1. Five identical conducting plates, 1, 2, 3,4 and 5 are fixed parallel plates equidistant from each other (see figure). A conductor connects plates 2 and 5 while another conductor joins 1 and 3. The junction of 1 and 3 and the plate 4 are connected to a source of constant emf V_0. Find

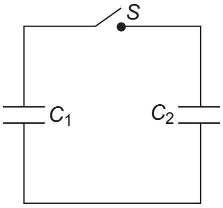


(a) the effective capacity of the system between the terminals of source.

(b) the charges on the plates 3 and 5.

Given, d = distance between any two successive plates and A = area of either face of each plate.

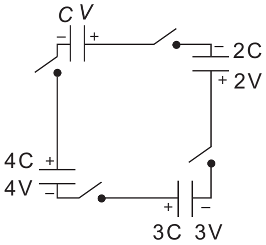
2. A $8\,\mu\text{F}$ capacitor C_1 is charged to $V_0 = 120\text{ V}$. The charging battery is then removed and the capacitor is connected in parallel to an uncharged $+ 4\,\mu\text{F}$ capacitor C_2.



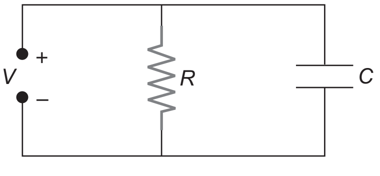
(a) what is the potential difference V across the combination?

(b) what is the stored energy before and after the switch S is closed?

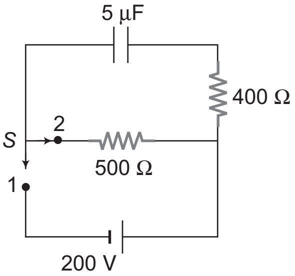
3. Condensers with capacities $C, 2C, 3C$ and $4C$ are charged to the voltage, $V, 2\,V, 3\,V$ and $4\,V$ correspondingly. The circuit is closed. Find the voltage on all condensers in the equilibrium.



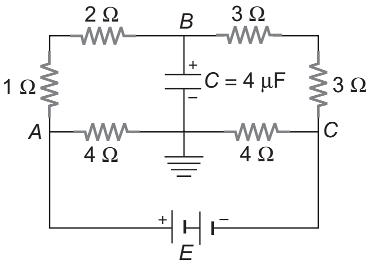
4. In the circuit shown, a time varying voltage $V = 2000t$ volt is applied where t is in second. At time $t = 5$ ms, determine the current through the resistor $R = 4\,\Omega$ and through the capacitor $C = 300\,\mu\text{F}$.



5. A capacitor of capacitance $5\,\mu\text{F}$ is connected to a source of constant emf of 200 V. Then, the switch was shifted to contact 2 from contact 1. Find the amount of heat generated in the $400\,\Omega$ resistance.



6. Analyze the given circuit in the steady state condition. Charge on the capacitor is $q_0 = 16\,\mu\text{C}$.

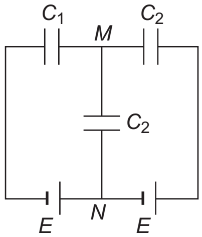


(a) Find the current in each branch

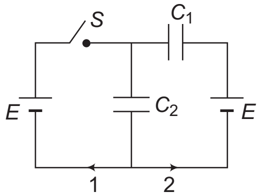
(b) Find the emf of the battery.

(c) If now the battery is removed and the points A and C are shorted. Find the time during which charge on the capacitor becomes $8\,\mu\text{C}$.

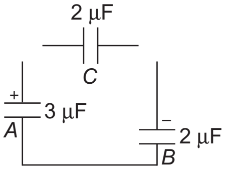
7. Find the potential difference between points M and N of the system shown in figure, if the emf is equal to $E = 110$ V and the capacitance ratio $\frac{C_1}{C_2}$ is 2.



8. In the given circuit diagram, find the charges which flow through directions 1 and 2 when switch S is closed.



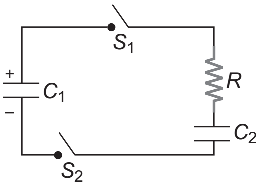
9. Two capacitors A and B with capacities $3\,\mu\text{F}$ and $2\,\mu\text{F}$ are charged to a potential difference of 100 V and 180 V, respectively. The plates of the capacitors are connected as shown in figure with one wire of each capacitor free. The upper plate of A is positive and that of B is negative. An uncharged $2\,\mu\text{F}$ capacitor C with lead wires falls on the free ends to complete the circuit. Calculate



(i) the final charge on the three capacitors,

(ii) the amount of electrostatic energy stored in the system before and after completion of the circuit.

10. The capacitor C_1 in the figure initially carries a charge q_0. When the switch S_1 and S_2 are closed, capacitor C_1 is connected to a resistor R and a second capacitor C_2, which initially does not carry any charge.



(a) Find the charges deposited on the capacitors in steady state and the current through R as a function of time.

(b) What is heat lost in the resistor after a long time of closing the switch?

11. A leaky parallel plate capacitor is filled completely with a material having dielectric constant $K = 5$ and electrical conductivity $\sigma = 7.4 \times 10^{-12}\ \Omega^{-1}\text{m}^{-1}$. If the charge on the capacitor at the instant $t = 0$ is $q_0 = 8.55\,\mu\text{C}$, then calculate the leakage current at the instant $t = 12$ s.

12. A parallel plate vacuum capacitor with plate area A and separation x has charges $+Q$ and $-Q$ on its plates. The capacitor is disconnected from the source of charge, so the charge on each plate remains fixed.

(a) What is the total energy stored in the capacitor?

(b) The plates are pulled apart an additional distance dx. What is the change in the stored energy?

(c) If F is the force with which the plates attract each other, then the change in the stored energy must equal the work $dW = Fdx$ done in pulling the plates apart. Find an expression for F.

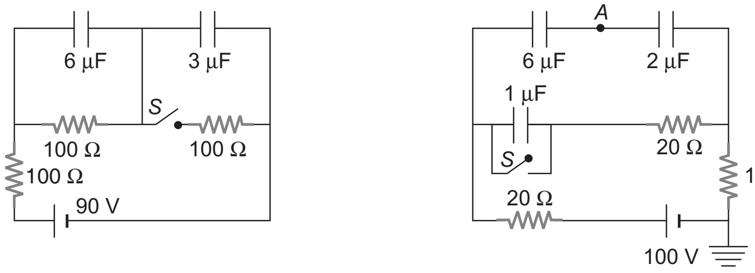
(d) Explain why F is not equal to QE, where E is the electric field between the plates.

13. A spherical capacitor has the inner sphere of radius 2 cm and the outer one of 4 cm. If the inner sphere is earthed and the outer one is charged with a charge of $2\,\mu\text{C}$ and isolated. Calculate

(a) the potential to which the outer sphere is raised.

(b) the charge retained on the outer surface of the outer sphere.

14. Calculate the charge on each capacitor and the potential difference across it in the circuits shown in figure for the cases :



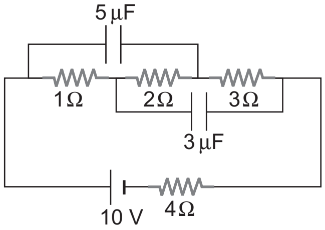
(a) (b)

(i) switch S is closed and

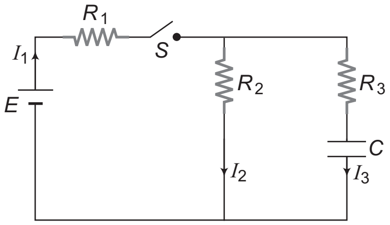
(ii) switch S is open.

(iii) In figure (b), what is the potential of point A when S is open?

15. In the shown network, find the charges on capacitors of capacitances $5\,\mu F$ and $3\,\mu F$, in steady state.



16. In the circuit shown, $E = 18\,kV$, $C = 10\,\mu F$, $R_1 = 4\,M\Omega$, $R_2 = 6\,M\Omega$, $R_3 = 3\,M\Omega$. With C completely uncharged, switch S is suddenly closed (at $t = 0$).

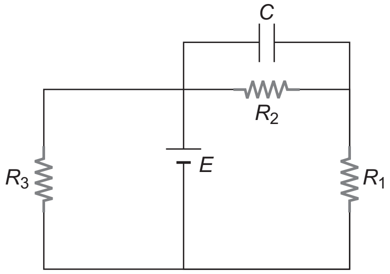


(a) Determine the current through each resistor for $t = 0$ and $t = \infty$.

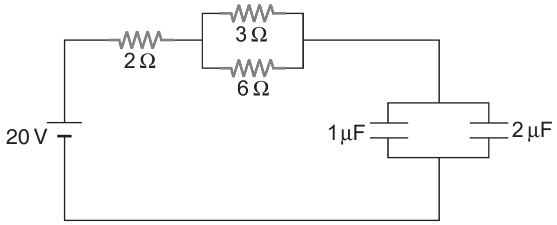
(b) What are the values of V_2 (potential difference across R_2) at $t = 0$ and $t = \infty$?

(c) Plot a graph of the potential difference V_2 *versus* t and determine the instantaneous value of V_2.

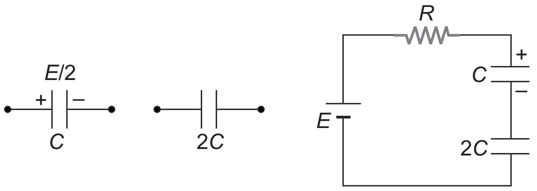
17. The charge on the capacitor is initially zero. Find the charge on the capacitor as a function of time t. All resistors are of equal value R.



18. The capacitors are initially uncharged. In a certain time the capacitor of capacitance $2\,\mu F$ gets a charge of $20\,\mu C$. In that time interval find the heat produced by each resistor individually.



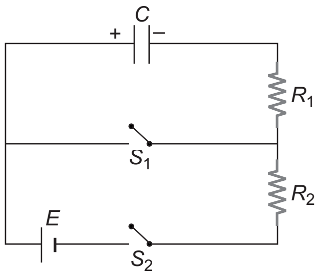
19. A capacitor of capacitance C has potential difference $E/2$ and another capacitor of capacitance $2C$ is uncharged. They are joined to form a closed circuit as shown in the figure.



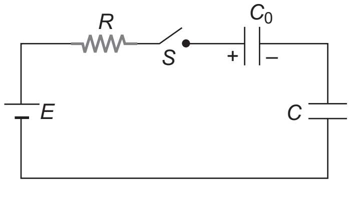
(a) Find the current in the circuit at $t = 0$.

(b) Find the charge on C as a function of time.

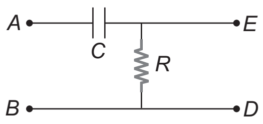
20. The capacitor shown in figure has been charged to a potential difference of V volt, so that it carries a charge CV with both the switches S_1 and S_2 remaining open. Switch S_1 is closed at $t = 0$. At $t = R_1C$ switch S_1 is opened and S_2 is closed. Find the charge on the capacitor at $t = 2R_1C + R_2C$.

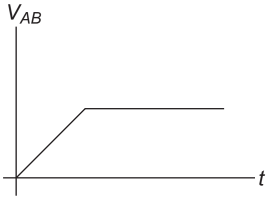


21. The switch S is closed at $t = 0$. The capacitor C is uncharged but C_0 has a charge $Q_0 = 2\,\mu\text{C}$ at $t = 0$. If $R = 100\,\Omega, C = 2\,\mu\text{F}, C_0 = 2\,\mu\text{F}, E = 4\text{ V}$. Calculate $i(t)$ in the circuit.



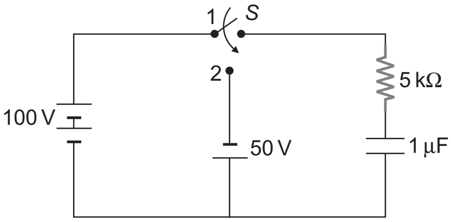
22. A time varying voltage is applied to the clamps A and B such that voltage across the capacitor plates is as shown in the figure. Plot the time dependence of voltage across the terminals of the resistance E and D.



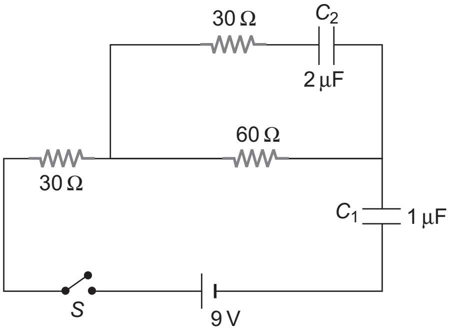


23. In the above problem if given graph is between V_{AB} and time. Then, plot graph between V_{ED} and time.

24. Initially, the switch is in position 1 for a long time. At $t = 0$, the switch is moved from 1 to 2. Obtain expressions for V_C and V_R for $t > 0$.



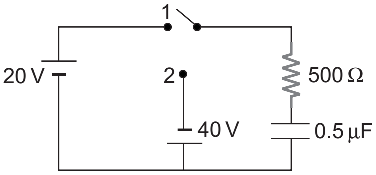
25. For the arrangement shown in the figure, the switch is closed at $t = 0$. C_2 is initially uncharged while C_1 has a charge of $2\,\mu C$.



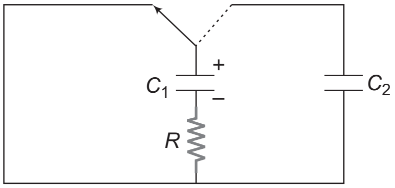
(a) Find the current coming out of the battery just after the switch is closed.

(b) Find the charge on the capacitors in the steady state condition.

26. In the given circuit, the switch is closed in the position 1 at $t = 0$ and then moved to 2 after $250\,\mu s$. Derive an expression for current as a function of time for $t > 0$. Also plot the variation of current with time.



27. A charged capacitor C_1 is discharged through a resistance R by putting switch S in position 1 of circuit shown in figure. When discharge current reduces to I_0, the switch is suddenly shifted to position 2. Calculate the amount of heat liberated in resistance R starting from this instant.



Answers

Introductory Exercise 25.1

1. $[\mathrm{M^{-1}L^{-2}T^{4}A^{2}}]$ **2.** False **3.** (a) -10 V (b) $-10\,\mu$C, $-20\,\mu$C (c) 3.0×10^{-4} J

Introductory Exercise 25.2

1. $\pm 182\,\mu$C **2.** (a) 604 V (b) 90.8 $\mathrm{cm^2}$ (c) 16.3 $\mu\mathrm{C/m^2}$ **3.** (a) 1.28 (b) 6.2×10^{-7} $\mathrm{C/m^2}$

Introductory Exercise 25.3

1. $q_{3\mu\mathrm{F}} = 30\,\mu$C, $q_{4\mu\mathrm{F}} = 20\,\mu$C, $q_{2\mu\mathrm{F}} = 10\,\mu$C **2.** $q_{4\mu\mathrm{F}} = 120\,\mu$C, $q_{9\mu\mathrm{F}} = 90\,\mu$C, $q_{3\mu\mathrm{F}} = 30\,\mu$C

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (a,b) **4.** (a,b) **5.** (d) **6.** (d) **7.** (b) **8.** (d) **9.** (b) **10.** (b)

Objective Questions

1. (c) **2.** (d) **3.** (c) **4.** (c) **5.** (d) **6.** (d) **7.** (c) **8.** (b) **9.** (b) **10.** (c)
11. (a) **12.** (b) **13.** (b) **14.** (c) **15.** (c) **16.** (a) **17.** (d) **18.** (b) **19.** (b) **20.** (b)
21. (a) **22.** (c) **23.** (d) **24.** (c) **25.** (d) **26.** (b) **27.** (b)

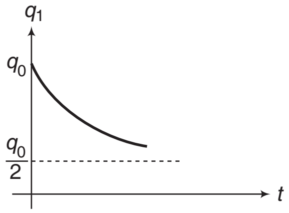
Subjective Questions

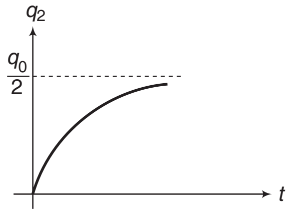
1. Starting from the left face the charges are, $3\,\mu$C, $7\,\mu$C, $-7\,\mu$C, $3\,\mu$C **2.** $\dfrac{\varepsilon_0 A}{d}, \dfrac{5dq}{2A\varepsilon_0}$

3. $10\,\mu$C, $20\,\mu$C, $30\,\mu$C **4.** $40\,\mu$C **5.** $24\,\mu$C

6. (a) $800\,\mu$C, 800 V, $800\,\mu$C, 400 V (b) $\dfrac{1600}{3}\,\mu$C, $\dfrac{3200}{3}\,\mu$C, $\dfrac{1600}{3}$ V **7.** $400\,\mu$F

9. 0.69 **10.** $q_1 = \dfrac{q_0}{2} + \dfrac{q_0}{2} e^{-2t/RC}, \quad q_2 = \dfrac{q_0}{2}(1 - e^{-2t/RC})$





11. $CE\,(1 - e^{-t/CR}) + q_0\,e^{-t/CR}$

12. (a) E/R_1 (b) $E/(R_1 + R_3)$ **13.** (a) $i_1 = E/R_1$, $i_2 = E/R_2$ (b) $i_1 = E/R_1$, $i_2 = 0$ (c) $\dfrac{1}{2}CE^2$ (d) $C(R_1 + R_2)$

14. (a) $\dfrac{5C}{3}$ (b) $\dfrac{4C}{3}$ (c) $2C$

15. (a) $4.0\,\mu$F : 2.64×10^{-3}C, 660 V, $6.0\,\mu$F : 3.96×10^{-3} C, 660 V
(b) $4.0\,\mu$F : 5.28×10^{-4}C, 132 V, $6.0\,\mu$F : 7.92×10^{-4}C, 132 V

16. 2.83×10^{-2} J/m^3 **17.** 0.0135 m^2 **18.** 40 mF, 10 mF **19.** 10 V, 5 V

20. (a) Five capacitors in series (b) Six rows of three capacitors in each row.

21. 0.16 μF, 0.24 μF **22.** (a) 120μC (b) 60μC (c) 480μC **23.** 19.6 J

24. (a) 3.54×10^{-9} F (b) ±35.4 μC (c) 2.0×10^{6} N/C

25. (a) 76 μC (b) 1.4 mJ (c) 11 V (d) 1.2 mJ **26.** $10\mu C, \frac{40}{3}\mu C$

27. (a) 2.5μF (b) $Q_1 = 5.5 \times 10^{-4}$ C, $V_1 = 66$ V, $Q_2 = 3.7 \times 10^{-4}$C, $V_2 = 88$ V, $Q_3 = Q_4 = 1.8 \times 10^{-4}$ C, $V_3 = V_4 = 44$ V, $Q_5 = Q_1, V_5 = V_1$

28. 78.68 V, 151.32 V **29.** 60μC **30.** (a) 3μF (b) 60μC (c) 30μC (d) 20μC (e) 20μC

31. (a) $q_1 = q_3 = 9\ \mu C$, $q_2 = q_4 = 16\ \mu C$ (b) $q_1 = 8.64\ \mu C$, $q_2 = 17.28\ \mu C$, $q_3 = 10.08\ \mu C$, $q_4 = 13.44\ \mu C$

32. $q_2 = q_3 = \dfrac{C_1V_0}{1 + \dfrac{C_1}{C_2} + \dfrac{C_1}{C_3}}$, $q_1 = C_1V_0 - \dfrac{C_1V_0}{1 + \dfrac{C_1}{C_2} + \dfrac{C_1}{C_3}}$

33. (a) 2 V (b) $U_i = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)V^2, U_f = \left(\frac{\varepsilon_0 A}{d}\right)V^2$ (c) $W = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)V^2$

34. (a) 1 mA, 1 mA, 0 (b) 2 mA, 1 mA, 3 mA **35.** (a) 18 V (b) *a* is at higher potential (c) 6 V (d) –36 μC on both the capacitors

36. (a) –6.0 V (b) *b* (c) 6.0 V (d) –54.0 μC

37. (a) $\frac{CV}{2}(1 - e^{-\alpha t})$ (b) $\frac{V}{2R} - \frac{V}{6R}e^{-\alpha t}, \frac{V}{2R}$ Here $\alpha = \frac{2}{3RC}$

LEVEL 2

Single Correct Option

1.(c) **2.**(b) **3.**(d) **4.**(c) **5.**(d) **6.**(d) **7.**(d) **8.**(b) **9.**(c) **10.**(d)
11.(d) **12.**(b) **13.**(d) **14.**(c) **15.**(c) **16.**(a) **17.**(b) **18.**(b) **19.**(b) **20.**(c)
21.(c) **22.**(d) **23.**(b) **24.**(d) **25.**(a) **26.**(b) **27.**(c) **28.**(a) **29.**(b) **30.**(b)
31.(d) **32.**(c) **33.**(b) **34.**(c)

More than One Correct Options

1. (a,c,d) **2.** (b,c,d) **3.** (b,c) **4.** (a,b,d) **5.** (b,c) **6.** (a,d) **7.** (a,b,d) **8.** (a,b,c) **9.** (a,b,c,d) **10.** (b)

Comprehension Based Questions

1.(b) **2.**(a) **3.**(b) **4.**(a)

Match the Columns

1. (a) → q (b) → p (c) → p (d) → p
2. (a) → s (b) → p (c) → r (d) → s
3. (a) → q (b) → p,r (c) → q (d) → p,s
4. (a) → r (b) → p (c) → s
5. (a) → s (b) → s (c) → s
6. (a) → p (b) → p,q (c) → s (d) → p,q,r

Subjective Questions

1. (a) $\frac{5}{3}\left(\frac{\varepsilon_0 A}{d}\right)$ (b) $q_3 = \frac{4}{3}\left(\frac{\varepsilon_0 A V_0}{d}\right), q_5 = \frac{2}{3}\left(\frac{\varepsilon_0 A V_0}{d}\right)$ **2.** (a) 80 V (b) 57.6 mJ, 38.4 mJ

3. $-\frac{19}{5}$V, $-\frac{2}{5}$V, $\frac{7}{5}$V, $\frac{14}{5}$V **4.** 2.5 A, 0.6 A **5.** 44.4 mJ

6. (a) 3 A, 2.67A (b) 24 V (c) 11.1 μs **7.** $V_N - V_M = \frac{110}{3}$ volt **8.** $q_1 = EC_2, q_2 = \frac{-EC_1C_2}{(C_1 + C_2)}$

9. (i) 90μC, 210μC, 150μC (ii) (a) 47.4 mJ (b) 18 mJ

10. (a) $q_1 = \left(\frac{C_1}{C_1 + C_2}\right) q_0$ and $q_2 = \left(\frac{C_2}{C_1 + C_2}\right) q_0$, $i = \frac{q_0}{RC_1} e^{-t/RC}$ (b) $\Delta H = \frac{q_0^2 C_2}{2C_1(C_1 + C_2)}$, here $C = \frac{C_1 C_2}{C_1 + C_2}$

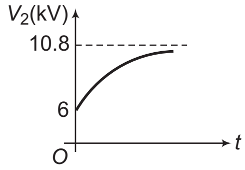
11. 0.193 μA **12.** (a) $\frac{Q^2 x}{2\varepsilon_0 A}$ (b) $\left(\frac{Q^2}{2\varepsilon_0 A}\right) \cdot dx$ (c) $\frac{Q^2}{2\varepsilon_0 A}$ **13.** (a) 2.25×10^5 V (b) +1 μC

14.

		Fig. (a)		**Fig. (b)**		
		6μF	**3**μF	**1**μF	**6**μF	**2**μF
(i)	PD (volts)	30	30	0	10	30
	charge (μC)	180	90	0	60	60
(ii)	PD (volts)	0	90	100	25	75
	charge (μC)	0	270	100	150	150
(iii)	$V_A = 75$ volt					

15. 15 μC, 15 μC

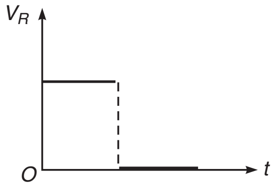
16. (a) At $t = 0$, $i_1 = 3$ mA, $i_2 = 1$ mA, $i_3 = 2$ mA At $t = \infty$,
$i_1 = i_2 = 1.8$ mA, $i_3 = 0$ (b) At $t = 0$, $V_2 = 6$ kV At $t = \infty$, $V_2 = 10.8$ kV
(c) $V_2 = (10.8 - 4.8\, e^{-t/54})$ kV



17. $q = \frac{CE}{2}\left(1 - e^{-\frac{2t}{CR}}\right)$ **18.** $H_2 = 0.075$ mJ, $H_3 = 0.05$ mJ, $H_6 = 0.025$ mJ

19. (a) $\frac{E}{2R}$ (b) $\frac{CE}{6}[5 - 2e^{-3t/2RC}]$ **20.** $EC\left(1 - \frac{1}{e}\right) + \frac{VC}{e^2}$ **21.** $(0.03\, e^{-10^4 t})$ A

22.



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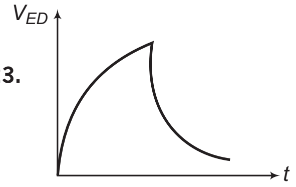

24. $V_C = 50\,(3\, e^{-200t} - 1)$, $V_R = 150\, e^{-200t}$ **25.** (a) $\frac{7}{50}$A or $\frac{11}{50}$ A (b) $Q_1 = 9\,\mu C$, $Q_2 = 0$

26. $i = (0.04 e^{-4000t})$ A for $t \le 250\,\mu s$, $= -\,(0.11\, e^{-4000t})$ A for $t \ge 250\,\mu s$

For *i* - *t* graph, see the hints.

27. $\frac{(I_0 R)^2 C_1 C_2}{2(C_1 + C_2)}$

26

Magnetics

Chapter Contents

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26.2 Magnetic force on a moving charge(F_m)
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26.1 Introduction

The fascinating attractive properties of magnets have been known since ancient times. The word magnet comes from ancient Greek place name Magnesia (the modern town Manisa in Western Turkey), where the natural magnets called lodestones were found. The fundamental nature of magnetism is the interaction of moving electric charges. Unlike electric forces which act on electric charges whether they are moving or not, magnetic forces act only on moving charges and current carrying wires.

We will describe magnetic forces using the concept of a field. A magnetic field is established by a permanent magnet, by an electric current or by other moving charges. This magnetic field, in turn, exerts forces on other moving charges and current carrying conductors. In this chapter, first we study the magnetic forces and torques exerted on moving charges and currents by magnetic fields, then we will see how to calculate the magnetic fields produced by currents and moving charges.

26.2 Magnetic Force on a Moving Charge ($\mathbf{F}_m$)

An unknown electric field can be determined by magnitude and direction of the force on a test charge q_0 at rest. To explore an unknown magnetic field (denoted by **B**), we must measure the magnitude and direction of the force on a moving test charge.

The magnetic force ($\mathbf{F}_m$) on a charge q moving with velocity **v** in a magnetic field **B** is given, both in magnitude and direction, by

$$\boxed{\mathbf{F}_m = q\,(\mathbf{v} \times \mathbf{B})} \qquad \ldots\text{(i)}$$

Following points are worthnoting regarding the above expression.

(i) The magnitude of $\mathbf{F}_m$ is

$$\boxed{F_m = Bqv \sin\theta}$$

where, θ is the angle between **v** and **B**.

(ii) $\mathbf{F}_m$ is zero when,

(a) $B = 0$, i.e. no magnetic field is present.

(b) $q = 0$, i.e. particle is neutral.

(c) $v = 0$, i.e. charged particle is at rest or

(d) $\theta = 0°$ or $180°$, i.e. $\mathbf{v} \uparrow\uparrow \mathbf{B}$ or $\mathbf{v} \uparrow\downarrow \mathbf{B}$

(iii) $\mathbf{F}_m$ is maximum at $\theta = 90°$ and this maximum value is Bqv.

(iv) The units of **B** must be the same as the units of F/qv. Therefore, the SI unit of B is equivalent to $\frac{\text{N-s}}{\text{C-m}}$. This unit is called the **tesla** (abbreviated as T), in honour of Nikola Tesla, the prominent Serbian-American scientist and inventor.

Thus,

$$\boxed{1 \text{ tesla} = 1\text{T} = \frac{1\,\text{N-s}}{\text{C-m}} = \frac{1\,\text{N}}{\text{A-m}}}$$

The CGS unit of **B**, the **gauss** ($1\text{G} = 10^{-4}$ T) is also in common use.

(v) In equation number (i) q is to be substituted with sign. If q is positive, magnetic force is along $\mathbf{v} \times \mathbf{B}$ and if q is negative, magnetic force is in a direction opposite to $\mathbf{v} \times \mathbf{B}$.

(vi) **Direction of $\mathbf{F}_m$** From the property of cross product we can infer that $\mathbf{F}_m$ is perpendicular to both **v** and **B** or it is perpendicular to the plane formed by **v** and **B**. The exact direction of $\mathbf{F}_m$ can be given by any of the following methods:

(a) Direction of $\mathbf{F}_m$ = (sign of q) (direction of $\mathbf{v} \times \mathbf{B}$) or, as we stated earlier also,

$$\mathbf{F}_m \uparrow\uparrow \mathbf{v} \times \mathbf{B} \quad \text{if } q \text{ is positive and}$$
$$\mathbf{F}_m \uparrow\downarrow \mathbf{v} \times \mathbf{B} \quad \text{if } q \text{ is negative.}$$

(b) Fleming's left hand rule According to this rule, the forefinger, the central finger and the thumb of the left hand are stretched in such a way that they are mutually perpendicular to each other. If the central finger shows the direction of velocity of positive charge ($\mathbf{v}_{+q}$) and forefinger shows the direction of magnetic field (**B**), then the thumb will give the direction of magnetic force ($\mathbf{F}_m$). If instead of positive charge we have the negative charge, then $\mathbf{F}_m$ is in opposite direction.

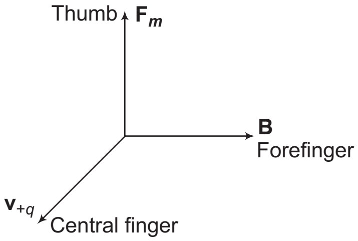


Fig. 26.1

(c) Right hand rule Wrap the fingers of your right hand around the line perpendicular to the plane of **v** and **B** as shown in figure, so that they curl around with the sense of rotation from **v** to **B** through the smaller angle between them. Your thumb then points in the direction of the force $\mathbf{F}_m$ on a positive charge. (Alternatively, the direction of the force $\mathbf{F}_m$ on a positive charge is the direction in which a right hand thread screw would advance if turned the same way).

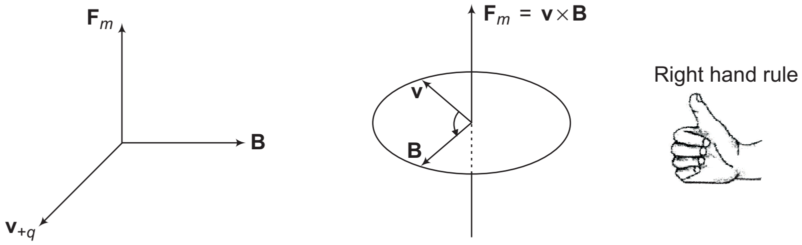


Fig. 26.2

(vii) $\mathbf{F}_m \perp \mathbf{v}$ or $\mathbf{F}_m \perp \dfrac{d\mathbf{s}}{dt}$. Therefore, $\mathbf{F}_m \perp d\mathbf{s}$ or the work done by the magnetic force in a static magnetic field is zero.

$$W_{\mathbf{F}_m} = 0$$

So, from work energy theorem KE and hence the speed of the charged particle remains constant in magnetic field. The magnetic force can change the direction only. It cannot increase or decrease the speed or kinetic energy of the particle.

Note *By convention the direction of magnetic field* **B** *perpendicular to the paper going inwards is shown by* ⊗ *and the direction perpendicular to the paper coming out is shown by* ⊙.

Direction of **B** perpendicular to paper inwards

Direction of **B** perpendicular to paper outwards

Fig. 26.3

Example 26.1 *A charged particle is projected in a magnetic field*

$$\mathbf{B} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \times 10^{-2}\ T$$

The acceleration of the particle is found to be

$$\mathbf{a} = (x\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ m/s^2$$

Find the value of x.

Solution As we have read, $\mathbf{F}_m \perp \mathbf{B}$

i.e. the acceleration $\mathbf{a} \perp \mathbf{B}$ or $\mathbf{a} \cdot \mathbf{B} = 0$

or $$(x\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \cdot (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \times 10^{-2} = 0$$

or $$(3x + 8) \times 10^{-2} = 0$$

∴ $$x = -\frac{8}{3}\ \text{m/s}^2$$ **Ans.**

Example 26.2 *When a proton has a velocity* $\mathbf{v} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times 10^6\ m/s$, *it experiences a force* $\mathbf{F} = -(1.28 \times 10^{-13}\ \hat{\mathbf{k}})\ N$. *When its velocity is along the z-axis, it experiences a force along the x-axis. What is the magnetic field?*

HOW TO PROCEED *In the second part of the question, it is given that magnetic force is along x-axis when velocity is along z-axis. Hence, magnetic field should be along negative y-direction. As in case of positive charge (here proton)*

$$\boxed{\mathbf{F}_m \uparrow\uparrow \mathbf{v} \times \mathbf{B}}$$

So, let $\mathbf{B} = -B_0\hat{\mathbf{j}}$

where, B_0 *= positive constant.*

Now, applying $\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B})$ *we can find value of* B_0 *from the first part of the question.*

Solution Substituting proper values in, $\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B})$

We have, $$-(1.28 \times 10^{-13}\ \hat{\mathbf{k}}) = (1.6 \times 10^{-19})[(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times (-B_0\hat{\mathbf{j}})] \times 10^6$$

∴ $$1.28 = 1.6 \times 2 \times B_0$$

or $$B_0 = \frac{1.28}{3.2} = 0.4$$

Therefore, the magnetic field is $\mathbf{B} = (-0.4\ \hat{\mathbf{j}})\ \mathbf{T}$ **Ans.**

Example 26.3 *A magnetic field of* $(4.0\times10^{-3}\,\hat{\mathbf{k}})$ *T exerts a force* $(4.0\,\hat{\mathbf{i}}+3.0\,\hat{\mathbf{j}})\times10^{-10}$ *N on a particle having a charge* 10^{-9} *C and moving in the x-y plane. Find the velocity of the particle.*

Solution Given, $\mathbf{B}=(4\times10^{-3}\,\hat{\mathbf{k}})$ T, $q=10^{-9}$ C

and magnetic force $\mathbf{F}_m=(4.0\,\hat{\mathbf{i}}+3.0\,\hat{\mathbf{j}})\times10^{-10}$ N

Let velocity of the particle in x-y plane be

$$\mathbf{v}=v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}}$$

Then, from the relation

$$\mathbf{F}_m=q\,(\mathbf{v}\times\mathbf{B})$$

We have

$$\begin{aligned}(4.0\,\hat{\mathbf{i}}+3.0\,\hat{\mathbf{j}})\times10^{-10}&=10^{-9}\,[(v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}})\times(4\times10^{-3}\,\hat{\mathbf{k}})]\\&=(4v_y\times10^{-12}\,\hat{\mathbf{i}}-4v_x\times10^{-12}\,\hat{\mathbf{j}})\end{aligned}$$

Comparing the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we have

$$4\times10^{-10}=4v_y\times10^{-12}$$

$$\therefore\quad v_y=10^2\text{ m/s}=100\text{ m/s}$$

and $$3.0\times10^{-10}=-\,4v_x\times10^{-12}$$

$$\therefore\quad v_x=-\,75\text{ m/s}$$

$$\therefore\quad \mathbf{v}=(-\,75\,\hat{\mathbf{i}}+100\,\hat{\mathbf{j}})\text{ m/s}$$ **Ans.**

INTRODUCTORY EXERCISE 26.1

1. Write the dimensions of E/B. Here, E is the electric field and B the magnetic field.
2. In the relation $\mathbf{F}=q\,(\mathbf{v}\times\mathbf{B})$, which pairs are always perpendicular to each other.
3. If a beam of electrons travels in a straight line in a certain region. Can we say there is no magnetic field?
4. A charge $q=-4\,\mu$C has an instantaneous velocity $\mathbf{v}=(2\,\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})\times10^6$ m/s in a uniform magnetic field $\mathbf{B}=(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-3\,\hat{\mathbf{k}})\times10^{-2}$ T. What is the force on the charge?
5. A particle initially moving towards south in a vertically downward magnetic field is deflected toward the east. What is the sign of the charge on the particle?
6. An electron experiences a magnetic force of magnitude 4.60×10^{-15} N, when moving at an angle of 60° with respect to a magnetic field of magnitude 3.50×10^{-3} T. Find the speed of the electron.
7. He^{2+} ion travels at right angles to a magnetic field of 0.80 T with a velocity of 10^5 m/s. Find the magnitude of the magnetic force on the ion.

26.3 Path of a Charged Particle in Uniform Magnetic Field

The path of a charged particle in uniform magnetic field depends on the angle θ (the angle between **v** and **B**). Depending on the different values of θ, the following three cases are possible.

Case 1 **When θ is 0° or 180°**

As we have seen in Art. 26.2, $\mathbf{F}_m = 0$, when θ is either $0°$ or $180°$. Hence, path of the charged particle is a straight line (undeviated) when it enters parallel or antiparallel to magnetic field.

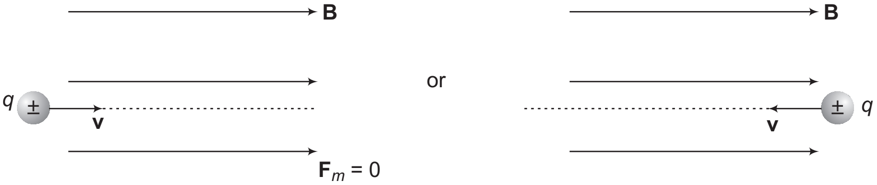


Fig. 26.4

Case 2 **When $\theta = 90°$**

When $\theta = 90°$, the magnetic force is $F_m = Bqv \sin 90° = Bqv$. This magnetic force is perpendicular to the velocity at every instant. Hence, path is a **circle**. The necessary centripetal force is provided by the magnetic force. Hence, if r be the radius of the circle, then

$$\frac{mv^2}{r} = Bqv$$

or
$$r = \frac{mv}{Bq}$$

This expression of r can be written in the following different ways

$$\boxed{r = \frac{mv}{Bq} = \frac{p}{Bq} = \frac{\sqrt{2Km}}{Bq} = \frac{\sqrt{2qVm}}{Bq}}$$

Here, p = momentum of particle

$$K = \text{KE of particle} = \frac{p^2}{2m} \quad \text{or} \quad p = \sqrt{2Km}$$

We also know that if the charged particle is accelerated by a potential difference of V volts, it acquires a KE given by

$$K = qV$$

Further, time period of the circular path will be

$$T = \frac{2\pi r}{v} = \frac{2\pi\left(\frac{mv}{Bq}\right)}{v} = \frac{2\pi m}{Bq}$$

or
$$\boxed{T = \frac{2\pi m}{Bq}}$$

or The angular speed (ω) of the particle is $\omega = \frac{2\pi}{T} = \frac{Bq}{m}$

$\therefore$ $\boxed{\omega = \frac{Bq}{m}}$

Frequency of rotation is $f = \frac{1}{T}$

or $\boxed{f = \frac{Bq}{2\pi m}}$

The following points are worthnoting regarding a circular path:

(i) The plane of the circle is perpendicular to magnetic field. If the magnetic field is along z-direction, the circular path is in x-y plane. The speed of the particle does not change in magnetic field.

Hence, if v_0 be the speed of the particle, then velocity of particle at any instant of time will be

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}$$

where, $v_x^2 + v_y^2 = v_0^2$

(ii) T, f and ω are independent of v while the radius is directly proportional to v.

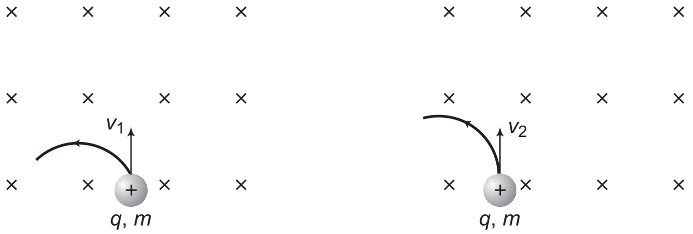


Fig. 26.5

Hence, if two charged particles of equal mass and charge enter in a magnetic field **B** with different speeds v_1 and v_2 $(> v_1)$ at right angles, then

$$\boxed{\begin{aligned} T_1 &= T_2 \\ r_2 &> r_1 \end{aligned}}$$

but

as shown in figure.

Note *Charge per unit mass $\frac{q}{m}$ is known as specific charge. It is sometimes denoted by α. So, in terms of α, the above formulae can be written as*

$$r = \frac{v}{B\alpha}, \quad T = \frac{2\pi}{B\alpha}, \quad f = \frac{B\alpha}{2\pi} \quad \text{and} \quad \omega = B\alpha$$

Case 3 When θ is other than 0°, 180° or 90°

In this case velocity can be resolved into two components, one along **B** and another perpendicular to **B**. Let the two components be $v_{||}$ and $v_{\perp}$.

Then,

$$v_{||} = v\cos\theta$$

and
$$v_{\perp} = v\sin\theta$$

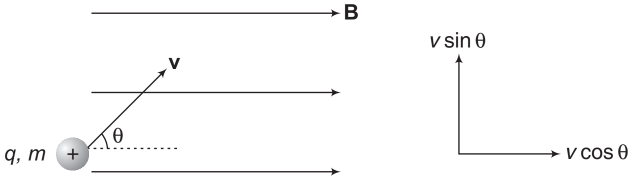


Fig. 26.6

The component perpendicular to field ($v_{\perp}$) gives a circular path and the component parallel to field ($v_{||}$) gives a straight line path. The resultant path is a helix as shown in figure.

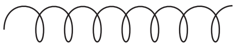


Fig. 26.7

The radius of this helical path is

$$\boxed{r = \frac{mv_{\perp}}{Bq} = \frac{mv\sin\theta}{Bq}}$$

Time period and frequency do not depend on velocity and so they are given by

$$\boxed{T = \frac{2\pi m}{Bq}} \quad \text{and} \quad \boxed{f = \frac{Bq}{2\pi m}}$$

There is one more term associated with a helical path, that is **pitch *(p)*** of the helical path. Pitch is defined as the distance travelled along magnetic field in one complete cycle.

i.e.
$$p = v_{||}T$$

or
$$p = (v\cos\theta)\frac{2\pi m}{Bq}$$

∴
$$\boxed{p = \frac{2\pi mv\cos\theta}{Bq}}$$

Example 26.4 *Two particles A and B of masses m_A and m_B respectively and having the same charge are moving in a plane. A uniform magnetic field exists perpendicular to this plane. The speeds of the particles are v_A and v_B respectively and the trajectories are as shown in the figure. Then,* **(JEE 2001)**

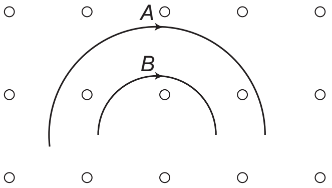


Fig. 26.8

(a) $m_A v_A < m_B v_B$

(b) $m_A v_A > m_B v_B$

(c) $m_A < m_B$ and $v_A < v_B$

(d) $m_A = m_B$ and $v_A = v_B$

Solution Radius of the circle $=\dfrac{mv}{Bq}$

or radius $\propto mv$ if B and q are same.

$$(\text{Radius})_A > (\text{Radius})_B$$

$\therefore$ $$m_A v_A > m_B v_B$$

$\therefore$ Correct option is (b).

Example 26.5 *A proton, a deuteron and an α-particle having the same kinetic energy are moving in circular trajectories in a constant magnetic field. If r_p, r_d and r_α denote respectively the radii of the trajectories of these particles, then*

(a) $r_\alpha = r_p < r_d$ *(b)* $r_\alpha > r_d > r_p$ **(JEE 1997)**

(c) $r_\alpha = r_d > r_p$ *(d)* $r_p = r_d = r_\alpha$

Solution Radius of the circular path is given by

$$r = \frac{mv}{Bq} = \frac{\sqrt{2Km}}{Bq}$$

Here, K is the kinetic energy to the particle.

Therefore, $r \propto \dfrac{\sqrt{m}}{q}$ if K and B are same.

$\therefore$ $$r_p : r_d : r_\alpha = \frac{\sqrt{1}}{1} : \frac{\sqrt{2}}{1} : \frac{\sqrt{4}}{2} = 1 : \sqrt{2} : 1$$

Hence, $$r_\alpha = r_p < r_d$$

$\therefore$ Correct option is (a).

Example 26.6 *Two particles X and Y having equal charges, after being accelerated through the same potential difference, enter a region of uniform magnetic field and describe circular paths of radii R_1 and R_2, respectively. The ratio of the mass of X to that of Y is* **(JEE 1988)**

(a) $(R_1/R_2)^{1/2}$ *(b)* R_2/R_1

(c) $(R_1/R_2)^2$ *(d)* R_1/R_2

Solution $R = \dfrac{\sqrt{2qVm}}{Bq}$

or $$R \propto \sqrt{m}$$

or $$\frac{R_1}{R_2} = \sqrt{\frac{m_X}{m_Y}}$$

or $$\frac{m_X}{m_Y} = \left(\frac{R_1}{R_2}\right)^2$$

$\therefore$ Correct option is (c).

INTRODUCTORY EXERCISE 26.2

1. A neutron, a proton, an electron and an α - particle enter a region of constant magnetic field with equal velocities. The magnetic field is along the inward normal to the plane of the paper. The tracks of the particles are labeled in figure. The electron follows track…… and the α - particle follows track…… **(JEE 1984)**

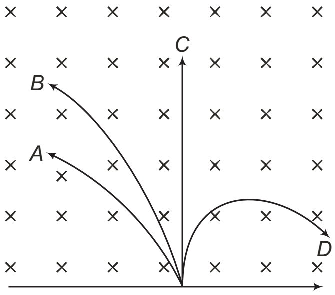


Fig. 26.9

2. An electron and a proton are moving with the same kinetic energy along the same direction. When they pass through a uniform magnetic field perpendicular to the direction of their motion, they describe circular path of the same radius. Is this statement true or false? **(JEE 1985)**
3. A charged particle enters a region of uniform magnetic field at an angle of 85° to the magnetic line of force. The path of the particle is a circle. Is this statement true or false? **(JEE 1983)**
4. Can a charged particle be accelerated by a magnetic field. Can its speed be increased?
5. An electron beam projected along positive *x*-axis deflects along the positive *y*-axis. If this deflection is caused by a magnetic field, what is the direction of the field?
6. An electron and a proton are projected with same velocity perpendicular to a magnetic field.
   (a) Which particle will describe the smaller circle?
   (b) Which particle will have greater frequency?
7. An electron is accelerated through a PD of 100 V and then enters a region where it is moving perpendicular to a magnetic field $B = 0.2$ T. Find the radius of the circular path. Repeat this problem for a proton.

26.4 Magnetic Force on a Current Carrying Conductor

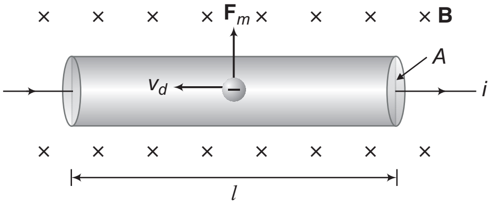


Fig. 26.10

A charged particle in motion experiences a magnetic force in a magnetic field. Similarly, a current carrying wire also experiences a force when placed in a magnetic field. This follows from the fact that the current is a collection of many charged particles in motion. Hence, the resultant force exerted by the field on the wire is the vector sum of the individual forces exerted on all the charged particles making up the current. The force exerted on the particles is transmitted to the wire when the particles collide with the atoms making up the wire.

Suppose a conducting wire carrying a current i is placed in a magnetic field **B**. The length of the wire is l and area of cross-section is A. The free electrons drift with a speed v_d opposite to the direction of current. The magnetic force exerted on the electron is

$$d\mathbf{F}_m = -e(\mathbf{v}_d \times \mathbf{B})$$

If n be the number of free electrons per unit volume of the wire, then total number of electrons in volume Al of the wire are, nAl. Therefore, total force on the wire is

$$\mathbf{F}_m = -e(nAl)(\mathbf{v}_d \times \mathbf{B})$$

If we denote the length l along the direction of the current by **l**, then the above equation becomes

$$\boxed{\mathbf{F}_m = i(\mathbf{l} \times \mathbf{B})} \qquad \text{...(i)}$$

where, $neAv_d = i$

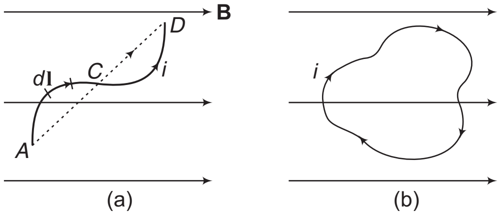
The following points are worthnoting regarding the above expression :

(i) Magnitude of $\mathbf{F}_m$ is, $F_m = ilB\sin\theta$, here θ is the angle between **l** and **B**. F_m is zero for $\theta = 0°$ or $180°$ and maximum for $\theta = 90°$.

(ii) Here, **l** is a vector that points in the direction of the current i and has a magnitude equal to the length.

(iii) The above expression applies only to a straight segment of wire in a uniform magnetic field.

(iv) For the magnetic force on an arbitrarily shaped wire segment, let us consider the magnetic force exerted on a small segment of vector length $d\mathbf{l}$.



(a) (b)

Fig. 26.11

$$d\mathbf{F}_m = i(d\mathbf{l} \times \mathbf{B}) \qquad \text{...(ii)}$$

To calculate the total force $\mathbf{F}_m$ acting on the wire shown in figure, we integrate Eq. (ii) over the length of the wire.

$$\mathbf{F}_m = i\int_A^D (d\mathbf{l} \times \mathbf{B}) \qquad \text{...(iii)}$$

Now, let us consider two special cases involving Eq. (iii). In both cases, the magnetic field is taken to be constant in magnitude and direction.

Case 1 A curved wire ACD as shown in Fig. (a) carries a current i and is located in a uniform magnetic field **B**. Because the field is uniform, we can take **B** outside the integral in Eq. (iii) and we obtain,

$$\mathbf{F}_m = i\left(\int_A^D d\mathbf{l}\right) \times \mathbf{B} \qquad \text{...(iv)}$$

But, the quantity $\int_A^D d\mathbf{l}$ represents the vector sum of all length elements from A to D. From the polygon law of vector addition, the sum equals the vector **l** directed from A to D. Thus,

$$\mathbf{F}_m = i\,(\mathbf{l} \times \mathbf{B})$$

or we can write

$$\boxed{\mathbf{F}_{ACD} = \mathbf{F}_{AD} = i\,(\mathbf{AD} \times \mathbf{B})} \text{ in uniform field.}$$

Case 2 An arbitrarily shaped closed loop carrying a current i is placed in a uniform magnetic field as shown in Fig. (b). We can again express the force acting on the loop in the form of Eq. (iv), but this time we must take the vector sum of the length elements $d\mathbf{l}$ over the entire loop,

$$\mathbf{F}_m = i\,(\oint d\mathbf{l}) \times \mathbf{B}$$

Because the set of length elements forms a closed polygon, the vector sum must be zero.

$\therefore$ $$\mathbf{F}_m = 0$$

Thus, **the net magnetic force acting on any closed current carrying loop in a uniform magnetic field is zero**.

(v) The direction of $\mathbf{F}_m$ can be given by Fleming's left hand rule as discussed in Art. 26.2. According to this rule, the forefinger, the central finger and the thumb of the left hand are stretched in such a way that they are mutually perpendicular to each other. If the central finger shows the direction of current (or $\mathbf{l}$) and forefinger shows the direction of magnetic field ($\mathbf{B}$), then the thumb will give the direction of magnetic force ($\mathbf{F}_m$).

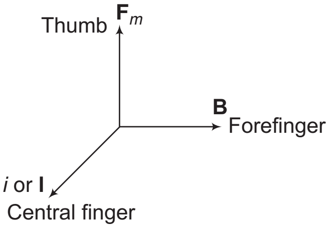


Fig. 26.12

Example 26.7 *A horizontal rod 0.2 m long is mounted on a balance and carries a current. At the location of the rod a uniform horizontal magnetic field has magnitude 0.067 T and direction perpendicular to the rod. The magnetic force on the rod is measured by the balance and is found to be 0.13 N. What is the current?*

Solution $$F = ilB \sin 90°$$

$\therefore$ $$i = \frac{F}{lB} = \frac{0.13}{0.2 \times 0.067}$$

$$= 9.7 \text{ A}$$ **Ans.**

Example 26.8 *A square of side 2.0 m is placed in a uniform magnetic field* $\mathbf{B} = 2.0\ T$ *in a direction perpendicular to the plane of the square inwards. Equal current* $i = 3.0\ A$ *is flowing in the directions shown in figure. Find the magnitude of magnetic force on the loop.*

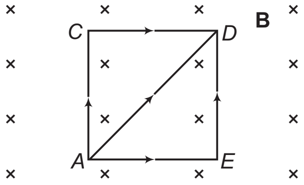


Fig. 26.13

Solution Force on wire ACD = Force on AD = Force on AED

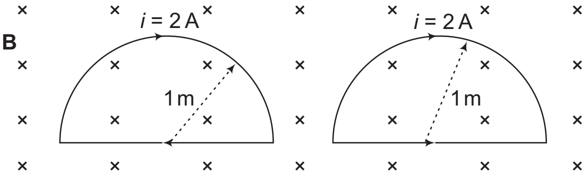
$\therefore$ Net force on the loop $= 3\,(\mathbf{F}_{AD})$

or $$F_{\text{net}} = 3\,(i)\,(AD)\,(B)$$

$$= (3)\,(3.0)\,(2\sqrt{2})\,(2.0) \text{ N} = 36\sqrt{2} \text{ N}$$

Direction of this force is towards EC.

Example 26.9 *In the figure shown a semicircular wire loop is placed in a uniform magnetic field* $B = 1.0\,T$*. The plane of the loop is perpendicular to the magnetic field. Current* $i = 2$ *A flows in the loop in the directions shown. Find the magnitude of the magnetic force in both the cases (a) and (b). The radius of the loop is* $1.0\ m$



(a) (b)

Fig. 26.14

Solution **Refer figure (a)** It forms a closed loop and the current completes the loop. Therefore, net force on the loop in uniform field should be zero. **Ans.**

Refer figure (b) In this case although it forms a closed loop, but current does not complete the loop. Hence, net force is not zero.

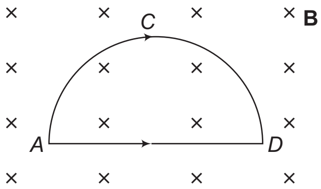


Fig. 26.15

$$\mathbf{F}_{ACD} = \mathbf{F}_{AD}$$

$$\therefore \qquad \mathbf{F}_{\text{loop}} = \mathbf{F}_{ACD} + \mathbf{F}_{AD} = 2\mathbf{F}_{AD}$$

$$\therefore \qquad |\mathbf{F}_{\text{loop}}| = 2\,|\mathbf{F}_{AD}|$$

$$= 2ilB\sin\theta \qquad [l = 2r = 2.0\text{ m}]$$

$$= (2)(2)(2)(1)\sin 90° = 8\text{ N}$$ **Ans.**

INTRODUCTORY EXERCISE 26.3

1. A wire of length l carries a current i along the x-axis. A magnetic field $\mathbf{B} = B_0(\hat{\mathbf{j}} + \hat{\mathbf{k}})$ exists in the space. Find the magnitude of the magnetic force acting on the wire.
2. In the above problem will the answer change if magnetic field becomes $\mathbf{B} = B_0(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$.
3. A wire along the x-axis carries a current of 3.50 A in the negative direction. Calculate the force (expressed in terms of unit vectors) on a 1.00 cm section of the wire exerted by these magnetic fields

   (a) $\mathbf{B} = -(0.65\text{ T})\hat{\mathbf{j}}$ (b) $\mathbf{B} = +(0.56\text{ T})\hat{\mathbf{k}}$ (c) $\mathbf{B} = -(0.31\text{ T})\hat{\mathbf{i}}$

   (d) $\mathbf{B} = +(0.33\text{ T})\hat{\mathbf{i}} - (0.28\text{ T})\hat{\mathbf{k}}$ (e) $\mathbf{B} = +(0.74\text{ T})\hat{\mathbf{j}} - (0.36\text{ T})\hat{\mathbf{k}}$
4. Find net force on the equilateral loop of side 4 m carrying a current of 2 A kept in a uniform magnetic field of 2 T as shown in figure.

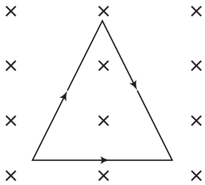


Fig. 26.16

26.5 Magnetic Dipole

Every current carrying loop is a magnetic dipole. It has two poles: south (S) and north (N). This is similar to a bar magnet. Magnetic field lines emanate from the north pole and after forming a closed path terminate on south pole. Each magnetic dipole has some magnetic moment (**M**). The magnitude of **M** is

$$\boxed{|\mathbf{M}| = NiA}$$

Here, N = number of turns in the loop

i = current in the loop and

A = area of cross-section of the loop.

For the direction of **M** any one of the following methods can be used:

(i) As in case of an electric dipole, the dipole moment **p** has a direction from negative charge to positive charge. In the similar manner, direction of **M** is from **south to north** pole. The south and north poles can be identified by the sense of current. The side from where the current seems to be clockwise becomes south pole and the opposite side from where it seems anti-clockwise becomes north pole.

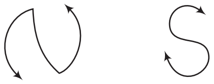


Fig. 26.17

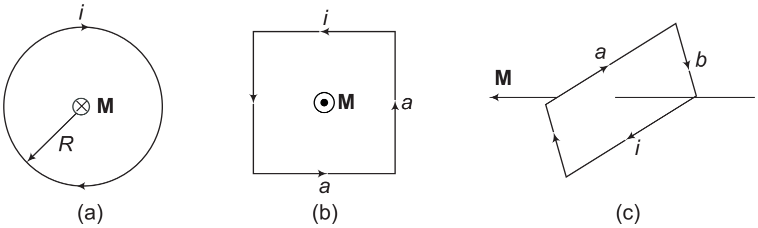


Fig. 26.18

Now, let us find the direction and magnitude of **M** in the three loops shown in Fig. 26.18.

Refer figure (a) In this case, current appears to be clockwise from outside the paper, so this side becomes the south pole. From the back of the paper it seems anti-clockwise. Hence, this side becomes the north pole. As the magnetic moment is from south to north pole. It is directed perpendicular to paper inwards. Further,

$$|\mathbf{M}| = NiA = \pi R^2 i$$

Refer figure (b) Here, opposite is the case. South pole is into the paper and north pole is outside the paper. Therefore, magnetic moment is perpendicular to paper in outward direction. The magnitude of **M** is

$$|\mathbf{M}| = a^2 i$$

Refer figure (c) In this case, south pole is on the right side of the loop and north pole on the left side. Hence, **M** is directed from right to left. The magnitude of magnetic moment is

$$|\mathbf{M}| = abi$$

(ii) Vector **M** is along the normal to the plane of the loop. The orientation (up or down along the normal) is given by the right hand rule. Wrap your fingers of the right hand around the perimeter of the loop in the direction of current as shown in figure. Then, extend your thumb so that it is perpendicular to the plane of the loop. The thumb points in the direction of **M**.

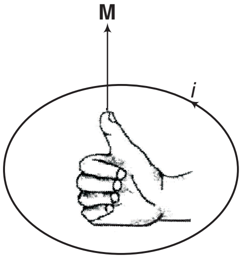


Fig. 26.19

Extra Points to Remember

- In addition to the method discussed above for finding **M** here are two more methods for calculating **M**.

  Method 1 This method is useful for calculating **M** for a rectangular or square loop.

  The magnetic moment (**M**) of the rectangular loop shown in figure is

  $$\mathbf{M} = i\,(\mathbf{AB}\times\mathbf{BC}) = i\,(\mathbf{BC}\times\mathbf{CD}) = i\,(\mathbf{CD}\times\mathbf{DA}) = i\,(\mathbf{DA}\times\mathbf{AB})$$

  Here, the cross product of any two consecutive sides (taken in order) gives the area as well as the correct direction of **M** also.

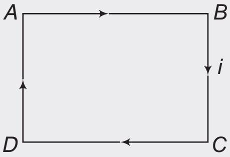
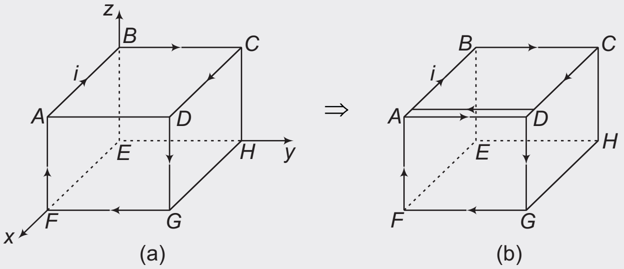


Fig 26.20

Note *If coordinates of vertices are known. Then, vector of any side can be written in terms of coordinates, e.g.*

$$\boxed{\mathbf{AB} = (x_B - x_A)\,\hat{\mathbf{i}} + (y_B - y_A)\,\hat{\mathbf{j}} + (z_B - z_A)\,\hat{\mathbf{k}}}$$

Method 2 Sometimes, a current carrying loop does not lie in a single plane. But by assuming two equal and opposite currents in one branch (which obviously makes no change in the given circuit) two (or more) closed loops are completed in different planes. Now, the net magnetic moment of the given loop is the vector sum of individual loops.



(a) (b)

Fig. 26.21

For example, in Fig. (a), six sides of a cube of side l carry a current i in the directions shown. By assuming two equal and opposite currents in wire AD, two loops in two different planes (xy and yz) are completed.

$$\mathbf{M}_{ABCDA} = -il^2\hat{\mathbf{k}}$$

$$\mathbf{M}_{ADGFA} = -il^2\,\hat{\mathbf{i}}$$

$$\therefore \quad \mathbf{M}_{\text{net}} = -il^2\,(\hat{\mathbf{i}} + \hat{\mathbf{k}})$$

Example 26.10 *A square loop OABCO of side l carries a current i. It is placed as shown in figure. Find the magnetic moment of the loop.*

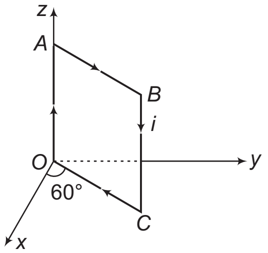


Fig. 26.22

Solution As discussed above, magnetic moment of the loop can be written as

$$\mathbf{M} = i\,(\mathbf{BC} \times \mathbf{CO})$$

Here, $\mathbf{BC} = -l\,\hat{\mathbf{k}}$, $\mathbf{CO} = -l\cos 60°\,\hat{\mathbf{i}} - l\sin 60°\,\hat{\mathbf{j}} = -\frac{l}{2}\hat{\mathbf{i}} - \frac{\sqrt{3}l}{2}\hat{\mathbf{j}}$

$\therefore$ $$\mathbf{M} = i\left[(-l\hat{\mathbf{k}}) \times \left(-\frac{l}{2}\hat{\mathbf{i}} - \frac{\sqrt{3}l}{2}\hat{\mathbf{j}}\right)\right]$$

or $$\mathbf{M} = \frac{il^2}{2}(\hat{\mathbf{j}} - \sqrt{3}\,\hat{\mathbf{i}})$$ **Ans.**

Example 26.11 *Find the magnitude of magnetic moment of the current carrying loop ABCDEFA. Each side of the loop is* 10 *cm long and current in the loop is i* = 2.0 *A.*

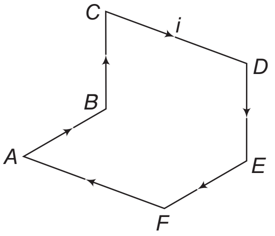


Fig. 26.23

Solution By assuming two equal and opposite currents in *BE*, two current carrying loops (*ABEFA* and *BCDEB*) are formed. Their magnetic moments are equal in magnitude but perpendicular to each other. Hence,

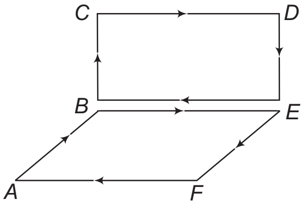


Fig. 26.24

$$M_{\text{net}} = \sqrt{M^2 + M^2} = \sqrt{2}\,M$$

where, $$M = iA = (2.0)(0.1)(0.1) = 0.02\ \text{A-m}^2$$

$\therefore$ $$M_{\text{net}} = (\sqrt{2})(0.02)\ \text{A-m}^2$$
$$= 0.028\ \text{A-m}^2$$ **Ans.**

26.6 Magnetic Dipole in Uniform Magnetic Field

Let us consider a rectangular ($a \times b$) current carrying loop $OACDO$ placed in xy-plane. A uniform magnetic field

$$\mathbf{B} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}$$

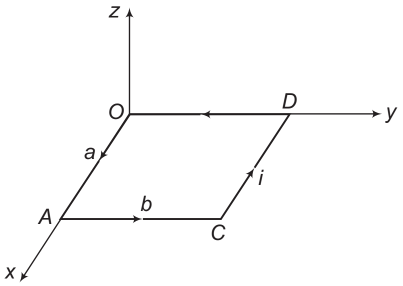


Fig. 26.25

exists in space.

We are interested in finding the net force and torque in the loop.

Force : Net force on the loop is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_{OA} + \mathbf{F}_{AC} + \mathbf{F}_{CD} + \mathbf{F}_{DO}\\ &= i\,[\,(\mathbf{OA} \times \mathbf{B}) + (\mathbf{AC} \times \mathbf{B}) + (\mathbf{CD} \times \mathbf{B}) + (\mathbf{DO} \times \mathbf{B})]\\ &= i\,[(\mathbf{OA} + \mathbf{AC} + \mathbf{CD} + \mathbf{DO}) \times \mathbf{B}]\\ &= \text{null vector}\end{aligned}$$

or $|\mathbf{F}| = 0$, as $\mathbf{OA} + \mathbf{AC} + \mathbf{CD} + \mathbf{DO}$ forms a null vector.

Torque : Using $\mathbf{F} = i\,(\mathbf{l} \times \mathbf{B})$, we have

$$\begin{aligned}\mathbf{F}_{OA} &= i\,(\mathbf{OA} \times \mathbf{B}) = i\,[(a\hat{\mathbf{i}}) \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})] = ia\,[B_y \hat{\mathbf{k}} - B_z \hat{\mathbf{j}}]\\ \mathbf{F}_{AC} &= i\,(\mathbf{AC} \times \mathbf{B}) = i\,[(b\hat{\mathbf{j}}) \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})] = ib\,[-B_x \hat{\mathbf{k}} + B_z \hat{\mathbf{i}}]\\ \mathbf{F}_{CD} &= i\,(\mathbf{CD} \times \mathbf{B}) = i\,[(-a\hat{\mathbf{i}}) \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})] = ia\,[-B_y \hat{\mathbf{k}} + B_z \hat{\mathbf{j}}]\\ \mathbf{F}_{DO} &= i\,(\mathbf{DO} \times \mathbf{B}) = i\,[(-b\hat{\mathbf{j}}) \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})] = ib\,[B_x \hat{\mathbf{k}} - B_z \hat{\mathbf{i}}]\end{aligned}$$

All these forces are acting at the centre of the wires. For example, $\mathbf{F}_{OA}$ will act at the centre of OA. When the forces are in equilibrium, net torque about any point remains the same. Let us find the torque about O.

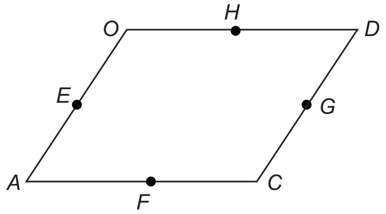


Fig. 26.26

E, F, G and H are the mid-points of OA, AC, CD and DO, respectively.

Using $\tau = \mathbf{r} \times \mathbf{F}$, we have

$$\tau_O = (\mathbf{OE} \times \mathbf{F}_{OA}) + (\mathbf{OF} \times \mathbf{F}_{AC}) + (\mathbf{OG} \times \mathbf{F}_{CD}) + (\mathbf{OH} \times \mathbf{F}_{DO})$$

$$= \left[\left(\frac{a}{2}\hat{\mathbf{i}}\right) \times \{ia\,(B_y\hat{\mathbf{k}} - B_z\hat{\mathbf{j}})\}\right] + \left[\left(a\hat{\mathbf{i}} + \frac{b}{2}\hat{\mathbf{j}}\right) \times \{ib\,(-B_x\hat{\mathbf{k}} + B_z\hat{\mathbf{i}})\}\right]$$

$$+ \left[\left(\frac{a}{2}\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right) \times \{ia\,(-B_y\hat{\mathbf{k}} + B_z\hat{\mathbf{j}})\}\right] + \left[\left(\frac{b}{2}\hat{\mathbf{j}}\right) \times ib\,(B_x\hat{\mathbf{k}} - B_z\hat{\mathbf{i}})\right]$$

$$= iab\,B_x\hat{\mathbf{j}} - iabB_y\hat{\mathbf{i}}$$

This can also be written as

$$\tau_O = (iab\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$

Here, $iab\hat{\mathbf{k}}$ = magnetic moment of the dipole **M**

and $B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} = \mathbf{B}$

$$\therefore \qquad \boxed{\tau = \mathbf{M} \times \mathbf{B}}$$

Note that although this formula has been derived for a rectangular loop, it comes out to be true for any shape of loop. The following points are worthnoting regarding the torque acting on the loop in uniform magnetic field.

(i) Magnitude of τ is $MB\sin\theta$ or $NiAB\sin\theta$. Here, θ is the angle between **M** and **B**. Torque is zero when $\theta = 0°$ or $180°$ and it is maximum at $\theta = 90°$.

(ii) If the loop is free to rotate in a magnetic field, the axis of rotation becomes an axis parallel to τ passing through the centre of mass of the loop.

The above equation for the torque is very similar to that of an electric dipole in an electric field. The similarity between electric and magnetic dipoles extends even further as illustrated in the table below.

Table 26.1

S.No.	Field of similarity	Electric dipole	Magnetic dipole
1.	Magnitude	$\lvert\mathbf{p}\rvert = q\,(2d)$	$\lvert\mathbf{M}\rvert = NiA$
2.	Direction	from $-q$ to $+q$	from S to N
3.	Net force in uniform field	zero	zero
4.	Torque	$\tau = \mathbf{p} \times \mathbf{E}$	$\tau = \mathbf{M} \times \mathbf{B}$
5.	Potential energy	$U = -\mathbf{p} \cdot \mathbf{E}$	$U = -\mathbf{M} \cdot \mathbf{B}$
6.	Work done in rotating the dipole	$W_{\theta_1 - \theta_2} = pE(\cos\theta_1 - \cos\theta_2)$	$W_{\theta_1 - \theta_2} = MB(\cos\theta_1 - \cos\theta_2)$
7	Field along axis	$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\mathbf{p}}{r^3}$	$\mathbf{B} = \frac{\mu_0}{4\pi} \cdot \frac{2\mathbf{M}}{r^3}$
8.	Field perpendicular to axis	$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{\mathbf{p}}{r^3}$	$\mathbf{B} = -\frac{\mu_0}{4\pi} \cdot \frac{\mathbf{M}}{r^3}$

Note *In last two points $r >>$ size of loop.*

Note that the expressions for the magnetic dipole can be obtained from the expressions for the electric dipole by replacing **p** by **M** and ε_0 by $\frac{1}{\mu_0}$. Here, μ_0 is called the permeability of free space. It is related with ε_0 and speed of light c as

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$$

and it has the value,

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ T-m/A}}$$

Dimensions of $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ are that of speed or $[\text{LT}^{-1}]$.

Hence,
$$\boxed{\left[\frac{1}{\sqrt{\varepsilon_0 \mu_0}}\right] = [\text{LT}^{-1}]}$$

Example 26.12 *A circular loop of radius $R = 20$ cm is placed in a uniform magnetic field $\mathbf{B} = 2\,T$ in xy-plane as shown in figure. The loop carries a current $i = 1.0$ A in the direction shown in figure. Find the magnitude of torque acting on the loop.*

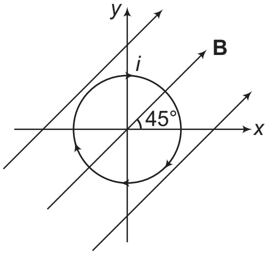


Fig. 26.27

Solution Magnitude of torque is given by

$$|\tau| = MB \sin\theta$$

Here, $M = NiA = (1)(1.0)(\pi)(0.2)^2$

$$= (0.04\,\pi)\text{ A-m}^2$$

$$B = 2\text{T}$$

and θ = angle between **M** and **B** = 90°

$\therefore$ $|\tau| = (0.04\,\pi)(2)\sin 90°$

$= 0.25$ N-m **Ans.**

Note **M** *is along negative z-direction (perpendicular to paper inwards) while* **B** *is in xy-plane. So, the angle between* **M** *and* **B** *is 90° not 45°. If the direction of torque is also desired, then we can write*

$$\mathbf{B} = 2\cos 45°\,\hat{\mathbf{i}} + 2\sin 45°\,\hat{\mathbf{j}} = \sqrt{2}\,(\hat{\mathbf{i}} + \hat{\mathbf{j}})\,T$$

and $\mathbf{M} = -(0.04\,\pi)\,\hat{\mathbf{k}}\ A\text{-}m^2$

$\therefore$ $\tau = \mathbf{M} \times \mathbf{B} = (0.04\,\sqrt{2}\pi)\,(-\hat{\mathbf{j}} + \hat{\mathbf{i}})$

or $\tau = 0.18\,(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ ***Ans.***

INTRODUCTORY EXERCISE 26.4

1. A charge q is uniformly distributed on a non-conducting disc of radius R. It is rotated with an angular speed ω about an axis passing through the centre of mass of the disc and perpendicular to its plane. Find the magnetic moment of the disc.

   [**Hint :** For any charge distribution : Magnetic moment $= \left(\dfrac{q}{2m}\right)$ (angular momentum)]

2. A circular loop of wire having a radius of 8.0 cm carries a current of 0.20 A. A vector of unit length and parallel to the dipole moment **M** of the loop is given by $0.60\,\hat{\mathbf{i}} - 0.80\,\hat{\mathbf{j}}$. If the loop is located in uniform magnetic field given by $\mathbf{B} = (0.25\text{ T})\,\hat{\mathbf{i}} + (0.30\text{ T})\,\hat{\mathbf{k}}$ find,

   (a) the torque on the loop and

   (b) the magnetic potential energy of the loop.

3. A length L of wire carries a current i. Show that if the wire is formed into a circular coil, then the maximum torque in a given magnetic field is developed when the coil has one turn only, and that maximum torque has the magnitude $\tau = L^2 iB / 4\pi$.

4. A coil with magnetic moment 1.45 A-m^2 is oriented initially with its magnetic moment antiparallel to a uniform 0.835 T magnetic field. What is the change in potential energy of the coil when it is rotated 180° so that its magnetic moment is parallel to the field?

26.7 Biot Savart Law

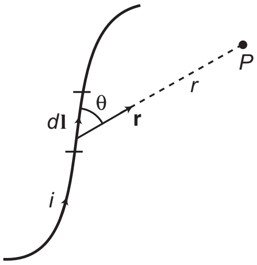


Fig. 26.28

In the preceding articles, we discussed the magnetic force exerted on a charged particle and current carrying conductor in a magnetic field. To complete the description of the magnetic interaction, this and the next article deals with the origin of the magnetic field. As in electrostatics, there are two methods of calculating the electric field at some point. One is Coulomb's law which gives the electric field due to a point charge and the another is Gauss's law which is useful in calculating the electric field of a highly symmetric configuration of charge. Similarly, in magnetics, there are basically two methods of calculating magnetic field at some point. One is **Biot Savart law** which gives the magnetic field due to an infinitesimally small current carrying wire at some point and the another is **Ampere's law**, which is useful in calculating the magnetic field of a highly symmetric configuration carrying a steady current.

We begin by showing how to use the law of Biot and Savart to calculate the magnetic field produced at some point in space by a small current element. Using this formalism and the principle of superposition, we then calculate the total magnetic field due to various current distributions.

From their experimental results, Biot and Savart arrived at a mathematical expression that gives the magnetic field at some point in space in terms of the current that produces the field. That expression is based on the following experimental observations for the magnetic field $d\mathbf{B}$ at a point P associated with a length element $d\mathbf{l}$ of a wire carrying a steady current i.

(i) The vector $d\mathbf{B}$ is perpendicular to both $d\mathbf{l}$ (which points in the direction of the current) and the unit vector $\hat{\mathbf{r}}$ directed from $d\mathbf{l}$ to P.

(ii) The magnitude of $d\mathbf{B}$ is inversely proportional to r^2, where r is the distance from $d\mathbf{l}$ to P.

(iii) The magnitude of $d\mathbf{B}$ is proportional to the current and to the magnitude dl of the length element $d\mathbf{l}$.

(iv) The magnitude of $d\mathbf{B}$ is proportional to $\sin\theta$ where θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{r}}$. These observations are summarized in mathematical formula known today as **Biot Savart law**

$$\boxed{d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i\,(d\mathbf{l}\times\hat{\mathbf{r}})}{r^2}} \qquad \ldots\text{(i)}$$

Here, $$\frac{\mu_0}{4\pi} = 10^{-7}\,\frac{\text{T-m}}{\text{A}}$$

It is important to note that $d\mathbf{B}$ in Eq. (i) is the field created by the current in only a small length element $d\mathbf{l}$ of the conductor. To find the total magnetic field $\mathbf{B}$ created at some point by a current of finite size, we must sum up contributions from all current elements that make up the current. That is, we must evaluate $\mathbf{B}$ by integrating Eq. (i).

$$\boxed{\mathbf{B} = \frac{\mu_0 i}{4\pi}\int\frac{d\mathbf{l}\times\hat{\mathbf{r}}}{r^2}}$$

where, the integral is taken over the entire current distribution. This expression must be handled with special care because the integrand is a cross product and therefore, a vector quantity.

The following points are worthnoting regarding the Biot Savart law.

(i) Magnitude of $d\mathbf{B}$ is given by

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2}$$

$|d\mathbf{B}|$ is zero at $\theta = 0°$ or $180°$ and maximum at $\theta = 90°$.

(ii) For the direction of $d\mathbf{B}$ either of the following methods can be employed.

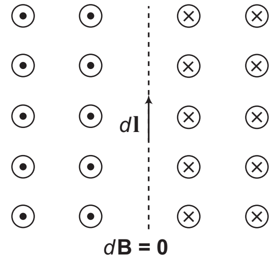


Fig. 26.29

(a) $d\mathbf{B} \uparrow\uparrow d\mathbf{l}\times\hat{\mathbf{r}}$. So, $d\mathbf{B}$ is along $d\mathbf{l}\times\mathbf{r}$.

(b) If $d\mathbf{l}$ is in the plane of paper. $d\mathbf{B} = 0$ at all points lying on the straight line passing through $d\mathbf{l}$. The magnetic field to the right of this line is in ⊗ direction and to the left of this line is in ⊙ direction.

26.8 Applications of Biot Savart Law

Let us now consider few applications of Biot Savart law.

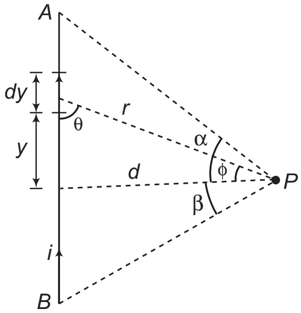


Fig. 26.30

Magnetic Field Surrounding a Thin, Straight Conductor

According to Biot Savart law,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{i d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \qquad \ldots\text{(i)}$$

As here every element of the wire contributes to **B** in the same direction (which is here ⊗).

Eq. (i) for this case becomes,

$$B = \frac{\mu_0}{4\pi}\int \frac{idl\sin\theta}{r^2} = \frac{\mu_0 i}{4\pi}\int \frac{dy\sin\theta}{r^2}$$

$$y = d\tan\phi \quad \text{or} \quad dy = (d\sec^2\phi)d\phi$$

$$r = d\sec\phi \quad \text{and} \quad \theta = 90° - \phi$$

$$B = \frac{\mu_0 i}{4\pi}\int_{\phi=-\beta}^{\phi=\alpha} \frac{\{(d\sec^2\phi)\,d\phi\}\sin(90° - \phi)}{(d\sec\phi)^2}$$

or $$B = \frac{\mu_0 i}{4\pi d}\int_{-\beta}^{\alpha}\cos\phi \cdot d\phi \quad \text{or} \quad \boxed{B = \frac{\mu_0}{4\pi}\frac{i}{d}(\sin\alpha + \sin\beta)}$$

Note down the following points regarding the above equation.

(i) For an infinitely long straight wire, $\alpha = \beta = 90°$

$$\therefore \qquad \sin\alpha + \sin\beta = 2 \quad \text{or} \quad \boxed{B = \frac{\mu_0}{2\pi}\frac{i}{d}}$$

(ii) The direction of magnetic field at a point P due to a long straight wire can be found by the right hand thumb rule. If we stretch the thumb of the right hand along the current and curl our fingers to pass through P, the direction of the fingers at P gives the direction of magnetic field there.

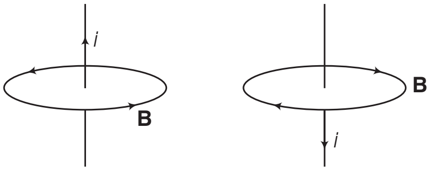


Fig. 26.31

(iii) $B \propto \frac{1}{d}$, i.e. B-d graph for an infinitely long straight wire is a rectangular hyperbola as shown in the figure.

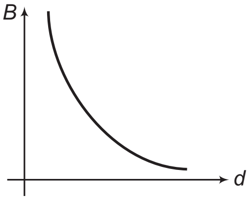


Fig. 26.32

Magnetic Field on the Axis of a Circular Coil

Suppose a current carrying circular loop has a radius R. Current in the loop is i. We want to find the magnetic field at a point P on the axis of the loop a distance z from the centre.

We can take the loop in xy-plane with its centre at origin and point P on the z-axis.

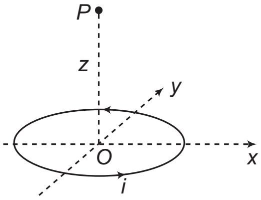


Fig. 26.33

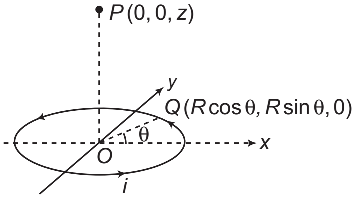


Fig. 26.34

Let us take a small current element at angle θ as shown.

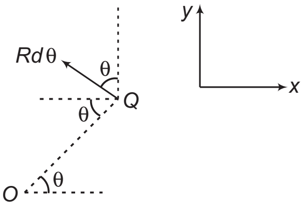


Fig. 26.35

$$P \equiv (0, 0, z)$$

$$Q \equiv (R\cos\theta, R\sin\theta, 0)$$

$$d\mathbf{l} = -(Rd\theta)\sin\theta\hat{\mathbf{i}} + (Rd\theta)\cos\theta\hat{\mathbf{j}}$$

$$\hat{\mathbf{r}} = \text{unit vector along } QP$$

$$= \frac{(-R\cos\theta\hat{\mathbf{i}} - R\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{r}$$

Here, $\quad r = \text{distance } QP = \sqrt{R^2 + z^2}$

Now, magnetic field at point P, due to current element $d\,\mathbf{l}$ at Q is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i}{r^2}(d\mathbf{l} \times \hat{\mathbf{r}})$$

$$= \frac{\mu_0}{4\pi}\frac{i}{r^3}[(-R\sin\theta\, d\theta\hat{\mathbf{i}} + R\cos\theta\, d\theta\hat{\mathbf{j}}) \times (-R\cos\theta\hat{\mathbf{i}} - R\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}})]$$

or $\quad$ $$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i}{r^3}[(zR\cos\theta\, d\theta)\hat{\mathbf{i}} + (zR\sin\theta\, d\theta)\hat{\mathbf{j}} + (R^2 d\theta)\hat{\mathbf{k}}]$$

$$= dB_x\hat{\mathbf{i}} + dB_y\hat{\mathbf{j}} + dB_z\hat{\mathbf{k}}$$

Here, $\quad$ $$dB_x = \frac{\mu_0}{4\pi}\frac{i}{r^3}(zR\cos\theta d\theta)$$

$$dB_y = \frac{\mu_0}{4\pi}\frac{i}{r^3}(zR\sin\theta d\theta)$$

and
$$dB_z = \frac{\mu_0}{4\pi}\frac{i}{r^3}(R^2 d\theta)$$

Integrating these differentials from $\theta = 0°$ to $\theta = 2\pi$ for the complete loop, we get

$$B_x = \frac{\mu_0}{4\pi}\frac{ziR}{r^3}\int_0^{2\pi}\cos\theta\, d\theta = 0$$

$$B_y = \frac{\mu_0}{4\pi}\frac{ziR}{r^3}\int_0^{2\pi}\sin\theta\, d\theta = 0$$

and
$$B_z = \frac{\mu_0}{4\pi}\frac{iR^2}{r^3}\int_0^{2\pi} d\theta = \frac{\mu_0}{2}\frac{iR^2}{r^3}$$

Substituting $r = (R^2 + z^2)^{1/2}$, we get $B_P = B_z = \dfrac{\mu_0 iR^2}{2(R^2 + z^2)^{3/2}}$

For N number of loops,
$$\boxed{B = \frac{\mu_0 NiR^2}{2(R^2 + z^2)^{3/2}}}$$

Note down the following points regarding a circular current carrying loop.

(i) At the centre of the loop, $z = 0$

and
$$\boxed{B\text{ (centre)} = \frac{\mu_0 Ni}{2R}}$$

(ii) For $z >> R$, $z^2 + R^2 \approx z^2$

$$\therefore \qquad B = \frac{\mu_0 NiR^2}{2z^3} = \left(\frac{\mu_0}{4\pi}\right)\frac{(2Ni\pi R^2)}{z^3} = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{2M}{z^3}\right)$$

Here, M = magnetic moment of the loop = $NiA = Ni\pi R^2$.

This result was expected as the magnetic field on the axis of a dipole is $\dfrac{\mu_0}{4\pi}\dfrac{2M}{r^3}$.

(iii) Direction of magnetic field on the axis of a circular loop can be obtained using the right hand thumb rule. If the fingers are curled along the current, the stretched thumb will point towards the magnetic field.

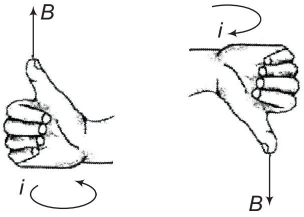


Fig. 26.36

(iv) The magnetic field at a point not on the axis is mathematically difficult to calculate. We have shown qualitatively in figure the magnetic field lines due to a circular current.

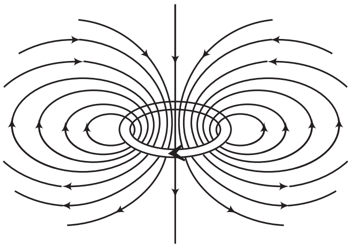


Fig. 26.37

(v) Magnetic field is maximum at the centre and decreases as we move away from the centre (on the axis of the loop). The B-z graph is somewhat like shown in figure.

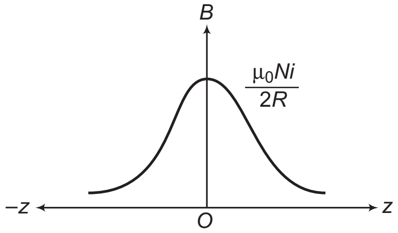


Fig. 26.38

(vi) Magnetic field due to an arc of a circle at the centre is

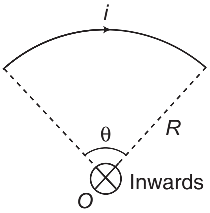


Fig. 26.39

$$B = \left(\frac{\theta}{2\pi}\right)\frac{\mu_0 i}{2R} = \frac{\mu_0}{4\pi}\left(\frac{i}{R}\right)\theta$$

or

$$\boxed{B = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{i}{R}\right)\theta}$$

Here, θ is to be substituted in radians.

Field Along the Axis of a Solenoid

The name solenoid was first given by Ampere to a wire wound in a closely spaced spiral over a hollow cylindrical non-conducting core. If n is the number of turns per unit length, each carries a current i uniformly wound round a cylinder of radius R, the number of turns in length dx are ndx. Thus, the magnetic field at the axial point O due to this element dx is

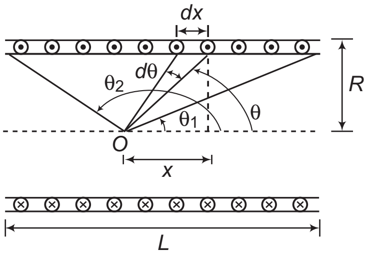


Fig. 26.40

$$dB = \frac{\mu_0}{2}\frac{(indx)R^2}{(R^2+x^2)^{3/2}} \qquad \ldots\text{(i)}$$

Its direction is along the axis of the solenoid. From the geometry, we know

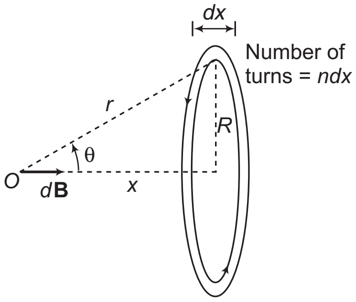


Fig. 26.41

$$x = R\cot\theta$$

$$dx = -R\,\text{cosec}^2\,\theta \cdot d\theta$$

Substituting these values in Eq. (i), we get

$$dB = -\frac{1}{2}\mu_0 ni\sin\theta\cdot d\theta$$

Total field B due to the entire solenoid is

$$B = \frac{1}{2}\mu_0 ni\int_{\theta_1}^{\theta_2}(-\sin\theta)\,d\theta$$

$$\therefore \qquad \boxed{B = \frac{\mu_0 ni}{2}(\cos\theta_2 - \cos\theta_1)}$$

If the solenoid is very long ($L >> R$) and the point O is chosen at the middle, i.e. if $\theta_1 = 180°$ and $\theta_2 = 0°$, then we get

$$\boxed{B\text{ (centre)} = \mu_0 ni} \qquad \text{[For } L >> R\text{]}$$

At the end of the solenoid,

$\theta_2 = 0°, \quad \theta_1 = 90°$ and we get

$$\boxed{B\text{ (end)} = \frac{1}{2}\mu_0 ni} \qquad \text{[For } L >> R\text{]}$$

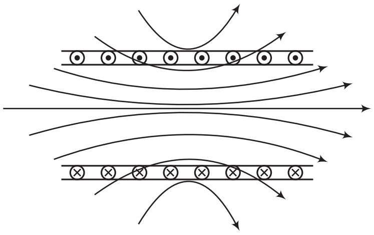
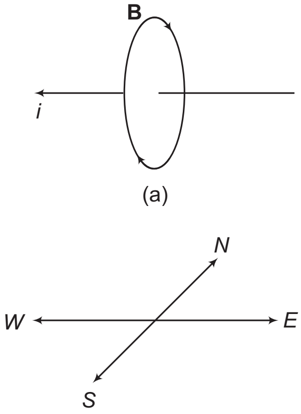


Fig. 26.42

Thus, the field at the end of a solenoid is just one half at the centre. The field lines are as shown in Fig. 26.42.

Example 26.13 *In a high tension wire electric current runs from east to west. Find the direction of magnetic field at points above and below the wire.*



(a)

(b)

Fig. 26.43

Solution When the current flows from east to west, magnetic field lines are circular round it as shown in figure (a). And so, the magnetic field above the wire is towards north and below the wire towards south.

Example 26.14 *A current path shaped as shown in figure produces a magnetic field at P, the centre of the arc. If the arc subtends an angle of* 30° *and the radius of the arc is* 0.6 *m, what are the magnitude and direction of the field produced at P if the current is* 3.0 *A.*

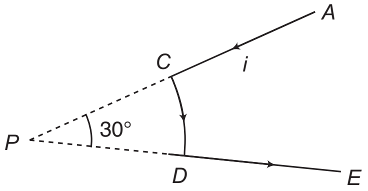


Fig. 26.44

Solution The magnetic field at P due to the straight segments AC and DE is zero.

CD is arc of circle.

$$\therefore \quad B = \left(\frac{\theta}{2\pi}\right)\left(\frac{\mu_0 i}{2R}\right) \qquad (N = 1)$$

or

$$B = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{i}{R}\right)\theta$$

$$\therefore \quad B = (10^{-7})\left(\frac{3.0}{0.6}\right)\left(\frac{\pi}{6}\right)$$

$$= 2.62 \times 10^{-7}\ \text{T}$$ **Ans.**

Example 26.15 *Figure shows a current loop having two circular arcs joined by two radial lines. Find the magnetic field B at the centre O.*

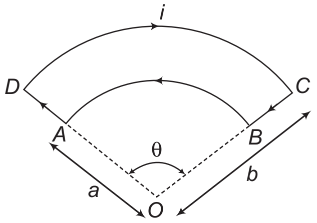


Fig. 26.45

Solution Magnetic field at point O, due to wires CB and AD will be zero.

Magnetic field due to wire BA will be

$$B_1 = \left(\frac{\theta}{2\pi}\right)\left(\frac{\mu_0 i}{2a}\right)$$

Direction of field $\mathbf{B}_1$ is coming out of the plane of the figure.

Similarly, field at O due to arc DC will be

$$B_2 = \left(\frac{\theta}{2\pi}\right)\left(\frac{\mu_0 i}{2b}\right)$$

Direction of field $\mathbf{B}_2$ is going into the plane of the figure. The resultant field at O is

$$B = B_1 - B_2 = \frac{\mu_0 i \theta (b - a)}{4\pi ab}$$ **Ans.**

Coming out of the plane.

Example 26.16 *The magnetic field B due to a current carrying circular loop of radius* 12 *cm at its centre is* 0.5×10^{-4} *T. Find the magnetic field due to this loop at a point on the axis at a distance of* 5.0 *cm from the centre.*

Solution Magnetic field at the centre of a circular loop is

$$B_1 = \frac{\mu_0 i}{2R}$$

and that at an axial point,

$$B_2 = \frac{\mu_0 i R^2}{2(R^2 + x^2)^{3/2}}$$

Thus,

$$\frac{B_2}{B_1} = \frac{R^3}{(R^2 + x^2)^{3/2}}$$

or

$$B_2 = B_1 \left[\frac{R^3}{(R^2 + x^2)^{3/2}}\right]$$

Substituting the values, we have

$$B_2 = (0.5 \times 10^{-4}) \left[\frac{(12)^3}{(144 + 25)^{3/2}}\right]$$

$$= 3.9 \times 10^{-5} \text{ T}$$ **Ans.**

INTRODUCTORY EXERCISE 26.5

1. (a) A conductor in the shape of a square of edge length $l = 0.4$ m carries a current $i = 10.0$ A. Calculate the magnitude and direction of magnetic field at the centre of the square.

   (b) If this conductor is formed into a single circular turn and carries the same current, what is the value of the magnetic field at the centre.

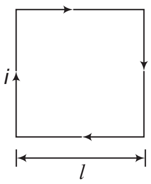


Fig. 26.46

2. Determine the magnetic field at point P located a distance x from the corner of an infinitely long wire bent at right angle as shown in figure. The wire carries a steady current i.

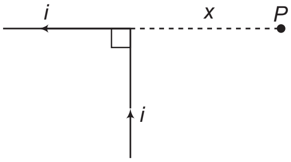


Fig. 26.47

3. A conductor consists of a circular loop of radius $R = 10$ cm and two straight, long sections as shown in figure. The wire lies in the plane of the paper and carries a current of $i = 7.00$ A. Determine the magnitude and direction of the magnetic field at the centre of the loop.

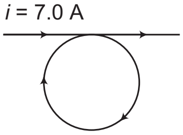


Fig. 26.48

4. The segment of wire shown in figure carries a current of $i = 5.0$ A, where the radius of the circular arc is $R = 3.0$ cm. Determine the magnitude and direction of the magnetic field at the origin. (Fig. 26.49)

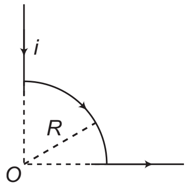


Fig. 26.49

5. Consider the current carrying loop shown in figure formed of radial lines and segments of circles whose centres are at point P. Find the magnitude and direction of **B** at point P. (Fig. 26.50)

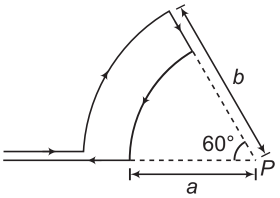


Fig. 26.50

26.9 Ampere's Circuital Law

The electrical force on a charge is related to the electric field (caused by other charges) by the equation,

$$\mathbf{F}_e = q\mathbf{E}$$

Just like the gravitational force, the static electrical force is a conservative force. This means that the work done by the static electric force around any closed path is zero.

$$q \oint \mathbf{E} \cdot d\mathbf{l} = 0 \text{ J}$$

Hence, we have

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \text{ V}$$

In other words, the integral of the static (time independent) electric field around a closed path is zero.

What about the integral of the magnetic field around a closed path? That is, we want to determine the value of

$$\oint \mathbf{B} \cdot d\mathbf{l}$$

Here, we have to be careful. The quantity $\mathbf{B} \cdot d\mathbf{l}$ does not represent some physical quantity, and certainly not work. Although the static magnetic force does no work on a moving charge, we cannot conclude that the path integral of the magnetic field around a closed path is zero. We are just curious about what this analogous line integral amounts to.

The line integral $\oint \mathbf{B} \cdot d\mathbf{l}$ of the resultant magnetic field along a closed, plane curve is equal to μ_0 times the total current crossing the area bounded by the closed curve provided the electric field inside the loop remains constant. Thus,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \, (i_{\text{net}}) \qquad \ldots\text{(i)}$$

This is known as **Ampere's circuital law**.

Eq. (i) in simplified form can be written as

$$\boxed{Bl = \mu_0 \, (i_{\text{net}})} \qquad \ldots\text{(ii)}$$

But this equation can be used only under the following conditions.

(i) At every point of the closed path $\mathbf{B} \parallel d\mathbf{l}$.

(ii) Magnetic field has the same magnitude B at all places on the closed path.

If this is not the case, then Eq. (i) is written as

$$B_1 dl_1 \cos\theta_1 + B_2 dl_2 \cos\theta_2 + \ldots = \mu_0 \, (i_{\text{net}})$$

Here, θ_1 is the angle between $\mathbf{B}_1$ and $d\mathbf{l}_1$, θ_2 the angle between $\mathbf{B}_2$ and $d\mathbf{l}_2$ and so on. Besides the Biot Savart law, Ampere's law gives another method to calculate the magnetic field due to a given current distribution. Ampere's law may be derived from the Biot Savart law and Bio Savart law may be derived from the Ampere's law. However, Ampere's law is more useful under certain symmetrical conditions. To illustrate the theory now let us take few applications of Ampere's circuital law.

Magnetic Field Created by a Long Current Carrying Wire

A long straight wire of radius R carries a steady current i that is uniformly distributed through the cross-section of the wire.

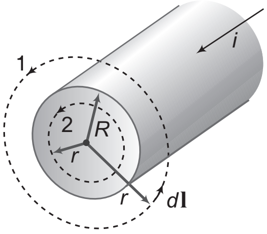


Fig. 26.51

For finding the behaviour of magnetic field due to this wire, let us divide the whole region into two parts. One is $r \geq R$ and the another is $r < R$. Here, r is the distance from the centre of the wire.

For $r \geq R$: Let us choose for our path of integration circle 1. From symmetry $\mathbf{B}$ must be constant in magnitude and parallel to $d\mathbf{l}$ at every point on this circle. Because the total current passing through the plane of the circle is i. Ampere's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{net}}$$

or $$Bl = \mu_0 i \qquad \text{[simplified form]}$$

or $$B(2\pi r) = \mu_0 i$$

$\therefore$ $$\boxed{B = \frac{\mu_0}{2\pi}\frac{i}{r}} \qquad [\text{for } r \geq R] \quad \dots\text{(iii)}$$

For $r < R$: Here, the current i' passing through the plane of circle 2 is less than the total current i. Because the current is uniform over the cross-section of the wire, the fraction of the current enclosed by circle 2 must equal the ratio of the area πr^2 enclosed by circle 2 to the cross-sectional area πR^2 of the wire.

$$\frac{i'}{i} = \frac{\pi r^2}{\pi R^2} \quad \Rightarrow \quad i' = \left(\frac{r^2}{R^2}\right) i$$

Then, the following procedure same as for circle 1, we apply Ampere's law to circle 2.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{net}}$$

$$Bl = \mu_0 i' \qquad \text{[simplified form]}$$

$\therefore$ $$B\,(2\pi r) = \mu_0 \left(\frac{r^2}{R^2}\right) i$$

$\therefore$ $$\boxed{B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r} \qquad [\text{For } r < R] \quad \dots\text{(iv)}$$

This result is similar in the form to the expression for the electric field inside a uniformly charged sphere. The magnitude of the magnetic field *versus* r for this configuration is plotted in figure. Note that inside the wire $B \to 0$ as $r \to 0$. Note also that Eqs. (iii) and (iv) give the same value of the magnetic field at $r = R$, demonstrating that the magnetic field is continuous at the surface of the wire.

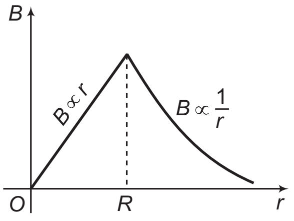


Fig. 26.52

Magnetic Field of a Solenoid

A solenoid is a long wire wound in the form of a helix. With this configuration, a reasonably uniform magnetic field can be produced in the space surrounded by the turns of wire, which we shall call the interior of the solenoid, when the solenoid carries a current. When the turns are closely spaced, each can be approximated as a circular loop, and the net magnetic field is the vector sum of the fields resulting from all the turns (as done in Art. 26.9). If the turns are closely spaced and the solenoid is of infinite length, the magnetic field lines are as shown in Fig. 26.53.

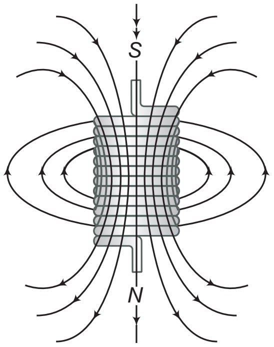


Fig. 26.53

One end of the solenoid behaves like the north pole (↺) and the opposite end behaves like the south pole (↻). As the length of the solenoid increases, the interior field becomes more uniform and the exterior field becomes weaker. An ideal solenoid is approached when the turns are closely spaced and the length is much greater than the radius of the turns.

In this case, the external field is zero, and the interior field is uniform over a great volume.

We can use Ampere's law to obtain an expression for the interior magnetic field in an ideal solenoid. Fig. 26.54 shows a longitudinal cross-section of part of such a solenoid carrying a current *i.* Because the solenoid is ideal, **B** in the interior space is uniform and parallel to the axis, and **B** in the exterior space is zero.

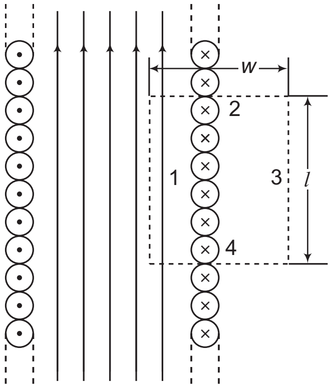


Fig. 26.54

Consider the rectangular path of length *l* and width *w* as shown in figure. We can apply Ampere's law to this path by evaluating the line integral $\mathbf{B}\cdot d\mathbf{l}$ over each side of the rectangle.

$$\int (\mathbf{B}\cdot d\mathbf{l})_{\text{side 3}} = 0 \qquad \text{as} \qquad B = 0$$

$$\int (\mathbf{B}\cdot d\mathbf{l})_{\text{side 2 and 4}} = 0 \qquad \text{as} \qquad \mathbf{B}\perp d\mathbf{l} \text{ or } B = 0 \text{ along these paths}$$

$$\int (\mathbf{B}\cdot d\mathbf{l})_{\text{side 1}} = Bl \qquad \text{as} \qquad \mathbf{B} \text{ is uniform and parallel to } d\mathbf{l}$$

The integral over the closed rectangular path is therefore,

$$\oint \mathbf{B}\cdot d\mathbf{l} = Bl$$

The right side of Ampere's law involves the total current passing through the area bounded by the path of integration.

In this case,

$$i_{\text{net}} = \text{(number of turns inside the area) (current through each turn)}$$

$$= (nl)\,(i) \qquad (n = \text{number of turns per unit length})$$

Using Ampere's law, $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_{\text{net}}$

or $Bl = (\mu_0)\,(nli)$ or $\boxed{B = \mu_0 ni}$ ...(v)

This result is same as obtained in Art. 26.9. Eq. (v) is valid only for points near the centre (that is far from the ends) of a very long solenoid. The field near each end is half the value given by Eq. (v).

Example 26.17 *A closed curve encircles several conductors. The line integral* $\int \mathbf{B}\cdot d\mathbf{l}$ *around this curve is* 3.83×10^{-7} *T-m.*

(a) What is the net current in the conductors?

(b) If you were to integrate around the curve in the opposite directions, what would be the value of the line integral?

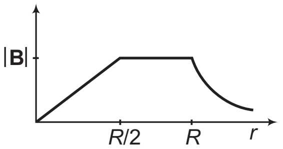
Solution (a) $\int \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_{\text{net}}$

$$\therefore \qquad i_{\text{net}} = \frac{\int \mathbf{B}\cdot d\mathbf{l}}{\mu_0} = \frac{3.83\times 10^{-7}}{4\pi \times 10^{-7}} = 0.3\,\text{A}$$

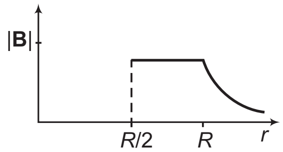
(b) In opposite direction, line integral will be negative.

Example 26.18 *An infinitely long hollow conducting cylinder with inner radius R/2 and outer radius R carries a uniform current density along its length. The magnitude of the magnetic field,* $|\mathbf{B}|$ *as a function of the radial distance r from the axis is best represented by* **(JEE 2012)**

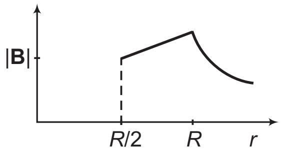
(a)



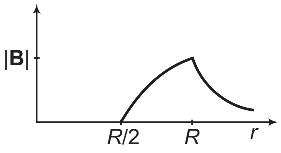
(b)



(c)



(d)



Solution

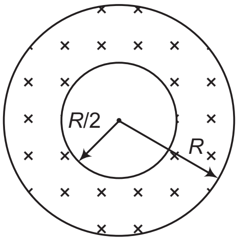


Fig. 26.55

r = distance of a point from centre

For $r \le R/2$ Using Ampere's circuital law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{net}}$$

or $$Bl = \mu_0 \, (I_{\text{in}})$$

or $$B\,(2\pi r) = \mu_0 \, (I_{\text{in}})$$

or $$B = \frac{\mu_0}{2\pi} \frac{I_{\text{in}}}{r} \qquad \dots\text{(i)}$$

Since, $$I_{\text{in}} = 0$$

∴ $$B = 0$$

For $\frac{R}{2} \le r \le R$ $$I_{\text{in}} = \left[\pi r^2 - \pi\left(\frac{R}{2}\right)^2\right]\sigma$$

Here, σ = current per unit area

Substituting in Eq. (i), we have

$$B = \frac{\mu_0}{2\pi} \frac{\left[\pi r^2 - \pi \frac{R^2}{4}\right]\sigma}{r}$$

$$= \frac{\mu_0 \sigma}{2r}\left(r^2 - \frac{R^2}{4}\right)$$

At $r = \frac{R}{2}$, $B = 0$

At $r = R$, $B = \frac{3\mu_0 \sigma R}{8}$

For $r \ge R$ $I_{\text{in}} = I_{\text{Total}} = I$ (say)

Therefore, substituting in Eq. (i), we have

$$B = \frac{\mu_0}{2\pi} \cdot \frac{I}{r} \quad \text{or} \quad B \propto \frac{1}{r}$$

∴ The correct graph is (d).

Example 26.19 *A device called a toroid (figure) is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring (a torus) made of a non-conducting material. For a toroid having N closely spaced turns of wire, calculate the magnetic field in the region occupied by the torus, a distance r from the centre.*

Solution To calculate this field, we must evaluate $\oint \mathbf{B} \cdot d\mathbf{l}$ over the circle of radius r. By symmetry we see that the magnitude of the field is constant on this circle and tangent to it.

So, $$\oint \mathbf{B} \cdot d\mathbf{l} = Bl = B\,(2\pi r)$$

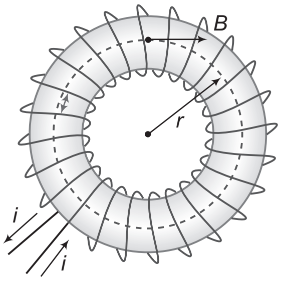


Fig. 26.56

Furthermore, the circular closed path surrounds N loops of wire, each of which carries a current i. Therefore, right side of Eq. (i) is $\mu_0 Ni$ in this case.

$\therefore$ $$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{net}}$$

or $$B(2\pi r) = \mu_0 Ni$$

or $$B = \frac{\mu_0 Ni}{2\pi r}$$

This result shows that $B \propto \frac{1}{r}$ and hence is non-uniform in the region occupied by torus. However, if r is very large compared with the cross-sectional radius of the torus, then the field is approximately uniform inside the torus. In that case,

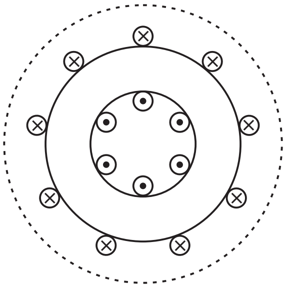


Fig. 26.57

$\frac{N}{2\pi r} = n =$ number of turns per unit length of torus

$\therefore$ $$B = \mu_0 ni$$

For an ideal toroid, in which turns are closely spaced, the external magnetic field is zero. This is because the net current passing through any circular path lying outside the toroid is zero. Therefore, from Ampere's law we find that $B = 0$, in the regions exterior to the torus.

INTRODUCTORY EXERCISE 26.6

1. Figure given in the question is a cross-sectional view of a coaxial cable. The centre conductor is surrounded by a rubber layer, which is surrounded by an outer conductor, which is surrounded by another rubber layer. The current in the inner conductor is 1.0 A out of the page, and the current in the outer conductor is 3.0 A into the page. Determine the magnitude and direction of the magnetic field at points a and b.

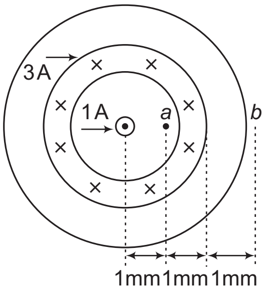


Fig. 26.58

2. Figure shows, in cross-section, several conductors that carry currents through the plane of the figure. The currents have the magnitudes $I_1 = 4.0$ A, $I_2 = 6.0$ A, and $I_3 = 2.0$ A, in the directions shown. Four paths labelled a to d, are shown. What is the line integral $\int \mathbf{B} \cdot d\mathbf{l}$ for each path? Each integral involves going around the path in the counter-clockwise direction.

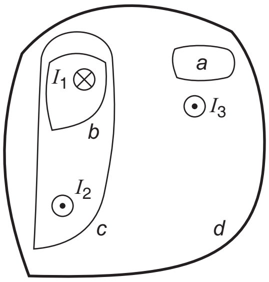


Fig. 26.59

3. A current I flows along the length of an infinitely long, straight, thin-walled pipe. Then, **(JEE 1993)**
   (a) the magnetic field at all points inside the pipe is the same, but not zero
   (b) the magnetic field at any point inside the pipe is zero
   (c) the magnetic field is zero only on the axis of the pipe
   (d) the magnetic field is different at different points inside the pipe

26.10 Force Between Parallel Current Carrying Wires

Consider two long wires 1 and 2 kept parallel to each other at a distance r and carrying currents i_1 and i_2 respectively in the same direction.

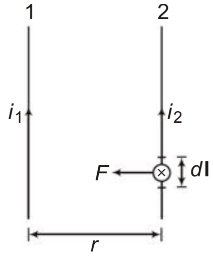


Fig. 26.60

Magnetic field on wire 2 due to current in wire 1 is, $B = \frac{\mu_0}{2\pi} \cdot \frac{i_1}{r}$ [in ⊗ direction]

Magnetic force on a small element dl of wire 2 due to this magnetic field is

$$d\mathbf{F} = i_2 (d\mathbf{l} \times \mathbf{B})$$

Magnitude of this force is $dF = i_2\, [(dl)\,(B) \sin 90°\,]$

$$= i_2\,(dl)\left(\frac{\mu_0}{2\pi}\,\frac{i_1}{r}\right) = \frac{\mu_0}{2\pi} \cdot \frac{i_1 i_2}{r} \cdot dl$$

Direction of this force is along $d\mathbf{l} \times \mathbf{B}$ or towards the wire 1.

The force per unit length of wire 2 due to wire 1 is

$$\boxed{\frac{dF}{dl} = \frac{\mu_0}{2\pi}\,\frac{i_1 i_2}{r}}$$

The same force acts on wire 1 due to wire 2. The wires attract each other if currents in the wires are flowing in the same direction and they repel each other if the currents are in opposite directions.

Example 26.20 *Two long parallel wires are separated by a distance of 2.50 cm. The force per unit length that each wire exerts on the other is $4.00 \times 10^{-5}\ N/m$, and the wires repel each other. The current in one wire is 0.600 A.*

(a) What is the current in the second wire?

(b) Are the two currents in the same direction or in opposite directions?

Solution (a) $\frac{F}{l} = \left(\frac{\mu_0}{2\pi}\right)\frac{i_1 i_2}{r}$

$$\therefore \quad 4 \times 10^{-5} = \frac{(2 \times 10^{-7})\,(0.6)\,i_2}{2.5 \times 10^{-2}}$$

$$\therefore \quad i_2 = 8.33 \text{ A}$$

(b) Wires repel each other if currents are in opposite directions.

Example 26.21 *Consider three long straight parallel wires as shown in figure. Find the force experienced by a 25 cm length of wire C.*

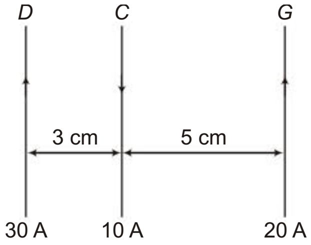


Fig. 26.61

Solution Repulsion by wire D, [towards right]

$$F_1 = \frac{\mu_0}{2\pi} \frac{i_1 i_2 l}{r}$$

$$= \frac{(2\times 10^{-7})(30\times 10)}{3\times 10^{-2}}(0.25)$$

$$= 5\times 10^{-4}\text{ N}$$

Repulsion by wire G, [towards left]

$$F_2 = \frac{(2\times 10^{-7})(20\times 10)}{5\times 10^{-2}}(0.25)$$

$$= 2\times 10^{-4}\text{ N}$$

$\therefore$

$$F_{\text{net}} = F_1 - F_2$$

$$= 3\times 10^{-4}\text{ N}$$ [towards right]

26.11 Magnetic Poles and Bar Magnets

In electricity, the isolated charge q is the simplest structure that can exist. If two such charges of opposite sign are placed near each other, they form an electric dipole characterized by an electric dipole moment **p**. In magnetism isolated magnetic 'poles' which would correspond to isolated electric charges do not exist. The simplest magnetic structure is the **magnetic dipole**, characterized by a magnetic dipole moment **M**. A current loop, a bar magnet and a solenoid of finite length are examples of magnetic dipoles.

When a magnetic dipole is placed in an external magnetic field **B**, a magnetic torque τ acts on it, which is given by

$$\tau = \mathbf{M}\times\mathbf{B}$$

Alternatively, we can measure **B** due to the dipole at a point along its axis a (large) distance r from its centre by the expression,

$$\boxed{B = \frac{\mu_0}{4\pi}\cdot\frac{2M}{r^3}}$$

A bar magnet might be viewed as two poles (North and South) separated by some distance. However, all attempts to isolate these poles fail. If a magnet is broken, the fragments prove to be dipoles and not isolated poles. If we break up a magnet into the electrons and nuclei that make up its atoms, it will be found that even these elementary particles are magnetic dipoles.

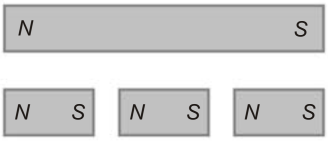


Fig. 26.62 If a bar magnet is broken, each fragment becomes a small dipole.

Each current carrying loop is just like a magnetic dipole, whose magnetic dipole moment is given by

$$\boxed{\mathbf{M} = ni\mathbf{A}}$$

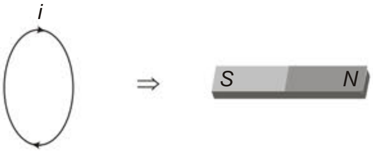


Fig. 26.63

Here, n is the number of turns in the loop, i is the current and $\mathbf{A}$ represents the area vector of the current loop.

The behaviour of a current loop can be described by the following hypothetical model:

(i) There are two magnetic charges; positive magnetic charge and negative magnetic charge. We call the positive magnetic charge a north pole and the negative magnetic charge as the south pole. Every pole has a pole strength m. The unit of pole strength is A-m.

(ii) A magnetic charge placed in a magnetic field experiences a force,

$$\boxed{\mathbf{F} = m\mathbf{B}}$$

The force on positive magnetic charge is along the field and a force on a negative magnetic charge is opposite to the field.

(iii) A magnetic dipole is formed when a negative magnetic charge $-m$ and a positive magnetic charge $+m$ are placed at a small separation d. The magnetic dipole moment is

$$\boxed{M = md}$$

The direction of $\mathbf{M}$ is from $-m$ to $+m$.

Geometrical Length and Magnetic Length

In case of a bar magnet, the poles appear at points which are slightly inside the two ends. The distance between the locations of the assumed poles is called the **magnetic length** of the magnet. The distance between the ends is called the **geometrical length.**

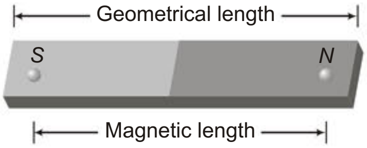


Fig. 26.64

The magnetic length of a bar magnet is written as $2l$. If m be the pole strength and $2l$ the magnetic length of a bar magnet, then its magnetic moment is

$$\boxed{M = 2ml}$$

Extra Points to Remember

- Current carrying loop, solenoid etc. are just like magnetic dipoles, whose dipole moment M is equal to NiA. Direction of **M** is from south pole (S) to north pole (N).
- The behaviour of a magnetic dipole (may be a bar magnet also) is similar to the behaviour of an electric dipole.
  The only difference is that the electric dipole moment **p** is replaced by magnetic dipole moment **M** and the constant $\frac{1}{4\pi\varepsilon_0}$ is replaced by $\frac{\mu_0}{4\pi}$.
- Table given below makes a comparison between an electric dipole and a magnetic dipole.

Table 26.2

S.No.	Physical quality to be compared	Electric dipole	Magnetic dipole
1.	Dipole moment	$p = q\,(2l)$	$M = m\,(2l)$
2.	Direction of dipole moment	From negative charge to the positive charge	From south to north pole
3.	Net force in uniform field	0	0
4.	Net torque in uniform field	$\tau = \mathbf{p} \times \mathbf{E}$	$\tau = \mathbf{M} \times \mathbf{B}$
5.	Field at far away point on the axis	$\frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3}$ (along **p**)	$\frac{\mu_0}{4\pi} \cdot \frac{2M}{r^3}$ (along **M**)
6.	Field at far away point on perpendicular bisector	$\frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{r^3}$ (opposite to **p**)	$\frac{\mu_0}{4\pi} \cdot \frac{M}{r^3}$ (opposite to **M**)
7.	Potential energy	$U_\theta = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta$	$U_\theta = -\mathbf{M} \cdot \mathbf{B} = -MB\cos\theta$
8.	Work done in rotating the dipole	$W_{\theta_1 - \theta_2} = pE\,(\cos\theta_1 - \cos\theta_2)$	$W_{\theta_1 - \theta_2} = MB\,(\cos\theta_1 - \cos\theta_2)$

Note *In the above table, θ is the angle between field (**E** or **B**) and dipole moment (**p** or **M**).*

Example 26.22 *Calculate the magnetic induction (or magnetic field) at a point 1 Å away from a proton, measured along its axis of spin. The magnetic moment of the proton is 1.4×10^{-26} A-m^2.*

Solution On the axis of a magnetic dipole, magnetic induction is given by

$$B = \frac{\mu_0}{4\pi} \cdot \frac{2M}{r^3}$$

Substituting the values, we get

$$B = \frac{(10^{-7})\,(2)\,(1.4 \times 10^{-26})}{(10^{-10})^3}$$

$$= 2.8 \times 10^{-3}\ \text{T}$$

$$= 2.8\ \text{mT}$$

Ans.

Example 26.23 *A bar magnet of magnetic moment 2.0 A-m^2 is free to rotate about a vertical axis through its centre. The magnet is released from rest from the east-west position. Find the kinetic energy of the magnet as it takes the north-south position. The horizontal component of the earth's magnetic field is $B = 25\ \mu T$. Earth's magnetic field is from south to north.*

Solution Gain in kinetic energy = loss in potential energy

Thus,

$$\text{KE} = U_i - U_f$$

As, $$U = -MB\cos\theta$$

∴ $$\text{KE} = -MB\cos\left(\frac{\pi}{2}\right) - (-MB\cos 0°)$$

$$= MB$$

Substituting the values, we have

$$\text{KE} = (2.0)(25 \times 10^{-6})\ \text{J}$$

$$= 50\ \mu\text{J}$$ **Ans.**

26.12 Earth's Magnetism

Our earth behaves as it has a powerful magnet within it. The value of magnetic field on the surface of earth is a few tenths of a gauss ($1\ \text{G} = 10^{-4}$ T). The earth's south magnetic pole is located near the north geographic pole and the earth's north magnetic pole is located near the south geographic pole. In fact, the configuration of the earth's magnetic field is very much like the one that would be achieved by burying a gigantic bar magnet deep in the interior of the earth.

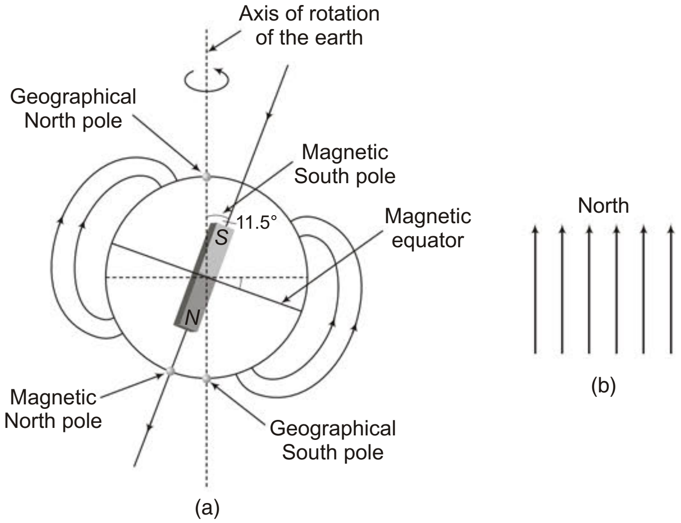


Fig. 26.65

The axis of earth's magnet makes an angle of 11.5° with the earth's rotational axis.

Theories Regarding the Origin of Earth's Magnetism

First Theory : Gilbert for the first time in 1600, gave the idea that there is a powerful magnet within the earth at its centre. Later on this theory was denied because the temperature in the interior of the earth is so high that it is impossible to retain its magnetism.

Second Theory : The second theory was put forward by Grover in 1849. He put the view that the earth magnetism is due to electric currents flowing near the outer surface of the earth. Hot air, rising from the region near equator, goes towards north and south hemispheres and become electrified. These currents magnetise the ferromagnetic material near the outer surface of the earth.

Third Theory : There are many conducting materials including iron and nickel in the molten state within the central core of the earth. Conventional currents are generated in this semifluid core due to earth's rotation about its axis. Due to these currents, magnetism is generated within the earth.

Till date not a single theory can explain all events regarding earth's magnetism.

Elements of Earth's Magnetism

There are three elements of earth's magnetism.

(i) **Angle of Declination (α)** At any point (say *P*) on earth's surface the longitude determines the north-south direction. The vertical plane in the direction of longitude or the vertical plane passing through the line joining the geographical north and south poles is called the **'geographical meridian'**. At point *P*, there also exists the magnetic field **B**. A vertical plane in the direction of **B** is called **'magnetic meridian'**.

At any place the acute angle between the magnetic meridian and the geographical meridian is called **'angle of declination 'α'.**

(ii) **Angle of Dip (θ)** 'The angle of dip (θ) at a place is the angle between the direction of earth's magnetic field and the horizontal at that place.'

Angle of dip at some place can be measured from a magnetic needle free to rotate in a vertical plane about a horizontal axis passing through centre of gravity of the needle. At earth's magnetic poles the magnetic field of earth is vertical, i.e. angle of dip is 90°, the freely suspended magnetic needle is vertical there. At magnetic equator field is horizontal, or angle of dip is 0°. The needle is horizontal. In northern hemisphere, the north pole of the magnetic needle inclines downwards, whereas in the southern hemisphere the south pole of the needle inclines downwards.

(iii) **Horizontal Component of Earth's Magnetic Field** Let B_e be the net magnetic field at some point. *H* and *V* be the horizontal and vertical components of B_e. Let θ is the angle of dip at the same place, then we can see that

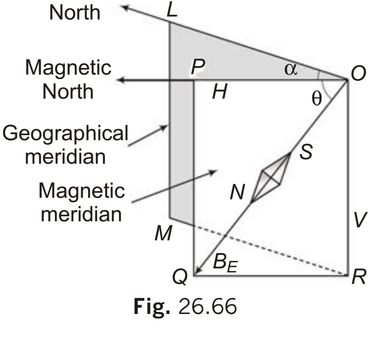


Fig. 26.66

$$H = B_e \cos\theta \qquad \ldots(\text{i})$$

and

$$V = B_e \sin\theta \qquad \ldots(\text{ii})$$

Squaring and adding Eqs. (i) and (ii), we get

$$B_e = \sqrt{H^2 + V^2}$$

Further, dividing Eq. (ii) by Eq. (i), we get

$$\theta = \tan^{-1}\left(\frac{V}{H}\right)$$

By knowing H and θ at some place we can find B_e and V at that place.

Neutral Points

When a magnet is placed at some point on earth's surface, there are points where horizontal component of earth's magnetic field is just equal and opposite to the field due to the magnet. Such points are called neutral points. If a magnetic compass is placed at a neutral point, no force acts on it and it may set in any direction.

Suppose a small bar magnet is placed such that north pole of the magnet is towards the magnetic south pole of the earth then neutral points are obtained both sides on the axis of the magnet. If distance of each neutral point from the middle point of a magnet be r, and the magnitude of the magnetic moment of the magnet be M, then

$$\frac{\mu_0}{4\pi} \cdot \frac{2M}{r^3} = H$$

When north pole of bar magnet is towards the magnetic north pole of the earth, the neutral points are obtained on perpendicular bisectors of the magnet. Let r be the distance of neutral points from centre, then

$$\frac{\mu_0}{4\pi} \cdot \frac{M}{r^3} = H$$

Example 26.24 *In the magnetic meridian of a certain place, the horizontal component of earth's magnetic field is* 0.26 G *and the dip angle is* 60°. *Find*

(a) vertical component of earth's magnetic field.

(b) the net magnetic field at this place.

Solution Given, $H = 0.26\,\text{G}$ and $\theta = 60°$

(a) $\tan\theta = \dfrac{V}{H}$

$$\therefore \qquad V = H\tan\theta = (0.26)\tan 60°$$

$$= 0.45\,\text{G} \qquad \textbf{Ans.}$$

(b) $H = B_e \cos\theta$

$$\therefore \qquad B_e = \frac{H}{\cos\theta} = \frac{0.26}{\cos 60°}$$

$$= 0.52\,\text{G} \qquad \textbf{Ans.}$$

Example 26.25 *A magnetic needle suspended in a vertical plane at 30° from the magnetic meridian makes an angle of 45° with the horizontal. Find the true angle of dip.*

Solution In a vertical plane at 30° from the magnetic meridian, the horizontal component is

$$H' = H \cos 30°$$

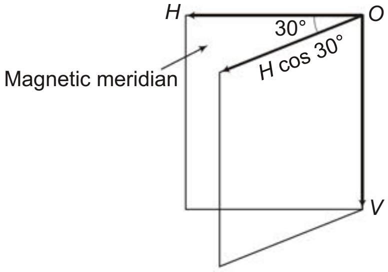


Fig. 26.67

While vertical component is still V. Therefore, apparent dip will be given by

$$\tan \theta' = \frac{V}{H'} = \frac{V}{H \cos 30°}$$

But, $$\frac{V}{H} = \tan \theta \qquad \text{(where, } \theta = \text{true angle of dip)}$$

$\therefore$ $$\tan \theta' = \frac{\tan \theta}{\cos 30°}$$

$\therefore$ $$\theta = \tan^{-1} [\tan \theta' \cos 30°]$$

$$= \tan^{-1} [(\tan 45°)(\cos 30°)]$$

$$\approx 41°$$ **Ans.**

26.13 Vibration Magnetometer

Vibration magnetometer is an instrument which is used for the following two purposes:

(i) To find magnetic moment of a bar magnet.

(ii) To compare magnetic fields of two magnets.

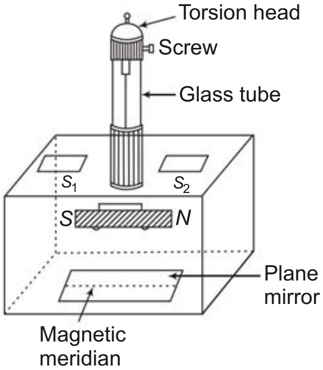


Fig. 26.68

The construction of a vibration magnetometer is as shown in figure.The magnet shown in figure is free to rotate in a horizontal plane. The magnet stays parallel to the horizontal component of earth's magnetic field. If the magnet is now displaced through an angle θ, a restoring torque of magnitude $MH \sin \theta$ acts on it and the magnet starts oscillating. From the theory of simple harmonic motion, we can find the time period of oscillations of the magnet.

Restoring torque in displaced position is

$$\tau = -MH \sin \theta \qquad \ldots\text{(i)}$$

Here, M = Magnetic moment of the magnet

and H = Horizontal component of earth's magnetic field.

Negative sign shows the restoring nature of torque. Now since, $\tau = I\alpha$ and $\sin\theta \approx \theta$ for small angular displacement.

Thus, Eq. (i) can be written as

$$I\alpha = -MH\theta$$

Since, α is proportional to $-\theta$. Therefore, motion is simple harmonic in nature, time period of which will be given by

$$T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} = 2\pi\sqrt{\frac{I}{MH}}$$

$$\therefore \qquad \boxed{T = 2\pi\sqrt{\frac{I}{MH}}} \qquad \text{...(ii)}$$

In the expression of T, I is the moment of inertia of the magnet about its axis of vibration.

(i) Measurement of Magnetic Moment : By finding time period T of vibrations of the given magnet, we can calculate magnetic moment M by the relation,

$$M = \frac{4\pi^2 I}{T^2 H}$$

(ii) Comparison of Two Magnetic Fields : Suppose we wish to compare the magnetic fields B_1 and B_2 at some point P due to two magnets. For this, vibration magnetometer is so placed that the centre of its magnet lies on P. Now, one of the given magnets is placed at some known distance from P in the magnetic meridian, such that point P lies on its axial line and its north pole points north. In this position, the field B_1 at P produced by the magnet will be in the direction of H. Hence, the magnet suspended in the magnetometer will vibrate in the resultant magnetic field $(H + B_1)$. Its period of vibration is noted, say it is T_1, then

$$T_1 = 2\pi\sqrt{\frac{I}{M(H + B_1)}}$$

Now, the first magnet is replaced by the second magnet and the second magnet is placed in the same position and again the time period is noted. If the field produced at P due to this magnet be B_2 and the new time period be T_2, then

$$T_2 = 2\pi\sqrt{\frac{I}{M(H + B_2)}}$$

Finally, the time period of the magnetometer under the influence of the earth's magnetic field alone is determined. Let it be T, then

$$T = 2\pi\sqrt{\frac{I}{MH}}$$

Solving above three equations for T, T_1 and T_2, we can show that

$$\boxed{\frac{B_1}{B_2} = \frac{(T^2 - T_1^2)\,T_2^2}{(T^2 - T_2^2)\,T_1^2}}$$

Example 26.26 *A short bar magnet is placed with its north pole pointing north. The neutral point is* 10 cm *away from the centre of the magnet. If* $H = 0.4$ G, *calculate the magnetic moment of the magnet.*

Solution When north pole of the magnet points towards magnetic north, null point is obtained on perpendicular bisector of the magnet. Simultaneously, magnetic field due to the bar magnet should be equal to the horizontal component of earth's magnetic field H.

Thus,
$$H = \frac{\mu_0}{4\pi} \cdot \frac{M}{r^3} \quad \text{or} \quad M = \frac{Hr^3}{(\mu_0/4\pi)}$$

Substituting the values, we have
$$M = \frac{(0.4 \times 10^{-4})(10 \times 10^{-2})^3}{10^{-7}} = 0.4 \text{ A-m}^2$$ **Ans.**

Example 26.27 *A magnetic needle performs* 20 *oscillations per minute in a horizontal plane. If the angle of dip be* 30°, *then how many oscillations per minute will this needle perform in vertical north-south plane and in vertical east-west plane?*

Solution In horizontal plane, the magnetic needle oscillates in horizontal component H.

∴
$$T = 2\pi \sqrt{\frac{I}{MH}}$$

In the vertical north-south plane (magnetic meridian), the needle oscillates in the total earth's magnetic field B_e, and in vertical east-west plane (plane perpendicular to the magnetic meridian) it oscillates only in earth's vertical component V. If its time period be T_1 and T_2, then
$$T_1 = 2\pi \sqrt{\frac{I}{MB_e}} \quad \text{and} \quad T_2 = 2\pi \sqrt{\frac{I}{MV}}$$

From above equations, we can find
$$\frac{T_1^2}{T^2} = \frac{H}{B_e} \quad \text{or} \quad \frac{n_1^2}{n^2} = \frac{B_e}{H}$$

Similarly,
$$\frac{n_2^2}{n^2} = \frac{V}{H}$$

Further,
$$\frac{B_e}{H} = \sec\theta = \sec 30° = \frac{2}{\sqrt{3}}$$

and
$$\frac{V}{H} = \tan\theta = \tan 30° = \frac{1}{\sqrt{3}}$$

∴
$$n_1^2 = (n)^2 \left(\frac{B_e}{H}\right) = (20)^2 \left(\frac{2}{\sqrt{3}}\right)$$

or
$$n_1 = 21.5 \text{ oscillations/min}$$ **Ans.**

and
$$n_2^2 = (n)^2 \left(\frac{V}{H}\right) = (20)^2 \left(\frac{1}{\sqrt{3}}\right)$$

∴
$$n_2 = 15.2 \text{ oscillations/min}$$ **Ans.**

26.14 Magnetic Induction and Magnetic Materials

We know that the electric lines of force change when a dielectric is placed between the parallel plates of a capacitor. Experiments show that magnetic lines also get modified due to the presence of certain materials in the magnetic field.

Few substances such as O_2, air, platinum, aluminium etc., show a very small increase in the magnetic flux passing through them, when placed in a magnetic field. Such substances are called **paramagnetic substances**. Few other substances such as H_2, H_2O, Cu, Zn, Sb etc. show a very small decrease in flux and are said to be **diamagnetic**. There are other substances like Fe, Co etc. through which the flux increases to a larger value and are known as **ferromagnetic substances**.

Magnetisation of Matter

A material body is consisting of large number of atoms and thus large number of electrons. Each electron produces orbital and spin magnetic moments and can be assumed as magnetic dipoles. In the absence of any external magnetic field, the dipoles of individual atoms are randomly oriented and the magnetic moments thus, cancel.

When we apply an external magnetic field to a substance, two processes may occur.

(i) All atoms which have non-zero magnetic moment are aligned along the magnetic field.

(ii) If the atom has a zero magnetic moment, the applied magnetic field distorts the electron orbit and thus, induces magnetic moment in opposite directions.

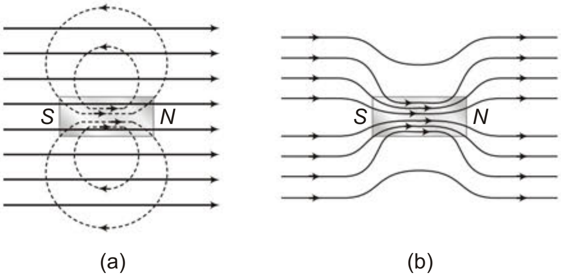
In diatomic substances, the individual atoms do not have a magnetic moment by its own. When an external field is applied, the second process occurs. The induced magnetic moment is thus set up in the direction opposite to **B**. In this case, the magnetic flux density in the interior of the body will be less than that of the external field **B**.

In paramagnetic substances, the constituent atoms have intrinsic magnetic moments. When an external magnetic field is applied, both of the above processes occur and the resultant magnetic moment is always in the direction of magnetic field **B** as the first effect predominates over the second.

26.15 Some Important Terms Used in Magnetism

Magnetic Induction (B)

When a piece of any substance is placed in an external magnetic field, the substance becomes magnetised. If an iron bar is placed in a uniform magnetic field, the magnetised bar produces its own magnetic field in the same direction as those of the original field inside the bar, but in opposite direction outside the bar. This results in a concentration of the lines of force within the bar. The magnetic flux density within the bar is increased whereas it becomes weak at certain places outside the bar.



(a) (b)

Fig. 26.69

“The number of magnetic lines of induction inside a magnetic substance crossing unit area normal to their direction is called the magnitude of magnetic induction, or magnetic flux density inside the substance. It is denoted by **B**. The SI unit of **B** is tesla (T) or weber/metre^2 (Wb/m^2). The CGS unit is gauss (G).

$$\boxed{1\,\text{Wb/m}^2 = 1\,\text{T} = 10^4\,\text{G}}$$

Intensity of Magnetisation (*I*)

"Intensity of magnetisation (***I***) is defined as the magnetic moment per unit volume of the magnetised substance." This basically represents the extent to which the substance is magnetised. Thus,

$$\boxed{I = \frac{M}{V}}$$

The SI unit of *I* is ampere/metre (A/m).

Magnetic Intensity or Magnetic Field Strength (H)

When a substance is placed in an external magnetic field, the actual magnetic field inside the substance is the sum of the external field and the field due to its magnetisation.

The capability of the magnetising field to magnetise the substance is expressed by means of a vector **H**, called the 'magnetic intensity' of the field. It is defined through the vector relation,

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{I}}$$

The SI unit of **H** is same as that of **I**, i.e. ampere/metre (A/m). The CGS unit is oersted.

Magnetic Permeability (μ)

"It is defined as the ratio of the magnetic induction **B** inside the magnetised substance to the magnetic intensity **H** of the magnetising field, i.e.

$$\boxed{\mu = \frac{\mathbf{B}}{\mathbf{H}}}$$

It is basically a measure of conduction of magnetic lines of force through it. The SI unit of magnetic permeability is weber/ampere-metre (Wb/A-m).

Relative Magnetic Permeability (μ_r)

It is the ratio of the magnetic permeability μ of the substance to the permeability of free space.

Thus,

$$\boxed{\mu_r = \frac{\mu}{\mu_0}}$$

μ_r is a pure ratio, hence, dimensionless. For vacuum its value is 1.

μ_r can also be defined as the ratio of the magnetic field B in the substance when placed in magnetic field B_0. Thus,

$$\boxed{\mu_r = \frac{B}{B_0}}$$

For paramagnetic substance, $\mu_r > 1$,
For diamagnetic substance, $\mu_r < 1$ and
For ferromagnetic substance, $\mu_r >> 1$.

Magnetic Susceptibility (χ_m)

We know that both diamagnetic and paramagnetic substances develop a magnetic moment depending on the applied field. Magnetic susceptibility is a measure of how easily a substance is magnetised in a magnetising field. For paramagnetic and diamagnetic substances, **I**, **H** and χ_m are related by the equation,

$$\mathbf{I} = \chi_m \mathbf{H}$$

or

$$\chi_m = \frac{I}{H}$$

Thus, the magnetic susceptibility χ_m may be defined as the ratio of the intensity of magnetisation to the magnetic intensity of the magnetising field.

Since, I and H have the same units, χ_m is unitless. It is a pure number.

By doing simple calculation, we can prove that μ_r and χ_m are related by

$$\mu_r = 1 + \chi_m$$

For paramagnetic substances χ_m is slightly positive. For diamagnetic substances, it is slightly negative and for ferromagnetic substances, χ_m is positive and very large.

26.16 Properties of Magnetic Materials

As discussed earlier, all substances (whether solid, liquid or gaseous) may be classified into three categories in terms of their magnetic properties. (i) paramagnetic, (ii) diamagnetic and (iii) ferromagnetic.

Paramagnetic Substances

Examples of such substances are platinum, aluminium, chromium, manganese, $CuSO_4$ solution etc. They have the following properties:

(i) The substances when placed in a magnetic field, acquire a feeble magnetisation in the same sense as the applied field. Thus, the magnetic inductance inside the substance is slightly greater than outside to it.

(ii) In a uniform magnetic field, these substances rotate until their longest axes are parallel to the field.

(iii) These substances are attracted towards regions of stronger magnetic field when placed in a non-uniform magnetic field.

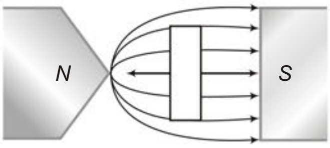


Fig. 26.70

Figure shows a strong electromagnet in which one of the pole pieces is sharply pointed while the other is flat. Magnetic field is much stronger near the pointed pole than near the flat pole. If a small piece of paramagnetic material is suspended in this region, a force can be observed in the direction of arrow.

(iv) If a paramagnetic liquid is filled in a narrow U-tube and one limb is placed in between the pole pieces of an electromagnet such that the level of the liquid is in line with the field, then the liquid will rise in the limb as the field is switched on.

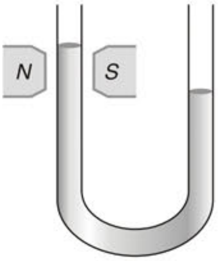


Fig. 26.71

(v) For paramagnetic substances, the relative permeability μ_r is slightly greater than one.

(vi) At a given temperature the magnetic susceptibility χ_m does not change with the magnetising field. However, it varies inversely as the absolute temperature. As temperature increases, χ_m decreases. At some higher temperature, χ_m becomes negative and the substance becomes diamagnetic.

Diamagnetic Substances

Examples of such substances are bismuth, antimony, gold, quartz, water, alcohol etc. They have the following properties:

(i) These substances when placed in a magnetic field, acquire feeble magnetisation in a direction opposite to that of the applied field. Thus, the lines of induction inside the substance is smaller than that outside to it.

(ii) In a uniform field, these substances rotate until their longest axes are normal to the field.

(iii) In a non-uniform field, these substances move from stronger to weaker parts of the field.

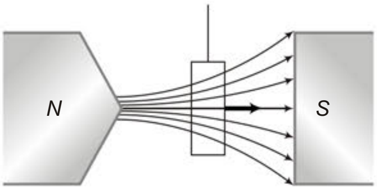


Fig. 26.72

(iv) If a diamagnetic liquid is filled in a narrow U-tube and one limb is placed in between the pole of an electromagnet, the level of liquid depresses when the field is switched on.

(v) The relative permeability μ_r is slightly less than 1.

(vi) The susceptibility χ_m of such substances is always negative. It is constant and does not vary with field or the temperature.

Ferromagnetic Substances

Examples of such substances are iron, nickel, steel, cobalt and their alloys. These substances resemble to a higher degree with paramagnetic substances as regard their behaviour. They have the following additional properties:

(i) These substances are strongly magnetised by even a weak magnetic field.

(ii) The relative permeability is very large and is of the order of hundreds and thousands.

(iii) The susceptibility is positive and very large.

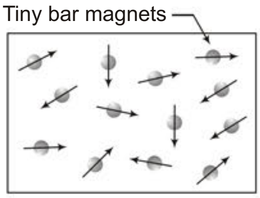
(iv) Susceptibility remains constant for very small values of **H**, increases for larger values of **H** and then decreases for very large values of **H**.

(v) Susceptibility decreases steadily with the rise of temperature. Above a certain temperature known as **Curie Temperature**, the ferromagnetic substances become paramagnetic. It is 1000°C for iron, 770°C for steel, 360°C for nickel and 1150°C for cobalt.

26.17 Explanation of Paramagnetism, Diamagnetism and Ferromagnetism

There are three properties of atoms that give rise to magnetic dipole moment.

1. The electrons moving around the nucleus in the orbits act as small current loops and contribute magnetic moments.
2. The spinning electron has an intrinsic magnetic dipole moment.
3. The nucleus contribute to magnetic moment due to the motion of charge within the nucleus. The magnitude of nuclear moments is about 10^{-3} times that of electronic moments or the spin magnetic moments, as the later two are of the same order. Still most of the magnetic moment of an atom is produced by electron spin, the net contribution of the orbital revolution is very small. This is because most of the electrons pair off in such a way that they produce equal and opposite orbital magnetic moment and they cancel out. Although, the electrons also try to pair up with their opposite spins but in case of spin motion of an electron it is not always possible to form equal and opposite pairs.



In the absence of external magnetic field

Fig. 26.73

Paramagnetism

The property of paramagnetism is found in those substances whose atoms or molecules have an excess of electrons spinning in the same direction.

Hence, atoms of paramagnetic substances have a permanent magnetic moment and behave like tiny bar magnets. In the absence of external magnetic field, the atomic magnets are randomly oriented and net magnetic moment is thus, zero.

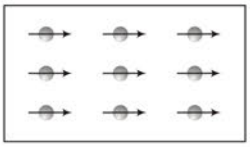


Fig. 26.74

When paramagnetic substance is placed in an external magnetic field, then each atomic magnet experiences a torque which tends to turn the magnet in the direction of the field. The atomic magnets are thus, aligned in the direction of the field. Thus, the whole substance is magnetised in the direction of the external magnetic field.

As the temperature of substance is increased, the thermal agitation disturbs the magnetic alignment of the atoms. Thus, we can say that paramagnetism is temperature dependent.

Curie's law

According to Curie's law, magnetic susceptibility of a paramagnetic substance is inversely proportional to absolute temperature T.

$$\boxed{\chi_m \propto \frac{1}{T}}$$

The exact law is beyond the scope of our course.

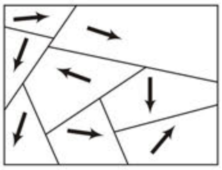
Diamagnetism

The property of diamagnetism is generally found in those substances whose atoms (or molecules) have even number of electrons which form pairs. "The net magnetic moment of an atom of a diamagnetic substance is thus zero." When a diamagnetic substance is placed in an external magnetic field, the spin motion of electrons is so modified that the electrons which produce the magnetic moments in the direction of external field slow down while the electrons which produce magnetic moments in opposite direction get accelerated. Thus, a net magnetic moment is induced in the opposite directions of applied magnetic field. Hence, the substance is magnetised opposite to the external field.

Note *That diamagnetism is temperature independent.*

Ferromagnetism

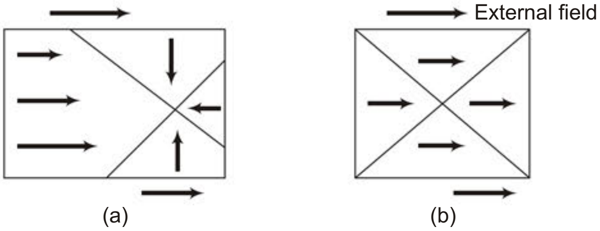
Iron like elements and their alloys are known as ferromagnetic substances. The susceptibility of these substances is in several thousands. Like paramagnetic substances, atoms of ferromagnetic substances have a permanent magnetic moment and behave like tiny magnets. But in ferromagnetic substances the atoms form innumerable small effective regions called 'domains'.



Unmagnetised

Fig. 26.75

The size of the domain vary from about 10^{-6}cm^3 to 10^{-2}cm^3. Each domain has 10^{17} to 10^{21} atoms whose magnetic moments are aligned in the same direction. In an unmagnetised ferromagnetic specimen, the domains are oriented randomly, so that their resultant magnetic moment is zero.



(a) (b)

Fig. 26.76

When the specimen is placed in a magnetic field, the resultant magnetisation may increase in two different ways.

(a) The domains which are oriented favourably with respect to the field increase in size. Whereas those oriented opposite to the external field are reduced.

(b) The domains rotate towards the field direction.

Note *That if the external field is weak, specimen gets magnetised by the first method and if the field is strong they get magnetised by the second method.*

Hysteresis : Retentivity and Coercivity

The distinguishing characteristics of a ferromagnetic material is not that it can be strongly magnetised but that the intensity of magnetisation **I** is not directly proportional to the magnetising field **H**. If a gradually increasing magnetic field **H** is applied to an unmagnetised piece of iron, its magnetisation increases non-linearly until it reaches a maximum.

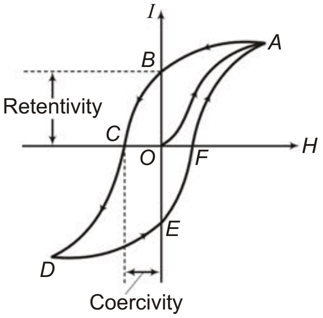


Fig. 26.77

If **I** is plotted against **H**, a curve like *OA* is obtained. This curve is known as magnetisation curve. At this stage all the dipoles are aligned and **I** has reached to a maximum or saturated value. If the magnetic field **H** is now decreased, the **I** does not return along magnetisation curve but follows path *AB*. At $\mathbf{H} = 0$, **I** does not come to its zero value but its value is still near the saturated value. The value of **I** at this point (i.e. *OB*) is known as **remanence, remanent magnetisation or retentivity**. The value of **I** at this point is known as residual induction. On applying a reverse field the value of **I** finally becomes zero. The abscissa *OC* represents the reversed magnetic field needed to demagnetise the specimen. This is known as **coercivity** of the material.

If the reverse field is further increased, a reverse magnetisation is set up which quickly reaches the saturation value. This is shown as *CD*. If **H** is now taken back from its negative saturation value to its original positive saturation value, a similar curve *DEFA* will be traced. The whole graph *ABCDEFA* thus, forms a closed loop, usually known as **hysteresis loop**. The whole process described above and the property of the iron characterized by it are called **hysteresis**. The energy lost per unit volume of a substance in a complete cycle is equal to the area. Thus, we can conclude the following three points from the above discussion:

(i) The retentivity of a substance is a measure of the magnetisation remaining in the substance when the magnetising field is removed.

(ii) The coercivity of a substance is a measure of the reverse magnetising field required to destroy the residual magnetism of the substance.

(iii) The energy loss per unit volume of a substance in a complete cycle of magnetisation is equal to the area of the hysteresis loop.

Demagnetisation

It is clear from the hysteresis loop that the intensity of magnetisation **I** does not reduce to zero on removing the magnetising field **H**. Further, **I** is zero when the magnetising field **H** is equal to the coercive field.

At these points the magnetic induction is not zero, and the specimen is not demagnetised. To demagnetise a substance, it is subjected to several cycles is magnetisation, each time with decreasing magnetising field and finally the field is reduced to zero. In this way, the size of the hysteresis curve goes on decreasing and the area finally reduces to zero.

Demagnetisation is obtained by placing the specimen in an alternating field of continuously diminishing amplitude. It is also obtained by heating. Ferromagnetic materials become practically non-magnetic at sufficiently high temperatures.

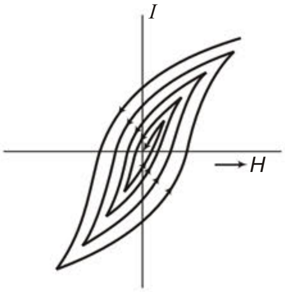


Fig. 26.78

Magnetic Properties of Soft Iron and Steel

A comparison of the magnetic properties of ferromagnetic substances can be made by the comparison of the shapes and sizes of their hysteresis loops.

Following three conclusions can be drawn from their hysteresis loops:

(i) Retentivity of soft iron is more than the retentivity of steel.

(ii) Coercivity of soft iron is less than the coercivity of steel.

(iii) Area of hysteresis loop (i.e. hysteresis loss) in soft iron is smaller than that in steel.

Choice of Magnetic Materials

The choice of a magnetic material for different uses is decided from the hysteresis curve of a specimen of the material.

(i) **Permanent Magnets** The materials for a permanent magnet should have

(a) high retentivity (so that the magnet is strong) and

(b) high coercivity (so that the magnetising is not wiped out by stray magnetic fields). As the material in this case is never put to cyclic changes of magnetisation, hence, hysteresis is immaterial. From the point of view of these facts **steel** is more suitable for the construction of permanent magnets than soft iron.

Modern permanent magnets are made of 'cobalt-steel', alloys 'ticonal'.

(ii) **Electromagnets** The materials for the construction of electromagnets should have

(a) high initial permeability

(b) low hysteresis loss

From the view point of these facts, **soft iron** is an ideal material for this purpose.

(iii) **Transformer Cores and Telephone Diaphragms** As the magnetic material used in these cases is subjected to cyclic changes. Thus, the essential requirements for the selection of the material are

(a) high initial permeability

(b) low hysteresis loss to prevent the breakdown

Electromagnet

As we know that a current carrying solenoid behaves like a bar magnet. If we place a soft iron rod in the solenoid, the magnetism of the solenoid increases hundreds of times and the solenoid is called an 'electromagnet'. It is a temporary magnet.

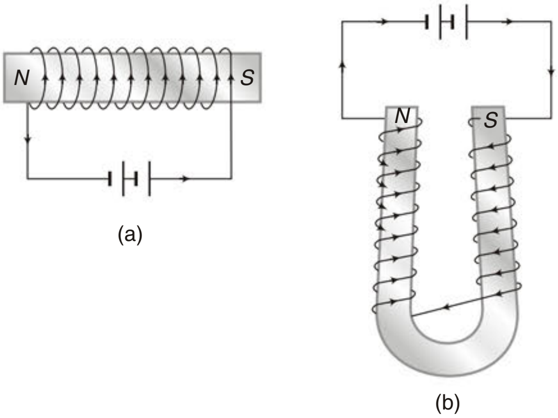


Fig. 26.79

An electromagnet is made by winding closely a number of turns of insulated copper wire over a soft iron straight rod or a horse shoe rod. On passing current through this solenoid, a magnetic field is produced in the space within the solenoid.

Applications of Electromagnets

(i) Electromagnets are used in electric bell, transformer, telephone diaphragms etc.
(ii) In medical field, they are used in extracting bullets from the human body.
(iii) Large electromagnets are used in cranes for lifting and transferring big machines and parts.

26.18 Moving Coil Galvanometer

The moving coil galvanometer is a device used to measure an electric current.

Principle

Action of a moving coil galvanometer is based upon the principle that when a current carrying coil is placed in a magnetic field, it experiences a torque whose magnitude depends on the magnitude of current.

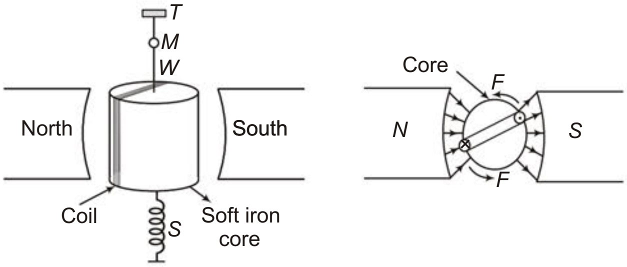


Fig. 26.80

Construction

The main parts of a moving coil galvanometer are shown in figure. The galvanometer consists of a coil, with many turns free to rotate about a fixed vertical axis in a uniform radial magnetic field. There is a cylindrical soft iron core, which not only makes the field radial but also increases the strength of magnetic field.

Theory

The current to be measured is passed through the galvanometer. As the coil is in the magnetic field (of constant magnitude) it experiences a torque given by

$$\begin{aligned}\tau &= MB\sin\theta \\ &= (NiA)\,B\sin\theta \qquad \text{...(i)}\end{aligned}$$

As shown in the figure, the pole pieces are made cylindrical, the magnetic field always remains parallel to the plane of the coil. Or angle between **B** and **M** always remains 90°. Therefore, Eq. (i) can be written as

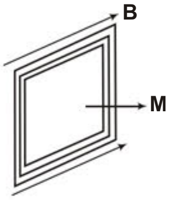


Fig. 26.81

$$\tau = NiAB \qquad (\text{as } \sin\theta = \sin 90^\circ = 1)$$

Here, N = total number of turns of the coil

i = current passing through the coil

A = area of cross-section of the coil and

B = magnitude of radial magnetic field.

This torque rotates the coil. The spring S shown in figure provides a counter torque $k\phi$ that balances the above torque $NiAB$. In equilibrium,

$$k\phi = NiAB \qquad \text{...(ii)}$$

Here, k is the torsional constant of the spring. With rotation of coil a small light mirror M (attached with phosphor bronze wire W) also rotates and equilibrium deflection ϕ can be measured by a lamp and scale arrangement.

The above Eq. (ii) can be written as

$$\boxed{i = \left(\frac{k}{NAB}\right)\phi} \qquad \text{...(iii)}$$

Hence, the current i is proportional to the deflection ϕ.

Galvanometer Constant

The constant $\frac{k}{NAB}$ in Eq. (iii) is called galvanometer constant.

Hence,

$$\boxed{\text{Galvanometer constant} = \frac{k}{NAB}} \qquad \text{...(iv)}$$

This constant may be found by passing a known current through the coil. Measuring the deflection ϕ and putting these values in Eq. (iii), we can find galvanometer constant.

Sensitivity of Galvanometer

Deflection per unit current (ϕ/i) is called sensitivity of galvanometer. From Eq. (iii), we can see that $\phi/i = NAB/k$. Hence,

$$\boxed{\text{Sensitivity} = \frac{\phi}{i} = \frac{NAB}{k}} \qquad \text{...(v)}$$

The sensitivity of a galvanometer can be increased by

(i) increasing the number of turns in the coil N or

(ii) increasing the magnitude of magnetic field.

Example 26.28 *A rectangular coil of area* 5.0×10^{-4} m^2 *and* 60 *turns is pivoted about one of its vertical sides. The coil is in a radial horizontal magnetic field of* 9×10^{-3} *T. What is the torsional constant of the spring connected to the coil if a current of* 0.20 *mA produces an angular deflection of* $18°$?

Solution From the equation,

$$i = \left(\frac{k}{NAB}\right)\phi$$

We find that torsional constant of the spring is given by

$$k = \frac{NABi}{\phi}$$

Substituting the values in SI units, we have

$$k = \frac{(60)(5.0 \times 10^{-4})(9 \times 10^{-3})(0.2 \times 10^{-3})}{18}$$

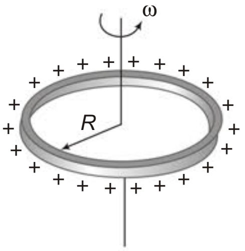
$$= 3 \times 10^{-9} \text{ N-m/degree}$$ **Ans.**

INTRODUCTORY EXERCISE 26.7

1. A coil of a moving coil galvanometer twists through $90°$ when a current of one microampere is passed through it. If the area of the coil is 10^{-4} m^2 and it has 100 turns, calculate the magnetic field of the magnet of the galvanometer. Given, $k = 10^{-8}$ N-m /degree.
2. A galvanometer coil 5 cm × 2 cm with 200 turns is suspended vertically in a field of 5×10^{-2} T. The suspension fibre needs a torque of 0.125×10^{-7} N-m to twist it through one radian. Calculate the strength of the current required to be maintained in the coil if we require a deflection of $6°$.

Final Touch Points

1. Sometimes, a non-conducting charged body is rotated with some angular speed. In this case, the ratio of magnetic moment and angular momentum is constant which is equal to $q/2m$, where q is the charge and m the mass of the body.



e.g. In case of a ring of mass m, radius R and charge q distributed on its circumference.

$$\text{Angular momentum, } L = I\omega = (mR^2)(\omega) \qquad \text{...(i)}$$

$$\text{Magnetic moment, } M = iA = (qf)(\pi R^2)$$

Here, f = frequency $= \dfrac{\omega}{2\pi}$

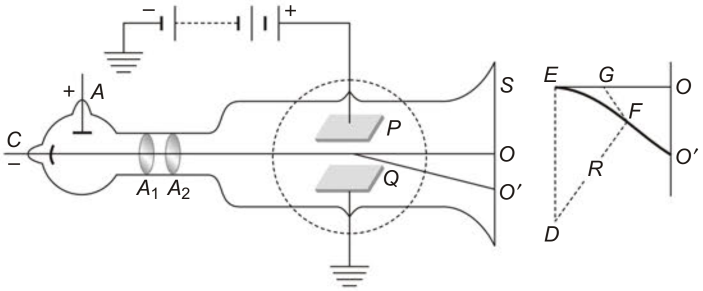
$$\therefore \qquad M = (q)\left(\frac{\omega}{2\pi}\right)(\pi R^2) = q\frac{\omega R^2}{2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\boxed{\frac{M}{L} = \frac{q}{2m}}$$

Although this expression is derived for simple case of a ring, it holds good for other bodies also. For example, for a disc or a sphere.

2. **Determination of e/m of an Electron (Thomson Method)** JJ Thomson in 1897, devised an experiment for the determination of e/m (specific charge) of the electron by using electric and magnetic fields in mutually perpendicular directions.



The discharge is maintained by the application of high PD between the cathode C and anode A of a discharge tube containing air at a very low pressure ($\sim 10^{-2}$ mm of Hg). The electrons so produced are allowed to pass through slits A_1 and A_2 also kept at the potential of A. The beam then passes along the axis of the tube and produces a spot of light at O on the fluorescent screen S. The electric field E is applied between two horizontal plates P and Q. The magnetic field B is applied in the direction perpendicular to the paper plane by passing the current through coils, in the region within the dotted circle. It is clear from Fleming's left hand rule, that F due to E is in upward direction, while due to B in downward direction.

Hence, fields E and B can be adjusted so that the electrons suffer no deflection and strike at point O on the screen. In this case,

or $$eE = evB \quad \text{or} \quad v = E/B \qquad \text{...(i)}$$

Now, electric field is switched off, the electrons thus, describe the circular arc and fall at O' on the screen. In this case, the force $F = Bev$ bends the electron beam in a circular arc, such that it is balanced by the centripetal force mv^2/R.

∴ $$Bev = mv^2/R$$

or $$v = BRe/m \qquad \text{...(ii)}$$

Combining Eqs. (i) and (ii), we get

$$e/m = E/RB^2 \qquad \text{...(iii)}$$

As E and B are known. To find R, consider arc EF of the circular path in the magnetic field region. From the geometry, we get

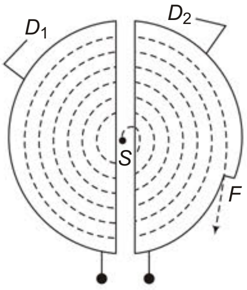
$$OO'/GO = EF/R$$

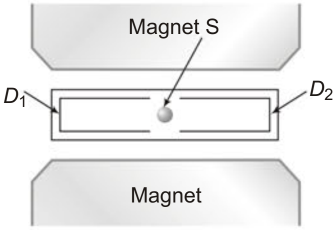
or $$R = EF \times GO/OO' \qquad \text{...(iv)}$$

Practically, EF is replaced by the width of the magnetic flux region and G is taken at the middle of the region. Thomson's value for e/m was 1.7×10^{11} C/kg, which is in excellent agreement with the modern value of 1.75890×10^{11} C/kg.

3. **Cyclotron** In 1932, Lawrence developed a machine named cyclotron, for the acceleration of charged particles, such as protons or deuterons. These particles (ions) are caused to move in circular orbits by magnetic field and are accelerated by the electric field.

   In its simplest form, it consists of two flat semicircular metal boxes, called dees because of their shape. These hollow chambers have their diametric edges parallel and slightly separated from each other. An alternating potential (with frequency of the order of megacycles per second) is applied between the dees. The dees are placed between the poles of a strong electromagnet which provides a magnetic field perpendicular to the plane of the dees.





Suppose that at any particular instant the alternating potential is in the direction which makes D_1 positive and D_2 negative. A positive ion of mass m, charge q starting from the source S (of positive ion) will be attracted by the dee D_2. Let its velocity while entering in dee D_2 is v. Due to magnetic field B, it will move in a circular path of radius r inside the dee D_2, where

$$r = \frac{mv}{Bq}$$

or $$\boxed{v = \frac{Bqr}{m}}$$

In the interior of the dee, the speed of the ion remains constant. After it has traversed half a cycle, the ion comes to the edge of D_2. If in the meantime, the potential difference between D_1 and D_2 has changed direction so that D_2 is now positive and D_1 negative, the positive ion will receive an additional acceleration, while going across the gap between the dees and speed of ion will increase. Then, it travels in a circular path of larger radius inside D_1 under the influence of magnetic field (because $r \propto v$). After traversing a half cycle in D_1, it will reach the edge of D_1 and receive an additional acceleration between the gaps because in the meantime the direction of potential difference between the dees has changed. The ion will continue travelling in a semicircle of increasing radii, the direction of potential difference changes every time the ion goes from D_1 to D_2 and from D_2 to D_1. The time taken by the charged particle to traverse the semicircular path in the dee is given by

$$t = \pi r / v = \frac{\pi m}{Bq} \qquad \ldots\text{(i)}$$

This relation indicates that time t is independent of the velocity of the particle and of the radius. For any given value of m/q, it is determined by the magnetic field intensity. By adjusting the magnetic field intensity the time can be made the same as that required to change the potentials. On the other hand, the oscillator frequency (of alternating potential) can also be adjusted to the nature of a given ion and to the strength of the magnetic field. The frequency of the oscillations required to keep the ion in phase is given by the relation

$$\boxed{f = \frac{1}{T} = \frac{1}{2t} = \frac{Bq}{2\pi m}} \qquad \ldots\text{(ii)}$$

If the oscillation frequency is adjusted to keep the charged ion always in phase, each time the ion crosses the gap it receives an additional energy and at the same time it describes a flat spiral of increasing radius. Eventually, the ion reaches the periphery of the dee, where it can be brought out of the chamber by means of a deflecting plate charged to a high negative potential. This attractive force draws the ion out of its spiral path and thus can be used easily. If R is the radius of the dee, kinetic energy of the ion emerging from the cyclotron is thus given by

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m\,(BqR/m)^2$$

$$K = B^2R^2q^2/2m \qquad \ldots\text{(iii)}$$

This relation indicates that the maximum energy attained by the ion is limited by the radius R, magnetic field B or the frequency of the alternating potential f. It is independent of the alternating voltage. It can be explained by the fact that when the voltage is low, the ion makes a large number of turns before reaching the periphery, but when the voltage is high the number of turns is small. The total energy remains same in both the cases provided B and R are unchanged.

Note *The cyclotron is used to bombard nuclei with energetic particles and study the resulting nuclear reactions. It is also used in hospitals to produce radioactive substances which can be used in diagnosis and treatment.*

Cyclotron is suitable only for accelerating heavy particles like proton, deuteron, α-particle etc. Electrons cannot be accelerated by the cyclotron because the mass of the electron is small and a small increase in energy of the electron makes the electrons move with a very high speed.

The uncharged particles (e.g., neutrons) cannot be accelerated by cyclotron.

Solved Examples

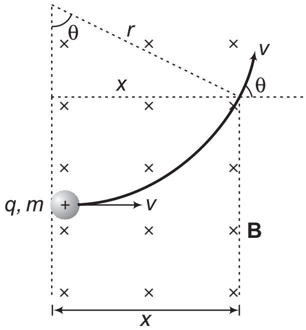
TYPED PROBLEMS

Type 1. *Based on deviation of charged particle in uniform magnetic field when $\theta = 90°$ or path is uniform circular*

Concept

Suppose a charged particle (q, m) enters a uniform magnetic field $\mathbf{B}$ at right angles with speed v as shown in figure. The magnetic field extends upto a length x. The path of the particle is a circle of radius r, where

$$r = \frac{mv}{Bq}$$



The speed of the particle in magnetic field does not change. But, it gets deviated in the magnetic field. The deviation θ can be found in two ways

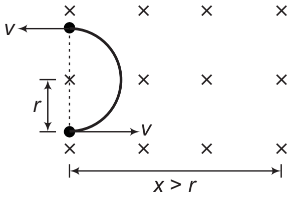
(i) After time t, deviation will be

$$\boxed{\theta = \omega t = \left(\frac{Bq}{m}\right) t} \qquad \left[\text{as } \omega = \frac{Bq}{m}\right]$$

(ii) In terms of the length of the magnetic field (i.e. when the particle leaves the magnetic field) the deviation will be

$$\boxed{\theta = \sin^{-1}\left(\frac{x}{r}\right)}$$

But, since, $\sin\theta \not> 1$, this relation can be used only when $x < r$. For $x \geq r$, the deviation will be $180°$ as shown in figure.



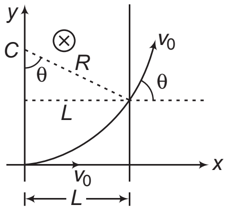
Example 1 *The region between $x = 0$ and $x = L$ is filled with uniform steady magnetic field $-B_0\hat{\mathbf{k}}$. A particle of mass m, positive charge q and velocity $v_0\hat{\mathbf{i}}$ travels along x-axis and enters the region of the magnetic field.* **[JEE 1999]**

Neglect the gravity throughout the question.

(a) Find the value of L if the particle emerges from the region of magnetic field with its final velocity at an angle 30° to its initial velocity.

(b) Find the final velocity of the particle and the time spent by it in the magnetic field, if the magnetic field now extends upto 2.1 L.

Solution (a) $\theta = 30°$



$$\sin\theta = \frac{L}{R}$$

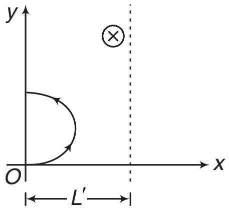
Here, $$R = \frac{mv_0}{B_0 q}$$

$\therefore$ $$\sin 30° = \frac{L}{\dfrac{mv_0}{B_0 q}}$$

or $$\frac{1}{2} = \frac{B_0 qL}{mv_0}$$

$\therefore$ $$L = \frac{mv_0}{2B_0 q}$$

(b) In part (a)



$$\sin 30° = \frac{L}{R} \quad \text{or} \quad \frac{1}{2} = \frac{L}{R}$$

or $$L = R/2$$

Now, when $L' = 2.1\ L$

or $$\frac{2.1}{2} R \Rightarrow \quad L' > R$$

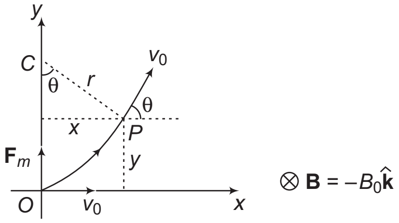
Therefore, deviation of the particle is $\theta = 180°$ as shown in figure.

$\therefore$ $$\mathbf{v}_f = -v_0 \hat{\mathbf{i}}$$

and $$t_{AB} = T/2 = \frac{\pi m}{B_0 q}$$

Type 2. *To find coordinates and velocity of particle at any time t in circular path*

Example 2 *A particle of specific charge* α *enters a uniform magnetic field* $\mathbf{B} = -B_0 \hat{\mathbf{k}}$ *with velocity* $\mathbf{v} = v_0 \hat{\mathbf{i}}$ *from the origin. Find the time dependence of velocity and position of the particle.*



OC = *CP* = radius of circle

HOW TO PROCEED *In such type of problems first of all see the angle between* **v** *and* **B**. *Because only this angle decides the path of the particle. Here, the angle is 90°. Therefore, the path is a circle. If it is a circle, see the plane of the circle (perpendicular to the magnetic field). Here, the plane is xy. Then, see the sense of the rotation.*

Here, it will be anti-clockwise as shown in figure, because at origin the magnetic force is along positive y-direction (which can be seen from Fleming's left hand rule). Find the deviation and radius of the particle.

$$\theta = \omega t = B_0 \alpha t \quad \textit{and} \quad r = \frac{v_0}{B_0 \alpha}$$

Now, according to the figure, find $\mathbf{v}(t)$ *and* $\mathbf{r}(t)$.

Solution Velocity of the particle at any time t is

$$\mathbf{v}(t) = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} = v_0 \cos\theta \hat{\mathbf{i}} + v_0 \sin\theta \hat{\mathbf{j}}$$

or $$\mathbf{v}(t) = v_0 \cos(B_0 \alpha t) \hat{\mathbf{i}} + v_0 \sin(B_0 \alpha t) \hat{\mathbf{j}}$$ **Ans.**

Position of particle at time t is

$$\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\sin\theta \hat{\mathbf{i}} + (r - r\cos\theta)\hat{\mathbf{j}}$$

Substituting the values of r and θ, we have

$$\mathbf{r}(t) = \frac{v_0}{B_0 \alpha} [\sin(B_0 \alpha t)\hat{\mathbf{i}} + \{1 - \cos(B_0 \alpha t)\}\hat{\mathbf{j}}]$$ **Ans.**

Type 3. *To find coordinates and velocity of particle at any time t in helical path*

Example 3 *A particle of specific charge* α *is projected from origin with velocity* $\mathbf{v} = v_0 \hat{\mathbf{i}} - v_0 \hat{\mathbf{k}}$ *in a uniform magnetic field* $\mathbf{B} = -B_0 \hat{\mathbf{k}}$. *Find time dependence of velocity and position of the particle.*

HOW TO PROCEED *Here, the angle between* **v** *and* **B** *is*

$$\theta = \cos^{-1} \frac{\mathbf{v} \cdot \mathbf{B}}{|\mathbf{v}||\mathbf{B}|} = \cos^{-1} \frac{B_0 v_0}{\sqrt{2} v_0 \cdot B_0} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

or $$\theta = 45°$$

Hence, the path is a helix. The axis of the helix is along z-axis (parallel to **B***) and plane of the circle of helix is xy (perpendicular to* **B***). So, in xy-plane, the velocity components and x and y-coordinates are same as that of the above problem. The only change is along z-axis. Velocity component in this direction will remain unchanged while the z-coordinate of particle at time t would be* $v_z t$.

Solution Velocity of particle at time t is

$$\mathbf{v}(t) = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$$

$$= v_0 \cos(B_0 \alpha t)\hat{\mathbf{i}} + v_0 \sin(B_0 \alpha t)\hat{\mathbf{j}} - v_0 \hat{\mathbf{k}}$$ **Ans.**

v_x and v_y can be found in the similar manner as done in Example 2.

The position of the particle at time t would be

$$\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

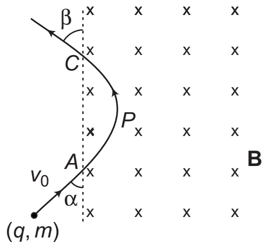
Here, $$z = v_z t = -v_0 t$$

and x and y are same as in Example 2.

Hence, $$\mathbf{r}\,(t) = \frac{v_0}{B_0\alpha}\,[\sin\,(B_0\alpha t)\hat{\mathbf{i}} + \{1 - \cos\,(B_0\alpha t)\}\hat{\mathbf{j}}] - v_0 t\hat{\mathbf{k}}$$ **Ans.**

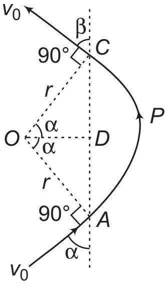
Type 4. *To find the time spent in magnetic field, deviation etc. if a charged particle enters from outside in uniform magnetic field (which extends upto large distance from point of entering)*

Example 4 *A charged particle (q, m) enters a uniform magnetic field $\mathbf{B}$ at angle α as shown in figure with speed v_0. Find*



(a) the angle β at which it leaves the magnetic field.
(b) time spent by the particle in magnetic field and
(c) the distance AC.

Solution (a) Here, velocity of the particle is in the plane of paper while the magnetic field is perpendicular to the paper inwards,. i.e. angle between $\mathbf{v}$ and $\mathbf{B}$ is 90°. So, the path is a circle. The radius of the circle is $r = \dfrac{mv_0}{Bq}$



O is the centre of the circle. In ΔAOC,

$$\angle OCD = \angle OAD$$

or $$90° - \beta = 90° - \alpha$$

$\therefore$ $$\beta = \alpha$$ **Ans.**

(b) $\angle COD = \angle DOA = \alpha$ (as $\angle OCD = \angle OAD = 90° - \alpha$)

$\therefore$ $$\angle AOC = 2\alpha$$

or $$\text{length } APC = r\,(2\alpha) = \frac{2mv_0}{Bq}.\alpha$$

$\therefore$ $$t_{APC} = \frac{APC}{v_0} = \frac{2m\alpha}{Bq}$$ **Ans.**

Alternate method

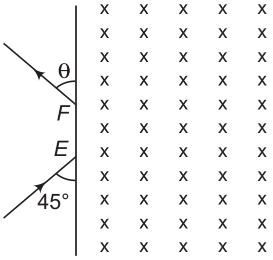
$$t_{APC} = \left(\frac{T}{2\pi}\right)(2\alpha) = \left(\frac{\alpha}{\pi}\right)\cdot T$$

$$= \left(\frac{\alpha}{\pi}\right)\left(\frac{2\pi m}{Bq}\right) = \frac{2\alpha m}{Bq}$$ **Ans.**

(c) Distance, $AC = 2\,(AD) = 2\,(r\sin\alpha)$

$$= \frac{2mv_0}{Bq}\sin\alpha$$ **Ans.**

Example 5 *A particle of mass* $m = 1.6 \times 10^{-27}$ *kg and charge* $q = 1.6 \times 10^{-19}\,C$ *enters a region of uniform magnetic field of strength* 1 *T along the direction shown in figure. The speed of the particle is* 10^7 *m/s.* **(JEE 1984)**

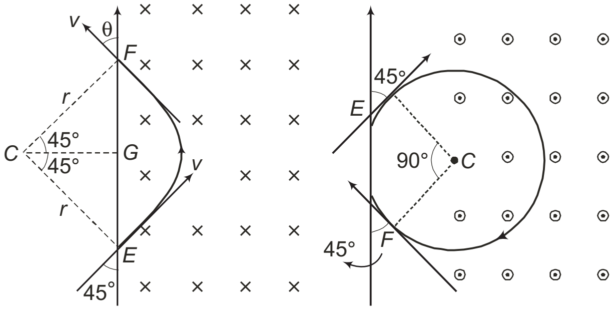


(a) *The magnetic field is directed along the inward normal to the plane of the paper. The particle leaves the region of the field at the point F. Find the distance EF and the angle θ.*

(b) *If the direction of the field is along the outward normal to the plane of the paper, find the time spent by the particle in the region of the magnetic field after entering it at E.*

Solution Inside a magnetic field, speed of charged particle does not change. Further, velocity is perpendicular to magnetic field in both the cases hence path of the particle in the magnetic field will be circular. Centre of circle can be obtained by drawing perpendiculars to velocity (or tangent to the circular path) at *E* and *F*. Radius and angular speed of circular path would be

$$r = \frac{mv}{Bq} \quad \text{and} \quad \omega = \frac{Bq}{m}$$



(i) (ii)

(a) Refer figure (i)

$$\angle CFG = 90° - \theta \text{ and } \angle CEG = 90° - 45° = 45°$$

Since, $CF = CE$

$\therefore$ $\angle CFG = \angle CEG$

or $90° - \theta = 45° \quad \text{or} \quad \theta = 45°$

Further, $FG = GE = r \cos 45°$

$\therefore$ $EF = 2FG = 2r \cos 45° = \dfrac{2mv \cos 45°}{Bq}$

$$= \frac{2\,(1.6 \times 10^{-27})\,(10^7)\left(\dfrac{1}{\sqrt{2}}\right)}{(1)\,(1.6 \times 10^{-19})} = 0.14 \text{ m}$$

Note *That in this case particle completes 1/4th of circle in the magnetic field because the angle rotated is 90°.*

(b) **Refer figure (ii)** In this case, particle will complete $\frac{3}{4}$th of circle in the magnetic field.

Hence, the time spent in the magnetic field :

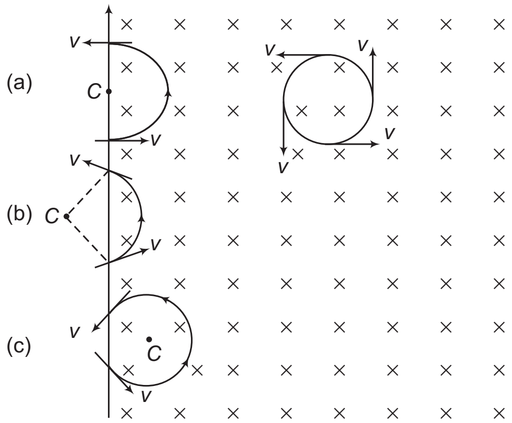
$$t = \frac{3}{4}\text{(time period of circular motion)}$$

$$= \frac{3}{4}\left(\frac{2\pi m}{Bq}\right) = \frac{3\pi m}{2Bq}$$

$$= \frac{(3\pi)(1.6\times10^{-27})}{(2)(1)(1.6\times10^{-19})}$$

$$= 4.712\times10^{-8}\text{ s}$$ **Ans.**

Note *From the above examples, we can see that particle never completes circular path if it enters from outside in uniform magnetic field at right angles (as in Examples 1, 4 and 5). Circle is completed if magnetic field extends all around (Example-2). Following figures explain these points more clearly. In all figures, particle is positively charged.*



In figure (a) *Centre of circular path is lying on the boundary line of magnetic field. Deviation of the particle is 180° and time spent in magnetic field* $t = \frac{T}{2}$.

In figure (b) *Centre of circular path lies outside the magnetic field. Deviation of the particle is less than 180° and time spent in magnetic field* $t < \frac{T}{2}$.

In figure (c) *Centre of circular path lies inside the magnetic field. Deviation of the particle is more than 180° and time spent in magnetic field* $t > \frac{T}{2}$.

Type 5. *Based on the concept of helical path*

Concept

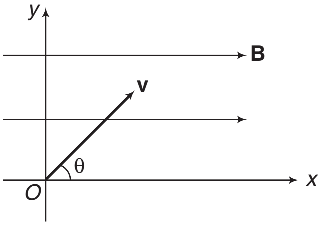
Following points are worthnoting in case of a helical path.

(i) The plane of the circle of the helix is perpendicular to the magnetic field.

(ii) The axis of the helix is parallel to magnetic field.

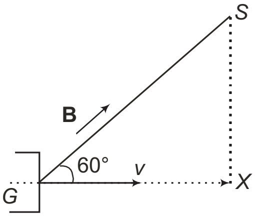
(iii) The particle while moving in helical path in magnetic field touches the line passing through the starting point parallel to the magnetic field after every pitch.

For example, a charged particle is projected from origin in a magnetic field (along x-direction) at angle θ from the x-axis as shown. As the velocity vector $\mathbf{v}$ makes an angle θ with $\mathbf{B}$, its path is a helix. The plane of the circle of the helix is yz (perpendicular to magnetic field) and axis of the helix is parallel to x-axis. The particle while moving in helical path touches the x-axis after every pitch, i.e. it will touch the x-axis at a distance

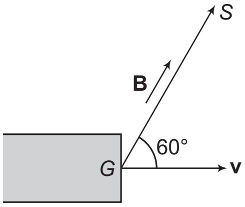


$$\boxed{x = np} \qquad \text{where, } n = 0, 1, 2 \ldots$$

Example 6 *An electron gun G emits electrons of energy 2 keV travelling in the positive x-direction. The electrons are required to hit the spot S where GS = 0.1 m, and the line GS makes an angle of 60° with the x-axis as shown in figure. A uniform magnetic field* **B** *parallel to GS exists in the region outside the electron gun.*



Find the minimum value of B needed to make the electrons hit S. **(JEE 1993)**



Solution Kinetic energy of electron,

$$K = \frac{1}{2}mv^2 = 2 \text{ keV}$$

$\therefore$ Speed of electron, $v = \sqrt{\dfrac{2K}{m}}$

$$v = \sqrt{\frac{2 \times 2 \times 1.6 \times 10^{-16}}{9.1 \times 10^{-31}}} \text{ m/s}$$

$$= 2.65 \times 10^7 \text{ m/s}$$

Since, the velocity ($\mathbf{v}$) of the electron makes an angle of $\theta = 60°$ with the magnetic field $\mathbf{B}$, the path will be a helix. So, the particle will hit S if

$$GS = np$$

Here, $n = 1, 2, 3 \ldots\ldots\ldots\ldots.$

$$p = \text{pitch of helix} = \frac{2\pi m}{qB} v\cos\theta$$

But for B to be minimum, $n = 1$

Hence, $$GS = p = \frac{2\pi m}{qB} v\cos\theta$$

$$B = B_{\min} = \frac{2\pi mv\cos\theta}{q(GS)}$$

Substituting the values, we have

$$B_{\min} = \frac{(2\pi)(9.1 \times 10^{-31})(2.65 \times 10^{7})\left(\frac{1}{2}\right)}{(1.6 \times 10^{-19})(0.1)}$$

or $$B_{\min} = 4.73 \times 10^{-3}\ \text{T}$$ **Ans.**

Type 6. *Based on calculation of magnetic field due to current carrying wires*

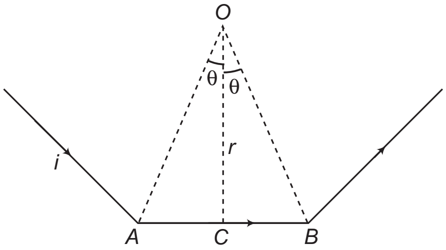
Example 7 *A wire shaped to a regular hexagon of side 2 cm carries a current of 2 A. Find the magnetic field at the centre of the hexagon.*

Solution $\because\ \theta = 30°$

$$\frac{BC}{OC} = \tan\theta \qquad (BC = 1\ \text{cm})$$

$$\therefore\quad \frac{1}{r} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\therefore\quad r = \sqrt{3}\ \text{cm}$$



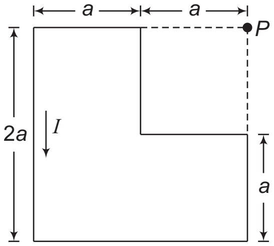
Net magnetic field at O is 6 times the magnetic field due to one side.

$$\therefore\quad B = 6\left[\frac{\mu_0}{4\pi}\frac{i}{r}(\sin\theta + \sin\theta)\right]$$

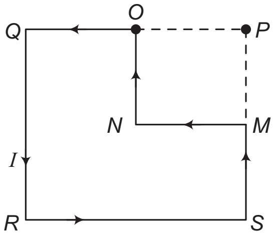
$$= \frac{6\,(10^{-7})\,(2)}{\sqrt{3} \times 10^{-2}}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= 6.9 \times 10^{-5}\ \text{T}$$ **Ans.**

Example 8 *Find the magnetic field* **B** *at the point P in figure.*



Solution Magnetic field at P due to SM and OQ is zero. Due to QR and RS are equal and outwards. Due to MN and NO are equal and inwards.



Due to QR and RS,

$$B_1 = 2\left[\frac{\mu_0}{4\pi}\frac{I}{2a}(\sin 0° + \sin 45°)\right]$$

$$= \frac{\mu_0 I}{4\sqrt{2}\pi a} \qquad \text{[outwards]}$$

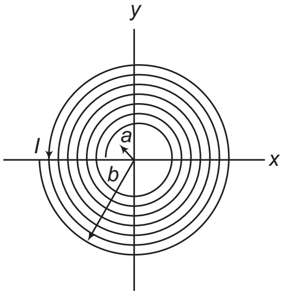
Due to MN and NO,

$$B_2 = 2\left[\frac{\mu_0}{4\pi}\frac{I}{a}(\sin 0° + \sin 45°)\right]$$

$$= \frac{\mu_0 I}{2\sqrt{2}\,\pi a} \qquad \text{[inwards]}$$

$$\therefore \qquad B_{\text{net}} = B_2 - B_1 = \frac{\mu_0 I}{4\sqrt{2}\,\pi a} \qquad \text{[inwards]}$$

Example 9 *A long insulated copper wire is closely wound as a spiral of N turns. The spiral has inner radius a and outer radius b. The spiral lies in the xy-plane and a steady current I flows through the wire. The z-component of the magnetic field at the centre of the spiral is* **(JEE 2011)**



(a) $\frac{\mu_0 NI}{2(b-a)}\ln\left(\frac{b}{a}\right)$

(b) $\frac{\mu_0 NI}{2(b-a)}\ln\left(\frac{b+a}{b-a}\right)$

(c) $\frac{\mu_0 NI}{2b}\ln\left(\frac{b}{a}\right)$

(d) $\frac{\mu_0 NI}{2b}\ln\left(\frac{b+a}{b-a}\right)$

Solution (a) If we take a small strip of dr at distance r from centre, then number of turns in this strip would be

$$dN = \left(\frac{N}{b-a}\right)dr$$

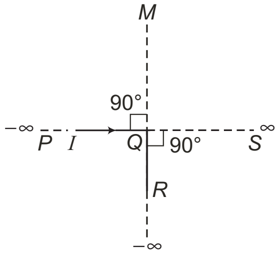
Magnetic field due to this element at the centre of the coil will be

$$dB = \frac{\mu_0 (dN) I}{2r} = \frac{\mu_0 NI}{(b-a)}\frac{dr}{2r}$$

$$\therefore \qquad B = \int_{r=a}^{r=b} dB = \frac{\mu_0 NI}{2(b-a)}\ln\left(\frac{b}{a}\right)$$

∴ Correct answer is (a).

Example 10 *An infinitely long conductor PQR is bent to form a right angle as shown in figure. A current I flows through PQR. The magnetic field due to this current at the point M is H_1. Now, another infinitely long straight conductor QS is connected at Q, so that current is I/2 in QR as well as in QS, the current in PQ remaining unchanged. The magnetic field at M is now H_2. The ratio H_1/H_2 is given by* **(JEE 2000)**



(a) 1/2 *(b)* 1 *(c)* 2/3 *(d)* 2

Solution H_1 = Magnetic field at M due to PQ + Magnetic field at M due to QR

But magnetic field at M due to $QR = 0$

∴ Magnetic field at M due to PQ (or due to current I in PQ)

$$= H_1$$

Now, H_2 = Magnetic field at M due to PQ (current I)

+ magnetic field at M due to QS (current I/2) + magnetic field at M due to QR

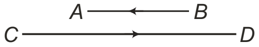
$$= H_1 + \frac{H_1}{2} + 0 = \frac{3}{2} H_1$$

$$\frac{H_1}{H_2} = \frac{2}{3}$$

Note *Magnetic field at any point lying on the current carrying straight conductor is zero.*

Type 7. *Based on the magnetic force on current carrying wire*

Example 11 *A long horizontal wire AB, which is free to move in a vertical plane and carries a steady current of 20 A, is in equilibrium at a height of 0.01 m over another parallel long wire CD which is fixed in a horizontal plane and carries a steady current of 30 A, as shown in figure. Show that when AB is slightly depressed, it executes simple harmonic motion. Find the period of oscillations.* **(JEE 1994)**



Solution Let m be the mass per unit length of wire AB. At a height x above the wire CD, magnetic force per unit length on wire AB will be given by

$$F_m = \frac{\mu_0}{2\pi} \frac{i_1 i_2}{x} \quad \text{(upwards)} \quad \ldots\text{(i)}$$

Weight per unit length of wire AB is

$$F_g = mg \quad \text{(downwards)}$$

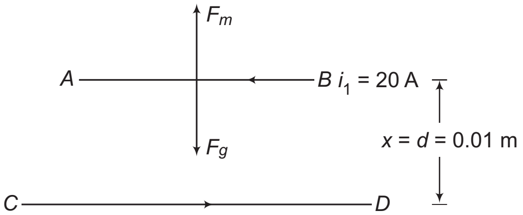
Here, m = mass per unit length of wire AB

At $x = d$, wire is in equilibrium, i.e.

$$F_m = F_g$$

or
$$\frac{\mu_0}{2\pi}\frac{i_1 i_2}{d} = mg$$

or
$$\frac{\mu_0}{2\pi}\frac{i_1 i_2}{d^2} = \frac{mg}{d} \qquad \text{...(ii)}$$



When AB is depressed, x decreases therefore, F_m will increase, while F_g remains the same. Change in magnetic force will become the net restoring force, Let AB is displaced by dx downwards.

Differentiating Eq. (i) w.r.t. x, we get

$$dF_m = -\frac{\mu_0}{2\pi}\frac{i_1 i_2}{x^2}\,.\,dx \qquad \text{...(iii)}$$

i.e. restoring force, $F = dF_m \propto -dx$

Hence, the motion of wire is simple harmonic.

From Eqs. (ii) and (iii), we can write

$$dF_m = -\left(\frac{mg}{d}\right)\cdot dx \qquad [\because\ x = d]$$

$\therefore$ Acceleration of wire, $a = -\left(\frac{g}{d}\right).\,dx$

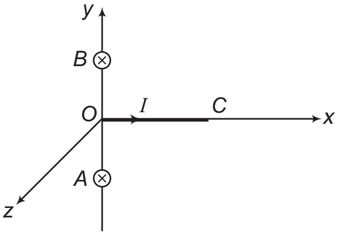
Hence, period of oscillation

$$T = 2\pi\sqrt{\frac{|\text{displacement}|}{|\text{acceleration}|}} = 2\pi\sqrt{\left|\frac{dx}{a}\right|}$$

or
$$T = 2\pi\sqrt{\frac{d}{g}} = 2\pi\sqrt{\frac{0.01}{9.8}}$$

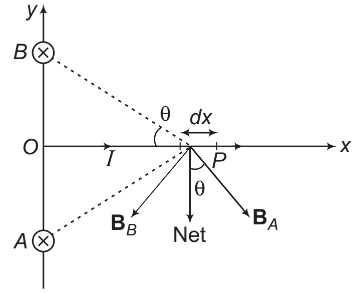
or
$$T = 0.2 \text{ s}$$ **Ans.**

Example 12 *A straight segment OC (of length L) of a circuit carrying a current I is placed along the x-axis. Two infinitely long straight wires A and B, each extending from $z = -\infty$ to $+\infty$, are fixed at $y = -a$ and $y = +a$ respectively, as shown in the figure. If the wires A and B each carry a current I into the plane of the paper, obtain the expression for the force acting on the segment OC. What will be the force on OC if the current in the wire B is reversed?*



(JEE 1992)

Solution (a) Let us assume a segment of wire OC at a point P, a distance x from the centre of length dx as shown in figure.



Magnetic field at P due to current in wires A and B will be in the directions perpendicular to AP and BP respectively as shown.

$$|\mathbf{B}| = \frac{\mu_0}{2\pi}\frac{I}{AP}$$

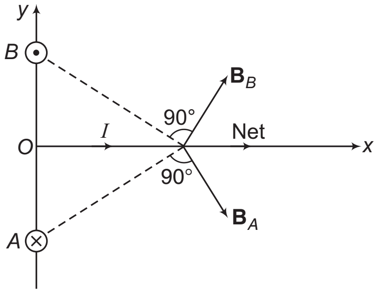
Therefore, net magnetic force at P will be along negative y-axis as shown below

$$B_{\text{net}} = 2|\mathbf{B}|\cos\theta$$

$$= 2\left(\frac{\mu_0}{2\pi}\right)\frac{I}{AP}\left(\frac{x}{AP}\right)$$

$$B_{\text{net}} = \left(\frac{\mu_0}{\pi}\right)\frac{Ix}{(AP)^2}$$

$$B_{\text{net}} = \frac{\mu_0}{\pi}\cdot\frac{Ix}{(a^2+x^2)}$$



Therefore, force on this element will be

$$dF = I\left\{\frac{\mu_0}{\pi}\frac{Ix}{a^2+x^2}\right\}dx \qquad \text{[in negative } z\text{-direction]}$$

∴ Total force on the wire will be

$$F = \int_{x=0}^{x=L} dF = \frac{\mu_0 I^2}{\pi}\int_0^L \frac{xdx}{x^2+a^2}$$

$$= \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{L^2+a^2}{a^2}\right) \qquad \text{[in negative } z\text{-axis]}$$

Hence,

$$\mathbf{F} = -\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{L^2+a^2}{a^2}\right)\hat{\mathbf{k}}$$

(b) When direction of current in B is reversed net magnetic field is along the current. Hence, force is zero.

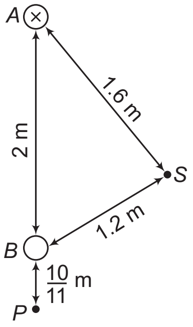
Example 13 *Two long straight parallel wires are 2 m apart, perpendicular to the plane of the paper. The wire A carries a current of 9.6 A, directed into the plane of the paper. The wire B carries a current such that the magnetic field of induction at the point P, at a distance of 10 / 11 m from the wire B, is zero.*

Find **(JEE 1997)**

(a) the magnitude and direction of the current in B.

(b) the magnitude of the magnetic field of induction at the point S.

(c) the force per unit length on the wire B.

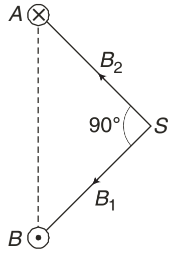


Solution (a) Direction of current at B should be perpendicular to paper outwards. Let current in this wire be i_B. Then,

$$\frac{\mu_0}{2\pi}\frac{i_A}{\left(2+\frac{10}{11}\right)}=\frac{\mu_0}{2\pi}\frac{i_B}{(10/11)}$$

or $$\frac{i_B}{i_A}=\frac{10}{32}$$

or $$i_B=\frac{10}{32}\times i_A=\frac{10}{32}\times 9.6=3\text{ A}$$



(b) Since, $AS^2+BS^2=AB^2$

$\therefore$ $$\angle ASB=90^\circ$$

At S : B_1 = Magnetic field due to i_A

$$=\frac{\mu_0}{2\pi}\frac{i_A}{1.6}=\frac{(2\times 10^{-7})(9.6)}{1.6}$$

$$=12\times 10^{-7}\text{ T}$$

B_2 = Magnetic field due to i_B

$$=\frac{\mu_0}{2\pi}\frac{i_B}{1.2}$$

$$=\frac{(2\times 10^{-7})(3)}{1.2}$$

$$=5\times 10^{-7}\text{ T}$$

Since, B_1 and B_2 are mutually perpendicular. Net magnetic field at S would be

$$B=\sqrt{B_1^2+B_2^2}$$

$$=\sqrt{(12\times 10^{-7})^2+(5\times 10^{-7})^2}$$

$$=13\times 10^{-7}\text{ T}$$

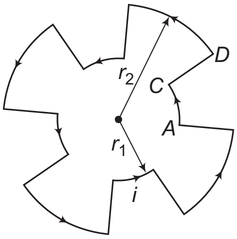
(c) Force per unit length on wire B :

$$\frac{F}{l}=\frac{\mu_0}{2\pi}\frac{i_Ai_B}{r}\qquad [\because\ r=AB=2\text{ m}]$$

$$=\frac{(2\times 10^{-7})(9.6\times 3)}{2}$$

$$=2.88\times 10^{-6}\text{ N/m}$$

Example 14 *A current of 10 A flows around a closed path in a circuit which is in the horizontal plane as shown in the figure. The circuit consists of eight alternating arcs of radii $r_1 = 0.08$ m and $r_2 = 0.12$ m. Each subtends the same angle at the centre.* **(JEE 2001)**



(a) Find the magnetic field produced by this circuit at the centre.

(b) An infinitely long straight wire carrying a current of 10 A is passing through the centre of the above circuit vertically with the direction of the current being into the plane of the circuit. What is the force acting on the wire at the centre due to the current in the circuit? What is the force acting on the arc AC and the straight segment CD due to the current at the centre?

Solution (a) Given, $i = 10$ A, $r_1 = 0.08$ m and $r_2 = 0.12$ m. Straight portions, i.e. *CD* etc, will produce zero magnetic field at the centre. Rest eight arcs will produce the magnetic field at the centre in the same direction, i.e. perpendicular to the paper outwards or vertically upwards and its magnitude is

$$B = B_{\text{inner arcs}} + B_{\text{outer arcs}}$$

$$= \frac{1}{2}\left\{\frac{\mu_0 i}{2r_1}\right\} + \frac{1}{2}\left\{\frac{\mu_0 i}{2r_2}\right\}$$

$$= \left(\frac{\mu_0}{4\pi}\right)(\pi i)\left(\frac{r_1 + r_2}{r_1 r_2}\right)$$

Substituting the values, we have

$$B = \frac{(10^{-7})(3.14)(10)(0.08 + 0.12)}{(0.08 \times 0.12)}$$

$$B = 6.54 \times 10^{-5} \text{ T}$$

(vertically upward or outward normal to the paper)

(b) Force on *AC*

Force on circular portions of the circuit, i.e. *AC* etc, due to the wire at the centre will be zero because magnetic field due to the central wire at these arcs will be tangential ($\theta = 180°$).

Force on *CD*

Current in central wire is also $i = 10$ A. Magnetic field at distance x due to central wire

$$B = \frac{\mu_0}{2\pi} \cdot \frac{i}{x}$$

∴ Magnetic force on element dx due to this magnetic field

$$dF = (i)\left(\frac{\mu_0}{2\pi} \cdot \frac{i}{x}\right) \cdot dx = \left(\frac{\mu_0}{2\pi}\right) i^2 \frac{dx}{x} \qquad [F = ilB \sin 90°]$$

Therefore, net force on CD is

$$F = \int_{x=r_1}^{x=r_2} dF = \frac{\mu_0 i^2}{2\pi}\int_{0.08}^{0.12}\frac{dx}{x} = \frac{\mu_0}{2\pi} i^2 \ln\left(\frac{3}{2}\right)$$

Substituting the values, $F = (2\times10^{-7})(10)^2 \ln(1.5)$

or $F = 8.1\times10^{-6}$ N (inwards)

Force on wire at the centre

Net magnetic field at the centre due to the circuit is in vertical direction and current in the wire in centre is also in vertical direction. Therefore, net force on the wire at the centre will be zero. ($\theta = 180°$). Hence,

(i) Force acting on the wire at the centre is zero.
(ii) Force on arc $AC = 0$.
(iii) Force on segment CD is 8.1×10^{-6} N (inwards).

Type 8. *Based on the magnetic force on a charged particle in electric and (or) magnetic field*

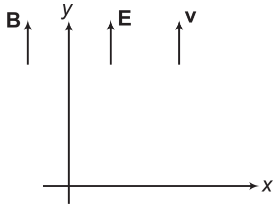
Example 15 *Consider the motion of a positive point charge in a region where there are simultaneous uniform electric and magnetic fields* $\mathbf{E} = E_0\hat{\mathbf{j}}$ *and* $\mathbf{B} = B_0\hat{\mathbf{j}}$. *At time* $t = 0$, *this charge has velocity* $\mathbf{v}$ *in the xy-plane making an angle* θ *with the x-axis. Which of the following option(s) is(are) correct for time* $t > 0$? **(JEE 2012)**

(a) If $\theta = 0°$, *the charge moves in a circular path in the xz-plane.*
(b) If $\theta = 0°$, *the charge undergoes helical motion with constant pitch along the y-axis*
(c) If $\theta = 10°$, *the charge undergoes helical motion with its pitch increasing with time along the y-axis.*
(d) If $\theta = 90°$, *the charge undergoes linear but accelerated motion along the y-axis.*

Solution Magnetic field will rotate the particle in a circular path (in xz-plane or perpendicular to **B**). Electric field will exert a constant force on the particle in positive y-direction. Therefore, resultant path is neither purely circular nor helical or the options (a) and (b) both are wrong.

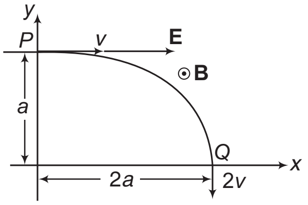
(c) $v_\perp$ and **B** will rotate the particle in a circular path in xz- plane (or perpendicular to **B**). Further, $v_{\parallel}$ and **E** will move the particle (with increasing speed) along positive y-axis (or along the axis of above circular path). Therefore, the resultant path is helical with increasing pitch along the y-axis (or along **B** and **E**). Therefore, option (c) is correct.

(d)



Magnetic force is zero, as θ between **B** and **v** is zero. But, electric force will act in y-direction. Therefore, motion is 1-D and uniformly accelerated (towards positive y-direction).

Therefore, option (d) is also correct.



Example 16 *A particle of charge +q and mass m moving under the influence of a uniform electric field* $E\,\hat{\mathbf{i}}$ *and uniform magnetic field* $B\,\hat{\mathbf{k}}$ *follows a trajectory from P to Q as shown in figure. The velocities at P and Q are* $v\,\hat{\mathbf{i}}$ *and* $-2\,\hat{\mathbf{j}}$. *Which of the following statement(s) is/are correct ?* **(JEE 1991)**

(a) $E = \dfrac{3}{4}\left[\dfrac{mv^2}{qa}\right]$

(b) *Rate of work done by the electric field at P is* $\dfrac{3}{4}\left[\dfrac{mv^3}{a}\right]$

(c) *Rate of work done by the electric field at P is zero*

(d) *Rate of work done by both the fields at Q is zero*

Solution Magnetic force does not do work. From work-energy theorem :

$$W_{F_e} = \Delta \text{KE}$$

or
$$(qE)\,(2a) = \frac{1}{2}m\,[4v^2 - v^2]$$

or
$$E = \frac{3}{4}\left(\frac{mv^2}{qa}\right)$$

$\therefore$ Option (a) is correct.

At P, rate of work done by electric field

$$= \mathbf{F}_e \cdot \mathbf{v} = (qE)\,(v)\cos 0^\circ$$
$$= q\left(\frac{3}{4}\,\frac{mv^2}{qa}\right)v = \frac{3}{4}\left(\frac{mv^3}{a}\right)$$

Therefore, option (b) is also correct.

Rate of work done at Q :

of electric field $= \mathbf{F}_e \cdot \mathbf{v} = (qE)\,(2v)\cos 90^\circ = 0$ and of magnetic field is always zero.

Therefore, option (d) is also correct.

Note that $\mathbf{F}_e = qE\hat{\mathbf{i}}$

Example 17 *A proton moving with a constant velocity passes through a region of space without any change in its velocity. If E and B represent the electric and magnetic fields, respectively. Then, this region of space may have* **(JEE 1985)**

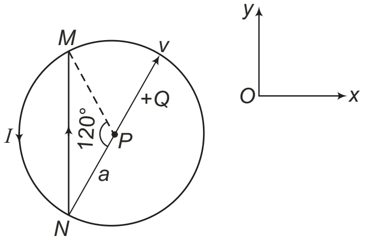
(a) $E = 0, B = 0$
(b) $E = 0, B \neq 0$
(c) $E \neq 0, B = 0$
(d) $E \neq 0, B \neq 0$

Solution If both E and B are zero, then $\mathbf{F}_e$ and $\mathbf{F}_m$ both are zero. Hence, velocity may remain constant. Therefore, option (a) is correct.

If $E = 0$, $B \neq 0$ but velocity is parallel or antiparallel to magnetic field, then also $\mathbf{F}_e$ and $\mathbf{F}_m$ both are zero. Hence, option (b) is also correct.

If $E \neq 0$, $B \neq 0$ but $\mathbf{F}_e + \mathbf{F}_m = 0$, then again velocity may remain constant or option (d) is also correct.

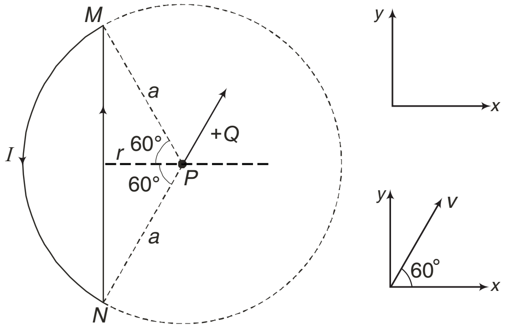
Example 18 *A wire loop carrying a current I is placed in the xy-plane as shown in figure.* **(JEE 1991)**



(a) *If a particle with charge +Q and mass m is placed at the centre P and given a velocity* **v** *along NP (see figure), find its instantaneous acceleration.*

(b) *If an external uniform magnetic induction field* $\mathbf{B} = B\hat{\mathbf{i}}$ *is applied, find the force and the torque acting on the loop due to this field.*

Solution (a) Magnetic field at P due to arc of circle,



Subtending an angle of 120° at centre would be

$$B_1 = \frac{1}{3}(\text{field due to circle}) = \frac{1}{3}\left(\frac{\mu_0 I}{2a}\right)$$

$$= \frac{\mu_0 I}{6a} \qquad \text{[outwards]}$$

$$= \frac{0.16\mu_0 I}{a} \qquad \text{[outwards]}$$

or $$\mathbf{B}_1 = \frac{0.16\mu_0 I}{a}\hat{\mathbf{k}}$$

Magnetic field due to straight wire NM at P,

$$B_2 = \frac{\mu_0}{4\pi}\frac{I}{r}(\sin 60° + \sin 60°)$$

Here, $$r = a\cos 60°$$

$\therefore$ $$B_2 = \frac{\mu_0}{4\pi}\frac{I}{a\cos 60°}(2\sin 60°)$$

or $$B_2 = \frac{\mu_0}{2\pi}\frac{I}{a}\tan 60°$$

$$= \frac{0.27\,\mu_0 I}{a} \qquad \text{(inwards)}$$

or $$\mathbf{B}_2 = -\frac{0.27\,\mu_0 I}{a}\,\hat{\mathbf{k}}$$

∴ $$\mathbf{B}_{\text{net}} = \mathbf{B}_1 + \mathbf{B}_2 = -\frac{0.11\,\mu_0\,I}{a}\,\hat{\mathbf{k}}$$

Now, velocity of particle can be written as

$$\mathbf{v} = v\cos 60^\circ\,\hat{\mathbf{i}} + v\sin 60^\circ\,\hat{\mathbf{j}}$$

$$= \frac{v}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}v}{2}\,\hat{\mathbf{j}}$$

Magnetic force,

$$\mathbf{F}_m = Q\,(\mathbf{v}\times\mathbf{B})$$

$$= \frac{0.11\,\mu_0 IQv}{2a}\,\hat{\mathbf{j}} - \frac{0.11\sqrt{3}\,\mu_0 IQv}{2a}\,\hat{\mathbf{i}}$$

∴ Instantaneous acceleration,

$$\mathbf{a} = \frac{\mathbf{F}_m}{m} = \frac{0.11\,\mu_0 IQv}{2am}\,(\hat{\mathbf{j}} - \sqrt{3}\,\hat{\mathbf{i}})$$

(b) In uniform magnetic field, force on a current loop is zero. Further, magnetic dipole moment of the loop will be

$$\mathbf{M} = (IA)\,\hat{\mathbf{k}}$$

Here, A is the area of the loop.

$$A = \frac{1}{3}(\pi a^2) - \frac{1}{2}[2\times a\sin 60^\circ]\,[a\cos 60^\circ]$$

$$= \frac{\pi a^2}{3} - \frac{a^2}{2}\sin 120^\circ$$

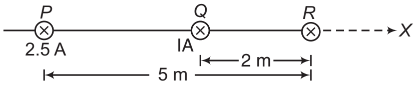
$$= 0.61\,a^2$$

∴ $$\mathbf{M} = (0.61\,Ia^2)\,\hat{\mathbf{k}}$$

Given, $$\mathbf{B} = B\hat{\mathbf{i}}$$

∴ $$\tau = \mathbf{M}\times\mathbf{B} = (0.61\,Ia^2B)\,\hat{\mathbf{j}}$$

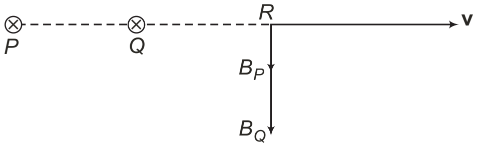
Example 19 *Two long parallel wires carrying currents 2.5 A and I (ampere) in the same direction (directed into the plane of the paper) are held at P and Q respectively such that they are perpendicular to the plane of paper. The points P and Q are located at a distance of 5 m and 2 m respectively from a collinear point R (see figure).* **(JEE 1990)**



(a) An electron moving with a velocity of 4×10^5 m/s along the positive x-direction experiences a force of magnitude 3.2×10^{-20} N at the point R. Find the value of I.

(b) Find all the positions at which a third long parallel wire carrying a current of magnitude 2.5 A may be placed, so that the magnetic induction at R is zero.

Solution (a) Magnetic field at R due to both the wires P and Q will be downwards as shown in figure.



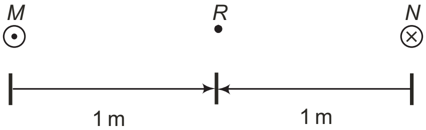
Therefore, net field at R will be sum of these two.

$$B = B_P + B_Q$$

$$= \frac{\mu_0}{2\pi}\frac{I_P}{5} + \frac{\mu_0}{2\pi}\frac{I_Q}{2} = \frac{\mu_0}{2\pi}\left(\frac{2.5}{5} + \frac{I}{2}\right)$$

$$= \frac{\mu_0}{4\pi}(I+1) = 10^{-7}(I+1)$$

Net force on the electron will be



$$F_m = Bqv \sin 90°$$

or $$(3.2 \times 10^{-20}) = (10^{-7})\,(I+1)\,(1.6 \times 10^{-19})\,(4 \times 10^5)$$

or $$I + 1 = 5$$

$\therefore$ $$I = 4\text{ A}$$

(b) Net field at R due to wires P and Q is

$$B = 10^{-7}(I+1)\text{ T}$$

$$= 5 \times 10^{-7}\text{ T}$$

Magnetic field due to third wire carrying a current of 2.5 A should be 5×10^{-7} T in upward direction, so that net field at R becomes zero. Let distance of this wire from R be r. Then,

$$\frac{\mu_0}{2\pi}\frac{2.5}{r} = 5 \times 10^{-7}$$

or $$\frac{(2 \times 10^{-7})\,(2.5)}{r} = 5 \times 10^{-7}\text{ m}$$

or $$r = 1\text{ m}$$

So, the third wire can be put at M or N as shown in figure.

If it is placed at M, then current in it should be outwards and if placed at N, then current be inwards.

Type 9. *Path of charged particle in both electric and magnetic fields*

Concept

Here, normally two cases are popular. In the first case, $\mathbf{E} \uparrow\uparrow \mathbf{B}$ and particle velocity is perpendicular to both of these fields. In the second case, $\mathbf{E} \perp \mathbf{B}$ and the particle is released from rest. Let us now consider both the cases separately.

Example 20 *When* $\mathbf{E} \uparrow\uparrow \mathbf{B}$ *and particle velocity is perpendicular to both of these fields.*

Solution Consider a particle of charge q and mass m released from the origin with velocity $\mathbf{v} = v_0\hat{\mathbf{i}}$ into a region of uniform electric and magnetic fields parallel to y-axis, i.e. $\mathbf{E} = E_0\hat{\mathbf{j}}$ and $\mathbf{B} = B_0\hat{\mathbf{j}}$. The electric field accelerates the particle in y-direction, i.e. y-component of velocity goes on increasing with acceleration,

$$a_y = \frac{F_y}{m} = \frac{F_e}{m} = \frac{qE_0}{m} \qquad \text{...(i)}$$

The magnetic field rotates the particle in a circle in xz-plane (perpendicular to magnetic field). The resultant path of the particle is a **helix with increasing pitch**. The axis of the plane is parallel to y-axis. Velocity of the particle at time t would be

$$\mathbf{v}(t) = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

Here,
$$v_y = a_y t = \frac{qE_0}{m}t$$

and
$$v_x^2 + v_z^2 = \text{constant} = v_0^2$$

$$\theta = \omega t = \frac{Bq}{m}t$$

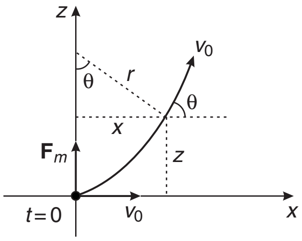
$$v_x = v_0\cos\theta = v_0\cos\left(\frac{Bqt}{m}\right)$$

and
$$v_z = v_0\sin\theta = v_0\sin\left(\frac{Bqt}{m}\right)$$

$\therefore$
$$\boxed{\mathbf{v}(t) = v_0\cos\left(\frac{Bqt}{m}\right)\hat{\mathbf{i}} + \left(\frac{qE_0}{m}t\right)\hat{\mathbf{j}} + v_0\sin\left(\frac{Bqt}{m}\right)\hat{\mathbf{k}}}$$

Similarly, position vector of particle at time t can be given by

$$\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$



Here,
$$y = \frac{1}{2}a_y t^2 = \frac{1}{2}\left(\frac{qE_0}{m}\right)t^2$$

$$x = r\sin\theta = \left(\frac{mv_0}{Bq}\right)\sin\left(\frac{Bqt}{m}\right)$$

and
$$z = r(1-\cos\theta) = \left(\frac{mv_0}{Bq}\right)\left[\left\{1-\cos\left(\frac{Bqt}{m}\right)\right\}\right]$$

$\therefore$
$$\boxed{\mathbf{r}(t) = \left(\frac{mv_0}{Bq}\right)\sin\left(\frac{Bqt}{m}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{qE_0}{m}\right)t^2\,\hat{\mathbf{j}} + \left(\frac{mv_0}{Bq}\right)\left[\left\{1-\cos\left(\frac{Bqt}{m}\right)\right\}\right]\hat{\mathbf{k}}}$$

Note *(i) While moving in helical path the particle touches the y-axis after every T or after,*

$$t = nT, \qquad \text{where } n = 0, 1, 2\ldots$$

Here,
$$T = \frac{2\pi m}{Bq}$$

(ii) At $t = 0$, *velocity is along positive x-axis and magnetic field is along y-axis. Therefore, magnetic force is along positive z-axis and the particle rotates in xz-plane as shown in figure.*

Example 21 *When* $\mathbf{E} \perp \mathbf{B}$ *and the particle is released at rest from origin.*

Solution Consider a particle of charge q and mass m emitted at origin with zero initial velocity into a region of uniform electric and magnetic fields. The field $\mathbf{E}$ is acting along x-axis and field $\mathbf{B}$ along y-axis, i.e.

$$\mathbf{E} = E_0\hat{\mathbf{i}}$$

and $$\mathbf{B} = B_0\hat{\mathbf{j}}$$

Electric field will provide the particle an acceleration (and therefore a velocity component) in x-direction and the magnetic field will rotate the particle in xz-plane (perpendicular to $\mathbf{B}$). Hence, at any instant of time its velocity (and hence, position) will have only x and z components. Let at time t its velocity be

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_z\hat{\mathbf{k}}$$

Net force on it at this instant is

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\,(\mathbf{v} \times \mathbf{B})$$

$$= q\,[E_0\hat{\mathbf{i}} + (v_x\hat{\mathbf{i}} + v_z\hat{\mathbf{k}}) \times (B_0\hat{\mathbf{j}})]$$

or $$\mathbf{F} = q\,(E_0 - v_zB_0)\,\hat{\mathbf{i}} + qv_xB_0\hat{\mathbf{k}}$$

$\therefore$ $$\mathbf{a} = \frac{\mathbf{F}}{m} = a_x\hat{\mathbf{i}} + a_z\hat{\mathbf{k}}$$

where, $$a_x = \frac{q}{m}\,(E_0 - v_zB_0) \qquad \text{...(i)}$$

and $$a_z = \frac{q}{m}\,v_xB_0 \qquad \text{...(ii)}$$

Differentiating Eq. (i) w.r.t. time, we have

$$\frac{d^2v_x}{dt^2} = -\frac{qB_0}{m}\left(\frac{dv_z}{dt}\right)$$

But, $$\frac{dv_z}{dt} = a_z = \frac{qB_0}{m}\,v_x$$

$\therefore$ $$\frac{d^2v_x}{dt^2} = -\left(\frac{qB_0}{m}\right)^2 v_x \qquad \text{...(iii)}$$

Comparing this equation with the differential equation of SHM $\left(\frac{d^2y}{dt^2} = -\omega^2 y\right)$, we get

$$\omega = \frac{qB_0}{m}$$

and the general solution of Eq. (iii) is

$$v_x = A\sin(\omega t + \phi) \qquad \text{...(iv)}$$

At time $t = 0$, $v_x = 0$, hence, $\phi = 0$

Again, $$\frac{dv_x}{dt} = A\omega\cos\omega t \qquad (\text{as } \phi = 0)$$

From Eq. (i), $a_x = \frac{qE_0}{m}$ at $t = 0$, as $v_z = 0$ at $t = 0$

$\therefore$ $$A\omega = \frac{qE_0}{m} \quad \text{or} \quad A = \frac{qE_0}{\omega m}$$

Substituting $\omega = \frac{qB_0}{m}$, we get $$A = \frac{E_0}{B_0}$$

Therefore, Eq. (iv) becomes

where,

$$\boxed{\begin{array}{c} v_x = \dfrac{E_0}{B_0}\sin\omega t \\[2ex] \omega = \dfrac{qB_0}{m} \end{array}}$$

Now substituting value of v_x in Eq. (ii), we get

$$\frac{dv_z}{dt} = \frac{qE_0}{m}\sin\omega t$$

$$\therefore \qquad \int_0^{v_z} dv_z = \frac{qE_0}{m}\int_0^t \sin\omega t\, dt$$

or

$$v_z = \frac{qE_0}{\omega m}(1 - \cos\omega t)$$

Substituting $\omega = \dfrac{qB_0}{m}$, we get

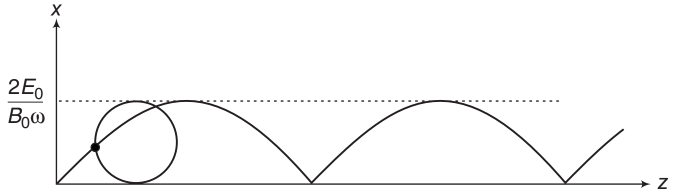
$$\boxed{v_z = \frac{E_0}{B_0}(1 - \cos\omega t)}$$

On integrating equations for v_x and v_z and knowing that at $t = 0$, $x = 0$ and $z = 0$, we get

and

$$\boxed{\begin{array}{c} x = \dfrac{E_0}{B_0\omega}(1 - \cos\omega t) \\[2ex] z = \dfrac{E_0}{B_0\omega}(\omega t - \sin\omega t) \end{array}}$$

These equations are the equations for a **cycloid** which is defined as the path generated by the point on the circumference of a wheel rolling on a ground.



In the present case, the radius of the rolling wheel is $\dfrac{E_0}{B_0\omega}$, the maximum displacement along x-direction is $\dfrac{2E_0}{B_0\omega}$. The x-displacement becomes zero at $t = 0, 2\pi/\omega, 4\pi/\omega$, etc.

Note *Path of a charged particle in uniform electric and magnetic field will remain unchanged if*

$$\mathbf{F}_{\text{net}} = 0$$

or

$$\mathbf{F}_e + \mathbf{F}_m = 0$$

or

$$q\mathbf{E} + q(\mathbf{v}\times\mathbf{B}) = 0$$

or

$$\mathbf{E} = -(\mathbf{v}\times\mathbf{E})$$

$$= (\mathbf{E}\times\mathbf{v})$$

Miscellaneous Examples

Example 22 *A cyclotron's oscillator frequency is* 10 *MHz. What should be the operating magnetic field for accelerating protons? If the radius of its dees is* 60 *cm, what is the kinetic energy (in MeV) of the proton beam produced by the accelerator?*

$$(e = 1.60 \times 10^{-19} C,\ m_p = 1.67 \times 10^{-27} kg,\ 1\ MeV = 1.6 \times 10^{-13} J)$$

Solution **Magnetic field** Cyclotron's oscillator frequency should be same as the proton's revolution frequency (in circular path)

$$\therefore \qquad f = \frac{Bq}{2\pi m}$$

$$\text{or} \qquad B = \frac{2\pi mf}{q}$$

Substituting the values in SI units, we have

$$B = \frac{(2)(22/7)(1.67 \times 10^{-27})(10 \times 10^6)}{1.6 \times 10^{-19}}$$

$$= 0.67 \text{ T}$$ **Ans.**

Kinetic energy Let final velocity of proton just after leaving the cyclotron is v. Then, radius of dee should be equal to

$$R = \frac{mv}{Bq} \quad \text{or} \quad v = \frac{BqR}{m}$$

$\therefore$ Kinetic energy of proton,

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{BqR}{m}\right)^2 = \frac{B^2 q^2 R^2}{2m}$$

Substituting the values in SI units, we have

$$K = \frac{(0.67)^2 (1.6 \times 10^{-19})^2 (0.60)^2}{2 \times 1.67 \times 10^{-27}}$$

$$= 1.2 \times 10^{-12} \text{ J}$$

$$= \frac{1.2 \times 10^{-12}}{(1.6 \times 10^{-19})(10^6)} \text{ MeV}$$

$$= 7.5 \text{ MeV}$$ **Ans.**

Example 23 *A charged particle carrying charge* $q = 1\,\mu C$ *moves in uniform magnetic field with velocity* $v_1 = 10^6$ *m/s at angle* 45° *with x-axis in the xy-plane and experiences a force* $F_1 = 5\sqrt{2}$ *mN along the negative z-axis. When the same particle moves with velocity* $v_2 = 10^6$ *m/s along the z-axis, it experiences a force* F_2 *in y-direction. Find*

(a) the magnitude and direction of the magnetic field

(b) the magnitude of the force F_2.

Solution F_2 is in y-direction when velocity is along z-axis. Therefore, magnetic field should be along x-axis. So let,

$$\mathbf{B} = B_0\hat{\mathbf{i}}$$

(a) Given, $$\mathbf{v}_1 = \frac{10^6}{\sqrt{2}}\hat{\mathbf{i}} + \frac{10^6}{\sqrt{2}}\hat{\mathbf{j}}$$

and $$\mathbf{F}_1 = -5\sqrt{2}\times 10^{-3}\,\hat{\mathbf{k}}$$

From the equation, $$\mathbf{F} = q\,(\mathbf{v}\times\mathbf{B})$$

We have $$(-5\sqrt{2}\times 10^{-3})\,\hat{\mathbf{k}} = (10^{-6})\left[\left(\frac{10^6}{\sqrt{2}}\hat{\mathbf{i}} + \frac{10^6}{\sqrt{2}}\hat{\mathbf{j}}\right)\times(B_0\hat{\mathbf{i}})\right]$$

$$= -\frac{B_0}{\sqrt{2}}\hat{\mathbf{k}}$$

∴ $$\frac{B_0}{\sqrt{2}} = 5\sqrt{2}\times 10^{-3}$$

or $$B_0 = 10^{-2}\text{ T}$$

Therefore, the magnetic field is

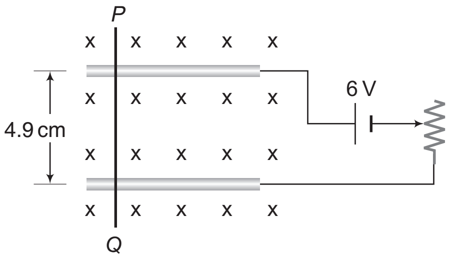
$$\mathbf{B} = (10^{-2}\,\hat{\mathbf{i}})\text{ T}$$ **Ans.**

(b) $$F_2 = B_0 q v_2 \sin 90°$$

As the angle between **B** and **v** in this case is 90°.

∴ $$F_2 = (10^{-2})(10^{-6})(10^{6})$$
$$= 10^{-2}\text{ N}$$ **Ans.**

Example 24 *A wire PQ of mass* 10 *g is at rest on two parallel metal rails. The separation between the rails is* 4.9 *cm. A magnetic field of* 0.80 *T is applied perpendicular to the plane of the rails, directed downwards. The resistance of the circuit is slowly decreased. When the resistance decreases to below* 20 Ω, *the wire PQ begins to slide on the rails. Calculate the coefficient of friction between the wire and the rails.*



Solution Wire PQ begins to slide when magnetic force is just equal to the force of friction, i.e.

$$\mu\, mg = ilB\sin\theta \qquad (\theta = 90°)$$

Here, $$i = \frac{E}{R} = \frac{6}{20} = 0.3\text{ A}$$

∴ $$\mu = \frac{ilB}{mg} = \frac{(0.3)(4.9\times 10^{-2})(0.8)}{(10\times 10^{-3})(9.8)}$$
$$= 0.12$$ **Ans.**

Example 25 *What is the value of B that can be set up at the equator to permit a proton of speed* 10^7 *m/s to circulate around the earth?*

$$[R = 6.4 \times 10^6 \, m, \; m_p = 1.67 \times 10^{-27} \, kg].$$

Solution From the relation

$$r = \frac{mv}{Bq},$$

We have

$$B = \frac{mv}{qr}$$

Substituting the values, we have

$$B = \frac{(1.67 \times 10^{-27})(10^7)}{(1.6 \times 10^{-19})(6.4 \times 10^6)}$$

$$= 1.6 \times 10^{-8} \text{ T}$$ **Ans.**

Example 26 *Deuteron in a cyclotron describes a circle of radius* 32.0 *cm. Just before emerging from the D's. The frequency of the applied alternating voltage is* 10 *MHz. Find*

(a) the magnetic flux density (i.e. the magnetic field).

(b) the energy and speed of the deuteron upon emergence.

Solution (a) Frequency of the applied emf = Cyclotron frequency

or $$f = \frac{Bq}{2\pi m}$$

∴ $$B = \frac{2\pi m f}{q}$$

$$= \frac{(2)(3.14)(2 \times 1.67 \times 10^{-27})(10 \times 10^6)}{1.6 \times 10^{-19}}$$

$$= 1.30 \text{ T}$$ **Ans.**

(b) The speed of deuteron on the emergence from the cyclotron,

$$v = \frac{2\pi R}{T} = 2\pi f R$$

$$= (2)(3.14)(10 \times 10^6)(32 \times 10^{-2})$$

$$= 2.01 \times 10^7 \text{ m/s}$$

∴ Energy of deuteron $$= \frac{1}{2} mv^2$$

$$= \frac{1}{2} \times (2 \times 1.67 \times 10^{-27})(2.01 \times 10^7)^2 \text{ J}$$

$$= 4.22 \text{ MeV}$$ **Ans.**

Note *1 MeV = 1.6* $\times 10^{-13}$ *J*

Example 27 *In the Bohr model of the hydrogen atom, the electron circulates around the nucleus in a path of radius* 5×10^{-11} *m at a frequency of* 6.8×10^{15} *Hz.*

(a) What value of magnetic field is set up at the centre of the orbit?

(b) What is the equivalent magnetic dipole moment?

Solution (a) An electron moving around the nucleus is equivalent to a current,

$$i = qf$$

Magnetic field at the centre,

$$B = \frac{\mu_0 i}{2R} = \frac{\mu_0 qf}{2R}$$

Substituting the values, we have

$$B = \frac{(4\pi \times 10^{-7})\,(1.6 \times 10^{-19})\,(6.8 \times 10^{15})}{2 \times 5.1 \times 10^{-11}}$$

$$= 13.4 \text{ T}$$ **Ans.**

(b) The current carrying circular loop is equivalent to a magnetic dipole with magnetic dipole moment,

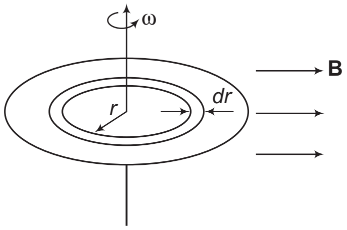
$$M = NiA = (Nqf\ \pi R^2)$$

Substituting the values, we have

$$M = (1)\,(1.6 \times 10^{-19})\,(6.8 \times 10^{15})\,(3.14)\,(5.1 \times 10^{-11})^2$$
$$= 8.9 \times 10^{-24} \text{ A-m}^2$$ **Ans.**

Example 28 *A flat dielectric disc of radius R carries an excess charge on its surface. The surface charge density is σ. The disc rotates about an axis perpendicular to its plane passing through the centre with angular velocity ω. Find the torque on the disc if it is placed in a uniform magnetic field B directed perpendicular to the rotation axis.*

Solution Consider an annular ring of radius r and of thickness dr on this disc. Charge within this ring,



$$dq = (\sigma)\,(2\pi r dr)$$

As ring rotates with angular velocity ω, the equivalent current is

$$i = (dq)\,(\text{frequency})$$
$$= (\sigma)\,(2\pi r dr)\left(\frac{\omega}{2\pi}\right) \quad \text{or} \quad i = \sigma\omega r dr$$

Magnetic moment of this annular ring,

$$M = iA = (\sigma\omega r dr)\,(\pi r^2)$$ (along the axis of rotation)

Torque on this ring,

$$d\tau = MB\sin 90° = (\sigma\omega\pi r^3 B)\,dr$$

∴ Total torque on the disc is $\tau = \int_0^R d\tau = (\sigma\omega\pi B)\int_0^R r^3\,dr$

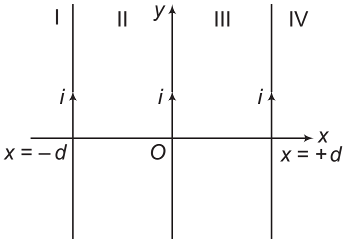
$$= \frac{\sigma\omega\pi B R^4}{4}$$ **Ans.**

Example 29 *Three infinitely long thin wires, each carrying current i in the same direction, are in the xy-plane of a gravity free space. The central wire is along the y-axis while the other two are along x = ± d.*

(i) Find the locus of the points for which the magnetic field B is zero.

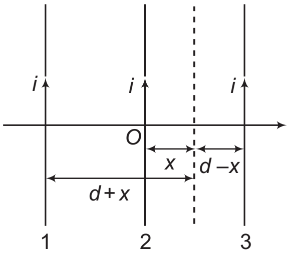
(ii) If the central wire is displaced along the z-direction by a small amount and released, show that it will execute simple harmonic motion. If the linear mass density of the wires is λ, find the frequency of oscillation.

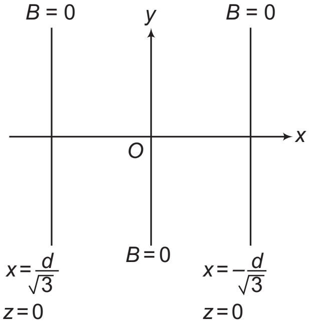
Solution (i) Magnetic field will be zero on the y-axis, i.e. $x = 0 = z$.



Magnetic field cannot be zero in region I and region IV because in region I magnetic field will be along positive z-direction due to all the three wires, while in region IV magnetic field will be along negative z-axis due to all the three wires. It can be zero only in region II and III.

Let magnetic field is zero on line $z = 0$ and $x = x$ (shown as dotted). The magnetic field on this line due to wires 1 and 2 will be along negative z-axis and due to wire 3 along positive z-axis. Thus,





$$B_1 + B_2 = B_3$$

or
$$\frac{\mu_0}{2\pi}\frac{i}{(d+x)} + \frac{\mu_0 i}{2\pi x} = \frac{\mu_0}{2\pi}\frac{i}{(d-x)}$$

or
$$\frac{1}{d+x} + \frac{1}{x} = \frac{1}{d-x}$$

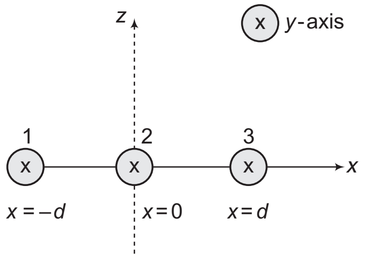
This equation gives
$$x = \pm \frac{d}{\sqrt{3}}$$

Hence, there will be two lines
$$x = \frac{d}{\sqrt{3}}$$

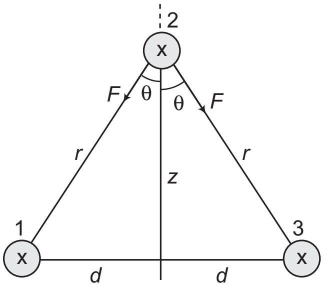
and
$$x = -\frac{d}{\sqrt{3}} \qquad (z = 0)$$

where, magnetic field is zero. **Ans.**

(ii) In this part, we change our coordinate axes system, just for better understanding.



There are three wires 1, 2 and 3 as shown in figure. If we displace the wire 2 towards the z-axis, then force of attraction per unit length between wires (1 and 2) and (2 and 3) will be given by



$$F = \frac{\mu_0}{2\pi}\frac{i^2}{r}$$

The components of F along x-axis will be cancelled out. Net resultant force will be towards negative z-axis (or mean position) and will be given by

$$F_{\text{net}} = 2F\cos\theta = 2\left\{\frac{\mu_0}{2\pi}\frac{i^2}{r}\right\}\frac{z}{r}$$

$$F_{\text{net}} = \frac{\mu_0}{\pi}\frac{i^2}{(z^2+d^2)}\cdot z \qquad (r^2 = z^2 + d^2)$$

If $z << d$, then

$$z^2 + d^2 \approx d^2$$

and
$$F_{\text{net}} = -\left(\frac{\mu_0}{\pi}\frac{i^2}{d^2}\right)\cdot z$$

Negative sign implies that F_{net} is restoring in nature.

Therefore, $F_{\text{net}} \propto -z$

i.e. the wire will oscillate simple harmonically.

Let a be the acceleration of wire in this position and λ the mass per unit length of wire, then

$$F_{\text{net}} = \lambda a = -\left(\frac{\mu_0}{\pi}\frac{i^2}{d^2}\right)z$$

or
$$a = -\left(\frac{\mu_0 i^2}{\pi\lambda d^2}\right)z$$

∴ Frequency of oscillation,

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{\text{acceleration}}{\text{displacement}}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\left|\frac{a}{z}\right|} = \frac{1}{2\pi}\frac{i}{d}\sqrt{\frac{\mu_0}{\pi\lambda}}$$

or $$f = \frac{i}{2\pi d}\sqrt{\frac{\mu_0}{\pi\lambda}}$$ **Ans.**

Example 30 *Uniform electric and magnetic fields with strength E and B are directed along the y-axis. A particle with specific charge q/m leaves the origin in the direction of x-axis with an initial velocity v_0. Find*

(a) the y-coordinate of the particle when it crosses the y-axis for nth time.

(b) the angle α between the particle's velocity vector and the y-axis at that moment.

Solution (a) As discussed in Type-9 path of the particle is a helix of increasing pitch. The axis of the helix is parallel to y-axis (parallel to **E**) and plane of circle of the helix is xz (perpendicular to **B**). The particle will cross the y-axis after time,

$$t = nT = n\left(\frac{2\pi m}{Bq}\right) = \frac{2\pi mn}{Bq}$$

The y-coordinate of particle at this instant is

$$y = \frac{1}{2}a_y t^2$$

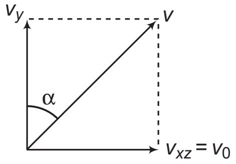
where, $$a_y = \frac{F_y}{m} = \frac{qE}{m}$$

∴ $$y = \frac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{2\pi mn}{Bq}\right)^2$$

$$= \frac{2n^2 mE\pi^2}{qB^2}$$ **Ans.**

(b) At this moment y-component of its velocity is

$$v_y = a_y t = \left(\frac{qE}{m}\right)\left(\frac{2\pi mn}{Bq}\right) = 2\pi n\left(\frac{E}{B}\right)$$



The angle α between particle's velocity vector and the y-axis at this moment is

$$\alpha = \tan^{-1}\left(\frac{v_{xz}}{v_y}\right)$$

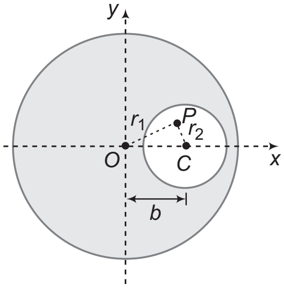
Here, $$v_{xz} = \sqrt{v_x^2 + v_z^2} = v_0$$

or $$\alpha = \tan^{-1}\left(\frac{Bv_0}{2\pi nE}\right)$$ **Ans.**

Example 31 *A current is passing through a cylindrical conductor with a hole (or cavity) inside it. Show that the magnetic field inside the hole is uniform and find its magnitude and direction.*

Solution Let us find the magnetic field at point P inside the cavity at a distance r_1 from O and r_2 from C.

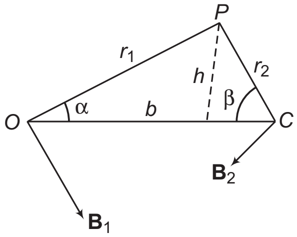
J = current per unit area
R = radius of cylinder
a = radius of cavity
i_1 = whole current from cylinder = $J\,(\pi R^2)$
i_2 = current from hole = $J\,(\pi a^2)$



At point P magnetic field due to i_1 is B_1 (perpendicular to OP) and is $\mathbf{B}_2$ due to i_2 (perpendicular to CP) in the directions shown. Although $\mathbf{B}_1$ and $\mathbf{B}_2$ are actually at P, but for better understanding they are drawn at O and C respectively. Let B_x be the x-component of resultant of $\mathbf{B}_1$ and $\mathbf{B}_2$ and B_y its y-component. Then,

$$\begin{aligned}B_x &= B_1\sin\alpha - B_2\sin\beta\\ &= \left(\frac{\mu_0}{2\pi}\frac{i_1}{R^2}r_1\right)\sin\alpha - \left(\frac{\mu_0}{2\pi}\frac{i_2}{a^2}\cdot r_2\right)\sin\beta\\ &= \left(\frac{\mu_0}{2\pi}\frac{J\pi R^2}{R^2}\cdot r_1\sin\alpha\right) - \left(\frac{\mu_0}{2\pi}\frac{J.\pi a^2}{a^2}r_2\sin\beta\right)\\ &= \frac{\mu_0 J}{2}(r_1\sin\alpha - r_2\sin\beta) = 0\end{aligned}$$

Because in ΔOPC $\frac{r_1}{\sin\beta} = \frac{r_2}{\sin\alpha} = h$ or $r_1\sin\alpha - r_2\sin\beta = 0$



Now,

$$\begin{aligned}B_y &= -(B_1\cos\alpha + B_2\cos\beta)\\ &= -\frac{\mu_0 J}{2}(r_1\cos\alpha + r_2\cos\beta)\end{aligned}$$

From ΔOPC, we can see that

$$r_1\cos\alpha + r_2\cos\beta = b \quad\text{or}\quad B_y = -\frac{\mu_0 Jb}{2} = \text{constant}$$

Thus, we can see that net magnetic field at point P is along negative y-direction and constant in magnitude. **Proved**

Note *(i) That ∠OPC is not necessarily 90°. At some point it may be 90°.*

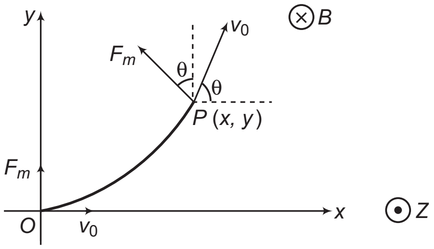
(ii) At point C magnetic field due to i_2 is zero (i.e. $B_2 = 0$) while that due to i_1 is $\frac{\mu_0}{2\pi}\frac{i_1}{R^2}b$ in negative y-direction. Substituting $i_1 = J\,(\pi R^2)$, we get

$$B = B_2 = \frac{\mu_0 Jb}{2} \qquad \textit{(along negative y-direction)}$$

This agrees with the result derived above.

Example 32 *A particle of charge q and mass m is projected from the origin with velocity* $\mathbf{v} = v_0 \hat{\mathbf{i}}$ *in a non-uniform magnetic field* $\mathbf{B} = -B_0 x \hat{\mathbf{k}}$. *Here,* v_0 *and* B_0 *are positive constants of proper dimensions. Find the maximum positive x-coordinate of the particle during its motion.*

Solution Magnetic field is along negative z-direction. So in the coordinate axes shown in figure, it is perpendicular to paper inwards. (⊗) Magnetic force on the particle at origin is along positive y-direction. So, it will rotate in xy-plane as shown. The path is not a perfect circle as the magnetic field is non-uniform. Speed of the particle in magnetic field remains constant. Magnetic force is always perpendicular to velocity. Let at point $P(x, y)$, its velocity vector makes an angle θ with positive x-axis. Then, magnetic force $\mathbf{F}_m$ will be at angle θ with positive y-direction. So,



$$a_y = \left(\frac{F_m}{m}\right)\cos\theta$$

$$\therefore \quad \frac{dv_y}{dt} = \frac{(B_0 x)(q v_0 \cos\theta)}{m} \qquad [F_m = Bqv_0 \sin 90°]$$

$$\therefore \quad \left(\frac{dv_y}{dx}\right)\cdot\left(\frac{dx}{dt}\right) = \left(\frac{B_0 q x}{m}\right)(v_0 \cos\theta)$$

Here,
$$\frac{dx}{dt} = v_x = v_0 \cos\theta$$

$$\therefore \quad \frac{dv_y}{dx} = \left(\frac{B_0 q}{m}\right) x$$

$$\therefore \quad \int_0^{v_0} dv_y = \left(\frac{B_0 q}{m}\right)\int_0^{x_{\max}} x\,dx$$

$$\therefore \quad v_0 = \left(\frac{B_0 q}{m}\right)\left(\frac{x_{\max}^2}{2}\right)$$

$$\therefore \quad x_{\max} = \sqrt{\frac{2mv_0}{B_0 q}}$$ **Ans.**

Note *At maximum x-displacement velocity is along positive y-direction.*

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

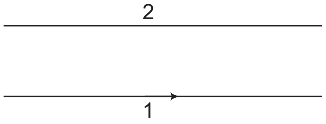
1. Assertion : Path of a charged particle in uniform magnetic field cannot be a parabola.

Reason : For parabolic path acceleration should be constant.

2. Assertion : A beam of protons is moving towards east in vertically upward magnetic field. Then, this beam will deflect towards south.

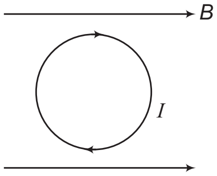
Reason : A constant magnetic force will act on the proton beam.

3. Assertion : Current in wire-1 is in the direction as shown in figure. The bottom wire is fixed. To keep the upper wire stationary, current in it should be in opposite direction.



Reason : Under the above condition, equilibrium of upper wire is stable.

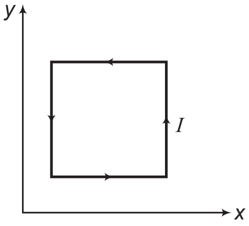
4. Assertion : A current carrying loop is placed in uniform magnetic field as shown in figure. Torque in the loop in this case is zero.



Reason : Magnetic moment vector of the loop is perpendicular to paper inwards.

5. Assertion : Force on current carrying loop shown in figure in magnetic field, $\mathbf{B} = (B_0 x)\,\hat{\mathbf{k}}$ is along positive x-axis. Here, B_0 is a positive constant.

Reason : A torque will also act on the loop.



6. Assertion : An electron and a proton are accelerated by same potential difference and then enter in uniform transverse magnetic field. The radii of the two will be different.

Reason : Charges on them are different.

7. Assertion : A charged particle moves along positive y-axis with constant velocity in uniform electric and magnetic fields. If magnetic field is acting along positive x-axis, then electric field should act along positive z-axis.

Reason : To keep the charged particle undeviated the relation $\mathbf{E} = \mathbf{B} \times \mathbf{v}$ must hold good.

8. Assertion : Power of a magnetic force on a charged particle is always zero.

Reason : Power of electric force on charged particle cannot be zero.

9. Assertion : If a charged particle enters from outside at right angles in uniform magnetic field. The maximum time spent in magnetic field may be $\frac{\pi m}{Bq}$.

Reason : It can complete only semi-circle in the magnetic field.

10. Assertion : A charged particle enters in a magnetic field $\mathbf{B} = B_0\hat{\mathbf{i}}$ with velocity $\mathbf{v} = v_0\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$, then minimum speed of charged particle may be v_0.

Reason : A variable acceleration will act on the charged particle.

11. Assertion : A charged particle is moving in a circle with constant speed in uniform magnetic field. If we increase the speed of particle to twice, its acceleration will become four times.

Reason : In circular path of radius R with constant speed v, acceleration is given by $\frac{v^2}{R}$.

Objective Questions

1. The universal property among all substances is

(a) diamagnetism (b) paramagnetism
(c) ferromagnetism (d) non-magnetism

2. A charged particle moves in a circular path in a uniform magnetic field. If its speed is reduced, then its time period will

(a) increase (b) decrease
(c) remain same (d) None of these

3. A straight wire of diameter 0.5 mm carrying a current 2 A is replaced by another wire of diameter 1 mm carrying the same current. The strength of magnetic field at a distance 2 m away from the centre is

(a) half of the previous value (b) twice of the previous value
(c) unchanged (d) quarter of its previous value

4. The path of a charged particle moving in a uniform steady magnetic field cannot be a

(a) straight line (b) circle
(c) parabola (d) None of these

5. The SI unit of magnetic permeability is

(a) $\text{Wb m}^{-2}\text{A}^{-1}$ (b) $\text{Wb m}^{-1}\text{A}$
(c) $\text{Wb m}^{-1}\text{A}^{-1}$ (d) Wb m A^{-1}

6. Identify the correct statement about the magnetic field lines.

(a) These start from the N-pole and terminate on the S-pole
(b) These lines always form closed loops
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

7. Identify the correct statement related to the direction of magnetic moment of a planar loop.

(a) It is always perpendicular to the plane of the loop
(b) It depends on the direction of current
(c) It is obtained by right hand screw rule
(d) All of the above

8. A non-planar closed loop of arbitrary shape carrying a current I is placed in uniform magnetic field. The force acting on the loop

(a) is zero only for one orientation of loop in magnetic field
(b) is zero for two symmetrically located positions of loop in magnetic field
(c) is zero for all orientations
(d) is never zero

9. The magnetic dipole moment of current loop is independent of

(a) number of turns
(b) area of loop
(c) current in the loop
(d) magnetic field in which it is lying

10. The acceleration of an electron at a certain moment in a magnetic field $\mathbf{B} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ is $\mathbf{a} = x\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$. The value of x is

(a) 0.5 (b) 1
(c) 2.5 (d) 1.5

11. Match the following and select the correct alternatives given below

(p) unit of magnetic induction B (q) dimensions of B
(r) unit of permeability (μ_0) (s) dimensions of μ_0
(t) dimensions of magnetic moment (u) $[MLT^{-2}A^{-2}]$
(v) $[ML^0T^{-2}A^{-1}]$
(x) Newton/amp-metre
(y) Newton/amp^2
(z) $[M^0L^2T^0A]$

(a) p-y, q-v, r-x, s-z, t-u (b) p-x, q-r, r-y, s-z, t-v
(c) p-x, q-v,. r-y, s-u, t-z (d) p-y, q-z, r-x, s-u, t-v

12. A closed loop carrying a current I lies in the xz-plane. The loop will experience a force if it is placed in a region occupied by uniform magnetic field along

(a) x-axis (b) y-axis
(c) z-axis (d) None of these

13. A stream of protons and α-particles of equal momenta enter a uniform magnetic field perpendicularly. The radii of their orbits are in the ratio

(a) 1 : 1 (b) 1 : 2
(c) 2 : 1 (d) 4 : 1

14. A loop of magnetic moment **M** is placed in the orientation of unstable equilibrium position in a uniform magnetic field **B**. The external work done in rotating it through an angle θ is

(a) $-MB(1-\cos\theta)$ (b) $-MB\cos\theta$
(c) $MB\cos\theta$ (d) $MB(1-\cos\theta)$

15. A current of 50 A is passed through a straight wire of length 6 cm, then the magnetic induction at a point 5 cm from the either end of the wire is (1 gauss = 10^{-4} T)

(a) 2.5 gauss
(b) 1.25 gauss
(c) 1.5 gauss
(d) 3.0 gauss

16. The magnetic field due to a current carrying circular loop of radius 3 m at a point on the axis at a distance of 4 m from the centre is 54 μT. What will be its value at the centre of the loop?

(a) 250 μT
(b) 150 μT
(c) 125 μT
(d) 75 μT

17. A conductor *ab* of arbitrary shape carries current *I* flowing from *b* to *a*. The length vector **ab** is oriented from *a* to *b*. The force **F** experienced by this conductor in a uniform magnetic field **B** is

(a) $\mathbf{F} = -I(\mathbf{ab} \times \mathbf{B})$
(b) $\mathbf{F} = I(\mathbf{B} \times \mathbf{ab})$
(c) $\mathbf{F} = I(\mathbf{ba} \times \mathbf{B})$
(d) All of these

18. When an electron is accelerated through a potential difference V, it experiences a force F through a uniform transverse magnetic field. If the potential difference is increased to 2 V, the force experienced by the electron in the same magnetic field is

(a) $2F$
(b) $2\sqrt{2}F$
(c) $\sqrt{2}F$
(d) $4F$

19. Two long straight wires, each carrying a current I in opposite directions are separated by a distance R. The magnetic induction at a point mid-way between the wires is

(a) zero
(b) $\frac{\mu_0 I}{\pi R}$
(c) $\frac{2\mu_0 I}{\pi R}$
(d) $\frac{\mu_0 I}{4\pi R}$

20. The magnetic field at a distance x on the axis of a circular coil of radius R is $\frac{1}{8}$th of that at the centre. The value of x is

(a) $\frac{R}{\sqrt{3}}$
(b) $\frac{2R}{\sqrt{3}}$
(c) $R\sqrt{3}$
(d) $R\sqrt{2}$

21. Electric field and magnetic field in a region of space is given by $\mathbf{E} = E_0\hat{\mathbf{j}}$ and $\mathbf{B} = B_0\hat{\mathbf{j}}$. A particle of specific charge α is released from origin with velocity $\mathbf{v} = v_0\hat{\mathbf{i}}$. Then, path of particle

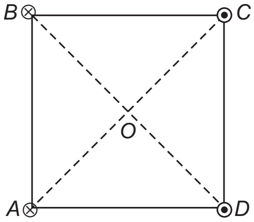
(a) is a circle
(b) is a helix with uniform pitch
(c) is a helix with non-uniform pitch
(d) is cycloid

Note *E_0, B_0 and v_0 are constant values.*

22. An electron having kinetic energy K is moving in a circular orbit of radius R perpendicular to a uniform magnetic induction. If kinetic energy is doubled and magnetic induction tripled, the radius will become

(a) $\frac{2R}{3}$
(b) $\frac{\sqrt{2}}{3}R$
(c) $\sqrt{\frac{2}{3}}R$
(d) $\frac{2}{\sqrt{3}}R$

23. Four long straight wires are located at the corners of a square *ABCD*. All the wires carry equal currents. Current in the wires *A* and *B* are inwards and in *C* and *D* are outwards. The magnetic field at the centre *O* is along

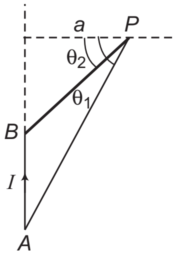


(a) *AD* (b) *CB*
(c) *AB* (d) *CD*

24. A charged particle of mass *m* and charge *q* is accelerated through a potential difference of *V* volts. It enters a region of uniform magnetic field *B* which is directed perpendicular to the direction of motion of the particle. The particle will move on a circular path of radius

(a) $\sqrt{\dfrac{Vm}{2qB^2}}$ (b) $\dfrac{2Vm}{qB^2}$ (c) $\sqrt{\dfrac{2Vm}{q}}\left(\dfrac{1}{B}\right)$ (d) $\sqrt{\dfrac{Vm}{q}}\left(\dfrac{1}{B}\right)$

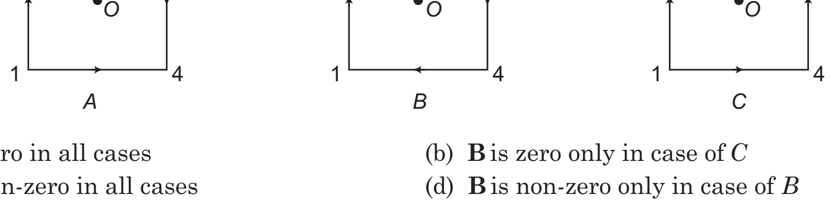
25. The straight wire *AB* carries a current *I*. The ends of the wire subtend angles θ_1 and θ_2 at the point *P* as shown in figure. The magnetic field at the point *P* is



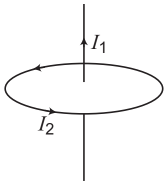
(a) $\dfrac{\mu_0 I}{4\pi a}(\sin\theta_1 - \sin\theta_2)$ (b) $\dfrac{\mu_0 I}{4\pi a}(\sin\theta_1 + \sin\theta_2)$

(c) $\dfrac{\mu_0 I}{4\pi a}(\cos\theta_1 - \cos\theta_2)$ (d) $\dfrac{\mu_0 I}{4\pi a}(\cos\theta_1 + \cos\theta_2)$

26. The figure shows three identical current carrying square loops *A*, *B* and *C*. Identify the correct statement related to magnetic field **B** at the centre *O* of the square loop. Current in each wire is *I*.

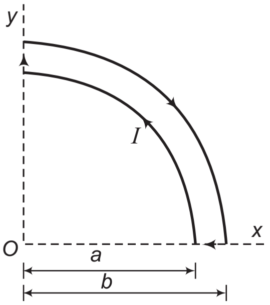


(a) **B** is zero in all cases (b) **B** is zero only in case of *C*
(c) **B** is non-zero in all cases (d) **B** is non-zero only in case of *B*



27. The figure shows a long straight wire carrying a current I_1 along the axis of a circular ring carrying a current I_2. Identify the correct statement.

(a) Straight wire attracts the ring
(b) Straight wire attracts a small element of the ring
(c) Straight wire does not attract any small element of the ring
(d) None of the above



28. The figure shows a wire frame in *xy*-plane carrying a current I. The magnetic field at the point O is

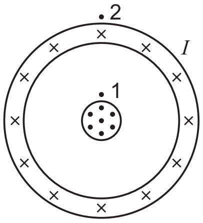
(a) $\frac{\mu_0 I}{8}\left[\frac{1}{a}-\frac{1}{b}\right]\hat{\mathbf{k}}$
(b) $\frac{\mu_0 I}{8}\left[\frac{1}{b}-\frac{1}{a}\right]\hat{\mathbf{k}}$
(c) $\frac{\mu_0 I}{4}\left[\frac{1}{a}-\frac{1}{b}\right]\hat{\mathbf{k}}$
(d) $\frac{\mu_0 I}{4}\left[\frac{1}{b}-\frac{1}{a}\right]\hat{\mathbf{k}}$

29. An electron moving in a circular orbit of radius R with frequency f. The magnetic field at the centre of the orbit is

(a) $\frac{\mu_0 ef}{2\pi R}$ (b) $\frac{\mu_0 ef}{2R}$
(c) $\frac{\mu ef^2}{2R}$ (d) zero

30. A square loop of side a carries a current I. The magnetic field at the centre of the loop is

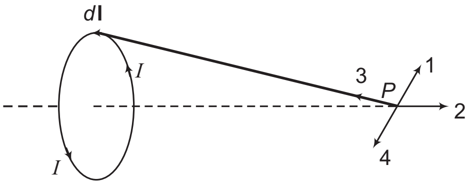
(a) $\frac{2\mu_0 I\sqrt{2}}{\pi a}$ (b) $\frac{\mu_0 I\sqrt{2}}{\pi a}$
(c) $\frac{4\mu_0 I\sqrt{2}}{\pi a}$ (d) $\frac{\mu_0 I}{\pi a}$



31. The figure shows the cross-section of two long coaxial tubes carrying equal currents I in opposite directions. If B_1 and B_2 are magnetic fields at points 1 and 2 as shown in figure, then

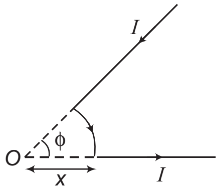
(a) $B_1 \neq 0; B_2 = 0$ (b) $B_1 = 0; B_2 = 0$
(c) $B_1 \neq 0; B_2 \neq 0$ (d) $B_1 = 0, B_2 \neq 0$

32. The figure shows a point P on the axis of a circular loop carrying current I. The correct direction of magnetic field vector at P due to $d\mathbf{l}$ is represented by



(a) 1 (b) 2
(c) 3 (d) 4

33. In figure, the curved part represents arc of a circle of radius x. If it carries a current I, then the magnetic field at the point O is

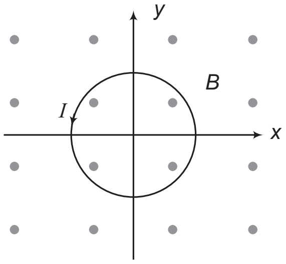


(a) $\frac{\mu_0 I\phi}{2\pi x}$ (b) $\frac{\mu_0 I\phi}{4\pi x}$ (c) $\frac{\mu_0 I\phi}{2x}$ (d) $\frac{\mu_0 I\phi}{4x}$

34. A cylindrical long wire of radius R carries a current I uniformly distributed over the cross-sectional area of the wire. The magnetic field at a distance x from the surface inside the wire is

(a) $\frac{\mu_0 I}{2\pi(R-x)}$ (b) $\frac{\mu_0 I}{2\pi x}$ (c) $\frac{\mu_0 I}{2\pi(R+x)}$ (d) None of these

35. A circular loop carrying a current I is placed in the xy-plane as shown in figure. A uniform magnetic field $\mathbf{B}$ is oriented along the positive z-axis. The loop tends to



(a) expand (b) contract (c) rotate about x-axis (d) rotate about y-axis

Subjective Questions

Note *You can take approximations in the answers.*

1. An electron has velocity $\mathbf{v} = (2.0 \times 10^6 \text{ m/s})\,\hat{\mathbf{i}} + (3.0 \times 10^6 \text{ m/s})\,\hat{\mathbf{j}}$. Magnetic field present in the region is $\mathbf{B} = (0.030 \text{ T})\,\hat{\mathbf{i}} - (0.15 \text{ T})\,\hat{\mathbf{j}}$.

(a) Find the force on electron.

(b) Repeat your calculation for a proton having the same velocity.

2. An electron moves through a uniform magnetic field given by $\mathbf{B} = B_x\hat{\mathbf{i}} + (3B_x)\hat{\mathbf{j}}$. At a particular instant, the electron has the velocity $\mathbf{v} = (2.0\,\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}})$ m/s and the magnetic force acting on it is $(6.4 \times 10^{-19} \text{ N})\,\hat{\mathbf{k}}$. Find B_x.

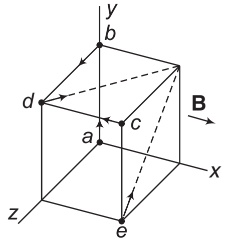
3. A particle with charge $7.80\,\mu\text{C}$ is moving with velocity $\mathbf{v} = -(3.80 \times 10^3 \text{ m/s})\,\hat{\mathbf{j}}$. The magnetic force on the particle is measured to be $\mathbf{F} = +(7.60 \times 10^{-3} \text{ N})\,\hat{\mathbf{i}} - (5.20 \times 10^{-3} \text{ N})\,\hat{\mathbf{k}}$.

(a) Calculate the components of the magnetic field you can find from this information.

(b) Are the components of the magnetic field that are not determined by the measurement of the force? Explain.

(c) Calculate the scalar product $\mathbf{B} \cdot \mathbf{F}$. What is the angle between $\mathbf{B}$ and $\mathbf{F}$?

4. Each of the lettered points at the corners of the cube in figure represents a positive charge q moving with a velocity of magnitude v in the direction indicated. The region in the figure is in a uniform magnetic field **B**, parallel to the x-axis and directed toward the right. Find the magnitude and direction of the force on each charge.



5. An electron in the beam of a TV picture tube is accelerated by a potential difference of 2.00 kV. Then, it passes through region of transverse magnetic field, where it moves in a circular arc with radius 0.180 m. What is the magnitude of the field?

6. A deuteron (the nucleus of an isotope of hydrogen) has a mass of 3.34×10^{-27} kg and a charge of $+e$. The deuteron travels in a circular path with a radius of 6.96 mm in a magnetic field with magnitude 2.50 T.

(a) Find the speed of the deuteron.

(b) Find the time required for it to make half of a revolution.

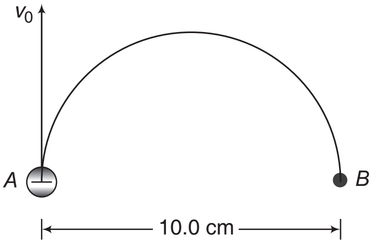
(c) Through what potential difference would the deuteron have to be accelerated to acquire this speed?

7. A neutral particle is at rest in a uniform magnetic field **B**. At time $t = 0$, it decays into two charged particles, each of mass m.

(a) If the charge of one of the particles is $+q$, what is the charge of the other?

(b) The two particles move off in separate paths, both of them lie in the plane perpendicular to **B**. At a later time, the particles collide. Express the time from decay until collision in terms of m, B and q.

8. An electron at point A in figure has a speed $v_0 = 1.41 \times 10^6$ m/s. Find



(a) the magnitude and direction of the magnetic field that will cause the electron to follow the semicircular path from A to B,

(b) the time required for the electron to move from A to B.

9. A proton of charge e and mass m enters a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{i}}$ with an initial velocity $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$. Find an expression in unit vector notation for its velocity at time t.

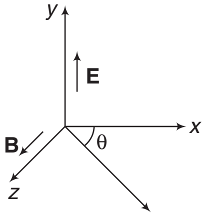
10. A proton moves at a constant velocity of 50 m/s along the x-axis, in uniform electric and magnetic fields. The magnetic field is $\mathbf{B} = (2.0\text{ mT})\hat{\mathbf{j}}$. What is the electric field?

11. A particle having mass m and charge q is released from the origin in a region in which electric field and magnetic field are given by

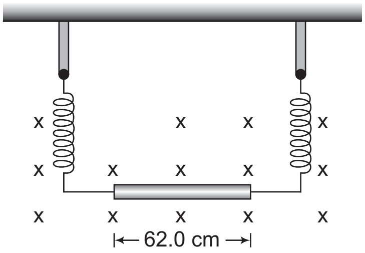
$$\mathbf{B} = -B_0\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{E} = E_0\hat{\mathbf{k}}$$

Find the y-component of the velocity and the speed of the particle as a function of its z-coordinate.

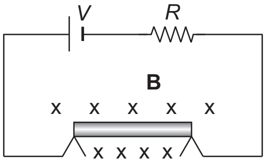
12. Protons move rectilinearly in the region of space where there are uniform mutually perpendicular electric and magnetic fields E and B. The trajectory of protons lies in the plane xz as shown in the figure and forms an angle θ with x-axis. Find the pitch of the helical trajectory along which the protons will move after the electric field is switched off.



13. A wire of 62.0 cm length and 13.0 g mass is suspended by a pair of flexible leads in a uniform magnetic field of magnitude 0.440 T in figure. What are the magnitude and direction of the current required to remove the tension in the supporting leads? Take $g = 10\ \text{m/s}^2$.



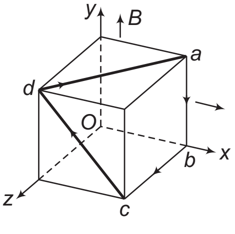
14. A thin, 50.0 cm long metal bar with mass 750 g rests on, but is not attached to, two metallic supports in a 0.450 T magnetic field as shown in figure. A battery and a resistance $R = 25.0\ \Omega$ in series are connected to the supports.



(a) What is the largest voltage the battery can have without breaking the circuit at the supports?

(b) The battery voltage has this maximum value calculated. Decreasing the resistance to $2.0\ \Omega$, find the initial acceleration of the bar.

15. In figure, the cube is 40.0 cm on each edge. Four straight segments of wire ab, bc, cd and da form a closed loop that carries a current $I = 5.00$ A, in the direction shown. A uniform magnetic field of magnitude $B = 0.020$ T is in the positive y-direction. Determine the magnitude and direction of the magnetic force on each segment.



16. Find the ratio of magnetic dipole moment and magnetic field at the centre of a disc. Radius of disc is R and it is rotating at constant angular speed ω about its axis. The disc is insulating and uniformly charged.

17. A magnetic dipole with a dipole moment of magnitude 0.020 J/T is released from rest in a uniform magnetic field of magnitude 52 mT. The rotation of the dipole due to the magnetic force on it is unimpeded. When the dipole rotates through the orientations where its dipole moment is aligned with the magnetic field, its kinetic energy is 0.80 mJ.

(a) What is the initial angle between the dipole moment and the magnetic field?

(b) What is the angle when the dipole is next (momentarily) at rest?

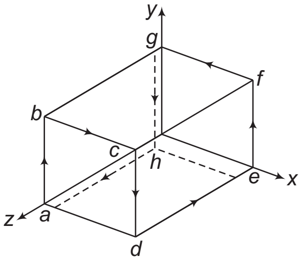
18. In the Bohr model of the hydrogen atom, in the lowest energy state the electron revolves round the proton at a speed of 2.2×10^6 m/s in a circular orbit of radius 5.3×10^{-11} m.

(a) What is the orbital period of the electron?

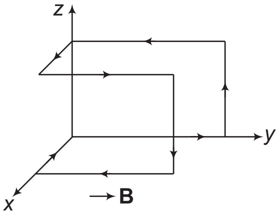
(b) If the orbiting electron is considered to be a current loop, what is the current I?

(c) What is the magnetic moment of the atom due to the motion of the electron?

19. A conductor carries a constant current I along the closed path *abcdefgha* involving 8 of the 12 edges each of length l. Find the magnetic dipole moment of the closed path.

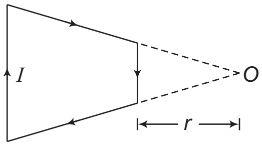


20. Given figure shows a coil bent with all edges of length 1 m and carrying a current of 1 A. There exists in space a uniform magnetic field of 2 T in positive y-direction. Find the torque on the loop.

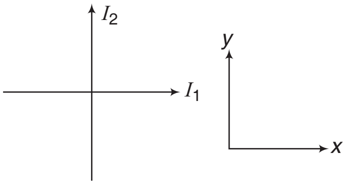


21. A very long wire carrying a current $I = 5.0$ A is bent at right angles. Find the magnetic induction at a point lying on a perpendicular normal to the plane of the wire drawn through the point of bending at a distance $l = 35$ cm from it.

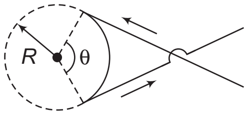
22. A current $I = \sqrt{2}$ A flows in a circuit having the shape of isosceles trapezium. The ratio of the bases of the trapezium is 2. Find the magnetic induction B at symmetric point O in the plane of the trapezium. The length of the smaller base of the trapezium is 100 mm and the distance $r = 50$ mm.



23. Two long mutually perpendicular conductors carrying currents I_1 and I_2 lie in one plane. Find the locus of points at which the magnetic induction is zero.



24. A wire carrying current i has the configuration as shown in figure. Two semi-infinite straight sections, both tangent to the same circle, are connected by a circular arc of central angle θ, along the circumference of the circle, with all sections lying in the same plane. What must θ be for B to be zero at the centre of the circle?

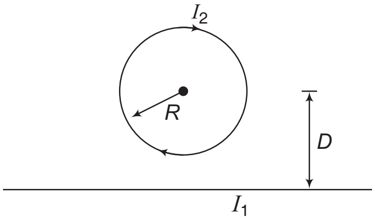


25. Two long parallel transmission lines 40.0 cm apart carry 25.0 A and 75.0 A currents. Find all locations where the net magnetic field of the two wires is zero if these currents are in

(a) the same direction (b) the opposite direction

26. A closely wound coil has a radius of 6.00 cm and carries a current of 2.50 A. How many turns must it have if, at a point on the coil axis 6.00 cm from the centre of the coil, the magnetic field is 6.39×10^{-4} T ?

27. A circular loop of radius R carries current I_2 in a clockwise direction as shown in figure. The centre of the loop is a distance D above a long, straight wire. What are the magnitude and direction of the current I_1 in the wire if the magnetic field at the centre of loop is zero?

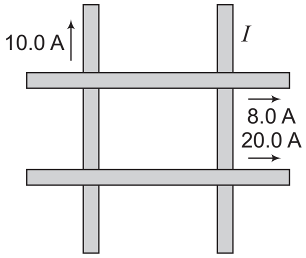


28. A closely wound, circular coil with radius 2.40 cm has 800 turns.

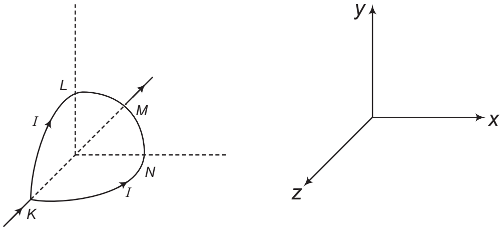
(a) What must the current in the coil be if the magnetic field at the centre of the coil is 0.0580 T?

(b) At what distance x from the centre of the coil, on the axis of the coil, is the magnetic field half its value at the centre?

29. Four very long, current carrying wires in the same plane intersect to form a square 40.0 cm on each side as shown in figure. Find the magnitude and direction of the current I so that the magnetic field at the centre of square is zero. Wires are insulated from each other.



30. A circular loop of radius R is bent along a diameter and given a shape as shown in figure. One of the semicircles (KNM) lies in the xz-plane and the other one (KLM) in the yz-plane with their centres at origin. Current I is flowing through each of the semicircles as shown in figure.



(a) A particle of charge q is released at the origin with a velocity $\mathbf{v} = -v_0\hat{\mathbf{i}}$. Find the instantaneous force $\mathbf{F}$ on the particle. Assume that space is gravity free.

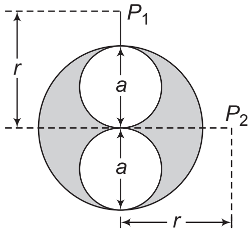
(b) If an external uniform magnetic field $B_0\hat{\mathbf{j}}$ is applied, determine the force $\mathbf{F}_1$ and $\mathbf{F}_2$ on the semicircles KLM and KNM due to the field and the net force $\mathbf{F}$ on the loop.

31. A regular polygon of n sides is formed by bending a wire of total length $2\pi r$ which carries a current i.

(a) Find the magnetic field B at the centre of the polygon.

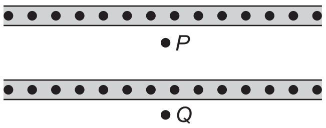
(b) By letting $n \to \infty$, deduce the expression for the magnetic field at the centre of a circular coil.

32. A long cylindrical conductor of radius a has two cylindrical cavities of diameter a through its entire length as shown in cross-section in figure. A current I is directed out of the page and is uniform throughout the cross-section of the conductor. Find the magnitude and direction of the magnetic field in terms of μ_0, I, r and a.

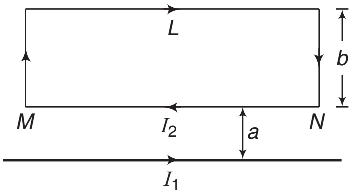


(a) at point P_1 and (b) at point P_2

33. Two infinite plates shown in cross-section in figure carry λ amperes of current out of the page per unit width of plate. Find the magnetic field at points P and Q.



34. For the situation shown in figure, find the force experienced by side MN of the rectangular loop. Also, find the torque on the loop.

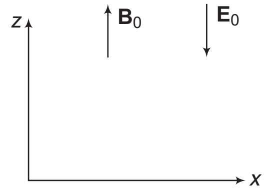


35. In a region of space, a uniform magnetic field B is along positive x-axis. Electrons are emitted from the origin with speed v at different angles. Show that the paraxial electrons are refocused on the x-axis at a distance $\frac{2\pi mv}{Be}$. Here, m is the mass of electron and e the charge on it.

36. A particle of mass m and charge q is projected into a region having a perpendicular magnetic field B. Find the angle of deviation of the particle as it comes out of the magnetic field if the width of the region is

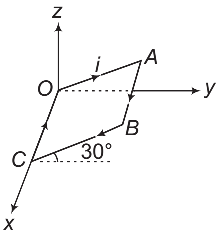
(a) $\frac{2mv}{Bq}$ (b) $\frac{mv}{Bq}$ (c) $\frac{mv}{2Bq}$

37. In a certain region, uniform electric field $\mathbf{E} = -E_0\hat{\mathbf{k}}$ and magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$ are present. At time $t = 0$, a particle of mass m and charge q is given a velocity $\mathbf{v} = v_0\hat{\mathbf{j}} + v_0\hat{\mathbf{k}}$. Find the minimum speed of the particle and the time when it happens so.



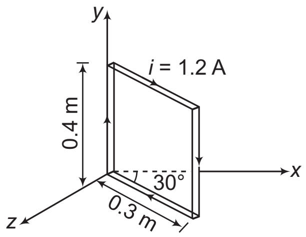
38. A particle of mass m and charge q is lying at the origin in a uniform magnetic field B directed along x-axis. At time $t = 0$, it is given a velocity v_0 at an angle θ with the y-axis in the xy-plane. Find the coordinates of the particle after one revolution.

39. Find the magnetic moment of the current carrying loop $OABCO$ shown in figure.

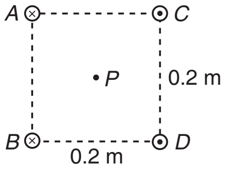


Given that, $i = 4.0$ A, $OA = 20$ cm and $AB = 10$ cm.

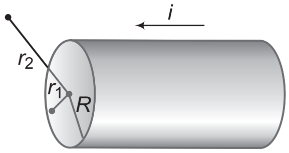
40. A rectangular loop consists of $N = 100$ closed wrapped turns and has dimensions (0.4 m × 0.3 m). The loop is hinged along the y-axis and its plane makes an angle $\theta = 30°$ with the x-axis. What is the magnitude of the torque exerted on the loop by a uniform magnetic field $B = 0.8$ T directed along the x-axis when current is $i = 1.2$ A in the direction shown. What is the expected direction of rotation of the loop?



41. Four long, parallel conductors carry equal currents of 5.0 A. The direction of the currents is into the page at points A and B and out of the page at C and D. Calculate the magnitude and direction of the magnetic field at point P, located at the centre of the square.



42. A long cylindrical conductor of radius R carries a current i as shown in figure. The current density J is a function of radius according to, $J = br$, where b is a constant. Find an expression for the magnetic field B



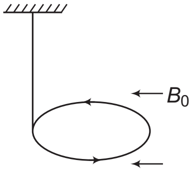
(a) at a distance $r_1 < R$ and

(b) at a distance $r_2 > R$, measured from the axis.

LEVEL 2

Single Correct Option

1. A uniform current carrying ring of mass m and radius R is connected by a massless string as shown. A uniform magnetic field B_0 exists in the region to keep the ring in horizontal position, then the current in the ring is



(a) $\dfrac{mg}{\pi R B_0}$ (b) $\dfrac{mg}{R B_0}$ (c) $\dfrac{mg}{3\pi R B_0}$ (d) $\dfrac{mg}{\pi R^2 B_0}$

2. A wire of mass 100 g is carrying a current of 2 A towards increasing x in the form of $y = x^2 (-2m \le x \le +2m)$. This wire is placed in a magnetic field $\mathbf{B} = -0.02\,\hat{\mathbf{k}}$ tesla. The acceleration of the wire (in m/s^2) is

(a) $-1.6\,\hat{\mathbf{j}}$ (b) $-3.2\,\hat{\mathbf{j}}$ (c) $1.6\,\hat{\mathbf{j}}$ (d) zero

3. A conductor of length l is placed perpendicular to a horizontal uniform magnetic field B. Suddenly, a certain amount of charge is passed through it, when it is found to jump to a height h. The amount of charge that passes through the conductor is

(a) $\dfrac{m\sqrt{gh}}{Bl}$ (b) $\dfrac{m\sqrt{gh}}{2Bl}$

(c) $\dfrac{m\sqrt{2gh}}{Bl}$ (d) None of these

4. A solid conducting sphere of radius R and total charge q rotates about its diametric axis with constant angular speed ω. The magnetic moment of the sphere is

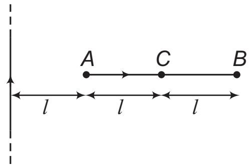
(a) $\frac{1}{3}qR^2\omega$ (b) $\frac{2}{3}qR^2\omega$ (c) $\frac{1}{5}qR^2\omega$ (d) $\frac{2}{5}qR^2\omega$

5. A charged particle moving along positive x-direction with a velocity v enters a region where there is a uniform magnetic field $\mathbf{B} = -B\hat{\mathbf{k}}$, from $x = 0$ to $x = d$. The particle gets deflected at an angle θ from its initial path. The specific charge of the particle is

(a) $\frac{Bd}{v\cos\theta}$ (b) $\frac{v\tan\theta}{Bd}$

(c) $\frac{B\sin\theta}{vd}$ (d) $\frac{v\sin\theta}{Bd}$

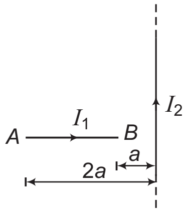
6. A current carrying rod AB is placed perpendicular to an infinitely long current carrying wire as shown in figure. The point at which the conductor should be hinged so that it will not rotate ($AC = CB$)



(a) A (b) somewhere between B and C

(c) C (d) somewhere between A and C

7. The segment AB of wire carrying current I_1 is placed perpendicular to a long straight wire carrying current I_2 as shown in figure. The magnitude of force experienced by the straight wire AB is



(a) $\frac{\mu_0 I_1 I_2}{2\pi}\ln 3$ (b) $\frac{\mu_0 I_1 I_2}{2\pi}\ln 2$

(c) $\frac{2\mu_0 I_1 I_2}{2\pi}$ (d) $\frac{\mu_0 I_1 I_2}{2\pi}$

8. A straight long conductor carries current along the positive x-axis. Identify the correct statement related to the four points $A(a, a, 0)$, $B(a, 0, a)$, $C(a, -a, 0)$ and $D(a, 0, -a)$.

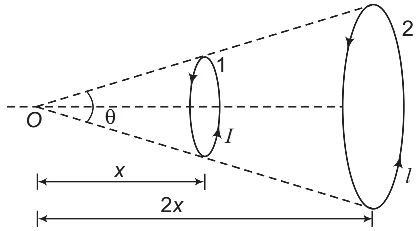
(a) The magnitude of magnetic field at all points is same

(b) Fields at A and B are mutually perpendicular

(c) Fields at A and C are antiparallel

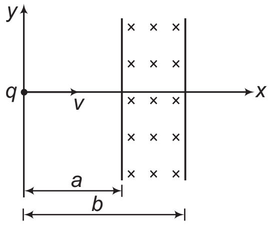
(d) All of the above

9. The figure shows two coaxial circular loops 1 and 2, which forms same solid angle θ at point O. If B_1 and B_2 are the magnetic fields produced at the point O due to loop 1 and 2 respectively, then



(a) $\frac{B_1}{B_2} = 1$ (b) $\frac{B_1}{B_2} = 2$ (c) $\frac{B_1}{B_2} = 8$ (d) $\frac{B_1}{B_2} = 4$

10. In the figure shown, a charge q moving with a velocity v along the x-axis enter into a region of uniform magnetic field. The minimum value of v so that the charge q is able to enter the region $x > b$



(a) $\frac{qBb}{m}$ (b) $\frac{qBa}{m}$

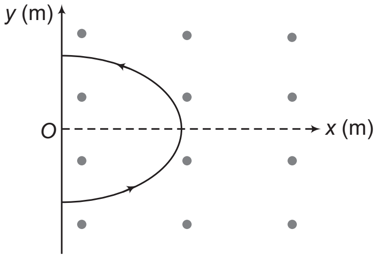
(c) $\frac{qB(b-a)}{m}$ (d) $\frac{qB(b+a)}{2m}$

11. An insulating rod of length l carries a charge q uniformly distributed on it. The rod is pivoted at one of its ends and is rotated at a frequency f about a fixed perpendicular axis. The magnetic moment of the rod is

(a) $\frac{\pi q f l^2}{12}$ (b) $\frac{\pi q f l^2}{2}$

(c) $\frac{\pi q f l^2}{6}$ (d) $\frac{\pi q f l^2}{3}$

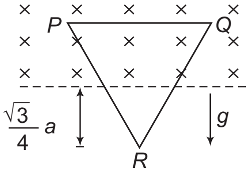
12. A wire carrying a current of 3 A is bent in the form of a parabola $y^2 = 4 - x$ as shown in figure, where x and y are in metre. The wire is placed in a uniform magnetic field $\mathbf{B} = 5\,\hat{\mathbf{k}}$ tesla. The force acting on the wire is



(a) $60\,\hat{\mathbf{i}}$ N (b) $-60\hat{\mathbf{i}}$ N

(c) $30\,\hat{\mathbf{i}}$ N (d) $-30\,\hat{\mathbf{i}}$ N

13. An equilateral triangle frame PQR of mass M and side a is kept under the influence of magnetic force due to inward perpendicular magnetic field B and gravitational field as shown in the figure. The magnitude and direction of current in the frame so that the frame remains at rest, is



(a) $I = \frac{2Mg}{aB}$; anti-clockwise (b) $I = \frac{2Mg}{aB}$; clockwise

(c) $I = \frac{Mg}{aB}$; anti-clockwise (d) $I = \frac{Mg}{aB}$; clockwise

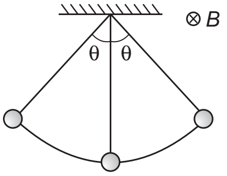
14. A tightly wound long solenoid has n turns per unit length, radius r and carries a current i. A particle having charge q and mass m is projected from a point on the axis in the direction perpendicular to the axis. The maximum speed for which particle does not strike the solenoid will be

(a) $\frac{\mu_0 qrni}{2m}$
(b) $\frac{\mu_0 qrni}{m}$
(c) $\frac{2\mu_0 qrni}{3m}$
(d) None of these

15. If the acceleration and velocity of a charged particle moving in a constant magnetic region is given by $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{k}}$, $\mathbf{v} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{k}}$. [$a_1$, a_2, b_1 and b_2 are constants], then choose the **wrong** statement.

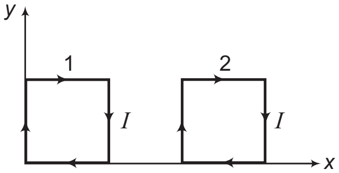
(a) Magnetic field may be along y-axis
(b) $a_1b_1 + a_2b_2 = 0$
(c) Magnetic field is along x-axis
(d) Kinetic energy of particle is always constant

16. A simple pendulum with a charged bob is oscillating as shown in the figure. Time period of oscillation is T and angular amplitude is θ. If a uniform magnetic field perpendicular to the plane of oscillation is switched on, then



(a) T will decrease but θ will remain constant
(b) T will remain constant but θ will decrease
(c) Both T and θ will remain the same
(d) Both T and θ will decrease

17. Magnetic field in a region is given by $\mathbf{B} = B_0 x\hat{\mathbf{k}}$. Two loops each of side a is placed in this magnetic region in the xy-plane with one of its sides on x-axis. If F_1 is the force on loop 1 and F_2 be the force on loop 2, then



(a) $F_1 = F_2 = 0$
(b) $F_1 > F_2$
(c) $F_2 > F_1$
(d) $F_1 = F_2 \neq 0$

18. Consider a coaxial cable which consists of an inner wire of radius a surrounded by an outer shell of inner and outer radii b and c, respectively. The inner wire carries a current I and outer shell carries an equal and opposite current. The magnetic field at a distance x from the axis where $b < x < c$ is

(a) $\frac{\mu_0 I(c^2 - b^2)}{2\pi x(c^2 - a^2)}$
(b) $\frac{\mu_0 I(c^2 - x^2)}{2\pi x(c^2 - a^2)}$
(c) $\frac{\mu_0 I(c^2 - x^2)}{2\pi x(c^2 - b^2)}$
(d) zero

19. A particle of mass 1×10^{-26} kg and charge $+1.6 \times 10^{-19}$ C travelling with a velocity of 1.28×10^{6} m/s along positive direction of x-axis enters a region in which a uniform electric field **E** and a uniform magnetic field **B** are present such that

$$E_z = -\,102.4 \text{ kV/m} \quad \text{and} \quad B_y = 8 \times 10^{-2} \text{ Wb/m}^2$$

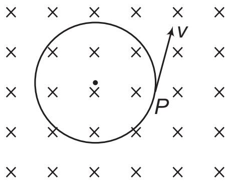
The particle enters this region at origin at time $t = 0$. Then,

(a) net force acts on the particle along the +ve z-direction
(b) net force acts on the particle along –ve z-direction
(c) net force acting on particle is zero
(d) net force acts in xz-plane

20. A wire lying along y-axis from $y = 0$ to $y = 1$ m carries a current of 2 mA in the negative y-direction. The wire lies in a non-uniform magnetic field given by $\mathbf{B} = (0.3 \text{ T/m})\, y\hat{\mathbf{i}} + (0.4 \text{ Tm})\, y\hat{\mathbf{j}}$. The magnetic force on the entire wire is

(a) $-3 \times 10^{-4}\,\hat{\mathbf{j}}$ N
(b) $6 \times 10^{-3}\,\hat{\mathbf{k}}$ N
(c) $-3 \times 10^{-4}\,\hat{\mathbf{k}}$ N
(d) $3 \times 10^{-4}\,\hat{\mathbf{k}}$ N

21. A particle having a charge of 20 μC and mass 20 μg moves along a circle of radius 5 cm under the action of a magnetic field $B = 0.1$ tesla. When the particle is at P, uniform transverse electric field is switched on and it is found that the particle continues along the tangent with a uniform velocity. Find the electric field



(a) 2 V/m
(b) 0.5 V/m
(c) 5 V/m
(d) 1.5 V/m

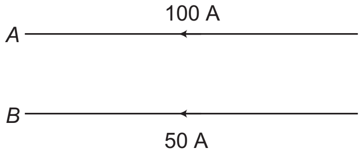
22. Two circular coils A and B of radius $\dfrac{5}{\sqrt{2}}$ cm and 5 cm carry currents 5 A and $5\sqrt{2}$ A, respectively. The plane of B is perpendicular to plane of A and their centres coincide. Magnetic field at the centre is

(a) 0
(b) $4\pi\sqrt{2} \times 10^{-5}$ T
(c) $4\pi \times 10^{-5}$ T
(d) $2\pi\sqrt{2} \times 10^{-5}$ T

23. A charged particle with specific charge s moves undeflected through a region of space containing mutually perpendicular and uniform electric and magnetic fields E and B. When the electric field is switched off, the particle will move in a circular path of radius

(a) $\dfrac{E}{Bs}$
(b) $\dfrac{Es}{B}$
(c) $\dfrac{Es}{B^2}$
(d) $\dfrac{E}{B^2 s}$

24. Two long parallel conductors are carrying currents in the same direction as shown in the figure. The upper conductor (A) carrying a current of 100 A is held firmly in position. The lower conductor (B) carries a current of 50 A and free to move up and down. The linear mass density of the lower conductor is 0.01 kg/m.

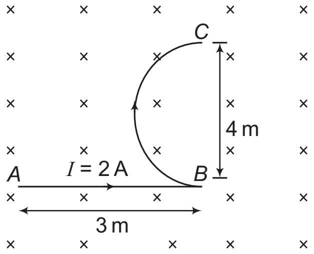


(a) Conductor B will be in equilibrium if the distance between the conductors is 0.1 m
(b) Equilibrium of conductor B is unstable
(c) Both (a) and (b) are wrong
(d) Both (a) and (b) are correct

25. Equal currents are flowing in three infinitely long wires along positive x, y and z-directions. The magnetic field at a point $(0, 0, -a)$ would be (i = current in each wire)

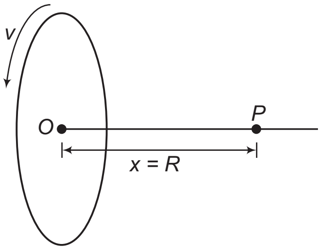
(a) $\frac{\mu_0 i}{2\pi a}(\hat{\mathbf{j}} - \hat{\mathbf{i}})$
(b) $\frac{\mu_0 i}{2\pi a}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$
(c) $\frac{\mu_0 i}{2\pi a}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$
(d) $\frac{\mu_0 i}{2\pi a}(-\hat{\mathbf{i}} - \hat{\mathbf{j}})$

26. In the figure, the force on the wire ABC in the given uniform magnetic field will be (B = 2 tesla)



(a) $4(3 + 2\pi)\ N$
(b) $20\ N$
(c) $30\ N$
(d) $40\ N$

27. A uniformly charged ring of radius R is rotated about its axis with constant linear speed v of each of its particles. The ratio of electric field to magnetic field at a point P on the axis of the ring distant $x = R$ from centre of ring is (c is speed of light)



(a) $\frac{c^2}{v}$
(b) $\frac{v^2}{c}$
(c) $\frac{v}{c}$
(d) $\frac{c}{v}$

More than One Correct Options

1. Two circular coils of radii 5 cm and 10 cm carry currents of 2 A. The coils have 50 and 100 turns respectively and are placed in such a way that their planes as well as their centres coincide. Magnitude of magnetic field at the common centre of coils is

(a) $8\pi \times 10^{-4}$ T if currents in the coils are in same sense
(b) $4\pi \times 10^{-4}$ T if currents in the coils are in opposite sense
(c) zero if currents in the coils are in opposite sense
(d) $8\pi \times 10^{-4}$ T if currents in the coils are in opposite sense

2. A charged particle enters into a gravity free space occupied by an electric field **E** and magnetic field **B** and it comes out without any change in velocity. Then, the possible cases may be

(a) $\mathbf{E} = 0$ and $\mathbf{B} \neq 0$
(b) $\mathbf{E} \neq 0$ and $\mathbf{B} = 0$
(c) $\mathbf{E} \neq 0$ and $\mathbf{B} \neq 0$
(d) $\mathbf{E} = 0, \mathbf{B} = 0$

3. A charged particle of unit mass and unit charge moves with velocity $\mathbf{v} = (8\hat{\mathbf{i}} + 6\hat{\mathbf{j}})$ m/s in a magnetic field of $\mathbf{B} = 2\hat{\mathbf{k}}$ T. Choose the correct alternative (s).

(a) The path of the particle may be $x^2 + y^2 - 4x - 21 = 0$

(b) The path of the particle may be $x^2 + y^2 = 25$

(c) The path of the particle may be $y^2 + z^2 = 25$

(d) The time period of the particle will be 3.14 s

4. When a current carrying coil is placed in a uniform magnetic field with its magnetic moment anti-parallel to the field, then

(a) torque on it is maximum

(b) torque on it is zero

(c) potential energy is maximum

(d) dipole is in unstable equilibrium

5. If a long cylindrical conductor carries a steady current parallel to its length, then

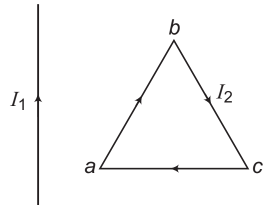
(a) the electric field along the axis is zero

(b) the magnetic field along the axis is zero

(c) the magnetic field outside the conductor is zero

(d) the electric field outside the conductor is zero

6. An infinitely long straight wire is carrying a current I_1. Adjacent to it there is another equilateral triangular wire having current I_2. Choose the **wrong** options.



(a) Net force on loop is leftwards

(b) Net force on loop is rightwards

(c) Net force on loop is upwards

(d) Net force on loop is downwards

7. A charged particle is moving along positive y-axis in uniform electric and magnetic fields

$$\mathbf{E} = E_0\hat{\mathbf{k}}$$

and

$$\mathbf{B} = B_0\hat{\mathbf{i}}$$

Here, E_0 and B_0 are positive constants. Choose the correct options.

(a) particle may be deflected towards positive z-axis

(b) particle may be deflected towards negative z-axis

(c) particle may pass undeflected

(d) kinetic energy of particle may remain constant

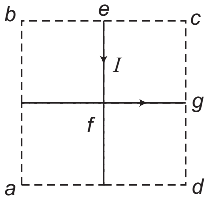
8. A charged particle revolves in circular path in uniform magnetic field after accelerating by a potential difference of V volts. Choose the correct options if V is doubled.

(a) kinetic energy of particle will become two times

(b) radius in circular path will become two times

(c) radius in circular path will become $\sqrt{2}$ times

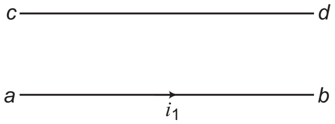
(d) angular velocity will remain unchanged



9. *abcd* is a square. There is a current I in wire *efg* as shown. Choose the correct options.

(a) Net magnetic field at a is inwards
(b) Net magnetic field at b is zero
(c) Net magnetic field at c is outwards
(d) Net magnetic field at d is inwards

10. There are two wires *ab* and *cd* in a vertical plane as shown in figure. Direction of current in wire *ab* is rightwards. Choose the correct options.



(a) If wire *ab* is fixed, then wire *cd* can be kept in equilibrium by the current in *cd* in leftward direction
(b) Equilibrium of wire *cd* will be stable equilibrium
(c) If wire *cd* is fixed, then wire *ab* can be kept in equilibrium by flowing current in *cd* in rightward direction
(d) Equilibrium of wire *ab* will be stable equilibrium

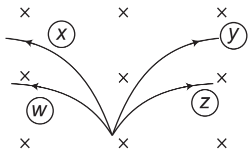
Match the Columns

1. An electron is moving towards positive x-direction. Match the following two columns for deflection of electron just after the fields are switched on. (E_0 and B_0 are positive constants)

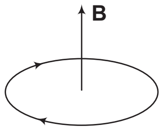
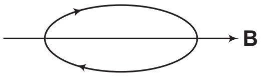
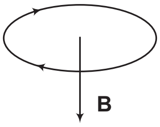
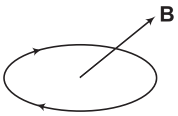
Column I	Column II
(a) If magnetic field $\mathbf{B} = B_0\hat{\mathbf{j}}$ is switched on	(p) Negative y-axis
(b) If magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$ is switched on	(q) Positive y-axis
(c) If magnetic field $\mathbf{B} = B_0\hat{\mathbf{i}}$ and electric field $\mathbf{E} = E_0\hat{\mathbf{j}}$ is switched on	(r) Negative z-axis
(d) If electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$ is switched on	(s) Positive z-axis

2. Four charged particles, $(-q, m)$, $(-3q, 4m)$, $(+q, m)$ and $(+2q, m)$ enter in uniform magnetic field (in inward direction) with same kinetic energy as shown in figure. Inside the magnetic field their paths are shown. Match the following two columns.

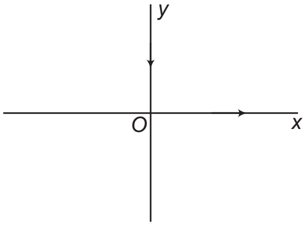
Column I	Column II
(a) Particle $(-q, m)$	(p) w
(b) Particle $(-3q, 4m)$	(q) x
(c) Particle $(+q, m)$	(r) y
(d) Particle $(+2q, m)$	(s) z



3. In Column I, a current carrying loop and a uniform magnetic field are shown. Match this with Column II.

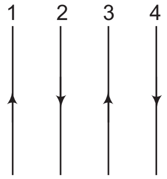
Column I	Column II
(a) 	(p) Force = 0
(b) 	(q) Maximum torque
(c) 	(r) Minimum potential energy
(d) 	(s) Positive potential energy

4. Equal currents are flowing in two infinitely long wires lying along x and y-axes in the directions shown in figure. Match the following two columns.



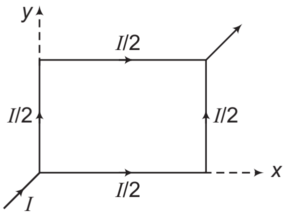
Column I	Column II
(a) Magnetic field at (a, a)	(p) along positive y-axis
(b) Magnetic field at $(-a, -a)$	(q) along positive z-axis
(c) Magnetic field at $(a, -a)$	(r) along negative z-axis
(d) Magnetic field at $(-a, a)$	(s) zero

5. Equal currents are flowing in four infinitely long wires. Distance between two wires is same and directions of currents are shown in figure. Match the following two columns.



Column I	Column II
(a) Force on wire-1	(p) inwards
(b) Force on wire-2	(q) leftwards
(c) Force on wire-3	(r) rightwards
(d) Force on wire-4	(s) zero

6. A square loop of uniform conducting wire is as shown in figure. A current I (in ampere) enters the loop from one end and exits the loop from opposite end as shown in figure.

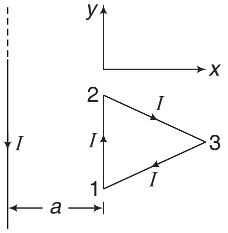


The length of one side of square loop is l metre. The wire has uniform cross-section area and uniform linear mass density.

Column I	Column II
(a) $\mathbf{B} = B_0\hat{\mathbf{i}}$ in tesla	(p) magnitude of net force on loop is $\sqrt{2}B_0Il$ newton
(b) $\mathbf{B} = B_0\hat{\mathbf{j}}$ in tesla	(q) magnitude of net force on loop is zero
(c) $\mathbf{B} = B_0(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ in tesla	(r) magnitude of force on loop is 2 B_0Il
(d) $\mathbf{B} = B_0\hat{\mathbf{k}}$ in tesla	(s) magnitude of net force on loop is B_0Il newton

Subjective Questions

1. An equilateral triangular frame with side a carrying a current I is placed at a distance a from an infinitely long straight wire carrying a current I as shown in the figure. One side of the frame is parallel to the wire. The whole system lies in the xy-plane. Find the magnetic force **F** acting on the frame.



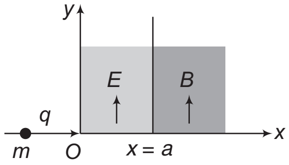
2. Find an expression for the magnetic dipole moment and magnetic field induction at the centre of a Bohr's hypothetical hydrogen atom in the nth orbit of the electron in terms of universal constants.

3. A square loop of side 6 cm carries a current of 30 A. Calculate the magnitude of magnetic field **B** at a point P lying on the axis of the loop and a distance $\sqrt{7}$ cm from centre of the loop.

4. A positively charged particle of charge 1 C and mass 40 g, is revolving along a circle of radius 40 cm with velocity 5 m/s in a uniform magnetic field with centre at origin O in xy-plane. At $t = 0$, the particle was at $(0, 0.4\text{ m}, 0)$ and velocity was directed along positive x-direction. Another particle having charge 1 C and mass 10 g moving uniformly parallel to z-direction with velocity $\frac{40}{\pi}$ m/s collides with revolving particle at $t = 0$ and gets stuck with it. Neglecting gravitational force and colombians force, calculate x, y and z-coordinates of the combined particle at $t = \frac{\pi}{40}$ sec.

5. A proton beam passes without deviation through a region of space where there are uniform transverse mutually perpendicular electric and magnetic fields with $E = 120\,\frac{\text{kV}}{\text{m}}$ and $B = 50$ mT. Then, the beam strikes a grounded target. Find the force imparted by the beam on the target if the beam current is equal to $I = 0.80$ mA.

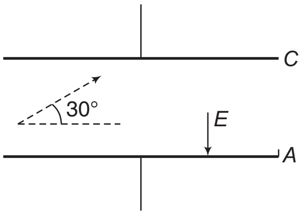
6. A positively charged particle having charge q is accelerated by a potential difference V. This particle moving along the x-axis enters a region where an electric field E exists. The direction of the electric field is along positive y-axis. The electric field exists in the region bounded by the lines $x = 0$ and $x = a$. Beyond the line $x = a$ (i.e. in the region $x \geq a$) there exists a magnetic field of strength B, directed along the positive y-axis. Find



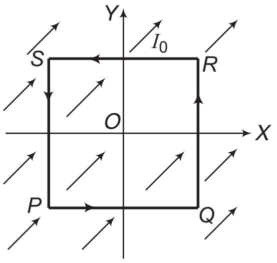
(a) at which point does the particle meet the line $x = a$

(b) the pitch of the helix formed after the particle enters the region $x \geq a$. Mass of the particle is m.

7. A charged particle having charge 10^{-6} C and mass of 10^{-10} kg is fired from the middle of the plate making an angle 30° with plane of the plate. Length of the plate is 0.17 m and it is separated by 0.1 m. Electric field $E = 10^{-3}$ N/C is present between the plates. Just outside the plates magnetic field is present. Find the velocity of projection of charged particle and magnitude of the magnetic field perpendicular to the plane of the figure, if it has to graze the plate at C and A parallel to the surface of the plate. (Neglect gravity)



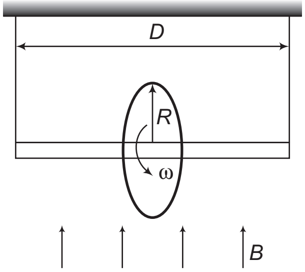
8. A uniform constant magnetic field B is directed at an angle of 45° to the x-axis in xy-plane. $PQRS$ is a rigid square wire frame carrying a steady current I_0, with its centre at the origin O. At time $t = 0$, the frame is at rest in the position shown in the figure with its sides parallel to x and y-axis. Each side of the frame has mass M and length L.



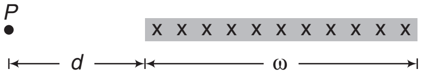
(a) What is the magnitude of torque τ acting on the frame due to the magnetic field?

(b) Find the angle by which the frame rotates under the action of this torque in a short interval of time Δt, and the axis about which the rotation occurs (Δt is so short that any variation in the torque during this interval may be neglected). Given : The moment of inertia of the frame about an axis through its centre perpendicular to its plane is $\frac{4}{3}ML^2$.

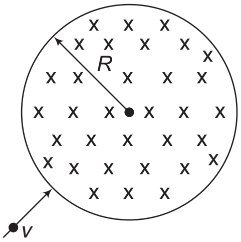
9. A ring of radius R having uniformly distributed charge Q is mounted on a rod suspended by two identical strings. The tension in strings in equilibrium is T_0. Now, a vertical magnetic field is switched on and ring is rotated at constant angular velocity ω. Find the maximum value of ω with which the ring can be rotated if the strings can withstand a maximum tension of $\frac{3T_0}{2}$.



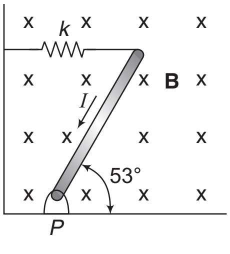
10. Figure shows a cross-section of a long ribbon of width ω that is carrying a uniformly distributed total current i into the page. Calculate the magnitude and direction of the magnetic field **B** at a point P in the plane of the ribbon at a distance d from its edge.



11. A particle of mass m having a charge q enters into a circular region of radius R with velocity v directed towards the centre. The strength of magnetic field is B. Find the deviation in the path of the particle.



12. A thin, uniform rod with negligible mass and length 0.2 m is attached to the floor by a frictionless hinge at point P. A horizontal spring with force constant k = 4.80 N/m connects the other end of the rod with a vertical wall. The rod is in a uniform magnetic field B = 0.340 T directed into the plane of the figure. There is current I = 6.50 A in the rod, in the direction shown.

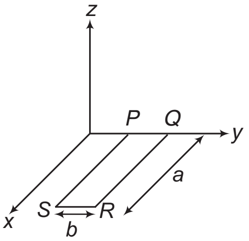


(a) Calculate the torque due to the magnetic force on the rod, for an axis at P. Is it correct to take the total magnetic force to act at the centre of gravity of the rod when calculating the torque?

(b) When the rod is in equilibrium and makes an angle of 53.0° with the floor, is the spring stretched or compressed?

(c) How much energy is stored in the spring when the rod is in equilibrium?

13. A rectangular loop $PQRS$ made from a uniform wire has length a, width b and mass m. It is free to rotate about the arm PQ, which remains hinged along a horizontal line taken as the y-axis (see figure). Take the vertically upward direction as the z-axis. A uniform magnetic field $\mathbf{B} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{k}})B_0$ exists in the region. The loop is held in the xy-plane and a current I is passed through it. The loop is now released and is found to stay in the horizontal position in equilibrium.



(a) What is the direction of the current I in PQ?

(b) Find the magnetic force on the arm RS.

(c) Find the expression for I in terms of B_0, a, b and m.

Answers

Introductory Exercise 26.1

1. $[LT^{-1}]$ **2.** $(\mathbf{F}, \mathbf{v}), (\mathbf{F}, \mathbf{B})$ **3.** No **4.** $(-0.16\,\hat{\mathbf{i}} - 0.32\,\hat{\mathbf{j}} - 0.64\,\hat{\mathbf{k}})$ N

5. Positive **6.** 9.47×10^6 m/s **7.** 2.56×10^{-14} N

Introductory Exercise 26.2

1. D, B **2.** False **3.** False **4.** Yes, No **5.** Along positive z-direction

6. (a) electron (b) electron **7.** 0.0167 cm, 0.7 cm

Introductory Exercise 26.3

1. $\sqrt{2}\,B_0\,il$ **2.** No

3. (a) $(0.023\text{ N})\,\hat{\mathbf{k}}$ (b) $(0.02\text{ N})\,\hat{\mathbf{j}}$ (c) zero (d) $(-0.0098\text{ N})\,\hat{\mathbf{j}}$ (e) $(-0.013\text{ N})\,\hat{\mathbf{j}} + (-0.026\text{ N})\,\hat{\mathbf{k}}$

4. 32 N upwards

Introductory Exercise 26.4

1. $\dfrac{qR^2\omega}{4}$ **2.** (a) $\tau = (-9.6\,\hat{\mathbf{i}} - 7.2\,\hat{\mathbf{j}} + 8.0\,\hat{\mathbf{k}}) \times 10^{-4}$ N-m (b) $U = -(6.0 \times 10^{-4})$ J

4. –2.42 J

Introductory Exercise 26.5

1. (a) 28.3 μT into the page (b) 24.7 μT into the page **2.** $\dfrac{\mu_0 i}{4\pi x}$ into the page

3. 58.0 μT into the page **4.** 26.2 μT into the page **5.** $\dfrac{\mu_0 i}{12}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$ out of the page

Introductory Exercise 26.6

1. 200 μT toward the top of the page, 133 μT toward the bottom of the page

2. (a) zero (b) -5.0×10^{-6} T-m (c) 2.5×10^{-6} T-m (d) 5.0×10^{-6} T-m

3. (b) the magnetic field at any point inside the pipe is zero

Introductory Exercise 26.7

1. 90 T **2.** 1.3×10^{-7} A

Exercises

LEVEL 1

Assertion and Reason

1. (c) **2.** (c) **3.** (b) **4.** (d) **5.** (c) **6.** (b) **7.** (a) **8.** (c) **9.** (a) **10.** (d)
11. (d)

Objective Questions

1. (a) **2.** (c) **3.** (c) **4.** (c) **5.** (c) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (a)
11. (c) **12.** (d) **13.** (c) **14.** (a) **15.** (c) **16.** (a) **17.** (d) **18.** (c) **19.** (c) **20.** (c)

21. (c) **22.** (b) **23.** (d) **24.** (c) **25.** (a) **26.** (b) **27.** (c) **28.** (a) **29.** (b) **30.** (a)
31. (a) **32.** (a) **33.** (b) **34.** (d) **35.** (a)

Subjective Questions

1. (a) $(6.24 \times 10^{-14}\ \text{N})\hat{\mathbf{k}}$ (b) $-(6.24 \times 10^{-14}\ \text{N})\hat{\mathbf{k}}$

2. $B_x = -2.0$ T

3. (a) $B_x = (-0.175)\text{T}, B_z = (-0.256)\text{T}$ (b) Yes, B_y (c) zero, 90°

4. (a) $-qvB\,\hat{\mathbf{k}}$ (b) $+qvB\,\hat{\mathbf{j}}$ (c) zero (d) $\frac{-qvB}{\sqrt{2}}\hat{\mathbf{j}}$ (e) $\left(-\frac{qvB}{\sqrt{2}}\right)(\hat{\mathbf{j}} + \hat{\mathbf{k}})$

5. 8.38×10^{-4} T

6. (a) 8.35×10^{5} m/s (b) 2.62×10^{-8} s (c) 7.26 kV

7. (a) –q (b) $\frac{\pi m}{Bq}$

8. (a) 1.6×10^{-4} T into the page (b) 1.11×10^{-7} s

9. $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y \cos\omega t\,\hat{\mathbf{j}} - v_y \sin\omega t\,\hat{\mathbf{k}}$, Here $\omega = \frac{Be}{m}$

10. $\mathbf{E} = (-0.1\ \text{V/m})\,\hat{\mathbf{k}}$

11. $v_y = 0, v = \sqrt{\frac{2qE_0z}{m}}$

12. $\frac{2\pi mE \tan\theta}{qB^2}$

13. 0.47 A from left to right

14. (a) 817.5 V (b) 112.8 m/s^2

15. $\mathbf{F}_{ab} = 0, \mathbf{F}_{bc} = (-0.04\ \text{N})\,\hat{\mathbf{i}},\ \mathbf{F}_{cd} = (-0.04\ \text{N})\,\hat{\mathbf{k}}, \mathbf{F}_{da} = (0.04\hat{\mathbf{i}} + 0.04\hat{\mathbf{k}})\,\text{N}$

16. $\frac{\pi R^3}{2\mu_0}$

17. (a) 76.7° (b) 76.7°

18. (a) 1.5×10^{-16} s (b) 1.1 mA (c) 9.3×10^{-24} A-m^2

19. $\mathbf{M} = 2Il^2\hat{\mathbf{j}}$

20. zero

21. 2.0 μT

22. 2×10^{-6} T

23. $y = \left(\frac{I_1}{I_2}\right)x$

24. 2 rad

25. (a) Between the wires, 30.0 cm from wire carrying 75.0 A
(b) 20.0 cm from wire carrying 25.0 A and 60.0 cm from wire carrying 75.0 A

26. 69

27. $I_1 = \left(\frac{\pi D}{R}\right) I_2$, towards right

28. (a) 2.77 A (b) 0.0184 m

29. 2.0 A toward bottom of page

30. (a) $\frac{-\mu_0 qv_0 I}{4R}\hat{\mathbf{k}}$ (b) $\mathbf{F}_1 = \mathbf{F}_2 = 2B_0IR\hat{\mathbf{i}}, \mathbf{F} = 4B_0IR\hat{\mathbf{i}}$

31. (a) $\frac{\mu_0 in^2 \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)}{2\pi^2 r}$ (b) $\frac{\mu_0 i}{2r}$

32. (a) $\frac{\mu_0 I}{\pi r}\left(\frac{2r^2 - a^2}{4r^2 - a^2}\right)$ to the left (b) $\frac{\mu_0 I}{\pi r}\left(\frac{2r^2 + a^2}{4r^2 + a^2}\right)$ towards the top of the page

33. $B_P = 0, B_Q = \mu_0\lambda$

34. $\frac{\mu_0 I_1 I_2 L}{2\pi a}$, zero

36. (a) π (b) π (c) $\frac{\pi}{6}$

37 $v_0, \frac{mv_0}{qE_0}$

38. $\left(\frac{2\pi\, m v_0 \sin\theta}{Bq},\ 0,\ 0\right)$

39. $(0.04\,\hat{\mathbf{j}} - 0.07\,\hat{\mathbf{k}})$ A-m^2

40. 9.98 N·m, clockwise as seen looking down from above.

41. 20.0 μT toward the bottom of the square

42. (a) $\frac{\mu_0 b\, r_1^2}{3}$ (b) $\frac{\mu_0 b\, R^3}{3r_2}$

LEVEL 2

Objective Questions

1.(a) **2.**(c) **3.**(c) **4.**(c) **5.**(d) **6.**(d) **7.**(b) **8.**(d) **9.**(b) **10.**(c)
11.(d) **12.**(a) **13.**(b) **14.**(a) **15.**(c) **16.**(c) **17.**(d) **18.**(c) **19.**(c) **20.**(d)
21.(b) **22.**(c) **23.**(d) **24.**(d) **25.**(a) **26.**(b) **27.**(a)

More than One Correct Options

1.(a,c) **2.**(a,c,d) **3.**(a,b,d) **4.**(b,c,d) **5.**(b,d) **6.**(a,b,c,d) **7.**(a,b,c,d) **8.**(a,c,d) **9.**(a,c,d) **10.**(a,b,c)

Match the Columns

1. (a) → r (b) → q (c) → p (d) → r

2. (a) → r (b) → s (c) → q (d) → p

3. (a) → p,s (b) → p,q (c) → p,r (d) → p,s

4. (a) → q (b) → r (c) → s (d) → s

5. (a) → q (b) → r (c) → q (d) → r

6. (a) → s (b) → s (c) → q (d) → p

Subjective Questions

1. $\mathbf{F} = \dfrac{\mu_0 I^2}{\pi}\left[\dfrac{1}{2} - \dfrac{1}{\sqrt{3}}\ln\left(\dfrac{2+\sqrt{3}}{2}\right)\right](\hat{\mathbf{i}})$

2. $M = \dfrac{neh}{4\pi m}$, $B = \dfrac{\mu_0 \pi m^2 e^7}{8\varepsilon_0^3 h^5 n^5}$

3. 2.7×10^{-4} T

4. (0.2 m, 0.2 m, 0.2 m)

5. 2×10^{-5} N

6. (a) $y = \dfrac{Ea^2}{4V}$ (b) $p = \dfrac{\pi Ea}{B}\sqrt{\dfrac{2m}{qV}}$

7. 2.0 m/s, 3.46 mT

8. (a) $I_0 L^2 B$ (b) $\dfrac{3}{4}\dfrac{I_0 B}{M}(\Delta t)^2$

9. $\omega_{\max} = \dfrac{DT_0}{BQR^2}$

10. $B = \dfrac{\mu_0}{2\pi}\dfrac{i}{\omega}\ln\left(\dfrac{d+\omega}{d}\right)$ (upwards)

11. $2\tan^{-1}\left(\dfrac{BqR}{mv}\right)$

12. (a) 0.0442 N-m, clockwise, yes (b) stretched (c) 7.8×10^{-3} J

13. (a) P to Q (b) $IbB_0(3\hat{\mathbf{k}} - 4\hat{\mathbf{i}})$ (c) $\dfrac{mg}{6bB_0}$

27

Electromagnetic Induction

Chapter Contents

27.1 Introduction
27.2 Magnetic field lines and magnetic flux
27.3 Faraday's law
27.4 Lenz's law
27.5 Motional electromotive force
27.6 Self inductance and inductors
27.7 Mutual inductance
27.8 Growth and decay of current in an L-R circuit
27.9 Oscillations in L-C circuit
27.10 Induced electric field

27.1 Introduction

Almost every modern device has electric circuits at its heart. We learned in the chapter of current electricity that an electromagnetic force (emf) is required for a current to flow in a circuit. But for most of the electric devices used in industry the source of emf is not a battery but an electrical generating station. In these stations other forms of energy are converted into electric energy. For example, in a hydroelectric plant gravitational potential energy is converted into electric energy. Similarly, in a nuclear plant nuclear energy is converted into electric energy.

But how this conversion is done? Or what is the physics behind this? The branch of physics, known as **electromagnetic induction** gives the answer to all these queries. If the **magnetic flux (ϕ_B)** through a circuit changes, an emf and a current are induced in the circuit. Electromagnetic induction was discovered in 1830. The central principle of electromagnetic induction is **Faraday's law**. This law relates induced emf to change in magnetic flux in any loop, including a closed circuit. We will also discuss Lenz's law, which helps us to predict the directions of induced emf and current.

27.2 Magnetic Field Lines and Magnetic Flux

Let us first discuss the concept of magnetic field lines and magnetic flux. We can represent any magnetic field by **magnetic field lines**. Unlike the electric lines of force, it is wrong to call them magnetic lines of force, because they do not point in the direction of the force on a charge. The force on a moving charged particle is always perpendicular to the magnetic field (or magnetic field lines) at the particle's position.

The idea of magnetic field lines is same as for the electric field lines as discussed in the chapter of electrostatics. The magnetic field at any point is tangential to the field line at that point. Where the field lines are close, the magnitude of field is large, where the field lines are far apart, the field magnitude is small. Also, because the direction of **B** at each point is unique, field lines never intersect. Unlike the electric field lines, magnetic lines form a closed loop.

Magnetic Flux

The flux associated with a magnetic field is defined in a similar manner to that used to define electric flux. Consider an element of area dS on an arbitrary shaped surface as shown in figure. If the magnetic field at this element is **B**, the magnetic flux through the element is

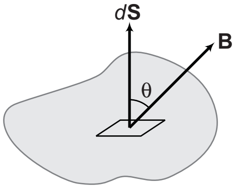


Fig. 27.1

$$\boxed{d\phi_B = \mathbf{B}\cdot d\mathbf{S} = BdS\cos\theta}$$

Here, $d\mathbf{S}$ is a vector that is perpendicular to the surface and has a magnitude equal to the area dS and θ is the angle between **B** and $d\mathbf{S}$ at that element. In general, $d\phi_B$ varies from element to element. The total magnetic flux through the surface is the sum of the contributions from the individual area elements.

$\therefore$ $$\boxed{\phi_B = \int BdS\cos\theta = \int \mathbf{B}\cdot d\mathbf{S}}$$

Note down the following points regarding the magnetic flux :

(i) Magnetic flux is a scalar quantity (dot product of two vector quantities is a scalar quantity)

(ii) The SI unit of magnetic flux is tesla-metre^2 (1 T-m^2). This unit is called **weber** (1 Wb).

$$\boxed{1\text{ Wb} = 1\text{ T-m}^2 = 1\text{ N-m/A}}$$

Thus, unit of magnetic field is also weber/m^2 (1 Wb/m^2).

or $$\boxed{1\text{ T} = 1\text{ Wb/m}^2}$$

(iii) In the special case in which **B** is uniform over a plane surface with total area S,

$$\phi_B = BS\cos\theta$$

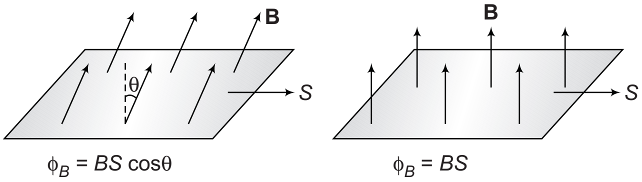


Fig. 27.2

If **B** is perpendicular to the surface, then $\cos\theta = 1$ and

$$\phi_B = BS$$

Gauss's Law for Magnetism

In Gauss's law, the total electric flux through a closed surface is proportional to the total electric charge enclosed by the surface. For example, if a closed surface encloses an electric dipole, the total electric flux is zero because the total charge is zero.

By analogy, if there were such as thing as a single magnetic charge (magnetic monopole), the total magnetic flux through a closed surface would be proportional to the total magnetic charge enclosed. But as no magnetic monopole has ever been observed, we conclude that **the total magnetic flux through a closed surface is zero**.

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{S} = 0}$$

Unlike electric field lines that begin and end on electric charges, magnetic field lines never have end points. Such a point would otherwise indicate the existence of a monopole. For a closed surface, the vector area element $d\mathbf{S}$ always points out of the surface. However, for an open surface we choose one of the possible sides of the surface to be the positive and use that choice consistently.

27.3 Faraday's Law

This law states that, "the induced emf in a closed loop equals the negative of the time rate of change of magnetic flux through the loop."

$$\boxed{e = -\frac{d\phi_B}{dt}}$$

If a circuit is a coil consisting of N loops all of the same area and if ϕ_B is the flux through one loop, an emf is induced in every loop, thus the total induced emf in the coil is given by the expression,

$$\boxed{e = -N\frac{d\phi_B}{dt}}$$

The negative sign in the above equations is of important physical significance, which we will discuss in article. 27.4.

Note down the following points regarding the Faraday's law:

(i) As we have seen, induced emf is produced only when there is a change in magnetic flux passing through a loop. The flux passing through the loop is given by

$$\phi = BS\cos\theta$$

This, flux can be changed in several ways:

(a) The magnitude of **B** can change with time. In the problems if magnetic field is given a function of time, it implies that the magnetic field is changing. Thus,

$$B = B(t)$$

(b) The current producing the magnetic field can change with time. For this, the current can be given as a function of time. Hence,

$$i = i(t)$$

(c) The area of the loop inside the magnetic field can change with time.This can be done by pulling a loop inside (or outside) a magnetic field.

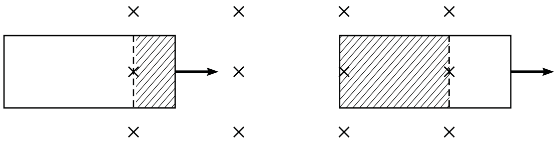


Fig. 27.3

(d) The angle θ between **B** and the normal to the loop (or **s**) can change with time.

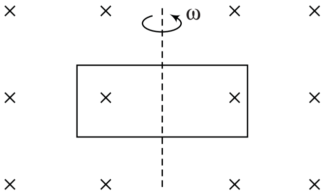


Fig. 27.4

This can be done by rotating a loop in a magnetic field.

(e) Any combination of the above can occur.

(ii) When the magnetic flux passing through a loop is changed, an induced emf and hence, an induced current is produced in the circuit. If R is the resistance of the circuit, then induced current is given by

$$i = \frac{e}{R} = \frac{1}{R}\left(\frac{-d\phi_B}{dt}\right)$$

Current starts flowing in the circuit, means flow of charge takes place. Charge flown in the circuit in time dt will be given by

$$dq = idt = \frac{1}{R}(-d\phi_B)$$

Thus, for a time interval Δt we can write the average values as,

$$\boxed{e = -\frac{\Delta\phi_B}{\Delta t}, \quad i = \frac{1}{R}\left(\frac{-\Delta\phi_B}{\Delta t}\right) \text{ and } \Delta q = \frac{1}{R}(-\Delta\phi_B)}$$

From these equations, we can see that e and i are inversely proportional to Δt while Δq is independent of Δt. It depends on the magnitude of change in flux, not the time taken in it. This can be explained by the following example.

Example 27.1 *A square loop ACDE of area* $20\ \text{cm}^2$ *and resistance* $5\ \Omega$ *is rotated in a magnetic field* $\mathbf{B} = 2T$ *through* 180°, *(a) in* 0.01 *s and (b) in* 0.02 *s*

Find the magnitudes of average values of e, i and Δq *in both the cases.*

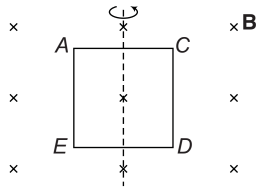


Fig. 27.5

Solution Let us take the area vector **S** perpendicular to plane of loop inwards. So initially, $\mathbf{S} \uparrow\uparrow \mathbf{B}$ and when it is rotated by 180°, $\mathbf{S} \uparrow\downarrow \mathbf{B}$. Hence, initial flux passing through the loop,

$$\phi_i = BS\cos 0° = (2)(20 \times 10^{-4})(1) = 4.0 \times 10^{-3}\ \text{Wb}$$

Flux passing through the loop when it is rotated by 180°,

$$\phi_f = BS\cos 180° = (2)(20 \times 10^{-4})(-1) = -4.0 \times 10^{-3}\ \text{Wb}$$

Therefore, change in flux,

$$\Delta\phi_B = \phi_f - \phi_i = -8.0 \times 10^{-3}\ \text{Wb}$$

(a) Given, $\Delta t = 0.01\text{s}, \quad R = 5\ \Omega$

$$\therefore \qquad |e| = \left|-\frac{\Delta\phi_B}{\Delta t}\right| = \frac{8.0 \times 10^{-3}}{0.01} = 0.8\ \text{V}$$

$$i = \frac{|e|}{R} = \frac{0.8}{5} = 0.16\ \text{A}$$

and $$\Delta q = i\Delta t = 0.16 \times 0.01 = 1.6 \times 10^{-3}\ \text{C}$$

(b) $\Delta t = 0.02\text{s}$

$$\therefore \qquad |e| = \left|-\frac{\Delta\phi_B}{\Delta t}\right| = \frac{8.0 \times 10^{-3}}{0.02} = 0.4\ \text{V}$$

$$i = \frac{|e|}{R} = \frac{0.4}{5} = 0.08\ \text{A}$$

and $$\Delta q = i\Delta t = (0.08)(0.02) = 1.6 \times 10^{-3}\ \text{C}$$

Note *Time interval* Δt *in part (b) is two times the time interval in part (a), so e and i are half while* Δq *is same.*

Example 27.2 *A coil consists of 200 turns of wire having a total resistance of 2.0 Ω. Each turn is a square of side 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly from 0 to 0.5 T in 0.80 s, what is the magnitude of induced emf and current in the coil while the field is changing?*

Solution From the Faraday's law,

$$\text{Induced emf, } |e| = \frac{N\Delta\phi}{\Delta t} = (NS)\frac{\Delta B}{\Delta t}$$

$$= \frac{(200)(18\times 10^{-2})^2\,(0.5-0)}{0.8}$$

$$= 4.05\text{ V}$$ **Ans.**

$$\text{Induced current, } i = \frac{|e|}{R} = \frac{4.05}{2} \approx 2.0\text{ A}$$ **Ans.**

Example 27.3 *The magnetic flux passing through a metal ring varies with time t as : $\phi_B = 3(at^3 - bt^2)\,T\text{-}m^2$ with $a = 2.00\ s^{-3}$ and $b = 6.00\ s^{-2}$. The resistance of the ring is 3.0 Ω. Determine the maximum current induced in the ring during the interval from t = 0 to t = 2.0 s.*

Solution Given, $\phi_B = 3(at^3 - bt^2)$

$$\therefore \qquad |e| = \left|\frac{d\phi_B}{dt}\right| = 9at^2 - 6bt$$

$$\therefore \text{ Induced current,} \qquad i = \frac{|e|}{R} = \frac{9at^2 - 6bt}{3} = 3at^2 - 2bt$$

For current to be maximum,

$$\frac{di}{dt} = 0$$

$$\therefore \qquad 6at - 2b = 0$$

$$\text{or} \qquad t = \frac{b}{3a}$$

i.e. at $t = \frac{b}{3a}$, current is maximum. This maximum current is

$$i_{\text{max}} = 3a\left(\frac{b}{3a}\right)^2 - 2b\left(\frac{b}{3a}\right)$$

$$= \frac{b^2}{3a} - \frac{2b^2}{3a} = \frac{b^2}{3a}$$

Substituting the given values of a and b, we have

$$i_{\text{max}} = \frac{(6)^2}{3(2)} = 6.0\text{ A}$$ **Ans.**

27.4 Lenz's Law

The negative sign in Faraday's equations of electromagnetic induction describes the direction in which the induced emf drives current around a circuit. However, that direction is most easily determined with the help of Lenz's law. This law states that:

"The direction of any magnetic induction effect is such as to oppose the cause of the effect."

For different types of problems, Lenz's law has been further subdivided into following concepts.

1. **Attraction and repulsion concept** If magnetic flux is changed by bringing a magnet and a loop (or solenoid etc.) closer to each other then direction of induced current is so produced, that the magnetic field produced by it always repels the two. Similarly, if they are moved away from each other then they are attracted towards each other. Following two examples will illustrate this.

Example 27.4 *A bar magnet is freely falling along the axis of a circular loop as shown in figure. State whether its acceleration a is equal to, greater than or less than the acceleration due to gravity g.*

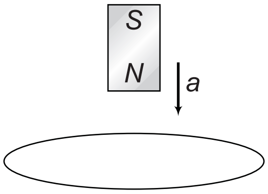
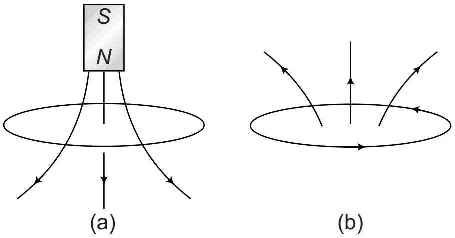


Fig. 27.6

Solution $a < g$. Because according to Lenz's law, whatever may be the direction of induced current, it will oppose the cause.



(a) (b)

Fig. 27.7

Here, the cause is, the free fall of magnet and so the induced current will oppose it and the acceleration of magnet will be less than the acceleration due to gravity g. This can also be explained in a different manner. When the magnet falls downwards with its north pole downwards.

The magnetic field lines passing through the coil in the downward direction increase. Since, the induced current opposes this, the upper side of the coil will become north pole, so that field lines of coil's magnetic field are upwards. Now, like poles repel each other. Hence, $a < g$.

Example 27.5 *A bar magnet is brought near a solenoid as shown in figure. Will the solenoid attract or repel the magnet?*

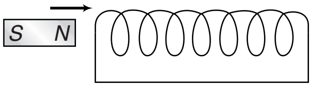


Fig. 27.8

Solution When the magnet is brought near the solenoid, according to Lenz's law, both repel each other. On the other hand, if the magnet is moved away from the solenoid, it attracts the magnet. When the magnet is brought near the solenoid, the nearer side becomes the same pole and when it is moved away it becomes the opposite pole as shown in figure.

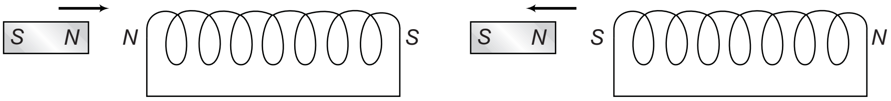


Fig. 27.9

It can also be explained by increasing or decreasing field lines as discussed in example 27.4.

2. **Cross or dot magnetic field increasing or decreasing concept**

If cross magnetic field passing through a loop increases then induced current will produce dot magnetic field. Similarly, if dot magnetic field passing through a loop decreases then dot magnetic field is produced by the induced current. Let us take some examples in support of it.

Example 27.6 *A circular loop is placed in magnetic field $B = 2t$. Find the direction of induced current produced in the loop.*

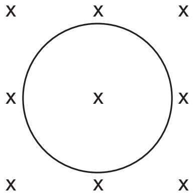


Fig. 27.10

Solution $B = 2t$, means $\otimes$ magnetic field (or we can also say cross magnetic flux) passing through the loop is increasing. So, induced current will produce dot magnetic field. To produce $\odot$ magnetic field, induced current from our side should be anti-clockwise.

Example 27.7 *A rectangular loop is placed to the left of large current carrying straight wire as shown in figure. Current varies with time as $I = 2t$. Find direction of induced current I_{in} in the square loop.*

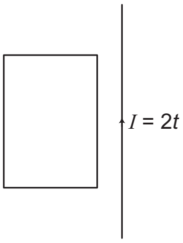


Fig. 27.11

Solution Current I will produce $\odot$ magnetic field passing through the loop. Current I is increasing, so dot magnetic field will also increase. Therefore, induced current should produce cross magnetic field. For producing cross magnetic field in the loop, induced current from our side should be clockwise.

3. **Situations where flux passing through the loop is always zero or change in flux is zero.**

Example 27.8 *A current carrying straight wire passes inside a triangular loop as shown in Fig. 27.12. The current in the wire is perpendicular to paper inwards. Find the direction of the induced current in the loop if current in the wire is increased.*

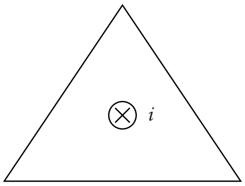


Fig. 27.12

Solution Magnetic field lines round the current carrying wire are as shown in Fig.27.13. Since, the lines are tangential to the loop ($\theta = 90^\circ$) the flux passing through the loop is always zero, whether the current is increased or decreased. Hence, change in flux is also zero. Therefore, induced current in the loop will be zero.

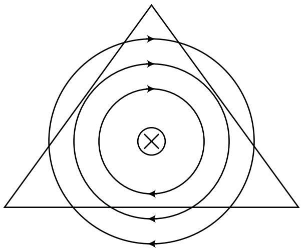


Fig. 27.13

Example 27.9 *A rectangular loop is placed adjacent to a current carrying straight wire as shown in figure. If the loop is rotated about an axis passing through one of its sides, find the direction of induced current in the loop.*

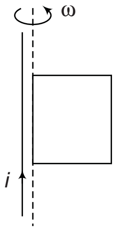


Fig. 27.14

Solution Magnetic field lines around the straight wire are circular. So, same magnetic lines will pass through loop under all conditions.

$$\Delta\phi = 0$$

$$\Rightarrow \quad \text{emf} = 0$$

$$\Rightarrow \quad i = 0$$

4. **Attraction or repulsion between two loops facing each other if current in one loop is changed**

Example 27.10 *Two loops are facing each other as shown in Fig. 27.15. State whether the loops will attract each other or repel each other if current I_1 is increased.*

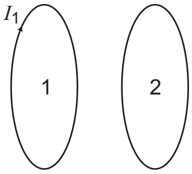


Fig. 27.15

Solution If current I_1 is increased then induced current in loop-2 (say I_2) will be in opposite direction. Now, two wires having currents in opposite directions repel each other. So, the loops will repel each other.

INTRODUCTORY EXERCISE 27.1

1. Figure shows a conducting loop placed near a long straight wire carrying a current i as shown. If the current increases continuously, find the direction of the induced current in the loop.

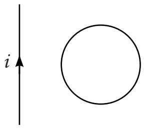


Fig. 27.16

2. A metallic loop is placed in a non-uniform steady magnetic field. Will an emf be induced in the loop?
3. Write the dimensions of $\frac{d\phi_B}{dt}$.
4. A triangular loop is placed in a dot ⊙ magnetic field as shown in figure. Find the direction of induced current in the loop if magnetic field is increasing.

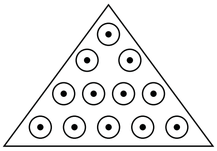


Fig. 27.17

5. Two circular loops lie side by side in the same plane. One is connected to a source that supplies an increasing current, the other is a simple closed ring. Is the induced current in the ring is in the same direction as that in the loop connected to the source or opposite? What if the current in the first loop is decreasing?
6. A wire in the form of a circular loop of radius 10 cm lies in a plane normal to a magnetic field of 100 T. If this wire is pulled to take a square shape in the same plane in 0.1 s, find the average induced emf in the loop.
7. A closed coil consists of 500 turns has area 4 cm^2 and a resistance of $50 \ \Omega$. The coil is kept with its plane perpendicular to a uniform magnetic field of 0.2 Wb/m^2. Calculate the amount of charge flowing through the coil if it is rotated through 180°.
8. The magnetic field in a certain region is given by $\mathbf{B} = (4.0 \ \hat{\mathbf{i}} - 1.8 \ \hat{\mathbf{k}}) \times 10^{-3}$ T. How much flux passes through a 5.0 cm^2 area loop in this region if the loop lies flat on the xy-plane?

27.5 Motional Electromotive Force

Till now, we have considered the cases in which an emf is induced in a stationary circuit placed in a magnetic field, when the field changes with time. In this section, we describe what is called **motional emf**, which is the emf induced in a conductor moving through a constant magnetic field.

The straight conductor of length l shown in figure is moving through a uniform magnetic field directed into the page. For simplicity we assume that the conductor is moving in a direction perpendicular to the field with constant velocity under the influence of some external agent. The electrons in the conductor experience a force

$$\mathbf{F}_m = -e(\mathbf{v} \times \mathbf{B})$$

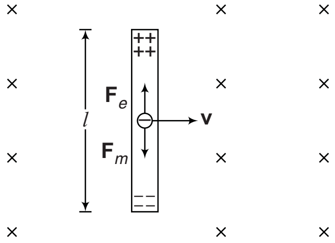


Fig. 27.18

Under the influence of this force, the electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. As a result of this charge separation, an electric field is produced inside the conductor. The charges accumulate at both ends untill the downward magnetic force evB is balanced by the upward electric force eE. At this point, electrons stop moving. The condition for equilibrium requires that,

$$eE = evB \quad \text{or} \quad E = vB$$

The electric field produced in the conductor (once the electrons stop moving and E is constant) is related to the potential difference across the ends of the conductor according to the relationship,

$$\Delta V = El = Blv$$

$$\boxed{\Delta V = Blv}$$

where the upper end is at a higher electric potential than the lower end. Thus,

"a potential difference is maintained between the ends of a straight conductor as long as the conductor continues to move through the uniform magnetic field."

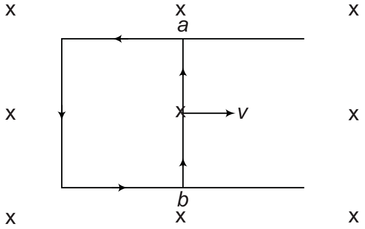


Fig. 27.19

Now, suppose the moving rod slides along a stationary U-shaped conductor forming a complete circuit. We call this a **motional electromagnetic force** denoted by e, we can write

$$\boxed{e = Bvl}$$

If R is the resistance of the circuit, then current in the circuit is

$$\boxed{i = \frac{e}{R} = \frac{Bvl}{R}}$$

$V_a > V_b$. Therefore, direction of current in the loop is anti-clockwise as shown in figure.

Extra Points to Remember

- The direction of motional emf or current can be given by right hand rule.

  Stretch your right hand.

  The stretched fingers point in the direction of magnetic field. Thumb is along the velocity of conductor. The upper side of the palm is at higher potential and lower side on lower potential. If the circuit is closed, the induced current within the conductor is along perpendicular to palm upwards.

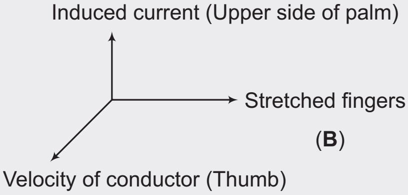

Induced current (Upper side of palm)
Stretched fingers
(**B**)
Velocity of conductor (Thumb)

Fig. 27.20

-

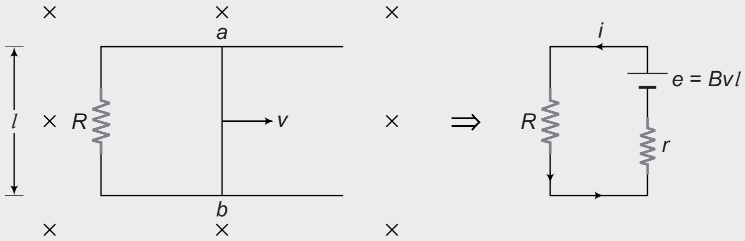


Fig. 27.21

In the Fig. 27.20, we can replace the moving rod *ab* by a battery of emf *Bvl* with the positive terminal at *a* and the negative terminal at *b*. The resistance *r* of the rod *ab* may be treated as the internal resistance of the battery. Hence, the current in the circuit is

$$i = \frac{e}{R + r} \quad \text{or} \quad \boxed{i = \frac{Bvl}{R + r}}$$

- **Induction and energy transfers** In the Fig. 27.21, if you move the conductor *ab* with a constant velocity *v*, the current in the circuit is

$$i = \frac{Bvl}{R} \qquad (r = 0)$$

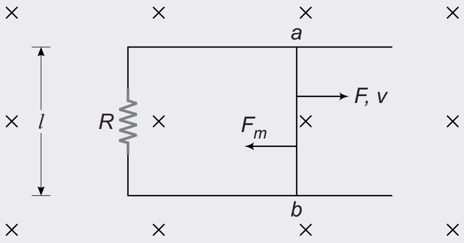


Fig. 27.22

A magnetic force $F_m = ilB = \frac{B^2l^2\ v}{R}$ acts on the conductor in opposite direction of velocity. So, to move the conductor with a constant velocity *v* an equal and opposite force *F* has to be applied in the conductor. Thus,

$$F = F_m = \frac{B^2l^2v}{R}$$

The rate at which work is done by the applied force is

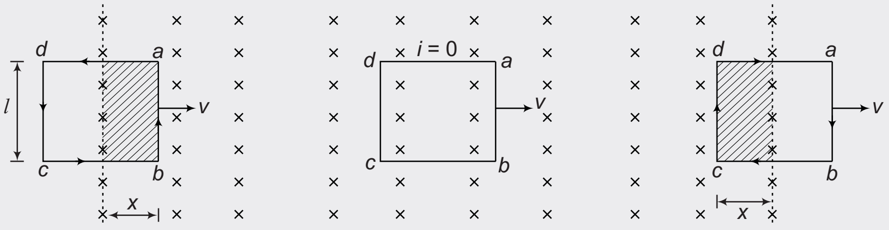
$$P_{\text{applied}} = Fv = \frac{B^2l^2v^2}{R}$$

and the rate at which energy is dissipated in the circuit is

$$P_{\text{dissipated}} = i^2R = \left(\frac{Bvl}{R}\right)^2 R = \frac{B^2l^2v^2}{R}$$

This is just equal to the rate at which work is done by the applied force.

- Motional emf is not a different kind of induced emf, it is exactly the induced emf described by Faraday's law, in the case in which there is a conductor moving in a magnetic field. Equation, $e = -\frac{d\phi_B}{dt}$ is best applied to problems in which there is a changing flux through a closed loop while $e = Bvl$ is applied to problems in which a conductor moves through a magnetic field. Note that, if a conductor is moving in a magnetic field but circuit is not closed, then only PD will be asked between two points of the conductor. If the circuit is closed, then current will be asked in the circuit. Now, let us see how these two are similar.



(a) (b) (c)

Fig. 27.23

Refer figure (a) A loop *abcd* enters a uniform magnetic field *B* at constant speed *v*.

Using Faraday's equation,

$$|e| = \left|-\frac{d\phi_B}{dt}\right| = \frac{d(BS)}{dt} = \frac{d(Blx)}{dt} = Bl\frac{dx}{dt} = Blv$$

For the direction of current, we can use Lenz's law. As the loop enters the field, ⊗ magnetic field passing through the loop increases, hence, induced current should produce ⊙ magnetic field or current in the loop is anti-clockwise. From the theory of motional emf, $e = Bvl$ and using right hand rule also, current in the circuit is anti-clockwise. Thus, we see that $e = -\frac{d\phi_B}{dt}$ and $e = Bvl$ give the same result. In the similar manner, we can show that current in the loop in figure (b) is zero and in figure (c) it is clockwise.

- We can generalize the concept of motional emf for a conductor with any shape moving in any magnetic field uniform or not. For an element $d\mathbf{l}$ of conductor the contribution de to the emf is the magnitude dl multiplied by the component of $\mathbf{v} \times \mathbf{B}$ parallel to $d\mathbf{l}$, that is

$$de = (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

For any two points *a* and *b* the motional emf in the direction from *b* to *a* is,

$$\boxed{e = \int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}}$$

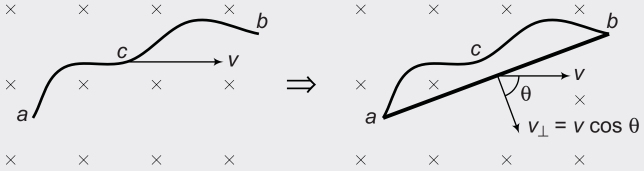


Fig. 27.24

In general, we can say that motional emf in wire *acb* in a uniform magnetic field is the motional emf in an imaginary straight wire *ab*. Thus,

$$e_{acb} = e_{ab} = (\text{length of } ab)\,(v_{\perp})\,(B)$$

Here, $v_{\perp}$ is the component of velocity perpendicular to both **B** and *ab*.

From right hand rule we can see that *b* is at higher potential and *a* at lower potential.

Hence, $$V_{ba} = V_b - V_a = (ab)\,(v\cos\theta)\,(B)$$

- **Motional emf induced in a rotating bar :** A conducting rod of length *l* rotates with a constant angular speed ω about a pivot at one end. A uniform magnetic field **B** is directed perpendicular to the plane of rotation as shown in figure. Consider a segment of rod of length *dr* at a distance *r* from *O*. This segment has a velocity,

$$v = r\omega$$

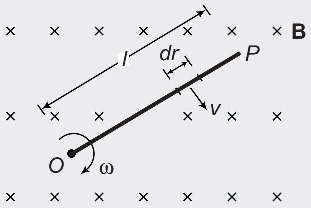


Fig. 27.25

The induced emf in this segment is $de = Bv\,dr = B\,(r\omega)\,dr$

Summing the emfs induced across all segments, which are in series, gives the total emf across the rod.

$$\therefore \qquad e = \int_0^l de = \int_0^l Br\omega\,dr = \frac{B\omega l^2}{2}$$

$$\therefore \qquad \boxed{e = \frac{B\omega l^2}{2}}$$

From right hand rule we can see that *P* is at higher potential than *O*. Thus,

$$V_P - V_O = \frac{B\omega l^2}{2}$$

- Note that in the problems of electromagnetic induction whenever you see a conductor moving in a magnetic field use the motional approach. It is easier than the other approach. But, if the conductor (or loop) is stationary, you have no choice. Use $e = -\dfrac{d\phi_B}{dt}$.

- Now onwards, the following integrations will be used very frequently.

If $$\int_0^x \frac{dx}{a - bx} = \int_0^t c\,dt$$

then, $$x = \frac{a}{b}\,(1 - e^{-bct})$$

and if $$\int_{x_0}^x \frac{dx}{a - bx} = \int_0^t c\,dt$$

then, $$x = \frac{a}{b} - \left(\frac{a}{b} - x_0\right) e^{-bct}$$

Here *a*, *b* and *c* are positive constants.

Note *In an electrical circuit, a moving or rotating wire may be assumed as a battery of emf* Bvl *or* $\frac{B\omega l^2}{2}$ *and then it can be solved with the help of Kirchhoff's laws. The following example will illustrate this concept.*

Example 27.11 *Two parallel rails with negligible resistance are* 10.0 *cm apart. They are connected by a* 5.0 Ω *resistor. The circuit also contains two metal rods having resistances of* 10.0 Ω *and* 15.0 Ω *along the rails. The rods are pulled away from the resistor at constant speeds* 4.00 *m/s and* 2.00 *m/s respectively. A uniform magnetic field of magnitude* 0.01 *T is applied perpendicular to the, plane of the rails. Determine the current in the* 5.0 Ω *resistor.*

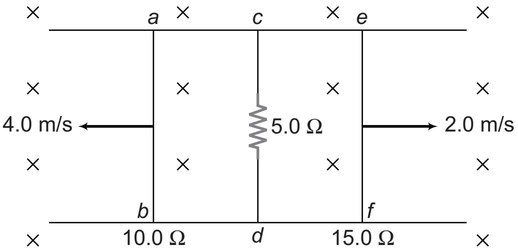


Fig. 27.26

HOW TO PROCEED *Here, two conductors are moving in a uniform magnetic field. So, we will use the motional approach. The rod ab will act as a source of emf,*

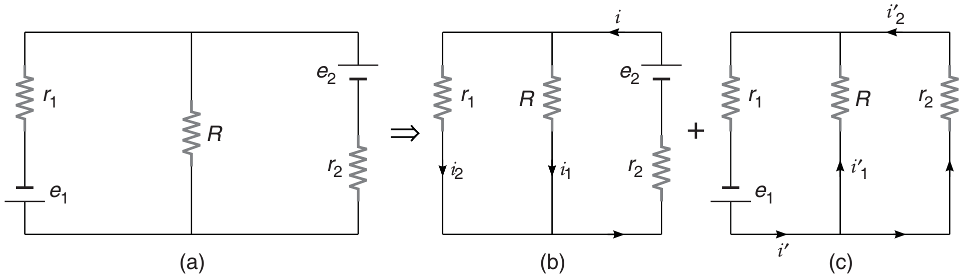
$$e_1 = Bvl = (0.01)(4.0)(0.1) = 4 \times 10^{-3}\ V$$

and internal resistance $r_1 = 10.0\ \Omega$
Similarly, rod ef will also act as a source of emf,

$$e_2 = (0.01)(2.0)(0.1) = 2.0 \times 10^{-3}\ V$$

and internal resistance $r_2 = 15.0\ \Omega$.
From right hand rule we can see that, $V_b > V_a$ *and* $V_e > V_f$
Now, either by applying Kirchhoff's laws or applying principle of superposition (discussed in the chapter of current electricity) we can find current through 5.0 Ω *resistor. We will here use the superposition principle. You solve it by using Kirchhoff's laws.*

Solution In the figures $R = 5.0\,\Omega$, $r_1 = 10\,\Omega$, $r_2 = 15\,\Omega$, $e_1 = 4 \times 10^{-3}$ V and $e_2 = 2 \times 10^{-3}$ V



(a) (b) (c)

Fig. 27.27

Refer figure (b) Net resistance of the circuit $= r_2 + \dfrac{Rr_1}{R + r_1} = 15 + \dfrac{10 \times 5}{10 + 5} = \dfrac{55}{3}\,\Omega$

$$\therefore \qquad \text{Current, } i = \frac{e_2}{\text{Net resistance}} = \frac{2 \times 10^{-3}}{55/3} = \frac{6}{55} \times 10^{-3}\ \text{A}$$

$\therefore$ Current through R,

$$i_1 = \left(\frac{r_1}{R + r_1}\right) i = \left(\frac{10}{10 + 5}\right)\left(\frac{6}{55} \times 10^{-3}\right)\text{A}$$

$$= \frac{4}{55} \times 10^{-3}\ \text{A} = \frac{4}{55}\ \text{mA}$$

Refer figure (c) Net resistance of the circuit $= r_1 + \dfrac{Rr_2}{R + r_2}$

$$= 10 + \frac{5 \times 15}{5 + 15} = \frac{55}{4}\,\Omega$$

$$\therefore \qquad \text{Current, } i' = \frac{e_1}{\text{Net resistance}}$$

$$= \frac{4 \times 10^{-3}}{55/4} = \frac{16}{55} \times 10^{-3}\ \text{A}$$

$$\therefore \text{ Current through } R, \quad i_1' = \left(\frac{r_2}{R + r_2}\right) i' = \left(\frac{15}{15 + 5}\right)\left(\frac{16}{55}\right) \times 10^{-3}\ \text{A}$$

$$= \frac{12}{55}\ \text{mA}$$

From superposition principle net current through 5.0 Ω resistor is

$$i_1' - i_1 = \frac{8}{55}\ \text{mA from } d \text{ to } c$$ **Ans.**

Example 27.12 *Figure shows the top view of a rod that can slide without friction. The resistor is* 6.0 Ω *and a* 2.5 T *magnetic field is directed perpendicularly downward into the paper. Let* $l = 1.20$ m.

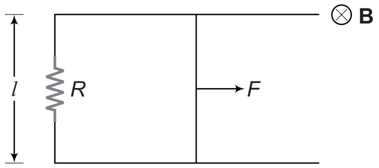


Fig. 27.28

(a) *Calculate the force* F *required to move the rod to the right at a constant speed of* 2.0 m/s.

(b) *At what rate is energy delivered to the resistor?*

(c) *Show that this rate is equal to the rate of work done by the applied force.*

Solution The motional emf in the rod, $e = Bvl$ or $e = (2.5)(2.0)(1.2)\ \text{V} = 6.0\ \text{V}$

The current in the circuit, $i = \frac{e}{R} = \frac{6.0}{6.0} = 1.0\ \text{A}$

(a) The magnitude of force F required will be equal to the magnetic force acting on the rod, which opposes the motion.

$\therefore$ $F = F_m = ilB$ or $F = (1.0)(1.2)(2.5)\ \text{N} = 3\ \text{N}$ **Ans.**

(b) Rate by which energy is delivered to the resistor is

$P_1 = i^2R = (1)^2\ (6.0) = 6\ \text{W}$ **Ans.**

(c) The rate by which work is done by the applied force is

$$P_2 = F \cdot v = (3)(2.0) = 6\ \text{W}$$

$P_1 = P_2$ **Hence proved.**

INTRODUCTORY EXERCISE 27.2

1. A horizontal wire 0.8 m long is falling at a speed of 5 m/s perpendicular to a uniform magnetic field of 1.1 T, which is directed from east to west. Calculate the magnitude of the induced emf. Is the north or south end of the wire positive?
2. As shown in figure, a metal rod completes the circuit. The circuit area is perpendicular to a magnetic field with $B = 0.15$ T. If the resistance of the total circuit is 3 Ω, how large a force is needed to move the rod as indicated with a constant speed of 2 m/s?

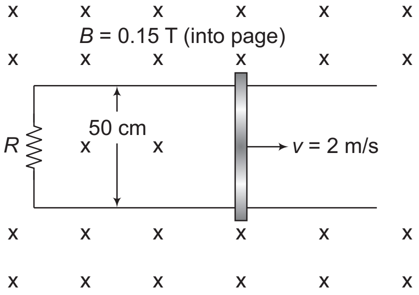


Fig. 27.29

3. A rod of length $3l$ is rotated with an angular velocity ω as shown in figure. The uniform magnetic field B is into the paper. Find
(a) $V_A - V_C$ (b) $V_A - V_D$

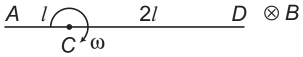


Fig. 27.30

4. As the bar shown in figure moves in a direction perpendicular to the field, is an external force required to keep it moving with constant speed.

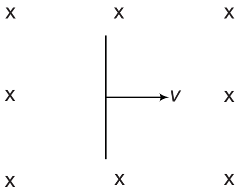


Fig. 27.31

27.6 Self-inductance and Inductors

Consider a single isolated circuit. When a current is present in the circuit, it sets up a magnetic field that causes a magnetic flux through the same circuit. This flux changes as the current in the circuit is changed. According to Faraday's law any change in flux in a circuit produces an induced emf in it. Such an emf is called a **self-induced emf**. The name is so called because the source of this induced emf is the change of current in the same circuit.

According to Lenz's law the self-induced emf always opposes the change in the current that caused the emf and so tends to make it more difficult for variations in current to occur. We will here like to define a term **self-inductance *L*** of a circuit which is of great importance in our proceeding discussions. It can be defined in the following two ways :

First Definition Suppose a circuit includes a coil with N turns of wire. It carries a current i. The total flux ($N\phi_B$) linked with the coil is directly proportional to the current (i) in the coil, i.e.

$$N\phi_B \propto i$$

When the proportionality sign is removed a constant L comes in picture, which depends on the dimensions and number of turns in the coil. This constant is called self-inductance. Thus,

$$N\phi_B = Li \qquad \text{or} \qquad \boxed{L = \frac{N\phi_B}{i}}$$

From here we can define self-inductance (L) of any circuit as, **the total flux per unit current**. The SI unit of self-inductance is henry (1H).

Second Definition If a current i is passed in a circuit and it is changed with a rate di/dt, the induced emf e produced in the circuit is directly proportional to the rate of change of current. Thus,

$$e \propto \frac{di}{dt}$$

When the proportionality constant is removed, the same constant L again comes here.

Hence, $$e = -L\frac{di}{dt}$$

The minus sign here is a reflection of Lenz's law. It says that the self-induced emf in a circuit opposes any change in the current in that circuit. From the above equation,

$$\boxed{L = \left|\frac{-e}{di/dt}\right|}$$

This equation states that, **the self-inductance of a circuit is the magnitude of self induced emf per unit rate of change of current**.

A circuit or part of a circuit, that is designed to have a particular inductance is called an **inductor**. The usual symbol for an inductor is

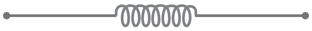


Fig. 27.32

Thus, an inductor is a circuit element which opposes the change in current through it. It may be a circular coil, solenoid etc.

Significance of Self-inductance and Inductor

Like capacitors and resistors, inductors are among the circuit elements of modern electronics. Their purpose is to oppose any variations in the current through the circuit. In a DC circuit, an inductor helps to maintain a steady state current despite fluctuations in the applied emf. In an AC circuit, an inductor tends to suppress variations of the current that are more rapid than desired. An inductor plays a dormant role in a circuit so far as current is constant. It becomes active when current changes in the circuit. Every inductor has some self-inductance which depends on the size, shape and the number of turns etc. For N turns close together, it is always proportional to N^2. It also depends on the magnetic properties of the material enclosed by the circuit. When the current passing through it is changed, an emf of magnitude $L\,di/dt$ is induced across it. Later in this article, we will explore the method of finding the self-inductance of an inductor.

Potential Difference Across an Inductor

We can find the polarities of self-induced emf across an inductor from Lenz's law.

The induced emf is produced whenever there is a change in the current in the inductor. This emf always acts to oppose this change. Figure shows three cases. Assume that the inductor has negligible resistance, so the PD, $V_{ab} = V_a - V_b$ between the inductor terminals a and b is equal in magnitude to the self-induced emf.

Refer figure (a) The current is constant, and there is no self-induced emf. Hence, $V_{ab} = 0$

Refer figure (b) The current is increasing, so $\dfrac{di}{dt}$ is positive. The induced emf e must oppose the increasing current, so it must be in the sense from b to a, a becomes the higher potential terminal and V_{ab} is positive. The direction of the emf is analogous to a battery with a as its positive terminal.

Refer figure (c) The current is decreasing and $\dfrac{di}{dt}$ is negative. The self-induced emf e opposes this decrease and V_{ab} is negative. This is analogous to a battery with b as its positive terminal.

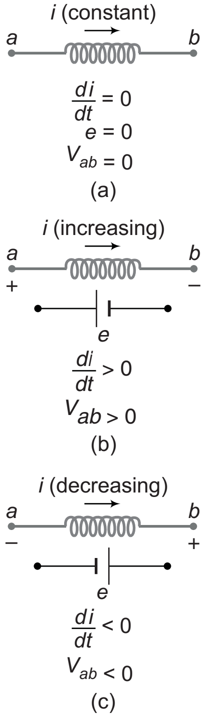


Fig. 27.33

In each case, we can write the PD, V_{ab} as

$$\boxed{V_{ab} = -e = L\frac{di}{dt}}$$

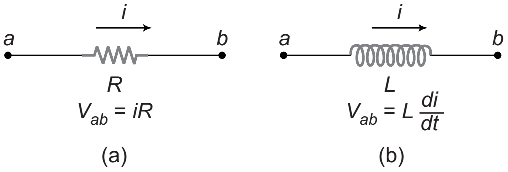


Fig. 27.34

The circuit's behaviour of an inductor is quite different from that of a resistor. While a resistor opposes the current i, an inductor opposes the change (di/dt) in the current.

Kirchhoff's potential law with an inductor In Kirchhoff's potential law when we go through an inductor in the same direction as the assumed current, we encounter a voltage drop equal to $L\,di/dt$, where di/dt is to be substituted with sign.

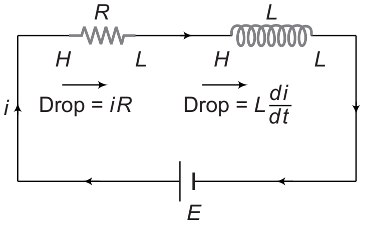


Fig. 27.35

For example in the loop shown in figure, Kirchhoff's second law gives the equation.

$$E - iR - L\frac{di}{dt} = 0$$

Example 27.13 *The inductor shown in the figure has inductance* $0.54\,H$ *and carries a current in the direction shown that is decreasing at a uniform rate* $\frac{di}{dt} = -0.03\ A/s$.

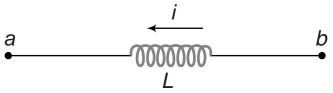


Fig. 27.36

(a) Find the self-induced emf.

(b) Which end of the inductor a or b is at a higher potential?

Solution (a) Self-induced emf,

$$e = -L\frac{di}{dt} = (-0.54)(-0.03)\ \text{V}$$

$$= 1.62 \times 10^{-2}\ \text{V}$$ **Ans.**

(b) $V_{ba} = L\frac{di}{dt} = -1.62 \times 10^{-2}\ \text{V}$

Since, $V_{ba}\ (= V_b - V_a)$ is negative. It implies that $V_a > V_b$ or a is at higher potential. **Ans.**

Example 27.14 *In the circuit diagram shown in figure,* $R = 10\ \Omega$, $L = 5H$, $E = 20\ V$, $i = 2\ A$. *This current is decreasing at a rate of* $-1.0\ A/s$. *Find* V_{ab} *at this instant.*

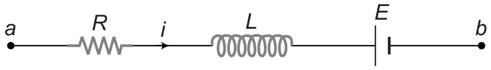


Fig. 27.37

Solution PD across inductor,

$$V_L = L\frac{di}{dt} = (5)(-1.0) = -5\ \text{V}$$

Now, $V_a - iR - V_L - E = V_b$

$\therefore$ $$V_{ab} = V_a - V_b = E + iR + V_L$$
$$= 20 + (2)(10) - 5 = 35\text{ V}$$ **Ans.**

Note *As the current is decreasing, the inductor can be replaced by a source of emf* $e = \left|L \cdot \frac{di}{dt}\right| = 5$ *V in such a manner that this emf supports the decreasing current, or it sends the current in the circuit in the same direction as the existing current. So, positive terminal of this source is towards b. Thus, the given circuit can be drawn as shown below,*

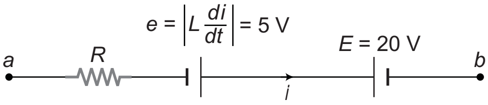


Fig. 27.38

Now, we can find V_{ab}.

Method of Finding Self-inductance of a Circuit

We use the equation, $L = N\phi_B / i$ to calculate the inductance of given circuit.

A good approach for calculating the self-inductance of a circuit consists of the following steps:

(a) Assume that there is a current i flowing through the circuit (we can call the circuit as inductor).

(b) Determine the magnetic field **B** produced by the current.

(c) Obtain the magnetic flux ϕ_B.

(d) With the flux known, the self-inductance can be found from $L = N\phi_B / i$.

To demonstrate this procedure, we now calculate the self-inductance of two inductors.

Inductance of a Solenoid

Let us find the inductance of a uniformly wound solenoid having N turns and length l. Assume that l is much longer than the radius of the windings and that the core of the solenoid is air.

We can assume that the interior magnetic field due to a current i is uniform and given by equation,

$$B = \mu_0 ni = \mu_0 \left(\frac{N}{l}\right) i$$

where, $n = \frac{N}{l}$ is the number of turns per unit length.

The magnetic flux through each turn is, $\phi_B = BS = \mu_0 \frac{NS}{l} i$

Here, S is the cross-sectional area of the solenoid. Now,

$$L = \frac{N\phi_B}{i} = \frac{N}{i}\left(\frac{\mu_0 NSi}{l}\right) = \frac{\mu_0 N^2 S}{l}$$

$\therefore$ $$\boxed{L = \frac{\mu_0 N^2 S}{l}}$$

This result shows that L depends on dimensions (S, l) and is proportional to the square of the number of turns.

$$\boxed{L \propto N^2}$$

Because $N = nl$, we can also express the result in the form,

$$L = \mu_0 \frac{(nl)^2}{l} S = \mu_0 n^2 Sl = \mu_0 n^2 V \quad \text{or} \quad \boxed{L = \mu_0 n^2 V}$$

Here, $V = Sl$ is the volume of the solenoid.

Inductance of a Rectangular Toroid

A toroid with a rectangular cross-section is shown in figure. The inner and outer radii of the toroid are R_1 and R_2 and h is the height of the toroid. Applying Ampere's law for a toroid, we can show that magnetic field inside a rectangular toroid is given by

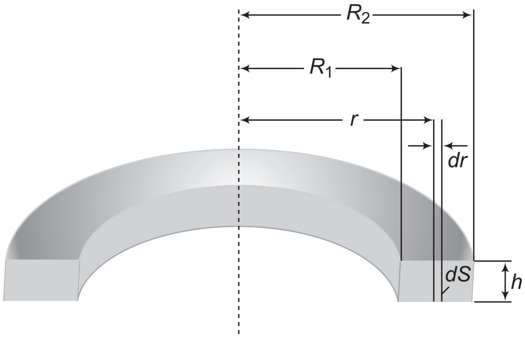


Fig. 27.39

$$B = \frac{\mu_0 Ni}{2\pi r}$$

where, r is the distance from the central axis of the toroid. Because the magnitude of magnetic field changes within the toroid, we must calculate the flux by integrating over the toroid's cross-section. Using the infinitesimal cross-sectional area element $dS = hdr$ shown in the figure, we obtain

$$\phi_B = \int B\, dS = \int_{R_1}^{R_2} \left(\frac{\mu_0 Ni}{2\pi r} \right) (hdr)$$

$$= \frac{\mu_0 Nhi}{2\pi} \ln \left(\frac{R_2}{R_1} \right)$$

Now, $$L = \frac{N\phi_B}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln \left(\frac{R_2}{R_1} \right)$$

or $$\boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln \left(\frac{R_2}{R_1} \right)}$$

As expected, the self-inductance is a constant determined by only the physical properties of the toroid.

Energy Stored in an Inductor

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability of storing energy in its magnetic field.

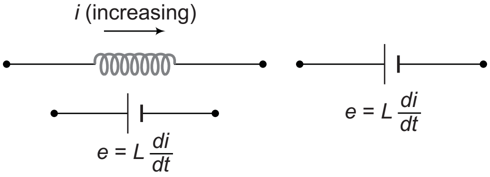


Fig. 27.40

An increasing current in an inductor causes an emf between its terminals.

The work done per unit time is power.

$$P = \frac{dW}{dt} = -\,ei = -\,Li\frac{di}{dt}$$

From $\qquad dW = -\,dU \quad \text{or} \quad P = \frac{dW}{dt} = -\frac{dU}{dt}$

We have, $\qquad \frac{dU}{dt} = Li\frac{di}{dt} \quad \text{or} \quad dU = Li\,di$

The total energy U supplied while the current increases from zero to a final value i is

$$U = L\int_0^i i\,di = \frac{1}{2}Li^2$$

$\therefore \qquad \boxed{U = \frac{1}{2}Li^2}$

This is the expression for the energy stored in the magnetic field of an inductor when a current i flows through it. The source of this energy is the external source of emf that supplies the current.

Note *(i) After the current has reached its final steady state value i, $di/dt = 0$ and no more energy is taken by the inductor.*

(ii) When the current decreases from i to zero, the inductor acts as a source that supplies a total amount of energy $\frac{1}{2}Li^2$ to the external circuit. If we interrupt the circuit suddenly by opening a switch, the current decreases very rapidly, the induced emf is very large and the energy may be dissipated as a spark across the switch.

(iii) If we compare the behaviour of a resistor and an inductor towards the current flow we can observe that energy flows into a resistor whenever a current passes through it. Whether the current is steady (constant) or varying this energy is dissipated in the form of heat. By contrast energy flows into an ideal, zero resistance inductor only when the current in the inductor increases. This energy is not dissipated, it is stored in the inductor and released when the current decreases.

(iv) As we said earlier also, the energy in an inductor is actually stored in the magnetic field within the coil. We can develop relations of magnetic energy density u (energy stored per unit volume) analogous to those we obtained in electrostatics. We will concentrate on one simple case of an ideal long cylindrical solenoid. For a long solenoid its magnetic field can be assumed completely of within the solenoid. The energy U stored in the solenoid when a current i is

$$U = \frac{1}{2}Li^2 = \frac{1}{2}(\mu_0 n^2 V)\,i^2 \quad \text{as} \quad L = \mu_0 n^2 V$$

The energy per unit volume is $u = \frac{U}{V}$

$\therefore$ $$u = \frac{U}{V} = \frac{1}{2}\mu_0 n^2 i^2 = \frac{1}{2}\frac{(\mu_0 ni)^2}{\mu_0} = \frac{1}{2}\frac{B^2}{\mu_0}$$

as $$B = \mu_0 ni$$

Thus, $$\boxed{u = \frac{1}{2}\frac{B^2}{\mu_0}}$$

This expression is similar to $u = \frac{1}{2}\varepsilon_0 E^2$ *used in electrostatics. Although, we have derived it for one special situation, it turns out to be correct for any magnetic field configuration.*

Example 27.15 *(a) Calculate the inductance of an air core solenoid containing* 300 *turns if the length of the solenoid is* 25.0 *cm and its cross-sectional area is* 4.00 *cm*^{2}.

(b) Calculate the self-induced emf in the solenoid if the current through it is decreasing at the rate of 50.0 *A/s.*

Solution (a) The inductance of a solenoid is given by

$$L = \frac{\mu_0 N^2 S}{l}$$

Substituting the values, we have

$$L = \frac{(4\pi \times 10^{-7})(300)^2\ (4.00 \times 10^{-4})}{(25.0 \times 10^{-2})}\ \text{H}$$

$$= 1.81 \times 10^{-4}\ \text{H}$$ **Ans.**

(b) $e = -L\frac{di}{dt}$

Here, $$\frac{di}{dt} = -\ 50.0\ \text{A/s}$$

$\therefore$ $$e = -\ (1.81 \times 10^{-4})(-50.0)$$

$$= 9.05 \times 10^{-3}\ \text{V}$$

or $$e = 9.05\ \text{mV}$$ **Ans.**

Example 27.16 *What inductance would be needed to store* 1.0 *kWh of energy in a coil carrying a* 200 *A current.* (1 *kWh* = 3.6 × 10^6 *J*)

Solution We have, $i = 200$ A

and $$U = 1\,\text{kWh} = 3.6 \times 10^6\ \text{J}$$

$\therefore$ $$L = \frac{2U}{i^2} \qquad \left(U = \frac{1}{2}Li^2\right)$$

$$= \frac{2(3.6 \times 10^6)}{(200)^2} = 180\ \text{H}$$ **Ans.**

Example 27.17 *(a) What is the magnetic flux through one turn of a solenoid of self-inductance* $8.0 \times 10^{-5}\,H$ *when a current of* 3.0 *A flows through it? Assume that the solenoid has* 1000 *turns and is wound from wire of diameter* 1.0 *mm. (b) What is the cross-sectional area of the solenoid?*

Solution Given, $L = 8.0 \times 10^{-5}$ H, $i = 3.0\,\text{A}$ and $N = 1000$ turns

(a) From the relation, $L = \dfrac{N\phi}{i}$

The flux linked with one turn,

$$\phi = \frac{Li}{N} = \frac{(8.0 \times 10^{-5})(3.0)}{1000}$$

$$= 2.4 \times 10^{-7}\,\text{Wb}$$

(b) This $\phi = BS = (\mu_0 ni)(S)$

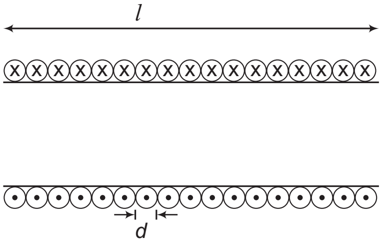


Fig. 27.41

Here, n = number of turns per unit length

$$= \frac{N}{l} = \frac{N}{Nd} = \frac{1}{d}$$

$\therefore$
$$\phi = \frac{\mu_0 iS}{d}$$

or
$$S = \frac{\phi d}{\mu_0 i} = \frac{(2.4 \times 10^{-7})\ (1.0 \times 10^{-3})}{(4\pi \times 10^{-7})\ (3.0)}$$

$$= 6.37 \times 10^{-5}\,\text{m}^2$$ **Ans.**

Example 27.18 *A* 10 *H inductor carries a current of* 20 *A. How much ice at* $0°C$ *could be melted by the energy stored in the magnetic field of the inductor? Latent heat of ice is* $22.6 \times 10^3\,J/kg$.

Solution Energy stored is $\dfrac{1}{2}Li^2$.

This energy is completely used in melting the ice.

Hence,
$$\frac{1}{2}Li^2 = mL_f$$

Here, L_f = latent heat of fusion

Hence, mass of ice melted, $m = \dfrac{Li^2}{2L_f}$

Substituting the values, we have

$$m = \frac{(10)(20)^2}{2(2.26 \times 10^3)}$$

$$= 0.88 \text{ kg}$$ **Ans.**

INTRODUCTORY EXERCISE 27.3

1. The current through an inductor of 1H is given by $i = 3t \sin t$. Find the voltage across the inductor.
2. In the figure shown $i = 10e^{-4t}$ A. Find V_L and V_{ab}.

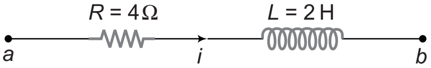


Fig. 27.42

3. The current (in Ampere) in an inductor is given by $I = 5 + 16t$, where t is in seconds. The self-induced emf in it is 10 mV. Find
   (a) the self-inductance, and
   (b) the energy stored in the inductor and the power supplied to it at $t = 1$s
4. (a) Calculate the self-inductance of a solenoid that is tightly wound with wire of diameter 0.10 cm, has a cross-sectional area 0.90 cm^2 and is 40 cm long.
   (b) If the current through the solenoid decreases uniformly from 10 A to 0 A in 0.10 s, what is the emf induced between the ends of the solenoid?

27.7 Mutual Inductance

We have already discussed in Chapter 26, the magnetic interaction between two wires carrying steady currents. The current in one wire causes a magnetic field, which exerts a force on the current in the second wire.

An additional interaction arises between two circuits when there is a changing current in one of the circuits.

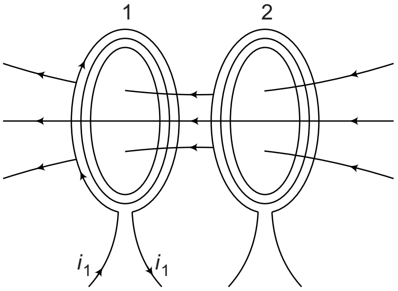


Fig. 27.43

Consider two neighbouring coils of wire as shown in Fig. 27.42. A current flowing in coil 1 produces magnetic field and hence, a magnetic flux through coil 2. If the current in coil 1 changes, the flux through coil 2 changes as well. According to Faraday's law this induces an emf in coil 2. In this way, a change in the current in one circuit can induce a current in a second circuit. This phenomenon is known as **mutual induction**. Like the self-inductance (L), two circuits have mutual inductance (M). It also have two definitions as under:

First Definition Suppose the circuit 1 has a current i_1 flowing in it. Then, total flux $N_2\phi_{B_2}$ linked with circuit 2 is proportional to the current in 1. Thus,

$$N_2\,\phi_{B_2} \propto i_1$$

or

$$N_2\phi_{B_2} = Mi_1$$

Here, the proportionality constant M is known as the mutual inductance M of the two circuits.

Thus,

$$\boxed{M = \frac{N_2\phi_{B_2}}{i_1}}$$

From this expression, M can be defined as the **total flux $N_2\phi_{B_2}$ linked with circuit 2 per unit current in circuit 1**.

Second Definition If we change the current in circuit 1 at a rate di_1/dt, an induced emf e_2 is developed in circuit 1, which is proportional to the rate di_1/dt. Thus,

$$e_2 \propto di_1/dt$$

or

$$e_2 = -\,Mdi_1/dt$$

Here, the proportionality constant is again M. Minus sign indicates that e_2 is in such a direction that it opposes any change in the current in circuit 1. From the above equation,

$$\boxed{M = \left|\frac{-e_2}{di_1/dt}\right|}$$

This equation states that, **the mutual inductance of two circuits is the magnitude of induced emf e_2 per unit rate of change of current di_1/dt**.

Note down the following points regarding the mutual inductance:

1. The SI unit of mutual inductance is **henry** (1H).
2. M depends upon closeness of the two circuits, their orientations and sizes and the number of turns etc.
3. **Reciprocity theorem :** $M_{21} = M_{12} = M$

$$e_2 = -\,M(di_1/dt)$$

and

$$e_1 = -\,M(di_2/dt)$$

$$M_{12} = \frac{N_2\phi_{B_2}}{i_1}$$

and

$$M_{21} = \frac{N_1\phi_{B_1}}{i_2}$$

4. A good approach for calculating the mutual inductance of two circuits consists of the following steps:
   (a) Assume any one of the circuits as primary (first) and the other as secondary (second).
   (b) Pass a current i_1 through the primary circuit.
   (c) Determine the magnetic field **B** produced by the current i_1.
   (d) Obtain the magnetic flux ϕ_{B_2}.
   (e) With this flux, the mutual inductance can be found from,

$$M = \frac{N_2\,\phi_{B_2}}{i_1}$$

Mutual Inductance of a Solenoid Surrounded by a Coil

Figure shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1, radius R_1 and number of turns N_1.

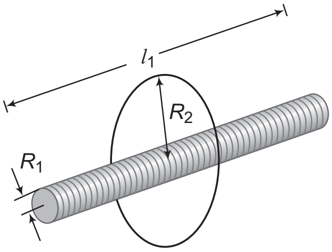


Fig. 27.44

To calculate M between them, let us assume a current i_1 in solenoid.

There is no magnetic field outside the solenoid and the field inside has magnitude,

$$B = \mu_0\left(\frac{N_1}{l_1}\right)i_1$$

and is directed parallel to the solenoid's axis. The magnetic flux ϕ_{B_2} through the surrounding coil is, therefore,

$$\phi_{B_2} = B(\pi R_1^2) = \frac{\mu_0 N_1 i_1}{l_1}\pi R_1^2$$

Now, $$M = \frac{N_2\phi_{B_2}}{i_1} = \left(\frac{N_2}{i_1}\right)\left(\frac{\mu_0 N_1 i_1}{l_1}\right)\pi R_1^2 = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}$$

∴ $$\boxed{M = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}}$$

Note that M is independent of the radius R_2 of the surrounding coil. This is because solenoid's magnetic field is confined to its interior. In principle, we can also calculate M by finding the magnetic flux through the solenoid produced by the current in the surrounding coil. This approach is much more difficult, because ϕ_{B_1} is so complicated. However, since $M_{12} = M_{21}$, we do know the result of this calculation.

Combination of Inductances

In series If several inductances are in series so that there are no interactions through mutual inductance.

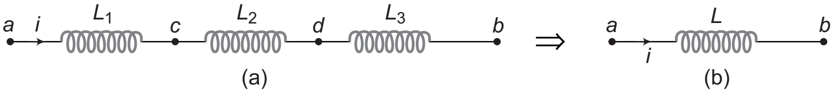


Fig. 27.45

Refer figure (a)

$$V_a - V_c = L_1 \frac{di}{dt}$$

$$V_c - V_d = L_2 \frac{di}{dt}$$

and $$V_d - V_b = L_3 \frac{di}{dt}$$

Adding all these equations, we have

$$V_a - V_b = (L_1 + L_2 + L_3) \frac{di}{dt} \qquad \text{...(i)}$$

Refer figure (b)

$$V_a - V_b = L \frac{di}{dt} \qquad \text{...(ii)}$$

Here, $L =$ equivalent inductance.

From Eqs. (i) and (ii), we have

$$\boxed{L = L_1 + L_2 + L_3}$$

In parallel

Refer figure (a)

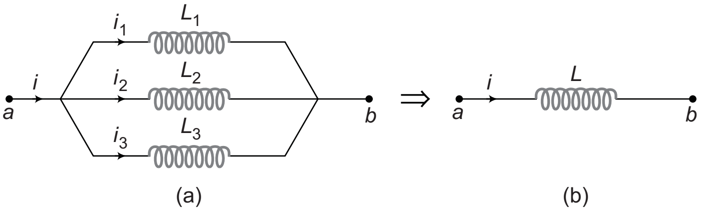


Fig. 27.46

$$i = i_1 + i_2 + i_3$$

or $$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} + \frac{di_3}{dt}$$

or $$\frac{di}{dt} = \frac{V_a - V_b}{L_1} + \frac{V_a - V_b}{L_2} + \frac{V_a - V_b}{L_3} \qquad \text{...(i)}$$

Refer figure (b)

$$\frac{di}{dt}=\frac{V_a-V_b}{L} \qquad \dots\text{(ii)}$$

From Eqs. (i) and (ii),

$$\frac{1}{L}=\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}$$

Example 27.19 *A straight solenoid has 50 turns per cm in primary and total 200 turns in the secondary. The area of cross-section of the solenoids is 4 cm^2. Calculate the mutual inductance. Primary is tightly kept inside the secondary.*

Solution The magnetic field at any point inside the straight solenoid of primary with n_1 turns per unit length carrying a current i_1 is given by the relation,

$$B=\mu_0 n_1 i_1$$

The magnetic flux through the secondary of N_2 turns each of area S is given as

$$N_2\phi_2=N_2(BS)=\mu_0 n_1 N_2 i_1 S$$

$$\therefore \qquad M=\frac{N_2\phi_2}{i_1}=\mu_0 n_1 N_2 S$$

Substituting the values, we get

$$M=(4\pi\times10^{-7})\left(\frac{50}{10^{-2}}\right)(200)(4\times10^{-4})$$

$$=5.0\times10^{-4}\text{ H}$$

Ans.

Example 27.20 *Two solenoids A and B spaced close to each other and sharing the same cylindrical axis have 400 and 700 turns, respectively. A current of 3.50 A in coil A produced an average flux of 300 μT-m^2 through each turn of A and a flux of 90.0 μT-m^2 through each turn of B.*

(a) Calculate the mutual inductance of the two solenoids.

(b) What is the self-inductance of A?

(c) What emf is induced in B when the current in A increases at the rate of 0.5 A/s?

Solution (a)

$$M=\frac{N_B\phi_B}{i_A}$$

$$=\frac{(700)(90\times10^{-6})}{3.5}$$

$$=1.8\times10^{-2}\text{ H}$$

Ans.

(b) $L_A=\dfrac{N_A\phi_A}{i_A}$

$$=\frac{(400)(300\times10^{-6})}{3.5}$$

$$=3.43\times10^{-2}\text{ H}$$

Ans.

(c)
$$e_B = M\left(\frac{di_A}{dt}\right)$$
$$= (1.8 \times 10^{-2})(0.5)$$
$$= 9.0 \times 10^{-3}\text{ V}$$
Ans.

INTRODUCTORY EXERCISE 27.4

1. Calculate the mutual inductance between two coils when a current of 4 A changes to 12 A in 0.5 s in primary and induces an emf of 50 mV in the secondary. Also, calculate the induced emf in the secondary if current in the primary changes from 3 A to 9 A is 0.02 s.
2. A coil has 600 turns which produces 5×10^{-3} Wb / turn of flux when 3 A current flows in the wire. This produced 6×10^{-3} Wb/turn in 1000 turns secondary coil. When the switch is opened, the current drops to zero in 0.2 s in primary. Find
   (a) mutual inductance,
   (b) the induced emf in the secondary,
   (c) the self-inductance of the primary coil.
3. Two coils have mutual inductance $M = 3.25 \times 10^{-4}$ H. The current i_1 in the first coil increases at a uniform rate of 830 A/s.
   (a) What is the magnitude of the induced emf in the second coil? Is it constant?
   (b) Suppose that the current described is in the second coil rather than the first. What is the induced emf in the first coil?

27.8 Growth and Decay of Current in an *L-R* Circuit

Growth of Current

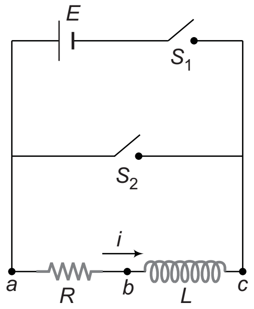


Fig. 27.47

Let us consider a circuit consisting of a battery of emf E, a coil of self-inductance L and a resistor R. The resistor R may be a separate circuit element, or it may be the resistance of the inductor windings. By closing switch S_1, we connect R and L in series with constant emf E. Let i be the current at some time t after switch S_1 is closed and di/dt be its rate of increase at that time. Applying Kirchhoff's loop rule starting at the negative terminal and proceeding counterclockwise around the loop

$$E - V_{ab} - V_{bc} = 0 \quad \text{or} \quad E - iR - L\frac{di}{dt} = 0$$

$$\therefore \qquad \int_0^i \frac{di}{E - iR} = \int_0^t \frac{dt}{L} \quad \text{or} \quad i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

By letting $E/R = i_0$ and $L/R = \tau_L$, the above expression reduces to

$$\boxed{i = i_0\,(1 - e^{-t/\tau_L})}$$

Here, $i_0 = E/R$ is the current at $t = \infty$. It is also called the steady state current or the maximum current in the circuit.

And $\tau_L = \dfrac{L}{R}$ is called **time constant** of the L-R circuit. At a time equal to one time constant the current has risen to $(1 - 1/e)$ or about 63% of its final value i_0.

The i-t graph is as shown in figure.

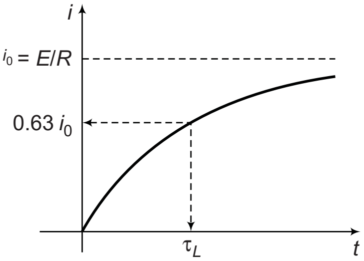


Fig. 27.48

Note that the final current i_0 does not depend on the inductance L, it is the same as it would be if the resistance R alone were connected to the source with emf E.

Let us have an insight into the behaviour of an L - R circuit from energy considerations.

The instantaneous rate at which the source delivers energy to the circuit ($P = Ei$) is equal to the instantaneous rate at which energy is dissipated in the resistor $(= i^2R)$ plus the rate at which energy is stored in the inductor $\left(= iV_{bc} = Li\dfrac{di}{dt}\right)$ or $\dfrac{d}{dt}\left(\dfrac{1}{2}Li^2\right) = Li\dfrac{di}{dt}$.

Thus,
$$\boxed{Ei = i^2R + Li\frac{di}{dt}}$$

Decay of Current

Now suppose switch S_1 in the circuit shown in figure has been closed for a long time and that the current has reached its steady state value i_0. Resetting our stopwatch to redefine the initial time we close switch S_2 at time $t = 0$ and at the same time we should open the switch S_1 to by pass the battery. The current through L and R does not instantaneously go to zero but decays exponentially. To apply Kirchhoff's loop rule and to find current in the circuit at time t, let us draw the circuit once more.

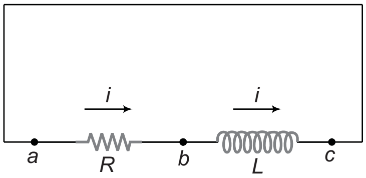


Fig. 27.49

Applying loop rule we have,

$$(V_a - V_b) + (V_b - V_c) = 0 \qquad (\text{as } V_a = V_c)$$

or
$$iR + L\left(\frac{di}{dt}\right) = 0$$

Note *Don't bother about the sign of* $\dfrac{di}{dt}$.

$$\therefore \quad \frac{di}{i} = -\frac{R}{L}dt$$

$$\therefore \quad \int_{i_0}^{i} \frac{di}{i} = -\frac{R}{L}\int_0^t dt$$

$$\therefore \quad \boxed{i = i_0 e^{-t/\tau_L}}$$

where, $\tau_L = \dfrac{L}{R}$, *is the time for current to decrease to* $1/e$ *or about 37% of its original value. The i-t graph is as shown in Fig. 27.49.`*

The energy that is needed to maintain the current during this decay is provided by energy stored in the magnetic field. Thus, the rate at which energy is dissipated in the resistor = rate at which the stored energy decreases in magnetic field of inductor

or
$$i^2R = -\frac{dU}{dt} = -\frac{d}{dt}\left(\frac{1}{2}Li^2\right) = Li\left(-\frac{di}{dt}\right)$$

or
$$i^2R = Li\left(\frac{-di}{dt}\right)$$

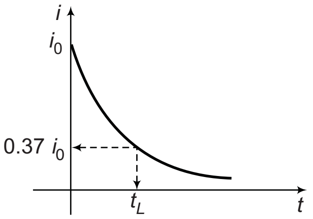


Fig. 27.50

Example 27.21 *A coil of resistance* 20 Ω *and inductance* 0.5 *H is switched to DC* 200 *V supply. Calculate the rate of increase of current*

(a) at the instant of closing the switch and

(b) after one time constant.

(c) Find the steady state current in the circuit.

Solution (a) This is the case of growth of current in an *L*-*R* circuit. Hence, current at time *t* is given by

$$i = i_0\,(1 - e^{-t/\tau_L})$$

Rate of increase of current,

$$\frac{di}{dt} = \frac{i_0}{\tau_L}e^{-t/\tau_L}$$

At $t = 0$,
$$\frac{di}{dt} = \frac{i_0}{\tau_L} = \frac{E/R}{L/R} = \frac{E}{L}$$

Substituting the value, we have

$$\frac{di}{dt} = \frac{200}{0.5} = 400\ \text{A/s}$$ **Ans.**

(b) At $t = \tau_L$,

$$\frac{di}{dt} = (400)e^{-1} = (0.37)(400)$$

$$= 148\ \text{A/s}$$ **Ans.**

(c) The steady state current in the circuit,

$$i_0 = \frac{E}{R} = \frac{200}{20} = 10\ \text{A}$$ **Ans.**

Example 27.22 *A 5 H inductor is placed in series with a 10 Ω resistor. An emf of 5 V being suddenly applied to the combination. Using these values prove the principle of conservation of energy, for time equal to the time constant.*

Solution At any instant t, current in L-R circuit is given as

$$i = i_0\,(1 - e^{-t/\tau_L})$$

Here,
$$i_0 = \frac{E}{R} \text{ and } \tau_L = \frac{L}{R}$$

After one time constant ($t = \tau_L$), current in the circuit is

$$i = \frac{E}{R}\left(1 - \frac{1}{e}\right) = \frac{5}{10}\left(1 - \frac{1}{e}\right) = 0.316\ \text{A}$$

The rate at which the energy is delivered by the battery is

$$P_1 = Ei = (5)\,(0.316) = 1.58\ \text{W} \qquad \ldots\text{(i)}$$

At this time rate by which energy is dissipated in the resistor is

$$P_2 = i^2 R = (0.316)^2\ (10) = 0.998\ \text{W} \qquad \ldots\text{(ii)}$$

The rate at which energy is stored in the inductor is

$$P_3 = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) = Li\left(\frac{di}{dt}\right)$$

Here,
$$\frac{di}{dt} = \frac{i_0}{\tau_L}e^{-1} = \frac{E}{eL} \qquad \text{(after one time constant)}$$

Substituting the values, we get

$$P_3 = (L)\,(i)\left(\frac{E}{eL}\right) = \frac{Ei}{e}$$

$$= \frac{5 \times 0.316}{2.718} = 0.582\ \text{W} \qquad \ldots\text{(iii)}$$

From Eqs. (i), (ii) and (iii), we have

$$P_1 = P_2 + P_3$$

It is the same as required by the principle of conservation of energy.

INTRODUCTORY EXERCISE 27.5

1. Show that $\frac{L}{R}$ has units of time.
2. A coil of inductance 2 H and resistance 10 Ω are in a series circuit with an open key and a cell of constant 100 V with negligible resistance. At time $t = 0$, the key is closed. Find
   (a) the time constant of the circuit.
   (b) the maximum steady current in the circuit.
   (c) the current in the circuit at $t = 1$s.
3. In the simple L-R circuit, can the emf induced across the inductor ever be greater than the emf of the battery used to produce the current?

27.9 Oscillations in *L-C* Circuit

If a charged capacitor C is short-circuited through an inductor L, the charge and current in the circuit start oscillating simple harmonically. If the resistance of the circuit is zero, no energy is dissipated as heat. We also assume an idealized situation in which energy is not radiated away from the circuit. With these idealizations-zero resistance and no radiation, the oscillations in the circuit persist indefinitely and the energy is transferred from the capacitor's electric field to the inductor's magnetic field and back. The total energy associated with the circuit is constant. This is analogous to the transfer of energy in an oscillating mechanical system from potential energy to kinetic energy and back, with constant total energy. Later, we will see that this analogy goes much further.

Let us now derive an equation for the oscillations in an L-C circuit.

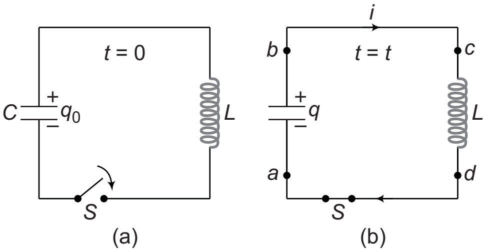


Fig. 27.51

Refer figure (a) A capacitor is charged to a PD, $V_0 = q_0 C$

Here, q_0 is the maximum charge on the capacitor. At time $t = 0$, it is connected to an inductor through a switch S. At time $t = 0$, switch S is closed.

Refer figure (b) When the switch is closed, the capacitor starts discharging. Let at time t charge on the capacitor is $q\ (< q_0)$ and since, it is further decreasing there is a current i in the circuit in the direction shown in figure. Later we will see that, as the charge is oscillating there may be a situation when q will be increasing, but in that case direction of current is also reversed and the equation remains unchanged.

The potential difference across capacitor = potential difference across inductor, or

$$V_b - V_a = V_c - V_d$$

$$\therefore \qquad \frac{q}{C} = L\left(\frac{di}{dt}\right) \qquad \text{...(i)}$$

Now, as the charge is decreasing,
$$i = \left(\frac{-dq}{dt}\right)$$

or
$$\frac{di}{dt} = -\frac{d^2q}{dt^2}$$

Substituting in Eq. (i), we get
$$\frac{q}{C} = -L\left(\frac{d^2q}{dt^2}\right)$$

or
$$\boxed{\frac{d^2q}{dt^2} = -\left(\frac{1}{LC}\right)q} \qquad \text{...(ii)}$$

This is the standard equation of simple harmonic motion $\left(\frac{d^2x}{dt^2} = -\omega^2 x\right)$.

Here,
$$\boxed{\omega = \frac{1}{\sqrt{LC}}} \qquad \text{...(iii)}$$

The general solution of Eq. (ii), is $\quad q = q_0 \cos(\omega t \pm \phi)$

For example in our case $\phi = 0$ as $q = q_0$ at $t = 0$.

Hence,
$$q = q_0 \cos \omega t \qquad \text{...(iv)}$$

Thus, we can say that charge in the circuit oscillates simple harmonically with angular frequency given by Eq. (iii). Thus,

$$\boxed{\omega = \frac{1}{\sqrt{LC}}, \; f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \quad \text{and} \quad T = \frac{1}{f} = 2\pi\sqrt{LC}}$$

The oscillations of the *L-C* circuit are an electromagnetic analog to the mechanical oscillations of a block-spring system.

Table below shows a comparison of oscillations of a mass-spring system and an *L-C* circuit.

Table 27.1

S.No.	Mass spring system	Inductor-capacitor circuit
1.	Displacement (x)	Charge (q)
2.	Velocity (v)	Current (i)
3.	Acceleration (a)	Rate of change of current $\left(\frac{di}{dt}\right)$
4.	$\frac{d^2x}{dt^2} = -\omega^2 x$, where $\omega = \sqrt{\frac{k}{m}}$	$\frac{d^2q}{dt^2} = -\omega^2 q$, where $\omega = \frac{1}{\sqrt{LC}}$
5.	$x = A\sin(\omega t \pm \phi)$ or $x = A\cos(\omega t \pm \phi)$	$q = q_0 \sin(\omega t \pm \phi)$ or $q = q_0 \cos(\omega t \pm \phi)$
6.	$v = \frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$	$i = \frac{dq}{dt} = \omega\sqrt{q_0^2 - q^2}$
7.	$a = \frac{dv}{dt} = -\omega^2 x$	Rate of change of current $= \frac{di}{dt} = -\omega^2 q$
8.	Kinetic energy $= \frac{1}{2}mv^2$	Magnetic energy $= \frac{1}{2}Li^2$
9.	Potential energy $= \frac{1}{2}kx^2$	Potential energy $= \frac{1}{2}\frac{q^2}{C}$
10.	$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant} = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2$	$\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = \text{constant} = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}Li_{\text{max}}^2$
11.	$\lvert v_{\text{max}} \rvert = A\omega$	$i_{\text{max}} = q_0\omega$
12	$\lvert a_{\text{max}} \rvert = \omega^2 A$	$\left\lvert \left(\frac{di}{dt}\right)_{\text{max}} \right\rvert = \omega^2 q_0$
13.	$\frac{1}{k}$	C
14.	m	L

A graphical description of the energy transfer between the inductor and the capacitor in an *L-C* circuit is shown in the figure. The right side of the figure shows the analogous energy transfer in the oscillating block-spring system.

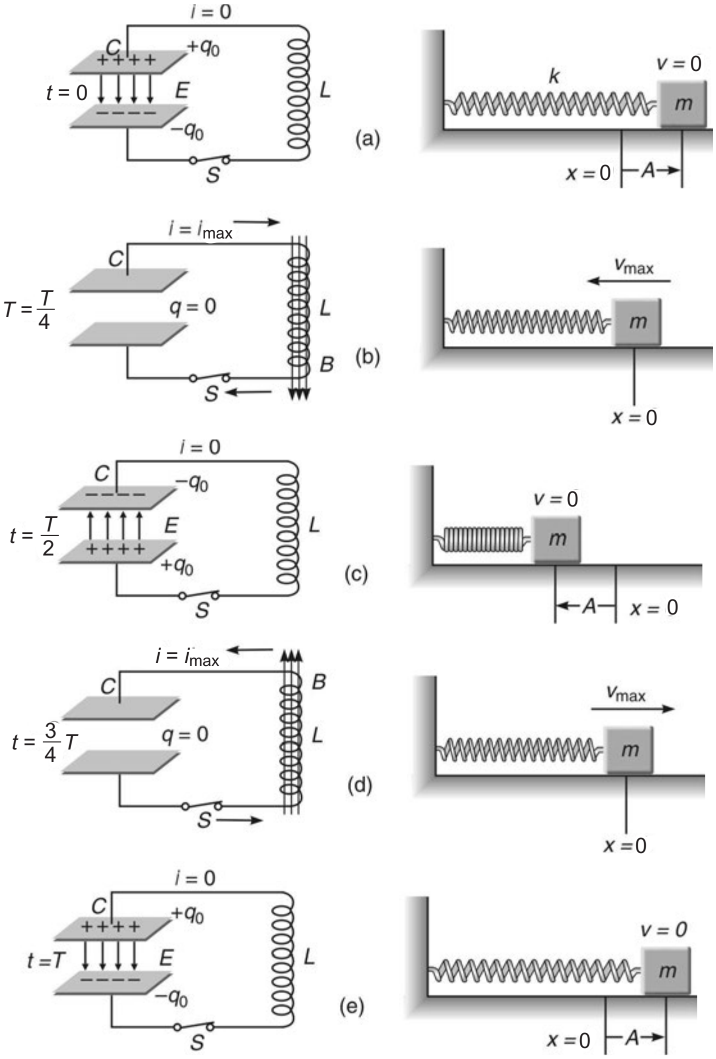


Fig. 27.52

Note *In L-C oscillations, q, i and* $\frac{di}{dt}$ *all oscillate simple harmonically with same angular frequency* ω. *But the phase difference between q and i or between i and* $\frac{di}{dt}$ *is* $\frac{\pi}{2}$, *while that between i and* $\frac{di}{dt}$ *is* π. *Their amplitudes are* q_0, $q_0\omega$ *and* $\omega^2 q_0$ *respectively. So, now suppose*

$$q = q_0 \cos \omega t, \text{ then}$$

$$i = \frac{dq}{dt} = -q_0\omega \sin \omega t \text{ and}$$

$$\frac{di}{dt} = -q_0\omega^2 \cos \omega t$$

Similarly, potential energy across capacitor (U_C) and across inductor (U_L) also oscillate with double the frequency 2ω but not simple harmonically. The different graphs are as shown in Fig. 27.52.

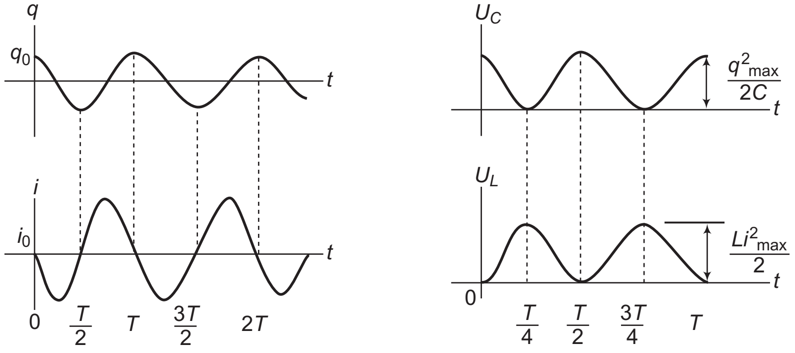


Fig. 27.53

Example 27.23 *A capacitor of capacitance 25 μF is charged to 300 V. It is then connected across a 10 mH inductor. The resistance in the circuit is negligible.*

(a) Find the frequency of oscillation of the circuit.

(b) Find the potential difference across capacitor and magnitude of circuit current 1.2 ms after the inductor and capacitor are connected.

(c) Find the magnetic energy and electric energy at $t = 0$ and $t = 1.2$ ms.

Solution (a) The frequency of oscillation of the circuit is

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Substituting the given values, we have

$$f = \frac{1}{2\pi\sqrt{(10\times 10^{-3})(25\times 10^{-6})}} = 318.3 \text{ Hz}$$ **Ans.**

(b) Charge across the capacitor at time t will be

$$q = q_0 \cos \omega t \quad \text{and} \quad i = -q_0\omega \sin \omega t$$

Here, $q_0 = CV_0 = (25\times 10^{-6})(300) = 7.5\times 10^{-3}$ C

Now, charge in the capacitor after $t = 1.2\times 10^{-3}$ s is

$$q = (7.5\times 10^{-3})\cos (2\pi \times 318.3)(1.2\times 10^{-3})\text{ C}$$

$$= -5.53\times 10^{-3} \text{ C}$$

$\therefore$ PD across capacitor, $V = \frac{|q|}{C} = \frac{5.53\times 10^{-3}}{25\times 10^{-6}} = 221.2$ volt **Ans.**

The magnitude of current in the circuit at $t = 1.2\times 10^{-3}$ s is

$$|i| = q_0\omega \sin \omega t$$

$$= (7.5\times 10^{-3})(2\pi)(318.3)\sin (2\pi \times 318.3)(1.2\times 10^{-3})\text{ A}$$

$$= 10.13 \text{ A}$$ **Ans.**

(c) **At $t = 0$** Current in the circuit is zero.

Hence, $U_L = 0$

Charge in the capacitor is maximum.

Hence, $$U_C = \frac{1}{2}\frac{q_0^2}{C}$$

or $$U_C = \frac{1}{2} \times \frac{(7.5 \times 10^{-3})^2}{(25 \times 10^{-6})}$$

$$= 1.125 \text{ J}$$ **Ans.**

$\therefore$ Total energy, $E = U_L + U_C = 1.125$ J

At $t = 1.2$ ms

$$U_L = \frac{1}{2} Li^2$$

$$= \frac{1}{2}(10 \times 10^{-3})(10.13)^2$$

$$= 0.513 \text{ J}$$

$\therefore$ $$U_C = E - U_L = 1.125 - 0.513$$

$$= 0.612 \text{ J}$$ **Ans.**

Otherwise U_C can be calculated as

$$U_C = \frac{1}{2}\frac{q^2}{C}$$

$$= \frac{1}{2} \times \frac{(5.53 \times 10^{-3})^2}{(25 \times 10^{-6})}$$

$$= 0.612 \text{ J}$$

INTRODUCTORY EXERCISE 27.6

1. Show that $\sqrt{LC}$ has units of time.
2. While comparing the *L*-*C* oscillations with the oscillations of spring-block system, with whom the magnetic energy can be compared and why?
3. In an *L*-*C* circuit, $L = 0.75$ H and $C = 18\ \mu$F,
   (a) At the instant when the current in the inductor is changing at a rate of 3.40 A/s, what is the charge on the capacitor?
   (b) When the charge on the capacitor is 4.2×10^{-4} C, what is the induced emf in the inductor?
4. An *L*-*C* circuit consists of a 20.0 mH inductor and a $0.5\ \mu$F capacitor. If the maximum instantaneous current is 0.1 A, what is the greatest potential difference across the capacitor?

27.10 Induced Electric Field

When a conductor moves in a magnetic field, we can understand the induced emf on the basis of magnetic forces on charges in the conductor as described in Art. 27.5. But an induced emf also occurs when there is a changing flux through a stationary conductor. What is it that pushes the charges around the circuit in this type of situation?

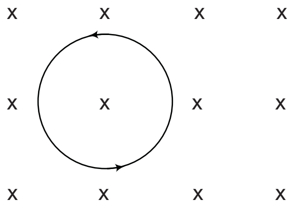


Fig. 27.54

As an example, let's consider the situation shown in figure. A conducting circular loop is placed in a magnetic field which is directed perpendicular to the paper inwards. When the magnetic field changes with time (suppose it increases with time) the magnetic flux ϕ_B also changes and according to Faraday's law the induced emf $e = -\dfrac{d\phi_B}{dt}$ is produced in the loop. If the total resistance of the loop is R, the induced current in the loop is given by

$$i = \frac{e}{R}$$

But what force makes the charges move around the loop? It can't be the magnetic force, because the charges are not moving in the magnetic field.

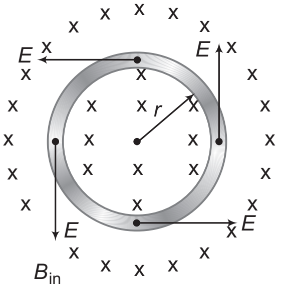


Fig. 27.55

Actually, there is an **induced electric field** in the conductor **caused by the changing magnetic flux.** *This electric field has the following important properties:*

1. It is non-conservative in nature. The line integral of **E** around a closed path is not zero. This line integral is given by

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi_B}{dt}} \qquad \text{...(i)}$$

   Note that this equation is valid only if the path around which we integrate is stationary.

2. Because of symmetry, the electric field **E** has the same magnitude at every point on the circle and is tangent to it at each point. The directions of **E** at several points on the loop are shown in figure.
3. Being a non-conservative field, the concept of potential has no meaning for such a field.
4. This field is different from the electrostatic field produced by stationary charges (which is conservative in nature).
5. The relation $\mathbf{F} = q\mathbf{E}$ is still valid for this field.
6. This field can vary with time.

So, **a changing magnetic field acts as a source of electric field of a sort that we cannot produce with any static charge distribution.** This may seen strange but its the way nature behaves.

Note *1. For symmetrical situations (as shown in figure) Eq. (i), in simplified form can be written as*

$$\boxed{El = \left|\frac{d\phi_B}{dt}\right| = S\left|\frac{dB}{dt}\right|}$$

Here, l is the length of closed loop in which electric field is to be calculated and S is the area in which magnetic field is changing.

2. Direction of electric field is the same as the direction of induced current.

Example 27.24 *The magnetic field at all points within the cylindrical region whose cross-section is indicated in the accompanying figure start increasing at a constant rate $\alpha\ T/s$. Find the magnitude of electric field as a function of r, the distance from the geometric centre of the region.*

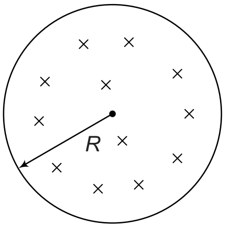


Fig. 27.56

Solution **For $r \le R$**

Using $El = S\left|\frac{dB}{dt}\right|$ or $E\,(2\pi r) = (\pi r^2)\,\alpha$

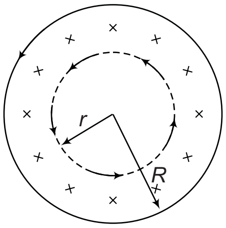


Fig. 27.57

$$\therefore \qquad E = \frac{r\alpha}{2}$$

$\therefore$ $E \propto r$, i.e. E-r graph is a straight line passing through origin.

At $r = R$, $E = \dfrac{R\alpha}{2}$

For $r \geq R$

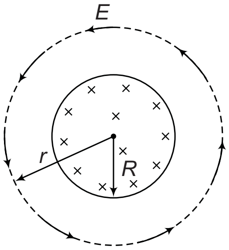


Fig. 27.58

Using $El = S\left|\dfrac{dB}{dt}\right|$,

$$\therefore \qquad E(2\pi r) = (\pi R^2)(\alpha)$$

$$\therefore \qquad E = \frac{\alpha R^2}{2r}$$

$\therefore$ $E \propto \dfrac{1}{r}$, i.e. E-r graph is a rectangular hyperbola.

The E-r graph is as shown in figure.

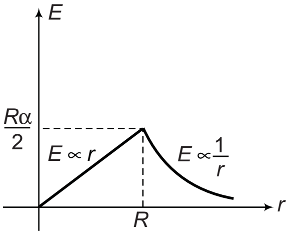


Fig. 27.59

The direction of electric field is shown in above figure.

Example 27.25 *A long thin solenoid has* 900 *turns/metre and radius* 2.50 *cm. The current in the solenoid is increasing at a uniform rate of* 60 *A/s. What is the magnitude of the induced electric field at a point?*

(a) 0.5 *cm from the axis of the solenoid.*

(b) 1.0 *cm from the axis of the solenoid.*

Solution $\because$ $B = \mu_0 ni$

$$\therefore \qquad \frac{dB}{dt} = \mu_0 n\frac{di}{dt}$$

$$= (4\pi \times 10^{-7})(900)(60)$$

$$= 0.068 \text{ T/s}$$

Using the result of electric field derived in above problem (as both points lie inside the solenoid).

$$E = \frac{r}{2}\left(\frac{dB}{dt}\right)$$

(a) $E = \left(\frac{0.5 \times 10^{-2}}{2}\right)(0.068) = 1.7 \times 10^{-4}$ V/m

(b) $E = \frac{(1.0 \times 10^{-2})}{2}(0.068) = 3.4 \times 10^{-4}$ V/m

INTRODUCTORY EXERCISE 27.7

1. A long solenoid of cross-sectional area 5.0 cm^2 is wound with 25 turns of wire per centimetre. It is placed in the middle of a closely wrapped coil of 10 turns and radius 25 cm as shown.

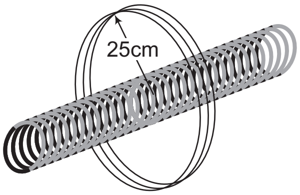


Fig. 27.60

(a) What is the emf induced in the coil when the current through the solenoid is decreasing at a rate −0.20 A/s?
(b) What is the electric field induced in the coil?

2. For the situation described in figure, the magnetic field changes with time according to

$$B = (2.00t^3 - 4.00t^2 + 0.8)\text{ T} \quad \text{and} \quad r_2 = 2R = 5.0 \text{ cm}$$

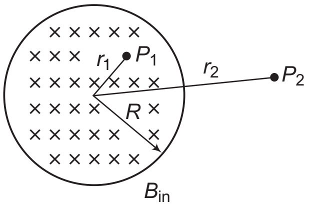


Fig. 27.61

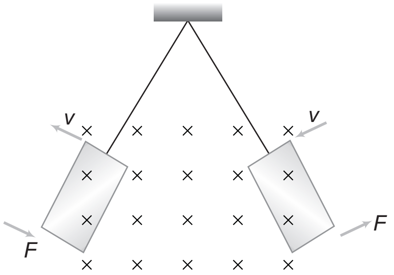
(a) Calculate the force on an electron located at P_2 at $t = 2.00$ s
(b) What are the magnitude and direction of the electric field at P_1 when $t = 3.00$ s and $r_1 = 0.02$ m.

Hint : For the direction, see whether the field is increasing or decreasing at given times.

Final Touch Points

1. **Eddy currents** When a changing magnetic flux is applied to a piece of conducting material, circulating currents called eddy currents are induced in the material. These eddy currents often have large magnitudes and heat up the conductor.

   When a metal plate is allowed to swing through a strong magnetic field, then in entering or leaving the field the eddy currents are set up in the plate which opposes the motion as shown in figure. The kinetic energy dissipates in the form of heat. The slowing down of the plate is called the **electromagnetic damping.**

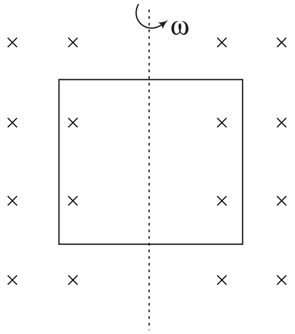


   The electromagnetic damping is used to damp the oscillations of a galvanometer coil or chemical balance and in braking electric trains. Otherwise, the eddy currents are often undesirable. To reduce the eddy currents some slots are cut into moving metallic parts of machinery. These slots intercept the conducting paths and decreases the magnitudes of the induced currents.

2. **Back EMF of Motors** An electric motor converts electrical energy into mechanical energy and is based on the fact that a current carrying coil in a uniform magnetic field experiences a torque. As the coil rotates in the magnetic field, the flux linked with the rotating coil will change and hence, an emf called back emf is produced in the coil.

   When the motor is first turned on, the coil is at rest and so there is no back emf. The 'start up' current can be quite large. To reduce 'start up' current a resistance called **'starter'** is put in series with the motor for a short period when the motor is started. As the rotation rate increases the back emf increases and hence, the current reduces.

3. **Electric Generator or Dynamo** A dynamo converts mechanical energy (rotational kinetic energy) into electrical energy. It consists of a coil rotating in a magnetic field. Due to rotation of the coil magnetic flux linked with it changes, so an emf is induced in the coil.



   Suppose at time $t = 0$, plane of coil is perpendicular to the magnetic field.

The flux linked with it at any time t will be given by

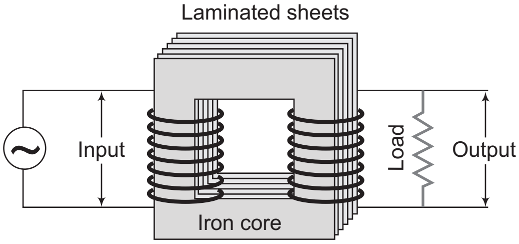
$$\phi = NBA \cos \omega t \qquad (N = \text{number of turns in the coil})$$

$$\therefore \quad e = -\frac{d\phi}{dt} = NBA\omega \sin \omega t$$

or $$e = e_0 \sin \omega t$$

where, $$e_0 = NBA\omega$$

4. **Transformer** It is a device which is either used to increase or decrease the voltage in AC circuits through mutual induction. A transformer consists of two coils wound on the same core.



The coil connected to input is called primary while the other connected to output is called secondary coil. An alternating current passing through the primary creates a continuously changing flux through the core. This changing flux induces an alternating emf in the secondary.

As magnetic lines of force are closed curves, the flux per turn of primary must be equal to flux per turn of the secondary. Therefore,

$$\frac{\phi_P}{N_P} = \frac{\phi_S}{N_S}$$

or $$\frac{1}{N_P}\cdot\frac{d\phi_P}{dt} = \frac{1}{N_S}\cdot\frac{d\phi_S}{dt} \qquad \left(\text{as } e \propto \frac{d\phi}{dt}\right)$$

$$\therefore \quad \frac{e_S}{e_P} = \frac{N_S}{N_P}$$

In an ideal transformer, there is no loss of power. Hence,

$$ei = \text{constant}$$

$$\therefore \quad \frac{e_S}{e_P} = \frac{N_S}{N_P} = \frac{i_P}{i_S}$$

Regarding a transformer, the following are few important points.

(i) In **step-up** transformer, $N_S > N_P$. It increases voltage and reduces current

(ii) In **step-down** transformer, $N_P > N_S$. It increases current and reduces voltage

(iii) It works only on AC

(iv) A transformer cannot increase (or decrease) voltage and current simultaneously. As, $ei = \text{constant}$

(v) Some power is always lost due to eddy currents, hysteresis, etc.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on Faraday's and Lenz's law*

Concept

Problems of induced emf or induced current can be solved by the following two methods.

Method 1 Magnitudes are given by

$$|e| = \left| N \frac{d\phi_B}{dt} \right| \quad \text{and} \quad |i| = \frac{|e|}{R}$$

Direction is given by Lenz's law.

Method 2 Magnitudes are given by

$$|e| = |Bvl| \quad \text{or} \quad |e| = \left| \frac{B\omega l^2}{2} \right| \quad \text{and} \quad |i| = \frac{|e|}{R}$$

Direction is given by right hand rule.

Note *In the first method, we have to first find the magnetic flux passing through the loop and then differentiate it with respect to time. Second method is simple but it can be applied if and only if some conductor is either in translational or rotational motion.*

Example 1 *Current in a long current carrying wire is*

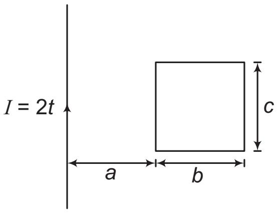
$$I = 2t$$

A conducting loop is placed to the right of this wire. Find

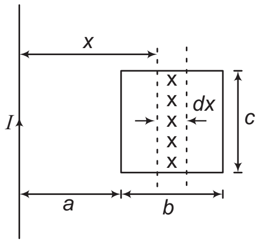
(a) magnetic flux ϕ_B passing through the loop.

(b) induced emf $|e|$ produced in the loop.

(c) if total resistance of the loop is R, then find induced current I_{in} in the loop.



Solution Here, no conductor is in motion. So, we can apply only method-1. Further, magnetic field of straight wire is non-uniform. Therefore, magnetic flux can be obtained by integration.



(a) At a distance x from the straight wire, magnetic field is

$$B = \frac{\mu_0}{2\pi}\frac{I}{x} \qquad [\text{in } \otimes \text{ direction}]$$

Let us take a small strip of width dx.

$\therefore$ Area of this strip is

$$dS = c\,(dx)$$

Now, $d\mathbf{S}$ can also be assumed inwards. Or, angle between $\mathbf{B}$ and $d\mathbf{S}$ may be assumed to be 0°. Therefore, small magnetic flux passing through the loop is

$$d\phi_B = BdS \cos 0^\circ$$

$$= \frac{\mu_0}{2\pi}\frac{I}{x}\,cdx$$

Total magnetic flux is

$$\phi_B = \int_{x=a}^{x=a+b} d\phi_B$$

$$= \int_a^{a+b}\left(\frac{\mu_0 Ic}{2\pi}\right)\frac{dx}{x}$$

$$= \frac{\mu_0 Ic}{2\pi}\ln\left(\frac{a+b}{a}\right)$$

Substituting the values of I, we get

$$\phi_B = \frac{\mu_0 ct}{\pi}\ln\left(\frac{a+b}{a}\right) \qquad \textbf{Ans.}$$

(b) $$|e| = \left|\frac{d\phi_B}{dt}\right| = \frac{d}{dt}\left[\frac{\mu_0 ct}{\pi}\ln\left(\frac{a+b}{a}\right)\right]$$

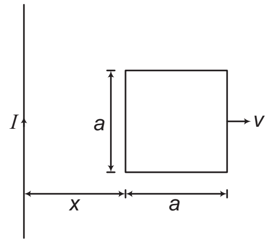
$$= \frac{\mu_0 c}{\pi}\ln\left(\frac{a+b}{a}\right) \qquad \textbf{Ans.}$$

(c) Induced current,

$$I_{\text{in}} = \frac{|e|}{R} = \frac{\mu_0 c}{\pi R}\ln\left(\frac{a+b}{a}\right)$$

Note *The main current $I\ (= 2t)$ is increasing with time. Hence, $\otimes$ magnetic field passing through the loop will also increase. So, induced current I_{in} will produce $\odot$ magnetic field. Or, induced current is anti-clockwise.*

Example 2 *A constant current I flows through a long straight wire as shown in figure. A square loop starts moving towards right with a constant speed v.*



(a) Find induced emf produced in the loop as a function of x.

(b) If total resistance of the loop is R, then find induced current in the loop.

Note *In this problem, loop is in motion therefore both methods can be applied.*

Solution **Method 1**

(a) Using the result of magnetic flux obtained in Example-1, we have

$$\phi_B = \frac{\mu_0 Ic}{2\pi} \ln\left(\frac{a+b}{a}\right)$$

Here, $a = x, b = c = a$

Substituting the values, we get

$$\phi_B = \frac{\mu_0 Ia}{2\pi} \ln\left(\frac{x+a}{x}\right) = \frac{\mu_0 Ia}{2\pi} \ln\left(1+\frac{a}{x}\right)$$

Now, $$|e| = \left|\frac{d\phi_B}{dt}\right| = \frac{\mu_0 Ia}{2\pi}\left(\frac{x}{x+a}\right)\left(\frac{a}{x^2}\right)\frac{dx}{dt}$$

Putting $\frac{dx}{dt} = v$, we have

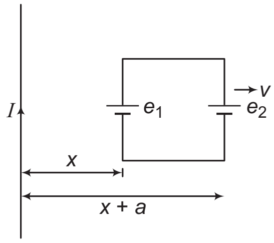
$$|e| = \frac{\mu_0 Ia^2}{2\pi x(x+a)} v$$ **Ans.**

(b) Induced current,

$$I_{\text{in}} = \frac{|e|}{R} = \frac{\mu_0 Ia^2 v}{2\pi Rx(x+a)}$$ **Ans.**

Note *Near the wire (towards right) value of ⊗ magnetic field is high. So, the loop is moving from higher magnetic field to lower magnetic field. or, ⊗ magnetic field passing through the loop is decreasing. Hence, induced current will produce ⊗ magnetic field or it should be clockwise..*

Method 2



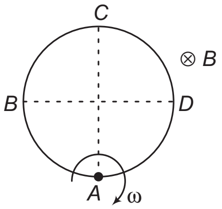
$$e_1 = B_1 vl = \frac{\mu_0}{2\pi}\frac{I}{x} va$$

$$e_2 = B_2 vl = \frac{\mu_0}{2\pi}\frac{I}{x+a} va$$

$$e_1 > e_2$$

$\therefore$ $$e_{\text{net}} = e_1 - e_2 = \frac{\mu_0 Iva}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right) = \frac{\mu_0 Iva^2}{2\pi x(x+a)}$$

This is the same result as was obtained in Method 1.



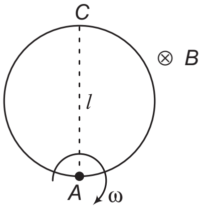
Example 3 *A conducting circular ring is rotated with angular velocity ω about point A as shown in figure. Radius of ring is a. Find*

(a) potential difference between points A and C

(b) potential difference between the points A and D.

Solution Here, the loop is rotating. So, we can applying $e = \frac{B\omega l^2}{2}$

(a)

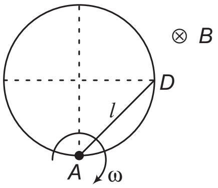


Using right hand rule, we can see that

$$V_C > V_A$$

$$\therefore \qquad V_C - V_A = \frac{B\omega l^2}{2} = \frac{B\omega (2a)^2}{2} = 2B\omega a^2 \qquad \textbf{Ans.}$$

(b)



Using right hand rule, we can see that

$$V_D > V_A$$

$$\therefore \qquad V_D - V_A = \frac{B\omega l^2}{2} = \frac{B\omega (\sqrt{2}\,a)^2}{2} = B\omega a^2 \qquad \textbf{Ans.}$$

Type 2. *Based on potential difference across an inductor*

Example 4 *Two different coils have self-inductances $L_1 = 8$ mH and $L_2 = 2$ mH. The current in one coil is increased at a constant rate. The current in the second coil is also increased at the same constant rate. At a certain instant of time, the power given to the two coils is the same. At that time, the current, the induced voltage and the energy stored in the first coil are i_1, V_1 and W_1 respectively. Corresponding values for the second coil at the same instant are i_2, V_2 and W_2 respectively. Then,* **(JEE 1994)**

(a) $\frac{i_1}{i_2} = \frac{1}{4}$

(b) $\frac{i_1}{i_2} = 4$

(c) $\frac{W_1}{W_2} = \frac{1}{4}$

(d) $\frac{V_1}{V_2} = 4$

Solution **Potential difference across an inductor :**

$V \propto L$, if rate of change of current is constant $\left(V = -L\dfrac{di}{dt}\right)$

$$\therefore \qquad \frac{V_2}{V_1} = \frac{L_2}{L_1} = \frac{2}{8} = \frac{1}{4}$$

$$\text{or} \qquad \frac{V_1}{V_2} = 4$$

Power given to the two coils is same, i.e.

$$V_1 i_1 = V_2 i_2$$

$$\text{or} \qquad \frac{i_1}{i_2} = \frac{V_2}{V_1} = \frac{1}{4}$$

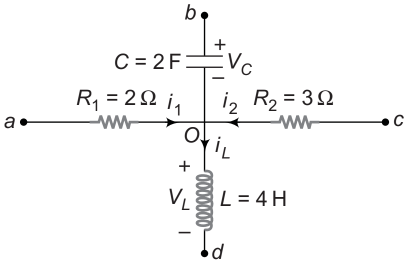
Energy stored, $W = \dfrac{1}{2}Li^2$

$$\therefore \qquad \frac{W_2}{W_1} = \left(\frac{L_2}{L_1}\right)\left(\frac{i_2}{i_1}\right)^2 = \left(\frac{1}{4}\right)(4)^2$$

$$\text{or} \qquad \frac{W_1}{W_2} = \frac{1}{4}$$

∴ The correct options are (a), (c) and (d).

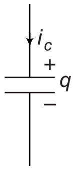
Example 5 *In the figure shown,* $i_1 = 10\,e^{-2t}$ *A,* $i_2 = 4$ *A and* $V_C = 3e^{-2t}$ *V. Determine* **(JEE 1992)**



(a) i_L *and* V_L *(b)* V_{ac}, V_{ab} *and* V_{cd}.

Solution (a) Charge stored in the capacitor at time t,

$$q = CV_C = (2)(3e^{-2t}) = 6e^{-2t}\ \text{C}$$



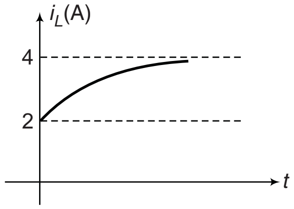
$$\therefore \qquad i_c = \frac{dq}{dt} = -\,12e^{-2t}\ \text{A}$$

(Direction of current is from b to O)

Applying junction rule at O,

$$i_L = i_1 + i_2 + i_c = 10e^{-2t} + 4 - 12e^{-2t}$$
$$= (4 - 2e^{-2t})\ \text{A}$$
$$= [2 + 2\,(1 - e^{-2t})]\ \text{A}$$ **Ans.**

i_L *versus* time graph is as shown in figure.

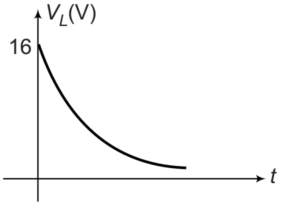


i_L increases from 2 A to 4 A exponentially.

$$V_L = V_{Od} = L\frac{di_L}{dt} = (4)\frac{d}{dt}(4 - 2e^{-2t})$$

$$= 16e^{-2t}\ \text{V}$$ **Ans.**

V_L *versus* time graph is as shown in figure.



V_L decreases exponentially from 16 V to 0.

(b) $\boldsymbol{V_{ac} = V_a - V_c}$

$$V_a - i_1R_1 + i_2R_2 = V_c$$

$\therefore$ $$V_a - V_c = V_{ac} = i_1R_1 - i_2R_2$$

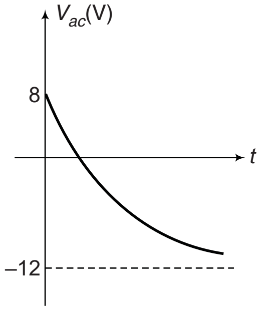
Substituting the values, we have

$$V_{ac} = (10e^{-2t})(2) - (4)(3)$$

$$V_{ac} = (20e^{-2t} - 12)\ \text{V}$$

At $t = 0$, $V_{ac} = 8$ V and at $t = \infty$, $V_{ac} = -12$V

Therefore, V_{ac} decreases exponentially from 8 V to –12 V.



$\boldsymbol{V_{ab} = V_a - V_b}$

$$V_a - i_1R_1 + V_C = V_b$$

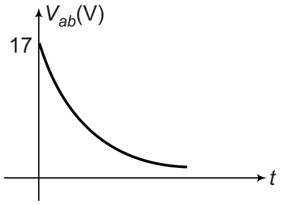
$\therefore$ $$V_a - V_b = V_{ab} = i_1R_1 - V_C$$

Substituting the values, we have

$$V_{ab} = (10e^{-2t})(2) - 3e^{-2t}$$

or $$V_{ab} = 17e^{-2t}\ \text{V}$$ **Ans.**

Thus, V_{ab} decreases exponentially from 17 V to 0.



V_{ab} *versus* t graph is shown in figure.

$V_{cd} = V_c - V_d$

$$V_c - i_2R_2 - V_L = V_d$$

$$\therefore \qquad V_c - V_d = V_{cd} = i_2R_2 + V_L$$

Substituting the values, we have

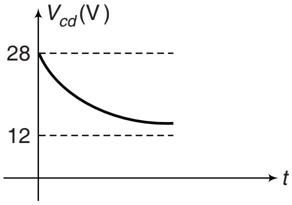
$$V_{cd} = (4)(3) + 16e^{-2t}$$

or $$V_{cd} = (12 + 16e^{-2t})\text{ V} \qquad \textbf{Ans.}$$

At $t = 0$, $V_{cd} = 28$ V and at $t = \infty$, $V_{cd} = 12$ V

i.e. V_{cd} decreases exponentially from 28 V to 12 V.

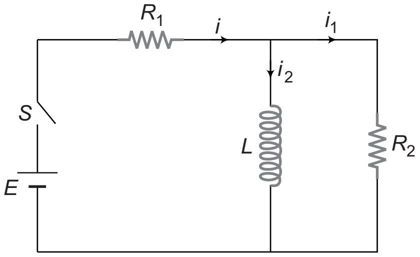
V_{cd} *versus* t graph is shown in figure.



Type 3. *Based on L - R circuit*

Concept

At time $t = 0$, when there is zero current in the circuit, an inductor offers infinite resistance and at $t = \infty$, when steady state is reached an ideal inductor (of zero resistance) offers zero resistance.



Thus, in the circuit shown, if switch S is closed at time $t = 0$, then

$$i_2 = 0$$

and $$i = i_1 = \frac{E}{R_1 + R_2} \quad \text{at} \quad t = 0$$

as initially the inductor offers infinite resistance and at $t = \infty$,

$$i_1 = 0, \quad \text{while} \quad i = i_2 = \frac{E}{R_1}$$

as in steady state the inductor offers zero resistance.

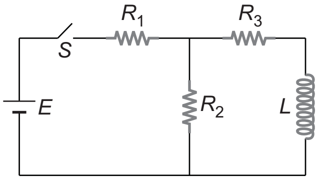
Example 6 *For the circuit shown in figure,* $E = 50$ V, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$ *and* $L = 2.0$ mH. *Find the current through* R_1 *and* R_2.

(a) Immediately after switch S is closed.

(b) A long time after S is closed.

(c) Immediately after S is reopened.

(d) A long time after S is reopened.



Solution (a) Resistance offered by inductor immediately after switch is closed will be infinite. Therefore, current through R_3 will be zero and

$$\text{current through } R_1 = \text{current through } R_2 = \frac{E}{R_1 + R_2}$$

$$= \frac{50}{10+20} = \frac{5}{3}\text{ A}$$ **Ans.**

(b) After long time of closing the switch, resistance offered by inductor will be zero.

In that case R_2 and R_3 are in parallel, and the resultant of these two is then in series with R_1. Hence,

$$R_{\text{net}} = R_1 + \frac{R_2R_3}{R_2 + R_3}$$

$$= 10 + \frac{(20)\,(30)}{20+30} = 22\ \Omega$$

Current through the battery (or through R_1)

$$= \frac{E}{R_{\text{net}}} = \frac{50}{22}\text{ A}$$ **Ans.**

This current will distribute in R_2 and R_3 in inverse ratio of resistance. Hence,

$$\text{Current through } R_2 = \left(\frac{50}{22}\right)\left(\frac{R_3}{R_2+R_3}\right)$$

$$= \left(\frac{50}{22}\right)\left(\frac{30}{30+20}\right) = \frac{15}{11}\text{ A}$$ **Ans.**

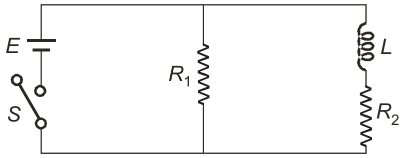
(c) Immediately after switch is reopened, the current through R_1 will become zero.

But current through R_2 will be equal to the steady state current through R_3, which is equal to,

$$\left(\frac{50}{22} - \frac{15}{11}\right)\text{A} = 0.91\text{ A}$$ **Ans.**

(d) A long after S is reopened, current through all resistors will be zero.

Example 7 *An inductor of inductance $L = 400$ mH and resistors of resistances $R_1 = 2\,\Omega$ and $R_2 = 2\,\Omega$ are connected to a battery of* emf *$E = 12$ V as shown in the figure. The internal resistance of the battery is negligible. The switch S is closed at time $t = 0$.*

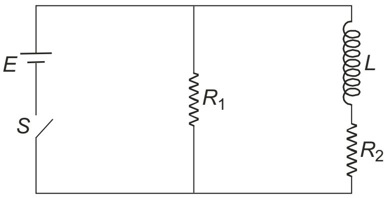


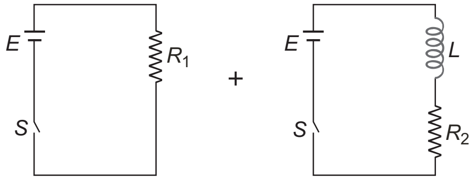
What is the potential drop across L as a function of time? After the steady state is reached, the switch is opened. What is the direction and the magnitude of current through R_1 as a function of time? **(JEE 2001)**

Solution (a) Given, $R_1 = R_2 = 2\ \Omega$, $E = 12$ V and $L = 400$ mH $= 0.4$ H.

Two parts of the circuit are in parallel with the applied battery.

So, the given circuit can be broken as :





(a) (b)

Now refer Fig. (b)

This is a simple *L*-*R* circuit, whose time constant

$$\tau_L = L/R_2 = \frac{0.4}{2} = 0.2\ \text{s}$$

and steady state current

$$i_0 = \frac{E}{R_2} = \frac{12}{2} = 6\ \text{A}$$

Therefore, if switch *S* is closed at time $t = 0$, then current in the circuit at any time *t* will be given by

$$i(t) = i_0\ (1 - e^{-t/\tau_L})$$

$$i\ (t) = 6(1 - e^{-t/0.2})$$

$$= 6(1 - e^{-5t}) = i \qquad \text{(say)}$$

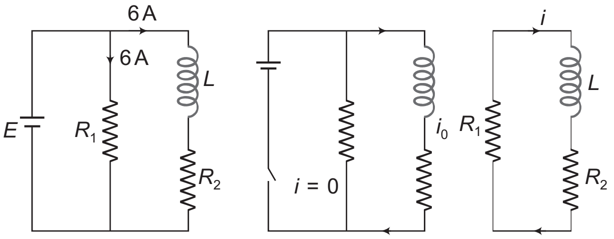
Therefore, potential drop across *L* at any time *t* is

$$V = \left| L\frac{di}{dt} \right| = L(30e^{-5t}) = (0.4)(30)\,e^{-5t} \quad \text{or} \quad V = 12\,e^{-5t}\ \text{volt}$$

(b) The steady state current in *L* or R_2 is

$$i_0 = 6\ \text{A}$$

Now, as soon as the switch is opened, current in R_1 is reduced to zero immediately. But in *L* and R_2 it decreases exponentially. The situation is as follows :



Steady state condition (c)

$t = 0$ *S* is open (d)

$t = t$ (e)

Refer figure (e)

Time constant of this circuit would be

$$\tau_L' = \frac{L}{R_1 + R_2} = \frac{0.4}{(2+2)} = 0.1 \text{ s}$$

∴ Current through R_1 at any time t is

$$i = i_0 e^{-t/\tau_{L'}} = 6\,e^{-t/0.1}$$

or $$i = 6\,e^{-10t} \text{ A}$$

Direction of current in R_1 is as shown in figure or clockwise.

Example 8 *A solenoid has an inductance of* 10 *H and a resistance of* 2 Ω. *It is connected to a* 10 *V battery. How long will it take for the magnetic energy to reach* 1/4 *of its maximum value?* **(JEE 1996)**

Solution $U = \frac{1}{2}Li^2$, i.e. $U \propto i^2$

U will reach $\frac{1}{4}$th of its maximum value when current is reached half of its maximum value. In L-R circuit, equation of current growth is written as

$$i = i_0(1 - e^{-t/\tau_L})$$

Here, i_0 = Maximum value of current

τ_L = Time constant = L/R

$$\tau_L = \frac{10 \text{ H}}{2\ \Omega} = 5 \text{ s}$$

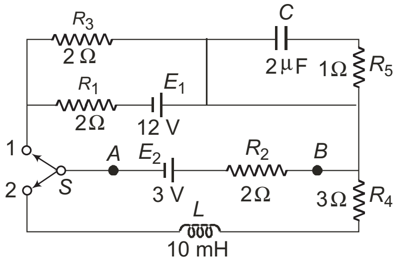
Therefore, $$i = i_0/2 = i_0(1 - e^{-t/5})$$

or $$\frac{1}{2} = 1 - e^{-t/5} \quad \text{or} \quad e^{-t/5} = \frac{1}{2}$$

or $$-t/5 = \ln\left(\frac{1}{2}\right) \quad \text{or} \quad t/5 = \ln(2) = 0.693$$

∴ $$t = (5)(0.693) \quad \text{or} \quad t = 3.465 \text{ s}$$

Example 9 *A circuit containing a two position switch S is shown in figure.*

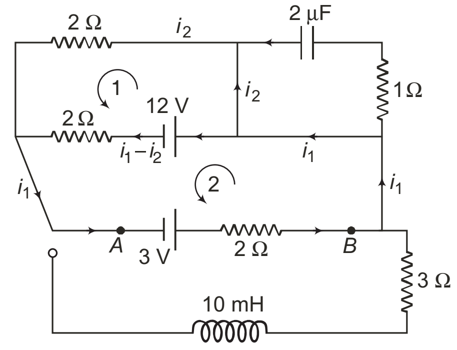


(a) *The switch S is in position* 1. *Find the potential difference* $V_A - V_B$ *and the rate of production of joule heat in* R_1.

(b) *If now the switch S is put in position* 2 *at* $t = 0$. *Find*

(i) *steady current in* R_4 *and* (ii) *the time when current in* R_4 *is half the steady value. Also calculate the energy stored in the inductor L at that time.*

Solution (a) In steady state, no current will flow through capacitor.



Applying Kirchhoff's second law in loop 1,

$$-2i_2+2\,(i_1-i_2)+12=0$$

$\therefore$ $$2i_1-4i_2=-12$$

or $$i_1-2i_2=-6 \qquad \text{...(i)}$$

Applying Kirchhoff's second law in loop 2,

$$-12-2\,(i_1-i_2)+3-2i_1=0$$

$\therefore$ $$4i_1-2i_2=-9 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$i_2=2.5\text{ A} \quad \text{and} \quad i_1=-1\text{ A}$$

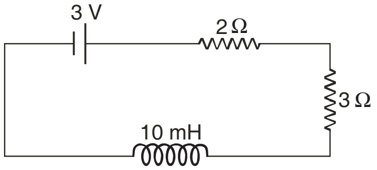
Now, $$V_A+3-2i_1=V_B \quad \text{or} \quad V_A-V_B=2i_1-3$$

$$=2\,(-1)-3=-5\text{ V}$$

$$P_{R_1}=(i_1-i_2)^2R_1=(-1-2.5)^2\,(2)$$

$$=24.5\text{ W}$$

(b) **In position 2** Circuit is as below



Steady current in R_4,

$$i_0=\frac{3}{3+2}=0.6\text{ A}$$

Time when current in R_4 is half the steady value,

$$i=i_0(1-e^{-t/\tau_L})$$

$$i=i_0/2 \text{ at } t=t_{1/2}, \text{ where}$$

$$t_{1/2}=\tau_L\,(\ln 2)=\frac{L}{R}\ln\,(2)=\frac{(10\times10^{-3})}{5}\ln\,(2)$$

$$=1.386\times10^{-3}\text{ s}$$

$$U=\frac{1}{2}Li^2=\frac{1}{2}\,(10\times10^{-3})\,(0.3)^2$$

$$=4.5\times10^{-4}\text{ J}$$

Type 4. *Based on L-C oscillations*

Example 10 *In an L-C circuit, $L = 3.3$ H and $C = 840$ pF. At $t = 0$, charge on the capacitor is 105 μC and maximum. Compute the following quantities at $t = 2.0$ ms:*

(a) The energy stored in the capacitor.

(b) The total energy in the circuit,

(c) The energy stored in the inductor.

Solution Given, $L = 3.3\ \text{H}$, $C = 840 \times 10^{-12}\ \text{F}$ and $q_0 = 105 \times 10^{-6}\ \text{C}$

The angular frequency of *L-C* oscillations is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{3.3 \times 840 \times 10^{-12}}}$$

$$= 1.9 \times 10^4\ \text{rad/s}$$

Charge stored in the capacitor at time t would be

$$q = q_0 \cos \omega t$$

(a) At $t = 2 \times 10^{-3}$ s,

$$q = (105 \times 10^{-6}) \cos [1.9 \times 10^4][2 \times 10^{-3}]$$

$$= 100.3 \times 10^{-6}\ \text{C}$$

∴ Energy stored in the capacitor,

$$U_C = \frac{1}{2}\frac{q^2}{C}$$

$$= \frac{(100.3 \times 10^{-6})^2}{2 \times 840 \times 10^{-12}}$$

$$= 6.0\ \text{J}$$ **Ans.**

(b) Total energy in the circuit,

$$U = \frac{1}{2}\frac{q_0^2}{C} = \frac{(105 \times 10^{-6})^2}{2 \times 840 \times 10^{-12}}$$

$$= 6.56\ \text{J}$$ **Ans.**

(c) Energy stored in inductor in the given time

= total energy in circuit – energy stored in capacitor

$= (6.56 - 6.0)$ J

$= 0.56$ J **Ans.**

Example 11 *An inductor of inductance 2.0 mH is connected across a charged capacitor of capacitance 5.0 μF and the resulting L-C circuit is set oscillating at its natural frequency. Let Q denotes the instantaneous charge on the capacitor and I the current in the circuit. It is found that the maximum value of Q is 200 μC.*

(JEE 1998)

(a) When $Q = 100\ \mu C$, what is the value of $|dI/dt|$?

(b) When $Q = 200\ \mu C$, what is the value of I?

(c) Find the maximum value of I.

(d) When I is equal to one-half of its maximum value, what is the value of $|Q|$?

Solution This is a problem of L-C oscillations.

Charge stored in the capacitor oscillates simple harmonically as

$$Q = Q_0 \sin\ (\omega t \pm \phi)$$

Here, Q_0= maximum value of $Q = 200\,\mu\text{C} = 2 \times 10^{-4}$ C

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2\times10^{-3})(5.0\times10^{-6})}} = 10^4\text{s}^{-1}$$

Let at $t = 0$, $Q = Q_0$, then

$$Q(t) = Q_0 \cos\omega t \qquad \ldots\text{(i)}$$

$$I(t) = \frac{dQ}{dt} = -Q_0\omega \sin\omega t \text{ and} \qquad \ldots\text{(ii)}$$

$$\frac{dI(t)}{dt} = -Q_0\omega^2 \cos\omega t \qquad \ldots\text{(iii)}$$

(a) $Q = 100\,\mu\text{C}$

or $$\frac{Q_0}{2} \text{ at } \cos\omega t = \frac{1}{2}$$

or $$\omega t = \frac{\pi}{3}$$

At $\cos\omega t = \frac{1}{2}$, from Eq. (iii) :

$$\left|\frac{dI}{dt}\right| = (2.0\times10^{-4}\text{C})(10^4\text{s}^{-1})^2\left(\frac{1}{2}\right)$$

$$\left|\frac{dI}{dt}\right| = 10^4 \text{ A/s}$$

(b) $Q = 200\,\mu\text{C}$ or Q_0 when $\cos\omega t = 1$, i.e. $\omega t = 0, 2\pi \ldots$

At this time $$I(t) = -Q_0\omega\ \sin\omega t$$

or $$I(t) = 0 \qquad (\sin 0° = \sin 2\pi = 0)$$

(c) $I(t) = -Q_0\omega \sin\omega t$

∴ Maximum value of I is $Q_0\,\omega$

$$I_{\max} = Q_0\,\omega = (2.0\times10^{-4})(10^4)$$

$$I_{\max} = 2.0 \text{ A}$$

(d) From energy conservation,

$$\frac{1}{2}LI_{\max}^2 = \frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}$$

or $$Q = \sqrt{LC(I_{\max}^2 - I^2)}$$

$$I = \frac{I_{\max}}{2} = 1.0 \text{ A}$$

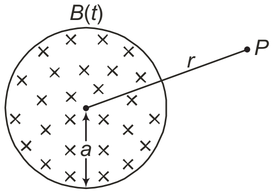
∴ $$Q = \sqrt{(2.0\times10^{-3})(5.0\times10^{-6})(2^2 - 1^2)}$$

$$Q = \sqrt{3}\times10^{-4} \text{ C}$$

or $$Q = 1.732\times10^{-4} \text{ C}$$

Type 5. *Based on induced electric field*

Example 12 *A uniform but time-varying magnetic field $B(t)$ exists in a circular region of radius a and is directed into the plane of the paper as shown. The magnitude of the induced electric field at point P at a distance r from the centre of the circular region* **(JEE 2000)**



(a) is zero *(b) decreases as $1/r$* *(c) increases as r* *(d) decreases as $1/r^2$*

Solution $\int \mathbf{E}\cdot d\mathbf{l} = \left|\dfrac{d\phi}{dt}\right| = S\left|\dfrac{dB}{dt}\right|$ or $E(2\pi r) = \pi a^2\left|\dfrac{dB}{dt}\right|$

For $r \ge a$,

$\therefore$
$$E = \frac{a^2}{2r}\left|\frac{dB}{dt}\right|$$

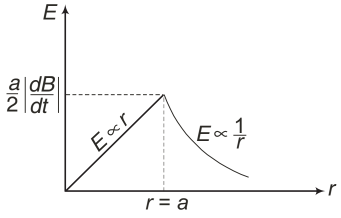
$\therefore$ Induced electric field $\propto 1/r$

For $r \le a$,

$$E(2\pi r) = \pi r^2\left|\frac{dB}{dt}\right| \quad \text{or} \quad E = \frac{r}{2}\left|\frac{dB}{dt}\right| \quad \text{or} \quad E \propto r$$

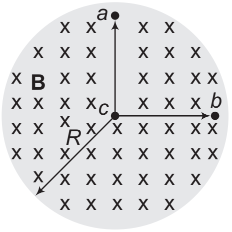
At $r = a$, $E = \dfrac{a}{2}\left|\dfrac{dB}{dt}\right|$

Therefore, variation of E with r (distance from centre) will be as follows



$\therefore$ The correct option is (b).

Example 13 *The magnetic field* $\mathbf{B}$ *at all points within a circular region of radius R is uniform in space and directed into the plane of the page in figure. If the magnetic field is increasing at a rate dB/dt, what are the magnitude and direction of the force on a stationary positive point charge q located at points a, b and c? (Point a is a distance r above the centre of the region, point b is a distance r to the right of the centre, and point c is at the centre of the region.)*



Solution Inside the circular region at distance r,

$$El = \frac{d\phi}{dt} = S\left(\frac{dB}{dt}\right)$$

$$\therefore \qquad E\,(2\pi r) = (\pi r^2)\cdot\frac{dB}{dt}$$

$$\therefore \qquad E = \frac{r}{2}\frac{dB}{dt}$$

$$F = qE = \frac{qr}{2}\frac{dB}{dt}$$

At points a and b, distance from centre is r.

$$\therefore \qquad F = \frac{qr}{2}\frac{dB}{dt}$$

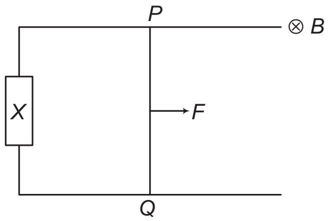
At point C, distance $r = 0$

$$\therefore \qquad F = 0$$

$\otimes$ magnetic field is increasing. Hence, induced current in an imaginary loop passing through a and b should produce $\odot$ magnetic field. Hence, induced current through an imaginary circular loop passing through a and b should be anti-clockwise. Force on positive charge is in the direction of induced current. Hence, force at a is towards left and force at b is upwards.

Type 6. *Based on motion of a wire in uniform magnetic field with other element like resistance, capacitor or an inductor*

Concept



A constant force F is applied on wire PQ of length l and mass m. There is an electrical element X in the box as shown in figure. There are the following three different cases :

Case 1 If X is a resistance, then velocity of the wire increases exponentially.

Case 2 If X is a capacitor, then wire moves with a constant acceleration a ($< F/m$).

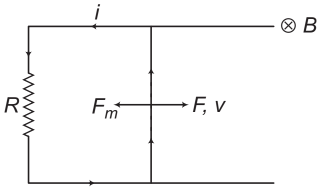
Case 3 If X is an inductor and instead of constant force F an initial velocity v_0 is given to the wire then the wire starts simple harmonic motion with v_0 as the maximum velocity ($= \omega A$) at mean position.

Example 14 *In the above case if X is a resistance R, then find velocity of wire as a function of time t.*

Solution At time t suppose velocity of wire is v, then due to motional emf a current i flows in the closed circuit in anti-clockwise direction.

$$i = \frac{e}{R} = \frac{Bvl}{R}$$

Due to this current magnetic force will act on the wire in the direction shown in figure,



$$F_m = ilB = \left(\frac{B^2l^2}{R}\right) v$$

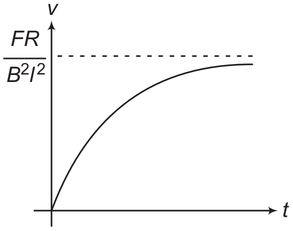
$$F_{\text{net}} = F - F_m = F - \left(\frac{B^2\,l^2}{R}\right) v = m\,\frac{dv}{dt}$$

$$\therefore \qquad \int_0^v \frac{dv}{F - \left(\frac{B^2\,l^2}{R}\right) v} = \frac{1}{m}\int_0^t dt$$

Solving this equation, we get

$$v = \frac{FR}{B^2l^2}\,(1 - e^{-\frac{B^2l^2}{mR}t})$$

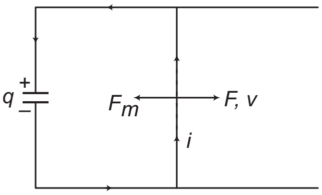
Thus, velocity of the wire increases exponentially. v- t graph is as shown below.



Example 15 *If X is a capacitor C, then find the constant acceleration a of the wire.*

Solution At time t *suppose velocity of wire is v. Then, due to motional emf $e = Bvl$ capacitor gets charged .*

$$q = CV = C\,(Bvl)$$



This charge is increasing as v will be increasing.

Hence, there will be a current in the circuit as shown in figure.

$$i = \frac{dq}{dt} = (BlC)\,\frac{dv}{dt}$$

or $$i = (BlC)\,a \qquad \left(\text{as } \frac{dv}{dt} = a\right)$$

Due to this current a magnetic force F_m will act in the direction shown in figure,

$$F_m = ilB = (B^2l^2C)\,a$$

Now,

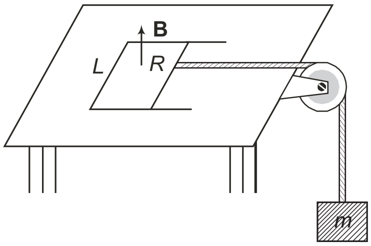
$$F_{\text{net}} = F - F_m$$

or $$ma = F - (B^2l^2C)\,a$$

$\therefore$ $$a = \frac{F}{m + B^2l^2C}$$ **Ans.**

Now, we can see that this acceleration is constant but less than F/m.

Example 16 *A pair of parallel horizontal conducting rails of negligible resistance shorted at one end is fixed on a table. The distance between the rails is L. A conducting massless rod of resistance R can slide on the rails frictionlessly. The rod is tied to a massless string which passes over a pulley fixed to the edge of the table. A mass m tied to the other end of the string hangs vertically. A constant magnetic field B exists perpendicular to the table. If the system is released from rest, calculate* **(JEE 1997)**



(a) the terminal velocity achieved by the rod and

(b) the acceleration of the mass at the instant when the velocity of the rod is half the terminal velocity.

Solution (a) Let v be the velocity of the wire (as well as block) at any instant of time t.

Motional emf, $e = BvL$

Motional current, $i = \frac{e}{r} = \frac{BvL}{R}$

and magnetic force on the wire

$$F_m = iLB = \frac{vB^2L^2}{R}$$

Net force on the system at this moment will be

$$F_{\text{net}} = mg - F_m = mg - \frac{vB^2L^2}{R}$$

or $$ma = mg - \frac{vB^2L^2}{R}$$

$$a = g - \frac{vB^2L^2}{mR} \qquad \text{...(i)}$$

Velocity will acquire its terminal value, i.e. $v = v_T$ when F_{net} or acceleration a of the particle becomes zero.

Thus,
$$0 = g - \frac{v_T B^2 L^2}{mR}$$

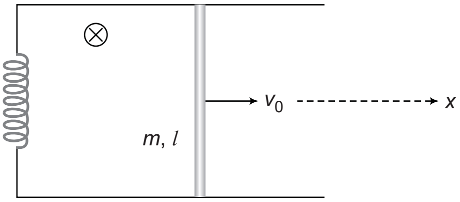
or
$$v_T = \frac{mgR}{B^2 L^2}$$

(b) When
$$v = \frac{v_T}{2} = \frac{mgR}{2B^2 L^2}$$

Then from Eq. (i), acceleration of the block,
$$a = g - \left(\frac{mgR}{2B^2L^2}\right)\left(\frac{B^2L^2}{mR}\right) = g - \frac{g}{2}$$

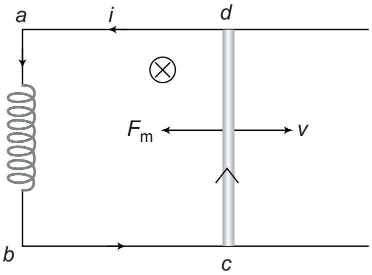
or
$$a = \frac{g}{2}$$

Example 17 *A loop is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field B. The distance between the conductors is l.*



At the moment $t = 0$, the rod is imparted an initial velocity v_0 directed to the right. Find the law of its motion x (t) if the electric resistance of the loop is negligible.

Solution Let at any instant of time, velocity of the rod is v towards right. The current in the circuit is i. In the figure,



$$V_a - V_b = V_d - V_c$$

or
$$L\frac{di}{dt} = Bvl = Bl\frac{dx}{dt} \qquad \left(\text{as } v = \frac{dx}{dt}\right)$$

i.e.
$$L\,di = Bl\,dx$$

Integrating on both sides, we get

$$Li = Blx$$

or $$i = \frac{Bl}{L}x \qquad \text{...(i)}$$

Magnetic force on the rod at this instant is

$$F_m = ilB = \frac{B^2l^2}{L}x \qquad \text{...(ii)}$$

Since, this force is in opposite direction of **v**, so from Newton's second law we can write

$$m\left(\frac{d^2x}{dt^2}\right) = -\frac{B^2l^2}{L}x$$

or $$\left(\frac{d^2x}{dt^2}\right) = -\frac{B^2l^2}{mL}x$$

Comparing this with equation of SHM,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

We have, $$\omega = \frac{Bl}{\sqrt{mL}}$$

Therefore, the rod will oscillate simple harmonically with angular frequency $\omega = \frac{Bl}{\sqrt{mL}}$. At time $t = 0$, rod was at $x = 0$ and it was moving towards positive x-axis. Hence, x-t equation of the rod is

$$x = A\sin\omega t \qquad \text{...(iii)}$$

To find A, we use the fact that at $t = 0$, v or $\frac{dx}{dt}$ has a value v_0. Hence,

$$\frac{dx}{dt} = v = A\omega\cos\omega t$$

or $$A\omega = v_0 \qquad \text{(at } t = 0)$$

or $$A = \frac{v_0}{\omega}$$

Substituting in Eq. (iii), we have

$$x = \frac{v_0}{\omega}\sin\omega t, \text{ where } \omega = \frac{Bl}{\sqrt{mL}}$$ **Ans.**

Alternate method of finding A

At $x = A$, $v = 0$, i.e. whole of its kinetic energy is converted into magnetic energy. Thus,

$$\frac{1}{2}Li^2 = \frac{1}{2}mv_0^2$$

Substituting value of i from Eq. (i), with $x = A$, we have

$$L\left(\frac{Bl}{L}A\right)^2 = mv_0^2$$

or $$A = \frac{\sqrt{mL}}{Bl}v_0 = \frac{v_0}{\omega}$$

as $$\omega = \frac{Bl}{\sqrt{mL}}$$ **Ans.**

Miscellaneous Examples

Example 18 *A sensitive electronic device of resistance* 175 Ω *is to be connected to a source of emf by a switch. The device is designed to operate with a current of* 36 *mA, but to avoid damage to the device, the current can rise to no more than* 4.9 *mA in the first* 58 μs *after the switch is closed. To protect the device it is connected in series with an inductor.*

(a) What emf must the source have?

(b) What inductance is required?

(c) What is the time constant?

Solution (a) Given, $R = 175\ \Omega$ and peak value current

$$i_0 = 36 \times 10^{-3}\ \text{A}$$

$$\text{Applied voltage, } V = i_0 R = (175)\,(36 \times 10^{-3})\ \text{volt} = 6.3\ \text{V}$$ **Ans.**

(b) From the relation,

$$i = i_0\,(1 - e^{-t/\tau_L})$$

We have, $$(4.9) = (36)\,[1 - e^{-t/\tau_L}]$$

or $$e^{-t/\tau_L} = 1 - \frac{4.9}{36} = 0.864$$

∴ $$\frac{t}{\tau_L} = -\ln(0.864) = 0.146$$

or $$\frac{t}{L/R} = 0.146$$

∴ $$\frac{Rt}{L} = 0.146$$

or $$L = \frac{Rt}{0.146} = \frac{(175)\,(58 \times 10^{-6})}{0.146}$$

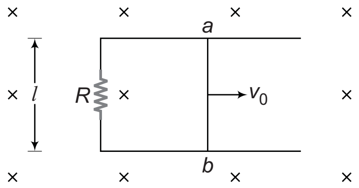
$$= 7.0 \times 10^{-2}\ \text{H}$$ **Ans.**

(c) Time constant of the circuit,

$$\tau_L = \frac{L}{R} = \frac{7.0 \times 10^{-2}}{175}$$

$$= 4.0 \times 10^{-4}$$ **Ans.**

Example 19 *A conducting rod shown in figure of mass m and length l moves on two frictionless horizontal parallel rails in the presence of a uniform magnetic field directed into the page. The rod is given an initial velocity* v_0 *to the right and is released at* $t = 0$. *Find as a function of time,*



(a) the velocity of the rod

(b) the induced current and

(c) the magnitude of the induced emf.

HOW TO PROCEED *The initial velocity will produce an induced emf and hence, an induced current in the circuit. The current carrying wire will now experience a magnetic force* ($\mathbf{F}_m$) *in opposite direction of its velocity. The force will retard the motion of the conductor. Thus,*

Initial velocity → motional emf → induced current → magnetic force → retardation.

Solution (a) Let v be the velocity of the rod at time t.

Current in the circuit at this moment is

$$i = \frac{Bvl}{R} \qquad \text{...(i)}$$

From right hand rule, we can see that this current is in counterclockwise direction.

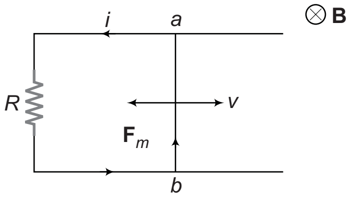
The magnetic force is,

$$F_m = -ilB = -\frac{B^2l^2}{R}v$$

Here, negative sign denotes that the force is to the left and retards the motion. This is the only horizontal force acting on the bar, and hence, Newton's second law applied to motion in horizontal direction gives

$$m\frac{dv}{dt} = F_m = -\frac{B^2l^2}{R}v$$

$$\therefore \qquad \frac{dv}{v} = -\left(\frac{B^2l^2}{mR}\right)dt$$



Integrating this equation using the initial condition that, $v = v_0$ at $t = 0$, we find that

$$\int_{v_0}^{v}\frac{dv}{v} = -\frac{B^2l^2}{mR}\int_0^t dt$$

Solving this equation, we find that

$$v = v_0e^{-t/\tau} \qquad \text{...(ii)} \quad \textbf{Ans.}$$

where, $$\tau = \frac{mR}{B^2l^2}$$

This expression indicates that the velocity of the rod decreases exponentially with time under the action of the magnetic retarding force.

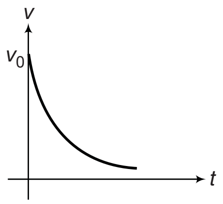
(b) $i = \dfrac{Bvl}{R}$

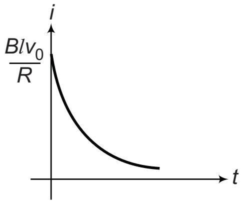
Substituting the value of v from Eq. (ii), we get

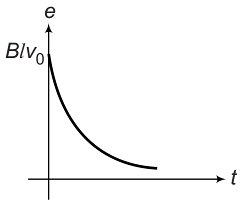
$$i = \frac{Blv_0}{R}e^{-t/\tau} \qquad \textbf{Ans.}$$

(c) $$e = iR = Blv_0e^{-t/\tau} \qquad \textbf{Ans.}$$

i and e both decrease exponentially with time. v-t, i-t and e-t graphs are as shown in figure







Alternate solution This problem can also be solved by energy conservation principle. Let at some instant velocity of the rod is v. As no external force is present. Energy is dissipated in the resistor at the cost of kinetic energy of the rod. Hence,

$$\left(-\frac{dK}{dt}\right) = \text{power dissipated in the resistor}$$

or
$$-\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{e^2}{R}$$

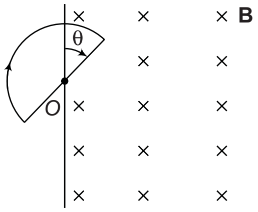
or
$$-mv\left(\frac{dv}{dt}\right) = \frac{B^2l^2v^2}{R} \qquad (\text{as } e = Bvl)$$

$\therefore$
$$\frac{dv}{v} = -\frac{B^2l^2}{mR}dt$$

$\therefore$
$$\int_{v_0}^{v}\frac{dv}{v} = -\frac{B^2l^2}{mR}\int_0^t dt$$

or
$$v = v_0e^{-t/\tau}, \quad \text{where } \tau = \frac{mR}{B^2l^2}$$

Example 20 *A wire loop enclosing a semicircle of radius R is located on the boundary of a uniform magnetic field B. At the moment t = 0, the loop is set into rotation with a constant angular acceleration α about an axis O coinciding with a line of vector* **B** *on the boundary. Find the emf induced in the loop as a function of time. Draw the approximate plot of this function. The arrow in the figure shows the emf direction taken to be positive.*



Solution
$$\theta = \frac{1}{2}\alpha t^2$$

$\therefore$
$$t = \sqrt{\frac{2\theta}{\alpha}} = \text{time taken to rotate an angle } \theta$$

where, $\theta = 0$ to $\pi, 2\pi$ to 3π, 4π to 5π etc.

⊗ magnetic field passing through the loop is increasing. Hence, current in the loop is anti-clockwise or induced emf is negative. And for, $\theta = \pi$ to 2π, 3π to 4π, 5π to 6π etc.

⊗ magnetic field passing through the loop is decreasing. Hence, current in the loop is clockwise or emf is positive.

So,

$$t_1 = \text{time taken to rotate an angle } \pi = \sqrt{\frac{2\pi}{\alpha}}$$

$$t_2 = \text{time taken to rotate an angle } 2\pi = \sqrt{\frac{4\pi}{\alpha}}$$

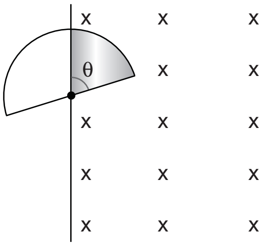
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$t_n = \text{time taken to rotate an angle } n\pi = \sqrt{\frac{2n\pi}{\alpha}}$$

Now, from 0 to t_1 emf is negative
t_1 to t_2 emf is positive
t_2 to t_3 emf is again negative

and so on.

Now, at time t, angle rotated is $\theta = \frac{1}{2}\alpha t^2$



Area inside the field is

$$S = (\pi R^2)\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}R^2\theta$$

or
$$S = \frac{1}{4}R^2\alpha t^2$$

So, flux passing through the loop, $\phi = BS = \frac{1}{4}BR^2\alpha t^2$

$$e = \left|\frac{d\phi}{dt}\right| = \frac{1}{2}BR^2\alpha t$$

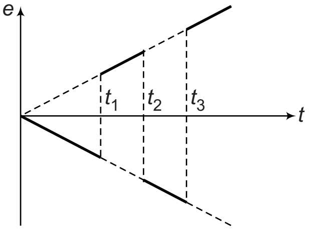
$$e \propto t$$

i.e. e-t graph is a straight line passing through origin. e-t equation with sign can be written as

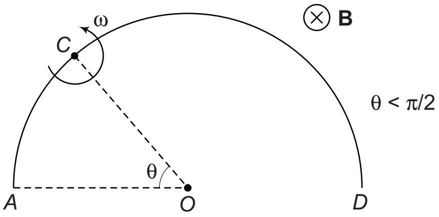
$$e = (-1)^n\left(\frac{1}{2}BR^2\alpha t\right)$$ **Ans.**

Here, $n = 1, 2, 3\ldots$ is the number of half revolutions that the loop performs at the given moment t.

The e-t graph is as shown in figure.



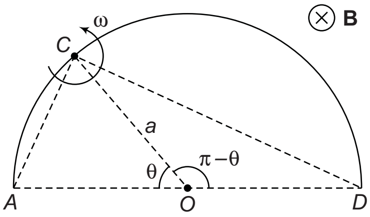
Example 21 *A uniform wire of resistance per unit length λ is bent into a semicircle of radius a. The wire rotates with angular velocity ω in a vertical plane about a horizontal axis passing through C. A uniform magnetic field B exists in space in a direction perpendicular to paper inwards.*



(a) Calculate potential difference between points A and D. Which point is at higher potential?

(b) If points A and D are connected by a conducting wire of zero resistance, find the potential difference between A and C.

Solution (a) Length of straight wire AC is $l_1 = 2a\sin\left(\frac{\theta}{2}\right)$



Therefore, the motional emf (or potential difference) between points C and A is

$$V_{CA} = V_C - V_A = \frac{1}{2}B\omega l_1^2 = 2a^2 B\omega\sin^2\left(\frac{\theta}{2}\right) \qquad \text{...(i)}$$

From right hand rule, we can see that $V_C > V_A$

Similarly, length of straight wire CD is

$$l_2 = 2a\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2a\cos\left(\frac{\theta}{2}\right)$$

Therefore, the PD between points C and D is

$$V_{CD} = V_C - V_D = \frac{1}{2}B\omega l_2^2 = 2a^2 B\omega\cos^2\left(\frac{\theta}{2}\right) \qquad \text{...(ii)}$$

with $V_C > V_D$

Eq. (ii) − Eq.(i) gives,

$$V_A - V_D = 2a^2 B\omega\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$

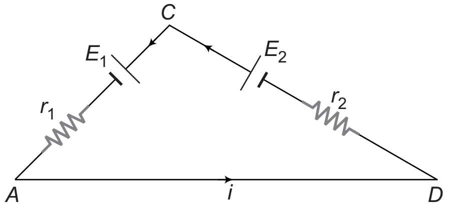
$$= 2a^2 B\omega\cos\theta \qquad \textbf{Ans.}$$

A is at higher potential.

(b) When A and D are connected from a wire current starts flowing in the circuit as shown in figure :

Resistance between A and C is r_1 = (length of arc AC) $\lambda = a\theta\lambda$

and between C and D is r_2 = (length of arc CD) $\lambda = (\pi - \theta)\,a\lambda$



In the figure, $E_1 = 2a^2 B\omega\sin^2\left(\frac{\theta}{2}\right)$ and $E_2 = 2a^2 B\omega\cos^2\left(\frac{\theta}{2}\right)$

with $E_2 > E_1$

∴ Current in the circuit is

$$i = \frac{E_2 - E_1}{r_1 + r_2} = \frac{2a^2 B\omega\cos\theta}{\pi a\lambda} = \frac{2aB\omega\cos\theta}{\pi\lambda}$$

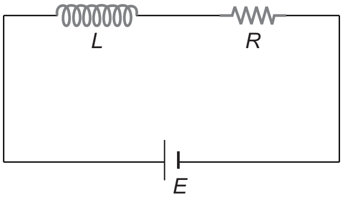
and potential difference between points C and A is

$$V'_{CA} = E_1 + ir_1 = 2a^2B\omega\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{2aB\omega\cos\theta}{\pi\lambda}\right)(a\theta\lambda)$$

$$= 2a^2B\omega\left(\sin^2\frac{\theta}{2} + \frac{\theta}{\pi}\cos\theta\right)$$ **Ans.**

Note *$V_{CA} = E_1$ when no current flows through the circuit and $V'_{CA} = E_1 + ir_1$ when a current i flows in the circuit.*

Example 22 *A battery of emf E and of negligible internal resistance is connected in an L-R circuit as shown in figure. The inductor has a piece of soft iron inside it. When steady state is reached the piece of soft iron is abruptly pulled out suddenly so that the inductance of the inductor decreases to nL with $n < 1$ with battery remaining connected. Calculate*



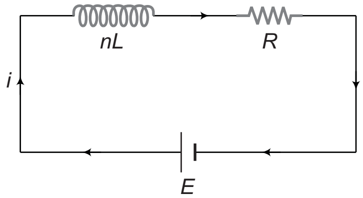
(a) current as a function of time assuming $t = 0$ at the instant when piece is pulled.
(b) the work done to pull out the piece.
(c) thermal power generated in the circuit as a function of time.
(d) power supplied by the battery as a function of time.

HOW TO PROCEED *When the inductance of an inductor is abruptly changed, the flux passing through it remains constant.*

$$\phi = \text{constant}$$

$$\therefore \qquad Li = \text{constant} \qquad \left(L = \frac{\phi}{i}\right)$$

Solution (a) At time $t = 0$, steady state current in the circuit is $i_0 = E/R$. Suddenly, L reduces to nL ($n < 1$), so current in the circuit at time $t = 0$ will increase to $\frac{i_0}{n} = \frac{E}{nR}$. Let i be the current at time t.



Applying Kirchhoff's loop rule, we have

$$E - nL\left(\frac{di}{dt}\right) - iR = 0$$

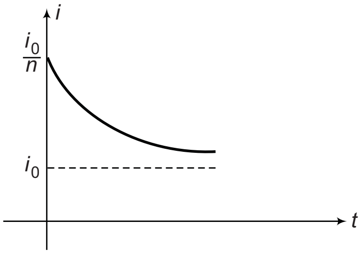
$$\therefore \qquad \frac{di}{E - iR} = \frac{1}{nL}dt$$

$$\therefore \qquad \int_{i_0/n}^{i}\frac{di}{E - iR} = \frac{1}{nL}\int_0^t dt$$

Solving this equation, we get $$i = i_0 - \left(i_0 - \frac{i_0}{n}\right) e^{-t/\tau_L}$$ **Ans.**

Here, $$i_0 = \frac{E}{R}$$

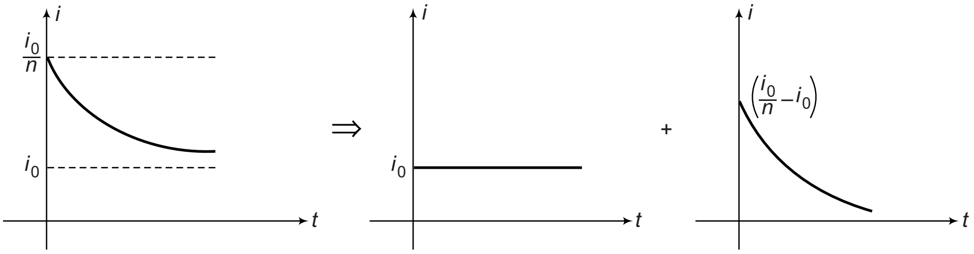
and $$\tau_L = \frac{nL}{R}$$



From the *i*-*t* equation, we get $i = \frac{i_0}{n}$ at $t = 0$ and $i = i_0$ at $t = \infty$

The *i*-*t* graph is as shown in figure.

Note *At $t = 0$, current in the circuit is $\frac{i_0}{n}$. Current in the circuit in steady state will be again i_0. So, it will decrease exponentially from $\frac{i_0}{n}$ to i_0. From the i-t graph, the equation can be formed without doing any calculation.*



$$\therefore \quad i = i_0 + \left(\frac{i_0}{n} - i_0\right) e^{-t/\tau_L}$$

(b) Work done to pull out the piece,

$$W = U_f - U_i = \frac{1}{2} L_f i_f^2 - \frac{1}{2} L_i i_i^2$$

$$= \frac{1}{2}(nL)\left(\frac{E}{nR}\right)^2 - \frac{1}{2}(L)\left(\frac{E}{R}\right)^2$$

$$= \frac{1}{2} L \left(\frac{E}{R}\right)^2 \left(\frac{1}{n} - 1\right)$$

$$= \frac{1}{2} L \left(\frac{E}{R}\right)^2 \left(\frac{1-n}{n}\right)$$ **Ans.**

(c) Thermal power generated in the circuit as a function of time is

$$P_1 = i^2 R$$ **Ans.**

Here, *i* is the current calculated in part (a).

(d) Power supplied by the battery as a function of time is

$$P_2 = Ei$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

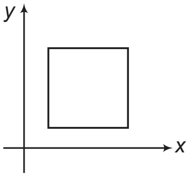
(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true; but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

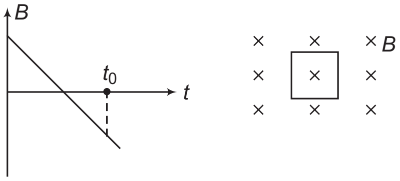
(*d*) *If **Assertion** is false, but the **Reason** is true.*

1. Assertion : A square loop is placed in x-y plane as shown in figure. Magnetic field in the region is $\mathbf{B} = -B_0 x\,\hat{\mathbf{k}}$. The induced current in the loop is anti-clockwise.



Reason : If inward magnetic field from such a loop increases, then current should be anti-clockwise.

2. Assertion : Magnetic field B (shown inwards) varies with time t as shown. At time t_0 induced current in the loop is clockwise.

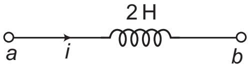


Reason : If rate of change of magnetic flux from a coil is constant, charge should flow in the coil at a constant rate.

3. Assertion : Electric field produced by a variable magnetic field can't exert a force on a charged particle.

Reason : This electric field is non-conservative in nature.

4. Assertion : Current flowing in the circuit is $i = 2t - 8$



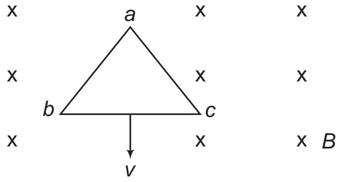
At $t = 1$ s, $V_a - V_b = +4$ V

Reason : $V_a - V_b$ is $+4$ V all the time.

5. **Assertion :** Angular frequency of L-C oscillations is 2 rad/s and maximum current in the circuit is 1 A. Then, maximum rate of change of current should be 2 A/s.

Reason : $\left(\frac{dI}{dt}\right)_{\max} = (I_{\max})\omega.$

6. **Assertion :** A conducting equilateral loop *abc* is moved translationally with constant speed v in uniform inward magnetic field B as shown. Then : $V_a - V_b = V_b - V_c$.



Reason : Point a is at higher potential than point b.

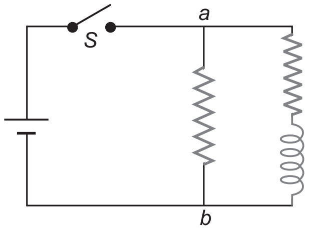
7. **Assertion :** Motional induced emf $e = Bvl$ can be derived from the relation $e = -\frac{d\phi}{dt}$.

Reason : Lenz's law is a consequence of law of conservation of energy.

8. **Assertion :** If some ferromagnetic substance is filled inside a solenoid, its coefficient of self induction L will increase.

Reason : By increasing the current in a coil, its coefficient of self induction L can be increased.

9. **Assertion :** In the circuit shown in figure, current in wire *ab* will become zero as soon as switch is opened.



Reason : A resistance does not oppose increase or decrease of current through it.

10. **Assertion :** In parallel, current distributes in inverse ratio of inductance

$$i \propto \frac{1}{L}$$

Reason : In electrical circuits, an inductor can be treated as a resistor.

Objective Questions

1. The dimensions of self inductance are

(a) $[MLT^{-2}A^{-2}]$ (b) $[ML^2T^{-1}A^{-2}]$
(c) $[ML^2T^{-2}A^{-2}]$ (d) $[ML^2T^{-2}A^{-1}]$

2. When the number of turns in the two circular coils closely wound are doubled (in both), their mutual inductance becomes

(a) four times (b) two times
(c) remains same (d) sixteen times

3. Two coils carrying current in opposite direction are placed co-axially with centres at some finite separation. If they are brought close to each other then, current flowing in them should

(a) decrease
(b) increase
(c) remain same
(d) become zero

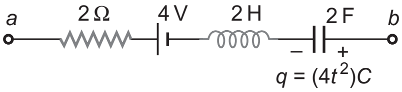
4. A current carrying ring is placed in a horizontal plane. A charged particle is dropped along the axis of the ring to fall under the influence of gravity

(a) the current in the ring may increase
(b) the current in the ring may decrease
(c) the velocity of the particle will increase till it reaches the centre of the ring
(d) the acceleration of the particle will decrease continuously till it reaches the centre of the ring

5. Identify the **incorrect** statement. Induced electric field

(a) is produced by varying magnetic field
(b) is non-conservative in nature
(c) cannot exist in a region not occupied by magnetic field
(d) None of the above

6. In the figure shown, V_{ab} at $t = 1$ s is



(a) 30 V
(b) – 30 V
(c) 20 V
(d) – 20 V

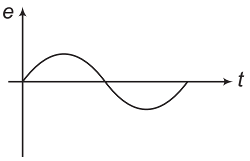
7. Two coils have a mutual inductance of 0.005 H. The current changes in the first coil according to equation $I = I_0 \sin \omega t$, where $I_0 = 10\,\text{A}$ and $\omega = 100\,\pi$ rad/s. The maximum value of emf (in volt) in the second coil is

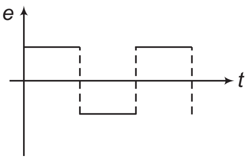
(a) 2π
(b) 5π
(c) π
(d) 4π

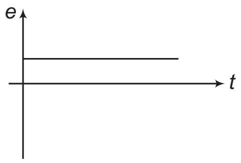
8. An inductance of 2 H carries a current of 2 A. To prevent sparking when the circuit is broken a capacitor of 4 μF is connected across the inductance. The voltage rating of the capacitor is of the order of

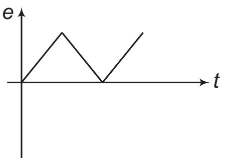
(a) 10^3 V
(b) 10 V
(c) 10^5 V
(d) 10^6 V

9. A conducting rod is rotated about one end in a plane perpendicular to a uniform magnetic field with constant angular velocity. The correct graph between the induced emf (e) across the rod and time (t) is

(a)


(b)


(c)


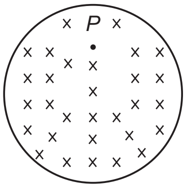
(d)


10. A magnet is taken towards a conducting ring in such a way that a constant current of 10 mA is induced in it. The total resistance of the ring is 0.5 Ω. In 5 s, the magnetic flux through the ring changes by

(a) 0.25 mWb (b) 25 mWb

(c) 50 mWb (d) 15 mWb

11. A uniform but increasing with time magnetic field exists in a cylindrical region. The direction of force on an electron at P is



(a) towards right

(b) towards left

(c) into the plane of paper

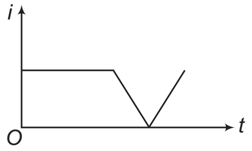
(d) out of the plane of paper

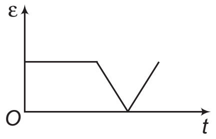
12. A magnetic flux through a stationary loop with a resistance R varies during the time interval τ as $\phi = at(\tau - t)$. Find the amount of heat generated in the loop during that time

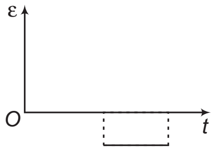
(a) $\frac{a\tau^2}{2R}$ (b) $\frac{a^2\tau^3}{3R}$

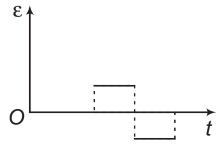
(c) $\frac{2a^2\tau^3}{3R}$ (d) $\frac{a\tau}{3R}$

13. The current i in an induction coil varies with time t according to the graph shown in the figure. Which of the following graphs shows the induced emf (ε) in the coil with time?



(a)


(b)


(c)


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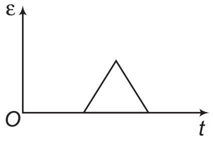
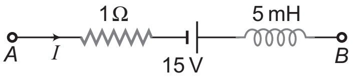


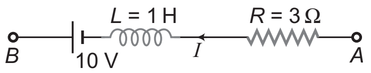
14. The network shown in the figure is a part of complete circuit. What is the potential difference $V_B - V_A$ when the current I is 5 A and is decreasing at a rate of 10^3 A/s?



(a) 5 V (b) 10 V

(c) 15 V (d) 20 V

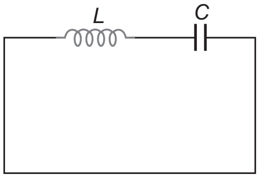
15. In the given branch AB of a circuit a current, $I = (10t + 5)$ A is flowing, where t is time in second. At $t = 0$, the potential difference between points A and $B (V_A - V_B)$ is



(a) 15 V (b) – 5 V

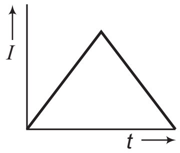
(c) – 15 V (d) 5 V

16. In an LC circuit, the capacitor has maximum charge q_0. The value of $\left(\frac{dI}{dt}\right)_{\max}$ is

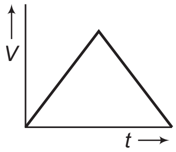


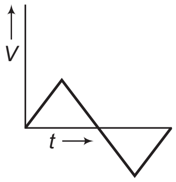
(a) $\frac{q_0}{LC}$
(b) $\frac{q_0}{\sqrt{LC}}$
(c) $\frac{q_0}{LC} - 1$
(d) $\frac{q_0}{LC} + 1$

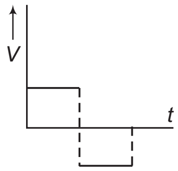
17. An alternating current I in an inductance coil varies with time t according to the graph as shown :

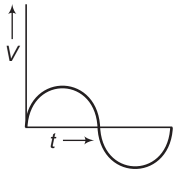


Which one of the following graphs gives the variation of voltage with time?

(a)


(b)


(c)


(d)


18. A loop of area 1 m^2 is placed in a magnetic field $B = 2$T, such that plane of the loop is parallel to the magnetic field. If the loop is rotated by 180°, the amount of net charge passing through any point of loop, if its resistance is 10 Ω, is

(a) 0.4 C
(b) 0.2 C
(c) 0.8 C
(d) 0 C

19. A rectangular loop of sides a and b is placed in xy-plane. A uniform but time varying magnetic field of strength $\mathbf{B} = 20\,t\hat{\mathbf{i}} + 10\,t^2\hat{\mathbf{j}} + 50\hat{\mathbf{k}}$ is present in the region. The magnitude of induced emf in the loop at time t is

(a) $20 + 20\,t$
(b) 20
(c) $20\,t$
(d) zero

20. The armature of a DC motor has 20 Ω resistance. It draws a current of 1.5 A when run by 200 V DC supply. The value of back emf induced in it will be

(a) 150 V
(b) 170 V
(c) 180 V
(d) 190 V

21. In a transformer, the output current and voltage are respectively 4 A and 20 V. If the ratio of number of turns in the primary to secondary is 2 : 1, what is the input current and voltage?

(a) 2 A and 40 V
(b) 8 A and 10 V
(c) 4 A and 10 V
(d) 8 A and 40 V

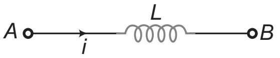
22. When a loop moves towards a stationary magnet with speed v, the induced emf in the loop is E. If the magnet also moves away from the loop with the same speed, then the emf induced in the loop is

(a) E
(b) $2E$
(c) $\frac{E}{2}$
(d) zero

23. A short magnet is allowed to fall from rest along the axis of a horizontal conducting ring. The distance fallen by the magnet in one second may be

(a) 5 m (b) 6 m
(c) 4 m (d) None of these

24. In figure, if the current i decreases at a rate α, then $V_A - V_B$ is



(a) zero (b) $-\alpha L$
(c) αL (d) No relation exists

25. A coil has an inductance of 50 mH and a resistance of 0.3 Ω. If a 12 V emf is applied across the coil, the energy stored in the magnetic field after the current has built up to its steady state value is

(a) 40 J (b) 40 mJ
(c) 20 J (d) 20 mJ

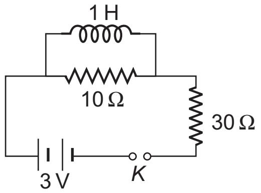
26. A constant voltage is applied to a series R-L circuit by closing the switch. The voltage across inductor ($L = 2$ H) is 20 V at $t = 0$ and drops to 5 V at 20 ms. The value of R in Ω is

(a) $100 \ln 2\Omega$ (b) $100\,(1 - \ln 2)\,\Omega$
(c) $100 \ln 4\Omega$ (d) $100(1 - \ln 4)$

27. A coil of area 10 cm^2 and 10 turns is in magnetic field directed perpendicular to the plane and changing at a rate of 10^8 gauss/s. The resistance of coil is 20 Ω. The current in the coil will be

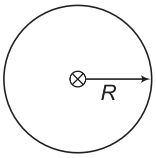
(a) 0.5 A (b) 5×10^{-3} A (c) 0.05 A (d) 5 A

28. In figure, final value of current in 10 Ω resistor, when plug of key K is inserted is



(a) $\frac{3}{10}$ A (b) $\frac{3}{20}$ A (c) $\frac{3}{11}$ A (d) zero

29. A circuit consists of a circular loop of radius R kept in the plane of paper and an infinitely long current carrying wire kept perpendicular to the plane of paper and passing through the centre of loop. The mutual inductance of wire and loop will be

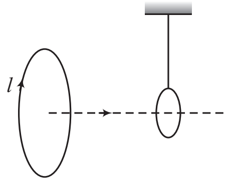


(a) $\frac{\mu_0 \pi R}{2}$ (b) 0
(c) $\mu_0 \pi R^2$ (d) $\frac{\mu_0 R^2}{2}$

30. A flat circular coil of n turns, area A and resistance R is placed in a uniform magnetic field B. The plane of coil is initially perpendicular to B. When the coil is rotated through an angle of 180° about one of its diameter, a charge Q_1 flows through the coil. When the same coil after being brought to its initial position, is rotated through an angle of 360° about the same axis a charge Q_2 flows through it. Then, Q_2/Q_1 is

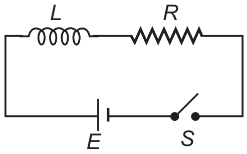
(a) 1 (b) 2 (c) 1/2 (d) 0

31. A small circular loop is suspended from an insulating thread. Another coaxial circular loop carrying a current I and having radius much larger than the first loop starts moving towards the smaller loop. The smaller loop will



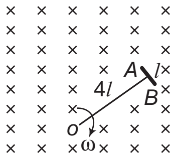
(a) be attracted towards the bigger loop
(b) be repelled by the bigger loop
(c) experience no force
(d) All of the above

32. In the circuit shown in figure, $L = 10\,\text{H}, R = 5\,\Omega, E = 15$ V. The switch S is closed at $t = 0$. At $t = 2$ s, the current in the circuit is



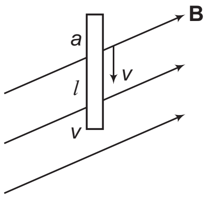
(a) $3\left(1-\frac{1}{e}\right)$ A
(b) $3\left(1-\frac{1}{e^2}\right)$ A
(c) $3\left(\frac{1}{e}\right)$ A
(d) $3\left(\frac{1}{e^2}\right)$ A

33. In the figure shown, a T-shaped conductor moves with constant angular velocity ω in a plane perpendicular to uniform magnetic field **B**. The potential difference $V_A - V_B$ is



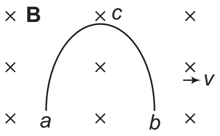
(a) zero
(b) $\frac{1}{2}B\omega l^2$
(c) $2B\omega l^2$
(d) $B\omega l^2$

34. A conducting rod of length l falls vertically under gravity in a region of uniform magnetic field **B**. The field vectors are inclined at an angle θ with the horizontal as shown in figure. If the instantaneous velocity of the rod is v, the induced emf in the rod ab is



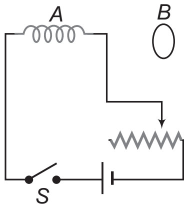
(a) Blv
(b) $Blv \cos\theta$
(c) $Blv \sin\theta$
(d) zero

35. A semi-circular conducting ring acb of radius R moves with constant speed v in a plane perpendicular to uniform magnetic field B as shown in figure. Identify the correct statement.



(a) $V_a - V_c = BRv$
(b) $V_b - V_c = BRv$
(c) $V_a - V_b = 0$
(d) None of these

36. The ring B is coaxial with a solenoid A as shown in figure. As the switch S is closed at $t = 0$, the ring B

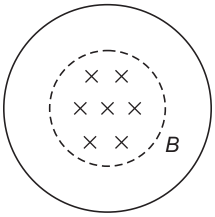


(a) is attracted towards A
(b) is repelled by A
(c) is initially repelled and then attracted
(d) is initially attracted and then repelled

37. If the instantaneous magnetic flux and induced emf produced in a coil is ϕ and E respectively, then according to Faraday's law of electromagnetic induction

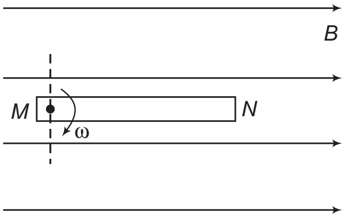
(a) E must be zero if $\phi = 0$
(b) $E \neq 0$ if $\phi = 0$
(c) $E \neq 0$ but ϕ may or may not be zero
(d) $E = 0$ then ϕ must be zero

38. The figure shows a conducting ring of radius R. A uniform steady magnetic field B lies perpendicular to the plane of the ring in a circular region of radius r ($< R$). If the resistance per unit length of the ring is λ, then the current induced in the ring when its radius gets doubled is



(a) $\frac{BR}{\lambda}$
(b) $\frac{2BR}{\lambda}$
(c) zero
(d) $\frac{Br^2}{4R\lambda}$

39. A metallic rod of length l is hinged at the point M and is rotating about an axis perpendicular to the plane of paper with a constant angular velocity ω. A uniform magnetic field of intensity B is acting in the region (as shown in the figure) parallel to the plane of paper. The potential difference between the points M and N



(a) is always zero
(b) varies between $\frac{1}{2}B\omega l^2$ to 0
(c) is always $\frac{1}{2}B\omega l^2$
(d) is always $B\omega l^2$

Subjective Questions

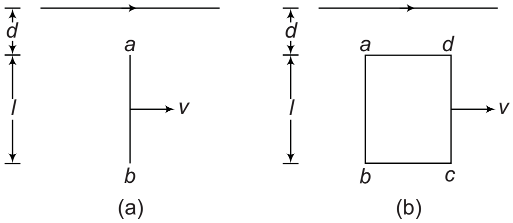
Note *You can take approximations in the answers.*

1. An inductor is connected to a battery through a switch. The emf induced in the inductor is much larger when the switch is opened as compared to the emf induced when the switch is closed. Is this statement true or false?

2. A coil formed by wrapping 50 turns of wire in the shape of a square is positioned in a magnetic field so that the normal to the plane of the coil makes an angle of 30°, with the direction of the field. When the magnetic field is increased uniformly from 200 μT to 600 μT in 0.4 s, an emf of magnitude 80.0 mV is induced in the coil. What is the total length of the wire?

3. A loop of wire enclosing an area S is placed in a region where the magnetic field is perpendicular to the plane. The magnetic field **B** varies with time according to the expression $B = B_0\, e^{-at}$ where a is some constant. That is, at $t = 0$. The field is B_0 and for $t > 0$, the field decreases exponentially. Find the induced emf in the loop as a function of time.

4. The long straight wire in figure (a) carries a constant current i. A metal bar of length l is moving at constant velocity **v** as shown in figure. Point a is a distance d from the wire.

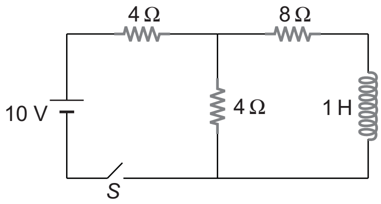


(a) Calculate the emf induced in the bar.

(b) Which point a or b is at higher potential?

(c) If the bar is replaced by a rectangular wire loop of resistance R, what is the magnitude of current induced in the loop?

5. The switch in figure is closed at time $t = 0$. Find the current in the inductor and the current through the switch as functions of time thereafter.

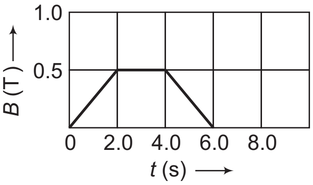


6. A small coil is introduced between the poles of an electromagnet so that its axis coincides with the magnetic field direction. The cross-sectional area of the coil is equal to $S = 3.0\ \text{mm}^2$, the number of turns is $N = 60$. When the coil turns through 180° about its diameter, a galvanometer connected to the coil indicates a charge $q = 4.5\ \mu\text{C}$ flowing through it. Find the magnetic induction magnitude between the poles, provided the total resistance of the electric circuit equals $R = 40\ \Omega$.

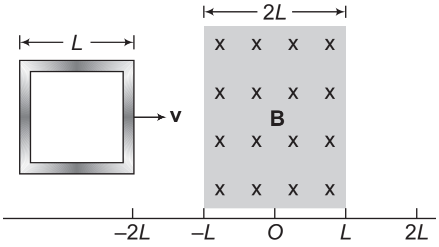
7. The magnetic field through a single loop of wire, 12 cm in radius and of 8.5 Ω resistance, changes with time as shown in figure. Calculate the emf in the loop as a function of time. Consider the time intervals

(a) $t = 0$ to $t = 2.0$ s (b) $t = 2.0$ s to $t = 4.0$ s (c) $t = 4.0$ s to $t = 6.0$ s.

The magnetic field is perpendicular to the plane of the loop.



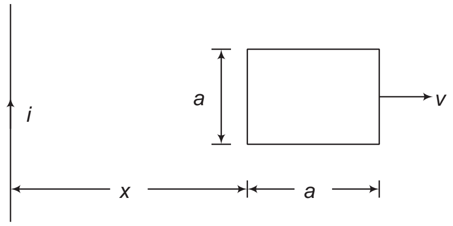
8. A square loop of wire with resistance R is moved at constant speed v across a uniform magnetic field confined to a square region whose sides are twice the lengths of those of the square loop.



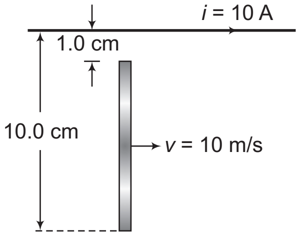
(a) Sketch a graph of the external force F needed to move the loop at constant speed, as a function of the coordinate x, from $x = -2L$ to $x = +2L$. (The coordinate x is measured from the centre of the magnetic field region to the centre of the loop. It is negative when the centre of the loop is to the left of the centre of the magnetic field region. Take positive force to be to the right).

(b) Sketch a graph of the induced current in the loop as a function of x. Take counterclockwise currents to be positive.

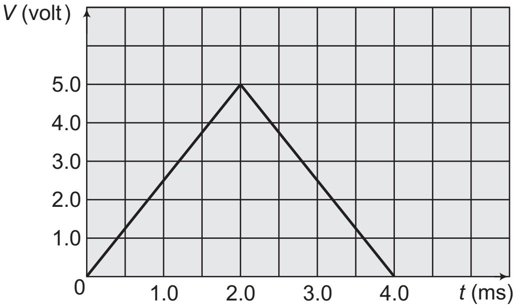
9. A square frame with side a and a long straight wire carrying a current i are located in the same plane as shown in figure. The frame translates to the right with a constant velocity v. Find the emf induced in the frame as a function of distance x.



10. In figure, a wire perpendicular to a long straight wire is moving parallel to the later with a speed $v = 10$ m/s in the direction of the current flowing in the later. The current is 10 A. What is the magnitude of the potential difference between the ends of the moving wire?



11. The potential difference across a 150 mH inductor as a function of time is shown in figure. Assume that the initial value of the current in the inductor is zero. What is the current when $t = 2.0$ ms? and $t = 4.0$ ms ?



12. At the instant when the current in an inductor is increasing at a rate of 0.0640 A/s, the magnitude of the self-induced emf is 0.0160 V.

(a) What is the inductance of the inductor?

(b) If the inductor is a solenoid with 400 turns, what is the average magnetic flux through each turn when the current is 0.720 A?

13. Two toroidal solenoids are wound around the same pipe so that the magnetic field of one passes through the turns of the other. Solenoid 1 has 700 turns and solenoid 2 has 400 turns. When the current in solenoid 1 is 6.52 A, the average flux through each turn of solenoid 2 is 0.0320 Wb.

(a) What is the mutual inductance of the pair of solenoids?

(b) When the current in solenoid 2 is 2.54 A, what is the average flux through each turn of solenoid 1?

14. A coil of inductance 1 H and resistance 10 Ω is connected to a resistanceless battery of emf 50 V at time $t = 0$. Calculate the ratio of the rate at which magnetic energy is stored in the coil to the rate at which energy is supplied by the battery at $t = 0.1$ s.

15. A 3.56 H inductor is placed in series with a 12.8 Ω resistor. An emf of 3.24 V is then suddenly applied across the *RL* combination.

(a) At 0.278 s after the emf is applied what is the rate at which energy is being delivered by the battery?

(b) At 0.278 s, at what rate is energy appearing as thermal energy in the resistor?

(c) At 0.278 s, at what rate is energy being stored in the magnetic field?

16. A 35.0 V battery with negligible internal resistance, a 50.0 Ω resistor, and a 1.25 mH inductor with negligible resistance are all connected in series with an open switch. The switch is suddenly closed

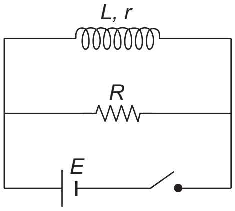
(a) How long after closing the switch will the current through the inductor reach one-half of its maximum value?

(b) How long after closing the switch will the energy stored in the inductor reach one-half of its maximum value?

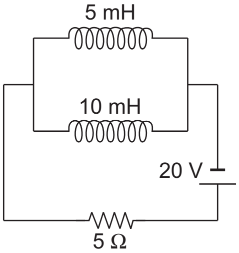
17. A solenoid of inductance L with resistance r is connected in parallel to a resistance R. A battery of emf E and of negligible internal resistance is connected across the parallel combination as shown in the figure. At time $t = 0$, switch S is opened, calculate

(a) current through the solenoid after the switch is opened.

(b) amount of heat generated in the solenoid



18. In the given circuit, find the current through the 5 mH inductor in steady state.

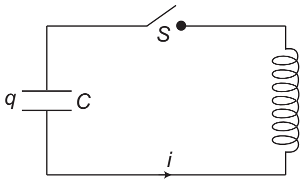


19. In an oscillating L-C circuit in which $C = 4.00\,\mu\text{F}$, the maximum potential difference across the capacitor during the oscillations is 1.50 V and the maximum current through the inductor is 50.0 mA.

(a) What is the inductance L?

(b) What is the frequency of the oscillations?

(c) How much time does the charge on the capacitor take to rise from zero to its maximum value?



20. In the L-C circuit shown, $C = 1\,\mu\text{F}$. With capacitor charged to 100 V, switch S is suddenly closed at time $t = 0$. The circuit then oscillates at 10^3 Hz.

(a) Calculate ω and T

(b) Express q as a function of time

(c) Calculate L

(d) Calculate the average current during the first quarter-cycle.

21. An L-C circuit consists of an inductor with $L = 0.0900$ H and a capacitor of $C = 4 \times 10^{-4}$ F. The initial charge on the capacitor is $5.00\,\mu\text{C}$, and the initial current in the inductor is zero.

(a) What is the maximum voltage across the capacitor?

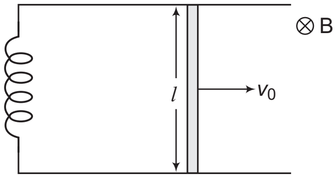
(b) What is the maximum current in the inductor?

(c) What is the maximum energy stored in the inductor?

(d) When the current in the inductor has half its maximum value, what is the charge on the capacitor and what is the energy stored in the inductor?

LEVEL 2

Single Correct Option

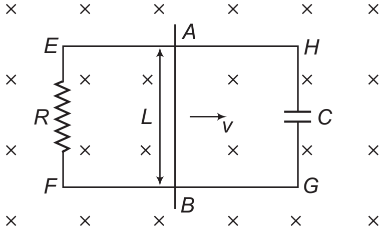


1. Two ends of an inductor of inductance L are connected to two parallel conducting wires. A rod of length l and mass m is given velocity v_0 as shown. The whole system is placed in perpendicular magnetic field B. Find the maximum current in the inductor. (Neglect gravity and friction)

(a) $\dfrac{mv_0}{L}$ (b) $\sqrt{\dfrac{m}{L}}v_0$

(c) $\dfrac{mv_0^2}{L}$ (d) None of these

2. A conducting rod is moving with a constant velocity v over the parallel conducting rails which are connected at the ends through a resistor R and capacitor C as shown in the figure. Magnetic field B is into the plane. Consider the following statements.



(i) Current in loop $AEFBA$ is anti-clockwise (ii) Current in loop $AEFBA$ is clockwise

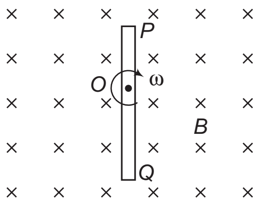
(iii) Current through the capacitor is zero (iv) Energy stored in the capacitor is $\dfrac{1}{2}CB^2L^2v^2$

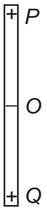
Which of the following options is correct?

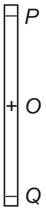
(a) Statements (i) and (iii) are correct (b) Statements (ii) and (iv) are correct

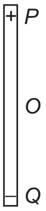
(c) Statements (i), (iii) and (iv) are correct (d) None of these

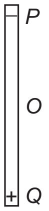
3. A rod is rotating with a constant angular velocity ω about point O (its centre) in a magnetic field B as shown. Which of the following figure correctly shows the distribution of charge inside the rod?



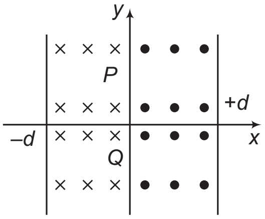
(a)


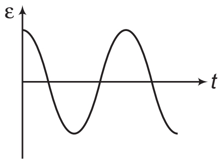
(b)


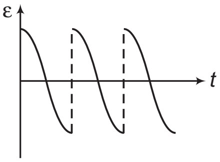
(c)


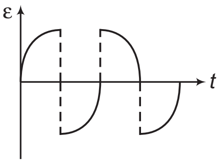
(d)


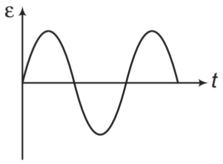
4. A straight conducting rod PQ is executing SHM in xy-plane from $x = -d$ to $x = +d$. Its mean position is $x = 0$ and its length is along y-axis. There exists a uniform magnetic field B from $x = -d$ to $x = 0$ pointing inward normal to the paper and from $x = 0$ to $= +d$ there exists another uniform magnetic field of same magnitude B but pointing outward normal to the plane of the paper. At the instant $t = 0$, the rod is at $x = 0$ and moving to the right. The induced emf (ε) across the rod PQ *vs* time (t) graph will be



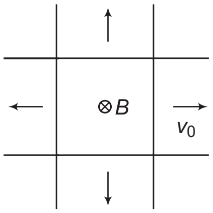
(a)


(b)


(c)


(d)


5. Two parallel long straight conductors lie on a smooth plane surface. Two other parallel conductors rest on them at right angles so as to form a square of side a. A uniform magnetic field B exists at right angles to the plane containing the conductors. Now, conductors start moving outward with a constant velocity v_0 at $t = 0$. Then, induced current in the loop at any time t is (λ is resistance per unit length of the conductors)



(a) $\dfrac{aBv_0}{\lambda\,(a + v_0 t)}$

(b) $\dfrac{aBv_0}{2\lambda}$

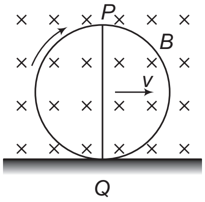
(c) $\dfrac{Bv_0}{\lambda}$

(d) $\dfrac{Bv_0}{2\lambda}$

6. A conducting square loop is placed in a magnetic field B with its plane perpendicular to the field. Now the sides of the loop start shrinking at a constant rate α. The induced emf in the loop at an instant when its side is a, is

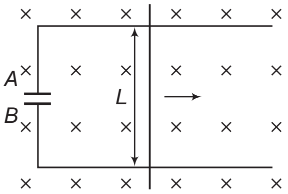
(a) $2a\alpha B$ (b) $a^2\alpha B$ (c) $2a^2\alpha B$ (d) $a\alpha B$

7. A conducting straight wire PQ of length l is fixed along a diameter of a non-conducting ring as shown in the figure. The ring is given a pure rolling motion on a horizontal surface such that its centre of mass has a velocity v. There exists a uniform horizontal magnetic field B in horizontal direction perpendicular to the plane of ring. The magnitude of induced emf in the wire PQ at the position shown in the figure will be



(a) Bvl (b) $2Bvl$ (c) $3Bvl/2$ (d) zero

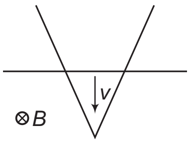
8. A conducting rod of length $L = 0.1$ m is moving with a uniform speed $v = 0.2$ m/s on conducting rails in a magnetic field $B = 0.5$ T as shown. On one side, the end of the rails is connected to a capacitor of capacitance $C = 20\,\mu$F. Then, the charges on the capacitor's plates are



(a) $q_A = 0 = q_B$ (b) $q_A = +20\,\mu\text{C}$ and $q_B = -20\,\mu\text{C}$

(c) $q_A = +0.2\,\mu\text{C}$ and $q_B = -0.2\,\mu\text{C}$ (d) $q_A = -0.2\,\text{C}$ and $q_B = -0.2\,\mu\text{C}$

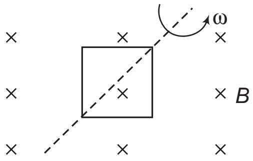
9. A wire is bent in the form of a V shape and placed in a horizontal plane. There exists a uniform magnetic field B perpendicular to the plane of the wire. A uniform conducting rod starts sliding over the V shaped wire with a constant speed v as shown in the figure. If the wire has no resistance, the current in rod will



(a) increase with time (b) decrease with time

(c) remain constant (d) always be zero

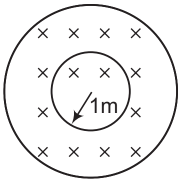
10. A square loop of side b is rotated in a constant magnetic field B at angular frequency ω as shown in the figure. What is the emf induced in it?



(a) $b^2B\omega \sin \omega t$ (b) $bB\omega \sin^2 \omega t$

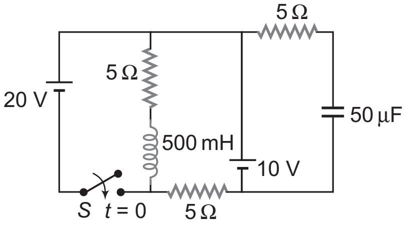
(c) $bB^2\omega \cos \omega t$ (d) $b^2B\omega$

11. A uniform but time varying magnetic field exists in a cylindrical region as shown in the figure. The direction of magnetic field is into the plane of the paper and its magnitude is decreasing at a constant rate of 2×10^{-3} T/s. A particle of charge 1 μC is moved slowly along a circle of radius 1m by an external force as shown in figure. The plane of the circle lies in the plane of the paper and it is concentric with the cylindrical region. The work done by the external force in moving this charge along the circle will be



(a) zero
(b) $2\pi \times 10^{-9}$ J
(c) $\pi \times 10^{-9}$ J
(d) $4\pi \times 10^{-6}$ J

12. Switch S is closed at $t = 0$, in the circuit shown. The change in flux in the inductor ($L = 500$ mH) from $t = 0$ to an instant when it reaches steady state is



(a) 2 Wb
(b) 1.5 Wb
(c) 0 Wb
(d) None of these

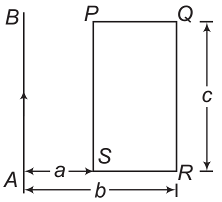
13. An L-R circuit is connected to a battery at time $t = 0$. The energy stored in the inductor reaches half its maximum value at time

(a) $\frac{R}{L} \ln\left[\frac{\sqrt{2}}{\sqrt{2}-1}\right]$
(b) $\frac{L}{R} \ln\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]$
(c) $\frac{L}{R} \ln\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)$
(d) $\frac{R}{L} \ln\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]$

14. Electric charge q is distributed uniformly over a rod of length l. The rod is placed parallel to a long wire carrying a current i. The separation between the rod and the wire is a. The force needed to move the rod along its length with a uniform velocity v is

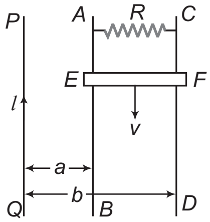
(a) $\frac{\mu_0 iqv}{2\pi a}$
(b) $\frac{\mu_0 iqv}{4\pi a}$
(c) $\frac{\mu_0 iqvl}{2\pi a}$
(d) $\frac{\mu_0 iqvl}{4\pi a}$

15. AB is an infinitely long wire placed in the plane of rectangular coil of dimensions as shown in the figure. Calculate the mutual inductance of wire AB and coil $PQRS$



(a) $\frac{\mu_0 b}{2\pi} \ln \frac{a}{b}$
(b) $\frac{\mu_0 c}{2\pi} \ln \frac{b}{a}$
(c) $\frac{\mu_0 abc}{2\pi (b-a)^2}$
(d) None of these

16. *PQ* is an infinite current carrying conductor. *AB* and *CD* are smooth conducting rods on which a conductor *EF* moves with constant velocity *v* as shown. The force needed to maintain constant speed of *EF* is



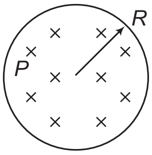
(a) $\frac{1}{vR}\left[\frac{\mu_0 Iv}{2\pi}\ln\frac{(b)}{(a)}\right]^2$

(b) $\frac{v}{R}\left[\frac{\mu_0 Iv}{2\pi}\ln\frac{(a)}{(b)}\right]^2$

(c) $\frac{v}{R}\left[\frac{\mu_0 Iv}{2\pi}\ln\frac{(b)}{(a)}\right]^2$

(d) None of these

17. The figure shows a circular region of radius R occupied by a time varying magnetic field $\mathbf{B}(t)$ such that $\frac{dB}{dt} < 0$. The magnitude of induced electric field at the point P at a distance $r < R$ is



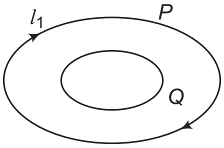
(a) decreasing with r

(b) increasing with r

(c) not varying with r

(d) varying as r^{-2}

18. Two circular loops P and Q are concentric and coplanar as shown in figure. The loop Q is smaller than P. If the current I_1 flowing in loop P is decreasing with time, then the current I_2 in the loop Q



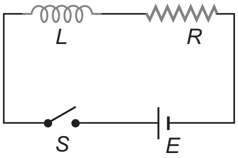
(a) flows in the same direction as that of P

(b) flows in the opposite direction as that of Q

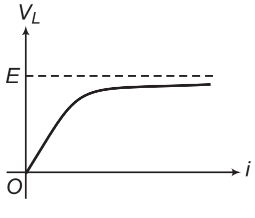
(c) is zero

(d) None of the above

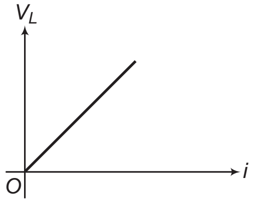
19. In the circuit shown in figure, the switch S is closed at $t = 0$. If V_L is the voltage induced across the inductor and i is the instantaneous current, the correct variation of V_L *versus* i is given by



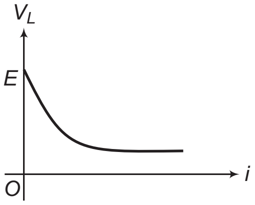
(a)



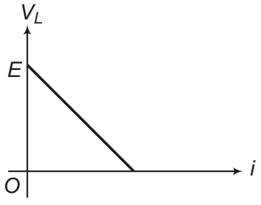
(b)



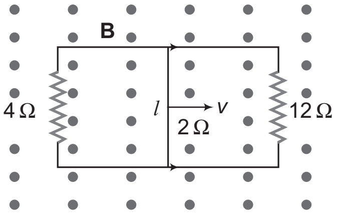
(c)



(d)

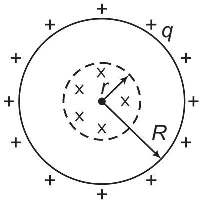


20. In the figure shown, a uniform magnetic field $|\mathbf{B}| = 0.5$ T is perpendicular to the plane of circuit. The sliding rod of length $l = 0.25$ m moves uniformly with constant speed $v = 4\ \text{ms}^{-1}$. If the resistance of the slides is 2 Ω, then the current flowing through the sliding rod is



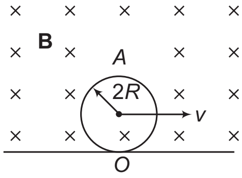
(a) 0.1 A
(b) 0.17 A
(c) 0.08 A
(d) 0.03 A

21. The figure shows a non-conducting ring of radius R carrying a charge q. In a circular region of radius r, a uniform magnetic field $\mathbf{B}$ perpendicular to the plane of the ring varies at a constant rate $\frac{dB}{dt} = \beta$. The torque acting on the ring is



(a) $\frac{1}{2}qr^2\beta$
(b) $\frac{1}{2}qR^2\beta$
(c) $qr^2\beta$
(d) zero

22. A conducting ring of radius $2R$ rolls on a smooth horizontal conducting surface as shown in figure. A uniform horizontal magnetic field B is perpendicular to the plane of the ring. The potential of A with respect to O is



(a) $2\,BvR$
(b) $\frac{1}{2}\,BvR$
(c) $8\,BvR$
(d) $4\,BvR$

23. A uniformly wound long solenoid of inductance L and resistance R is cut into two parts in the ratio $\eta:1$, which are then connected in parallel. The combination is then connected to a cell of emf E. The time constant of the circuit is

(a) $\frac{L}{R}$
(b) $\frac{L}{(\eta+1)R}$
(c) $\left(\frac{\eta}{\eta+1}\right)\frac{L}{R}$
(d) $\left(\frac{\eta+1}{\eta}\right)\frac{L}{R}$

24. When a choke coil carrying a steady current is short-circuited, the current in it decreases to $\beta\,(< 1)$ times its initial value in a time T. The time constant of the choke coil is

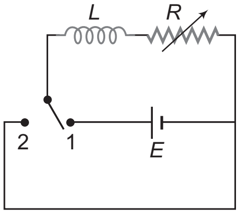
(a) $\dfrac{T}{\beta}$ (b) $\dfrac{T}{\ln\left(\dfrac{1}{\beta}\right)}$ (c) $\dfrac{T}{\ln \beta}$ (d) $T \ln \beta$

25. In the steady state condition, the rate of heat produced in a choke coil is P. The time constant of the choke coil is τ. If now the choke coil is short-circuited, then the total heat dissipated in the coil is

(a) $P\tau$ (b) $\dfrac{1}{2}P\tau$

(c) $\dfrac{P\tau}{\ln 2}$ (d) $P\tau \ln 2$

26. In the circuit shown in figure initially the switch is in position 1 for a long time, then suddenly at $t = 0$, the switch is shifted to position 2. It is required that a constant current should flow in the circuit, the value of resistance R in the circuit



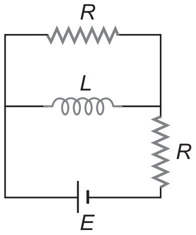
(a) should be decreased at a constant rate

(b) should be increased at a constant rate

(c) should be maintained constant

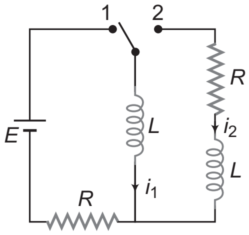
(d) Not possible

27. The figure shows an L-R circuit, the time constant for the circuit is



(a) $\dfrac{L}{2R}$ (b) $\dfrac{2L}{R}$ (c) $\dfrac{2R}{L}$ (d) $\dfrac{R}{2L}$

28. In figure, the switch is in the position 1 for a long time, then the switch is shifted to position 2 at $t = 0$. At this instant the value of i_1 and i_2 are



(a) $\dfrac{E}{R}, 0$ (b) $\dfrac{E}{R}, \dfrac{-E}{R}$

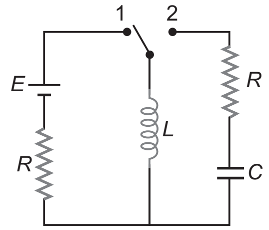
(c) $\dfrac{E}{2R}, \dfrac{-E}{2R}$ (d) None of these

29. In a decaying L-R circuit, the time after which energy stored in the inductor reduces to one-fourth of its initial value is

(a) $(\ln 2)\dfrac{L}{R}$ (b) $0.5\dfrac{L}{R}$

(c) $\sqrt{2}\dfrac{L}{R}$ (d) $\left(\dfrac{\sqrt{2}}{\sqrt{2}-1}\right)\dfrac{L}{R}$

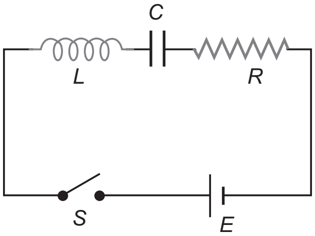
30. Initially, the switch is in position 1 for a long time and then shifted to position 2 at $t = 0$ as shown in figure. Just after closing the switch, the magnitude of current through the capacitor is



(a) zero (b) $\dfrac{E}{2R}$

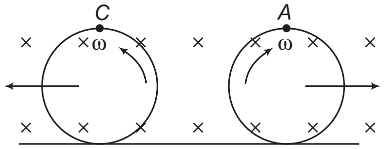
(c) $\dfrac{E}{R}$ (d) None of these

31. When the switch S is closed at $t = 0$, identify the correct statement just after closing the switch as shown in figure



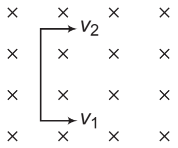
(a) The current in the circuit is maximum
(b) Equal and opposite voltages are dropped across inductor and resistor
(c) The entire voltage is dropped across inductor
(d) All of the above

32. Two metallic rings of radius R are rolling on a metallic rod. A magnetic field of magnitude B is applied in the region. The magnitude of potential difference between points A and C on the two rings (as shown), will be



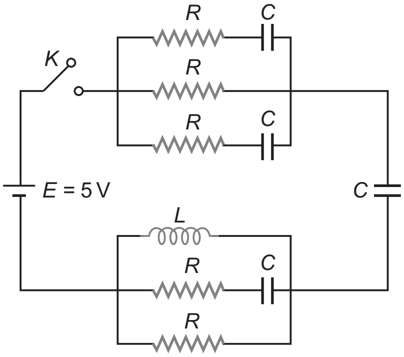
(a) 0 (b) $4\,B\omega R^2$
(c) $8\,B\omega R^2$ (d) $2\,B\omega R^2$

33. In the figure, magnetic field points into the plane of paper and the conducting rod of length l is moving in this field such that the lowest point has a velocity v_1 and the topmost point has the velocity $v_2(v_2 > v_1)$. The emf induced is given by



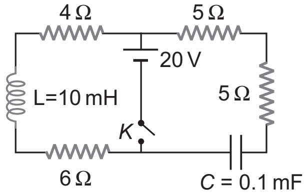
(a) $Bv_1 l$
(b) $Bv_2 l$
(c) $\frac{1}{2}B(v_2 + v_1)l$
(d) $\frac{1}{2}B(v_2 - v_1)l$

34. Find the current passing through battery immediately after key (K) is closed. It is given that initially all the capacitors are uncharged. (Given that $R = 6\,\Omega$ and $C = 4\,\mu\text{F}$)



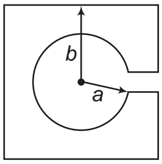
(a) 1 A
(b) 5 A
(c) 3 A
(d) 2 A

35. In the circuit shown, the key (K) is closed at $t = 0$, the current through the key at the instant $t = 10^{-3}\ln 2$, is



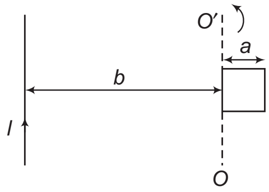
(a) 2 A
(b) 8 A
(c) 4 A
(d) zero

36. A loop shown in the figure is immersed in the varying magnetic field $B = B_0 t$, directed into the page. If the total resistance of the loop is R, then the direction and magnitude of induced current in the inner circle is



(a) clockwise $\frac{B_0(\pi a^2 - b^2)}{R}$
(b) anti-clockwise $\frac{B_0\pi\,(a^2 + b^2)}{R}$
(c) clockwise $\frac{B_0(\pi a^2 + 4b^2)}{R}$
(d) clockwise $\frac{B_0(4b^2 - \pi a^2)}{R}$

37. A square loop of side a and a straight long wire are placed in the same plane as shown in figure. The loop has a resistance R and inductance L. The frame is turned through 180° about the axis OO'. What is the electric charge that flows through the loop?



(a) $\dfrac{\mu_0 Ia}{2\pi R}\ln\left(\dfrac{2a+b}{b}\right)$

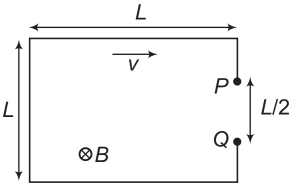
(b) $\dfrac{\mu_0 Ia}{2\pi R}\ln\left(\dfrac{b}{b^2-a^2}\right)$

(c) $\dfrac{\mu_0 Ia}{2\pi R}\ln\left(\dfrac{a+2b}{b}\right)$

(d) None of these

More than One Correct Options

1. The loop shown moves with a velocity v in a uniform magnetic field of magnitude B, directed into the paper. The potential difference between points P and Q is e. Then,



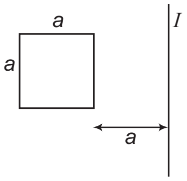
(a) $e=\dfrac{1}{2}BLv$

(b) $e=BLv$

(c) P is positive with respect to Q

(d) Q is positive with respect to P

2. An infinitely long wire is placed near a square loop as shown in figure. Choose the correct options.



(a) The mutual inductance between the two is $\dfrac{\mu_0 a}{2\pi}\ln(2)$

(b) The mutual inductance between the two is $\dfrac{\mu_0 a^2}{2\pi}\ln(2)$

(c) If a constant current is passed in the straight wire in upward direction and loop is brought close to the wire, then induced current in the loop is clockwise

(d) In the above condition, induced current in the loop is anti-clockwise

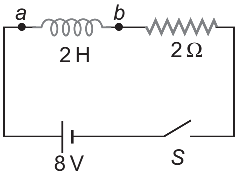
3. Choose the correct options.

(a) SI unit of magnetic flux is henry-ampere

(b) SI unit of coefficient of self-inductance is J/A

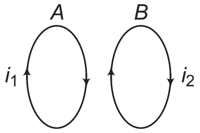
(c) SI unit of coefficient of self-inductance is $\dfrac{\text{volt-second}}{\text{ampere}}$

(d) SI unit of magnetic induction is weber



4. In the circuit shown in figure, circuit is closed at time $t = 0$. At time $t = \ln(2)$ second

(a) rate of energy supplied by the battery is 16 J/s
(b) rate of heat dissipated across resistance is 8 J/s
(c) rate of heat dissipated across resistance is 16 J/s
(d) $V_a - V_b = 4$ V



5. Two circular coils are placed adjacent to each other. Their planes are parallel and currents through them i_1 and i_2 are in same direction. Choose the correct options.

(a) When A is brought near B, current i_2 will decrease
(b) In the above process, current i_2 will increase
(c) When current i_1 is increased, current i_2 will decrease
(d) In the above process, current i_2 will increase

6. A coil of area 2 m^2 and resistance 4 Ω is placed perpendicular to a uniform magnetic field of 4 T. The loop is rotated by 90° in 0.1 second. Choose the correct options.

(a) Average induced emf in the coil is 8 V
(b) Average induced current in the circuit is 20 A
(c) 2 C charge will flow in the coil in above period
(d) Heat produced in the coil in the above period can't be determined from the given data

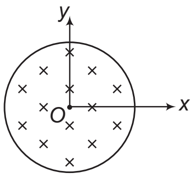
7. In L-C oscillations,

(a) time period of oscillation is $\frac{2\pi}{\sqrt{LC}}$

(b) maximum current in circuit is $\frac{q_0}{\sqrt{LC}}$

(c) maximum rate of change of current in circuit is $\frac{q_0}{LC}$

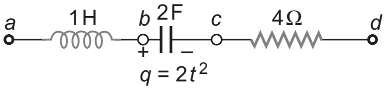
(d) maximum potential difference across the inductor is $\frac{q_0}{2C}$. Here, q_0 is maximum charge on capacitor



8. Magnetic field in a cylindrical region of radius R in inward direction is as shown in figure.

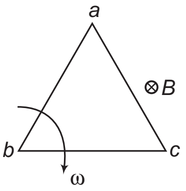
(a) an electron will experience no force kept at $(2R, 0, 0)$ if magnetic field increases with time
(b) in the above situation, electron will experience the force in negative y-axis
(c) If a proton is kept at $\left(0, \frac{R}{2}, 0\right)$ and magnetic field is decreasing, then it will experience the force in positive x-direction
(d) if a proton is kept at $(-R, 0, 0)$ and magnetic field is increasing, then it will experience force in negative y-axis

9. In the figure shown, q is in coulomb and t in second. At time $t = 1$ s



(a) $V_a - V_b = 4$ V
(b) $V_b - V_c = 1$ V
(c) $V_c - V_d = 16$ V
(d) $V_a - V_d = 20$ V

10. An equilateral triangular conducting frame is rotated with angular velocity ω in a uniform magnetic field B as shown. Side of triangle is l. Choose the correct options.

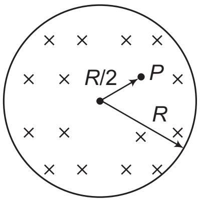


(a) $V_a - V_c = 0$
(b) $V_a - V_c = \frac{B\omega l^2}{2}$
(c) $V_a - V_b = \frac{B\omega l^2}{2}$
(d) $V_c - V_b = -\frac{B\omega l^2}{2}$

Comprehension Based Questions

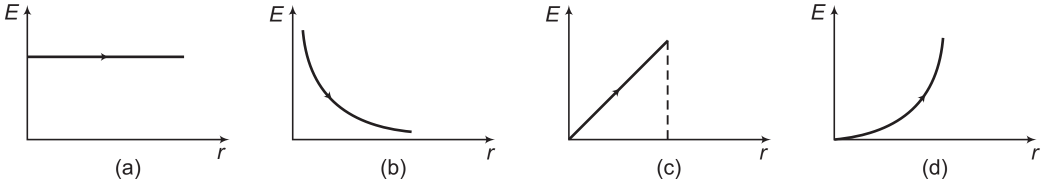
Passage I (Q. No. 1 to 3)

A uniform but time varying magnetic field $B = (2t^3 + 24t)\,T$ *is present in a cylindrical region of radius* $R = 2.5$ *cm as shown in figure.*



1. The force on an electron at P at $t = 2.0$ s is
(a) 96×10^{-21} N
(b) 48×10^{-21} N
(c) 24×10^{-21} N
(d) zero

2. The variation of electric field at any instant as a function of distance measured from the centre of cylinder in first problem is



3. In the previous problem, the direction of circular electric lines at $t = 1$ s is
(a) clockwise
(b) anti-clockwise
(c) no current is induced
(d) cannot be predicted

Passage II (Q. No. 4 to 7)

A thin non-conducting ring of mass m, radius a carrying a charge q can rotate freely about its own axis which is vertical. At the initial moment, the ring was at rest in horizontal position and no magnetic field was present. At instant t = 0, a uniform magnetic field is switched on which is vertically downward and increases with time according to the law $B = B_0 t$. *Neglecting magnetism induced due to rotational motion of ring.*

4. The magnitude of induced emf on the closed surface of ring will be
(a) $\pi a^2 B_0$
(b) $2a^2 B_0$
(c) zero
(d) $\frac{1}{2}\pi a^2 B_0$

5. The magnitude of an electric field on the circumference of the ring is
(a) aB_0
(b) $2aB_0$
(c) $\frac{1}{2}aB_0$
(d) zero

6. Angular acceleration of ring is

(a) $\frac{qB_0}{2m}$ (b) $\frac{qB_0}{4m}$

(c) $\frac{qB_0}{m}$ (d) $\frac{2qB_0}{m}$

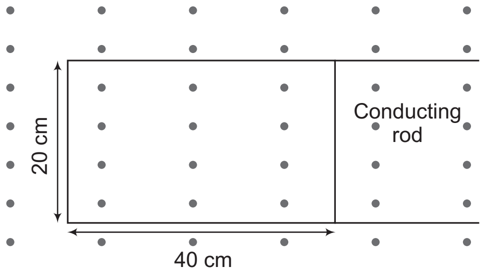
7. Find instantaneous power developed by electric force acting on the ring at $t = 1$ s.

(a) $\frac{2q^2B_0^2a^2}{14m}$ (b) $\frac{q^2B_0^2a^2}{8m}$

(c) $\frac{3q^2B_0^2a^2}{m}$ (d) $\frac{q^2B_0^2a^2}{4m}$

Passage III (Q. No. 8 to 10)

Figure shows a conducting rod of negligible resistance that can slide on smooth U-shaped rail made of wire of resistance $1\,\Omega/m$. *Position of the conducting rod at* $t = 0$ *is shown. A time dependent magnetic field* $B = 2t$ *tesla is switched on at* $t = 0$.



8. The current in the loop at $t = 0$ due to induced emf is

(a) 0.16 A, clockwise (b) 0.08 A, clockwise

(c) 0.16 A, anti-clockwise (d) zero

9. At $t = 0$, when the magnetic field is switched on, the conducting rod is moved to the left at constant speed 5 cm/s by some external means. At $t = 2$ s, net induced emf has magnitude

(a) 0.12 V (b) 0.08 V

(c) 0.04 V (d) 0.02 V

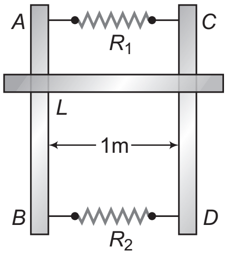
10. The magnitude of the force required to move the conducting rod at constant speed 5 cm/s at the same instant $t = 2$s, is equal to

(a) 0.096 N (b) 0.12 N

(c) 0.08 N (d) 0.064 N

Passage IV (Q. No. 11 to 13)

Two parallel vertical metallic rails AB and CD are separated by $1\ m$. *They are connected at the two ends by resistances* R_1 *and* R_2 *as shown in the figure. A horizontal metallic bar L of mass* $0.2\ kg$ *slides without friction, vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of* $0.6\ T$ *perpendicular to the plane of the rails. It is observed that when the terminal velocity is attained, the powers dissipated in* R_1 *and* R_2 *are* $0.76\ W$ *and* $1.2\ W$ *respectively* $(g = 9.8\ m/s^2)$



11. The terminal velocity of the bar L will be

(a) 2 m/s (b) 3 m/s (c) 1 m/s (d) None of these

12. The value of R_1 is

(a) 0.47 Ω (b) 0.82 Ω

(c) 0.12 Ω (d) None of these

13. The value of R_2 is

(a) 0.6 Ω (b) 0.5 Ω

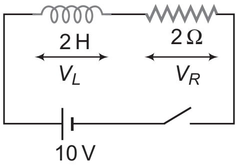
(c) 0.4 Ω (d) 0.3 Ω

Match the Columns

1. Match the following two columns.

Column I	Column II
(a) Magnetic induction	(p) $[MT^{-2}A^{-1}]$
(b) Coefficient of self-induction	(q) $[L^2T^{-2}]$
(c) LC	(r) $[ML^2T^{-2}A^{-2}]$
(d) Magnetic flux	(s) None of these

2. In the circuit shown in figure, switch is closed at time $t = 0$. Match the following two columns.

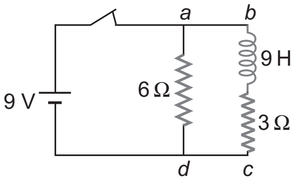


Column I	Column II
(a) V_L at $t = 0$	(p) zero
(b) V_R at $t = 0$	(q) 10 V
(c) V_L at $t = 1$ s	(r) $\frac{10}{e}$ V
(d) V_R at $t = 1$ s	(s) $\left(1 - \frac{1}{e}\right)$ 10 V

3. In an L-C oscillation circuit, $L = 1\,\text{H}$, $C = \frac{1}{4}$ F and maximum charge in the capacitor is 4 C. Match the following two columns. Note that in Column II, all values are in SI units.

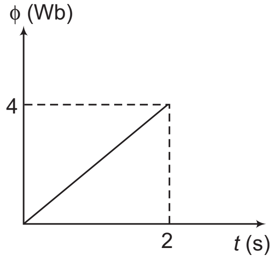
Column I	Column II
(a) Maximum current in the circuit	(p) 16
(b) Maximum rate of change of current in the circuit	(q) 4
(c) Potential difference across inductor when $q = 2C$	(r) 2
(d) Potential difference across capacitor when rate of change of current is half its maximum value	(s) 8

4. In the circuit shown in figure, switch remains closed for long time. It is opened at time $t = 0$. Match the following two columns at $t = (\ln 2)$ second.



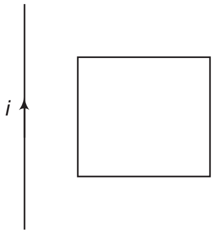
Column I	Column II
(a) Potential differences across inductor	(p) 9 V
(b) Potential difference across 3 Ω resistance	(q) 4.5 V
(c) Potential difference across 6 Ω resistance	(r) 6 V
(d) Potential difference between points *b* and *c*	(s) None of these

5. Magnetic flux passing through a coil of resistance 2 Ω is as shown in figure. Match the following two columns. In Column II all physical quantities are in SI units.



Column I	Column II
(a) Induced emf produced	(p) 4
(b) Induced current	(q) 1
(c) Charge flow in 2 s	(r) 8
(d) Heat generation in 2 s	(s) 2

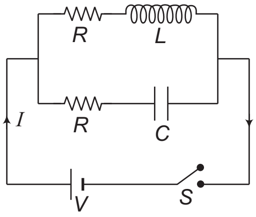
6. A square loop is placed near a long straight current carrying wire as shown. Match the following two columns.



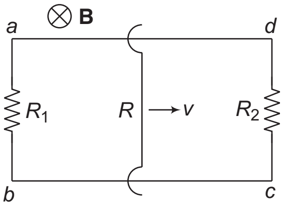
Column I	Column II
(a) If current is increased	(p) induced current in loop is clockwise
(b) If current is decreased	(q) induced current in loop is anti-clockwise
(c) If loop is moved away from the wire	(r) wire will attract the loop
(d) If loop is moved towards the wire	(s) wire will repel the loop

Subjective Questions

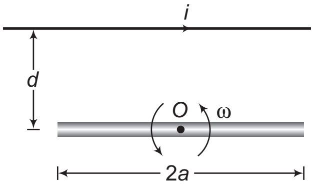
1. In the circuit diagram shown, initially there is no energy in the inductor and the capacitor. The switch is closed at $t = 0$. Find the current I as a function of time if $R = \sqrt{L/C}$.



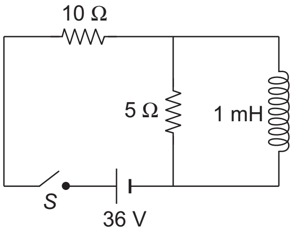
2. A rectangular loop with a sliding connector of length l is located in a uniform magnetic field perpendicular to the loop plane. The magnetic induction is equal to B. The connector has an electric resistance R, the sides ab and cd have resistances R_1 and R_2. Neglecting the self-inductance of the loop, find the current flowing in the connector during its motion with a constant velocity v.



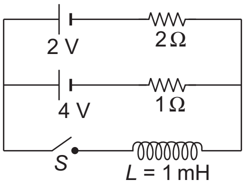
3. A rod of length $2a$ is free to rotate in a vertical plane, about a horizontal axis O passing through its mid-point. A long straight, horizontal wire is in the same plane and is carrying a constant current i as shown in figure. At initial moment of time, the rod is horizontal and starts to rotate with constant angular velocity ω, calculate emf induced in the rod as a function of time.



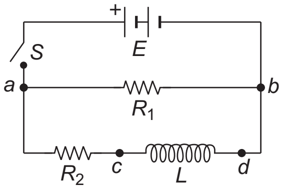
4. In the circuit arrangement shown in figure, the switch S is closed at $t = 0$. Find the current in the inductance as a function of time? Does the current through 10 Ω resistor vary with time or remains constant.



5. In the circuit shown, switch S is closed at time $t = 0$. Find the current through the inductor as a function of time t.



6. In the circuit shown in figure, $E = 120$ V, $R_1 = 30.0\ \Omega$, $R_2 = 50.0\ \Omega$, and $L = 0.200$ H. Switch S is closed at $t = 0$. Just after the switch is closed.

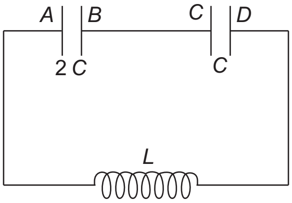


(a) What is the potential difference V_{ab} across the inductor R_1?
(b) Which point, a or b, is at higher potential?
(c) What is the potential difference V_{cd} across the inductor L?
(d) Which point, c or d, is at a higher potential?

The switch is left closed for a long time and then is opened. Just after the switch is opened

(e) What is the potential difference V_{ab} across the resistor R_1?
(f) Which point a or b, is at a higher potential?
(g) What is the potential difference V_{cd} across the inductor L?
(h) Which point, c or d, is at a higher potential?

7. Two capacitors of capacitances $2C$ and C are connected in series with an inductor of inductance L. Initially, capacitors have charge such that $V_B - V_A = 4V_0$ and $V_C - V_D = V_0$. Initial current in the circuit is zero. Find

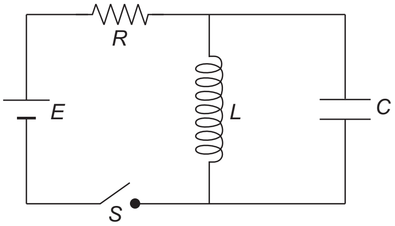


(a) maximum current that will flow in the circuit,
(b) potential difference across each capacitor at that instant,
(c) equation of current flowing towards left in the inductor.

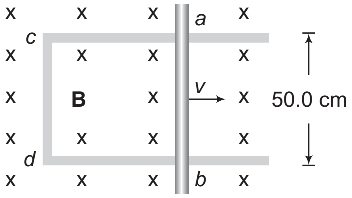
8. A 1.00 mH inductor and a 1.00 μF capacitor are connected in series. The current in the circuit is described by $i = 20\,t$, where t is in second and i is in ampere. The capacitor initially has no charge. Determine

(a) the voltage across the inductor as a function of time,
(b) the voltage across the capacitor as a function of time,
(c) the time when the energy stored in the capacitor first exceeds that in the inductor.

9. In the circuit shown in the figure, $E = 50.0$ V, $R = 250\ \Omega$ and $C = 0.500\ \mu$F. The switch S is closed for a long time, and no voltage is measured across the capacitor. After the switch is opened, the voltage across the capacitor reaches a maximum value of 150 V. What is the inductance L?



10. The conducting rod ab shown in figure makes contact with metal rails ca and db. The apparatus is in a uniform magnetic field 0.800 T, perpendicular to the plane of the figure.



(a) Find the magnitude of the emf induced in the rod when it is moving toward the right with a speed 7.50 m/s.

(b) In what direction does the current flow in the rod?

(c) If the resistance of the circuit $abdc$ is 1.50 Ω (assumed to be constant), find the force (magnitude and direction) required to keep the rod moving to the right with a constant speed of 7.50 m/s. You can ignore friction.

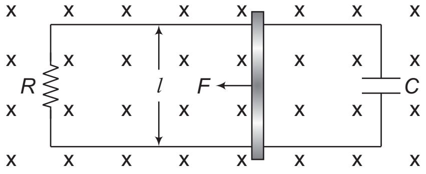
(d) Compare the rate at which mechanical work is done by the force (Fv) with the rate at which thermal energy is developed in the circuit (I^2R).

11. A non-conducting ring of mass m and radius R has a charge Q uniformly distributed over its circumference. The ring is placed on a rough horizontal surface such that plane of the ring is parallel to the surface. A vertical magnetic field $B = B_0 t^2$ tesla is switched on. After 2 s from switching on the magnetic field the ring is just about to rotate about vertical axis through its centre.

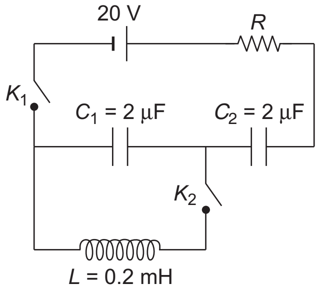
(a) Find friction coefficient μ between the ring and the surface.

(b) If magnetic field is switched off after 4 s, then find the angle rotated by the ring before coming to stop after switching off the magnetic field.

12. Two parallel long smooth conducting rails separated by a distance l are connected by a movable conducting connector of mass m. Terminals of the rails are connected by the resistor R and the capacitor C as shown in figure. A uniform magnetic field B perpendicular to the plane of the rail is switched on. The connector is dragged by a constant force F. Find the speed of the connector as a function of time if the force F is applied at $t = 0$. Also find the terminal velocity of the connector.



13. A circuit containing capacitors C_1 and C_2, shown in the figure is in the steady state with key K_1 closed and K_2 opened. At the instant $t = 0$, K_1 is opened and K_2 is closed.

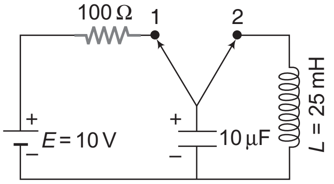


(a) Find the angular frequency of oscillations of L-C circuit.

(b) Determine the first instant t, when energy in the inductor becomes one third of that in the capacitor.

(c) Calculate the charge on the plates of the capacitor at that instant.

14. Initially, the capacitor is charged to a potential of 5 V and then connected to position 1 with the shown polarity for 1 s. After 1 s it is connected across the inductor at position 2.

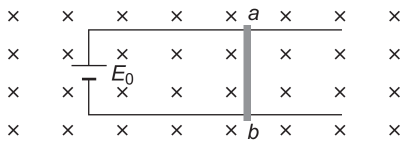


(a) Find the potential across the capacitor after 1 s of its connection to position 1.

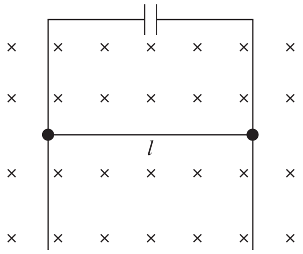
(b) Find the maximum current flowing in the L-C circuit when capacitor is connected across the inductor. Also, find the frequency of LC oscillations.

15. A rod of mass m and resistance R slides on frictionless and resistanceless rails a distance l apart that include a source of *emf* E_0. (see figure). The rod is initially at rest. Find the expression for the

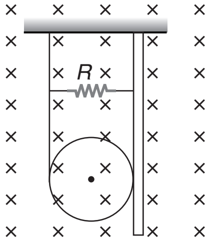
(a) velocity of the rod $v(t)$. (b) current in the loop $i(t)$.



16. Two metal bars are fixed vertically and are connected on the top by a capacitor C. A sliding conductor of length l and mass m slides with its ends in contact with the bars. The arrangement is placed in a uniform horizontal magnetic field directed normal to the plane of the figure. The conductor is released from rest. Find the displacement $x(t)$ of the conductor as a function of time t.



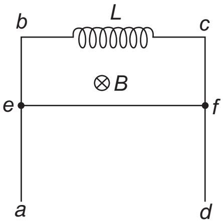
17. A conducting light string is wound on the rim of a metal ring of radius r and mass m. The free end of the string is fixed to the ceiling. A vertical infinite smooth conducting plane is always tangent to the ring as shown in the figure. A uniform magnetic field B is applied perpendicular to the plane of the ring. The ring is always inside the magnetic field. The plane and the string are connected by a resistance R. When the ring is released, find



(a) the current in the resistance R as a function of time.

(b) the terminal velocity of the ring.

18. A conducting frame *abcd* is kept in a vertical plane. A conducting rod *ef* of mass m and length l can slide smoothly on it remaining always horizontal. The resistance of the loop is negligible and inductance is constant having value L. The rod is left from rest and allowed to fall under gravity and inductor has no initial current. A magnetic field of constant magnitude B is present throughout the loop pointing inwards. Determine

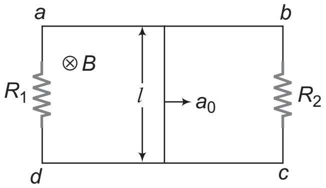


(a) position of the rod as a function of time assuming initial position of the rod to be $x = 0$ and vertically downward as the positive x-axis.

(b) the maximum current in the circuit.

(c) maximum velocity of the rod

19. A rectangular loop with a sliding conductor of length l is located in a uniform magnetic field perpendicular to the plane of loop. The magnetic induction perpendicular to the plane of loop is equal to B. The part *ad* and *bc* has electric resistance R_1 and R_2, respectively. The conductor starts moving with constant acceleration a_0 at time $t = 0$. Neglecting the self-inductance of the loop and resistance of conductor. Find

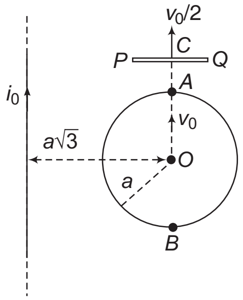


(a) the current through the conductor during its motion.

(b) the polarity of *abcd* terminal.

(c) external force required to move the conductor with the given acceleration.

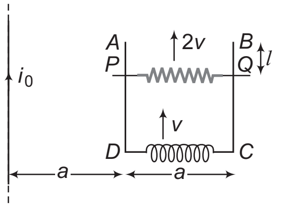
20. A conducting circular loop of radius a and resistance per unit length R is moving with a constant velocity v_0, parallel to an infinite conducting wire carrying current i_0. A conducting rod of length $2a$ is approaching the centre of the loop with a constant velocity $\frac{v_0}{2}$ along the direction of the current. At the instant $t = 0$, the rod comes in contact with the loop at A and starts sliding on the loop with the constant velocity. Neglecting the resistance of the rod and the self-inductance of the circuit, find the following when the rod slides on the loop.



(a) The current through the rod when it is at a distance of $\left(\frac{a}{2}\right)$ from the point A of the loop.

(b) Force required to maintain the velocity of the rod at that instant.

21. U-frame $ABCD$ and a sliding rod PQ of resistance R, start moving with velocities v and $2v$ respectively, parallel to a long wire carrying current i_0. When the distance $AP = l$ at $t = 0$, determine the current through the inductor of inductance L just before connecting rod PQ loses contact with the U-frame.



Answers

Introductory Exercise 27.1

1. Anti-clockwise

2. No

3. $[ML^2A^{-1}T^{-3}]$

4. Clockwise

5. Same direction, opposite direction.

6. 6.74 V

7. 1600 μC

8. 9.0×10^{-7} Wb

Introductory Exercise 27.2

1. 4.4 V, north

2. 0.00375 N

3. (a) $\frac{B\omega l^2}{2}$ (b) $\frac{-3B\omega l^2}{2}$

4. No

Introductory Exercise 27.3

1. $3(t \cos t + \sin t)$

2. $-80e^{-4t}$, $-40e^{-4t}$

3. (a) 0.625 mH (b) 0.13 J, 0.21 J/s

4. (a) 4.5×10^{-5} H (b) 4.5×10^{-3} V

Introductory Exercise 27.4

1. 3.125 mH, 0.9375 V

2. (a) 2 H (b) 30 V (c) 1 H

3. (a) 0.27 V, Yes (b) 0.27 V

Introductory Exercise 27.5

2. (a) 0.2 s (b) 10 A (c) 9.93 A

3. No

Introductory Exercise 27.6

2. With KE as $v \Leftrightarrow i$ and $m \Leftrightarrow L$. Therefore, $\frac{1}{2}mv^2 = \frac{1}{2}Li^2$ **3.** (a) 45.9 μC (b) 23.3 V **4.** 20.0 V

Introductory Exercise 27.7

1. (a) 3.1×10^{-6} V (b) 2.0×10^{-6} V/m

2. (a) 8.0×10^{-21} N (downward and to the right perpendicular to r_2)

(b) 0.36 V/m (upwards and to the left perpendicular to r_1)

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (d) **4.** (a,b) **5.** (a) **6.** (d) **7.** (b) **8.** (c) **9.** (d) **10.** (c)

Objective Questions

1. (c) **2.** (a) **3.** (b) **4.** (c) **5.** (c) **6.** (b) **7.** (b) **8.** (a) **9.** (c) **10.** (b)
11. (a) **12.** (b) **13.** (c) **14.** (c) **15.** (a) **16.** (a) **17.** (c) **18.** (a) **19.** (d) **20.** (b)
21. (a) **22.** (d) **23.** (c) **24.** (b) **25.** (a) **26.** (c) **27.** (d) **28.** (d) **29.** (b) **30.** (d)
31. (b) **32.** (a) **33.** (a) **34.** (d) **35.** (c) **36.** (b) **37.** (c) **38.** (c) **39.** (a)

Subjective Questions

1. True

2. 272 m

3. $e = aSB_0e^{-at}$

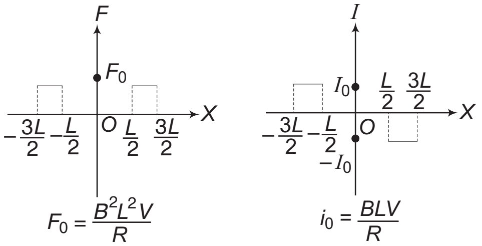
4. (a) $\frac{\mu_0\, iv}{2\pi}\ln\left(1+\frac{l}{d}\right)$ (b) a (c) zero

5. (a) $(0.5)(1 - e^{-10t})A$ (b) $1.50\text{ A} - (0.25\text{ A})e^{-10t}$

6. 0.5 T

7. (a) 0.011 V (b) zero (c) – 0.011 V

8.



$F_0 = \frac{B^2L^2V}{R}$

(a)

$i_0 = \frac{BLV}{R}$

(b)

9. $e = \frac{\mu_0}{4\pi}\frac{2ia^2v}{x(x+a)}$

10. $(2\times10^{-5})\ln(10)$ V

11. 3.33×10^{-2} A, 6.67×10^{-2} A

12. (a) 0.25 H (b) 4.5×10^{-4} Wb

13. (a) 1.96 H (b) 7.12×10^{-3} Wb

14. 0.37

15. (a) 518 mW (b) 328 mW (c) 191 mW

16. (a) 17.3 μs (b) 30.7 μs

17. (a) $\frac{E}{r}e^{-\left(\frac{R+r}{L}\right)t}$ (b) $\frac{E^2L}{2r(R+r)}$

18. $\frac{8}{3}$ A

19. (a) 3.6 mH (b) 1.33 kHz (c) 0.188 ms

20. (a) 6.28×10^3 rad/s, 10^{-3} s (b) $10^{-4}\cos(6.28\times10^3t)$ (c) 0.0253 H (d) 0.4 A

21. (a) 1.25×10^{-2} V (b) 8.33×10^{-4} A (c) 3.125×10^{-8} J (d) 4.33×10^{-6} C, 7.8×10^{-7} J

LEVEL 2

Single Correct Option

1.(b) **2.**(c) **3.**(a) **4.**(b) **5.**(c) **6.**(a) **7.**(a) **8.**(c) **9.**(c) **10.**(a)
11.(b) **12.**(b) **13.**(c) **14.**(a) **15.**(b) **16.**(a) **17.**(b) **18.**(a) **19.**(d) **20.**(a)
21.(a) **22.**(d) **23.**(a) **24.**(b) **25.**(b) **26.**(d) **27.**(b) **28.**(b) **29.**(a) **30.**(c)
31.(c) **32.**(b) **33.**(c) **34.**(a) **35.**(a) **36.**(d) **37.**(d)

More than One Correct Options

1. (a,c) **2.** (a,c) **3.** (a,c) **4.** (a,b,d) **5.** (a,c) **6.** (b,c,d) **7.** (b,c) **8.** (b,c,d) **9.** (a,b,c) **10.** (a,c)

Comprehension Based Questions

1.(b) **2.**(c) **3.**(b) **4.**(a) **5.**(c) **6.**(a) **7.**(d) **8.**(a) **9.**(b) **10.**(c)
11.(c) **12.**(a) **13.**(d)

Match the Columns

1. (a) → p (b) → r (c) → s (d) → s
2. (a) → q (b) → p (c) → r (d) → s
3. (a) → s (b) → p (c) → s (d) → s
4. (a) → s (b) → q (c) → p (d) → p
5. (a) → s (b) → q (c) → s (d) → p
6. (a) → q, s (b) → p, r (c) → p, r (d) → q, s

Subjective Questions

1. $I = \dfrac{V}{R}$ **2.** $i = \dfrac{Bvl}{R + R'}$ where $R' = \dfrac{R_1 R_2}{R_1 + R_2}$ **3.** $e = \dfrac{\mu_0 i \omega}{2\pi \sin\omega t}\left[\dfrac{d}{\sin\omega t} \ln\left(\dfrac{d - a\sin\omega t}{d + a\sin\omega t}\right) + 2a\right]$

4. $3.6\,(1 - e^{-t/\tau_L})$A. Here, $\tau_L = 300\,\mu$s. Current through 10 Ω resistor varies with time.

5. $i = 5(1 - e^{-2000t/3})$ A

6. (a) 120 V (b) a (c) 120 V (d) c (e) –72 V (f) b (g) –192 V (h) d

7. (a) $q_0\omega$ (b) $3V_0 : 3V_0$ (c) $i = q_0 \sin\omega t$. Here, $q_0 = 2CV_0$ and $\omega = \sqrt{\dfrac{3}{2LC}}$

8. (a) 20 mV (b) $10^7 t^2$ V (c) 63.2 μs

9. 0.28 H

10. (a) 3 V (b) b to a (c) 0.8 N towards right (d) both are 6 W

11. (a) $\dfrac{2B_0QR}{mg}$ (b) $\dfrac{B_0Q}{m}$

12. $V = \dfrac{FR}{B^2l^2}(1 - e^{-\alpha t})$ Here, $\alpha = \dfrac{B^2l^2}{mR + RB^2l^2C}$, $v_T = \dfrac{FR}{B^2l^2}$

13. (a) 5×10^4 rad/s (b) 1.05×10^{-5} s (c) $10\sqrt{3}\,\mu$C

14. (a) 8.16 V (b) 5.16 A, 10 Hz

15. (a) $v = \dfrac{E_0}{Bl}\left(1 - e^{-\frac{B^2l^2}{mR}t}\right)$ (b) $i = \dfrac{E_0 - Blv}{R}$

16. $x = \dfrac{mgt^2}{2(m + CB^2l^2)}$

17. (a) $i = \dfrac{mg}{2Br}\left(1 - e^{-\frac{-2B^2r^2}{mR}t}\right)$ (b) $v_T = \dfrac{mgR}{4B^2r^2}$

18. (a) $x = \dfrac{v_0}{\omega}(1 - \cos\omega t)$, $\omega = \dfrac{Bl}{\sqrt{mL}}$ (b) $i_{\max} = \dfrac{2mg}{Bl}$ (c) $v_{\max} = v_0 = \dfrac{g\sqrt{mL}}{Bl}$

19. (a) $i = \dfrac{Bla_0t}{R_1R_2}(R_1 + R_2)$ (b) Polarity of a, b is positive and polarity of c, d is negative

(c) $F_{\text{ext}} = a_0\left[m + \dfrac{B^2l^2t}{R_1R_2}(R_1 + R_2)\right]$

20. (a) $i = \dfrac{9v_0\mu_0 i_0}{16aR\pi^2}\ln(3)$ (b) $\dfrac{9\,\mu_0^2 i_0^2 v_0}{32aR\pi^3}(\ln 3)^2$ **21.** $i = \left(\dfrac{e}{R}\right)[1 - e^{-lR/Lv}]$, where $e = \dfrac{\mu_0 i_0 v}{2\pi}\ln(2)$

28

Alternating Current

Chapter Contents

28.1 Introduction
28.2 Alternating currents and phasors
28.3 Current and potential relations
28.4 Phasor algebra
28.5 Series L-R circuit
28.6 Series C-R circuit
28.7 Series L-C-R circuit
28.8 Power in an AC circuit

28.1 Introduction

A century ago, one of the great technological debates was whether the electrical distribution system should be AC or DC. Thomas Edison favoured direct current (DC), that is, steady current that does not vary with time. George Westinghouse favoured alternating current (AC), with sinusoidally varying voltages and currents. He argued that transformers can be used to step the voltage up or down with AC but not with DC. Low voltages are safer for consumer use, but high voltages and correspondingly low currents are best for long distances power transmission to minimize i^2R losses in the cables. Eventually, Westinghouse prevailed, and most present day household and industrial power distribution systems operate with alternating current.

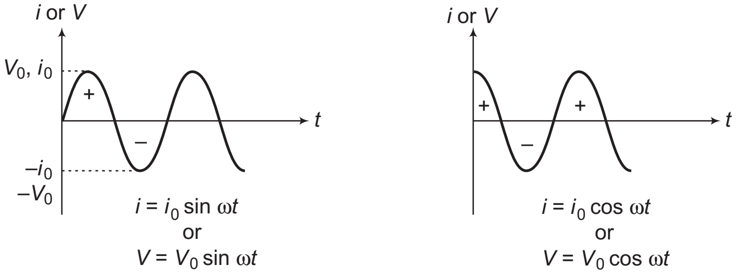


Fig. 28.1

A time varying current or voltage may be periodic and non-periodic. In case of periodic current or voltage, the current or voltage is said to be alternating if its amplitude is constant and alternate half cycle is positive and half negative. If the current or voltage varies periodically as sine or cosine function of time, the current or voltage is said to be sinusoidal and is what we usually mean by it.

28.2 Alternating Currents and Phasors

The basic principle of the AC generator is a direct consequence of Faraday's law of induction. When a conducting loop is rotated in a magnetic field at constant angular frequency ω a sinusoidal voltage (emf) is induced in the loop. This instantaneous voltage is

$$\boxed{V = V_0 \sin \omega t} \qquad \text{...(i)}$$

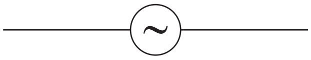


Fig. 28.2

The usual circuit diagram symbol for an AC source is shown in Fig. 28.2. In Eq. (i), V_0 is the maximum output voltage of the AC generator or the **voltage amplitude** and ω is the **angular frequency** equal to 2π times the frequency f.

$$\boxed{\omega = 2\pi f}$$

The frequency of AC in India is 50 Hz, i.e.

$$f = 50 \text{ Hz}$$

So, $$\omega = 2\pi f \approx 314 \text{ rad/s}$$

The time of one cycle is known as **time period *T***, the number of cycles per second the frequency f.

$$\boxed{T = \frac{1}{f}} \quad \text{or} \quad \boxed{T = \frac{2\pi}{\omega}}$$

A sinusoidal current might be described as

$$i = i_0 \sin \omega t$$

If an alternating current is passed through an ordinary ammeter or voltmeter, it will record the mean value for the complete cycle, as the quantity to be measured varies with time. The **average** value of current for one cycle is

$$\langle i \rangle_{\text{One cycle}} = (i_{\text{av}})_T = \frac{\int_0^T i\,dt}{\int_0^T dt} = \frac{\int_0^{2\pi/\omega} (i_0 \sin \omega t)\,dt}{\int_0^{2\pi/\omega} dt} = 0$$

Thus, $\boxed{\langle i \rangle_{\text{One cycle}} = 0}$

Similarly, the average value of the voltage (or emf) for one cycle is zero.

$$\boxed{\langle V \rangle_{\text{One cycle}} = 0}$$

Since, these averages for the whole cycle are zero, the DC instrument will indicate zero deflection. In AC, the average value of current is defined as its average taken over half the cycle. Hence,

$$\langle i \rangle_{\text{Half cycle}} = (i_{\text{av}})_{T/2} = \frac{\int_0^{T/2} i\,dt}{\int_0^{T/2} dt} = \frac{\int_0^{\pi/\omega} (i_0 \sin \omega t)\,dt}{\int_0^{\pi/\omega} dt} = \frac{2}{\pi} i_0$$

This is sometimes simply written as i_{av}. Hence,

$$i_{\text{av}} = \langle i \rangle_{\text{Half cycle}} = \frac{2}{\pi} i_0 \approx 0.637\, i_0$$

Similarly, $$V_{\text{av}} = \frac{2}{\pi} V_0 \approx 0.637\, V_0$$

A DC meter can be used in an AC circuit if it is connected in the full wave rectifier circuit. The average value of the rectified current is the same as the average current in any half cycle, i.e. $\frac{2}{\pi}$ times the maximum current i_0. A more useful way to describe a quantity is the **root mean square (rms)** value. We square the instantaneous current, take the average (mean) value of i^2 and finally take the square root of that average. This procedure defines the root-mean-square current denoted as i_{rms}. Even when i is negative, i^2 is always positive so i_{rms} is never zero (unless i is zero at every instant). Hence,

$$\langle i^2 \rangle_{\text{One cycle}} = \frac{\int_0^T i^2\,dt}{\int_0^T dt} = \frac{\int_0^{2\pi/\omega} (i_0^2 \sin^2 \omega t)\,dt}{\int_0^{2\pi/\omega} dt} = \frac{i_0^2}{2}$$

∴ $$i_{\text{rms}} = \sqrt{\langle i^2 \rangle_{\text{One cycle}}} = \frac{i_0}{\sqrt{2}} \approx 0.707\, i_0$$

Thus, $$\boxed{i_{\text{rms}} = \frac{i_0}{\sqrt{2}} \approx 0.707\, i_0}$$

Similarly, we get

$$\boxed{V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \approx 0.707\, V_0}$$

The square root of the mean square value is called the **virtual value** and is the value given by AC instruments.

Thus, when we speak of our house hold power supply as 220 V AC, this means that the rms voltage is 220 V and its voltage amplitude is

$$V_0 = \sqrt{2}\, V_{\text{rms}} = 311 \text{ V}$$

Form Factor

The ratio, $= \dfrac{\text{rms value}}{\text{average value}} = \dfrac{V_0/\sqrt{2}}{2V_0/\pi} = \dfrac{\pi}{2\sqrt{2}} = 1.11$

is known as form factor.

The different values i_0, i_{av} and i_{rms} are shown in Fig. 28.3.

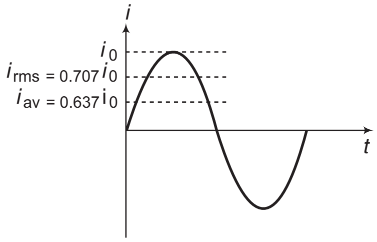


Fig. 28.3

Note *(1) The average value of* $\sin \omega t$, $\cos \omega t$, $\sin 2\omega t$, $\cos 2\omega t$, *etc., is zero because it is positive in half of the time and negative in rest half of the time. Thus,*

$$\langle \sin \omega t \rangle = \langle \cos \omega t \rangle = \langle \sin 2\,\omega t \rangle = \langle \cos 2\,\omega t \rangle = 0$$

If $$i = i_0 \sin \omega t$$

then $$\langle i \rangle = \langle i_0 \sin \omega t \rangle = i_0 \langle \sin \omega t \rangle = 0$$

(2) The average value of $\sin^2 \omega t$ *and* $\cos^2 \omega t$ *is* $\frac{1}{2}$.

or $$\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2}$$

If $$i^2 = i_0^2 \sin^2 \omega t$$

then $$\langle i^2 \rangle = \langle i_0^2 \sin^2 \omega t \rangle = i_0^2 \langle \sin^2 \omega t \rangle = \frac{i_0^2}{2}$$

(3) Like SHM, general expressions of current/voltage in an sinusoidal AC are

$$i = i_0 \sin(\omega t \pm \phi)$$

$$V = V_0 \sin(\omega t \pm \phi)$$

or $$i = i_0 \cos(\omega t \pm \phi)$$

and $$V = V_0 \cos(\omega t \pm \phi)$$

(4) Average value of current or voltage over a half cycle can be zero also. This depends on the time interval (of course $T/2$*) over which average value is to be found. Think why?*

Phasors

If an AC generator is connected to a series circuit containing resistors, inductors and capacitors and we want to know the amplitude and time characteristics of the alternating current. To simplify our analysis of circuits containing two or more elements, we use graphical constructions called **phasor diagrams.** In these constructions, alternating (sinusoidal) quantities, such as current and voltage are rotating vectors called **phasors**.

In these diagrams, the instantaneous value of a quantity that varies sinusoidally with time is represented by the projection onto a vertical axis (if it is a sine function) or onto a horizontal axis (if it is a cosine function) of a vector with a length equal to the amplitude (i_0) of the quantity. The vector rotates counterclockwise with constant angular velocity ω.

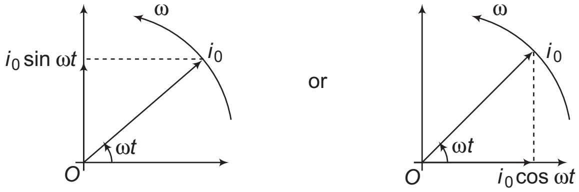


Fig. 28.4

A phasor is not a real physical quantity with a direction in space, such as velocity, momentum or electric field. Rather, it is a geometric entity that helps us to describe and analyze physical quantities that vary sinusoidally with time.

Example 28.1 *Show that average heat produced during a cycle of AC is same as produced by DC with $i = i_{rms}$.*

Solution For an AC, $i = i_0 \sin \omega t$

Therefore, instantaneous value of heat produced in time dt across a resistance R is

$$dH = i^2 R dt = i_0^2 R \sin^2 \omega t \, dt$$

∴ Average value of heat produced during a cycle,

$$H_{av} = \frac{\int_0^T dH}{\int_0^T dt} = \frac{\int_0^{2\pi/\omega} (i_0^2 R \sin^2 \omega t) \, dt}{\int_0^{2\pi/\omega} dt}$$

$$= \frac{i_0^2}{2} R \left(\frac{2\pi}{\omega}\right) = i_{rms}^2 RT$$

i.e. AC produces same heating effects as DC of value $i = i_{rms}$.

Example 28.2 *If the current in an AC circuit is represented by the equation,*

$$i = 5 \sin (300t - \pi/4)$$

Here, t is in second and i in ampere. Calculate

(a) peak and rms value of current (b) frequency of AC (c) average current

Solution (a) As in case of AC,

$$i = i_0 \sin (\omega t \pm \phi)$$

∴ The peak value, $i_0 = 5 \text{ A}$ **Ans.**

and $i_{rms} = \frac{i_0}{\sqrt{2}} = \frac{5}{\sqrt{2}} = 3.535 \text{ A}$ **Ans.**

(b) Angular frequency, $\omega = 300 \text{ rad/s}$

∴ $f = \frac{\omega}{2\pi} = \frac{300}{2\pi} \approx 47.75 \text{ Hz}$ **Ans.**

(c) $i_{av} = \left(\frac{2}{\pi}\right) i_0 = \left(\frac{2}{\pi}\right)(5) = 3.18 \text{ A}$ **Ans.**

28.3 Current and Potential Relations

In this section, we will derive voltage current relations for individual circuit elements carrying a sinusoidal current. We will consider resistors, inductors and capacitors.

Resistor in an AC Circuit

Consider a resistor with resistance R through which there is a sinusoidal current given by

$$i = i_0 \sin \omega t \qquad \ldots(\text{i})$$

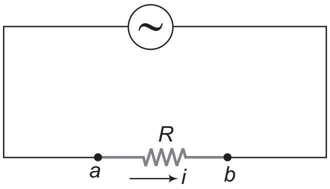


Fig. 28.5

Here, i_0 is the current amplitude (maximum current). From Ohm's law, the instantaneous PD between points a and b is

$$V_R = iR = (i_0R) \sin \omega t$$

We can write as

$$i_0R = V_0, \quad \text{the voltage amplitude}$$

$$\therefore \qquad V_R = V_0 \sin \omega t \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we can see that current and voltage are **in phase** if only resistance is in the circuit. Fig. 28.6 shows graphs of i and V_R as functions of time.

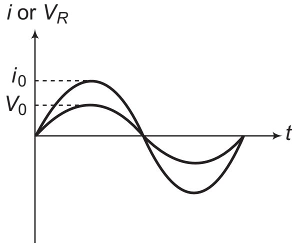


Fig. 28.6

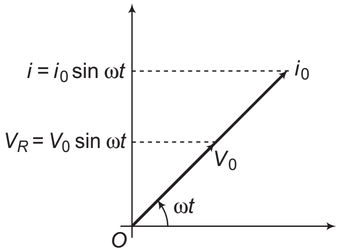


Fig. 28.7

The corresponding phasor diagram is shown in Fig. 28.7.

Because i and V_R are in phase and have the same frequency, the current and voltage phasors rotate together, they are parallel at each instant. Their projection on vertical axis represents the instantaneous current and voltage respectively.

Note *Direction of an alternating current is not shown in a circuit, as it keeps on changing. In the figure, the direction of instantaneous current is only shown.*

Capacitor in an AC Circuit

If a capacitor of capacitance C is connected across the alternating source, the instantaneous charge on the capacitor is

$$q = CV_C = CV_0 \sin \omega t$$

and the instantaneous current i passing through it is given by

$$i = \frac{dq}{dt} = CV_0\omega \cos \omega t$$

$$= \frac{V_0}{1/\omega C} \sin (\omega t + \pi/2)$$

or $$i = i_0 \sin (\omega t + \pi/2)$$

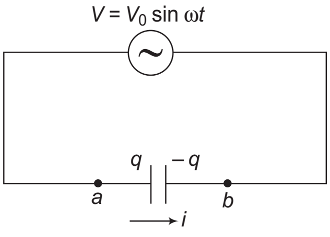


Fig. 25.8

Here, $$\boxed{V_0 = \frac{i_0}{\omega C}}$$

This relation shows that the quantity $\frac{1}{\omega C}$ is the effective AC resistance or the **capacitive reactance** of the capacitor and is represented as X_C. It has unit as ohm. Thus,

$$\boxed{X_C = \frac{1}{\omega C}}$$

It is clear that the **current leads the voltage by 90°** or the potential drop across the capacitor lags the current passing it by 90°.

Fig. 28.9 shows V and i as functions of time t.

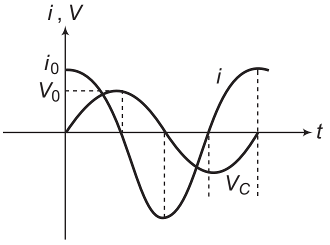


Fig. 28.9

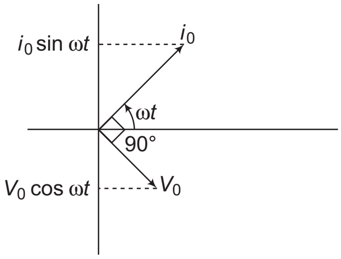


Fig. 28.10

The phasor diagram 28.10 shows that voltage phasor is behind the current phasor by a quarter cycle or 90°.

Inductor in an AC Circuit

Consider a pure inductor of self-inductance L and zero resistance connected to an alternating source. Again we assume that an instantaneous current $i = i_0 \sin \omega t$ flows through the inductor. Although, there is no resistance, there is a potential difference V_L between the inductor terminals a and b because the current varies with time giving rise to a self-induced emf.

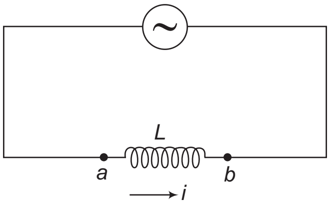


Fig. 28.11

$$V_L = V_{ab} = -\text{ (induced emf)} = -\left(-L\frac{di}{dt}\right)$$

or $$V_L = L\frac{di}{dt} = Li_0\omega \cos \omega t$$

or $$V_L = V_0 \sin\left(\omega t + \frac{\pi}{2}\right) \quad \text{...(i)}$$

Here, $$\boxed{V_0 = i_0(\omega L)} \quad \text{...(ii)}$$

or $$i_0 = \frac{V_0}{\omega L}$$

$\therefore$ $$\boxed{i = \frac{V_0}{\omega L}\sin \omega t} \quad \text{...(iii)}$$

Eq. (iii) shows that effective AC resistance, i.e. **inductive reactance of inductor** is

$$\boxed{X_L = \omega L}$$

and the maximum current,

$$\boxed{i_0 = \frac{V_0}{X_L}}$$

The unit of X_L is also ohm.

From Eqs. (i) and (iii), we see that **the voltage across the inductor leads the current passing through it by 90°**.

Fig. 28.12 shows V_L and i as functions of time.

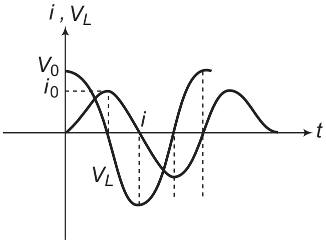


Fig. 28.12

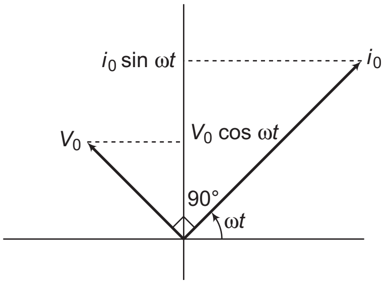


Fig. 28.13

Phasor diagram in Fig. 28.13 shows that V_L leads the current i by 90°.

✓ *Extra Points to Remember*

- Circuit elements with AC

Circuit elements	Amplitude relation	Circuit quantity	Phase of V
Resistor	$V_0 = i_0 R$	R	in phase with i
Capacitor	$V_0 = i_0 X_C$	$X_C = \frac{1}{\omega C}$	lags i by 90°
Inductor	$V_0 = i_0 X_L$	$X_L = \omega L$	leads i by 90°

- In DC, $\omega = 0$, therefore, $X_L = 0$ and $X_C = \infty$

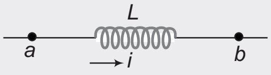


Fig. 28.14

- The potential of point *a* with respect to point *b* is given by $V_L = +L\frac{di}{dt}$, the negative of the induced emf. This expression gives the correct sign of V_L in all cases.
- If an oscillating voltage of a given amplitude V_0 is applied across an inductor, the resulting current will have a smaller amplitude i_0 for larger value of ω. Since, X_L is proportional to frequency, a high frequency voltage applied to the inductor gives only a small current while a lower frequency voltage of the same amplitude gives rise to a larger current. Inductors are used in some circuit applications, such as power supplies and radio interference filters to block high frequencies while permitting lower frequencies to pass through. A circuit device that uses an inductor for this purpose is called a **low pass filter**.
- The capacitive reactance of a capacitor is inversely proportional to the capacitance C and the angular frequency ω. The greater the capacitance and the higher the frequency, the smaller is the capacitive reactance X_C. Capacitors tend to pass high frequency current and to block low frequency current, just the opposite of inductors. A device that passes signals of high frequency is called a **high pass filter**.
- Figure shows the graphs of R, X_L and X_C as functions of angular frequency ω.

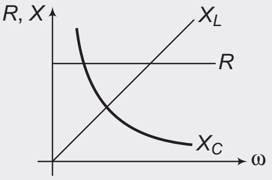


Fig. 28.15

- Remember that we can write, $V_R = iR$, $(V_0)_R = i_0R$, $(V_0)_L = i_0X_L$ and $(V_0)_C = i_0X_C$ but can't write (for instantaneous voltages).

$$V_L = iX_L$$

or

$$V_C = iX_C$$

This is because there is a phase difference between the voltage and current in both an inductor and a capacitor.

Example 28.3 *A 100 Ω resistance is connected in series with a 4 H inductor. The voltage across the resistor is $V_R = (2.0\,V)\sin(10^3\ rad/s)\,t$:*

(a) Find the expression of circuit current

(b) Find the inductive reactance

(c) Derive an expression for the voltage across the inductor.

Solution (a)

$$i = \frac{V_R}{R} = \frac{(2.0\text{ V})\sin(10^3\text{ rad/s})t}{100}$$

$$= (2.0\times10^{-2}\text{ A})\sin(10^3\text{ rad/s})t$$ **Ans.**

(b)

$$X_L = \omega L = (10^3\text{ rad/s})(4\text{ H})$$

$$= 4.0\times10^3\ \Omega$$ **Ans.**

(c) The amplitude of voltage across inductor,

$$V_0 = i_0X_L = (2.0\times10^{-2}\text{ A})(4.0\times10^3\ \Omega)$$

$$= 80\text{ V}$$ **Ans.**

In an AC, voltage across the inductor leads the current by 90° or $\pi/2$ rad. Hence,

$$\begin{aligned} V_L &= V_0 \sin(\omega t + \pi/2) \\ &= (80\text{ V}) \sin\left\{(10^3 \text{ rad/s})\, t + \frac{\pi}{2} \text{ rad}\right\} \end{aligned}$$

Note *That the amplitude of voltage across the resistor* $(= 2.0\ V)$ *is not same as the amplitude of the voltage across the inductor* $(= 80\ V)$*, even though the amplitude of the current through both devices is the same.*

28.4 Phasor Algebra

The complex quantities normally employed in AC circuit analysis, can be added and subtracted like coplanar vectors. Such coplanar vectors which represent sinusoidally time varying quantities are known as **phasors**.

In Cartesian form, a phasor **A** can be written as

$$\boxed{\mathbf{A} = a + jb}$$

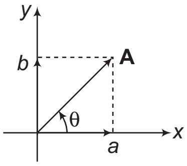


Fig. 28.16

where, a is the x-component and b is the y-component of phasor **A**.

The magnitude of **A** is $$\boxed{|\mathbf{A}| = \sqrt{a^2 + b^2}}$$

and the angle between the direction of phasor **A** and the positive x-axis is

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

When a given phasor **A**, the direction of which is along the x-axis is multiplied by the operator j, a new phasor j**A** is obtained which will be 90° anti-clockwise from **A**, i.e. along y-axis. If the operator j is multiplied now to the phasor j**A**, a new phasor j^2**A** is obtained which is along $-x$-axis and having same magnitude as of **A**. Thus,

$$j^2 \mathbf{A} = -\mathbf{A}$$

$$\therefore \qquad \boxed{j^2 = -1 \quad \text{or} \quad j = \sqrt{-1}}$$

Now, using the j operator, let us discuss different circuits of an AC.

28.5 Series *L-R* Circuit

As we know, potential difference across a resistance in AC is in phase with current and it leads in phase by 90° with current across the inductor.

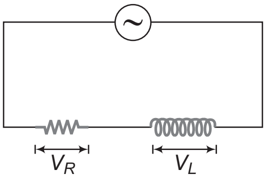


Fig. 28.17

Suppose in phasor diagram current is taken along positive x-direction. Then, V_R is also along positive x-direction and V_L along positive y-direction, so, we can write

$$\begin{aligned} V &= V_R + jV_L = iR + j(iX_L) \\ &= iR + j(i\omega L) \qquad \text{(as } X_L = \omega L\text{)} \\ &= i\mathbf{Z} \end{aligned}$$

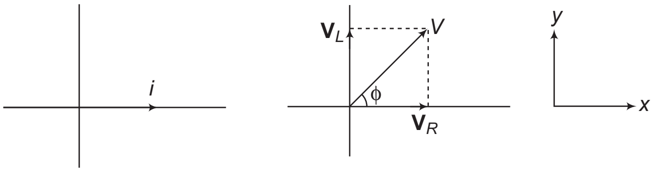


Fig. 28.18

Here, $\mathbf{Z} = R + jX_L = R + j(\omega L)$ is called as **impedance** of the circuit. Impedance plays the same role in AC circuits as the ohmic resistance does in DC circuits. The modulus of impedance is

$$|\mathbf{Z}| = \sqrt{R^2 + (\omega L)^2}$$

The potential difference leads the current by an angle,

$$\phi = \tan^{-1}\frac{|\mathbf{V}_L|}{|\mathbf{V}_R|} = \tan^{-1}\left(\frac{X_L}{R}\right)$$

or

$$\boxed{\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)}$$

28.6 Series *C-R* Circuit

Potential difference across a capacitor in AC lags in phase by 90° with the current in the circuit.

Suppose in phasor diagram current is taken along positive *x*-direction. Then, V_R is also along positive *x*-direction but V_C is along negative *y*-direction. So, we can write

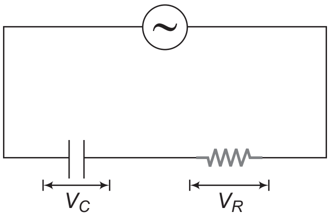


Fig. 28.19

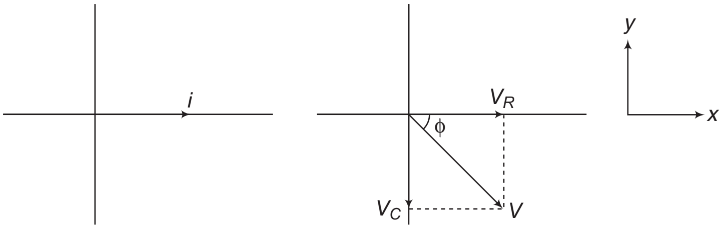


Fig. 28.20

$$\begin{aligned} V &= V_R - jV_C = iR - j(iX_C) \\ &= iR - j\left(\frac{i}{\omega C}\right) = i\boldsymbol{Z} \end{aligned}$$

Here, impedance is $\boldsymbol{Z} = R - j\left(\frac{1}{\omega C}\right)$

The modulus of impedance is

$$\boxed{|\mathbf{Z}| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

and the potential difference lags the current by an angle,

$$\phi = \tan^{-1}\left|\frac{V_C}{V_R}\right| = \tan^{-1}\frac{X_C}{R} = \tan^{-1}\left(\frac{1/\omega C}{R}\right)$$

or $$\boxed{\phi = \tan^{-1}\left(\frac{1}{\omega RC}\right)}$$

28.7 Series *L-C-R* Circuit

Potential difference across an inductor leads the current by 90° in phase while that across a capacitor, it lags in phase by 90°.

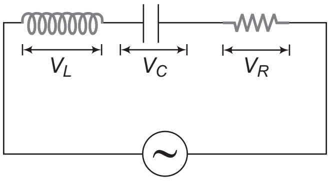


Fig. 28.21

Suppose in a phasor diagram current is taken along positive *x*-direction. Then, V_R is along positive *x*-direction, V_L along positive *y*-direction and V_C along negative *y*-direction.

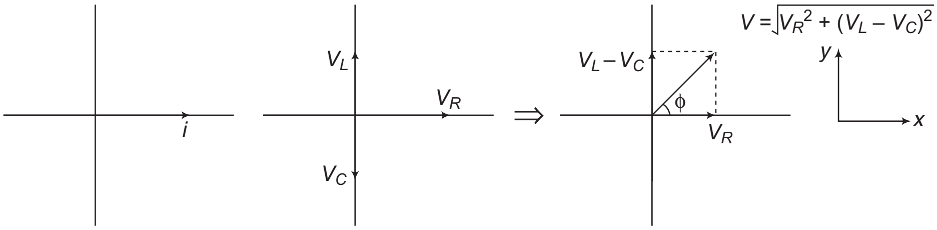


Fig. 28.22

Let us assume that $X_L > X_C$ or $V_L > V_C$

So, we can write $$V = V_R + jV_L - jV_C = iR + j(iX_L) - j(iX_C)$$
$$= iR + j[i(X_L - X_C)] = iZ$$

Here, impedance is $$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The modulus of impedance is $$|\mathbf{Z}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and the potential difference leads the current by an angle,

$$\phi = \tan^{-1}\left|\frac{V_L - V_C}{V_R}\right| = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

or $$\boxed{\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)}$$

Note *Let us take the most general case of a series L-C-R circuit in an AC.*

$$|\mathbf{Z}| = \sqrt{R^2 + (X_L \sim X_C)^2}$$

If $X_L = X_C$ *or* $\omega L = \dfrac{1}{\omega C}$ *or* $\omega = \dfrac{1}{\sqrt{LC}}$ *or* $f = \dfrac{1}{2\pi\sqrt{LC}}$ *the modulus of impedance*

$$|\mathbf{Z}| = R$$

and the current is in phase with voltage, i.e. if $V = V_0 \sin \omega t$, *then*

$$i = i_0 \sin \omega t$$

where, $$i_0 = \frac{V_0}{|\mathbf{Z}|} = \frac{V_0}{R}$$

Such a condition is known as resonance and frequency known as resonance frequency and is given by

$$\boxed{f = \frac{1}{2\pi\sqrt{LC}}}$$

The current in such a case is maximum.

If $X_L > X_C$, *then the modulus of the impedance*

$$|\mathbf{Z}| = \sqrt{R^2 + (X_L - X_C)^2}$$

and the voltage leads the current by an angle given by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

i.e. if $V = V_0 \sin \omega t$, *then* $$i = i_0 \sin(\omega t - \phi)$$

where, $$i_0 = \frac{V_0}{|\mathbf{Z}|}$$

If $X_C > X_L$, *then the modulus of the impedance is*

$$|\mathbf{Z}| = \sqrt{R^2 + (X_C - X_L)^2}$$

and the current leads the voltage by an angle given by

$$\phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

i.e. if $V = V_0 \sin \omega t$, *then* $$i = i_0 \sin(\omega t + \phi)$$

where, $$i_0 = \frac{V_0}{|\mathbf{Z}|}$$

Extra Points to Remember

- $i_0 = \dfrac{V_0}{|\mathbf{Z}|}$, $i_{\text{rms}} = \dfrac{V_{\text{rms}}}{|\mathbf{Z}|}$. But in general $i \neq \dfrac{V}{|\mathbf{Z}|}$.
- In *L-C-R* circuit, whenever voltage across various elements is asked, find rms values unless stated in the question for the peak or instantaneous value.

  The rms values are

  $$V_R = i_{\text{rms}} R, \quad V_L = i_{\text{rms}} X_L$$

  and $$V_C = i_{\text{rms}} X_C$$

  The peak values can be obtained by multiplying the rms values by $\sqrt{2}$. The instantaneous values across different elements is rarely asked.

- **Voltage magnification in series resonance circuit** At resonance $\left(f = \frac{1}{2\pi\sqrt{LC}}\right)$, the PD across the inductor and the capacitor are equal and 180° out of phase and therefore, cancel out. Hence, the applied emf is merely to overcome the resistance opposition only. If an inductance or capacitance of very large reactance (X_L or X_C) is connected with $X_L = X_C$ (at resonance) then PD across them increases to a very high value. The ratio is known as **voltage magnification** and is given by,

$$\frac{\text{PD across inductance (or capacitance)}}{\text{Applied emf}} = \frac{i_{\text{rms}}(\omega L)}{i_{\text{rms}}(R)} = \frac{\omega L}{R} \quad \text{or} \quad \frac{i_{\text{rms}}\left(\frac{1}{\omega C}\right)}{i_{\text{rms}}(R)} = \frac{1}{\omega CR}$$

This ratio is greater than unity.

- **Response curves of series circuit** The impedance of an L-C-R circuit depends on the frequency. The dependence is shown in figure. The frequency is taken on logarithmic scale because of its wide range.

From the figure, we can see that at resonance,

(i) $X_L = X_C$ or $\omega = \frac{1}{\sqrt{LC}}$

(ii) $Z = Z_{\min} = R$ and

(iii) i is maximum.

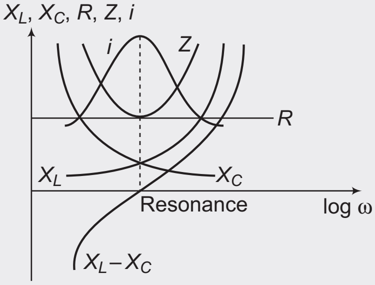


Fig. 28.23

Note *Here, by Z we mean the modulus of Z and i means* i_{rms}.

- **Acceptor circuit** If the frequency of the AC supply can be varied (e.g. in radio or television signal), then in series L-C-R circuit, at a frequency $f = 1/2\pi\sqrt{LC}$ maximum current flows in the circuit and have a maximum PD across its inductance (or capacitance). This is the method by which a radio or television set is tuned at a particular frequency. The circuit is known as **acceptor circuit**.

Example 28.4 *An alternating emf 200 virtual volts at 50 Hz is connected to a circuit of resistance 1 Ω and inductance 0.01 H. What is the phase difference between the current and the emf in the circuit? Also, find the virtual current in the circuit.*

Solution In case of an L-R AC circuit, the voltage leads the current in phase by an angle,

$$\phi = \tan^{-1}\left(\frac{X_L}{R}\right)$$

Here, $X_L = \omega L = (2\pi f L)$

$= (2\pi)(50)(0.01) = \pi\ \Omega$

and $R = 1\ \Omega$

$\therefore$ $\phi = \tan^{-1}(\pi) \approx 72.3°$ **Ans.**

Further, $i_{\text{rms}} = \frac{V_{\text{rms}}}{|\mathbf{Z}|} = \frac{V_{\text{rms}}}{\sqrt{R^2 + X_L^2}}$

Substituting the values, we have

$$i_{\text{rms}} = \frac{200}{\sqrt{(1)^2 + (\pi)^2}} = 60.67\ \text{A}$$ **Ans.**

Example 28.5 *A resistance and inductance are connected in series across a voltage,*

$$V = 283 \sin 314t$$

The current is found to be $4 \sin (314t - \pi/4)$. *Find the values of the inductance and resistance.*

Solution In L-R series circuit, current lags the voltage by an angle,

$$\phi = \tan^{-1}\left(\frac{X_L}{R}\right)$$

Here, $$\phi = \frac{\pi}{4}$$

$\therefore$ $$X_L = R \quad \text{or} \quad \omega L = R \qquad (\omega = 314 \text{ rad/s})$$

$\therefore$ $$314\, L = R \qquad \ldots\text{(i)}$$

Further, $$V_0 = i_0 |\mathbf{Z}|$$

$\therefore$ $$283 = 4\sqrt{R^2 + X_L^2}$$

or $$R^2 + (\omega L)^2 = \left(\frac{283}{4}\right)^2 = 5005.56$$

or $$2R^2 = 5005.56 \qquad (\text{as } \omega L = R)$$

$\therefore$ $$R \approx 50\,\Omega$$ **Ans.**

and from Eq. (i), $$L = 0.16 \text{ H}$$ **Ans.**

Example 28.6 *Find the voltage across the various elements, i.e. resistance, capacitance and inductance which are in series and having values* $1000\ \Omega$, $1\ \mu F$ *and* $2.0\ H$, *respectively. Given emf is*

$$V = 100\sqrt{2} \sin 1000\, t \text{ volt}$$

Solution The rms value of voltage across the source,

$$V_{\text{rms}} = \frac{100\sqrt{2}}{\sqrt{2}} = 100 \text{ V}$$

$\therefore$ $$\omega = 1000 \text{ rad/s}$$

$\therefore$ $$i_{\text{rms}} = \frac{V_{\text{rms}}}{|\mathbf{Z}|} = \frac{V_{\text{rms}}}{\sqrt{R^2 + (X_L \sim X_C)^2}} = \frac{V_{\text{rms}}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{100}{\sqrt{(1000)^2 + \left(1000 \times 2 - \frac{1}{1000 \times 1 \times 10^{-6}}\right)^2}}$$

$$= 0.0707 \text{ A}$$

The current will be same every where in the circuit, therefore,

PD across resistor, $V_R = i_{\text{rms}} R = 0.0707 \times 1000 = 70.7\text{ V}$

PD across inductor, $V_L = i_{\text{rms}} X_L = 0.0707 \times 1000 \times 2 = 141.4\text{ V}$ and

PD across capacitor, $V_C = i_{\text{rms}} X_C = 0.0707 \times \dfrac{1}{1 \times 1000 \times 10^{-6}}$

$= 70.7\text{ V}$ **Ans.**

Note *The rms voltages do not add directly as `$V_R + V_L + V_C = 282.8$ V which is not the source voltage 100 V. The reason is that these voltages are not in phase and can be added by vector or by phasor algebra.*

$$V = \sqrt{V_R^2 + (V_L \sim V_C)^2}$$

INTRODUCTORY EXERCISE 28.1

1. (a) What is the reactance of a 2.00 H inductor at a frequency of 50.0 Hz?
   (b) What is the inductance of an inductor whose reactance is 2.00 Ω at 50.0 Hz?
   (c) What is the reactance of a 2.00 μF capacitor at a frequency of 50.0 Hz?
   (d) What is the capacitance of a capacitor whose reactance is 2.00 Ω at 50.0 Hz?
2. An electric lamp which runs at 100 V DC and consumes 10 A current is connected to AC mains at 150 V, 50 Hz cycles with a choke coil in series. Calculate the inductance and drop of voltage across the choke. Neglect the resistance of choke.
3. A circuit operating at $\dfrac{360}{2\pi}$ Hz contains a 1μF capacitor and a 20 Ω resistor. How large an inductor must be added in series to make the phase angle for the circuit zero? Calculate the current in the circuit if the applied voltage is 120 V.

28.8 Power in an AC Circuit

In case of a steady current, the rate of doing work is given by

$$P = Vi$$

In an alternating circuit, current and voltage both vary with time and also they differ in time. So, we cannot use $P = Vi$ for the power generated.

Suppose in an AC, the voltage is leading the current by an angle ϕ. Then, we can write

$$V = V_0 \sin \omega t \quad \text{and} \quad i = i_0 \sin (\omega t - \phi)$$

The instantaneous value of power in that case is

$$P = Vi = V_0 i_0 \sin \omega t \sin (\omega t - \phi)$$

or

$$P = V_0 i_0 \left[\sin^2 \omega t \cos \phi - \frac{1}{2} \sin 2\omega t \sin \phi \right] \qquad \ldots\text{(i)}$$

Now, the average rate of doing work (power) in one cycle will be

$$\langle P \rangle_{\text{One cycle}} = \frac{\int_0^{T = 2\pi/\omega} P dt}{\int_0^{T = 2\pi/\omega} dt} \qquad \ldots\text{(ii)}$$

Substituting the value of P from Eq. (i) in Eq. (ii) and then integrating it with proper limits, we get

$$\langle P\rangle_{\text{One cycle}}=\frac{1}{2}V_0 i_0\cos\phi=\frac{V_0}{\sqrt{2}}\cdot\frac{i_0}{\sqrt{2}}\cos\phi$$

or
$$\boxed{\langle P\rangle_{\text{One cycle}}=V_{\text{rms}}i_{\text{rms}}\cos\phi}$$

Here, the term $\cos\phi$ is known as **power factor**.

It is said to be leading if current leads voltage, lagging if current lags voltage. Thus, a power factor of 0.5 lagging means current lags the voltage by 60° (as $\cos^{-1}0.5=60°$). The product of V_{rms} and i_{rms} gives the **apparent power**. While the **true power** is obtained by multiplying the apparent power by the power factor $\cos\phi$. Thus,

$$\boxed{\begin{array}{l}\text{Apparent power}=V_{\text{rms}}\times i_{\text{rms}}\\ \text{True power}=\text{apparent power}\times\text{power factor}\end{array}}$$

and

For $\phi=0°$, the current and voltage are in phase. The power is thus, maximum $(=V_{\text{rms}}\times i_{\text{rms}})$. For $\phi=90°$, the power is zero. The current is then stated **wattless**. Such a case will arise when resistance in the circuit is zero. The circuit is purely inductive or capacitive. The case is similar to that of a frictionless pendulum, where the total work done by gravity upon the pendulum in a cycle is zero.

Extra Points to Remember

Let us consider a choke coil (used in tube lights) of large inductance L and low resistance R. The power factor for such a coil is given by

$$\cos\phi=\frac{R}{Z}=\frac{R}{\sqrt{R^2+\omega^2L^2}}\approx\frac{R}{\omega L}\qquad(\text{as } R<<\omega L)$$

As $R<<\omega L$, $\cos\phi$ is very small. Thus, the power absorbed by the coil $V_{\text{rms}}\,i_{\text{rms}}\cos\phi$ is very small. On account of its large impedance $Z=\sqrt{R^2+\omega^2L^2}$, the current passing through the coil is very small. Such a coil is used in AC circuits for the purpose of adjusting current to any required value without waste of energy. The only loss of energy is due to hysteresis in the iron core, which is much less than the loss of energy in the resistance that can also reduce the current instead of a choke coil.

Example 28.7 *A 750 Hz, 20 V source is connected to a resistance of 100 Ω, an inductance of 0.1803 H and a capacitance of 10 μF all in series. Calculate the time in which the resistance (thermal capacity 2 J/°C) will get heated by 10° C.*

Solution The impedance of the circuit,

$$Z=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{R^2+\left\{(2\pi fL)-\frac{1}{(2\pi fC)}\right\}^2}$$

$$=\sqrt{(100)^2+\left\{(2\times 3.14\times 750\times 0.1803)-\frac{1}{(2\times 3.14\times 750\times 10^{-5})}\right\}^2}$$

$$=834\ \Omega$$

In case of an AC,

$$\langle P\rangle = V_{\text{rms}}\, i_{\text{rms}} \cos\phi = (V_{\text{rms}})\left(\frac{V_{\text{rms}}}{Z}\right)\left(\frac{R}{Z}\right) = \left(\frac{V_{\text{rms}}}{Z}\right)^2 R$$

$$= \left(\frac{20}{834}\right)^2 \times 100 = 0.0575 \text{ J/s}$$

Now, $\langle P\rangle \times t = S\Delta\theta$

Here, S = thermal capacity

$\therefore$ $$t = \frac{S\Delta\theta}{\langle P\rangle} = \frac{2\times 10}{0.0575} = 348 \text{ s}$$ **Ans.**

Note *In AC, the whole energy or power is consumed by resistance.*

Example 28.8 *In an L-C-R series circuit, $R = 150\ \Omega$, $L = 0.750\ H$ and $C = 0.0180\ \mu F$. The source has voltage amplitude $V = 150\ V$ and a frequency equal to the resonance frequency of the circuit.*

(a) What is the power factor?

(b) What is the average power delivered by the source?

(c) The capacitor is replaced by one with $C = 0.0360\ \mu F$ and the source frequency is adjusted to the new resonance value. Then, what is the average power delivered by the source?

Solution (a) At resonance frequency,

$$X_L = X_C,\ Z = R \text{ and power factor } \cos\phi = \frac{R}{Z} = 1.0$$

(b) $$P = \frac{V_{\text{rms}}^2}{R} = \frac{(150/\sqrt{2})^2}{150} = 75 \text{ W}$$

(c) Again, $$P = \frac{V_{\text{rms}}^2}{R} = 75 \text{ W}$$

INTRODUCTORY EXERCISE 28.2

1. If a 0.03 H inductor, a 10 Ω resistor and a 2 μF capacitor are connected in series. At what frequency will they resonate? What will be the phase angle at resonance?
2. An arc lamp consumes 10 A at 40 V. Calculate the power factor when it is connected with a suitable value of choke coil required to run the arc lamp on AC mains of 200 V (rms) and 50 Hz.

Final Touch Points

1. Frequency of AC in India is 50 Hz.
2. The AC is converted into DC with the help of rectifier while DC is converted into AC with the help of inverter.
3. An AC cannot produce electroplating or electrolysis.
4. The AC is measured by hot wire ammeter.
5. An AC can be stepped up or down with the help of a transformer.
6. An AC can be transmitted over long distances without much power loss.
7. An AC can be regulated by using choke coil without any significant waste of energy.
8. In an AC (sinusoidal), current or voltage can have the following four values
   - (i) instantaneous value
   - (ii) peak value (i_0 or V_0)
   - (iii) rms value (i_{rms} or V_{rms})
   - (iv) average value : In full cycle, average value is zero while in half cycle it is non-zero.

Note *That in sinusoidal AC the average value in half cycle can also be zero. It depends on the time interval over which half average value is desired.*

9. In an series *L-C-R* circuit,
   - (i) Capacitive reactance, $X_C = \dfrac{1}{\omega C}$
   - (ii) Inductive reactance, $X_L = \omega L$
   - (iii) Impedance, $Z = \sqrt{R^2 + (X_C - X_L)^2}$
   - (iv) If $X_C > X_L$, current leads and if $X_L > X_C$, voltage leads by an angle ϕ given by

$$\cos\phi = \frac{R}{Z} \quad \text{and} \quad \tan\phi = \frac{X_C \sim X_L}{R}$$

   - (v) Instantaneous power = instantaneous current × instantaneous voltage
   - (vi) Average power $= V_{\text{rms}}\, i_{\text{rms}} \cos\phi$, where

$$\cos\phi = \frac{R}{Z} = \text{power factor.}$$

Note *Power is also equal to* $P = i_{rms}^2 R$

But this is not equal to $P \neq \dfrac{V_{rms}^2}{R}$

This is because $V_{rms} = i_{rms} Z$ *and* $\cos\phi = \dfrac{R}{Z}$. *If we substitute in* $P = V_{rms}\, i_{rms} \cos\phi$, *then we get the first relation but not the second one. This implies that power is consumed only across resistance.*

   - (vii) $i_0 = \dfrac{V_0}{Z}$ or $i_{\text{rms}} = \dfrac{V_{\text{rms}}}{Z}$
   - (viii) $(V_C)_{\text{rms}} = (i_{\text{rms}})\, X_C$, $(V_L)_{\text{rms}} = (i_{\text{rms}})\, X_L$ and $(V_R)_{\text{rms}} = (i_{\text{rms}})\, R$
   - (ix) $V = \sqrt{V_R^2 + (V_C \sim V_L)^2}$

     Here, V is the rms value of applied voltage
     V_R is the rms value of voltage across resistance.
     V_C across capacitor and V_L across inductor etc.

10. $\omega = \omega_r = \dfrac{1}{\sqrt{LC}}$ is called resonance frequency.

11. At $\omega = \omega_r$,

(i) $X_L = X_C$

(ii) Z = minimum value = R

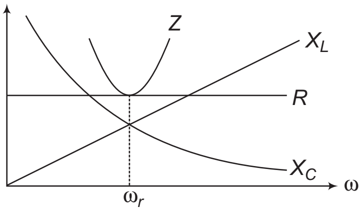
(iii) i_{rms} = maximum value $= \dfrac{V_{\text{rms}}}{Z_{\min}} = \dfrac{V_{\text{rms}}}{R}$

(iv) i_0 = maximum value $= \dfrac{V_0}{Z_{\min}} = \dfrac{V_0}{R}$

(v) Power factor $\cos\phi = 1$

12. In one complete cycle, power is consumed only by resistance. No power is consumed by a capacitor or an inductor.

13.



$$X_C = \frac{1}{\omega C} \Rightarrow X_C \propto \frac{1}{\omega}$$

$$X_L = \omega L \Rightarrow X_L \propto \omega$$

R does not depend on ω. It is a constant.

At $\omega = \omega_r$: $X_C = X_L$ and $Z = Z_{\min} = R$

14. For $\omega > \omega_r$, $X_L > X_C$. Hence, voltage will lead the current or circuit is inductive.

For $\omega < \omega_r$, $X_C > X_L$. Hence, current will lead the voltage function or circuit is capacitive.

At $\omega = \omega_r$, $X_C = X_L$. Hence, current function and voltage function are in same phase.

15.

Conditions	Phase angle	Power factor
$R = 0$	90°	0
$X_C = X_L \neq 0$ $R \neq 0$	0°	1
$X_C = X_L = 0$ $R \neq 0$	0°	1
$\omega = \omega_r$	0°	1

In all other cases, phase difference between current function and voltage function is

$$0° < \phi < 90°$$

If $\quad X_C > X_L, \phi = \tan^{-1}\left(\dfrac{X_C - X_L}{R}\right)$ or $\cos^{-1}\left(\dfrac{R}{Z}\right)$

If $\quad X_L > X_C, \phi = \tan^{-1}\left(\dfrac{X_L - X_C}{R}\right)$ or $\cos^{-1}\left(\dfrac{R}{Z}\right)$

Solved Examples

TYPED PROBLEMS

Type 1. *Based on a real inductor*

Concept

An ideal inductor wire has zero resistance ($R = 0$). So, in an AC, only X_L is the impedance. But, a real inductor has some resistance also. In DC, total resistance is only R. In AC, total resistance called impedance is $Z = \sqrt{R^2 + X_L^2}$

Therefore,

$$I_{\text{DC}} = \frac{V_{\text{DC}}}{R}$$

and
$$I_{\text{AC}} = \frac{V_{\text{AC}}}{Z} = \frac{V_{\text{AC}}}{\sqrt{R^2 + X_L^2}}$$

If
$$V_{\text{AC}} = V_{\text{DC}}, \quad \text{then} \quad I_{\text{AC}} < I_{\text{DC}}$$

Example 1 *A current of 4 A flows in a coil when connected to a 12 V DC source. If the same coil is connected to a 12 V, 50 rad/s AC source, a current of 2.4 A flows in the circuit. Determine the inductance of the coil. Also, find the power developed in the circuit if a 2500 μF capacitor is connected in series with the coil.*

Solution (i) A coil consists of an inductance (L) and a resistance (R).

In DC, only resistance is effective. Hence,

$$R = \frac{V}{i} = \frac{12}{4} = 3\ \Omega$$

In AC,
$$i_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{\sqrt{R^2 + \omega^2 L^2}}$$

$\therefore$
$$L^2 = \frac{1}{\omega^2}\left[\left(\frac{V_{\text{rms}}}{i_{\text{rms}}}\right)^2 - R^2\right]$$

$\therefore$
$$L = \frac{1}{\omega}\sqrt{\left(\frac{V_{\text{rms}}}{i_{\text{rms}}}\right)^2 - R^2}$$

Substituting the values, we have

$$L = \frac{1}{50}\sqrt{\left(\frac{12}{2.4}\right)^2 - (3)^2}$$

$$= 0.08\ \text{H}$$ **Ans.**

(ii) When capacitor is connected to the circuit, the impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Here, $R = 3\ \Omega$

$$X_L = \omega L = (50)(0.08) = 4\ \Omega$$

and $X_C = \dfrac{1}{\omega C} = \dfrac{1}{(50)(2500 \times 10^{-6})} = 8\ \Omega$

$\therefore$ $Z = \sqrt{(3)^2 + (4-8)^2} = 5\ \Omega$

Now, $\langle P \rangle = V_{\text{rms}} i_{\text{rms}} \cos\phi$

$$= V_{\text{rms}} \times \frac{V_{\text{rms}}}{Z} \times \frac{R}{Z}$$

$$= \left(\frac{V_{\text{rms}}}{Z}\right)^2 \times R$$

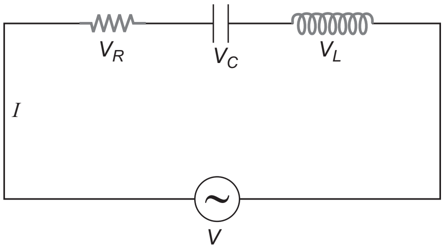
Substituting the values, we have

$$\langle P \rangle = \left(\frac{12}{5}\right)^2 \times 3$$

$$= 17.28\ \text{W}$$ **Ans.**

Type 2. *Different time functions in AC*

Concept



In an L-C-R series circuit, there are total five functions of time, V, I, V_R, V_C and V_L. Now, the following points are important in these functions.

(i) V and I have a phase difference of ϕ where

$$0° \le \phi \le 90°$$

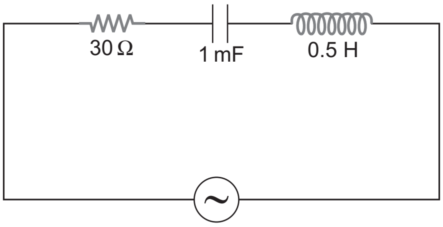
(ii) V_R and I are in same phase

(iii) V_C lags behind I by 90°

(iv) V_L leads I by 90°

(v) The functions, $V = V_L + V_C + V_L$ (all the time)

Example 2



$V = 200 \sin(100\,t + 30°)$

In the diagram shown in figure, V function is given. Find other four functions of time I, V_C, V_R and V_L. Also, find power consumed in the circuit, V is given in volts and ω in rad/s.

Solution Given, $\omega = 100$ rad/s

$$X_L = \omega L = 50\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{100 \times 10^{-3}} = 10\ \Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(30)^2 + (50 - 10)^2}$$

$$= 50\ \Omega$$

Current function

Maximum value of current,

$$I_0 = \frac{V_0}{Z} = \frac{200}{50} = 4\ \text{A}$$

$X_L > X_C$, therefore voltage leads the current by a phase difference ϕ where,

$$\cos\phi = \frac{R}{Z} = \frac{30}{50} = \frac{3}{5}$$

or $$\phi = 53^\circ$$

$\therefore$ $$I = 4\sin(100\,t + 30^\circ - 53^\circ)$$

or $$I = 4\sin(100\,t - 23^\circ)$$ **Ans.**

V_R, V_C and V_L functions

Maximum value of $V_R = I_0 R = 4 \times 30 = 120$ volt, V_R and I are in same phase.

Therefore,

$$V_R = 120\sin(100\,t - 23^\circ)$$ **Ans.**

Maximum value of $V_C = I_0 X_C = 4 \times 10 = 40$ volt

Now, V_C function lags the current function by 90°.

Therefore,

$$V_C = 40\sin(100\,t - 23^\circ - 90^\circ)$$

or $$V_C = 40\sin(100\,t - 113^\circ)$$ **Ans.**

Maximum value of $V_L = I_0 X_L = 4 \times 50 = 200$ volt, V_L function leads the current function by 90°. Therefore,

$$V_L = 200\sin(100\,t - 23^\circ + 90^\circ)$$

or $$V_L = 200\sin(100\,t + 67^\circ)$$ **Ans.**

Note *We can check at any time that,*

$$V = V_R + V_L + V_C$$

Power Power is consumed in an AC circuit only across a resistance and this power is given by

$$P = V_{\text{rms}}\, I_{\text{rms}} \cos\phi$$

$$= I_{\text{rms}}^2 R$$

Let us use the first formula,

$$P = \left(\frac{200}{\sqrt{2}}\right)\left(\frac{4}{\sqrt{2}}\right)\left(\frac{3}{5}\right)$$

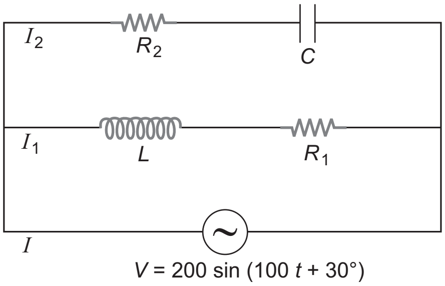
$$= 240\ \text{watt}$$ **Ans.**

Type 3. *Parallel circuits*

Concept

Two or more than two sine or cosine functions of same ω can be added by vector method. Actually, their amplitudes are added by vectors method.

Example 3 *In the circuit shown in figure,*



$$R_1 = 30\ \Omega,\ R_2 = 40\ \Omega,\ L = 0.4\ H \text{ and } C = \frac{1}{3}\ mF.$$

Find seven functions of time $I, I_1, I_2, V_{R_1}, V_L, V_{R_2}$ and V_C*. Also, find total power consumed in the circuit. In the given potential function, V is in volts and ω in rad/s.*

Solution **Circuit 1 (containing L and R_1)**

I_1 :

$$X_L = \omega L = 100 \times 0.4 = 40\ \Omega$$

$$R_1 = 30\ \Omega$$

$$\therefore \quad Z_1 = \sqrt{R_1^2 + X_L^2} = \sqrt{(30)^2 + (40)^2}$$

$$= 50\ \Omega$$

Maximum value of current, $I_1 = \dfrac{V_0}{Z_1} = \dfrac{200}{50} = 4\text{ A}$

Since, there is only X_L, so voltage function will lead the current function by an angle ϕ_1, where

$$\cos \phi_1 = \frac{R_1}{Z_1} = \frac{30}{50} = \frac{3}{5}$$

$$\therefore \quad \phi_1 = 53°$$

$$\therefore \quad I_1 = 4 \sin (100\,t + 30° - 53°)$$

or $$I_1 = 4 \sin (100\,t - 23°)$$ **Ans.**

V_{R_1} : V_{R_1} function is in phase with I_1 function.

$$\text{Maximum value of } V_{R_1} = (\text{maximum value of } I_1)\,(R_1)$$

$$= (4)\,(30)$$

$$= 120 \text{ volt}$$

$$\therefore \quad V_{R_1} = 120 \sin (100\,t - 23°)$$ **Ans.**

V_L : V_L function is 90° ahead of I_1 function.

$$\text{Maximum value of } V_L = (\text{maximum value of } I_1)\,(X_L)$$

$$= (4)(40) = 160 \text{ volt}$$

$\therefore$ $$V_L = 160 \sin (100\,t - 23° + 90°)$$

or $$V_L = 160 \sin (100\,t + 67°)$$ **Ans.**

Power In this circuit, power will be consumed only across R_1. This power is given by

$$P_{R_1} = (\text{ rms value of } I_1)^2 R_1$$
$$= \left(\frac{4}{\sqrt{2}}\right)^2 (30)$$
$$= 240 \text{ watt}$$

Circuit 2 (containing C and R_2)

I_2 : $$X_C = \frac{1}{\omega C} = \frac{1}{100 \times \frac{1}{3} \times 10^{-3}} = 30\ \Omega$$

$$R_2 = 40\ \Omega$$

$\therefore$ $$Z_2 = \sqrt{R_2^2 + X_C^2}$$
$$= \sqrt{(40)^2 + (30)^2}$$
$$= 50\ \Omega$$

Maximum value of $I_2 = \frac{V_0}{Z_2} = \frac{200}{50} = 4$ A

Since, there is only X_C, so I_2 function will lead the V function by an angle ϕ_2, where

$$\cos \phi_2 = \frac{R_2}{Z_2} = \frac{40}{50} = \frac{4}{5}$$

$\therefore$ $$\phi_2 = 37°$$

$\therefore$ $$I_2 = 4 \sin (100\,t + 30° + 37°) = 4 \sin (100\,t + 67°)$$ **Ans.**

V_{R_2} : V_{R_2} function is in phase with I_2 function.

$$\text{Maximum value of } V_{R_2} = (\text{ maximum value of } I_2)(R_2)$$
$$= 4 \times 40 = 160 \text{ volt}$$

$\therefore$ $$V_{R_2} = 160 \sin (100\,t + 67°)$$ **Ans.**

V_C : V_C function lags I_2 function by 90°

$$\text{Maximum value of } V_C = (\text{Maximum value of } I_2)(X_C)$$
$$= 4 \times 30$$
$$= 120 \text{ volt}$$

$\therefore$ $$V_C = 120 \sin (100\,t + 67° - 90°)$$

or $$V_C = 120 \sin (100\,t - 23°)$$ **Ans.**

Power In this circuit, power will be consumed only across R_2 and this power is given by

$$P_{R_2} = (\text{rms value of } I_2)^2 R_2$$
$$= \left(\frac{4}{\sqrt{2}}\right)^2 (40)$$
$$= 320 \text{ W}$$

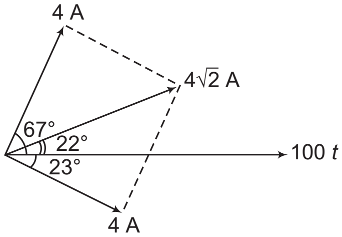
$\therefore$ Total power consumed in the circuit,

$$P = P_{R_1} + P_{R_2}$$
$$= (240 + 320) \text{ W}$$
$$= 560 \text{ W}$$ **Ans.**

I : $I = I_1 + I_2$

$I = 4\sin(100\,t - 23°) + 4\sin(100\,t + 67°)$

Now, the amplitudes can be added by vector method.



Resultant of 4 A and 4 A at 90° is $4\sqrt{2}$ A at 45° from both currents or at 22° from 100 t line.

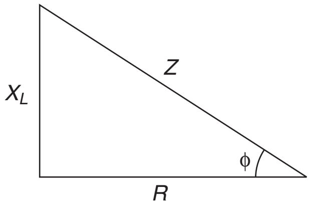
$\therefore$ $I = 4\sqrt{2}\sin(100\,t + 22°)$ **Ans.**

Miscellaneous Examples

Example 4 *An AC circuit consists of a* 220 Ω *resistance and a* 0.7 *H choke. Find the power absorbed from* 220 V *and* 50 *Hz source connected in this circuit if the resistance and choke are joined*

(a) in series *(b) in parallel.*

Solution (a) In series, the impedance of the circuit is



$$Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + (2\pi f L)^2}$$

$$= \sqrt{(220)^2 + (2 \times 3.14 \times 50 \times 0.7)^2}$$

$$= 311\ \Omega$$

$\therefore$ $$i_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{220}{311} = 0.707\ \text{A}$$

and $$\cos\phi = \frac{R}{Z} = \frac{220}{311} = 0.707$$

$\therefore$ The power absorbed in the circuit,

$$P = V_{\text{rms}} i_{\text{rms}} \cos\phi$$

$$= (220)\,(0.707)\,(0.707)$$

$$= 110.08\ \text{W}$$ **Ans.**

(b) When the resistance and choke are in parallel, the entire power is absorbed in resistance, as the choke (having zero resistance) absorbs no power.

$\therefore$ $$P = \frac{V_{\text{rms}}^2}{R} = \frac{(220)^2}{220} = 220\ \text{W}$$ **Ans.**

Example 5 *A sinusoidal voltage of frequency* 60 *Hz and peak value* 150 V *is applied to a series L-R circuit, where* $R = 20\ \Omega$ *and* $L = 40$ *mH.*

(a) Compute T, ω, X_L, Z *and* ϕ

(b) Compute the amplitudes of current, V_R *and* V_L

Solution (a) $$T = \frac{1}{f} = \frac{1}{60}\text{ s}$$ **Ans.**

$$\omega = 2\pi f = (2\pi)(60) = 377\text{ rad/s}$$ **Ans.**

$$X_L = \omega L = (377)(0.040)$$

$$= 15.08\ \Omega$$ **Ans.**

$$Z = \sqrt{X_L^2 + R^2}$$

$$= \sqrt{(15.08)^2 + (20)^2}$$

$$= 25.05\ \Omega$$ **Ans.**

$$\phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{15.08}{20}\right) = \tan^{-1}(0.754)$$

$$= 37°$$ **Ans.**

(b) Amplitudes (maximum value) are

$$i_0 = \frac{V_0}{Z} = \frac{150}{25.05} \approx 6\text{ A}$$ **Ans.**

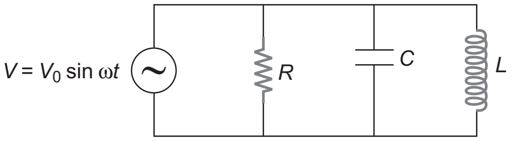
$$(V_0)_R = i_0 R = (6)(20) = 120\text{ V}$$ **Ans.**

$$(V_0)_L = i_0 X_L$$

$$= (6)(15.08) = 90.5\text{ V}$$ **Ans.**

Note $V_0 = \sqrt{(V_0)_R^2 + (V_0)_L^2}$

Example 6 *For the circuit shown in figure, find the instantaneous current through each element.*



Solution The three current equations are

$$V = i_R R, \quad V = L\frac{di_L}{dt}$$

and $$V = \frac{q}{C} \Rightarrow \frac{dV}{dt} = \frac{1}{C} i_C \qquad \text{...(i)}$$

The steady state solutions of Eq. (i) are

$$i_R = \frac{V_0}{R}\sin \omega t \equiv (i_0)_R \sin \omega t$$

$$i_L = -\frac{V_0}{\omega L}\cos \omega t \equiv -\frac{V_0}{X_L}\cos \omega t \equiv -(i_0)_L \cos \omega t$$

and $$i_C = V_0 \omega C \cos \omega t \equiv \frac{V_0}{X_C}\cos \omega t \equiv (i_0)_C \cos \omega t$$

where, the reactances X_L and X_C are as defined.

Example 7 *In the above problem find the total instantaneous current through the source, and find expressions for phase angle of this current and the impedance of the circuit.*

Solution For the total current, we have

$$i = i_R + i_L + i_C$$

$$= V_0\left[\frac{1}{R}\sin\omega t + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)\cos\omega t\right]$$

Using the trigonometric identity,

$$A\sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\sin(\theta + \phi)$$

where, $\phi = \tan^{-1}(B/A)$

We can write, $i \equiv i_0 \sin(\omega t + \phi)$

Here, $i_0 = \dfrac{V_0}{Z}$

where, $\dfrac{1}{Z} = \sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\dfrac{1}{X_C} - \dfrac{1}{X_L}\right)^2}$

and $\tan\phi = \dfrac{\left(\dfrac{1}{X_C} - \dfrac{1}{X_L}\right)}{(1/R)}$

Example 8 *An L-C-R series circuit with* 100 Ω *resistance is connected to an AC source of* 200 V *and angular frequency* 300 rad/s. *When only the capacitance is removed, the current lags behind the voltage by* 60°. *When only the inductance is removed, the current leads the voltage by* 60°. *Calculate the current and the power dissipated in the L-C-R circuit*

Solution When capacitance is removed, then

$$\tan\phi = \frac{X_L}{R}$$

or $\tan 60° = \dfrac{X_L}{R}$

∴ $X_L = \sqrt{3}R$ ...(i)

When inductance is removed, then

$$\tan\phi = \frac{X_C}{R}$$

or $\tan 60° = \dfrac{X_C}{R}$

∴ $X_C = \sqrt{3}\,R$ ...(ii)

From Eqs. (i) and (ii), we see that $X_C = X_L$

So, the *L-C-R* circuit is in resonance.

Hence, $Z = R$

$$\therefore \qquad i_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{200}{100} = 2\text{ A}$$ **Ans.**

$$\langle P \rangle = V_{\text{rms}}\, i_{\text{rms}} \cos\phi$$

At resonance current and voltage are in phase,

or $$\phi = 0°$$

$$\therefore \qquad \langle P \rangle = (200)\,(2)\,(1) = 400\text{ W}$$ **Ans.**

Example 9 *A series L-C-R circuit containing a resistance of* $120\ \Omega$ *has resonance frequency* 4×10^5 *rad/s. At resonance the voltages across resistance and inductance are* $60\ V$ *and* $40\ V$, *respectively. Find the values of L and C. At what angular frequency the current in the circuit lags the voltage by* $\pi/4$?

Solution At resonance, $X_L - X_C = 0$

and $$Z = R = 120\ \Omega$$

$$\therefore \qquad i_{\text{rms}} = \frac{(V_R)_{\text{rms}}}{R} = \frac{60}{120} = \frac{1}{2}\text{ A}$$

Also, $$i_{\text{rms}} = \frac{(V_L)_{\text{rms}}}{\omega L}$$

$$\therefore \qquad L = \frac{(V_L)_{\text{rms}}}{\omega i_{\text{rms}}} = \frac{40}{(4 \times 10^5)\left(\frac{1}{2}\right)}$$

$$= 2.0 \times 10^{-4}\text{ H}$$

$$= 0.2\text{ mH}$$ **Ans.**

The resonance frequency is given by

$$\omega = \frac{1}{\sqrt{LC}} \quad \text{or} \quad C = \frac{1}{\omega^2 L}$$

Substituting the values, we have

$$C = \frac{1}{(4 \times 10^5)^2\,(2.0 \times 10^{-4})}$$

$$= 3.125 \times 10^{-8}\text{ F}$$ **Ans.**

Current lags the voltage by 45°, when

$$\tan 45° = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

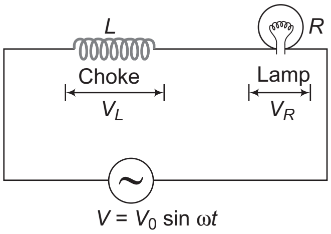
Substituting the values of *L, C, R* and tan 45°, we get

$$\omega = 8 \times 10^5\text{ rad/s}$$ **Ans.**

Example 10 *A choke coil is needed to operate an arc lamp at* $160\ V$ *(rms) and* 50 *Hz. The lamp has an effective resistance of* $5\ \Omega$ *when running at* 10 *A (rms). Calculate the inductance of the choke coil. If the same arc lamp is to be operated on* $160\ V$ *(DC), what additional resistance is required? Compare the power loses in both cases.*

Solution For lamp,

$$(V_{\text{rms}})_R = (i_{\text{rms}})(R) = 10 \times 5 = 50 \text{ V}$$



In series,

$$(V_{\text{rms}})^2 = (V_{\text{rms}})_R^2 + (V_{\text{rms}})_L^2$$

$\therefore$
$$(V_{\text{rms}})_L = \sqrt{(V_{\text{rms}})^2 - (V_{\text{rms}})_R^2}$$
$$= \sqrt{(160)^2 - (50)^2}$$
$$= 152 \text{ V}$$

As,
$$(V_{\text{rms}})_L = (i_{\text{rms}})X_L = (i_{\text{rms}})(2\pi f L)$$

$\therefore$
$$L = \frac{(V_{\text{rms}})_L}{(2\pi f)(i_{\text{rms}})}$$

Substituting the values, we get

$$L = \frac{152}{(2\pi)(50)(10)}$$
$$= 4.84 \times 10^{-2} \text{ H}$$ **Ans.**

Now, when the lamp is operated at 160 V, DC and instead of choke let an additional resistance R' is put in series with it, then

$$V = i(R + R')$$

or
$$160 = 10(5 + R')$$

$\therefore$
$$R' = 11\ \Omega$$ **Ans.**

In case of AC, as the choke has no resistance, power loss in choke is zero.

In case of DC, the loss in additional resistance R' is

$$P = i^2R' = (10)^2(11)$$
$$= 1100 \text{ W}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : In an AC circuit, potential difference across the capacitor may be greater than the applied voltage.

Reason : $V_C = IX_C$, whereas $V = IZ$ and X_C can be greater than Z also.

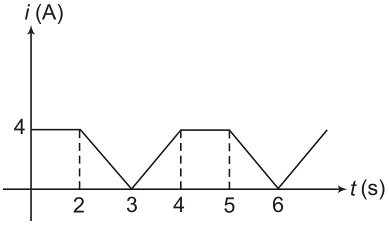
2. Assertion : In series L-C-R circuit, voltage will lead the current function for frequency greater than the resonance frequency.

Reason : At resonance frequency, phase difference between current function and voltage function is zero.

3. Assertion : Resonance frequency will decrease in L-C-R series circuit if a dielectric slab is inserted in between the plates of the capacitor.

Reason : By doing so, capacity of capacitor will increase.

4. Assertion : Average value of current in the given graph is 3 A.



Reason : Average value can't be greater than the peak value of any function.

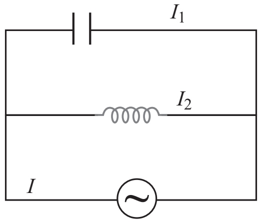
5. Assertion : In series L-C-R circuit, if a ferromagnetic rod is inserted inside an inductor, current in the circuit may increase or decrease.

Reason : By doing so X_L will increase.

6. Assertion : Potential difference across, resistor, capacitor and inductor each is 10 V. Then, voltage function and current functions should be in phase.

Reason : At this condition current in the circuit should be maximum.

7. Assertion : At some given instant I_1 and I_2 both are 2 A each. Then, I at this instant should be zero.



Reason : There is a phase difference of π between I_1 and I_2 functions.

8. **Assertion :** Peak value of current in AC through a resistance of 10 Ω is 2 A. Then, power consumed by the resistance should be 20 W.

Reason : Power in AC is $P = I^2_{\text{rms}}R$

9. **Assertion :** An inductor coil normally produces more current with DC source compared to an AC source of same value of rms voltage.

Reason : In DC source, applied voltage remains constant with time.

10. **Assertion :** In an L-R series circuit in AC, current in the circuit will decrease with increase in frequency.

Reason : Phase difference between current function and voltage function will increase with increase in frequency.

11. **Assertion :** In series L-C-R, AC circuit, current and voltage are in same phase at resonance.

Reason : In series L-C-R, AC circuit, resonant frequency does not depend on the value of resistance. Hence, current at resonance does not depend on resistance.

Objective Questions

1. The term cos ϕ in an AC circuit is called

(a) form factor
(b) phase factor
(c) power factor
(d) quality factor

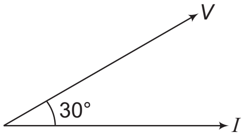
2. A DC ammeter cannot measure alternating current because

(a) AC changes its direction
(b) DC instruments will measure the average value
(c) AC can damage the DC instrument
(d) AC produces more heat

3. As the frequency of an alternating current increases, the impedance of the circuit

(a) increases continuously
(b) decreases continuously
(c) remains constant
(d) None of these

4. Phasor diagram of a series AC circuit is shown in figure. Then,



(a) The circuit must be containing resistor and capacitor only
(b) The circuit must be containing resistor and inductor only
(c) The circuit must be containing all three elements L, C and R
(d) The circuit cannot have only capacitor and inductor

5. The rms value of an alternating current

(a) is equal to 0.707 times peak value
(b) is equal to 0.636 times peak value
(c) is equal to $\sqrt{2}$ times the peak value
(d) None of the above

6. In an AC circuit, the applied potential difference and the current flowing are given by

$$V = 200 \sin 100t \text{ volt}, \; I = 5 \sin\left(100t - \frac{\pi}{2}\right) \text{ amp}$$

The power consumption is equal to

(a) 1000 W
(b) 40 W
(c) 20 W
(d) zero

7. The impedance of a series L-C-R circuit in an AC circuit is

(a) $\sqrt{R + (X_L - X_C)}$
(b) $\sqrt{R^2 + (X_L^2 - X_C^2)}$
(c) R
(d) None of these

8. If V_0 and I_0 are the peak current and voltage across the resistor in a series L-C-R circuit, then the power dissipated in the circuit is (Power factor = $\cos\theta$)

(a) $\frac{V_0 I_0}{2}$
(b) $\frac{V_0 I_0}{\sqrt{2}}$
(c) $V_0 I_0 \cos\theta$
(d) $\frac{V_0 I_0}{2} \cos\theta$

9. A generator produces a time varying voltage given by $V = 240 \sin 120\, t$, where t is in second. The rms voltage and frequency are

(a) 170 V and 19 Hz
(b) 240 V and 60 Hz
(c) 170 V and 60 Hz
(d) 120 V and 19 Hz

10. An L-C-R series circuit has a maximum current of 5 A. If $L = 0.5$ H and $C = 8\,\mu$F, then the angular frequency of AC voltage is

(a) 500 rad/s
(b) 5000 rad/s
(c) 400 rad/s
(d) 250 rad/s

11. The current and voltage functions in an AC circuit are

$$i = 100 \sin 100t \text{ mA}, V = 100 \sin\left(100t + \frac{\pi}{3}\right) \text{V}$$

The power dissipated in the circuit is

(a) 10 W
(b) 2.5 W
(c) 5 W
(d) 5 kW

12. A capacitor becomes a perfect insulator for

(a) alternating current
(b) direct current
(c) both (a) and (b)
(d) None of these

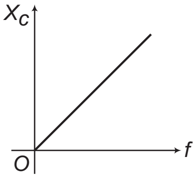
13. For an alternating voltage $V = 10 \cos 100\,\pi t$ volt, the instantaneous voltage at $t = \frac{1}{600}$ s is

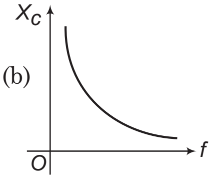
(a) 1 V
(b) 5 V
(c) $5\sqrt{3}$ V
(d) 10 V

14. In a purely resistive AC circuit,

(a) voltage leads current
(b) voltage lags current
(c) voltage and current are in same phase
(d) nothing can be said

15. Identify the graph which correctly represents the variation of capacitive reactance X_C with frequency

(a)


(b)


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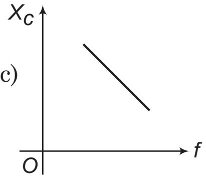

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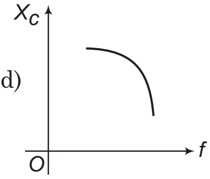


16. In an AC circuit, the impedance is $\sqrt{3}$ times the reactance, then the phase angle is

(a) 60° (b) 30°

(c) zero (d) None of these

17. Voltage applied to an AC circuit and current flowing in it is given by

$$V = 200\sqrt{2}\sin\left(\omega t + \frac{\pi}{4}\right) \text{ and } i = -\sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right)$$

Then, power consumed in the circuit will be

(a) 200 W (b) 400 W

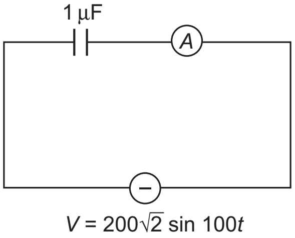
(c) $200\sqrt{2}$ W (d) None of these

18. When 100 volt DC source is applied across a coil, a current of 1 A flows through it. When 100 V AC source of 50 Hz is applied to the same coil, only 0.5 A current flows. Calculate the inductance of the coil.

(a) $(\pi/\sqrt{3})$ H (b) $(\sqrt{3}/\pi)$ H

(c) $(2/\pi)$ H (d) None of these

19. In the circuit shown in figure, the reading of the AC ammeter is



(a) $20\sqrt{2}$ mA (b) $40\sqrt{2}$ mA (c) 20 mA (d) 40 mA

20. An AC voltage is applied across a series combination of L and R. If the voltage drop across the resistor and inductor are 20 V and 15 V respectively, then applied peak voltage is

(a) 25 V (b) 35 V

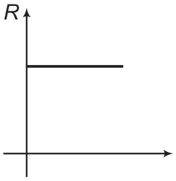
(c) $25\sqrt{2}$ V (d) $5\sqrt{7}$ V

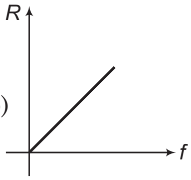
21. For wattless power in an AC circuit, the phase angle between the current and voltage is

(a) 0° (b) 90°

(c) 45° (d) Not possible

22. The correct variation of resistance R with frequency f is given by

(a)


(b)


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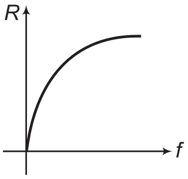

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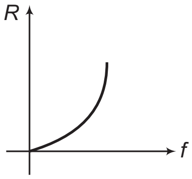


23. If L and R be the inductance and resistance of the choke coil, then identify the correct statement.

(a) L is very high compared to R (b) R is very high compared to L

(c) Both L and R are high (d) Both L and R are low

24. When an AC signal of frequency 1 kHz is applied across a coil of resistance 100 Ω, then the applied voltage leads the current by 45°. The inductance of the coil is

(a) 16 mH
(b) 12 mH
(c) 8 mH
(d) 4 mH

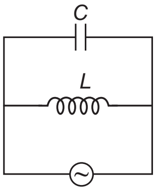
25. The frequency of an alternating current is 50 Hz. The minimum time taken by it in reaching from zero to peak value is

(a) 5 ms
(b) 10 ms
(c) 20 ms
(d) 50 ms

26. An alternating voltage is applied across the R-L combination. $V = 220 \sin 120\,t$ and the current $I = 4 \sin (120t - 60°)$ develops. The power consumption is

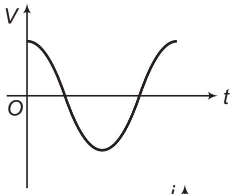
(a) zero
(b) 100 W
(c) 220 W
(d) 440 W

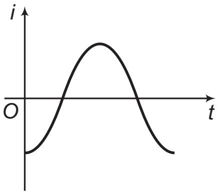
27. In the AC network shown in figure, the rms current flowing through the inductor and capacitor are 0.6 A and 0.8 A, respectively. Then, the current coming out of the source is



(a) 1.0 A
(b) 1.4 A
(c) 0.2 A
(d) None of the above

28. The figure represents the voltage applied across a pure inductor. The diagram which correctly represents the variation of current i with time t is given by



(a)


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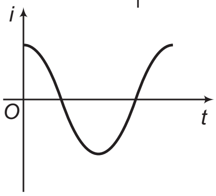

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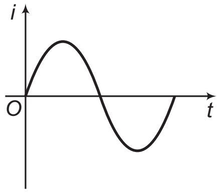
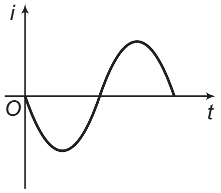
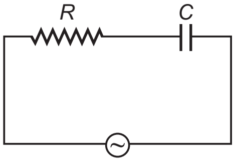


(d)


29. A steady current of magnitude I and an AC current of peak value I are allowed to pass through identical resistors for the same time. The ratio of heat produced in the two resistors will be

(a) 2 : 1
(b) 1 : 2
(c) 1 : 1
(d) None of these

30. A 50 Hz AC source of 20 V is connected across R and C as shown in figure. The voltage across R is 12 V. The voltage across C is



(a) 8 V
(b) 16 V
(c) 10 V
(d) Not possible to determine unless value of R and C are given

Subjective Questions

Note *You can take approximations in the answers.*

1. A 300 Ω resistor, a 0.250 H inductor, and a 8.00 μF capacitor are in series with an AC source with voltage amplitude 120 V and angular frequency 400 rad/s.

(a) What is the current amplitude?

(b) What is the phase angle of the source voltage with respect to the current? Does the source voltage lag or lead the current?

(c) What are the voltage amplitudes across the resistor, inductor, and capacitor?

2. A series circuit has an impedance of 60.0 Ω and a power factor of 0.720 at 50.0 Hz. The source voltage lags the current.

(a) What circuit element, an inductor or a capacitor, should be placed in series with the circuit to raise its power factor?

(b) What size element will raise the power factor to unity?

3. Voltage and current for a circuit with two elements in series are expressed as

$$V(t) = 170 \sin(6280t + \pi/3) \text{ volt}$$
$$i(t) = 8.5 \sin(6280t + \pi/2) \text{ amp}$$

(a) Plot the two waveforms.

(b) Determine the frequency in Hz.

(c) Determine the power factor stating its nature.

(d) What are the values of the elements?

4. A 5.00 H inductor with negligible resistance is connected across an AC source. Voltage amplitude is kept constant at 60.0 V but whose frequency can be varied. Find the current amplitude when the angular frequency is

(a) 100 rad/s

(b) 1000 rad/s

(c) 10000 rad/s

5. A 300 Ω resistor is connected in series with a 0.800 H inductor. The voltage across the resistor as a function of time is $V_R = (2.50 \text{ V}) \cos[(950 \text{ rad/s})t]$.

(a) Derive an expression for the circuit current.

(b) Determine the inductive reactance of the inductor.

(c) Derive an expression for the voltage V_L across the inductor.

6. An L-C-R series circuit with $L = 0.120$ H, $R = 240\ \Omega$, and $C = 7.30\ \mu$F carries an rms current of 0.450 A with a frequency of 400 Hz.

(a) What are the phase angle and power factor for this circuit?

(b) What is the impedance of the circuit?

(c) What is the rms voltage of the source?

(d) What average power is delivered by the source?

(e) What is the average rate at which electrical energy is converted to thermal energy in the resistor?

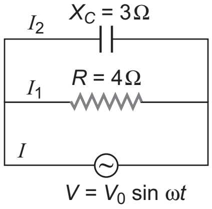
(f) What is the average rate at which electrical energy is dissipated (converted to other forms) in the capacitor?

(g) In the inductor?

LEVEL 2

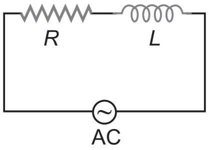
Single Correct Option

1. A capacitor and resistor are connected with an AC source as shown in figure. Reactance of capacitor is $X_C = 3\,\Omega$ and resistance of resistor is $4\,\Omega$. Phase difference between current I and I_1 is $\left[\tan^{-1}\left(\frac{3}{4}\right) = 37°\right]$



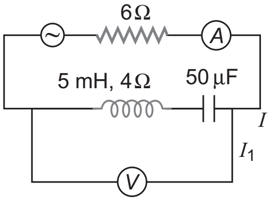
(a) 90°
(b) zero
(c) 53°
(d) 37°

2. A circuit contains resistance R and an inductance L in series. An alternating voltage $V = V_0 \sin \omega t$ is applied across it. The currents in R and L respectively will be



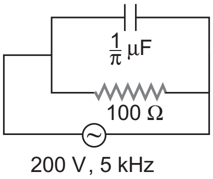
(a) $I_R = I_0 \cos \omega t, I_L = I_0 \cos \omega t$
(b) $I_R = -I_0 \sin \omega t, I_L = I_0 \cos \omega t$
(c) $I_R = I_0 \sin \omega t, I_L = -I_0 \cos \omega t$
(d) None of the above

3. In the circuit shown in figure, the AC source gives a voltage $V = 20 \cos (2000t)$. Neglecting source resistance, the voltmeter and ammeter readings will be



(a) 0 V, 2.0 A
(b) 0 V, 1.4 A
(c) 5.6 V, 1.4 A
(d) 8 V, 2.0 A

4. A signal generator supplies a sine wave of 200 V, 5 kHz to the circuit shown in the figure. Then, choose the **wrong** statement.



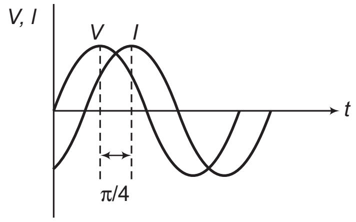
(a) The current in the resistive branch is 0.2 A
(b) The current in the capacitive branch is 0.126 A
(c) Total line current is ≈ 0.283 A
(d) Current in both the branches is same

5. A complex current wave is given by $i = (5 + 5 \sin 100\,\omega t)$ A. Its average value over one time period is given as

(a) 10 A (b) 5 A

(c) $\sqrt{50}$ A (d) 0

6. An AC voltage $V = V_0 \sin 100\,t$ is applied to the circuit, the phase difference between current and voltage is found to be $\pi/4$, then



(a) $R = 100\ \Omega, C = 1\ \mu\text{F}$ (b) $R = 1\ \text{k}\Omega, C = 10\ \mu\text{F}$

(c) $R = 10\ \text{k}\Omega, L = 1$ H (d) $R = 1\ \text{k}\Omega, L = 10$ H

7. In series *L-C-R* circuit, voltage drop across resistance is 8 V, across inductor is 6 V and across capacitor is 12 V. Then,

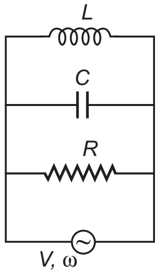
(a) voltage of the source will be leading in the circuit

(b) voltage drop across each element will be less than the applied voltage

(c) power factor of the circuit will be 3 /4

(d) None of the above

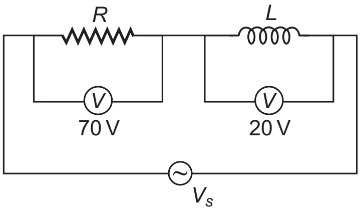
8. Consider an *L-C-R* circuit as shown in figure with an AC source of peak value V_0 and angular frequency ω. Then, the peak value of current through the AC source is



(a) $\dfrac{V_0}{\sqrt{\dfrac{1}{R^2} + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$ (b) $V_0\left[\dfrac{1}{R^2} + \left(\omega C - \dfrac{1}{\omega L}\right)\right]^2$

(c) $\dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$ (d) None of these

9. The adjoining figure shows an AC circuit with resistance *R*, inductance *L* and source voltage V_s. Then,



(a) the source voltage $V_s = 72.8$ V

(b) the phase angle between current and source voltage is $\tan^{-1}(7/2)$

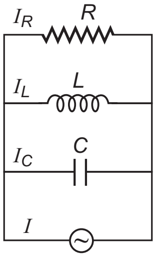
(c) Both (a) and (b) are correct

(d) Both (a) and (b) are wrong

10. When an alternating voltage of 220 V is applied across a device P, a current of 0.25 A flows through the circuit and it leads the applied voltage by an angle $\pi/2$ radian. When the same voltage source is connected across another device Q, the same current is observed in the circuit but in phase with the applied voltage. What is the current when the same source is connected across a series combination of P and Q?

(a) $\frac{1}{4\sqrt{2}}$ A lagging in phase by $\pi/4$ with voltage
(b) $\frac{1}{4\sqrt{2}}$ A leading in phase by $\pi/4$ with voltage
(c) $\frac{1}{\sqrt{2}}$ A leading in phase by $\pi/4$ with voltage
(d) $\frac{1}{4\sqrt{2}}$ A leading in phase by $\pi/2$ with voltage

11. In a parallel L-C-R circuit as shown in figure if I_R, I_L, I_C and I represent the rms values of current flowing through resistor, inductor, capacitor and the source, then choose the appropriate correct answer.

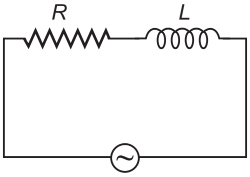


(a) $I = I_R + I_L + I_C$
(b) $I = I_R + I_L - I_C$
(c) I_L or I_C may be greater than I
(d) None of these

12. In a series L-C-R circuit, current in the circuit is 11 A when the applied voltage is 220 V. Voltage across the capacitor is 200 V. If the value of resistor is 20 Ω, then the voltage across the unknown inductor is

(a) zero
(b) 200 V
(c) 20 V
(d) None of these

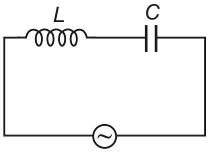
13. In the circuit shown in figure, the power consumed is



$V = V_0 \sin \omega t$

(a) zero
(b) $\frac{V_0^2}{2R}$
(c) $\frac{V_0^2 R}{2(R^2 + \omega^2 L^2)}$
(d) None of these

14. In a series L-C circuit, the applied voltage is V_0. If ω is very low, then the voltage drop across the inductor V_L and capacitor V_C are

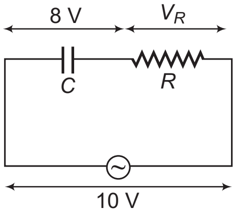


(a) $V_L = \frac{V_0}{2}; V_C = \frac{V_0}{2}$
(b) $V_L = 0; V_C = V_0$
(c) $V_L = V_0; V_C = 0$
(d) $V_L = -V_C = \frac{V_0}{2}$

15. A coil, a capacitor and an AC source of rms voltage 24 V are connected in series. By varying the frequency of the source, a maximum rms current of 6 A is observed. If coil is connected to a DC battery of emf 12 volt and internal resistance 4 Ω, then current through it in steady state is

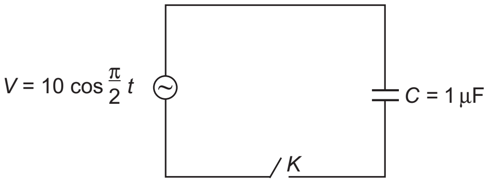
(a) 2.4 A
(b) 1.8 A
(c) 1.5 A
(d) 1.2 A

16. In a series *C*-*R* circuit shown in figure, the applied voltage is 10 V and the voltage across capacitor is found to be 8 V. The voltage across *R*, and the phase difference between current and the applied voltage will respectively be



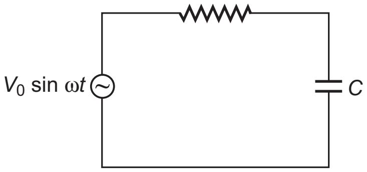
(a) 6 V, $\tan^{-1}\left(\frac{4}{3}\right)$
(b) 3 V, $\tan^{-1}\left(\frac{3}{4}\right)$
(c) 6 V, $\tan^{-1}\left(\frac{3}{4}\right)$
(d) None of these

17. An AC voltage source described by $V = 10\cos(\pi/2)t$ is connected to a 1 μF capacitor as shown in figure. The key *K* is closed at $t = 0$. The time ($t > 0$) after which the magnitude of current *I* reaches its maximum value for the first time is



(a) 1 s
(b) 2 s
(c) 3 s
(d) 4 s

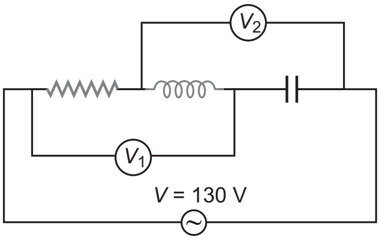
18. An AC voltage source $V = V_0 \sin \omega t$ is connected across resistance *R* and capacitance *C* as shown in figure. It is given that $R = 1/\omega C$. The peak current is I_0. If the angular frequency of the voltage source is changed to $\omega/\sqrt{3}$, then the new peak current in the circuit is



(a) $\frac{I_0}{2}$
(b) $\frac{I_0}{\sqrt{2}}$
(c) $\frac{I_0}{\sqrt{3}}$
(d) $\frac{I_0}{3}$

More than One Correct Options

1. In a R-L-C series circuit shown, the readings of voltmeters V_1 and V_2 are 100 V and 120 V. Choose the correct statement(s).



(a) Voltage across resistor, inductor and capacitor are 50 V, 86.6 V and 206.6 V respectively
(b) Voltage across resistor, inductor and capacitor are 10 V, 90 V and 30 V respectively
(c) Power factor of the circuit is $\frac{5}{13}$
(d) Circuit is capacitive in nature

2. Current in an AC circuit is given by $i = 3 \sin \omega t + 4 \cos \omega t$, then
(a) rms value of current is 5 A
(b) mean value of this current in positive one-half period will be $\frac{6}{\pi}$
(c) if voltage applied is $V = V_m \sin \omega t$, then the circuit may contain resistance and capacitance
(d) if voltage applied is $V = V_m \cos \omega t$, then the circuit may contain resistance and inductance only

3. A tube light of 60 V, 60 W rating is connected across an AC source of 100 V and 50 Hz frequency. Then,
(a) an inductance of $\frac{2}{5\pi}$ H may be connected in series
(b) a capacitor of $\frac{250}{\pi}$ μF may be connected in series to it
(c) an inductor of $\frac{4}{5\pi}$ H may be connected in series
(d) a resistance of 40 Ω may be connected in series

4. In an AC circuit, the power factor
(a) is unity when the circuit contains an ideal resistance only
(b) is unity when the circuit contains an ideal inductance only
(c) is zero when the circuit contains an ideal resistance only
(d) is zero when the circuit contains an ideal inductance only

5. In an AC series circuit, $R = 10\ \Omega$, $X_L = 20\ \Omega$ and $X_C = 10\ \Omega$. Then, choose the correct options
(a) Voltage function will lead the current function
(b) Total impedance of the circuit is $10\sqrt{2}\ \Omega$
(c) Phase angle between voltage function and current function is 45°
(d) Power factor of circuit is $\frac{1}{\sqrt{2}}$

6. In the above problem further choose the correct options.

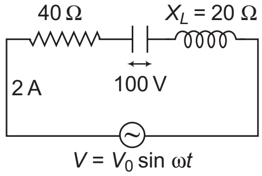
(a) The given values are at frequency less than the resonance frequency

(b) The given values are at frequency more than the resonance frequency

(c) If frequency is increased from the given value, impedance of the circuit will increase

(d) If frequency is decreased from the given value, current in the circuit may increase or decrease

7. In the circuit shown in figure,



(a) $V_R = 80$ V (b) $X_C = 50\ \Omega$

(c) $V_L = 40$ V (d) $V_0 = 100$ V

8. In *L-C-R* series AC circuit,

(a) If *R* is increased, then current will decrease (b) If *L* is increased, then current will decrease

(c) If *C* is increased, then current will increase (d) If *C* is increased, then current will decrease

Comprehension Based Questions

Passage I (Q. No. 1 to 3)

A student in a lab took a coil and connected it to a 12 *V DC source. He measures the steady state current in the circuit to be* 4 *A. He then replaced the* 12 *V DC source by a* 12 *V,* $(\omega = 50\ rad/s)$ *AC source and observes that the reading in the AC ammeter is* 2.4 *A. He then decides to connect a* 2500 μF *capacitor in series with the coil and calculate the average power developed in the circuit. Further he also decides to study the variation in current in the circuit (with the capacitor and the battery in series).*

Based on the readings taken by the student, answer the following questions.

1. The value of resistance of the coil calculated by the student is

(a) 3 Ω (b) 4 Ω

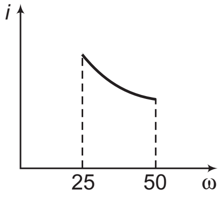
(c) 5 Ω (d) 8 Ω

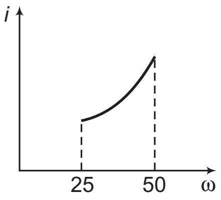
2. The power developed in the circuit when the capacitor of 2500 μF is connected in series with the coil is

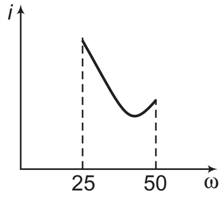
(a) 28.8 W (b) 23.04 W

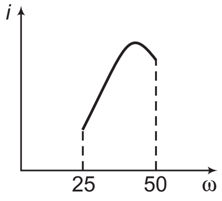
(c) 17.28 W (d) 9.6 W

3. Which of the following graph roughly matches the variations of current in the circuit (with the coil and capacitor connected in the series) when the angular frequency is decreased from 50 rad/s to 25 rad/s?

(a)


(b)


(c)


(d)


Passage II (Q. No. 4 to 6)

It is known to all of you that the impedance of a circuit is dependent on the frequency of source. In order to study the effect of frequency on the impedance, a student in a lab took 2 impedance boxes P and Q and connected them in series with an AC source of variable frequency. The emf of the source is constant at 10 V. Box P contains a capacitance of 1 μF in series with a resistance of 32 Ω. And the box Q has a coil of self-inductance 4.9 mH and a resistance of 68 Ω in series. He adjusted the frequency so that the maximum current flows in P and Q. Based on his experimental set up and the reading by him at various moment, answer the following questions.

4. The angular frequency for which he detects maximum current in the circuit is

(a) $10^5/7$ rad/s (b) 10^4 rad/s
(c) 10^5 rad/s (d) $10^4/7$ rad/s

5. Impedance of box P at the above frequency is

(a) 70 Ω (b) 77 Ω
(c) 90 Ω (d) 100 Ω

6. Power factor of the circuit at maximum current is

(a) 1/2 (b) 1
(c) 0 (d) $1/\sqrt{2}$

Match the Columns

1. Match the following two columns for a series AC circuit.

Column I	Column II
(a) Only C in the circuit	(p) current will lead
(b) Only L in the circuit	(q) voltage will lead
(c) Only R in the circuit	(r) $\phi = 90°$
(d) R and C in the circuit	(s) $\phi = 0°$

2. Applied AC voltage is given as

$$V = V_0 \sin \omega t$$

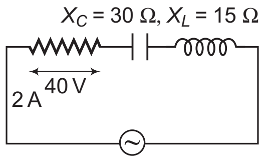
Corresponding to this voltage, match the following two columns.

Column I	Column II
(a) $I = I_0 \sin \omega t$	(p) only R circuit
(b) $I = -I_0 \cos \omega t$	(q) only L circuit
(c) $I = I_0 \sin (\omega t + \pi/6)$	(r) may be C-R circuit
(d) $I = I_0 \sin (\omega t - \pi/6)$	(s) may be L-C-R circuit

3. For an L-C-R series AC circuit, match the following two columns.

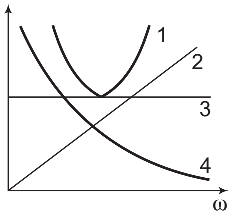
Column I	Column II
(a) If resistance is increased	(p) current will increase
(b) If capacitance is increased	(q) current will decrease
(c) If inductance is increased	(r) current may increase or decrease
(d) If frequency is increased	(s) power may decrease or increase

4. In the circuit shown in figure, match the following two columns. In Column II, quantities are given in SI units.



Column I	Column II
(a) Value of resistance R	(p) 60
(b) Potential difference across capacitor	(q) 20
(c) Potential difference across inductor	(r) 30
(d) Applied potential difference	(s) None of the above

5. Corresponding to the figure shown, match the two columns.



Column I	Column II
(a) Resistance	(p) 4
(b) Capacitive reactance	(q) 1
(c) Inductive reactance	(r) 2
(d) Impedance	(s) 3

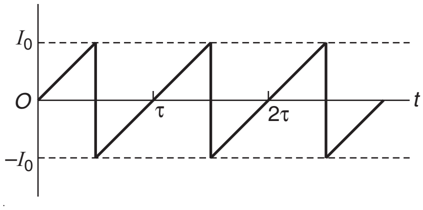
Subjective Questions

Note *Power factor leading means current is leading.*

1. A coil is in series with a 20 μF capacitor across a 230 V, 50 Hz supply. The current taken by the circuit is 8 A and the power consumed is 200 W. Calculate the inductance of the coil if the current in the circuit is

(a) leading (b) lagging

2. The current in a certain circuit varies with time as shown in figure. Find the average current and the rms current in terms of I_0.

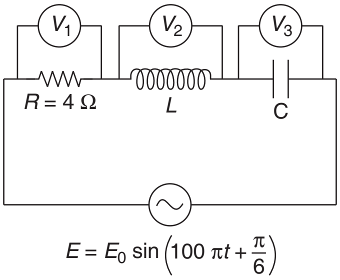


3. Two impedances Z_1 and Z_2 when connected separately across a 230 V, 50 Hz supply consume 100 W and 60 W at power factor of 0.5 lagging and 0.6 leading respectively. If these impedances are now connected in series across the same supply, find

(a) total power absorbed and overall power factor

(b) the value of reactance to be added in series so as to raise the overall power factor to unity.

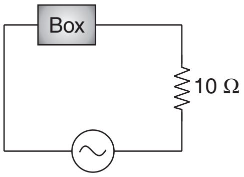
4. In the figure shown, the reading of voltmeters are $V_1 = 40$ V, $V_2 = 40$ V and $V_3 = 10$ V. Find



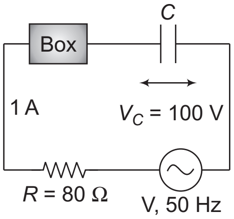
(a) the peak value of current (b) the peak value of emf

(c) the value of L and C

5. In the circuit shown in figure power factor of box is 0.5 and power factor of circuit is $\sqrt{3}/2$. Current leading the voltage. Find the effective resistance of the box.



6. A circuit element shown in the figure as a box is having either a capacitor or an inductor. The power factor of the circuit is 0.8, while current lags behind the voltage. Find



(a) the source voltage V,

(b) the nature of the element in box and find its value.

7. The maximum values of the alternating voltages and current are 400 V and 20 A respectively in a circuit connected to 50 Hz supply and these quantities are sinusoidal. The instantaneous values of the voltage and current are $200\sqrt{2}$ V and 10 A, respectively. At $t = 0$, both are increasing positively.

(a) Write down the expression for voltage and current at time t.

(b) Determine the power consumed in the circuit.

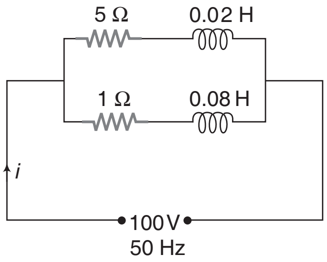
8. An L-C circuit consists of an inductor coil with $L = 5.00$ mH and a $20.0\,\mu$F capacitor. There is negligible resistance in the circuit. The circuit is driven by a voltage source with $V = V_0 \cos\omega t$. If $V_0 = 5.00$ mV and the frequency is twice the resonance frequency, determine

(a) the maximum charge on the capacitor

(b) the maximum current in the circuit

(c) the phase relationship between the voltages across the inductor, the capacitor and the source.

9. A coil having a resistance of 5 Ω and an inductance of 0.02 H is arranged in parallel with another coil having a resistance of 1 Ω and an inductance of 0.08 H. Calculate the power absorbed when a voltage of 100 V at 50 Hz is applied.



10. A circuit takes a current of 3 A at a power factor of 0.6 lagging when connected to a 115 V – 50 Hz supply. Another circuit takes a current of 5 A at a power factor of 0.707 leading when connected to the same supply. If the two circuits are connected in series across a 230 V, 50 Hz supply, then calculate

(a) the current (b) the power consumed and (c) the power factor

Answers

Introductory Exercise 28.1

1. (a) 628 Ω (b) 6.37 mH (c) 1.59 kΩ (d) 1.59 mF **2.** 0.036 H, 111.8 V **3.** 7.7 H, 6 A

Introductory Exercise 28.2

1. 650 Hz, 0 **2.** 0.2

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (b) **3.** (a) **4.** (b) **5.** (a or b) **6.** (b) **7.** (a) **8.** (a,b) **9.** (b) **10.** (b)
11. (c)

Objective Questions

1. (c) **2.** (b) **3.** (d) **4.** (d) **5.** (a) **6.** (d) **7.** (d) **8.** (d) **9.** (a) **10.** (a)
11. (b) **12.** (b) **13.** (c) **14.** (c) **15.** (b) **16.** (d) **17.** (d) **18.** (b) **19.** (c) **20.** (c)
21. (b) **22.** (a) **23.** (a) **24.** (a) **25.** (a) **26.** (c) **27.** (c) **28.** (c) **29.** (a) **30.** (b)

Subjective Questions

1. (a) 0.326 A (b) 35.3°, lagging (c) 97.8 V, 32.6 V, 102 V

2. (a) Inductor (b) 0.133 H

3. (b) 1000 Hz (c) $\frac{\sqrt{3}}{2}$, leading (d) $R = 17.32\,\Omega, C = 15.92\,\mu\text{F}$

4. (a) 0.12 A (b) 1.2×10^{-2} A (c) 1.2×10^{-3} A

5. (a) (8.33 mA)cos (950 rad/s)t (b) 760 Ω (c) – (6.33 V) sin (950 rad/s)t

6. (a) 45.8°, voltage leads the curren,t 0.697 (b) 343 Ω (c) 155 V (d) 48.6 W (e) 48.6 W (f) 0 (g) 0

LEVEL 2

Single Correct Option

1.(c) **2.**(d) **3.**(c) **4.**(b) **5.**(b) **6.**(b) **7.**(d) **8.**(b) **9.**(a) **10.**(b)
11.(c) **12.**(b) **13.**(c) **14.**(b) **15.**(c) **16.**(a) **17.**(a) **18.**(b)

More than One Correct Options

1.(a,c,d) **2.**(c,d) **3.**(c,d) **4.**(a,d) **5.**(a,b,c,d) **6.**(b,c,d) **7.**(a,b,c) **8.**(a)

Comprehension Based Questions

1.(a) **2.**(c) **3.**(b) **4.**(a) **5.**(b) **6.**(b)

Match the Columns

1. (a) → p,r (b) → q,r (c) → s (d) → p
2. (a) → p,s (b) → q (c) → r,s (d) → s
3. (a) → q,s (b) → r,s (c) → r,s (d) → r,s
4. (a) → q (b) → p (c) → r (d) → s
5. (a) → s (b) → p (c) → r (d) → q

Subjective Questions

1. (a) 0.416 H (b) 0.597 H **2.** zero, $\frac{I_0}{\sqrt{3}}$ **3.** (a) 99 W, 0.92 leading (b) 194.2 Ω

4. (a) $10\sqrt{2}$A (b) $50\sqrt{2}$V (c) $\frac{1}{25\pi}$H, (d) $\frac{1}{100\pi}$F **5.** 5 Ω

6. (a) 100 V (b) inductor, $L = \frac{1.6}{\pi}$H

7. (a) $V = 400\sin(100\pi t + \pi/4)$, $i = 20\sin(100\pi t + \pi/6)$ (b) $P = 3864$ W

8. (a) 33.4 nC (b) 0.211 mA
(c) Source and inductor voltages in phase. Capacitor voltage lags by 180°.

9. 797 W **10.** (a) 5.5 A (b) 1.188 kW (c) 0.939 lag

Hints & Solutions

23. Current Electricity

INTRODUCTORY EXERCISE 23.1

1. $\because \quad i = \dfrac{q}{t} = \dfrac{ne}{t}$

$$\therefore \qquad n = \frac{it}{e} = \frac{(0.7)\,(1)}{1.6 \times 10^{-19}}$$
$$= 4.375 \times 10^{18}$$

2. $\because \quad q = it$

$$= (3.6)\,(3 \times 60 \times 60)$$
$$= 38880 \text{ C}$$

3. (a) $\because \quad q = it = (7.5)\,(45) = 337.5 \text{ C}$

(b) $n = \dfrac{q}{e} = \dfrac{337.5}{1.6 \times 10^{-19}}$

$$= 2.1 \times 10^{21}$$

4. $f = \dfrac{v}{2\pi r} = \dfrac{2.2 \times 10^{6}}{(2\pi)\,(5.3 \times 10^{-11})}$

$$= 6.6 \times 10^{15} \text{ Hz}$$
$$I = qf$$
$$= (1.6 \times 10^{-19})\,(6.6 \times 10^{15})$$
$$= 1.06 \times 10^{-3} \text{ A}$$
$$= 1.06 \text{ mA}$$

5. $\Delta q = \int_0^{10} i\,dt = \int_0^{10} (10 + 4t)\,dt$

$$= 300 \text{ C}$$

6. Current due to both is from left to right. So, the two currents are additive.

INTRODUCTORY EXERCISE 23.2

1. False. Only under electrostatic conditions (when $i = 0$) all points of a conductor are at same potential.

INTRODUCTORY EXERCISE 23.3

1. $i = neAv_d$

i.e. $v_d \propto i$

When current has increased from $i = 1.2$ A to $i = 6.0$ A, i.e five times, then drift velocity will also increase to five times.

2. From $i = neAv_d$

We have $\qquad v_d = \dfrac{i}{neA}$

or $\qquad v_d = \dfrac{1}{8.5 \times 10^{28} \times 1.6 \times 10^{-19} \times 10^{-4}}$

$$= 0.735 \times 10^{-6} \text{ m/s}$$
$$= 0.735\,\mu\text{ m/s}$$

$$t = \frac{l}{v_d}$$
$$= \frac{10 \times 10^{3}}{0.735 \times 10^{-6}} \text{ s}$$
$$= \frac{10 \times 10^{3}}{0.735 \times 10^{-6} \times 60 \times 60 \times 24 \times 365} \text{ yr}$$
$$= 431.4 \text{ yr}$$

INTRODUCTORY EXERCISE 23.4

1. $R = \rho\,\dfrac{l}{A} = \dfrac{(1.72 \times 10^{-8})\,(35)}{(\pi/4)\,(2.05 \times 10^{-3})^2} = 0.18\ \Omega$

2. $\rho = \dfrac{1}{\sigma} \quad \Rightarrow \quad \rho\sigma = \text{constant}$

3. $R = \dfrac{\rho l}{A}$

$$\therefore \qquad A = \frac{\rho l}{R}$$

$m = (Vd)$, where V = volume and d = density

$$\therefore \qquad m = (Ald)$$
$$= \frac{\rho l^2}{R}\,d$$
$$= \frac{(1.72 \times 10^{-8})\,(3.5)^2\,(8.9 \times 10^{3})}{0.125}$$
$$= 15 \times 10^{-3} \text{ kg}$$
$$= 15 \text{ g}$$

4. $R = \dfrac{\rho\,(L)}{A} = \dfrac{\rho L}{tL} = \dfrac{\rho}{t}$

i.e. R is independent of L.

Hence, the correct option is (c).

INTRODUCTORY EXERCISE 23.5

1. Copper is metal and germanium is semiconductor. Resistance of a metal decreases and that of a semiconductor increases with decrease in temperature.

$\therefore$ Correct option is (d).

2. $4.1[1+4.0\times10^{-3}(\theta-20)]$

$$= 3.9[1+5.0\times10^{-3}(\theta-20)]$$

Solving we get,

$$\theta \approx 85°\text{C}$$

INTRODUCTORY EXERCISE 23.6

1. PD across each resistance is 10 V.

$$\therefore \quad i_{2\Omega} = \frac{10}{2} = 5\text{A}$$

$$i_{4\Omega} = \frac{10}{4} = 2.5\text{A}$$

2. $V_A = 0$ V (as it is earthed)

$$V_C - V_A = 5 \text{ V}$$

$$\therefore \quad V_C = 5 \text{ V}$$

$$V_B - V_A = 2 \text{ V}$$

$$\therefore \quad V_B = 2 \text{ V}$$

$$V_D - V_C = 10 \text{ V}$$

$$\therefore \quad V_D = 10 + V_C = 15 \text{ V}$$

$$i_{1\Omega} = \frac{V_C - V_B}{1} = 3\text{A from } C \text{ to } B \text{ as } V_C > V_B$$

$$i_{2\Omega} = \frac{V_D - V_A}{2} = 7.5\text{A from } D \text{ to } A \text{ as } V_D > V_A$$

3. $V_A = V_B$

$$\therefore \quad V_{AB} = 0$$

or

$$E - ir = 0$$

$$\therefore \quad E - \left(\frac{15+E}{8}\right)(2) = 0$$

Solving this equation, we get

$$E = 5 \text{ V}$$

4. Net emf $= (n - 2m)E$

$$= (10 - 2\times2)(1)$$

$$= 6 \text{ V}$$

$$i = \frac{\text{Net emf}}{\text{Net resistance}}$$

$$= \frac{6}{10+2} = 0.5 \text{ A}$$

5. $V_{R_1} = 0$

$$\therefore \quad i_{R_1} = 0$$

$$V_{R_2} = V_{R_3} = 10 \text{ V}$$

$$\therefore \quad i_{R_2} = i_{R_3} = \frac{10}{10} = 1 \text{ A}$$

INTRODUCTORY EXERCISE 23.7

1. Applying loop law equation in upper loop, we have

$$E + 12 - ir - 1 = 0 \quad \text{...(i)}$$

Applying loop law equation in lower loop, we have where

$$i = 1 + 2 = 3\text{A}$$

$$E + 6 - 1 = 0 \quad \text{...(ii)}$$

Solving these two equations, we get

$$E = -5 \text{ V and } r = 2\Omega$$

2. Power delivered by a battery $= Ei$

$$= 12 \times 3$$

$$= 36 \text{ W}$$

Power dissipated in resistance

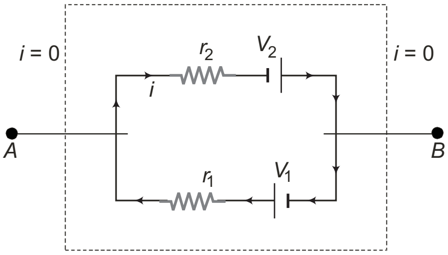
$$= i^2R = (3)(2)^2$$

$$= 12 \text{ W}$$

INTRODUCTORY EXERCISE 23.8

1. **(a) Equivalent emf (V) of the battery**

PD across the terminals of the battery is equal to its emf when current drawn from the battery is zero. In the given circuit,



Current in the internal circuit,

$$i = \frac{\text{Net emf}}{\text{Total resistance}} = \frac{V_1 + V_2}{r_1 + r_2}$$

Therefore, potential difference between A and B would be

$$V_A - V_B = V_1 - ir_1$$

$$\therefore \quad V_A - V_B = V_1 - \left(\frac{V_1+V_2}{r_1+r_2}\right)r_1 = \frac{V_1r_2 - V_2r_1}{r_1+r_2}$$

So, the equivalent emf of the battery is

$$V = \frac{V_1r_2 - V_2r_1}{r_1+r_2}$$

Note that if $V_1r_2 = V_2r_1 : V = 0$

If $V_1r_2 > V_2r_1 : V_A - V_B$ = Positive i.e. A side of the equivalent battery will become the positive terminal and *vice-versa.*

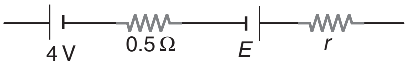
(b) **Internal resistance (r) of the battery**

r_1 and r_2 are in parallel. Therefore, the internal resistance r will be given by

$$1/r = 1/r_1 + 1/r_2$$

or $$r = \frac{r_1r_2}{r_1 + r_2}$$

2.



$$E = \frac{\Sigma\,(E/r)}{\Sigma\,(1/r)} = \frac{(6/1) - (2/1)}{(1/1) + (1/1)} = 2\text{ V}$$

Now, net emf of E and 4 V is 2V as they are oppositely connected.

3.
$$E_{\text{eq}} = \frac{\Sigma\,(E/r)}{\Sigma\,(1/r)}$$
$$= \frac{(10/1) + (4/2) + (6/2)}{(1/1) + (1/2) + (1/2)}$$
$$= 7.5\text{V}$$
$$\frac{1}{r} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2}$$
$$\therefore \quad r = 0.5\ \Omega$$

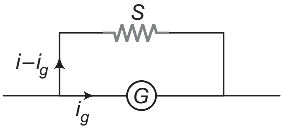
INTRODUCTORY EXERCISE 23.9

1. $V = i_g\,(G + R)$

$\therefore\ R = \frac{V}{i_g} - G$ = series resistance connected with galvanometer

$$= \left(\frac{5}{5 \times 10^{-3}}\right) - 1 = 999\ \Omega$$

2.



$$\frac{i_g}{i - i_g} = \frac{S}{G}$$
$$\therefore \quad S = \left(\frac{i_g}{i - i_g}\right) G$$
$$= \left[\frac{(50 \times 10^{-6})}{(5 \times 10^{-3} - 50 \times 10^{-6})}\right](100) \approx 1.0\ \Omega$$

3. $V = i_g\,G$
$$\therefore \quad i_g = \frac{V}{G}$$
Now, $$nV = i_g\,(G + R) = \frac{V}{G}\,(G + R)$$
$$\therefore \quad R = (n - 1)\,G$$

INTRODUCTORY EXERCISE 23.10

1.
$$r = R\left(\frac{l_1}{l_2} - 1\right)$$
$$= 5\left(\frac{0.52}{0.4} - 1\right)$$
$$= 1.5\ \Omega$$

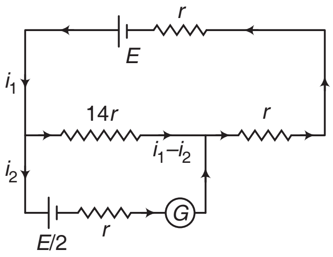
2. (a) $V_{AJ} = \frac{E}{2}$ or emf of lower battery

$$\therefore \quad i\,R_{AJ} = \frac{E}{2}$$
or $$\left(\frac{E}{15r + r}\right)\left(\frac{15r}{600}\right)(l) = \frac{E}{2}$$

Solving this equation, we get
$$l = 320\text{ cm}$$

(b) Resistance of 560 cm $= \left(\frac{15r}{600}\right)(560)$
$$= 14r$$

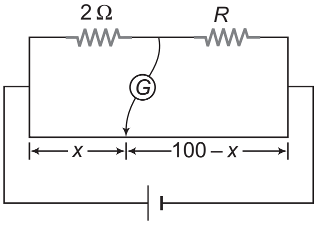
Now the circuit is as under,

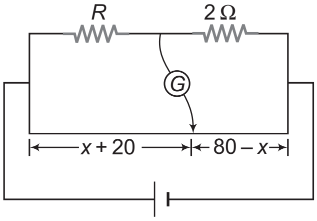


Applying loop law in upper loop we have,
$$E - 14r\,(i_1 - i_2) - i_1r - i_1r = 0 \qquad \ldots\text{(i)}$$
Applying loop law in lower law loop we have,
$$-\frac{E}{2} - i_2r + (i_1 - i_2)\,(14r) = 0 \qquad \ldots\text{(ii)}$$
Solving these two equations we get, $$i_2 = \frac{3E}{22r}$$

INTRODUCTORY EXERCISE 23.11

1. $R > 2\,\Omega \quad \Rightarrow \quad 100 - x > x$





Applying $\dfrac{P}{Q} = \dfrac{R}{S}$

We have
$$\frac{2}{R} = \frac{x}{100 - x} \qquad \ldots\text{(i)}$$
$$\frac{R}{2} = \frac{x + 20}{80 - x} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get $R = 3\,\Omega$

∴ Correct option is (a).

2. Using the concept of balanced, Wheatstone bridge, we have,
$$\frac{P}{Q} = \frac{R}{S} \quad \Rightarrow \quad \frac{X}{(52 + 1)} = \frac{10}{(48 + 2)}$$

$$\therefore \qquad X = \frac{10 \times 53}{50} = 10.6\,\Omega$$

∴ Correct option is (b).

3. Slide wire bridge is most sensitive when the resistance of all the four arms of bridge is same. Hence, B is the most accurate answer.

INTRODUCTORY EXERCISE 23.12

1. $\dfrac{P}{Q} = \dfrac{R}{X} \quad \Rightarrow \quad X = \left(\dfrac{Q}{P}\right) R = \left(\dfrac{1}{10}\right) R$

R lies between 142 Ω and 143 Ω.

Therefore, the unknown resistance X lies between 14.2 Ω and 14.3 Ω.

2. Experiment can be done in similar manner but now K_2 should be pressed first then K_1.

3. BC, CD and BA are known resistances.

The unknown resistance is connected between A and D.

INTRODUCTORY EXERCISE 23.13

1. Yellow → 4

Red → 2

Orange → 10^3

Gold → 5

$\therefore \quad R = (4.2 \times 10^3 \pm 5\%)\,\Omega$

2. 2 → Red

4 → Yellow

10^6 → Blue

5% → Gold

Exercises

LEVEL 1

Assertion and Reason

1. If PD between two terminals of a resistance is zero, then current through resistance is zero, this is confirmed. But PD between any two points of a circuit is zero, this does not mean current is zero.

2. In parallel, V = constant

∴ From the equation
$$P = \frac{V^2}{R} \quad \Rightarrow \quad P \propto \frac{1}{R}$$

3. $R = \dfrac{\rho l}{A}$ or $\dfrac{R}{l}$ = resistance per unit length $= \dfrac{\rho}{A} \propto \dfrac{1}{A}$

Near A, area of cross-section is less. Therefore, resistance per unit length will be more. Hence from the equation, $H = i^2Rt$, heat generation near A will be more.

Current density, $J = \dfrac{i}{A}$ or $J \propto \dfrac{1}{A}$ (as i is same)

4. Since net resistance decreases, therefore main current increases. Hence, net potential difference across voltmeter also increases.

5. Even if ammeter is non-ideal, its resistance should be small and net parallel resistance is less than the smallest individual resistance.

$\therefore$ $R_{\text{net}} <$ resistance of ammeter in the changed situation. Hence, net resistance of the circuit will decrease. So, main current will increase. But maximum percentage of main current will pass through ammeter (in parallel combination) as its resistance is less. Hence, reading of ammeter will increase.

Initial voltmeter reading = emf of battery

Final voltmeter reading = emf of battery

− potential drop across shown resistance.

Hence, voltmeter reading will decrease.

6. If current flows from a to b, then equation will become

$$V_a - ir - E = V_b \quad \text{or} \quad V_a - V_b = E + ir$$

So, $V_a - V_b$ is always positive. Hence, V_a is always greater than V_b.

7. Current in the circuit will be maximum when $R = 0$.

8. Resistance will increase with temperature on heating. Hence current will decrease.

Further $\quad P = \dfrac{V^2}{R}$ or $P \propto \dfrac{1}{R}$

Resistance is increasing. Hence, power consumed across R should decrease.

$V = IR$ is just an equation between P D across a resistance current passing through it and its resistance. This is not Ohm's law.

9. Electrons get accelerated by the electric. Then, suddenly collision takes place. Then, again accelerated and so on.

11. $$E_{\text{eq}} = \frac{E_1/r_1 + E_2/r_2}{(1/r_1) + (1/r_2)}$$

$$= E_1\left[\frac{(1/r_1) + (E_2/E_1)\,(1/r_2)}{(1/r_1) + (1/r_2)}\right]$$

So, E_{eq} may be greater than E_1 also, if $E_2/E_1 > 1$

r_1 and r_2 are in parallel. Hence, r_{eq} is less than both r_1 and r_2 individually.

Objective Questions

3. $H = I^2 Rt$

$$\therefore \quad [R] = \left[\frac{H}{I^2 t}\right]$$

$$= \left[\frac{\text{ML}^2\text{T}^{-2}}{\text{I}^2\text{T}}\right] = [\text{ML}^2\text{T}^{-3}\text{I}^{-2}]$$

4. $$R = \frac{l}{\sigma A}$$

$$\therefore \quad \sigma = \frac{l}{RA} = \frac{\text{m}}{\text{ohm - m}^2} = \text{ohm}^{-1}\text{-m}^{-1}$$

6. $$0.5 = \frac{E}{r + 3.75} \qquad \text{...(i)}$$

$$0.4 = \frac{E}{r + 4.75} \qquad \text{...(ii)}$$

Solving these two equations, we get

$$E = 2\text{V}$$

7. In parallel current distributes in increase ratio of resistance

$$\therefore \quad \frac{I_G}{I_S} = \frac{S}{G}$$

$$\therefore \quad G = \left(\frac{I_S}{I_G}\right)(S)$$

$$= \left(\frac{50 - 20}{20}\right)(12)$$

$$= 18\ \Omega$$

8. $$\frac{I_G}{I_S} = \frac{S}{G}$$

$$\therefore \quad S = \left(\frac{I_G}{I_S}\right)G$$

$$= \left(\frac{2}{98}\right)G = \frac{G}{49}$$

9. $$P = \frac{V^2}{R} \quad \text{or} \quad P \propto \frac{1}{R}$$

$$\frac{P_2}{P_1} = \frac{R_1}{R_2} = \frac{l_1}{l_2} \qquad (\text{as } R \propto l)$$

$$\therefore \quad P_2 = \left(\frac{l_1}{l_2}\right)P_1 = \left(\frac{l}{0.9\,l}\right)P_1 = 1.11\,P_1$$

So, power will increase by 11%.

10. By symmetry, $V_A = V_B$

or $\quad V_{AB} = 0$

11. $$r = R\left(\frac{l_1}{l_2} - 1\right) = 10\left(\frac{75}{60} - 1\right)$$

$$= 2.5\ \Omega$$

12. Let $V_0 = V$

Now, $\quad I_{AO} + I_{BO} = I_{OC}$

$$\therefore \quad \frac{6 - V}{6} + \frac{3 - V}{3} = \frac{V - 2}{2}$$

Solving this equation, we get

$$V = 3\ \text{V}$$

13. $v_d = \frac{i}{neA} = \frac{i}{ne\,(\pi r^2)} \quad \Rightarrow \quad v_d \propto \frac{i}{r^2}$

14. For making voltmeter of higher range, more resistance is required.

15. $V_{20\Omega} = V_{\text{Total}}$

$$\therefore \qquad (20)\,(0.3) = (R_{\text{Total}})\,(0.8)$$

$$\therefore \qquad R_{\text{Total}} = \frac{30}{4}\,\Omega$$

$$\therefore \qquad \frac{4}{30} = \frac{1}{R_1} + \frac{1}{20} + \frac{1}{15}$$

Solving we get $R_1 = 60\Omega$

16. Net resistance will decrease by increasing the parallel resistors. Therefore, main current i will increase, further,

$$(\text{PD})_{\text{Voltmeter}} = V_{\text{Total}} - (\text{PD})_{\text{Ammeter}}$$

$$= V_{\text{Total}} - i \quad (\text{resistance of ammeter})$$

Since, i has increased. Hence, PD across voltmeter will decrease.

17. $P = \frac{V^2}{R} \quad \text{or} \quad P \propto \frac{1}{R} \qquad \ldots\text{(i)}$

Resistor is cut in n equal parts. Therefore, each resistance will become $\frac{R}{n}$. Now, these are connected in parallel. Therefore, net resistance will become $\frac{1}{n}$ times $\frac{R}{n}$. or $\frac{R}{n^2}$.

Now, from Eq. (i), power will become n^2 times.

18. If A is fused, then complete circuit is broken.

19. $E - ir = 0$

$$\therefore \qquad 3 - \left(\frac{3+15}{1+2+R}\right)(1) = 0$$

Solving this equation, we get

$$R = 3\Omega$$

20. $100 = (5)\left[\frac{(2500\,R)}{2500 + R}\right]$

Solving this equation, we get

$$R = 20\Omega$$

21. Five parallel combination, each of value

$$\frac{R}{10} + \frac{R}{10} = \frac{R}{5}$$

22. $\frac{P}{Q} = \frac{20}{100 - 20} \quad \text{or} \quad Q = 4P \qquad \ldots\text{(i)}$

$$\therefore \qquad P < Q$$

Now, $$\frac{P+15}{Q} = \frac{40}{100-40}$$

or $$\frac{P+15}{Q} = \frac{2}{3} \qquad \ldots\text{(ii)}$$

Solving these two equations, we get

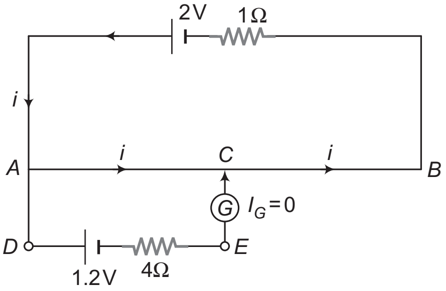
$$P = 9\Omega$$

23. Total potential of 10V equally distributes between 50 Ω and other parallel combination of 100 Ω and voltmeter. Hence, their net resistance should be same. Or

$$\frac{100 \times R}{100 + R} = 50$$

$\therefore \quad R = 100\ \Omega =$ resistance of voltmeter

24. $V_{AC} = V_{DE}$



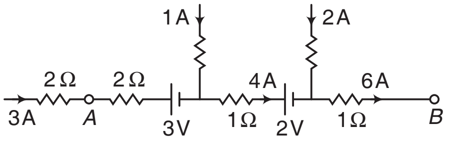
$$i\,(R_{AC}) = E = 1.2$$

$$\therefore \qquad \left(\frac{2}{4+1}\right)\left(\frac{4}{100} \times l\right) = 1.2$$

Solving this equation, we get

$$l = 75\ \text{cm}$$

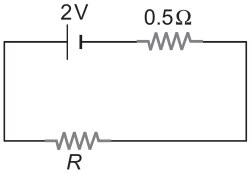
25.



$$V_A - 3 \times 2 - 3 - 1 \times 4 + 2 - 1 \times 6 = V_B$$

$$\therefore \qquad V_A - V_B = 17\ \text{V}$$

26.



Equivalent simple circuit is given as maximum power across R is obtained

When $\qquad R = r = 0.5\ \Omega$

$$i = \frac{2}{R + r} = 2\text{A}$$

$$\therefore \qquad = i^2R = (2)^2\,(0.5) = 2\text{W}$$

27. $r = R\left(\frac{E}{V} - 1\right) = 5\left(\frac{2.2}{1.8} - 1\right) = \frac{10}{9}\ \Omega$

28. $i = \frac{10 - 5}{2.5 + 2.5 + 40} = \frac{1}{9}\ \text{A}$ (clockwise)

$$V_B - 15\,i - 25\,i = V_A$$

$$\therefore \quad V_A - V_B = 40\,i = -\frac{40}{9}\ \text{V}$$

29. Potential drop across potentiometer wire

$$= (0.2 \times 10^{-3})\,(100) = 0.02\ \text{V}$$

Now given resistance and potentiometer wire are in series with given battery. So, potential will drop in direct ratio of resistance.

$$\therefore \quad \frac{0.02}{2 - 0.02} = \frac{R}{490}$$

$$\therefore \quad R = 4.9\Omega$$

30. When K is open

$$R_{\text{net}} = 3R/2$$

$$\therefore \quad i_1 = E/(3R/2) = \frac{2E}{3R}$$

When K is closed

$$R_{\text{net}} = 2\left[\frac{R \times 2R}{R + 2R}\right] = \frac{4}{3}R$$

$$\therefore \quad i_2 = E/(4R/3) = \frac{3E}{4R}$$

$$\therefore \quad \frac{i_1}{i_2} = \frac{8}{9}$$

31. $\frac{I_G}{I_S} = \frac{S}{G}$

$$\therefore \quad S = \left(\frac{I_G}{I_S}\right) G$$

$$= \frac{(1/34)}{(33/34)} \times 3663 = 111\ \Omega$$

32. Simple series and parallel grouping of resistors.

33. Two balanced Wheatstone bridges in parallel.

34. $R_{ab} = \frac{(3 \times 15)}{3 + 15} = 2.5\ \Omega$

As $\quad R_{60°} = \left(\frac{18}{360°}\right)(60°) = 3\Omega$

35. $R_{AB} = 2\,[\text{Net resistance of infinite series}] + 1$

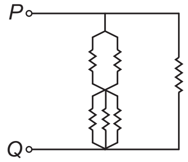
In parallel net resistance is always less than the smallest one. Hence, net resistance of infinite series is less than 1Ω.

$$\therefore \quad 1\Omega < R_{AB} < 3\Omega$$

36. $R_0 = R_{AB} = R + \frac{(R + R_0)\,R}{(R + R_0) + R}$

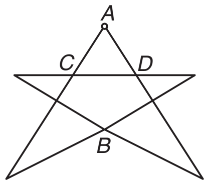
Solving this equation, we get $R = \frac{R_0}{\sqrt{3}}$

37. Simple circuit is as shown in figure,

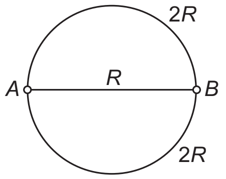


Each → R

38. Wheatstone (balanced) between A and B. So, resistance between C and D can removed.



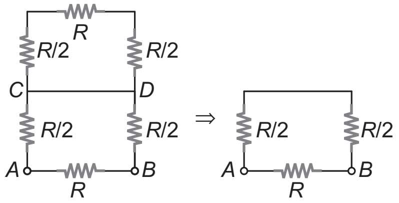
39.



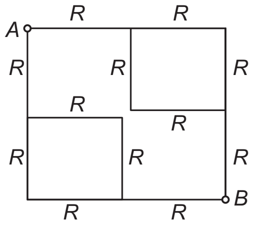
$$R = \left(\frac{4}{2\pi r}\right)(2r) = \frac{4}{\pi}$$

Now 2Ω, 2Ω and R are in parallel.

40.



41.



Connection can be removed from centre. $3R$ and $3R$ from two sides of AB are in parallel.

Subjective Questions

1. Under electrostatic conditions (when no current flows), $E = 0$. When current is non-zero, then electric field is also non-zero.

2. There is random or thermal motion of free electrons in the absence of potential difference.

3.
$$i = qf = q\left(\frac{v}{2\pi R}\right)$$
$$= \frac{(1.6\times10^{-19})\,(2.2\times10^{6})}{(2\pi)\,(5\times10^{-11})}$$
$$= 1.12\times10^{-3}\ \text{A} = 1.12\ \text{mA}$$

4. $P = \dfrac{V^2}{R}$

$$\therefore \qquad R = \frac{V^2}{P}$$
$$R_1 = \frac{(120)^2}{40} = 360\ \Omega$$
$$R_2 = \frac{(120)^2}{60} = 240\ \Omega$$
$$R_3 = \frac{(120)^2}{75} = 192\ \Omega$$

Now, all these resistors are in parallel.

5. (a) $i = \dfrac{12-6}{4+8} = 0.5$ A

(b) $P_{R_1} = i^2R_1 = 1\text{W}$

$\Rightarrow \quad P_{R_2} = i^2R_2 = 2\text{W}$

(c) Power supplied by $E_1 = E_1\, i = 6$ W

and power consumed by $E_2 = E_2 i = 3\text{W}$

6. $\dfrac{8}{12} = \dfrac{l}{40-l}$

Solving this equation, we get

$$l = 16\ \text{cm}$$

7. (a) Ideal voltmeter means infinite resistance.

$\therefore \qquad i = 0$

(b) $\qquad V = E \qquad$ (if $i = 0$)

$\qquad = 5$ V

(c) Reading of voltmeter $= E = 5$ V

8. (a) $E_1 > E_2$

Therefore, net current is anti-clockwise or from B to A.

(b) Current through E_1 is normal. Hence, it is doing the positive work.

(c) Current flows from B to A

$\therefore \qquad V_B > V_A$

9. $i = \dfrac{150-50}{2+3} = 20$ A (anti-clockwise)

$$V_Q + 150 - 20\times2 = V_P$$
$$\therefore \qquad V_Q = V_P - 110 = -10\ \text{V}$$

10. $\rho = 8.89\times10^3\ \text{kg/m}^3$

Mass of $1\text{m}^3 = 8.89\times10^3$ kg

$= 8.89\times10^3\ \text{kg} = 8.89\times10^6\text{g}$

$\therefore$ Number of gram moles $= \dfrac{8.89\times10^6}{63.54} = 1.4\times10^5$

Number of atoms $= 1.4\times10^5\times6.02\times10^{23}$

$= 8.42\times10^{28}$

One atom emits one conduction electron. Therefore, number of free electrons in unit volume (or 1m^3 volume)

$$n = 8.42\times10^{28}\ \text{per m}^3$$

Now, $\qquad i = neA\,v_d$

$$\therefore \qquad v_d = \frac{i}{neA} = \frac{i}{ne\,\pi r^2}$$
$$= \frac{2.0}{(8.42\times10^{28})\,(1.6\times10^{-19})(\pi)\,(0.5\times10^{-3})^2}$$
$$= 1.9\times10^{-4}\ \text{m/s}$$

11. (a) In 1 m, potentials difference,

$V = 0.49 \quad \therefore \quad V = iR$

$$\therefore \qquad i = \frac{0.49}{R} = \frac{(0.49)\,A}{\rho l}$$
$$= \frac{(0.49)\,(\pi/4)\,(0.84\times10^{-3})^2}{(2.75\times10^{-8})\,(1)}$$
$$= 9.9\ \text{A}$$

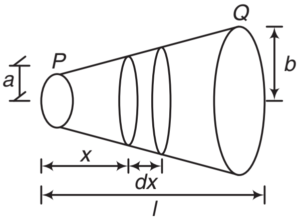
(b) PD between two points, 12 m apart

$$= (0.49\ \text{V/m})\,(12\,\text{m}) = 5.88\ \text{V}$$

(c) $R = \dfrac{V}{i} = \dfrac{5.88}{9.9} = 0.6\ \Omega$

12. Radius at distance x from end P,

$$r = a + \left(\frac{b-a}{l}\right)x$$



Resistance of element of thickness dx is

$$dR = \frac{\rho\,(dx)}{\pi r^2} \qquad \left(\text{Using } R = \frac{\rho l}{A}\right)$$

$$\therefore \qquad R = \int_{X=0}^{X=l} dR$$

13. $i = \dfrac{E}{R + r} \quad \Rightarrow \quad P = \text{power across } R = i^2 R$

$$P = \left(\frac{E}{R + r}\right)^2 R \qquad \ldots\text{(i)}$$

For power to be maximum,

$$\frac{dP}{dR} = 0$$

By putting $\dfrac{dP}{dR} = 0$ we get, $R = r$

Further, by putting $R = r$ in Eq. (i)

We get, $\qquad P_{\max} = \dfrac{E^2}{4r}$

14. As derived in the above question,

$$P_{\max} = \frac{E^2}{4r}$$

Here, $\quad E =$ net emf $= 2 + 2 = 4$ V

and $\quad r =$ net internal resistance

$$= 1 + 1 = 2\Omega$$

$$\therefore \qquad P_{\max} = \frac{(4)^2}{(4)\,(2)} = 2\text{W}$$

15. In series,

$$\alpha_{\text{eq}} = \frac{R_{01}\,\alpha_1 + R_{02}\,\alpha_2}{R_{01} + R_{02}}$$

$$= \frac{(600)\,(0.001) + (300)\,(0.004)}{600 + 300}$$

$$= 0.002 \text{ per}\,^\circ\text{C}$$

Now, $R_t = R_0\,[1 + \alpha\,\Delta\theta]$

$$= (600 + 300)\,[1 + 0.002 \times 30] = 954\Omega$$

16. In parallel current distributes in inverse ratio of resistance 1 → Aluminium 2 → Copper

$$\frac{R_1}{R_2} = \frac{i_2}{i_1}$$

$$\frac{\rho_1 l_1 / A_1}{\rho_2 l_2 / A_2} = \frac{2}{3}$$

$$\therefore \qquad \frac{\rho_1 l_1 d_2^2}{\rho_2 l_2 d_1^2} = \frac{2}{3}$$

$$\therefore \qquad d_2 = \left(\sqrt{\frac{2\rho_2 l_2}{3\rho_1 l_1}}\right) d_1$$

$$= \left(\sqrt{\frac{2 \times 0.017 \times 6}{3 \times 0.028 \times 7.5}}\right) (1 \text{ mm})$$

$$= 0.569 \text{ mm}$$

17. (a) $E = \dfrac{V}{l} = \dfrac{0.938}{0.75} = 1.25$ V/m

(b) $E = J\rho$

$$\therefore \qquad \rho = \frac{E}{J} = \frac{1.25}{4.4 \times 10^7}$$

$$= 2.84 \times 10^{-8}\ \Omega\text{-m}$$

18. (a) $J = \dfrac{i}{A} = \dfrac{V}{RA} = \dfrac{V}{\left(\dfrac{\rho l}{A}\right) A}$

$$J = \frac{V}{\rho l} \qquad \ldots\text{(i)}$$

or $\qquad J \propto \dfrac{1}{l}$

$l_{\min} = d$. So, J is maximum. Hence, potential difference should be applied across the face $(2d \times 3d)$

From Eq. (i),

$$J_{\max} = \frac{V}{\rho d}$$

(b) $i = \dfrac{V}{R} = \dfrac{V}{(\rho l / A)} = \dfrac{VA}{\rho l}$

or $\qquad i \propto \dfrac{A}{l}$

Across face $(2d \times 3d)$, area of cross-section is maximum and l is minimum. Hence, current is maximum.

$$i_{\max} = \frac{V\,(2d \times 3d)}{\rho\,(d)} = \frac{6Vd}{\rho}$$

19. (a) $\rho = \dfrac{RA}{l} = \dfrac{(0.104)\,(\pi/4)\,(2.5 \times 10^{-3})^2}{14}$

$$= 3.65 \times 10^{-8}\ \Omega\text{-m}$$

(b) $V = El = 1.28 \times 14 = 17.92$ V

$$\therefore \qquad i = \frac{V}{R} = \frac{17.92}{0.104} = 172.3 \text{ A}$$

(c) $v_d = \dfrac{i}{neA}$

$$= \frac{172.3}{(8.5 \times 10^{28})\,(1.6 \times 10^{-19})\left(\dfrac{\pi}{4}\right)(2.5 \times 10^{-3})^2}$$

$$= 2.58 \times 10^{-3} \text{ m/s}$$

20. $$R_1 + R_2 = 20 \qquad \ldots\text{(i)}$$

$$\alpha_{\text{eq}} = \frac{R_1\alpha_1 + R_2\alpha_2}{R_1 + R_2} \qquad \text{(in series)}$$

$$0 = \frac{R_1(-0.5\times10^{-3}) + R_2(5.0\times10^{-3})}{20}$$

$$\therefore \quad R_1 = 10R_2 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$R_2 = R_{\text{Fe}} = \frac{20}{11}\,\Omega$$

$$= 1.82\Omega$$

$$R_1 = R_{\text{Cu}}$$

$$= 10R_2$$

$$= 18.18\ \Omega$$

21. 8Ω and 12Ω resistors are in parallel.

$$\therefore \quad R_{\text{net}} = \frac{8\times12}{8+12} = 4.8\ \Omega$$

$$\therefore \quad i = \frac{24}{4.8}$$

$$= 5\text{A}$$

22. All four resistors are in parallel

$$\therefore \quad \frac{1}{R} = \frac{1}{8} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$$

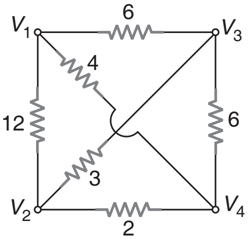
$$R = \frac{8}{5}\,\Omega$$

$$\therefore \quad i = \frac{24}{8/5}$$

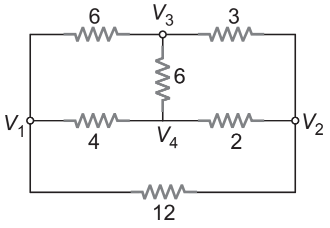
$$= 15\ \text{A}$$

23. All these resistors are in parallel.

24. The given network is as shown below.



The simple circuit is as shown below.



Now, this is a balanced Wheatstone bridge in parallel with 12 Ω resistance.

25. **First case**

$$i = \frac{12+6}{1+2+3} = 3\text{A} \qquad \text{(clockwise)}$$

Now, $\quad V_A - V_G = 12\ \text{V}$

$$\therefore \quad V_A = 12\ \text{V}, \text{ as } V_G = 0$$

$$V_A - V_B = 1\times3 = 3\ \text{V}$$

$$\therefore \quad V_B = V_A - 3 = 9\ \text{V}$$

$$V_B - V_C = 2\times3 = 6\ \text{V}$$

$$\therefore \quad V_C = V_B - 6 = 3\ \text{V}$$

$$V_G - V_D = 6\ \text{V}$$

$$\therefore \quad V_D = -6\ \text{V}, \text{ as } V_G = 0$$

In the second case,

$$i = \frac{12-6}{1+2+3} = 1\ \text{A}$$

Rest procedure is same.

26. $$i = \frac{200}{5+10+25} = 5\ \text{A} \qquad \text{(anti-clockwise)}$$

$$V_3 - V_G = 25\times5 = 125$$

$$\therefore \quad V_3 = 125\ \text{V as } V_G = 0$$

$$V_G - V_2 = 10\times5 = 50$$

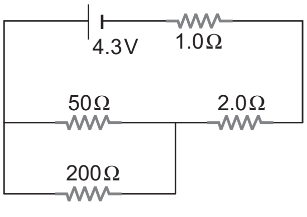
$$\therefore \quad V_2 = -50\text{V}$$

$$V_2 - V_1 = 5\times5 = 25\ \text{V}$$

$$\therefore \quad V_1 = V_2 - 25 = -75\ \text{V}$$

Now, $\quad V_{3-2} = V_3 - V_2$

27. (a)



$$R_{\text{net}} = 1.0 + 2.0 + \frac{50\times200}{50+200}$$

$$= 43\ \Omega$$

$$\therefore \quad i = \frac{4.3}{43} = 0.1\ \text{A}$$

= Readings of ammeter

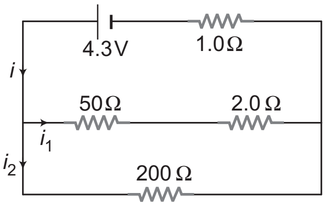
Readings of voltmeter

= (i) net resistance of 50 Ω and 200 Ω

$$= (0.1)\left(\frac{50\times200}{50+200}\right)$$

$$= 4\ \text{V}$$

(b)



$$R_{\text{net}} = 1.0 + \frac{52 \times 200}{52 + 200} = 42.27\ \Omega$$

$$\therefore \quad i = \frac{4.3}{42.27} \approx 0.1\ \text{A}$$

$$\text{Now} \quad \frac{i_1}{i_2} = \frac{200}{52}$$

$$\therefore \quad i_1 = \left(\frac{200}{252}\right)(0.1) = 0.08\ \text{A}$$

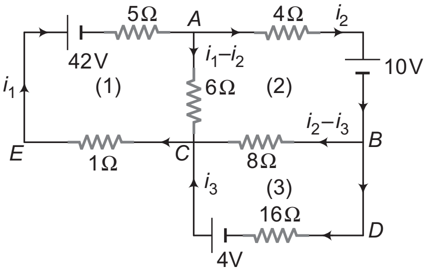
= Reading of ammeter

∴ Reading to voltmeter

= Potential difference across 50 Ω and 2.0 Ω

$= 0.08 \times 52 \approx 4.2$ V

28.



Loop 1

$$-42 - 6(i_1 - i_2) - 5i_1 - i_1 = 0 \qquad \text{...(i)}$$

Loop 2

$$-4i_2 - 10 - 8(i_2 - i_3) + 6(i_1 - i_2) = 0 \qquad \text{...(ii)}$$

Loop 3

$$8(i_2 - i_3) = -16\, i_3 + 4 = 0 \qquad \text{...(iii)}$$

Solving these equations, we get

$$i_1 = 4\ \text{A},\ i_2 = 1.0\ \text{A and } i_3 = 0.5\ \text{A}$$

29. Net resistance of voltmeter ($R = 400\ \Omega$) and 400 Ω will be 200 Ω. Now, we are getting a balanced Wheatstone bridge with 100 Ω and 200 Ω resistors on each side. Potential difference across each side will be 10 V which will distribute in direct ratio of resistors 100 Ω and 200 Ω.

$$\therefore \quad \frac{V_{100\,\Omega}}{V_{200\,\Omega}} = \frac{100}{200} = \frac{1}{2}$$

$$\text{or} \quad V_{200\,\Omega} = \left(\frac{2}{3}\right)(10) = \frac{20}{3}\ \text{V}$$

30. (a) (i) When switch S is open, V_1 and V_2 are in series, connected to 200 V battery. Potential will drop in direct ratio of their resistors.

$$\therefore \quad V_1 : V_2 = R_{V_1} : R_{V_2} = 3000 : 2000$$

$$= 3 : 2$$

$$\therefore \quad V_1 = \frac{3}{5} \times 200 = 120\ \text{V}$$

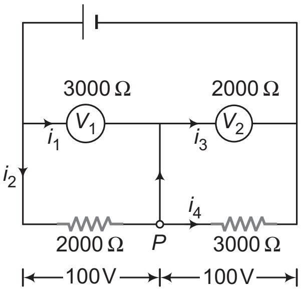
$$V_2 = \frac{2}{5} \times 200 = 80\ \text{V}$$

(ii) When S is closed then V_1 and R_1 are in parallel. Similarly, V_2 and R_2 are also in parallel. Now, they are in series and they come out to be equal. So, 200 V will equally distribute between them.

$$\therefore \quad V_1 = V_2 = \frac{200}{2} = 100\ \text{V each}$$

$$\text{(b)} \quad i_2 = \frac{100}{2000} = \frac{1}{20}\ \text{A}$$

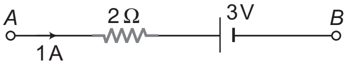
$$i_4 = \frac{100}{3000} = \frac{1}{30}\ \text{A}$$



If we apply junction law at P, then current through switch

$$= i_2 - i_4 = \frac{1}{60}\ \text{A in upward direction.}$$

31. Power absorbed by resistor is i^2R or 2W. Therefore, remaining 3W is absorbed by the battery (= Ei). Hence, E is 3 V and current of 1 A enters from the position terminal as shown below.



$$V_{AB} = E + ir$$

$$= E + iR \qquad \text{(Here, } r = R\text{)}$$

$$= 3 + (1)(2) = 5\ \text{V}$$

32. $8.4 = E - 1.5\,r$ ...(i)

$9.4 = E + 3.5\,r$ ...(ii)

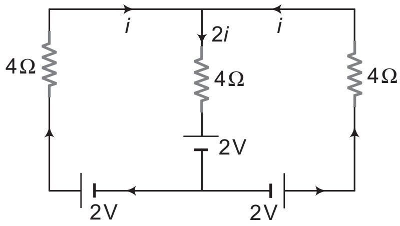
Solving these two equations, we get

$$r = 0.2\ \Omega \quad \text{and} \quad E = 8.7\ \text{V}$$

33. During charging,

$$V = E + ir = 2 + (5)(0.1) = 2.5 \text{ V}$$

34. Simple circuit is as shown below



By symmetry, currents on two sides will be same (let i)

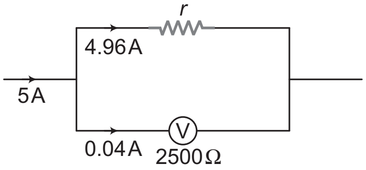
Now if we apply loop law in any of the closed loop, we will get $i = 0$.

35. Net resistance should remain unchanged.

$$\therefore \quad R + G = R' + \frac{GS}{G+S}$$

$$\therefore \quad R' - R = G - \frac{GS}{G+S} = \frac{G^2}{G+S}$$

36. Current through voltmeter



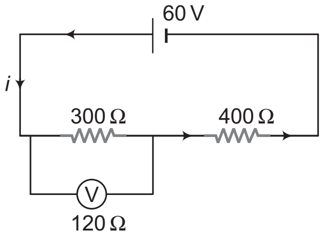
$$= \frac{V}{R} = \frac{100}{2500} = 0.04 \text{ A}$$

In parallel current distribution in inverse ratio of resistors. Hence,

$$\frac{4.96}{0.04} = \frac{2500}{r}$$

$$\therefore \quad r = 20.16\ \Omega$$

37. Voltmeter reads 30 V, half of 60 V. Hence, resistance of 400 Ω and voltmeter is also equal to 300 Ω.



$$\therefore \quad 300 = \left(\frac{400 \times R}{400 + R}\right)$$

where, R = resistance of voltmeter.

Solving the above equation, we get

$$R = 1200\ \Omega$$

In the new situation,

$$R_{\text{net}} = 400 + \frac{(300)(1200)}{300 + 1200} = 640\ \Omega$$

$$\therefore \quad i = \frac{60}{640} = 0.09375 \text{ A}$$

Now voltage drop across

Voltmeter = 60 – potential drop across 400 Ω resistor

= 60 – (400) i

= 60 – (400) (0.09375)

= 22.5 V

38. $R_{\text{net}} = 60 + \dfrac{(60)(120)}{60 + 120} = 100\ \Omega$

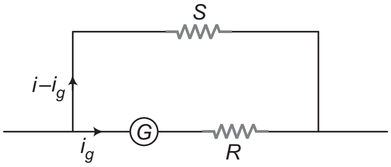
$$\therefore \quad i = \frac{120}{100} = 1.2 \text{ A}$$

Now, reading of voltmeter

= 120 – potential drop across R_1

= 120 – (60) (1.2) = 48 V

39. In parallel current distributes in inverse ratio of resistance.



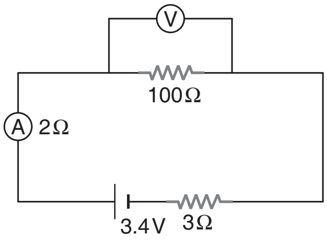
$$\therefore \quad \left(\frac{i - i_g}{i_g}\right) = \frac{G + R}{S}$$

$$\therefore \quad R = \left(\frac{i - i_g}{i_g}\right) S - G$$

$$= \left(\frac{20}{10^{-3}}\right)(0.005) - 20 = 80\ \Omega$$

Note *In calculations, we have taken* $i - i_g \approx i$.

40.



Reading of voltmeter = 3.4 – Voltage drop across ammeter and 3Ω resistance

$$= (3.4) - 0.04 \times 2 - 0.04 \times 3$$

$$= 3.2 \text{ V}$$

Now, $\qquad 3.2 \text{ V} = \dfrac{(0.04)\,(100\,R)}{(100 + R)}$

where, R = resistance of voltmeter

$\therefore \qquad R = 400\ \Omega$

If voltmeter is ideal, then

$$i = \frac{3.4}{2 + 100 + 3} = 0.03238 \text{ A}$$

Reading of voltmeter $= 100i = 3.238$ V

41. (a) $V = E - ir$

$$= E - \left(\frac{E}{r + R_V}\right) r$$

$$V = \left(\frac{ER_V}{r + R_V}\right) \qquad \ldots\text{(i)}$$

(b) $V = \dfrac{E}{100}$

Substituting in Eq. (i), we get

$$R_V = 4.5 \times 10^{-3}\ \Omega$$

(c) $V = E\left(\dfrac{1}{1 + \dfrac{r}{R_V}}\right)$

If R_V is increased from this value, V will increase.

42. (a) $\qquad I_A = \dfrac{\varepsilon}{R + R_A + r}$

$\therefore \qquad \varepsilon = (R + R_A + r)\, I_A$

Now, $\qquad I'_A = \dfrac{\varepsilon}{R + r} \qquad \ldots\text{(i)}$

Substituting the value of ε, we get

$$I'_A = I_A\left(1 + \frac{R_A}{R_A + r}\right)$$

If $\qquad R_A \to 0,\ I'_A \to I_A$

(b) In Eq. (i) substituting

$I_A = 0.99\, I'_A$ and the given values, we get

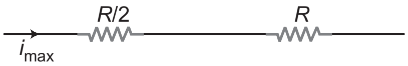
$R_A = 0.0045\ \Omega$

(c) $I_A = \dfrac{I'_A}{1 + \dfrac{R_A}{R_A + r}} = \dfrac{I'_A}{1 + \dfrac{1}{1 + \dfrac{r}{R_A}}}$

If R_A is decreased from this value, then I_A will increase from 99% of I'_A.

43. $P = i^2 R$

$\therefore \qquad i_{\max} = \sqrt{\dfrac{P_{\max}}{R}} = \sqrt{\dfrac{36}{2.4}} = \sqrt{15}$ A



Total maximum power $= (i_{\max})^2 \left(\dfrac{3R}{2}\right)$

$$= (15)\,(1.5)\,(2.4) = 54 \text{ W}$$

44. $V = E - ir$

$\therefore \qquad r = \dfrac{E - V}{i} = \dfrac{2.6 - 2}{1}$

$$= 0.6\ \Omega$$

Now, power generated in the battery

$$P = i^2 r$$

$$= (1)^2\,(0.6) = 0.6 \text{ W}$$

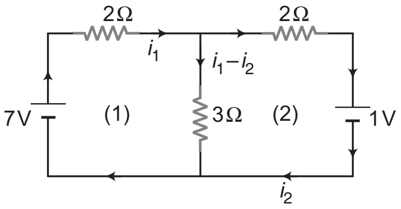
Power supplied by the battery $= Ei$

$$= 2.6 \text{ W}$$

$\therefore$ Net power supplied for external circuit

$$= 2.6 - 0.6 = 2.0 \text{ W}$$

45.



Loop equation in loop (1)

$$+7 - 2i_1 - 3\,(i_1 - i_2) = 0 \qquad \ldots\text{(i)}$$

Loop equation in loop (2)

$$-1 + 3\,(i_1 - i_2) - 2i_2 = 0 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$i_1 = 2 \text{A} \quad \text{and} \quad i_2 = 1 \text{ A}$$

Power supplied by $E_1 = E_1 i_1 = 14$ W

But power consumed by $E_2 = E_2 i_2 = 1$W

46. (a) $\qquad i = \dfrac{12}{5 + 1} = 2$ A

$\Rightarrow \qquad P_1 = Ei = 24$ W

(b) $P_2 = i^2 r = (2)^2\,(1) = 4$W

(c) $P = P_1 - P_2 = 20$ W

47. (b) $i = \dfrac{V}{R}$ in each resistance

(e) $P = \dfrac{V^2}{R}$ in each resistance

(f) $$P = \frac{V^2}{R}$$

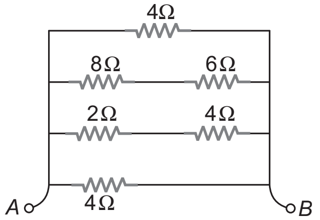
or $$P \propto \frac{1}{R} \qquad \text{(as } V \text{ is same)}$$

48. (a) $P = \dfrac{V^2}{R}$

$\therefore \quad V = \sqrt{PR} = \sqrt{5 \times 15 \times 10^3} = 273.8 \text{ V}$

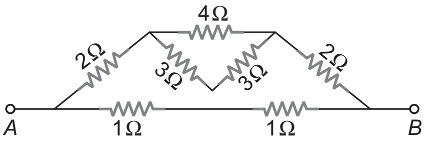
(b) $P = \dfrac{V^2}{R} = \dfrac{(120)^2}{9 \times 10^3} = 1.6 \text{ W}$

49. (a)

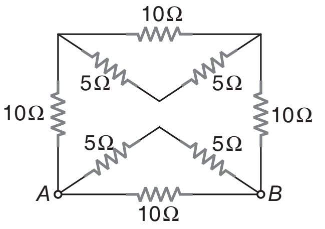


(b) A balanced Wheatstone bridge in parallel with R.

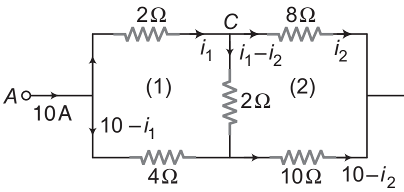
(c)



(d)



(e)



Let us take a current of 10 A between A and B

Loop equation in loop (1)

$$-2i_1 - 2(i_1 - i_2) + 4(10 - i_1) = 0 \qquad \text{...(i)}$$

Loop equation in loop (2)

$$-8i_2 + 10(10 - i_2) + 2(i_1 - i_2) = 0 \qquad \text{...(ii)}$$

Solving these two equations, we get

$$i_1 = 6.53 \text{ A} \quad \text{and} \quad i_2 = 6.11 \text{ A}$$

Now let us find V_{AB} across path ACB,

$$V_{AB} = 2i_1 + 8i_2 = 61.94 \text{ V}$$

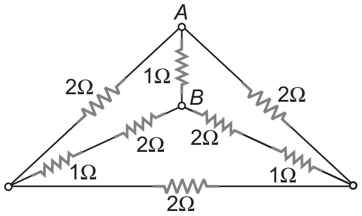
Now, $V_{AB} = i_{\text{net}}\ R_{\text{net}} = (10)\ R_{\text{net}}$

$\therefore \quad 61.94 = 10\ R_{\text{net}}$

or $\quad R_{\text{net}} = 6.194\ \Omega$

(f) Two resistors in vertical middle wire can be removed.

(g) Now balanced Wheatstone bridge, in parallel with 1Ω resistance between points A and B. The encircled resistance of 2Ω can be removed from the Wheatstone bridge.



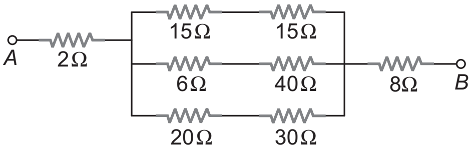
One resistor of 2Ω has already removed from the original circuit given the question. As its two ends will be the same potential (by symmetry).

50. $R_i + 5R$

If connected by two equal resistors between B and D and between C and E, the combination is a balanced Wheatstone bridge and two resistors in series.

$\therefore \quad R_f = R + R + R = 3R = 0.6\ R_i$

51.



52. (b) Three resistors are in parallel. Then, one resistor in series with this combination.

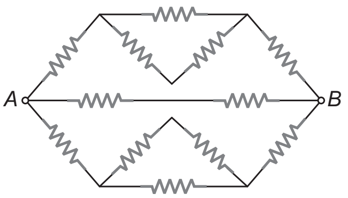
(c) Balanced Wheatstone bridge. Hence, two resistors in vertical wire can be removed.

(d) All four resistors are in parallel

(e) A balanced Wheatstone bridge.

53. (a) A balanced Wheatstone bridge with one resistance in parallel.

(b)



LEVEL 2

Single Correct Option

1. No current will flow through voltmeter. As it is ideal (infinite resistance). Current through two batteries

$$i = \frac{1.5 - 1.3}{r_1 + r_2} = \frac{0.2}{r_1 + r_2}$$

Now, $V = E_2 - ir_2$

$$\therefore \quad 1.45 = 1.5 - \left(\frac{0.2}{r_1 + r_2}\right)(r_2)$$

Solving this equation, we get

$$r_1 = 3r_2$$

2. In series, PD distributes in direct ratio of resistance.

In first case,

$$\frac{198}{V_{AB} - 198} = \frac{900}{R_1} \qquad \ldots\text{(i)}$$

In second case, $$\frac{180}{V_{AB} - 180} = \frac{900}{2R_1} \qquad \ldots\text{(ii)}$$

Solving these two equations, we get

$$V_{AB} = 220\text{ V}$$

3. Maximum current will pass through A.

$$P = i^2R$$

or $$P \propto i^2 \qquad (R\text{ is same})$$

4. $$4\,(R + R_A) = 20\text{ V}$$

$$\therefore \quad R = 5 - R_A$$

where, R_A = resistance of ammeter

5. $$r = R\left(\frac{l_1}{l_2} - 1\right) = 132.4\left(\frac{70}{60} - 1\right) \approx 22.1\,\Omega$$

6. Initial current, $i_1 = \dfrac{E_1 + E_2}{R + r_1 + r_2}$

Final current, when second battery is short circuited is

$$i_2 = \frac{E_1}{R + r_1}$$

$$i_2 > i_1 \text{ if } \frac{E_1}{R + r_1} > \frac{E_1 + E_2}{R + r_1 + r_2}$$

or $$E_1R + E_1r_1 + E_1r_2 > E_1R + E_1r_1 + E_2R + E_2\,r_1$$

or $$E_1r_2 > E_2\,(R + r_1)$$

7. B and C are in parallel

$$\therefore \quad V_B = V_C$$

Further $$R_A = R$$

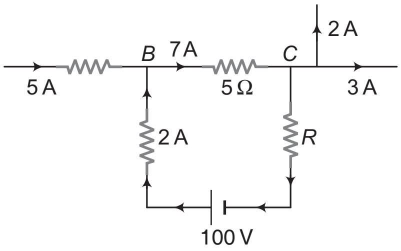
$$R_{BC} = \frac{(1.5R)\,(3R)}{1.5\,R + 3R} = R$$

or $$R_A = R_{BC}$$

$$\therefore \quad V_A = V_{BC}$$

Because $$iR_A = iR_{BC}$$

8.



Applying loop equation in closed loop we have,

$$+\,100 - 30 - 35 - 2R = 0$$

$$\therefore \quad 2R = 35\text{ V} = V_R$$

$$V_{5\Omega} = 7 \times 5 = 35\text{ V}$$

$$\therefore \quad \frac{V_{5\Omega}}{V_R} = 1$$

9. $$r = R\left(\frac{l_1}{l_2} - 1\right)$$

$$= R\left(\frac{y}{x} - 1\right) = \left(\frac{y - x}{x}\right)R$$

10. Let $R = at + b$

At $t = 10$ s, $R = 20\,\Omega$

$$\therefore \quad 20 = 10\,a + b \qquad \ldots\text{(i)}$$

At $$t = 30a + b \qquad \ldots\text{(ii)}$$

Solving these two equations, we get

$$a = 1.0\,\Omega/\text{s}$$

and $$b = 10\,\Omega$$

$$\therefore \quad R = (t + 10)$$

$$i = \frac{E}{R} = \frac{10}{t + 10}$$

$$\Delta q = \int_{10}^{30} i\,dt$$

$$= \int_{10}^{30}\left(\frac{10}{t + 10}\right)dt$$

$$= 10\log_e(2)$$

11. Suppose n (< 1) fraction of length is stretched to m times.

Then, $$(1 - n)\,l + (nl)\,m = 1.5l$$

or $$nm - n = 0.5 \qquad \ldots\text{(i)}$$

$$R = \frac{\rho l}{A} = \frac{\rho l}{(V/l)} \qquad (V = \text{volume})$$

$$= \frac{\rho l^2}{V}$$

or $\quad R \propto l^2 \qquad$ (if V = constant)

Now, the second condition is

$$(1-n)\,R + (nR)\,m^3 = 4R$$

$\therefore \quad nm^2 - n = 3 \qquad \ldots$(iii)

Solving these two equations, we get

$$n = \frac{1}{8}$$

12. Initially, $\quad \dfrac{l_1}{100 - l_1} = \dfrac{X}{R} = \dfrac{2}{3}$

$\therefore \quad l_1 = \dfrac{2}{5} \times 100$

$= 40$ cm

Finally, $\quad \dfrac{l_2}{100 - l_2} = \dfrac{X}{R'} = \dfrac{12}{8} = \dfrac{3}{2}$

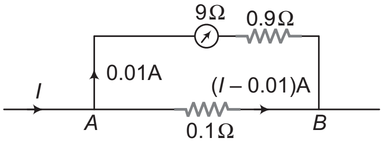
$\therefore \quad l_2 = \dfrac{3}{5} \times 100$

$= 60$ cm

$\therefore \quad J$ is displaced by

$$l_2 - l_1 = 20 \text{ cm}$$

13. In parallel, current distributes in inverse ratio of resistance.



$\therefore \quad \dfrac{0.01}{I - 0.01} = \dfrac{0.1}{9 + 0.9}$

Solving we get, $\quad I = 1$ A

14. Equivalent emf of two batteries ε_1 and ε_2 is

$$\varepsilon = \frac{\varepsilon_1/r_1 + \varepsilon_2/r_2}{1/r_1 + 1/r_2} = \frac{(2/2) + (4/6)}{(1/2) + (1/6)}$$

$= 2.5$ V

Now, $\quad V_{AN} = \varepsilon$

$\therefore \quad (I_{AN})\,(R_{AN}) = \varepsilon$

or $\quad \left(\dfrac{12}{4 + 4 \times 4}\right)(4)\,(l) = 2.5$

Solving this equation, we get

$$l = \frac{25}{24} \text{ m}$$

15. When K_1 and K_2 both are closed R_1 is short-circuited,

$$R_{\text{net}} = (50 + r)\ \Omega$$

When K_1 is open and K_2 is closed, current remains half.

Therefore, net resistance of the circuit becomes two times.

or $\quad (50 + r) + R_1 = 2\,(50 + r)$

Of the given options, the above equation is satisfied if

$$r = 0 \quad \text{and} \quad R_1 = 50\ \Omega$$

16. 100 Ω, 25 Ω and 20 Ω are in parallel.

Their, net resistance is 10 Ω

$\therefore \quad R_{\text{net}} = 4\,\Omega + 10\Omega + 6\Omega = 20\Omega$

$V = i\,R_{\text{net}} = 80$ V

17. All these resistors are in parallel.

$\therefore \quad R_{\text{net}} = \dfrac{R}{3} + r = 4\,\Omega$

Hence, the main current

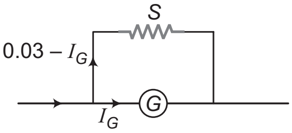
$$i = \frac{E}{R_{\text{net}}} = 1 \text{ A}$$

Current through either of the resistance

is $\quad \dfrac{i}{3} \quad$ or $\quad \dfrac{1}{3}$ A

$\therefore \quad V = iR = \left(\dfrac{1}{3}\right)(9) = 3$ V

18.



In parallel, current distributers in inverse ratio of resistance.

$$\frac{0.03 - I_G}{I_G} = \frac{G}{S} = \frac{r}{(r/4)} = 4$$

Solving this equation, we get

$$I_G = 0.006 \text{ A}$$

19. $\quad \dfrac{8\Omega}{6\Omega} = \dfrac{4\,\Omega}{3\Omega}$

$\therefore \quad V_A = V_B$

or $\quad V_{AB} = 0$

20. $V = iR$

$\therefore \quad V \propto R \qquad$ (as i = constant)

$\therefore \quad \dfrac{V_A}{V_B} = \left(\dfrac{\rho l_A}{\pi r_A^2}\right)\left(\dfrac{\pi r_B^2}{\rho l_B}\right)$

or $$\frac{r_B}{r_A} = \sqrt{\frac{V_A}{V_B} \times \frac{l_B}{l_A}}$$

$$= \sqrt{\frac{3}{2} \times \frac{1}{6}} = \frac{1}{2}$$

21. Current decreases $\frac{20}{30}$ times or $\frac{2}{3}$ times. Therefore, net resistance should become $\frac{3}{2}$ times.

$$\therefore \quad R + 50 = \frac{3}{2}(2950 + 50)$$

Solving we get, $R = 4450\,\Omega$

22. $$E_0 = V_{AC} = (i)_{AC}\,(R)_{AC}$$

$$= \left(\frac{E}{10}\right)\left(\frac{10}{1} \times 0.2\right)$$

$$= \frac{E}{5} \qquad \text{...(i)}$$

In second case,

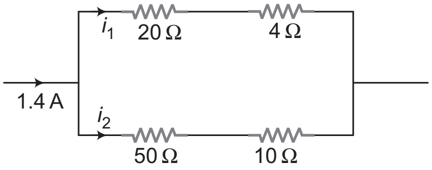
$$E_0 = \left(\frac{E}{10 + x}\right)\left(\frac{10}{1} \times 0.3\right) \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x = 5\,\Omega$$

23. V and V_0 are oppositely connected.

24. Balanced Wheatstone bridge. Hence, 1.5 Ω resistance can be removed from the circuit.



$$\frac{i_1}{i_2} = \frac{50 + 10}{20 + 4} = \frac{2.5}{1}$$

$$\therefore \quad i_1 = \left(\frac{2.5}{2.5 + 1}\right)(1.4) = 1\text{ A}$$

25. Resistance between A and B can be removed due to balanced Wheatstone bridge concept. Now, R_{DE} and R_{GH} are in series and they are connected in parallel with 10 V battery.

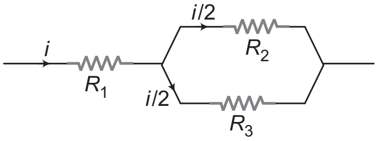
$$\therefore \quad I_{DE} = \frac{10}{R_{DE} + R_{HG}} = \frac{10}{2 + 2}$$

$$= 2.5\text{ A}$$

26. Net resistance of 3 kΩ and voltmeter is also 2 kΩ. Now, the applied 10 V is equally distributed between 2 kΩ and 2 kΩ. Hence, reading of voltmeter

$$= \frac{10}{2} = 5\text{ V}$$

27. $R_2 = R_3$ as $P = \frac{V^2}{R}$ and in parallel V is same.



Hence, $P_{R_2} = P_{R_3}$

If $R_2 = R_3$

Now current through R_1 is double so R_1 should be $\frac{1}{4}$ th of R_2 or R_3 for same power. As $P = i^2R$.

More Than One Correct Options

1. $H = \frac{V^2}{R_1} t_1 \quad \Rightarrow \quad R_1 = \frac{V^2 t_1}{H}$

Similarly, $R_2 = \frac{V^2 t_2}{H}$

In series, $$H = \left(\frac{V^2}{R_1 + R_2}\right) t$$

$$t = \frac{H\,(R_1 + R_2)}{V^2}$$

Substituting the values of R_1 and R_2, we get

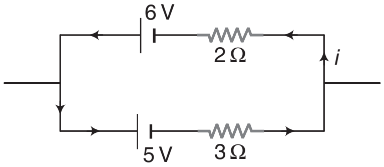
$$t = t_1 + t_2$$

In parallel, $$H = \frac{V^2}{R_{\text{net}}}\, t = V^2 t\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= V^2 t\left(\frac{H}{V^2 t_1} + \frac{H}{V^2 t_2}\right)$$

Solving we get, $t = \frac{t_1 t_2}{t_1 + t_2}$

2.



$$i = \frac{6 - 5}{2 + 3} = 0.2\text{ A}$$

$$V_1 = E_1 - ir_1 = 6 - 0.2 \times 2$$
$$= 5.6\text{ V}$$

3. (a) In series, current is same.

$$\therefore \qquad I_A = I_B$$

(b) $$V_A + V_B = V_C$$

$$\therefore \qquad I_A R_A + I_B R_B = I_C R_C$$

(d) In parallel, current distributes in inverse ratio of resistance.

$$\therefore \qquad \frac{I_B}{I_C} = \frac{I_A}{I_C} = \frac{R_C}{R_A + R_B}$$

4. Same as above.

5. (a) $V = iR$

In series i is same. Hence, V is also same as R is given same.

(b) $R = \frac{\rho l}{A}$

R is same. Hence, A should be smaller in first wire. Secondly, $v_d = \frac{i}{ne\,A}$ or $v_d \propto \frac{1}{A}$

A of first wire is less. Hence, its drift velocity should be more.

(c) $E = \frac{V}{l}$ or $E \propto \frac{1}{l}$ $\qquad (V \to \text{same})$

7. If switch S is open,

$$i_1 \lambda l = E_2$$

where, i_1 = current in upper circuit and λ is resistance per unit length of potentiometer wire.

$$\therefore \qquad \text{Null point length, } l = \frac{E_2}{i_1 \lambda}$$

(a) If jockey is shifted towards right, resistance in upper circuit will increase. So, current i_1 will decrease. Hence, l will increase.

(b) If E_1 is increased, i_1 will also increase. So, l will decrease.

(c) $l \propto E_2$

(d) If switch is closed, then null point will be obtained corresponding to

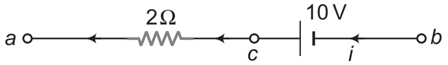
$$V_2 = E_2 - i_2 r_2$$

which is less than E_2. Hence, null point length will decrease.

8. By closing S_1, net external resistance will decrease. So, main current will increase.

By closing S_2, net emf will remain unchanged but net internal resistance will decrease. Hence, main current will increase.

9.



$$V_b + 10 - 2i = V_a$$

$$V_b - V_a = 2i - 10 = 2\text{ V}$$

$$\therefore \qquad i = 6\text{A}$$

Now, $\quad V_C - V_a = 2 \times 6 = 12\text{ V}$

10. Between a and c, balanced Wheatstone bridge is formed. Across all other points simple series and parallel grouping of resistors.

Comprehension Based Questions

1 and **2.** $$r = R\left(\frac{l_1}{l_2} - 1\right)$$

$$\therefore \qquad 10 = R\left(\frac{500}{490} - 1\right)$$

Solving this equation, we get

$$R = 490\ \Omega$$

Further $\qquad r = R\left(\frac{E}{V} - 1\right)$

or $\qquad 10 = 490\left(\frac{2}{V} - 1\right)$

Solving, we get $\quad V = 1.96\text{ V}$

Match the Columns

1. Let potential of point e is V volts. Then,

$$I_{ae} + I_{be} + I_{ce} + I_{de} = 0$$

$$\therefore \quad \left(\frac{2-V}{1}\right) + \left(\frac{4-V}{2}\right) + \left(\frac{6-V}{1}\right) + \left(\frac{4-V}{2}\right) = 0$$

or $\qquad V = 4\text{ V}$

Now current through any wire can be obtained by the equation,

$$I = \frac{\text{PD}}{R}$$

2. $i_1 = i_2$ or i is same at both sections.

$$A_1 < A_2$$

(a) Current density $= \frac{i}{A} \propto \frac{1}{A}$

(c) $\frac{\text{Resistance}}{\text{length}} = \frac{\rho}{A} \propto \frac{1}{A}$

(d) and (b) E or potential difference per unit length = (i) (Resistance per unit length)

$$= (i)\left(\frac{\rho}{A}\right) \propto \frac{1}{A}$$

3. By introducing parallel resistance R_3 in the circuit, total resistance of the circuit will decrease. Hence, main current i will increase.

Now, $\quad V_{R1} = E - V_{R_2} = E - iR_2$

Since, i is increasing, so V_{R_2} will increase. Hence, V_{R_1} or current passing through R_1 will decrease.

4. (a) $R = \dfrac{H}{i^2 t}$

$\therefore \quad [R] = [\mathrm{ML^2T^{-2}}]/[\mathrm{A^2T}] = [\mathrm{ML^2T^{-3}A^{-2}}]$

(b) $V = iR$

$\therefore \quad [\mathrm{V}] = [\mathrm{A}][\mathrm{ML^2T^{-3}A^{-2}}] = [\mathrm{ML^2T^{-3}A^{-1}}]$

(c) $\rho = \dfrac{RA}{l}$

$\therefore \; [\rho] = \dfrac{[\mathrm{ML^2T^{-3}A^{-2}}][\mathrm{L^2}]}{[\mathrm{L}]} = [\mathrm{ML^3T^{-3}A^{-2}}]$

(d) $[\sigma] = \left[\dfrac{1}{\rho}\right] = [\mathrm{M^{-1}L^{-3}T^3A^2}]$

5. $i = \dfrac{4-1}{1+1+1} = 1\mathrm{A}$ (anti-clockwise)

(a) $V_A = E - ir = 4 - 1 \times 1 = 3\ \mathrm{V}$

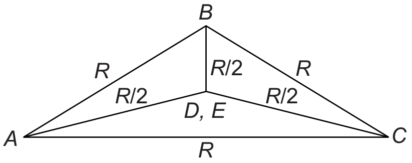
(b) $V_B = E + ir = 1 + 1 \times 1 = 2\ \mathrm{V}$

(c) $|P_A| = Ei - i^2 r = (4 \times 1) - (1)^2\,(1) = 3\mathrm{W}$

(d) $|P_B| = Ei + i^2 r = (1)\,(1) + (1)^2\,(1) = 2\mathrm{W}$

Subjective Questions

1. (a) Points *D* and *E* are symmetrically located with respect to points *A* and *C*. The circuit can be redrawn as shown in figure.

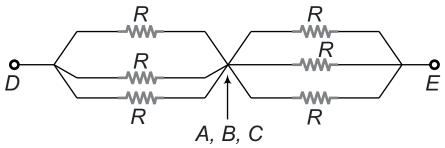


This is a combination of a balanced Wheatstone bridge in parallel with a resistance *R*. So, the resistance between *B* and *D* (or *E*) can be removed.

$$\frac{1}{R_{AC}} = \frac{1}{R} + \frac{1}{\frac{R}{2} + \frac{R}{2}} + \frac{1}{2R}$$

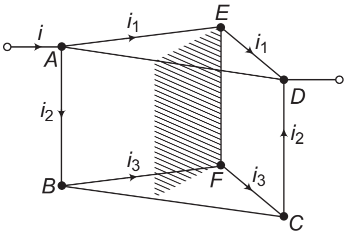
or $\quad R_{AC} = \dfrac{2R}{5}$ **Ans.**

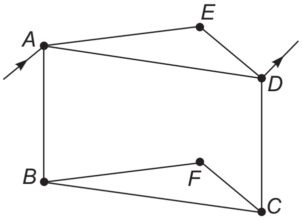
(b) With respect to *D* and *E*, points *A*, *B* and *C* all are symmetrically located. Hence, the simplified circuit can be drawn as shown in figure.



$\therefore \quad R_{DE} = \dfrac{R}{3} + \dfrac{R}{3} = \dfrac{2R}{3}$ **Ans.**

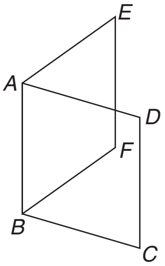
2. (a) Due to symmetry about the shaded plane, current distribution on either side of the plane will be identical and points *E* and *F* will be at same potential and no current will flow through it.





$$\therefore \quad R_{AD} = \frac{\frac{2}{3}r \times \frac{8}{3}r}{\frac{2}{3}r + \frac{8}{3}r} = \frac{8}{15}r$$ **Ans.**

(b) Redrawing the given arrangement for resistance across *AB*. Potentials $V_D = V_E$



$V_C = V_F$

$\therefore$ No current flows through *DE* and *CF*.

$$\therefore \quad R_{AB} = \frac{\frac{3}{2}r \times r}{\frac{3}{2}r + r} = \frac{3}{5}r$$ **Ans.**

3. (a) Current flowing through resistance 5Ω is 11 A

Power dissipated = $i^2 R$

$= (121)5 = 605\ \mathrm{W}$

(b) $V_B + 8V + 3V + 12V - 12V - 5V = V_C$

$V_B + 11V - 5V = V_C$

$6V = V_C - V_B$

(c) Both batteries are being charged. **Ans.**

4. (a) $V_A - V_B = 6\text{ V}$

Since, $V_B = 0$

$\therefore$ $V_A = 6\text{ V}$

$V_A - V_C = 4\text{ V} \Rightarrow V_C = V_A - 4 = 2\text{ V}$

(b) $V_A - V_D = V_A - V_C = 4\text{ V}$

From unitary method, we can find that,

$$AD = \left(\frac{100}{6}\right)(4) = 66.67\text{ cm}$$

(c) Since, they are at same potential, no current will flow through it.

(d) $V_A - V_B$ is still 6 V $\therefore$ $V_A = 6\text{ V}$

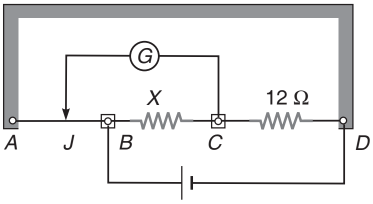
Further, $V_A - V_C = 7.5\text{ V}$

$\therefore$ $V_C = -1.5\text{ V}$

Since, EMF of the battery in lower circuit is more than the EMF of the battery in upper circuit. No such point will exist.

5. (a) There are no positive and negative terminals on the galvanometer because only zero deflection is needed.

(b)



(c) $AJ = 60\text{ cm} \Rightarrow BJ = 40\text{ cm}$

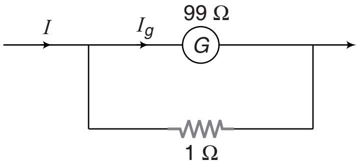
If no deflection is taking place. Then, the Wheatstone bridge is said to be balanced. Hence,

$$\frac{X}{12} = \frac{R_{BJ}}{R_{AJ}} \quad \text{or} \quad \frac{X}{12} = \frac{40}{60} = \frac{2}{3}$$

or $X = 8\,\Omega$ **Ans.**

6. For ammeter $99I_g = (I - I_g)\,1$

or $I = 100\,I_g$ ...(i)



I_g is the full scale deflection current of the galvanometer and I the range of ammeter.

For the circuit in Fig.1, given in the question

$$\frac{12\text{V}}{2 + r + \dfrac{99 \times 1}{99 + 1}} = 3\text{ A} \Rightarrow r = 1.01\,\Omega$$ **Ans.**

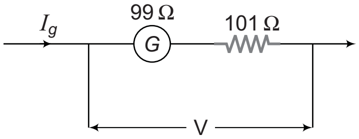
For voltmeter, range

$V = I_g\,(99 + 101)$

$V = 200\,I_g$ ...(ii)

Also resistance of the voltmeter

$= 99 + 101 = 200\,\Omega$



In Fig. 2, resistance across the terminals of the battery

$$R_1 = r + \frac{200 \times 2}{200 + 2} = 2.99\,\Omega$$

$\therefore$ Current drawn from the battery,

$$I_1 = \frac{12}{2.99} = 4.01\text{ A}$$

$\therefore$ Voltmeter reading

$$\frac{4}{5}V = 12 - I_1 r = 12 - 4.01 \times 1.01$$

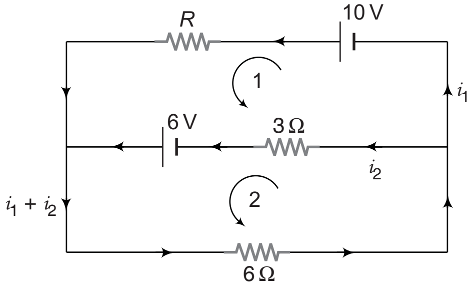
$$V = 7.96 \times \frac{5}{4} = 9.95\text{ V}$$ **Ans.**

Using Eq. (ii), $I_g = \dfrac{9.95}{200} = 0.05\text{ A}$

Using Eq.(i), range of the ammeter

$I = 100\,I_g = 5\text{ A}$ **Ans.**

7. Applying Kirchhoff's laws in two loops we have,



$10 - i_1R - 6 + 3i_2 = 0$ ...(i)

$6 - 6(i_1 + i_2) - 3i_2 = 0$ ...(ii)

Solving these two equations, we get

$$i_1 = \frac{6}{R + 2}$$

Power developed in R,

$$P = i_1^2 R = \frac{36}{(R + 2)^2}R$$ ...(iii)

For power to be maximum,

$$\frac{dP}{dR}=0$$

$\therefore \quad (R+2)^2(36)-(36R)(2)(R+2)=0$

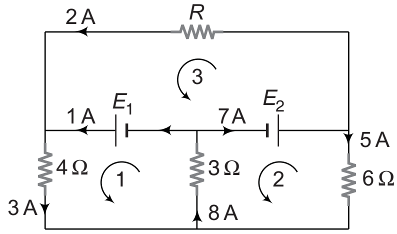
or $\quad R+2-2R=0$

$\therefore \quad R=2\,\Omega$ **Ans.**

For maximum power from Eq. (iii), we have

$P_{\text{max}}=4.5\text{ W}$ **Ans.**

8. Applying loop law in loops 1, 2 and 3, we have



$$E_1-12-24=0$$

$\therefore \quad E_1=36\text{ V}$

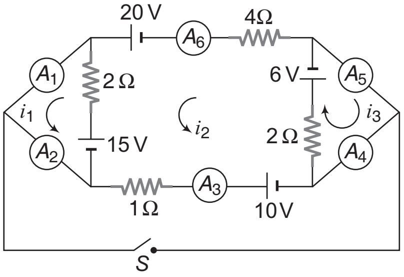
$$-E_2+24+30=0 \quad \text{or} \quad E_2=54\text{ V}$$

$$-2R-E_1+E_2=0$$

or $\quad R=\dfrac{E_2-E_1}{2}$

$$=9\,\Omega$$

9. Using the loop current method,



(a) $-2(i_1-i_2)+15=0$ …(i)

$$-4i_2+20-2(i_2-i_1)-15-i_2$$

$$-10-2(i_2+i_3)-6=0$$

or $\quad 2i_1-9i_2-2i_3-11=0$ …(ii)

$$-2(i_2+i_3)-6=0$$

or $\quad i_2+i_3+3=0$ …(iii)

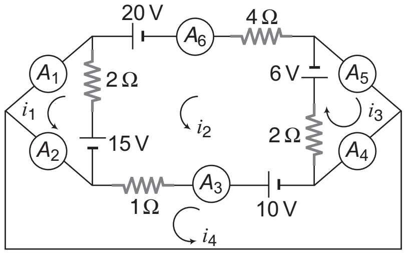
Solving these equations, we get

$i_1=9.5$ A, $i_2=2$ A and $i_3=-5$ A

Ammeter	A_1	A_2	A_3	A_4
Reading (amp)	9.5	9.5	2	5

PD across switch $=10+(1)i_2=10+2=12$ V **Ans.**

(b)



When switch is closed

$-2(i_1-i_2)+15=0$ …(i)

$2i_1-9i_2-2i_3-11+i_4=0$ …(ii)

$i_2+i_3+3=0$ …(iii)

$10-(i_4-i_2)=0$ …(iv)

Solving these four equation, we get

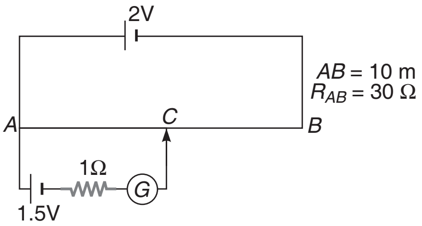
$i_1=12.5$ A, $i_2=5.0$ A , $i_3=-8.0$ A

$i_4=15$ A

Ammeter	A_1	A_2	A_3	A_4
Reading (amp)	12.5	2.5	10	7

And the current through switch is 15 A. **Ans.**

10.



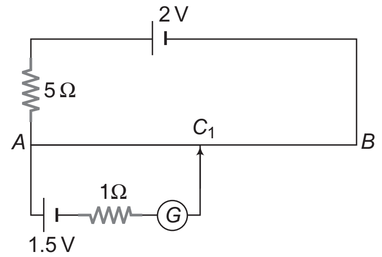
Potential gradient across wire,

$$AB=\frac{2}{10}=0.2\text{ V/m}$$

Now, $\quad V_{AC}=1.5\text{ V} \quad \text{or} \quad (0.2)(AC)=1.5$

$\therefore \quad AC=7.5\text{ m}$ **Ans.**

(a) $V_{AB}=\left(\dfrac{R_{AB}}{R_{AB}+5}\right)\times 2=\left(\dfrac{30}{30+5}\right)\times 2=\dfrac{12}{7}\text{ V}$



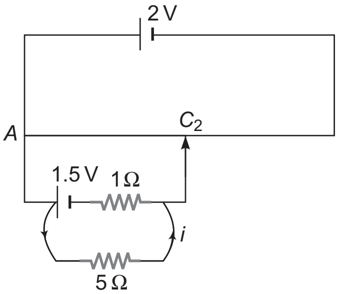
∴ Potential gradient across

$$AB = \frac{12}{70}\ \text{V/m}$$

Now, $V_{AC} = 1.5\ \text{V}$

$$\therefore \quad \left(\frac{12}{70}\right)(AC_1) = 1.5$$

$$\therefore \quad AC_1 = 8.75\ \text{m}$$ **Ans.**

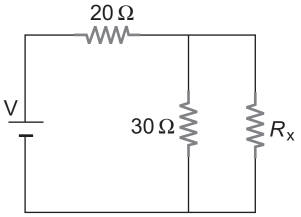


(b) $V_{AC_2} = V$

or $$(0.2)(AC_2) = \left(\frac{5}{5+1}\right)(1.5)$$

or $$AC_2 = 6.25\ \text{m}$$ **Ans.**

11. V = constant



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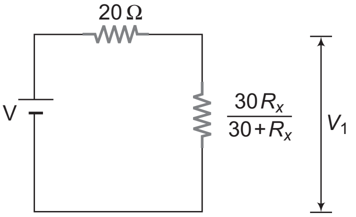


$$V_1 = \left(\frac{\dfrac{30R_x}{30+R_x}}{\dfrac{30R_x}{30+R_x}+20}\right)V$$

$$= \left(\frac{30R_x}{50R_x+600}\right)V$$

∴ Power generated in R_x is

$$P = \frac{V_1^2}{R_x} = \frac{900R_x V^2}{(50R_x+600)^2}$$

For P to be constant,

$$\frac{dP}{dR_x} = 0$$

or $$\frac{(50R_x+600)^2(900V^2) - 1800\times 50\times R_x V^2(50R_x+600)}{(50R_x+600)^4} = 0$$

or $50R_x + 600 - 100R_x = 0$

$$\therefore \quad R_x = 12\ \Omega$$ **Ans.**

12. The two batteries are in parallel. Thermal power generated in R will be maximum when,

total internal resistance = total external resistance

or $$R = \frac{R_1R_2}{R_1+R_2}$$

$$E_{\text{eq}} = \frac{\left(\dfrac{E_1}{R_1}+\dfrac{E_2}{R_2}\right)}{\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)}$$

$$= \left(\frac{E_1R_2+E_2R_1}{R_1+R_2}\right)$$

$$R_{\text{net}} = \frac{2R_1R_2}{R_1+R_2}$$

$$\therefore \quad i = \frac{E_{\text{eq}}}{R_{\text{net}}} = \frac{E_1R_2+E_2R_1}{2R_1R_2}$$

Maximum power through R

$$P_{\max} = i^2R = \frac{(E_1R_2+E_2R_1)^2}{4R_1R_2(R_1+R_2)}$$ **Ans.**

13. $$\frac{V^2}{R} = k(T-T_0) + C\left(\frac{dT}{dt}\right)$$

or $$\frac{dT}{\dfrac{V^2}{R} - k(T-T_0)} = \frac{dt}{C}$$

or $$\int_{T_0}^{T} \frac{dT}{\dfrac{V^2}{R} - k(T-T_0)} = \int_0^t \frac{dt}{C}$$

(at $t = 0$, temperature of conductor $T = T_0$)

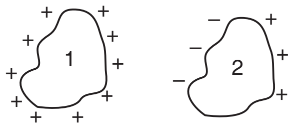
Solving this equation, we get

$$T = T_0 + \frac{V^2}{kR}(1 - e^{-kt/C})$$ **Ans.**

24. Electrostatics

INTRODUCTORY EXERCISE 24.1

1. Due to induction effect, a charged body can attract a neutral body as shown below.



Body-1 is positively charged and body-2 is neutral. But we can see that due to distance factor attraction is more than the repulsion.

4. Number of atoms in 3 gram-mole of hydrogen atom = number of electrons in it

$= 3\,N_0 = (3 \times 6.02 \times 10^{23})$

where, N_0 = Avogadro number

$\therefore$ Total charge

$$= -(1.6 \times 10^{-19})\,(3 \times 6.02 \times 10^{23})$$

$$= -2.89 \times 10^{5}\,\mathrm{C}$$

INTRODUCTORY EXERCISE 24.2

1. $F_e = \dfrac{1}{4\pi\,\varepsilon_0}\,\dfrac{q_1 q_2}{r^2}$

and $\quad F_g = G\,\dfrac{m_1\,m_2}{r^2}$

$$\therefore \quad \frac{F_e}{F_g} = \frac{(1/4\pi\,\varepsilon_0)\,q_1 q_2}{G\,m_1 m_2}$$

$$= \frac{(9 \times 10^{+9})\,(1.6 \times 10^{-19})^2}{(6.67 \times 10^{-11})\,(9.11 \times 10^{-31})\,(1.67 \times 10^{-27})}$$

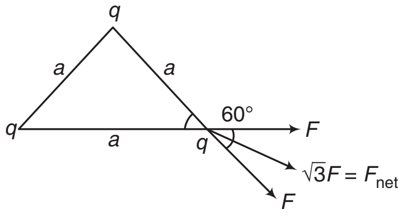
$$= 2.27 \times 10^{39}$$

2.
$$F = \frac{1}{4\pi\,\varepsilon_0}\,\frac{q_1 q_2}{r^2}$$

$$\therefore \quad \varepsilon_0 = \frac{q_1 q_2}{4\pi F r^2}$$

Units and dimensions can be found by above equation.

3.



F = Force between two point charges

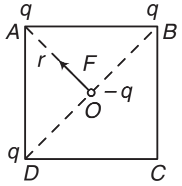
$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q \times q}{a^2}\right)$$

$$F_{\text{net}} = \sqrt{F^2 + F^2 + 2FF\cos 60^\circ}$$

$$= \sqrt{3}F$$

$$= \left(\frac{\sqrt{3}}{4\pi\varepsilon_0}\right)\left(\frac{q}{a}\right)^2$$

4.



Net force on $-q$ from the charges at B and D is zero.

So, net force on $-q$ is only due to the charge at A.

$$F = \left(\frac{1}{4\pi\,\varepsilon_0}\right)\frac{q \times q}{r^2}$$

where, $\quad r = \dfrac{\sqrt{2}a}{2} = \dfrac{a}{\sqrt{2}}$

$$\therefore \quad F = \left(\frac{1}{4\pi\,\varepsilon_0}\right)\frac{q^2}{(a/\sqrt{2})^2}$$

$$= \left(\frac{1}{2\pi\,\varepsilon_0}\right)\left(\frac{q}{a}\right)^2$$

5. The charged body attracts the natural body because attraction (due to the distance factor) is more than the repulsion.

7. $F = \dfrac{1}{4\pi\,\varepsilon_0}\cdot\dfrac{q_1 q_2}{r^2}$

$$(q_1)_{\min} = (q_2)_{\min} = e_2$$

$$\therefore \quad F_{\min} = \frac{1}{4\pi\,\varepsilon_0}\,\frac{e^2}{r^2}$$

9. Two forces are equal and opposite.

INTRODUCTORY EXERCISE 24.3

1. Electric field lines are not parallel and equidistant.

2. Electric lines flow higher potential to lower potential.

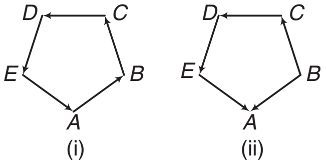
$$\therefore \quad V_A > V_B$$

3. If charged particle is positive, and at rest. Electric field lines are straight then only it will move in the direction of electric field.

4. See the hint of above question.

5. Electric field lines start from positive charge and terminate on negative charge.

6. In case of five charges at five vertices of regular pentagon net electric field at centre is zero. Because five vectors of equal magnitudes from a closed regular pentagon as shown in Fig. (i).



Where one charge is removed. Then, one vector (let **AB**) is deceased hence the net resultant is equal to magnitude of one vector

$$= |\mathbf{BA}| = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}$$

7. $\mathbf{E} = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q}{r^3}\right)(\mathbf{r}_p - \mathbf{r}_q)$

Here, $r = \sqrt{(3)^2 + (4)^2} = 5\text{m}$

$$\therefore \quad \mathbf{E} = \frac{(9\times10^9)(-2\times10^{-6})}{(5)^3}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$$

$$= -(4.32\hat{\mathbf{i}} + 5.76\hat{\mathbf{j}})\times10^2 \text{ N/C}$$

INTRODUCTORY EXERCISE 24.4

1. $K_i + U_i = K_f + U_f$

$$\therefore \quad \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_i} = \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_f}$$

$$\therefore v = \sqrt{\frac{2}{m}\left(\frac{1}{4\pi\varepsilon_0}\right)(q_1q_2)\left(\frac{1}{r_i} - \frac{1}{r_f}\right)}$$

$$= \sqrt{\frac{2}{10^{-4}}(9\times10^9)(-2\times10^{-12})\left(\frac{1}{1.0} - \frac{1}{0.5}\right)}$$

$$= 18.97 \text{ m/s}$$

2. Work done by electrostatic forces

$$= -\Delta U$$

$$= U_i - U_f$$

$$= \left(\frac{1}{4\pi\,\varepsilon_0}\right)(q_1q_2)\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= (9\times10^9)(-2\times10^{-12})\left(\frac{1}{1.0} - \frac{1}{2.0}\right)$$

$$= -9\times10^{-3}\text{J}$$

$$= -9 \text{ mJ}$$

3. Work done by electrostatic forces

$$W = -\Delta U = U_i - U_f$$

$$\therefore \quad U_f = U_i - W$$

$$= (-6.4\times10^{-8}) - (4.2\times10^{-8})$$

$$= -10.6\times10^{-8}\text{J}$$

4. $U_\infty = 0$

$$U_r = \frac{1}{4\pi\,\varepsilon_0}\cdot\frac{q_1q_2}{r} \qquad \text{(For two charges)}$$

$$U_r \neq U_\infty$$

For $\quad r \neq \infty$

$$U_r = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_3q_1}{r_{31}}\right]$$

Now, U_r can be equal to U_∞ for finite value of r.

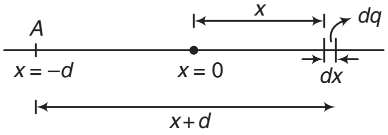
INTRODUCTORY EXERCISE 24.5

1. $W_{ab} = \Delta U_{a-b} = U_b - U_a = q\,(V_b - V_a) = q\,V_{ba}$

$$\therefore \; V_{ba} = \frac{W_{ab}}{q} = \frac{12}{10^{-2}} = 1200 \text{ V}$$

2. (a) $\alpha = \frac{\lambda}{x} = \frac{\text{C/m}}{\text{m}} = \frac{\text{C}}{\text{m}^2}$

(b)



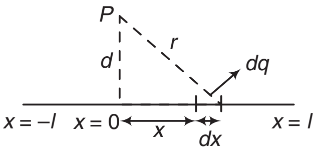
$$dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{dq}{x+d}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\lambda dx}{x+d}\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\alpha x\,dx}{x+d}\right)$$

$$\therefore \quad V = \int_{x=0}^{x=L} dV = \frac{\alpha}{4\pi\varepsilon_0}\left[L - d\ln\left(1 + \frac{L}{d}\right)\right]$$

3.



$$dq = \left(\frac{q}{2l}\right) dx$$

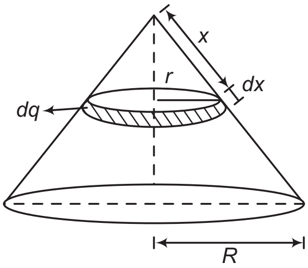
At point P,

$$dV = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{dq}{r}\right)$$

$$= \left(\frac{1}{4\pi\,\varepsilon_0}\right)\left(\frac{\frac{q}{2l}\cdot dx}{\sqrt{d^2 + x^2}}\right)$$

$\therefore \qquad V = 2\int_{x=0}^{x=l} dV$

4.



$$W = -\Delta U = U_{\text{apex}} - U_{\infty}$$

$\therefore \qquad W = qV_{\text{apex}} \qquad \ldots\text{(i)}$

as $\qquad U_{\infty} = 0$

$\mathbf{V}_{apex} \qquad \frac{r}{x} = \frac{R}{L}$

$\therefore \qquad r = \left(\frac{R}{L}\right) x$

Surface charge density,

$$\sigma = \frac{Q}{\pi Rl}$$

$$dq = (\sigma)\,(dA)$$

$$= \left(\frac{Q}{\pi Rl}\right)(2\pi r)\,dx$$

$$= \left(\frac{\theta}{\pi RL}\right)(2\pi)\left(\frac{R}{L}x\right)dx$$

$$= \left(\frac{2Q}{L^2}\right) x\,dx$$

Now, $\qquad dV = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{dq}{x}\right)$

$$= \left(\frac{2Q}{4\pi\varepsilon_0 L^2}\right) dx$$

$\therefore \qquad V = \int_0^L dV = \frac{Q}{2\pi\varepsilon_0 L}$

Substituting in Eq. (i), we have

$$W = \frac{Qq}{2\pi\varepsilon_0 L}$$

INTRODUCTORY EXERCISE 24.6

1. (a) $\mathbf{E} = -\left[\frac{\partial V}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\,\hat{\mathbf{j}}\right]$

$$= -a\,[(2x)\,\hat{\mathbf{i}} - (2y)\,\hat{\mathbf{j}}] = -2a\,[x\hat{\mathbf{i}} - y\hat{\mathbf{j}}]$$

(b) Again

$$\mathbf{E} = -\left[\frac{\partial V}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\,\hat{\mathbf{j}}\right]$$

$$= -a\,[\,y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\,]$$

2. $E = -\frac{dV}{dx} = -$ Slope of V-x graph.

From $x = -2\text{m}$ to $x = 0$, slope $= +\ 5$ V/m

$\therefore \qquad E = -\ 5$ V/m

From $x = 0$ to $x = 2$m, slope $= 0$,

$\therefore \qquad E = 0$

From $x = 2$m to $x = 4$m, slope $= +\ 5$V/m,

$\therefore \qquad E = -\ 5$ V/m

From $x = 4$ m to $x = 8$m, slope $\equiv -\ 5$ V/m

$\therefore \qquad E = +\ 5$ V/m

Corresponding E-x graph is as shown in answer.

3. $\frac{\partial V}{\partial x} = \frac{-\,50}{5} = -\,10$ V/m

Now, $\qquad |\mathbf{E}| = \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2}$

No information is given about $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. Hence,

$$|\mathbf{E}| \geq \left|\frac{\partial V}{\partial x}\right| \quad \text{or} \quad |\mathbf{E}| \geq 10 \text{ V/m}$$

4. $V_A = V_D$ and $V_B = V_C$ as the points A and D or B and C are lying on same equipotential surface ($\perp$ to electric field lines). Further, V_A or $V_D > V_B$ or V_C as electric lines always flow from higher potential to lower potential

$$V_A - V_B = V_D - V_C = Ed$$

$$= (20)\,(1) = 20 \text{ V}$$

INTRODUCTORY EXERCISE 24.7

1. (a) Given surface is a closed surface. Therefore, we can directly apply the result.

$$\phi = \frac{q_{\text{in}}}{\varepsilon_0} = 0 \quad \text{as} \quad q_{\text{in}} = 0$$

(b) Again given surface is a closed surface. Hence, we can directly apply the result.

$$\phi = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \quad \text{as} \quad q_{\text{in}} = q$$

(c) Given surface is not closed surface. Hence, we cannot apply the direct result of Gauss's theorem. If we draw a complete sphere, then

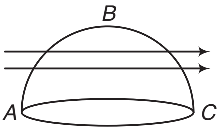
$$\phi \text{ through complete sphere} = \frac{q}{\varepsilon_0}$$

$$\therefore \quad \phi \text{ through hemisphere} = \frac{1}{2}\left(\frac{q}{\varepsilon_0}\right)$$

2. Net charge from any closed surface in uniform electric field = 0

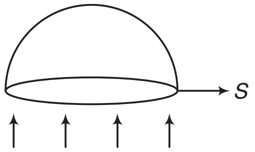
$\therefore$ Net charge inside any closed surface in uniform electric field = 0

3. (a)



Net flux entering from AB = net flux entering from BC.

(b)



$$\phi = ES = E(\pi R^2)$$

4. Given electric field is uniform electric field. Net flux from any closed surface in uniform electric field = 0.

Exercises

LEVEL 1

Assertion and Reason

1. An independent negative charge moves from lower potential to higher potential. In this process, electrostatic potential energy decreases and kinetic energy increases.

2. Two unlike charges come together when left freely.

3.
$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right)$$

$$\therefore \quad |\mathbf{E}| = \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2}$$

$$\therefore \quad |\mathbf{E}| \geq \frac{\partial V}{\partial x}$$

$$\text{or} \quad = 10 \text{ V/m}$$

4. $V = \frac{kq}{R}$ or $kq = VR$

For inside points ($r \leq R$),

$$E = \frac{kq}{R^3} r$$

$$\text{or} \quad E \propto r$$

At distance $r = \frac{R}{2}$,

$$E = \frac{(VR)}{R^3}\left(\frac{R}{2}\right) = \frac{V}{2R}$$

6.
$$V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{r}$$

$$= -\int_{(0,4)}^{(4,0)} (4\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}})$$

$$= -\int_{(0,4)}^{(4,0)} (4dx + 4dy) = 0$$

$$\therefore \quad V_A = V_B$$

7. At stable equilibrium position, potential energy is minimum.

8. In uniform electric field, net force on an electric dipole = 0

Therefore, no work is done in translational motion of the dipole.

Electric lines also flow from higher potential to lower potential. Electrostatical force on positive charge acts in the direction of electric field. Therefore, work done is positive.

9. Charge on shell does not contribute in electric field just inside the shell. But it contributes in the electric field just outside it. So, there is sudden change in electric field just inside and just outside it. Hence, it is discontinuous.

10. $|E| = 0$ minimum at centre and $|V| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$ is maximum at centre.

Objective Questions

1. $\phi = ES \rightarrow \left(\frac{\text{V}}{\text{m}}\right)(\text{m}^2) \rightarrow$ volt-m

2. $g_e = g - \left(\frac{qE}{m}\right)$

or g_e will decrease. Hence, $T = 2\pi\sqrt{\frac{l}{g_e}}$ will increase.

3. Electric lines terminate on negative charge.

4.
$$\begin{aligned} W &= \Delta U_i \\ &= U_f - U_i \\ &= 3\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{qq}{2l}\right] - 3\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{qq}{l}\right] \\ &= \left(-\frac{3}{2}\right)\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{l^2}\right) l = -\frac{3}{2} Fl \end{aligned}$$

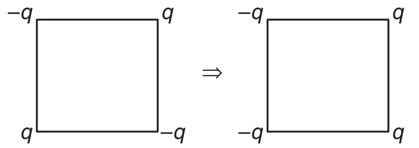
5.
$$\begin{aligned} v &= \sqrt{\frac{2qV}{m}} \\ v_p &= \sqrt{\frac{2eV}{m}} \\ v_d &= \sqrt{\frac{2e\,(2V)}{2m}} \\ v_\alpha &= \sqrt{\frac{2\,(2e)\,(4V)}{4m}} \end{aligned}$$

The ratio is $1 : 1 : \sqrt{2}$.

6. $V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}$

$$V' = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{2q}{4r}\right) = \frac{V}{2}$$

7.



$$W = \Delta U = U_f - U_i$$

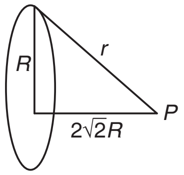
$$= \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{-q^2}{a} - \frac{q^2}{\sqrt{2}\,a} + \frac{q^2}{a} + \frac{q^2}{a} - \frac{q^2}{\sqrt{2}\,a} - \frac{q^2}{a}\right) - \left(\frac{-q^2}{a} + \frac{q^2}{\sqrt{2}\,a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}\,a} - \frac{q^2}{a}\right)\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}[4 - 2\sqrt{2}]$$

8.
$$\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{3R} + \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_2}{3R} \\ &= \frac{q_{\text{net}}}{(4\pi\varepsilon_0)\,(3R)} \end{aligned}$$

$$\therefore \quad \begin{aligned} R &= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q_{\text{net}}}{3V}\right) \\ &= \frac{(9\times10^9)\,(3\times10^{-6})}{(3)\,(9000)} \\ &= 1\text{ m} \end{aligned}$$

9.

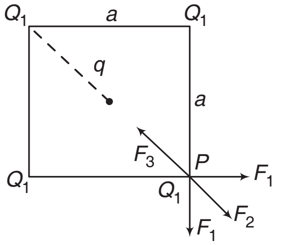


$$\begin{aligned} r &= 3R \\ V_p &= \frac{k\,q_{\text{net}}}{r} = \frac{k\,(3q)}{3R} = \frac{k\,q}{R} \end{aligned}$$

10. $\frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} = mg\sin\theta$

$$\therefore \quad \begin{aligned} r &= \sqrt{\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q^2}{mg\sin\theta}\right)} \\ &= \sqrt{\frac{(9\times10^9)\,(2.0\times10^{-6})^2}{(0.1)\,(9.8)\sin 30^\circ}} \\ &= 27\times10^{-2}\text{m} \\ &= 27\text{ cm} \end{aligned}$$

11.



$Q_1 = (2\sqrt{2} - 1)\,Q$

F_1 = Force between Q_1 and Q_1 at distance a

F_2 = Force between Q_1 and Q_1 at distance $\sqrt{2}a$

F_3 = Force between Q_1 and q at distance $\frac{a}{\sqrt{2}}$

For F_3 to be in the shown direction, q and Q_1 should have opposite signs. For net charge to be zero on Q_1 placed at P.

$|F_3|$ = Resultant of F_1, F_2 and F_1

$$\therefore \frac{kQ_1q}{(a/\sqrt{2})^2} = \sqrt{2}F_1 + F_2$$

$$= \sqrt{2}\left[\frac{kQ_1Q_1}{a^2}\right] + \frac{kQ_1Q_1}{(\sqrt{2}a)^2}$$

$$\therefore \quad |q| = \frac{Q_1}{2}\left[\sqrt{2} + \frac{1}{2}\right] = \left(\frac{2\sqrt{2}+1}{4}\right)Q_1$$

$$= \left(\frac{2\sqrt{2}+1}{4}\right)(2\sqrt{2}-1)Q$$

$$= \frac{7}{4}Q \quad \text{or} \quad q = -\frac{7}{4}Q$$

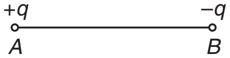
12. $$|a| = \left|\frac{qE}{m}\right| = \left|\frac{q\sigma}{2\varepsilon_0 m}\right|$$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{4s\,\varepsilon_0\,m}{|q||\sigma|}}$$

$$= \sqrt{\frac{4 \times 0.1 \times 8.86 \times 10^{-12} \times 1.67 \times 10^{-27}}{1.6 \times 10^{-19} \times 2.21 \times 10^{-9}}}$$

$$= 4 \times 10^{-6}\ \text{s}$$

13.

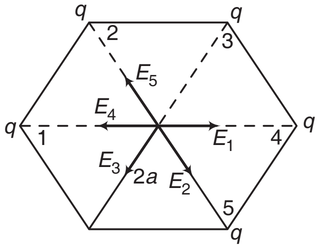


Between A and B two forces on third charge will act in same direction. So, this charge cannot remain in equilibrium.

To the right of B or left of A forces are in opposite directions but their magnitudes are different. Because charges have equals magnitudes but distances are different.

14. Between $2q$ and $-q$, two electric fields are in same direction. So their resultant can't be zero. To the right of $2q$ left of $-q$ they are in opposite directions. So, net field will be zero nearer to charge having small magnitude.

15.



E_1 and E_4 are cancelled.

E_2 and E_5 are cancelled.

$$\therefore \quad E_{\text{net}} = E_3 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{(2a)^2} = \frac{q}{16\pi\varepsilon_0 a^2}$$

16. $$F_1 = \frac{kq_1q_2}{r^2} \qquad \left(k = \frac{1}{4\pi\varepsilon_0}\right)$$

$$F_2 = \frac{kq'_1q'_2}{r^2}$$

where, $$q'_1 = q'_2 = \frac{q_1+q_2}{2}$$

$$\therefore \quad F_2 = \frac{k(q_1+q_2/2)^2}{r^2}$$

$$\left(\frac{q_1+q_2}{2}\right)^2 > q_1q_2$$

This is because,

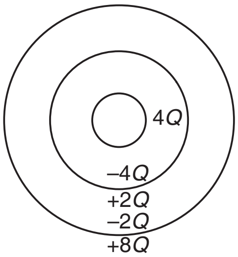
$$(q_1+q_2)^2 = (q_1-q_2)^2 + 4q_1q_2$$

or $$(q_1+q_2)^2 > 4q_1q_2$$

or $$\left(\frac{q_1+q_2}{2}\right)^2 > q_1q_2$$

$$\therefore \quad F_2 > F_1$$

17.



18. $$1000\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3 \Rightarrow R = 10r$$

i.e. radius has become 10 times.

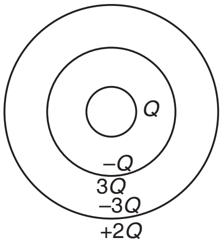
Charge will become 1000 times.

$$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{(\text{Charge})}{(\text{Radius})} \quad \text{or} \quad V \propto \frac{\text{Charge}}{\text{Radius}}$$

Hence, potential will become 100 times.

19. $$\text{PD} = \frac{q_{\text{in}}}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{2R}\right) \quad \text{or} \quad \text{PD} \propto q_{\text{in}}$$

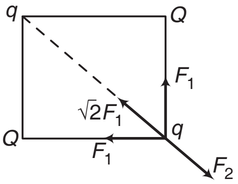
20.



The desired ratio is $-\dfrac{3Q}{2Q} = -\dfrac{3}{2}$

21. $\mathbf{S} = (1)\,\hat{\mathbf{i}} \quad \Rightarrow \quad \phi = \mathbf{B}\cdot\mathbf{S} = 5$ V-m

22.



$F_2 =$ Force between q and q

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q \times q}{(\sqrt{2}\,a)^2}$$

$F_1 =$ Force between Q and q. For net force on q to be zero .

$$F_2 = \sqrt{2}\,F_1$$

$$\therefore \qquad \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{2a^2} = \sqrt{2}\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{qQ}{a^2}\right]$$

$$\therefore \qquad |q| = 2\sqrt{2}Q$$

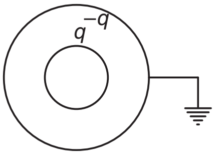
with sign, $\quad q = -\,2\sqrt{2}\,Q$

23. $V_B = 0$

$$\therefore \qquad \frac{kq_A}{r_B} + \frac{kq_B}{r_B} = 0$$

$$\therefore \qquad q_B = -\,q_A = -\,q$$

Charge distribution is as shown below.



From Gauss's theorem, electric field at any point is given by

$$E = \frac{kq_{\text{in}}}{r^2}$$

q_{in} inside A and outside B is zero. Therefore, $E = 0$

24. $\frac{kq}{r} = \frac{3}{2}\left(\frac{kq}{R}\right)\Big/2 \quad$ or $\quad r = \frac{4}{3}R$

$\therefore$ Distance from surface $= r = R$

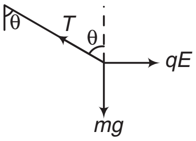
$$= \frac{R}{3}$$

25. $\phi = \frac{q_{\text{ in}}}{\varepsilon_0}$

$$q_{\text{in}} = 0$$

$$\therefore \qquad \phi = 0$$

26.



$$T\,\sin\theta = qE$$

or $\qquad T = \frac{qE}{\sin\theta}$

Similarly, $\quad T\,\cos\theta = mg$

27. $E = \frac{\sigma}{\varepsilon_0}$

$$E_1 = E_2$$

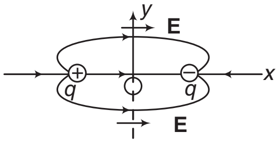
$$\therefore \qquad \sigma_1 = \sigma_2$$

$$V = \frac{\sigma R}{\varepsilon_0}$$

or $\qquad V \propto R \qquad$ (as σ → same)

$$\therefore \qquad \frac{V_1}{V_2} = \frac{R_1}{R_2} = \frac{a}{b}$$

28.



Potential is zero at infinite and at origin. Therefore, PD = 0. Hence, the work done asked in part (c) is also zero.

29. According to principle of generator PD in this case only depends on the charge on inner shell.

30.

$$500 = \frac{k\,|q|}{r^2} \qquad \text{...(i)}$$

$$-\,3000 = \frac{k\,(-\,q)}{r} \qquad \text{...(ii)}$$

Solving these two equations, we get

$$r = 6 \text{ m}$$

$$\therefore \qquad |q| = \frac{500\,r^2}{k}$$

$$= \frac{(500)\,(6)^2}{9 \times 10^9}$$

$$= 2 \times 10^{-6}\text{C}$$

$$= 2\,\mu\text{C}$$

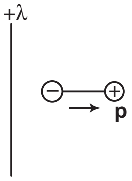
31. $\frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1q_2}{r_1^2} = \frac{1}{4\pi\varepsilon_0 k}\cdot\frac{q_1q_2}{r_2^2}$

$$\therefore \qquad r_2 = \frac{r_1}{\sqrt{k}} = \frac{50}{\sqrt{5}}$$

$$= 22.36 \text{ m}$$

32. $$W_{A\to B} = q\,(V_B - V_A)$$
$$= q\left[-\int_A^B E dr\right] = q\int_B^A E\,dr$$
$$= q\int_{2a}^{a} \frac{\lambda}{2\pi\varepsilon_0 r}\,dr = \frac{q\lambda}{2\pi\varepsilon_0}\ln\left(\frac{1}{2}\right)$$

33.



Negative charge of dipole is near to positively charge line charge. Hence, attraction is more.

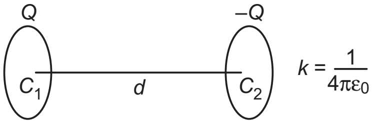
34. $$r = \sqrt{(4-1)^2 + (2-2)^2 + (0-4)^2}$$
$$= 5 \text{ m}$$
$$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}$$
$$= \frac{(9\times10^9)(2\times10^{-8})}{5} = 36 \text{ V}$$

Field is in the direction of $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_q$

35. $$V = \frac{kq}{R}$$
$$\therefore \quad kq = VR$$
$$E = \frac{Kq}{r^2} = \frac{VR}{r^2}$$

36. $$W = Fs\cos\theta = qEs\cos\theta$$
$$\therefore \quad E = \frac{W}{qs\cos\theta}$$
$$= \frac{4}{0.2\times2\times\cos 60°}$$
$$= 20 \text{ N/C}$$

37.



$$V_{C_1} = \frac{kQ}{R} - \frac{kQ}{\sqrt{R^2+d^2}}$$
$$V_{C_2} = \frac{kQ}{\sqrt{R^2+d^2}} - \frac{kQ}{R}$$
$$\therefore \quad V_{C_1} - V_{C_2} = \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+d^2}}\right]$$

38. $E = 0$, inside a hallow charged spherical conducting shell.

39. $$W = \mathbf{F}\cdot\mathbf{r}$$
$$= (Q\mathbf{E})\cdot\mathbf{r} = Q\,(\mathbf{E}\cdot\mathbf{r})$$
$$= Q\,(E_1 a + E_2 b)$$

Subjective Questions

1. $$F = \frac{kq\,(Q-q)}{r^2} \qquad \left(\text{where, } k = \frac{1}{4\pi\varepsilon_0}\right)$$

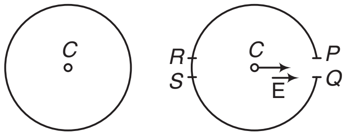
For F to be maximum, $\frac{dF}{dq} = 0$

By putting $\frac{dF}{dq} = 0$, we get

$$q = \frac{Q}{2}$$

3. $$E = \frac{\sigma}{2\varepsilon_0}$$
$$\therefore \quad \sigma = (2E\,\varepsilon_0)$$
$$= 2\times3.0\times8.86\times10^{-12}$$
$$= 5.31\times10^{-11} \text{ C/m}^2$$

4.



(i) (ii)

If loop is complete, then net electric field at centre C is zero. Because equal and opposite pair of electric field vectors are cancelled.

If PQ portion is removed as shown in figure, then electric field due to portion RS is not cancelled. Hence, electric field is only due to the option RS.

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q_{RS}}{R^2} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{(q/2\pi R)\,x}{R^2}$$
$$= \frac{qx}{8\pi^2\,\varepsilon_0 R^3}$$

5. Let q_1 and q_2 are the initial charges. After they are connected by a conducting wire, final charge on them become.

$$q'_1 = q'_2 = \left(\frac{q_1+q_2}{2}\right)$$

Now, given that

$$0.108 = \frac{(9\times10^9)(q_1)(q_2)}{(0.5)^2} \qquad \ldots\text{(i)}$$

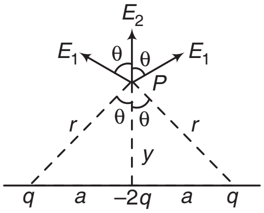
$$0.036 = \frac{(9\times 10^9)\left(\dfrac{q_1+q_2}{2}\right)^2}{(0.5)^2} \qquad \ldots\text{(ii)}$$

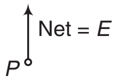
Solving Eqs. (i) and (ii), we can find q_1 and q_2.

6. Since, net force on electric dipole in uniform electric field is zero. Hence, torque can be calculated about point. This comes out to be a constant quantity given by

$$\tau = \mathbf{p}\times\mathbf{E}$$

7.





$E_1 =$ Electric field at P due to q

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q}{r^2}$$

$E_2 =$ Electric field at P due to $-2q$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{2q}{y^2}$$

∴ Net electric field at P,

$$E = 2E_1\cos\theta + E_2$$

$$= 2\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q}{r^2}\right)\left(\frac{y}{r}\right) + \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{2q}{y^2}\right)$$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\frac{y}{r^3}\right] + \left[\frac{1}{y^2}\right]$$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\frac{y}{(a^2+y^2)^{3/2}} + \frac{1}{y^2}\right]$$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\frac{y}{y^3} + \left(1+\frac{a^2}{y^2}\right)^{-3/2} + \frac{1}{y^2}\right]$$

Applying binomial expression for $\frac{a^2}{y^2} << 1$ we get,

$$E = \frac{2q}{4\pi\varepsilon_0}\left(-\frac{3}{2}\frac{a^3}{y^4}\right)$$

$$= -\frac{3qa^2}{4\pi\varepsilon_0 y^4}$$

or $$\mathbf{E} = -\frac{3qa^2}{4\pi\varepsilon_0 y^4}\,\hat{\mathbf{j}}$$

8. If we make a bigger cube comprising of eight small cubes of size given in the question with charge at centre (or at D).

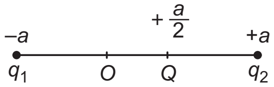
Then, total flux through large closed cube $= \frac{q}{\varepsilon_0}$.

There are 24 symmetrical faces like $EFGH$ on outermost surface of this bigger cube.

Total flux from these 24 faces is $\frac{q}{\varepsilon_0}$. Hence, flux from anyone force $= \frac{1}{24}\left(\frac{q}{\varepsilon_0}\right)$.

Electric lines are tangential to face $AEHD$. Hence, flux is zero.

9. (a)

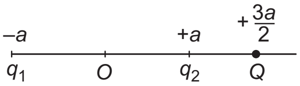


q_1 and q_2 should be of same sign.

Further, $$\frac{Kq_1Q}{(3a/2)^2} = \frac{Kq_2Q}{(a/2)^2}$$

∴ $$q_1 = 9q_2$$

(b)



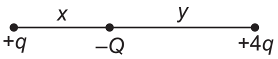
Therefore, q_1 and q_2 should be of opposite signs. Further,

$$\frac{kq_1Q}{(5a/2)^2} = \frac{kq_2Q}{(a/2)^2}$$

Or magnitude wise $q_1 = 25\,q_2$ with sign

$$q_1 = -25q_2$$

10.



$$x + y = L \qquad \ldots\text{(i)}$$

Let force on $-Q$ charge should be zero.

∴ $$\frac{kQ.\,q}{x^2} = \frac{kQ\,(4q)}{y^2}$$

∴ $$\frac{x}{y} = \frac{1}{2} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x = \frac{L}{3} \text{ and } y = \frac{2L}{3}$$

Net force on $4q$ should be zero.

∴ $$\frac{k\,|Q|\,(4q)}{(2L/3)^2} = \frac{kq\,(4q)}{L^2}$$

$$\therefore \qquad |Q| = \frac{4}{9}q$$

With sign,

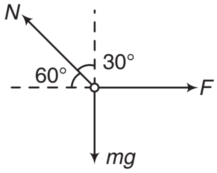
$$Q = -\frac{4}{9}q$$

Similarly net force on q should be zero.

$$\therefore \quad \frac{k\,|Q|\,q}{(L/3)^2} = \frac{kq\,(4q)}{L^2} \quad \text{or} \quad |Q| = \frac{4}{9}q$$

If we, sightly displace $-Q$ towards $4q$, attraction between these charges will increase, hence $-Q$ will move towards $+4q$ and it will not return back. Hence, equilibrium is unstable.

11.



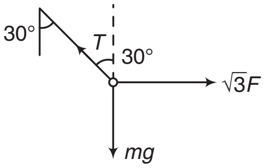
Using Lami's theorem, we have

$$\frac{mg}{\sin(90° + 30°)} = \frac{F}{\sin(90° + 60°)}$$

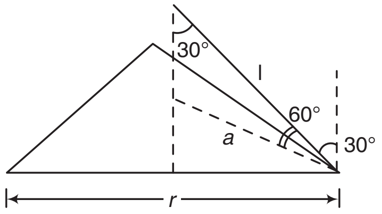
$$\therefore \quad F = \frac{mg}{\sqrt{3}} \quad \text{or} \quad \frac{1}{4\pi\varepsilon_0}\frac{(q)(q)}{R^2} = \frac{mg}{\sqrt{3}}$$

$$\therefore \quad q = \sqrt{\frac{4\pi\,\varepsilon_0\, mg\, R^2}{\sqrt{3}}}$$

12.



F = Electrostatic force between two charged balls.
$\sqrt{3}F$ = Resultant of electrostatic force on any one ball from rest two balls.



$$a = l\cos 60° = 10\text{ cm}$$

$$r = 2\,a\cos 30°$$

$$= (2)\,(10)\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}\text{ cm}$$

Now, applying Lami's theorem for the equilibrium of ball we have

$$\frac{mg}{\sin(90° + 30°)} = \frac{\sqrt{3}F}{\sin(90° + 60°)}$$

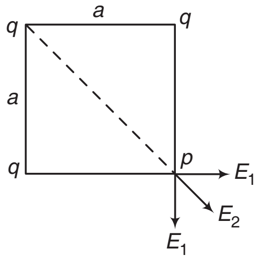
$$F = \frac{mg}{3} = \frac{1}{4\pi\varepsilon_0}\frac{(q)\,(q)}{r^2} = \frac{mg}{3}$$

$$\therefore \quad q = \sqrt{\frac{mgr^2}{3\,(1/4\,\pi\varepsilon_0)}}$$

$$= \sqrt{\frac{(0.1\times10^{-3})\,(9.8)\,(10\sqrt{3}\times10^{-2})^2}{3\times9\times10^9}}$$

$$= 3.3\times10^{-8}\text{ C}$$

13.



E_1 = Electric field due to charge q at distance a

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q}{a^2}\right)$$

E_2 = Electric field due to charge at distance $\sqrt{2}a$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q}{(\sqrt{2}\,a)^2} = \frac{E_1}{2}$$

Net electric field at P,

$$E_{\text{net}} = \sqrt{2}\,E_1 + \frac{E_2}{2} \quad \text{(In the direction of } E_2\text{)}$$

$$= \sqrt{2}E_1 + \frac{E_1}{2}$$

$$= \left(\sqrt{2} + \frac{1}{2}\right)E_1$$

$$= \frac{(2\sqrt{2} + 1)q}{8\pi\varepsilon_0 a^2}$$

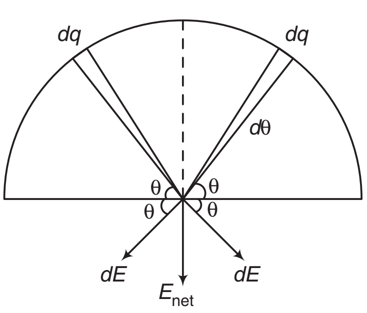
14. $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}(\mathbf{r}_p - \mathbf{r}_q)$$

$$r = \sqrt{(1.2)^2 + (1.6)^2} = 2\text{m}$$

$$\therefore \quad \mathbf{E} = \frac{(9\times10^9)\,(-8\times10^{-9})}{(2)^3}(1.2\hat{\mathbf{i}} - 10.8\hat{\mathbf{i}})$$

$$= (14.4\hat{\mathbf{j}} - 10.8\hat{\mathbf{j}})\text{ N/C}$$

15.

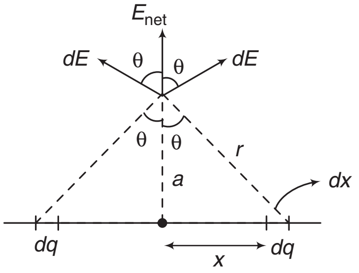


$$E_{\text{net}} = 2\int_0^{90^\circ} dE \sin\theta$$

$$= 2\int_0^{90^\circ} \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{dq}{R^2}\right)\sin\theta$$

$$= 2\int_0^{90^\circ} \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\lambda R d\theta}{R^2}\right)\sin\theta = \frac{\lambda}{2\pi\varepsilon_0 R}$$

16.



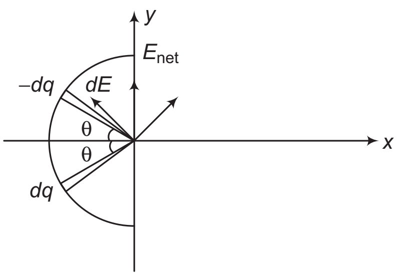
$$E_{\text{net}} = 2\int_{x=0}^{x=L/2} dE \cos\theta s$$

$$= 2\int_0^{L/2} \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{dq}{r^2}\right)\cdot\left(\frac{a}{r}\right)$$

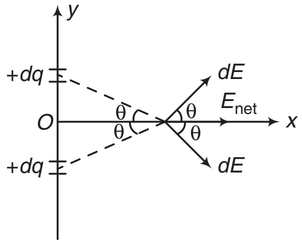
$$= 2\int_0^{L/2} \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\frac{Q}{L}\cdot dx}{a^2+x^2}\right)\frac{a}{\sqrt{a^2+x^2}}$$

Solving this integration, we find the result.

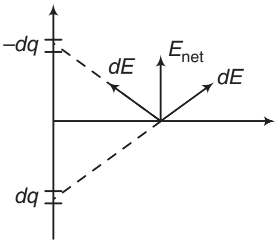
17. (a)



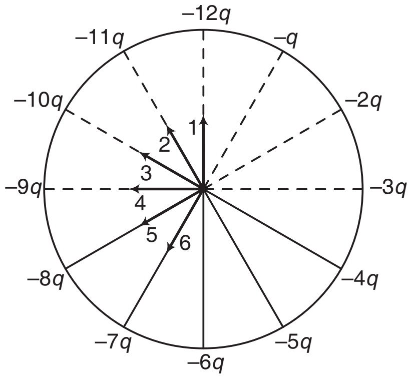
(b)



(c)



18. Six vectors of equal magnitudes are as shown in figure.



Now, resultant of two vectors of equal magnitudes (= E say) at 120° is also E and passing through their bisector line.

So, resultant of 1 and 5 is also E in the direction of 3. Similarly, resultant of 2 and 6 is also E in the direction of 4.

Finally, resultant of $2E$ in the direction of 3 and $2E$ in the direction of 4 passes through the bisector line of 3 and 4 (or 9.30)

19. $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times 25\times\sin 45^\circ}{10} = \dfrac{5}{\sqrt{2}}\,\text{s}$

$$R = S_x = u_x T + \frac{1}{2}a_x T^2$$

$$= (u\cos 45^\circ)(T) + \frac{1}{2}\left(\frac{qE}{m}\right)T^2$$

$$= (25)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) + \left(\frac{2\times10^{-6}\times2\times10^{7}}{1}\right)\left(\frac{5}{\sqrt{2}}\right)^2$$

$$= 62.5 + 250 = 312.5 \text{ m}$$

20. (a) $$g_e = g + \frac{qE}{m}$$

$$= 10 + \frac{(1.6\times10^{-19})(720)}{1.67\times10^{-27}}$$

$$\approx 6.9\times10^{10} \text{ m/s}^2$$

Now, $$R = \frac{u^2 \sin 2\theta}{g_e}$$

$$\therefore\ (1.27\times10^{-3}) = \frac{(9.55\times10^3)^2 \sin 2\theta}{6.9\times10^{10}}$$

Solving this equation, we get

$$\theta = 37° \text{ and } 53°$$

(b) Apply $T = \dfrac{2u\sin\theta}{g_e}$

21. (a) $$\mathbf{a} = \frac{q\mathbf{E}}{m} = -\frac{(1.6\times10^{-19})(120\hat{\mathbf{j}})}{(9.1\times10^{-31})}$$

$$= -(2.1\times10^{13}\hat{\mathbf{j}}) \text{ m/s}^2$$

(b) $$t = \frac{X}{v_x} = \frac{2\times10^{-2}}{1.5\times10^5} = \frac{4}{3}\times10^{-7}\text{ s}$$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$= (1.5\times10^5\,\hat{\mathbf{i}} + 3.0\times10^6\,\hat{\mathbf{j}}) - \left(2.1\times10^{13}\times\frac{4}{3}\times10^{-7}\right)\hat{\mathbf{j}}$$

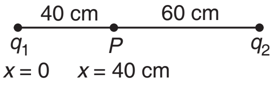
$$= (1.5\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\times10^5 \text{ m/s}$$

22. $$\frac{kq_1}{r_1} = \frac{k|q_2|}{r_2}$$

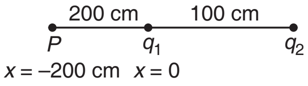
$$\therefore\ \frac{r_1}{r_2} = \frac{q_1}{|q_2|} = \frac{2}{3}$$

where, r_1 = distance from q_1 and

r_2 = distance from q_2

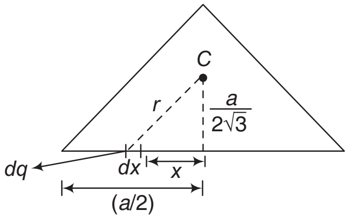


First point is at x = 40 cm where, $\dfrac{r_1}{r_2} = \dfrac{2}{3}$



Second point is at x = – 200 cm, where $\dfrac{r_1}{r_2} = \dfrac{2}{3}$.

23.



$$dq = \lambda dx$$

Potential at C, due to charge dq is

$$dV = \frac{k(dq)}{r} \qquad \left(\because k = \frac{1}{4\pi\varepsilon_0}\right)$$

$$\therefore\ \text{Total potential} = 6\int_{x=0}^{x=a/2} dV$$

24. (a) $$\Delta U = -q(\Delta V)$$

$$= -(12\times10^{-6})(50)$$

$$= -6.0\times10^{-4}\text{J}$$

(b) $$\Delta V = Ed = E(\Delta x)$$

$$= (250)(0.2)$$

$$= 50 \text{ V}$$

25. $k_A + U_A = k_B + U_B$

$$\therefore\ \frac{1}{2}mv_A^2 + qV_A = \frac{1}{2}mv_B^2 + qV_B$$

$$\therefore\ v_B = \sqrt{v_A^2 + \frac{2}{m}q(V_A - V_B)}$$

$$= \sqrt{(5)^2 + \frac{2\times(-5\times10^{-6})}{2\times10^{-4}}(200-800)}$$

$$= 7.42 \text{ m/s}$$

$v_B - v_A$ as the negative charge is moving (freely) from lower potential at A to higher potential at B. So, its electrostatic PE will decrease and kinetic energy will increase.

26. (a) $$V_C = \frac{k.\,q_{\text{net}}}{R} = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{-5Q}{R}\right)$$

(b) $$V_p = \frac{k\,q_{\text{net}}}{r}$$

where, r = distance of P from any point on circumference

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{-5Q}{\sqrt{R^2+z^2}}\right)$$

27. $$W = -\Delta U = U_i - U_f = q_2V_i - q_2V_f$$

$$= q_2\left(\frac{kq_1}{r_i}\right) - q_2\left(\frac{kq_1}{r_f}\right)$$

$$= kq_1 q_2 \left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= (9 \times 10^9)\,(2.4 \times 10^{-6})\,(-\,4.3 \times 10^6) \left(\frac{1}{0.15} - \frac{1}{0.25\sqrt{2}}\right)$$

$$= -\,0.356 \text{ J}$$

28. (a) $U = k\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1\, q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$ …(i)

where, $k = \frac{1}{4\pi\varepsilon_0}$

(b) Suppose q_3 is placed at coordinate x (> 0.2 m or 20 cm), then in Eq. (i) of part (a), put

$U = 0,\ r_{12} = 0.2 \text{ m},\ r_{13} = x$

and $r_{23} = (x - 0.2)$

Now, solving Eq. (i) we get the desired value of x.

29. $U = 0$

$\therefore \quad k\left(\frac{q \times q}{a} + \frac{q \times Q}{a} + \frac{q \times Q}{a}\right) = 0$

or $\quad Q = -\frac{q}{2}$

30. Apply $V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{r}$ …(i)

E is given in the question.

and $\quad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$

$\therefore \quad \mathbf{E} \cdot d\mathbf{r} = (5dx - 3dy)$

$\therefore \quad -\int \mathbf{E} \cdot d\mathbf{r} = (3y - 5x)$

With limits answer comes out to be

$V_B - V_A = 3\,(y_f - y_i) - 5\,(x_f - x_i)$

31. Procedure is same as work done in the above question. The only difference is, electric field is

$\mathbf{E} = (400\hat{\mathbf{j}}) \text{ V/m}$

$\therefore \quad V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{r} = -\,400\,(y_f - y_i)$

32. Similar to above two problems. But electric field here is

$\mathbf{E} = (20\hat{\mathbf{j}}) \text{ N/C}$

$\therefore \quad V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{r} = -\,20\,(x_f - x_i)$

33. (a) $[A] = \frac{[V]}{[xy]} = \frac{[\text{ML}^2\text{T}^{-3}\text{A}^{-1}]}{[\text{L}][\text{L}]}$

$= [\text{MT}^{-3}\ \text{A}^{-1}]$

(b) $\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$

(c) Substituting, $A = 10,\ x = 1,\ y = 1$ and $z = 1$ in the expression of part (b) we have

$\mathbf{E} = (-\,20\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 20\hat{\mathbf{k}}) \text{ N/C}$

$\therefore \quad |\mathbf{E}| = \sqrt{(-\,20)^2 + (-\,20)^2 + (-\,20)^2}$

$= 20\sqrt{3} \text{ N/C}$

34. See the hints of Q No. 31 to Q No. 33.

$V_f - V_i = V_B - V_A = -\,20\,(x_f - x_i) - 30\,(y_f - y_i)$

or $\quad V_f = -\,20\,(2 - 0) - 30\,(2 - 0)$

$= -\,100 \text{ V}$ (as $V_i = 0$)

35. (a) $\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right)$

$= -[(Ay - 2Bx)\,\hat{\mathbf{i}} + (Ax + C)\hat{\mathbf{j}}]$

$\therefore \quad E_x = (-\,Ay + 2Bx)$

and $\quad E_y = (-\,Ax - C)$

(b) **E** = 0 if E_x and E_y are separately zero.

$\because \quad -\,Ay + 2Bx = 0$ …(i)

and $\quad -\,Ax - C = 0$ …(ii)

Solving these two equations

We get

$x = -\frac{C}{A}$ and $y = -\frac{2BC}{A^2}$

36. Sphere is a closed surface. Therefore, Gauss's theorem can be applied directly on this

$\therefore \quad \phi_{\text{total}} = \frac{q_{\text{in}}}{\varepsilon_0}$

or $\quad q_{\text{in}} = (\phi_{\text{total}})\,(\varepsilon_0)$

$= (360)\,(8.86 \times 10^{-12})$

$= 3.19 \times 10^{-9} \text{ C}$

$= 3.19 \text{ nC}$

37. (a) $\phi_{\text{total}} = \frac{q_{\text{in}}}{\varepsilon_0}$

(b) $q_{\text{in}} = (\phi_{\text{total}})\,(\varepsilon_0)$

38. $\mathbf{S} = (0.2)\,\hat{\mathbf{i}}$

$\therefore \quad \phi = \mathbf{B} \cdot \mathbf{S} = 0.2\left(\frac{3}{5}E_0\right)$

$= 0.2 \times 0.6 \times 2 \times 10^3$

$= 240 \text{ N-m}^2\text{/C}$

39. Electric flux enters from the surface parallel to y - z plane at $x = 0$. But **E** = 0 at $x = 0$.

Hence, flux entering the cube = 0.

Flux leaves the cube from the surface parallel to y - z plane at $x = a$.

Flux leaving the cube = ES

$$= \left(\frac{E_0 x}{l}\right)(a^2) = \left(\frac{E_0 a}{l}\right)(a^2) \qquad (\text{at } x = a)$$

Substituting the values, we get

$$\phi = \frac{(5\times10^3)(10^{-2})^3}{(2\times10^{-2})}$$

$$= 2.5\times10^{-1}$$

$$= 0.25\ \text{N-m}^2/\text{C}$$

At all other four surfaces, electric lines are tangential. Hence, flux is zero.

$$\therefore \qquad \phi_{\text{net}} = (0.25)\ \text{N-m}^2/\text{C} = \frac{q_{\text{in}}}{\varepsilon_0}$$

$$\therefore \qquad q_{\text{in}} = (0.25)(\varepsilon_0)$$

$$= (0.25)(8.86\times10^{-12})$$

$$= 2.2\times10^{-12}\ \text{C}$$

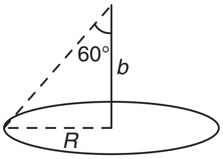
40. See the hint of Example-1 of section solved examples for miscellaneous examples. We have

$$\phi = \frac{(1-\cos\theta)\,Q}{2\varepsilon_0}$$

But, $\dfrac{Q}{\varepsilon_0} = \phi_{\text{total}}$

$$\therefore \qquad \phi = (\phi_{\text{total}})\left(\frac{1-\cos\theta}{2}\right) \qquad \text{...(i)}$$

Given that $\phi = \dfrac{1}{4}(\phi_{\text{total}})$



Substituting in Eq. (i),

we get, $\theta = 60°$

$$\frac{R}{b} = \tan 60° = \sqrt{3}$$

$$\therefore \qquad R = \sqrt{3}b$$

41. (a) $\mathbf{S}_\mathbf{j} = (L^2)(-\hat{\mathbf{j}})$

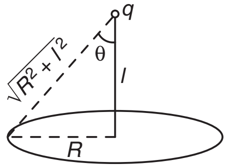
$$\therefore \qquad \phi s_1 = \mathbf{E}\cdot\mathbf{S}_\mathbf{j} = -CL^2$$

Similarly, we can find flux from other surfaces.

Note *Take area vector in outward direction of the cube.*

(b) Total flux from any closed surface in uniform electric field is zero.

42. We have



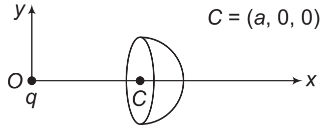
$$\phi = \frac{q\,(1-\cos\theta)}{2\,\varepsilon_0}$$

Here, flux due to + q and − q are in same direction.

$$\therefore \qquad \phi_{\text{total}} = 2\phi = \frac{q\,(1-\cos\theta)}{\varepsilon_0}$$

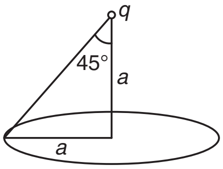
$$= \frac{q}{\varepsilon_0}\left[1 - \frac{l}{\sqrt{R^2+l^2}}\right]$$

43.



Flux passing through hemisphere = flux passing through circular surface of the hemisphere.

For finding flux through circular surface of hemisphere we can again use the concept used in above problem.

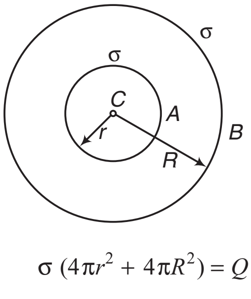


$$\phi = \frac{q\,(1-\cos\theta)}{2\varepsilon_0}$$

$$= \frac{q}{2\varepsilon_0}(1-\cos 45°)$$

$$= \frac{q}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{2}}\right)$$

44.



$$\sigma\,(4\pi r^2 + 4\pi R^2) = Q$$

$$\therefore \qquad \sigma = \frac{Q}{4\pi\,(r^2 + R^2)}$$

Potential at centre

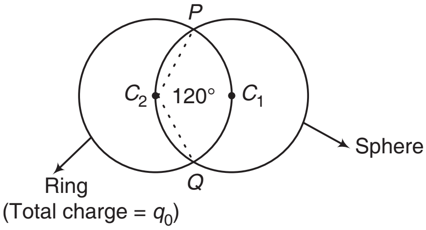
= potential due to A + potential due to B

$$= \frac{\sigma r}{\varepsilon_0} + \frac{\sigma R}{\varepsilon_0}$$

$$= \frac{\sigma}{\varepsilon_0}(r + R)$$

$$= \left(\frac{Q}{4\pi\varepsilon_0}\right)\left(\frac{r + R}{r^2 + R^2}\right)$$

45.



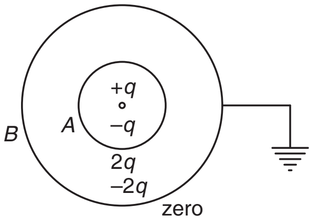
Charge on arc PQ of ring $= \dfrac{q_0}{3}$

This is also the charge lying inside the closed sphere.

$$\therefore \quad \phi \text{ through closed sphere} = \frac{q_{\text{in}}}{\varepsilon_0}$$

$$= \frac{(q_0/3)}{\varepsilon_0} = \frac{q_0}{3\varepsilon_0}$$

46.



If outermost shell is earthed. Then, charge on outer surface of outermost shell in this case is always zero.

47. (a) $V_A = \dfrac{\sigma a}{\varepsilon_0} - \dfrac{\sigma b}{\varepsilon_0} + \dfrac{\sigma c}{\varepsilon_0}$

$$= \frac{\sigma}{\varepsilon_0}(a - b + c)$$

$$V_B = \left(\frac{\sigma a}{\varepsilon_0}\right)\left(\frac{a}{b}\right) - \frac{\sigma b}{\varepsilon_0} + \frac{\sigma c}{\varepsilon_0}$$

$$= \frac{\sigma}{\varepsilon_0}\left[\frac{a^2}{b} - b + c\right]$$

$$V_C = \left(\frac{\sigma a}{\varepsilon_0}\right)\left(\frac{a}{c}\right) - \left(\frac{\sigma b}{\varepsilon_0}\right)\left(\frac{b}{c}\right) + \frac{\sigma c}{\varepsilon_0}$$

$$= \frac{\sigma}{\varepsilon_0}\left(\frac{a^2}{c} - \frac{b^2}{c} + c\right)$$

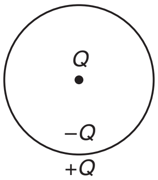
(c) $W_{A\to C} = 0$

$$\therefore \qquad V_A = V_C$$

$$\text{or} \qquad (a - b + c) = \left(\frac{a^2 - b^2}{c}\right) + c$$

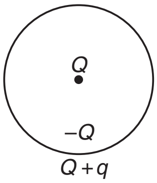
$$\text{or} \qquad a + b = c$$

48. (a)



$$\sigma_i = \frac{-Q}{4\pi a^2} \quad \text{and} \quad \sigma_0 = \frac{Q}{4\pi a^2}$$

(b)



$$\sigma_i = \frac{-Q}{4\pi a^2}$$

$$\text{and} \qquad \sigma_0 = \frac{(Q + q)}{4\pi a^2}$$

(c) According to Gauss's theorem,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{in}}}{r^2}$$

For $x \le R$, $q_{\text{in}} = Q$

and $r = x$ both cases.

$$\therefore \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}$$

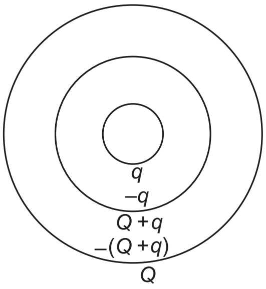
49. Let Q is the charge on shell B (which comes from earth)

$$V_B = 0$$

$$\therefore \qquad \frac{kq}{b} + \frac{kQ}{b} - \frac{kq}{c} = 0$$

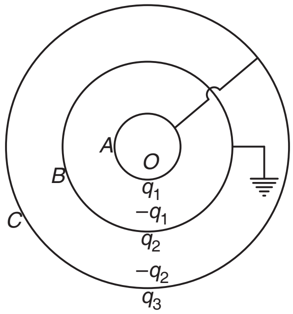
$$\therefore \qquad Q = \left(\frac{b}{c} - 1\right)q$$

Charges appearing on different faces are as shown below.



$$Q + q = \left(\frac{b}{c}\right) q$$

50.



Total charge on $A + C$ is $3q$

$$\therefore \quad q_1 - q_2 + q_3 = 3q \qquad \ldots\text{(i)}$$

$$V_B = 0$$

$$\therefore \quad \frac{kq_1}{2R} + k\left(\frac{q_2 - q_1}{2R}\right) + k\left(\frac{q_3 - q_2}{3R}\right) = 0 \qquad \ldots\text{(ii)}$$

$$V_A = V_C$$

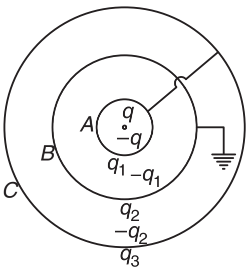
$$\therefore \quad \frac{kq_1}{R} + k\left(\frac{q_2 - q_1}{2R}\right) + k\left(\frac{q_3 - q_2}{3R}\right)$$

$$= \frac{kq_1}{3R} + k\left(\frac{q_2 - q_1}{3R}\right) + k\left(\frac{q_3 - q_2}{3R}\right) \qquad \ldots\text{(iii)}$$

In the above equations, $k = \dfrac{1}{4\pi\varepsilon_0}$.

Solving these three equations, we can find the asked charges.

51.



Total charge on $A + C$ is $3q$. Therefore,

$$-q + q_1 - q_2 + q_3 = 3q \qquad \ldots\text{(i)}$$

$$V_B = 0$$

$$\therefore \quad k\left(\frac{q_1 - q}{2R}\right) + k\left(\frac{q_2 - q_1}{2R}\right) + k\left(\frac{q_3 - q_2}{3R}\right) = 0$$

$$\ldots\text{(ii)}$$

$$V_A = V_C$$

$$\therefore \quad k\left(\frac{q_1 - q}{R}\right) + k\left(\frac{q_2 - q_1}{2R}\right) + k\left(\frac{q_3 - q_2}{3R}\right)$$

$$= k\left(\frac{q_1 - q}{3R}\right) + k\left(\frac{q_2 - q_1}{3R}\right) + k\left(\frac{q_3 - q_2}{3R}\right) \qquad \ldots\text{(iii)}$$

In the above equations,

$$k = \frac{1}{4\pi\varepsilon_0}$$

Solving above there equations, we can find q_1, q_2 and q_3.

52. (a) From Gauss's theorem

$$E = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q_{\text{in}}}{r^2}\right) = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{2Q}{r^2}\right)$$

$$= \frac{Q}{2\pi\varepsilon_0 r^2}$$

(b) According to principle of generator, potential difference depends only on q_{in}.

$$\therefore \quad \text{PD} = \frac{2\theta}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{3R}\right) = \frac{Q}{3\pi\varepsilon_0 R}$$

(c) According to principle of generator, whole inner charge transfers to outer sphere.

(d) $V_{\text{in}} = 0$

$$\therefore \quad \frac{k\,q_{\text{in}}}{R} - \frac{kQ}{3R} = 0 \;\Rightarrow\; q_{\text{in}} = \frac{Q}{3}$$

53. (a) At $r = R$,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{R} - \frac{2Q}{2R} + \frac{3Q}{3R}\right]$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{Q}{R}\right)$$

At $r = 3R$

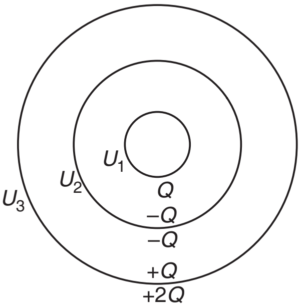
$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{3R} - \frac{2Q}{3R} + \frac{3Q}{3R}\right] = \frac{Q}{6\pi\varepsilon_0 R}$$

(b) $E = \left(\dfrac{1}{4\pi\varepsilon_0}\right)\left(\dfrac{q_{\text{in}}}{r^2}\right)$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left[\frac{Q - 2Q}{(5R/2)}\right] = \frac{-Q}{25\pi\varepsilon_0 R^2}$$

Minus sign implies that this electric field is radially towards centre.

(c)



$$U_T = U_1 + U_2 + U_3$$

where, $U_1 = \frac{1}{2}\frac{Q^2}{C_1}$

where, $C_1 = 4\pi\varepsilon_0 \Big/ \left(\frac{1}{R} - \frac{1}{2R}\right)$

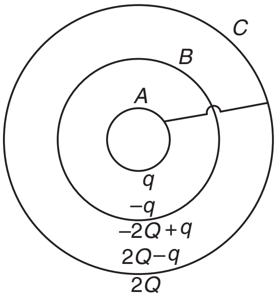
$$U_2 = \frac{1}{2}\frac{Q^2}{C_2}$$

where, $C_2 = 4\pi\varepsilon_0 \Big/ \left(\frac{1}{2R} - \frac{1}{3R}\right)$

and $U_3 = \frac{1}{2}\frac{(2Q)^2}{C_3}$

where, $C_3 = 4\pi\varepsilon_0 \Big/ \left(\frac{1}{3R} - \frac{1}{\infty}\right)$

(d)



Total charge on $(A + C)$ is $3Q + Q = 4Q$.

Now, $V_A = V_C$

$$\therefore \quad \frac{kq}{R} - \frac{k\,(2Q)}{2R} + \frac{k\,(4Q-q)}{3R} = \frac{k\,(Q - 2Q + 3Q)}{3R} \qquad \ldots\text{(i)}$$

Solving this equation, we get

$$q = \frac{Q}{2}$$

Now, $q_A = q = \frac{Q}{2}$

$$q_C = 4Q - q = \frac{7}{2}Q$$

(e) $E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_{\text{in}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q-2Q}{(5R/2)^2}\right] = \frac{-3Q}{50\pi\varepsilon_0 R^2}$

Minus sign indicates that electric field is radially inwards.

LEVEL 2

Single Correct Option

1. Let us conserve angular momentum of $+2q$ about the point at $+Q$.

$$mv_1r_1 \sin\theta_1 = mv_2\, r_2 \sin\theta_2$$

$$(m)(v)(R)\sin 150^\circ = m\left(\frac{v}{\sqrt{3}}\right) r_{\min} \sin 90^\circ$$

$$\therefore \quad r_{\min} = \frac{\sqrt{3}}{2}R$$

2. $(v_A)_y = v \;\Rightarrow\; (v_B)_y = 2v \sin 30^\circ = v$

Since, y-component of velocity remains unchanged. Hence electric field is along $(-\hat{\mathbf{i}})$ direction. Work done by electrostatic force in moving from A to B = change in its kinetic energy

$$\therefore \quad (eE)\,(2a - a) = \frac{1}{2}m\,(4v^2 - v^2)$$

$$E = \frac{mv^2}{2ea}$$

or $\mathbf{E} = -\frac{3mv^2}{2ea}\hat{\mathbf{i}}$

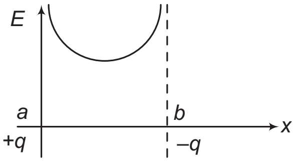
Rate of doing work done = power

$$= Fv\cos\theta$$

$$= \left(\frac{3mv^2}{2ea}\right)^2 (e)\,(2v)\cos 30^\circ$$

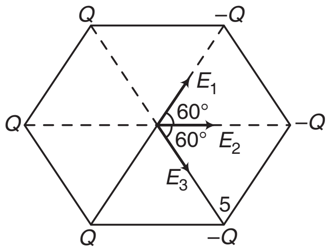
$$= \frac{3\sqrt{3}}{2}\frac{mv^2}{a}$$

3.



Just to the right of a, electric field is along ab ($\therefore$positive) and tending to infinite. Similarly, electric field just to the left of b electric field is again along ab ($\therefore$positive) and tending to infinite.

4.



$$E = E_1 = E_2 = E_3 = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{P}{x^3}\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{2Qa}{x^3}$$

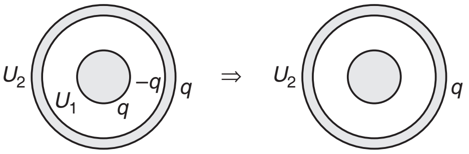
Resultant of E_1 and E_3 is also equal to E along E_2

$$\therefore \qquad E_{\text{net}} = 2E \qquad \text{(along } E_2)$$

$$= \frac{Qa}{\pi\varepsilon_0 x^3}$$

5. From centre to the surface of inner shell, potential will remain constant = 10 V (given).

6. By closing the switch whole inner charge transfers to outer shell.



$$\text{Heat produced} = U_i - U_f$$

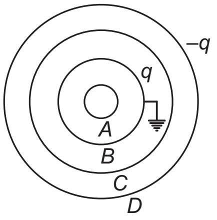
$$= (U_1 + U_2) - U_2$$

$$= U_1 = \frac{1}{2}\frac{q^2}{C}$$

where, $\quad C = \dfrac{4\pi\varepsilon_0}{(1/a - 1/2)a} = 8\pi\varepsilon_0 a$

$$\therefore \qquad \text{Heat} = \frac{q^2}{16\pi\varepsilon_0 a} = \frac{kq^2}{4a}$$

7.



Let Q charge comes on shell-C from earth. Then,

$$V_C = 0$$

$$\therefore \qquad \frac{kq}{3a} + \frac{kQ}{3a} - \frac{kq}{4a} = 0$$

Solving, we get

$$Q = -\frac{q}{4}$$

Now, $\qquad V_A = \dfrac{kq}{2a} - \dfrac{kq/4}{3a} - \dfrac{kq}{4a} = \dfrac{kq}{6a}$

and $\qquad V_C = 0$

$$\therefore \qquad V_A - V_C = \frac{kq}{6a}$$

8. $\rho = \dfrac{q}{(4/3)\,\pi R^3}$

$$\therefore \qquad q = \frac{4}{3}\pi\rho R^3$$

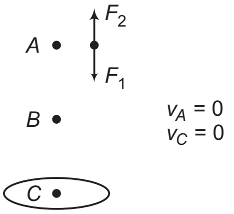
$$V_C - V_S = \frac{3}{2}\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}\right) - \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$$

$$= \frac{q}{8\pi\varepsilon_0 R}$$

Substituting the value of q, we have

$$V_C - V_S = \frac{\rho R^3}{6\varepsilon_0}$$

9.



Hence, in between A and C there is a point B, where speed of the particle should be maximum.

$$F_1 = mg = \text{constant}$$

F_2 = electrostatic repulsion (which increases as the particle moves down)

From A to B kinetic energy of the particle increases and potential energy decreases. Then, from B to C kinetic energy decreases and potential energy increases.

10. Over Q_1, potential is $+\alpha$. Hence, Q_1 is positive.

$V_A = 0$ and A point is nearer to Q_2. Therefore, Q_2 should be negative and $|Q_1| > |Q_2|$.

At A and B, potential is zero, not the force.

Equilibrium at C will depend on the nature of charge which is kept at C.

11. V_1 is positive and V_2 is negative. Hence at all points,

$$V_1 > V_2$$

12. Just to the right of q_1, electric field is $+\alpha$ or in positive direction (away from q_1). Hence, q_1 is positive. Just to the left of q_2, electric field is $-\alpha$ or towards left (or away from q_2). Hence, q_2 is also positive.

Further $E = 0$ near q_1. Hence,

$$q_1 < q_2$$

13. Electric lines of forces of q will not penetrate the conductor.

14. $\mathbf{E} = 400\cos 45°\,\hat{\mathbf{i}} + 400\sin 45°\,\hat{\mathbf{j}}$

$$V_A - V_B = -\int_B^A \mathbf{E}\cdot d\mathbf{r}$$

where, $\quad d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$

15. q_A will remain unchanged.

Hence, according to principle of generator potential difference will remain unchanged.

$$V'_A - V'_B = V_A - V_B$$

or $\quad V'_A = V_A - V_B \qquad (\text{as } V'_B = 0)$

16. $W_T = 0$

$W_{F_e} = (F_e)$ (displacement in the direction of force)

= Kinetic energy of the particle.

$$\therefore \quad \frac{1}{2}mv^2 = qE\left[\frac{l}{2} - \frac{l}{2}\cos 60°\right]$$

$$\therefore \quad v = \sqrt{\frac{qEl}{m}}$$

17. $L = mv\,r_\perp = m\,(at)\,(x_0)$

$$= m\left(\frac{qE_0}{m}\right)t\,(x_0) \quad \text{or} \quad L \propto t$$

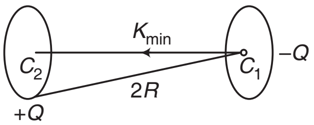
18. $U_i + K_i = U_f + K_f$

or $\quad qV_i + \frac{1}{2}mv_{\min}^2 = qV_f + 0$

or $\quad q\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{Q}{R}\right) + \frac{1}{2}mv_{\min}^2 + q\left[\frac{3}{2}\times\frac{Q}{4\pi\varepsilon_0 R}\right]$

From here, we can find $v_{\min}$.

19.



$$k_{C_1} + U_{C_1} = k_{C_2} + U_{C_2}$$

$$k_{\min} + qV_{C_1} = 0 + qV_{C_2} \qquad \ldots\text{(i)}$$

$$V_{C_1} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2R} - \frac{Q}{R}\right)$$

$$V_{C_2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} - \frac{Q}{2R}\right)$$

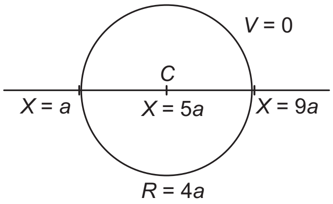
Substituting these values in Eq. (i), we can find $K_{\min}$.

20. $\mathbf{E} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}$

Now we can use, $\quad \int dV = -\int \mathbf{E}\cdot d\mathbf{r}$

two times and can find values of E_x and E_y.

21. Let $P = (x, y)$



$$r_1 = \sqrt{(x+3a)^2 + y^2}$$

$$r_2 = \sqrt{(x-3a)^2 + y^2}$$

$$V_p = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_2} - \frac{2Q}{r_1}\right] = 0 \qquad \ldots\text{(i)}$$

Substituting values of r_1 and r_2 in Eq. (i), we can see that equation is of a circle of radius $4a$ and centre at $5a$.

22. $F_x = 0$

$$\therefore \quad a_x = 0$$

$$F_y = qE$$

$$\therefore \quad a_y = \frac{qE}{m}$$

$$x = vt \text{ and}$$

$$y = \frac{1}{2}a_y t^2 = \frac{1}{2}\left(\frac{qE}{m}\right)t^2$$

Substituting $t = \frac{x}{v}$ in expression of y, we get

$$y = \frac{1}{2}\left(\frac{qEx^2}{mv^2}\right)$$

$$\text{KE} = \frac{1}{2}m\,(v_x^2 + v_y^2)$$

where, $v_x = v$ and $v_y = a_y t = \frac{qE}{m}t$

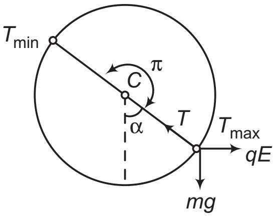
23. $qE = \frac{mv^2}{r}$

$$q\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right) = \frac{mv^2}{r}$$

$$\therefore \qquad v = \sqrt{\frac{g\lambda}{2\pi\varepsilon_0 m}} = \sqrt{\frac{2kq\lambda}{m}}$$

Now, $T = \frac{2\pi r}{v}$

24.



$$T \sin\alpha = qE$$
$$T \cos\alpha = mg$$
$$\therefore \qquad \alpha = \tan^{-1}\left(\frac{qE}{mg}\right)$$

Minimum tension will be obtained at $\alpha + \pi$.

25. Energy required = $\Delta U = U_f - U_i$

$$= \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q^2}{a}\right) - \left(\frac{q^2}{a} - \frac{q}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} - \frac{q^2}{\sqrt{2}a} + \frac{q^2}{a}\right)\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}[\sqrt{2} + 1]$$

26. On both sides of the positive charge $V = +\infty$ just over the charge.

27. $U = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{a}$ (a = side of triangle)

$$W = U_f - U_i = 3\left(\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{a}\right) - U$$

$$= 3U - U = 2U$$

28. $U_i + K_i = U_f + K_f$

$$0 + \frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Qq}{r} + 0 \quad \text{or} \quad r \propto \frac{1}{v^2}$$

If v is doubled, the minimum distance r will remain $\frac{1}{4}$ th.

29. See the hint of Sample Example 24.9

$$K = \frac{\rho}{\rho - \sigma} = \frac{1.6}{1.6 - 0.8} = 2$$

30. $V_p = 2\left(\frac{1}{4\pi\varepsilon_0}\,\frac{q}{a}\right) - 2\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\sqrt{b^2 + a^2}}\right]$

$$= \frac{q}{4\pi\varepsilon_0 a}\left[2 - 2\left(1 + \frac{b^2}{a^2}\right)^{-1/2}\right]$$

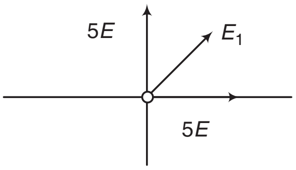
Since, $b << a$, we can apply binomial expansion

$$\therefore \qquad V_p = \frac{q}{4\pi\varepsilon_0 a}\left[2 - 2\left(1 - \frac{b^2}{2a^2}\right)\right]$$

$$= \frac{qb^2}{4\pi\varepsilon_0 a^3}$$

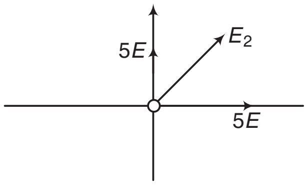
31. Let E = magnitude of electric field at origin due to charge ± q. Then,

(i)



$$E_1 = \sqrt{(5E)^2 + (5E)^2} = 5\sqrt{2}E$$

(ii)



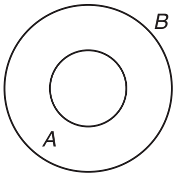
E_2 is again $5\sqrt{2}E$.

Similarly, we can find E_3 and E_4 also.

32. $U = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{1}{4\pi\varepsilon_0}\,\frac{q}{R^2}\right)^2$

$$= \frac{1}{2}\varepsilon_0\left[\frac{9\times10^9\times\frac{1}{9}\times10^{-9}}{(1)^2}\right]^2 = \frac{\varepsilon_0}{2}\ \text{J/m}^3$$

33.



They have a common potential in the beginning. This implies that only B has the charge in the beginning.

$\therefore \qquad V = \dfrac{kq_B}{b} \quad \text{or} \quad kq_B = Vb$

Now, suppose q_A charge is given to A. Then,

$$V_A = \frac{kq_A}{a} + \frac{kq_B}{b} = 0$$

or $\qquad kq_A = -a\left(\dfrac{kq_B}{b}\right) = -aV$

Now, $\qquad V_B = \dfrac{kq_A}{b} + \dfrac{kq_B}{b}$

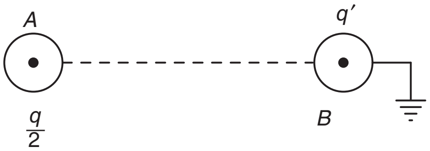
$$= -\frac{a}{b}V + V = V\left(1 - \frac{a}{b}\right)$$

34. Let $\mathbf{E} = E_x\,\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$

Apply $\displaystyle\int dV = -\int \mathbf{E}\cdot d\mathbf{v}$

three times and find values of E_x, E_y and E_z. Then, again apply the same equation for given point.

35.



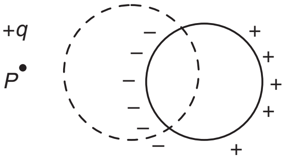
Let charge on B is q'

$$V_B = 0$$

$\therefore \qquad \dfrac{(k(q/2))}{d} + \dfrac{kq'}{r} = 0$

$\therefore \qquad q' = -\dfrac{qr}{2d}$

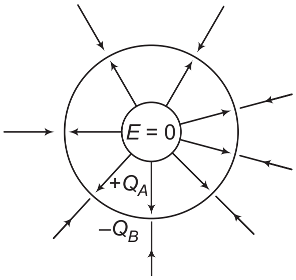
36.



The induced charges on conducting sphere due to $+q$ charge at P are as shown in figure.

Now, net charge inside the closed dotted surface is negative. Hence, according to Gauss's theorem net flux is zero.

37.



Since $|Q_B| > Q_A$, electric field outside sphere B is inwards (say negative). From A to B enclosed charge is positive. Hence, electic field is radially outwards (positive).

38. $\mathbf{E} = -\left[\dfrac{\partial V}{\partial x}\,\hat{\mathbf{i}} + \dfrac{\partial V}{\partial y}\,\hat{\mathbf{j}}\right] = -[(-ky)\,\hat{\mathbf{i}} + (-kx)\,\hat{\mathbf{j}}]$

$\therefore \qquad |\mathbf{E}| = \sqrt{(ky)^2 + (kx)^2}$

$$= k\sqrt{x^2 + y^2} = kr$$

$\therefore \qquad |\mathbf{E}| \propto r$

More than One Correct Options

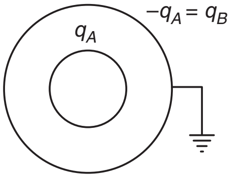
1. (a) $V_A = 2V = \dfrac{kq_A}{R} + \dfrac{kq_B}{2R}$

$$V_B = \frac{3}{2}V = \frac{kq_A}{2R} + \frac{kq_B}{2R}$$

Solving these two equations, we get

$$\frac{q_A}{q_B} = \frac{1}{2}$$

(b)



$$\frac{q'_A}{q'_B} = \frac{q_A}{-q_A} = -1$$

(c) & (d) Potential difference between A and B will remain unchanged as by earthing B, charge on will not changed.

$\therefore \qquad V'_A - V'_B = V_A - V_B$

$$= 2V - \frac{3}{2}V = \frac{V}{2}$$

$\therefore \qquad V'_A = \dfrac{V}{2}$

as $\qquad V'_B = 0$

2. $T = \dfrac{2u_y}{g} = \dfrac{2\times 10}{10} = 2\text{ s}$

$$H = \frac{u_y^2}{2g} = \frac{(10)^2}{20} = 5\text{ m}$$

$$R = \frac{1}{2}a_xT^2 = \frac{1}{2}\left(\frac{qE}{m}\right)T^2$$

$$= \frac{1}{2}\left(\frac{10^{-3} \times 10^{4}}{2}\right)(2)^2$$

$$= 10 \text{ m}$$

3. $100 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{(R + 0.05)}$ …(i)

$75 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{(R + 0.1)}$ …(ii)

Solving these equations, we get

$$q = \frac{5}{3} \times 10^{-9} \text{ C}$$

and $R = 0.1$ m

(a) $V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$

$$= \frac{(9 \times 10^9)\left(\frac{5}{3} \times 10^{-9}\right)}{0.1}$$

$$= 150 \text{ V}$$

(c) $E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2} = \frac{V}{R} = \frac{150}{0.1}$

$$= 1500 \text{ V/m}$$

(d) $V_{\text{centre}} = 1.5\, V_{\text{surface}}$

5. Electric field at any point depends on both charges Q_1 and Q_2. But electric flux passing from any closed surface depends on the charged enclosed by that closed surface only.

6. Flux from any closed surface $= \frac{q_{\text{in}}}{\varepsilon_0}$,

$q_{\text{in}} = 0$, due to a dipole.

8. $E = \frac{k\, q_{\text{in}}}{r^2}$ $\left(\text{Here, } k = \frac{1}{4\pi\varepsilon_0}\right)$

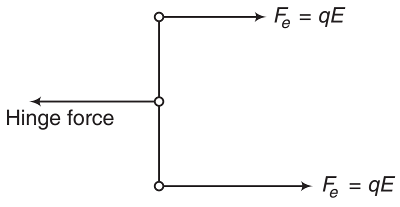
$\therefore$ $E_A = E_c = 0$

but, $E_B \neq 0$

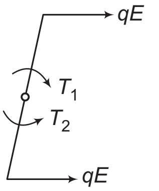
$V = \frac{kq}{R}$ $(r \leq R)$

$V = \frac{kq}{r}$ $(r \geq R)$

9.



Higher force $= 2qE$ (towards left)



If we displace the rod, $\tau_1 = \tau_2$ or $\tau_{\text{net}} = 0$ in displaced position too. Hence, equilibrium is neutral.

10. Along the line AB, charge q is at unstable equilibrium position at B (When displaced from B along AB, net force on it is away from B, whereas force at B is zero). Hence, potential energy at B is maximum.

Along CD equilibrium of q is stable. Hence, potential energy at B is minimum along CD.

Comprehension Based Questions

1. $V_{\text{outer}} = 0$

$\therefore \quad \frac{kQ}{2r} + \frac{kQ_1}{2r} = 0$

$\therefore \quad Q_1 = -Q =$ charge on outer shell

2. $V_{\text{inner}} = 0$

$\therefore \quad \frac{kQ_2}{r} + \frac{kQ_1}{2r} = 0$

$\therefore \quad Q_2 = -\frac{Q_1}{2} = \frac{Q}{2} =$ charge on inner shell

Charge flown through S_2 = initial charge on inner shell – final charge on it

$$= Q - Q_2 = \frac{Q}{2}$$

3. After two steps charge on inner shell remains $\frac{Q}{2}$ or half.

So, after n-times

$$q_{\text{in}} = \frac{Q}{(2)^n}$$

Now, according to the principle of generator, potential difference depends on the inner charge only.

$\therefore \quad \text{PD} = \frac{q_{\text{in}}}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{2r}\right]$

$$= \frac{1}{2^{n+1}}\left[\frac{Q}{4\pi\varepsilon_0 r}\right]$$

4. According to Gauss's theorem,

$$E = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q_{\text{in}}}{r^2}\right) \qquad \ldots\text{(i)}$$

For $r \le R$

$$q_{\text{in}} = \int_0^r (4\pi r^2)\cdot dr\cdot\rho$$

$$= \int_0^r (4\pi r^2)\,(\rho_0)\left(1-\frac{r}{R}\right)dr$$

$$= 4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^4}{4R}\right)$$

Substituting in Eq. (i), we get

$$E = \frac{\rho_0}{\varepsilon}\left[\frac{r}{3}-\frac{r^2}{4r}\right]$$

5. For outside the ball,

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{q_{\text{total}}}{r^2} \qquad \ldots\text{(i)}$$

where, $q_{\text{total}} = \int_0^R (4\pi r^2)\,(\rho_0)\left(1-\frac{r}{R}\right)dr$

Substituting this value in Eq. (i), we get

$$E = \frac{\rho_0 R^3}{12\,\varepsilon r^2}$$

6. For outside the ball, electric field will continuously decrease.
Hence, it will be maximum somewhere inside the ball. For maximum value,

$$\frac{dE}{dr} = 0$$

$$\therefore \quad \frac{d}{dr}\left[\frac{\rho_0}{\varepsilon}\left(\frac{r}{3}-\frac{r^2}{4R}\right)\right] = 0$$

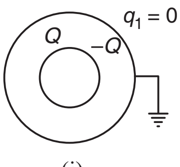
Solving, we get $r = \frac{2R}{3}$

7. Submitting $r = \frac{2R}{3}$ in the same expression of electric field, we get its maximum value.

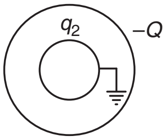
8. Potential difference in such situation depends on inner charge only. So, potential difference will remain unchanged. Hence,

$$\Delta V = V_a - V_b$$

9.



(i)



(ii)

$V_{\text{inner}} = 0$ when solid sphere is earthed

$$\because \quad \frac{kq_2}{a} - \frac{kQ}{b} = 0$$

$$\because \quad q_2 = Q\left(\frac{a}{b}\right)$$

10. Whole inner charge transfers to shell.

$\therefore$ Total charge on shell $= q_2 - Q$

$$= Q\left(\frac{a}{b}-1\right)$$

Match the Columns

1. (a) $\mathbf{E_C}$ and $\mathbf{E_F}$ are cancelled. $\mathbf{E_E}$ and $\mathbf{E_D}$ at 60°

(b) $\mathbf{E_B}$ and $\mathbf{E_E}$ are cancelled. $\mathbf{E_F}$ and $\mathbf{E_D}$ at 120°.

(c) $\mathbf{E_B}$ and $\mathbf{E_E}$ are cancelled. Similar, $\mathbf{E_F}$ and $\mathbf{E_C}$ are cancelled.

(d) $\mathbf{E_F}$ and $\mathbf{E_D}$ at 120°. So, their resultant is E in the direction of $\mathbf{E_E}$. Hence, net is $2E$.

2.

$$\int dV = -\int \mathbf{E}\cdot d\mathbf{r}$$

$$\therefore \quad V_A - V_B = -\int_B^A \mathbf{E}\cdot d\mathbf{r}$$

3. $V = \frac{kq}{R} \Rightarrow kq = VR$

(a) $$V = \frac{kq}{R^3}(1.5R^2 - 0.5r^2)$$

$$= \frac{VR}{R^3}\left[\frac{3}{2}R^2 - \frac{1}{2}\left(\frac{R}{2}\right)^2\right]$$

$$= \frac{11}{8}V$$

(b) $V = \frac{kq}{r} = \frac{VR}{2R} = \frac{V}{2}$

(c) $E = \frac{kq}{R^3}\cdot r$

$$= \frac{(V_R)}{(R^3)}\left(\frac{R}{2}\right) = \frac{V}{2R} = \frac{V}{2} \qquad (\text{if } R = 1\text{ m})$$

(d) $$E = \frac{kq}{r^2} = \frac{VR}{(2R)^2}$$

$$= \frac{V}{4R} = \frac{V}{4} \text{ for } R = 1\text{ m}$$

Subjective Questions

1. (a) By comparing this problem with spring-block system problem suspended vertically.

Here, $mg \equiv qE = 50 \times 10^{-6} \times 5 \times 10^{5} = 25$ N

$$X_{\max} = 2\ mg/K$$

Here, $X_{\max} = 2\ qE/K = \dfrac{2 \times 25}{100} = 0.5$ m

or $= 50$ cm **Ans.**

(b) Equilibrium position will be at $x = mg/K$.

Here, it will be at $x = qE/K = \dfrac{25}{100} = 0.25$ m

or 25 cm **Ans.**

(c) Force $Q\mathbf{E}$ is constant force, which does not affect the period of oscillation of SHM.

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{4}{100}}$$

$$= \frac{2\pi}{5}\ \text{s} = 1.26\ \text{s}$$ **Ans.**

(d) $\mu mg = 0.2 \times 4 \times 10 = 8$ N

Therefore, here constant force will be

$qE - \mu mg = 25 - 8 = 17$ N $= F$ (say)

$$X_{\max} = \frac{2F}{K} = \frac{2 \times 7}{100}$$

$$= 0.34\ \text{m}$$ **Ans.**

2. Total charge on ring $= \lambda(2\pi a) = q$ (say)

Electric field at distance x from the centre of ring.

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qx}{(a^2 + x^2)^{3/2}} = \frac{\lambda a x}{2\varepsilon_0 (a^2 + x^2)^{3/2}}$$

Restoring force on $-Q$ charge in this position would be

$$F = -QE = -\left[\frac{\lambda a Q x}{2\varepsilon_0 (a^2 + x^2)^{3/2}}\right]$$

For $x << a$,

$$F = -\left[\frac{\lambda a Q}{2\varepsilon_0 a^3}\right] x = -\left[\frac{\lambda Q}{2\varepsilon_0 a^2}\right] x$$

Comparing with $F = -kx$,

$$k = \frac{\lambda Q}{2\varepsilon_0 a^2}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2\varepsilon_0 m a^2}{\lambda Q}}$$ **Ans.**

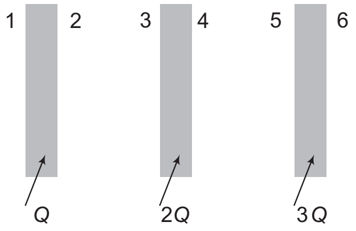
3. $q_1 = q_6 = \dfrac{q_{\text{net}}}{2} = 3Q$

$$q_2 = Q - q_1 = -2Q$$

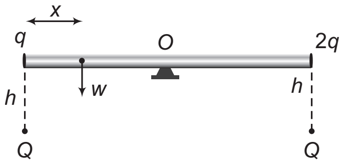
$$q_3 = -q_2 = +2Q$$

$$q_4 = 2Q - q_3 = 0$$

$$q_5 = -q_4 = 0$$



4.



(a) Net torque on the rod about $O = 0$

$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q(2q)}{h^2}\left(\frac{L}{2}\right) + w\left(\frac{L}{2} - x\right)$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q.q}{h^2}\left(\frac{L}{2}\right)$$

$$x = \frac{L}{2}\left[1 + \frac{Qq}{(4\pi\varepsilon_0) w h^2}\right]$$ **Ans.**

(b) There will be no force from the bearing it,

$w =$ net electrostatic repulsion from both the charges.

$$\therefore \quad w = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q(3q)}{h^2}$$

or $$h = \sqrt{\frac{3Qq}{4\pi\varepsilon_0 w}}$$ **Ans.**

5. $$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}}\right) = (-3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})\ \text{N/C}$$

$$\mathbf{a} = \frac{q\mathbf{E}}{m}$$

$$= \frac{10^{-6}}{10}(-3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$$

$$= (-3 \times 10^{-7}\ \hat{\mathbf{i}} - 4 \times 10^{-7}\ \hat{\mathbf{j}})\ \text{m/s}^2$$

When particle crosses x-axis, $y = 0$.

Initial y-coordinate was 3.2 m.

and $a_y = -4 \times 10^{-7}$ m/s^2

$$\therefore \quad y = 0 \text{ at time } t = \sqrt{\frac{2 \times 3.2}{4 \times 10^{-7}}} = 4000\ \text{s}$$

At this instant x-coordinate will be

$$x = x_i + \frac{1}{2}a_x t^2$$

$$= 2 + \frac{1}{2}(-3\times10^{-7})(4000)^2 = -0.4 \text{ m}$$

Now, $V_i = (3\times 2) + (4\times 3.2) = 18.8$ V

$V_f = (3)(-0.4) = -1.2$ V

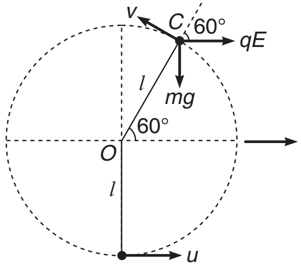
$\Delta V = 20$ V

$\therefore$ Speed, $v = \sqrt{\dfrac{2q\Delta V}{m}}$

$$= \sqrt{\frac{2\times10^{-6}\times 20}{10}}$$

$$= 2.0\times10^{-3} \text{ m/s} \qquad \textbf{Ans.}$$

6. From work-energy theorem,



$$\frac{1}{2}m(v^2 - u^2) = -mgl(1+\sin 60°) + qE\,l\cos 60°$$

Substituting the values, we get

$$u^2 - v^2 = 32.32 \qquad \ldots\text{(i)}$$

Further, at C tension in the string is zero.

Hence, $\dfrac{mv^2}{l} = mg\sin 60° - qE\cos 60°$

or $\qquad v^2 = 3.66 \qquad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$u = 6 \text{ m/s} \qquad \textbf{Ans.}$$

7. There are total 28 pairs of charges.

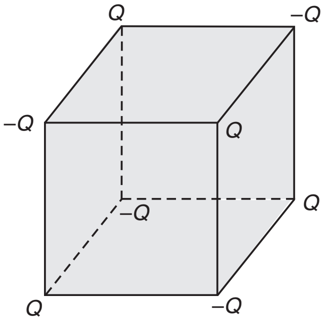
12 pairs $\rightarrow Q$ and $-Q \rightarrow$ distance L

12 pairs $\rightarrow (Q$ and $Q)$ or $(-Q$ and $-Q) \rightarrow \sqrt{2}L$

4 pairs $\rightarrow Q$ and $-Q \rightarrow \sqrt{3}L$

$$\therefore \quad U = 12\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{-Q^2}{L}\right) + 12\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q^2}{\sqrt{2}L}\right) + 4\left(\frac{1}{4\pi\,\varepsilon_0}\right)\left(\frac{-Q^2}{\sqrt{3}L}\right)$$

$$= -\frac{Q^2}{\pi\varepsilon_0 L}\left(\frac{3\sqrt{6}+\sqrt{2}-3\sqrt{3}}{\sqrt{6}}\right)$$



with decrease in L, potential energy will decrease. Therefore, cube should shrink as the conservative forces act in the direction of decreasing potential energy.

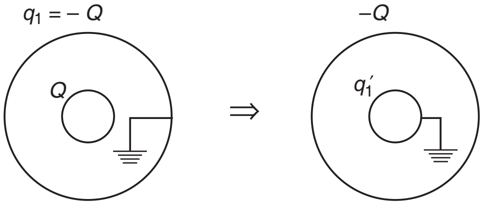
Increase in KE of the system = decrease in PE

or $\qquad 8\left(\dfrac{1}{2}mv^2\right) = U_i - U_f$

$$= \frac{Q^2}{\pi\varepsilon_0}\left(\frac{3\sqrt{6}+\sqrt{2}-3\sqrt{3}}{\sqrt{6}}\right)\left(\frac{1}{nL}-\frac{1}{L}\right)$$

or $\qquad v = \sqrt{\dfrac{Q^2(1-n)(3\sqrt{6}+\sqrt{2}-3\sqrt{3})}{4nm\pi L\varepsilon_0\sqrt{6}}} \qquad$ **Ans.**

8. Let charge q_1 comes from the earth on outer shell



$$V_{\text{outer}} = 0$$

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{2r}+\frac{q_1}{2r}\right] = 0$$

or $\qquad q_1 = -Q$

When S_2 is closed and opened,

$$V_{\text{inner}} = 0$$

$$\therefore \quad \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1'}{r}-\frac{Q}{2r}\right] = 0$$

or $\qquad q_1' = \dfrac{Q}{2}$

Proceeding in the similar manner after n such operations we get,

Charge on the inner shell,

$$q_n' = \frac{Q}{(2)^n}$$

and the potential difference between the shells,

$$\Delta V = \frac{q'_n}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{2r}\right)$$

$$\Delta V = \frac{1}{(2)^{n+1}}\left[\frac{Q}{4\pi\varepsilon_0 r}\right] \qquad \textbf{Ans.}$$

9. (a) Over charge Q_2, field intensity is infinite along negative x-axis. Therefore, Q_2 is negative.

Beyond $x > (l + a)$, field intensity is positive. Therefore, Q_1 is positive.

(b) At $x = l + a$, field intensity is zero.

$$\therefore \quad \frac{kQ_1}{(l+a)^2} = \frac{kQ_2}{a^2} \text{ or } \left|\frac{Q_1}{Q_2}\right| = \left(\frac{l+a}{a}\right)^2$$

(c) Intensity at distance x from charge 2 would be

$$E = \frac{kQ_1}{(x+l)^2} - \frac{kQ_2}{x^2}$$

For E to be maximum $\frac{dE}{dx} = 0$

$$\text{or} \quad -\frac{2kQ_1}{(x+l)^3} + \frac{2kQ_2}{x^3} = 0$$

$$\text{or} \quad \left(1+\frac{l}{x}\right)^3 = \frac{Q_1}{Q_2} = \left(\frac{l+a}{a}\right)^2$$

$$\text{or} \quad 1 + \frac{l}{x} = \left(\frac{l+a}{a}\right)^{2/3}$$

$$\text{or} \quad x = \frac{l}{\left(\frac{l+a}{a}\right)^{2/3} - 1}$$

$$\text{or} \quad b = \frac{l}{\left(\frac{l+a}{a}\right)^{2/3} - 1} \qquad \textbf{Ans.}$$

10. Capacities of conducting spheres are in the ratio of their radii. Let C_1 and C_2 be the capacities of S_1 and S_2, then

$$\frac{C_2}{C_1} = \frac{R}{r}$$

(a) Charges are distributed in the ratio of their capacities. Let in the first contact, charge acquired by S_2, is q_1. Therefore, charge on S_1 will be $Q - q_1$. Say it is q'_1.

$$\therefore \quad \frac{q_1}{q'_1} = \frac{q_1}{Q-q_1} = \frac{C_2}{C_1} = \frac{R}{r}$$

It implies that Q charge is to be distributed in S_2 and S_1 in the ratio of R/r.

$$\therefore \quad q_1 = Q\left(\frac{R}{R+r}\right) \qquad \ldots\text{(i)}$$

In the second contact, S_1 again acquires the same charge Q.

Therefore, total charge in S_1 and S_2 will be

$$Q + q_1 = Q\left(1 + \frac{R}{R+r}\right)$$

This charge is again distributed in the same ratio. Therefore, charge on S_2 in second contact,

$$q_2 = Q\left(1+\frac{R}{R+r}\right)\left(\frac{R}{R+r}\right)$$

$$= Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2\right]$$

Similarly,

$$q_3 = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \left(\frac{R}{R+r}\right)^3\right]$$

and

$$q_n = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \ldots + \left(\frac{R}{R+r}\right)^n\right]$$

$$\text{or} \quad q_n = Q\frac{R}{r}\left[1 - \left(\frac{R}{R+r}\right)^n\right] \qquad \ldots\text{(ii)}$$

$$\left[S_n = \frac{a(1-r^n)}{(1-r)}\right]$$

Therefore, electrostatic energy of S_2 after n such contacts

$$U_n = \frac{a_n^2}{2C} = \frac{q_n^2}{2(4\pi\varepsilon_0 R)} \quad \text{or} \quad U_n = \frac{q_n^2}{8\pi\varepsilon_0 R}$$

where, q_n can be written from Eq. (ii).

(b) $$q_n = \frac{QR}{R+r}\left[1 + \frac{R}{R+r} + \ldots + \ldots + \left(\frac{R}{R+r}\right)^{n-1}\right]$$

as $n \to \infty$

$$q_\infty = \frac{QR}{R+r}\left(\frac{1}{1 - \frac{R}{R+r}}\right)$$

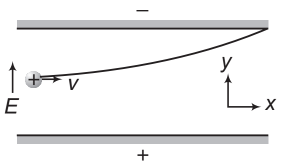
$$= \frac{QR}{R+r}\left(\frac{R+r}{r}\right) = Q\frac{R}{r} \qquad \left[S_\infty = \frac{a}{1-r}\right]$$

$$\therefore \quad U_\infty = \frac{q_\infty^2}{2C} = \frac{Q^2R^2/r^2}{8\pi\varepsilon_0 R}$$

$$\text{or} \quad U_\infty = \frac{Q^2R}{8\pi\varepsilon_0 r^2} \qquad \textbf{Ans.}$$

11. $v_x = v = \sqrt{\left(\dfrac{2qV}{m}\right)},\quad t = \dfrac{l}{v_x} = l\sqrt{\dfrac{m}{2qV}}$

$$a_y = \frac{qE}{m} = \frac{qat}{m} = \frac{dv_y}{dt}$$



Integrating both sides, we get

$$v_y = \frac{qat^2}{2m}$$

or $$v_y = \left(\frac{qa}{2m}\right)l^2\left(\frac{m}{2qV}\right) = \frac{al^2}{4V}$$

Now, angle of deviation

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{al^2}{4V}\Big/\sqrt{\frac{2qV}{m}}\right)$$

$$= \tan^{-1}\left(\frac{al^2}{4V}\sqrt{\frac{m}{2eV}}\right)$$ **Ans.**

12. From energy conservation,

$$U_C + K_C = U_D + K_D$$

or $$2\left[\frac{9\times10^9\times(5\times10^{-5})(-5\times10^{-5})}{5}\right] + 4$$

$$= 2\left[\frac{9\times10^9\times(5\times10^{-5})(-5\times10^{-5})}{AD}\right] + 0$$

Solving we get $AD = 9$ m

∴ Maximum distance,

$$OD = \sqrt{(9)^2 - (3)^2}$$

$$= \sqrt{72}\text{ m}$$ **Ans.**

13. From conservation of energy,

$$U_i + K_i = U_f + K_f$$

or $$qV_i + K_i = qV_f + K_f$$

or $$q\left(\frac{Q}{4\pi\varepsilon_0 r}\right) + \frac{1}{2}mv^2$$

$$= q\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)\left(1.5R^2 - 0.5\frac{R^2}{4}\right) + 0$$

or $$\frac{1}{2}mv^2 = \frac{11\,Qq}{3\pi\varepsilon_0 R} - \frac{Qq}{4\pi\varepsilon_0 r}$$

or $$v = \sqrt{\frac{Qq}{2\pi\varepsilon_0 mR}\left(\frac{r-R}{r} + \frac{3}{8}\right)}$$ **Ans.**

14. From energy conservation principle,

$$K_i + U_i = K_f + U_f$$

or $$\frac{1}{2}mv^2 + (+q)V_i = 0 + (+q)V_f$$

or $$v = \sqrt{\frac{2q}{m}(-V_i + V_f)}$$

$$= \sqrt{\frac{2q}{m}\left[\frac{-Q}{\sqrt{10}R} + \frac{8Q}{5R} + \frac{Q}{R} - \frac{8Q}{4R}\right]\frac{1}{4\pi\varepsilon_0}}$$

$$= \sqrt{\frac{Qq}{2\pi\varepsilon_0 mR}\left(\frac{3\sqrt{10}-5}{5\sqrt{10}}\right)}$$ **Ans.**

15. (a) $E = \dfrac{kQx}{(R^2+x^2)^{3/2}}$,

$$\frac{dE}{dx} = kQ\left[\frac{(R^2+x^2)^{3/2} - x.\frac{3}{2}(R^2+x^2)^{1/2}(2x)}{(R^2+x^2)^3}\right]$$

or $$\frac{dE}{dx} = kQ\left[\frac{R^2+x^2-3x^2}{(R^2+x^2)^{5/2}}\right]$$

or $$dE = \frac{Q}{4\pi\varepsilon_0}\left[\frac{R^2-2x^2}{(R^2+x^2)^{5/2}}\right]dx$$

$$\therefore\ |\Delta E| = \frac{Q}{4\pi\varepsilon_0}\left[\frac{R^2-2x^2}{(R^2+x^2)^{5/2}}\right]\Delta x$$ **Ans.**

Here, $\Delta x = 2a$

$$\therefore\ F = |q\Delta E| = \frac{Qqa}{2\pi\varepsilon_0}\left[\frac{R^2-2x^2}{(R^2+x^2)^{5/2}}\right]$$

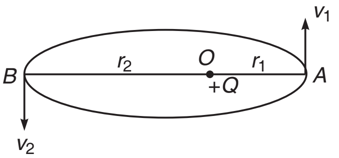
(b) $W = U_f - U_i$

$$= -pE\cos 180^\circ + pE\cos 0^\circ = 2pE$$

$$= 2(q)(2a)\left[\frac{1}{4\pi\varepsilon_0}\,\frac{Qx}{(R^2+x^2)^{3/2}}\right]$$

$$= \frac{aqQx}{\pi\varepsilon_0(R^2+x^2)^{3/2}}$$ **Ans.**

16. From conservation of mechanical energy and conservation of angular momentum about point O, we have



$$\frac{1}{2}mv_1^2 - \frac{1}{4\pi\varepsilon_0}.\frac{Qq}{r_1} = \frac{1}{2}mv_2^2 - \frac{1}{4\pi\varepsilon_0}.\frac{Q.q}{r_2} \quad \text{...(i)}$$

and $\quad mv_1r_1 \sin 90° = mv_2r_2 \sin 90°$

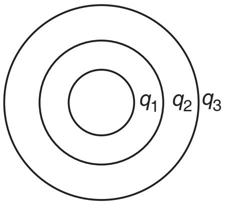
or $\quad v_1r_1 = v_2r_2 \qquad$ ...(ii)

Solving these two equations, we have

$$v_1 = \sqrt{\frac{Qqr_2}{2\pi\varepsilon_0 mr_1(r_1 + r_2)}}$$

and $$\sqrt{\frac{Qqr_1}{2\pi\varepsilon_0 mr_2(r_1 + r_2)}} = v_2$$ **Ans.**

17. Let $q_1 : q_2$ and q_3 be the respective charges. Then,



$$10 = \frac{9 \times 10^9}{10^{-2}}\left[\frac{q_1}{1} + \frac{q_2}{2} + \frac{q_3}{4}\right]$$

$$0 = \frac{9 \times 10^9}{10^{-2}}\left[\frac{q_1}{2} + \frac{q_2}{2} + \frac{q_3}{4}\right]$$

and $$40 = \frac{9 \times 10^9}{10^{-2}}\left[\frac{q_1}{4} + \frac{q_2}{4} + \frac{q_3}{4}\right]$$

Solving these equations, we get

$q_1 = +\dfrac{200}{9} \times 10^{-12}$ C, $q_2 = -200 \times 10^{-12}$ C and

$q_3 = \dfrac{3200}{9} \times 10^{-12}$ C

(a) **At $r = 1.25$ cm**

$$V = \frac{9 \times 10^9}{10^{-2}}\left[\frac{(200/9) \times 10^{-12}}{1.25} - \frac{200 \times 10^{-12}}{2} + \frac{(3200/9) \times 10^{-12}}{4}\right]$$

$= 6$ V **Ans.**

(b) **Potential at $r = 2.5$ cm**

$$V = \frac{9 \times 10^9}{10^{-2}}\left[\frac{(200/9) \times 10^{-12}}{2.5} - \frac{200 \times 10^{-12}}{2.5} + \frac{(3200/9) \times 10^{-12}}{4}\right]$$

$= 16$ V **Ans.**

(c) Electric field at $r = 1.25$ cm will be due to charge q_1 only.

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{r^2}$$

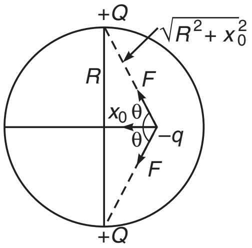
$$= \frac{9 \times 10^9 \times (200/9) \times 10^{-12}}{(1.25 \times 10^{-2})^2}$$

$= 1.28 \times 10^3$ V/m **Ans.**

18. (a) $F_{\text{net}} = 2F \cos\theta$

$$= \frac{2kQ.q}{(\sqrt{R^2 + x_0^2})^2}\cdot\frac{x_0}{\sqrt{R^2 + x_0^2}} \quad \left(\text{Here, } k = \frac{1}{4\pi\varepsilon_0}\right)$$

$$= \frac{2kQqx_0}{(R^2 + x_0^2)^{3/2}}$$



We can generalised the force by putting $x_0 = x$, we have

$$F = -\frac{2kQqx}{(R^2 + x^2)^{3/2}}$$ **Ans.**

(b) Motion of bead will be periodic between

$x = \pm x_0$ **Ans.**

(c) For $\dfrac{x}{R} << 1, R^2 + x^2 \approx R^2$

or $\quad F = -\left(\dfrac{2kQq}{R^3}\right)x$ or $a = \dfrac{F}{m} = -\left(\dfrac{2kQq}{mR^3}\right)x$

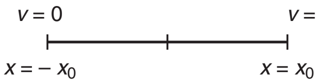
Since $a \propto -x$, motion will be simple harmonic in nature.

Comparing with $a = -\omega^2 x$, $\omega = \sqrt{\dfrac{2kQq}{mR^3}}$

$x = x_0 \cos\omega t$ (as the particle starts from extreme position)

$$v = \frac{dx}{dt} = -\omega\, x_0 \sin\omega t$$ **Ans.**

(d) Velocity will become zero at $t = T/2 = \pi/\omega$



or $$t = \pi\sqrt{\frac{mR^3}{2kQq}}$$ **Ans.**

19. Net charge between $r = a$ to $r = r$ would be

$$Q = \int_a^r \rho(4\pi r^2)\,dr = \int_a^r \frac{C}{r}(4\pi r^2)\,dr$$

$$Q = 2\pi C(r^2 - a^2)$$

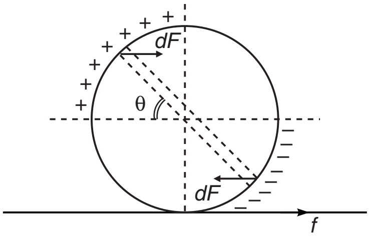
$$E_r = \frac{k(Q+q)}{r^2} = \frac{k[2\pi C(r^2 - a^2) + q]}{r^2}$$

From this expression, we can see that it we put

$$C = \frac{q}{2\pi a^2}$$

$$E_r = \frac{kq}{a^2} = \text{constant} \qquad \textbf{Ans.}$$

20. $dF = \lambda(Rd\theta)E_0$



Perpendicular distance between two equal and opposite pairs of dF will be

$$r_\perp = 2R\sin\theta$$

$$\therefore \quad d\tau = dFr_\perp = 2\lambda R^2 E_0 \sin\theta d\theta$$

$$\therefore \quad \tau = \int_0^{\pi/2} d\tau = 2\lambda R^2 E_0 \qquad \text{(clockwise)}$$

These pairs of forces will not provide net force.

Let force of friction on ring is f in forward direction.

For pure rolling to take place,

$$a = R\alpha$$

$$\text{or} \quad \frac{f}{m} = R\left[\frac{\tau - fR}{mR^2}\right]$$

$$\text{or} \quad f = \frac{\tau}{R} - f$$

$$\text{or} \quad f = \frac{\tau}{2R} = \lambda R E_0 \qquad \textbf{Ans.}$$

21. Two forces will act on the tank.

(a) Electrostatic force,

(b) Thrust force.

Let v be the velocity at any instant. Then,

$$F_{\text{net}} = QE - mnv$$

$$\text{or} \quad (m_0 + mnt)\frac{dv}{dt} = QE - mnv$$

$$\text{or} \quad \int_0^v \frac{dv}{QE - mnv} = \int_0^t \frac{dt}{m_0 + mnt}$$

$$\text{or} \quad \ln\left(\frac{QE}{QE - mnv}\right) = \ln\left(\frac{m_0 + mnt}{m_0}\right)$$

$$\text{or} \quad \frac{QE}{QE - mnv} = \frac{m_0 + mnt}{m_0}$$

$$\text{or} \quad v = QE\left(\frac{t}{m_0 + mnt}\right) \qquad \textbf{Ans.}$$

22. $qE = 30$ N, vertical component of electric force $= 30 \sin 30° = 15$ N and horizontal component of electric force $= 30\cos 30° = 15\sqrt{3}$ N

$$a_y = \frac{mg - 15}{m} = \frac{30 - 15}{3} = 5 \text{ m/s}^2 \text{ (downwards)}$$

$$a_x = \frac{15\sqrt{3}}{3} = 5\sqrt{3} \text{ m/s}^2$$

$$T_1 = \frac{2u_y}{a_y} = \frac{2 \times 20 \sin 30°}{5} = 4 \text{ s}$$

$$T_2 = eT_1 = 2 \text{ s}$$

Horizontal velocity after first drop

$$= (20\cos 30°) + a_x T_1$$

$$= (10\sqrt{3}) + (5\sqrt{3})4$$

$$= 30\sqrt{3} \text{ m/s}$$

$\therefore$ Horizontal distance travelled between first drop and second drop

$$= (30\sqrt{3})T_2 + \frac{1}{2}a_x T_2^2$$

$$= (30\sqrt{3})(2) + \frac{1}{2}(5\sqrt{3})(2)^2$$

$$= 70\sqrt{3} \text{ m} \qquad \textbf{Ans.}$$

25. Capacitors

INTRODUCTORY EXERCISE 25.1

1. $U = \frac{1}{2}\frac{q^2}{C}$

$\therefore \qquad [C] = \left[\frac{q^2}{U}\right] = \left[\frac{\mathrm{A^2T^2}}{\mathrm{ML^2T^{-2}}}\right]$

$= [\mathrm{M^{-1}L^{-2}T^4A^2}]$

2. Charge does not flow if their potentials are same.

3. $q_1 = C_1V_1 = 10\,\mu\mathrm{C}$

$q_2 = C_2V_2 = -40\,\mu\mathrm{C}$

(a) $V = \frac{q_{\text{Total}}}{C_{\text{Total}}} = \frac{-30\,\mu\mathrm{C}}{3\,\mu\mathrm{F}} = -10$ volt

(b) $q_1' = C_1V$ and $q_2' = C_2V$

(c) $\Delta U = \frac{C_1C_2}{2(C_1 + C_2)}(V_1 - V_2)^2$

INTRODUCTORY EXERCISE 25.2

1. $q = CV$

2. (a) $V = \frac{q}{C}$

(b) $C = \frac{\varepsilon_0 A}{d}$

$\therefore \qquad A = \frac{Cd}{\varepsilon_0}$

(c) $\qquad \sigma = \frac{q}{A}$

3. (a) $\qquad K = \frac{E_0}{E} = \frac{3.20 \times 10^5}{2.50 \times 10^5}$

$= 1.28$

(b) $\sigma_i = \sigma_0\left(1 - \frac{1}{K}\right)$

$= (E_0\varepsilon_0)\left(1 - \frac{1}{K}\right)$

$= (3.20 \times 10^5)(8.86 \times 10^{-12})\left(1 - \frac{1}{1.28}\right)$

$= 6.2 \times 10^{-7}\ \mathrm{C/m^2}$

INTRODUCTORY EXERCISE 25.3

1. $C_{\text{net}} = 2\,\mu\mathrm{F}$, $q = CV = 2 \times 15 = 30\,\mu\mathrm{C}$

Now, this q will be distributed between 4 μF and 2 μF in direct ratio of their capacities.

2. $C_{\text{net}} = 3\,\mu\mathrm{F}$, $q = CV = 3 \times 40 = 120\,\mu\mathrm{C}$.

Now, this q will be distributed between 9 μF and 3 μF in the direct ratio of their capacities.

Exercises

LEVEL 1

Assertion and Reason

1. Capacitance of conductor depends on the dimension of the conductor and the medium in which this conductor is kept.

3. Energy supplied by the battery is

$$\Delta qV = (CV)(V) = CV^2.$$

Energy stored in the capacitor is $\frac{1}{2}CV^2$.

4. In graph-1, discharging is slow.

Hence, $\tau_{C_1} > \tau_{C_2}$

Further,

$$\tau_C = CR$$

$\therefore \qquad \tau_C \propto R \qquad$ (as C = constant)

5. Charges are same, if initially the capacitors are uncharged.

Further, $\qquad V = \frac{q}{C}$

Hence, $\qquad V \propto \frac{1}{C}$

if q is same.

6. Charge (or current) will not flow in the circuit as they have already the same potential, which is a condition of parallel grouping.

Further,

$$\frac{q_1}{q_2} = \frac{C_1V}{C_2V} = \frac{C_1}{C_2} = \frac{1}{2}$$

7. Capacitor and R_2 are short-circuited. Hence, current through R_2 is zero and capacitor is not charged.

8. Capacitor and resistance in its own wire are directly connected with the battery. Hence, time constant during charging is CR.

9. $U = \frac{1}{2}\frac{q^2}{C}$ or $U \propto \frac{1}{C}$ as q is same in capacitors (if initially they are uncharged)

10. By inserting dielectric slab, value of C_2 will increase. In series, potential difference distributes in inverse ratio of capacitance. If capacitance C_2 is increased PD across C_2 will decrease. If C_2 is increased, charge on capacitors will also increase.

So, positive charge or current flows in clockwise direction.

Objective Questions

1. $F = qE = q\left(\frac{\sigma}{2\varepsilon_0}\right) = q\left(\frac{q}{2A\varepsilon_0}\right)$ q will not change.

$$\therefore \quad F = \text{constant}$$

3.
$$\begin{aligned} C &= C_1 + C_2 \\ &= (4\pi\varepsilon_0 a) + (4\pi\varepsilon_0 b) \\ &= 4\pi\varepsilon_0 (a + b) \end{aligned}$$

4.
$$\begin{aligned} V_{\text{net}} &= V_1 + V_2 + \ldots \qquad \text{(in series)} \\ &= V + V + \ldots \\ &= nV \end{aligned}$$

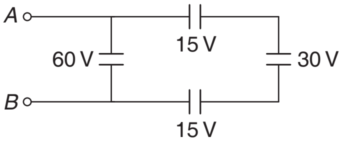
5. $V_{32} = V_5 = 6\text{ V}$

$$\therefore \quad q_5 = CV = 30\,\mu\text{C}$$

$$q_{32} = \left(\frac{3 \times 2}{3 + 2}\right) \times 6 = 7.2\,\mu\text{C}$$

$$\therefore \quad \frac{q_5}{q_2} = \frac{30}{7.2}$$

6.

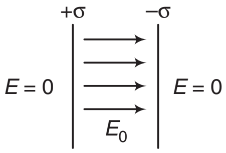


7.
$$\because \quad qE = mg$$

$$q\left(\frac{V}{d}\right) = \left(\frac{4}{3}\pi r^3 \rho\right) g$$

$$V \propto \frac{r^3}{q}$$

8.



$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_0} = \text{constant}$$

9. At $t = 0$, when capacitor is under charged, equivalent resistance of capacitor = 0

In this case, 6 Ω and 3 Ω are parallel (equivalent = 2 Ω)

$$\therefore \quad R_{\text{net}}\ (1 + 2)\ \Omega = 3\ \Omega$$

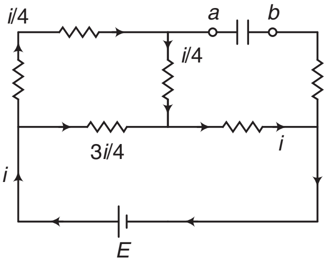
$$\therefore \quad \text{Current from battery} = \frac{12}{3} = 4\text{ A}$$

= Current through 1 Ω resistor

10. Final potential difference = E

$\therefore$ Final charge = EC

11.



$$V_a - \frac{i}{4}R - iR = V_b$$

$$\therefore \quad V_a - V_b = \frac{5}{4}\,iR = 10 \qquad \ldots\text{(i)}$$

$$R_{\text{net}} = R + \frac{(3R)(R)}{(3R + R)}$$

$$= \frac{7}{4}R$$

$$\therefore \quad i = \frac{E}{(7/4)R} = \left(\frac{4E}{7R}\right)$$

Substituting in Eq. (i), we have

$$\left(\frac{5}{4}\right)\left(\frac{4E}{7R}\right) \times R = 10$$

$$\therefore \quad E = 14\text{ V}$$

12. All capacitors have equal capacitance. Hence, equal potential drop (= 2.5 V) will take place across all capacitors.

$$V_N - V_B = 2.5\text{ V}$$

$$0 - V_B = 2.5\text{ V}$$

$$\therefore \quad V_B = -\,2.5\text{ V}$$

Further, $V_A - V_N = 3\,(2.5)\text{ V}$

$$= 7.5\text{ V}$$

$$\therefore \quad V_A = +\,7.5\text{ V} \qquad (\text{as } V_N = 0)$$

13. $q = CV = 200\,\mu\text{C}$

In parallel, the common potential is given by

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

$$= \frac{200\ \mu\text{C}}{(2+2)\ \mu\text{F}} = 50\ \text{V}$$

Heat loss $= U_i - U_f$

$$= \frac{1}{2}(2\times10^{-6})(100)^2 - \frac{1}{2}(4\times10^{-6})(50)^2$$

$$= 5\times10^{-3}\ \text{J}$$

$$= 5\ \text{mJ}$$

14. $$P = i^2R = (i_0e^{-t/\eta})^2\ R$$

$$= (i_0^2R)\ e^{-2t/\eta}$$

$$= P_0e^{-t/(\eta/2)}$$

Hence, the time constant is $\frac{\eta}{2}$.

15. Common potential in parallel grouping

$$= \frac{\text{Total charge}}{\text{Total capacity}}$$

$$= \frac{EC}{2C} = \frac{E}{2}$$

16. $V_A - 6 - 3\times2 + \frac{9}{1} - 3\times3 = V_B$

$\therefore \qquad V_A - V_B = 12\ \text{V}$

17. In steady state condition, current flows from outermost loop.

$$i = \frac{12}{6+2} = 1.5\ \text{A}$$

Now, $V_C = V_{6\Omega} = iR$

$$= 1.5\times6 = 9\ \text{V}$$

$\therefore \qquad q = CV_C = 18\ \mu\text{C}$

19. Horizontal range,

$$R = \frac{2u_x\times u_y}{g} = l \qquad \text{…(i)}$$

Maximum height, $H = \frac{u_y^2}{2g} = d \qquad$ …(ii)

Dividing Eq. (ii) by Eq. (i), we have

$$\frac{1}{4}\left(\frac{u_y}{u_x}\right) = \frac{d}{l}$$

or $$\frac{u_y}{u_x} = \frac{u\sin\theta}{u\cos\theta}$$

$$= \tan\theta = \frac{4d}{l}$$

20. $V = Ed = \left(\frac{\sigma}{2\varepsilon_0}\right)d$

$$\therefore \qquad d = \frac{2\varepsilon_0V}{\sigma}$$

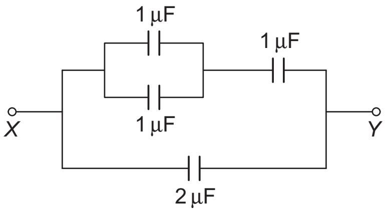
$$= 2\,\frac{(8.86\times10^{-12})\,(5)}{10^{-7}}$$

$$= 0.88\times10^{-3}\,\text{m}$$

$$= 0.88\ \text{mm}$$

21. Three capacitors (consisting of two loops are short-circuited).

22. The equivalent circuit is as shown below.



23. $C_1 = C_{\text{RHS}} + C_{\text{LHS}}$

$$= \frac{K_2\,\varepsilon_0(A/2)}{d} + \frac{K_1\varepsilon_0(A/2)}{d}$$

$$= \frac{\varepsilon_0A}{2d}(K_1+K_2) = \frac{5\varepsilon_0A}{2d}$$

$$C_2 = \frac{\varepsilon_0A}{d - d/2 - d/2 + \frac{d/2}{K_1} + \frac{d/2}{K_2}}$$

$$= \frac{2\varepsilon_0A}{d}\left(\frac{K_1K_2}{K_1+K_2}\right)$$

$$= \frac{12\,\varepsilon_0A}{5d}$$

$$\therefore \qquad \frac{C_1}{C_2} = \frac{25}{24}$$

24. A balanced Wheatstone bridge is parallel with C.

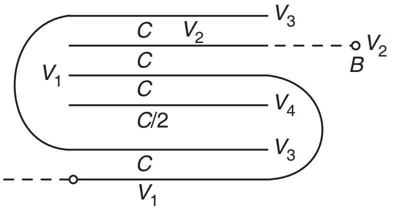
25. First three circuits are balanced Wheatstone bridge circuits.

26. $C = C_{\text{LHS}} + C_{\text{RHS}}$

$$= \frac{K_1\varepsilon_0\,(A/2)}{d} + \frac{\varepsilon_0(A/2)}{d - d/2 - d/2 + \frac{d/2}{K_2} + \frac{d/2}{K_3}}$$

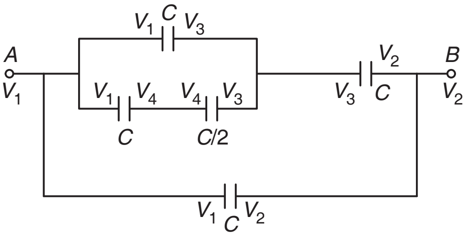
$$= \frac{\varepsilon_0A}{d}\left[\frac{K_1}{2} + \frac{K_2K_3}{K_2+K_3}\right]$$

27.



$$C = \frac{\varepsilon_0 A}{d} = 7\,\mu F$$

The equivalent circuit is as shown in figure.



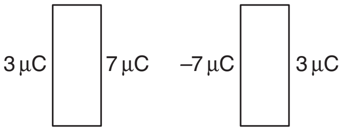
$$C_{AB} = \frac{11}{7}C = \frac{11}{7}(7\,\mu F)$$
$$= 11\,\mu F$$

Subjective Questions

1. Charge on outermost surfaces

$$= \frac{q_{\text{total}}}{2} = \frac{(10-4)\,\mu C}{2}$$
$$= 3\,\mu C$$

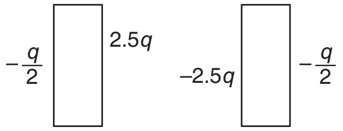
Hence, charges are as shown below.



2. Charge on outermost surfaces

$$= \frac{q_{\text{total}}}{2} = \frac{2q-3q}{2} = -\frac{q}{2}$$

Hence, charge on different faces are as shown below.



Electric field and hence potential difference between the two plates is due to ± 2.5 q.

$$\text{PD} = Ed$$
$$= \left(\frac{\sigma}{\varepsilon_0}\right)d = \left(\frac{2.5\,q\,d}{\varepsilon_0 A}\right)$$

Capacitance, $C = \frac{\varepsilon_0 A}{d}$

3. All three capacitors are in parallel with the battery. PD across each of them is 10 V. So, apply $q = CV$ for all of them.

4. Capacitor and resistor both are in parallel with the battery. PD across capacitor is 10 V. Now, apply $q = CV$.

5. In steady state, current flows in lower loop of the circuit.

$$i = \frac{30}{6+4} = 3\text{A}$$

Now, potential difference across capacitor = potential difference across 4 Ω resistance.

$$= iR$$
$$= (3)\,(4) = 12\text{ V}$$
$$\therefore \quad q = CV = (2\,\mu F)\,(12\text{ V})$$
$$= 24\,\mu C$$

6. (a) $$C_{\text{net}} = \frac{C_1 C_2}{C_1 + C_2} = \frac{2}{3}\,\mu F$$

$$q_{\text{net}} = C_{\text{net}} V$$
$$= \left(\frac{2}{3}\,\mu F\right)(1200\text{ V})$$
$$= 800\,\mu C$$

In series, q remains same.

$$\therefore \quad q_1 = q_2 = 800\,\mu C$$
$$V_1 = \frac{q_1}{C_1} = 800\text{ V}$$

and $$V_2 = \frac{q_2}{C_2} = 400\text{ V}$$

(b) Now, total charge will become 1600 μC. This will now distribute in direct ratio of capacity.

$$\therefore \quad \frac{q_1}{q_2} = \frac{C_1}{C_2} = \frac{1}{2}$$
$$q_1 = \left(\frac{1}{3}\right)(1600) = \frac{1600}{3}\,\mu C$$
$$q_2 = \left(\frac{2}{3}\right)(1600) = \left(\frac{3200}{3}\right)\mu C$$

They will have a common potential (in parallel) given by

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

$$= \frac{1600\,\mu\text{C}}{3\,\mu\text{F}}$$

$$= \frac{1600}{3}\ \text{V}$$

7. Charge, $q = CV = 10^4\,\mu\text{C}$

In parallel, common potential is given by

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

$$20 = \frac{(10^4\,\mu\text{C})}{(C+100)\,\mu\text{C}}$$

Solving this equation, we get

$$C = 400\,\mu\text{F}$$

8. Charge supplied by the battery,

$$q = CV$$

Energy supplied by the battery,

$$E = qV = CV^2$$

Energy stored in the capacitor,

$$U = \frac{1}{2}CV^2$$

$\therefore$ Energy dissipated across R in the form of heat

$$= E - U = \frac{1}{2}CV^2 = U$$

9. $i = i_0\, e^{-t/\tau_C}$

Putting $i = \frac{i_0}{2}$, we get

$$t = (\ln 2)\,\tau_C = (0.693)\,\tau_C$$

10. Both capacitors have equal capacitance. Hence, half-half charge distribute over both the capacitors.

$$q_1 = q_2 = \frac{q_0}{2}$$

q_1 decreases exponentially from q_0 to $\frac{q_0}{2}$ while q_2 increases exponentially from 0 to $\frac{q_0}{2}$.

Corresponding graphs and equation are given in the answer.

Time constant of two exponential equations will be

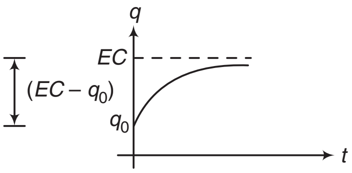
$$\tau_C = (C_{\text{net}})$$

$$R = \left(\frac{C}{2}\right)R = \frac{CR}{2}$$

11. $q_i = q_0$

$q_f = EC$

Now, charge on capacitor changes from q_i to q_f exponentially.



$$\therefore \quad q = q_0 + (EC - q_0)\,(1 - e^{-t/\tau_C})$$

$$= EC\,(1 - e^{-t/\tau_C}) + q_0 e^{-t/\tau_C}$$

Here, $\tau_C = CR$

12. (a) Immediately after the switch is closed whole current passes through C_1.

$$\therefore \quad i = E/R_1$$

(b) Long after switch is closed no current will pass through C_1 and C_2.

$$\therefore \quad i = \frac{E}{R_1 + R_3}$$

13. (a) At $t = 0$ equivalent resistance of capacitor is zero. R_1 and R_2 are in parallel across the battery PD across each is E.

$$\therefore \quad i_{R_1} = E/R_1$$

$$i_{R_2} = E/R_2$$

(b) In steady state, no current flow through capacitor wire. PD across R_1 is E.

$$\therefore \quad i_{R1} = E/R_1 \quad \text{and} \quad i_{R2} = 0$$

(c) In steady state, potential difference across capacitor is E.

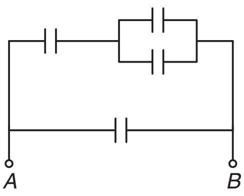
$$\therefore \quad U = \frac{1}{2}CV^2 = \frac{1}{2}CE^2$$

(d) When switch is opened, capacitor is discharged through resistors R_1 and R_2.

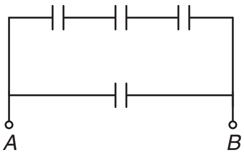
$$\tau_C = CR_{\text{net}}$$

$$= C\,(R_1 + R_2)$$

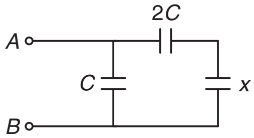
14. (a) Simple circuit is as shown below.



(b) The simple circuit is as shown below.



(c) Let $C_{AB} = x$. Then,



$$\text{Now,} \quad C_{AB} = C + \frac{(2C)(x)}{2C + x}$$

$$\text{or} \quad x = C + \frac{2Cx}{2C + x}$$

Solving this equation, we get

$$x = 2C$$

15. (a) $V = 660$ V across each capacitor

Now, $q = CV$ for both

(b) $q_{\text{net}} = q_1 + q_2$

$$= (3.96 - 2.64) \times 10^{3} \text{ C}$$

$$= 1.32 \times 10^{-3} \text{C}$$

Now, common potential

$$V = \frac{\text{Total charge}}{\text{Total capacity}} = \frac{13.2 \times 10^{-3}}{10 \times 10^{-6}}$$

$$= 132 \text{ V}$$

Now, apply $q = CV$ for both capacitors.

16. $u = \frac{1}{2} \varepsilon_0 E^2$

$$= \frac{1}{2} \varepsilon_0 \left(\frac{V}{d}\right)^2$$

17. $d = \frac{V_{\max}}{E_{\max}}$

$$C = \frac{K\varepsilon_0 A}{d}$$

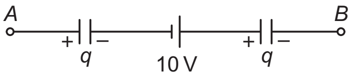
$$\therefore \quad A = \frac{dC}{K\,\varepsilon_0} = \frac{(V_{\max})(C)}{(K)(E_{\max})(\varepsilon_0)}$$

18. $0.1 = \frac{1}{2}(C_1 + C_2)(2)^2$ ...(i)

$$1.6 \times 10^{-2} = \frac{1}{2}\left(\frac{C_1 C_2}{C_1 + C_2}\right)(2)^2 \quad \text{...(ii)}$$

Solving these two equations, we can find C_1 and C_2.

19.



q shown in figure is in μC.

$$\text{Now,} \quad V_A - \frac{q}{1} + 10 - \frac{q}{2} = V_B$$

$$\text{or} \quad V_A - V_B = \frac{3q}{2} - 10 = 5$$

$$\therefore \quad q = 10 \text{ μC}$$

Now, $V = \frac{q}{C}$ across each capacitor.

20. See the answer.

21. In series, potential difference distributes in inverse ratio of capacitance.

$$\therefore \quad \frac{V_A}{V_B} = \frac{C_B}{C_A} = \frac{C_2}{C_1} = \frac{60}{40} = \frac{3}{2}$$

$$\therefore \quad C_2 = 1.5\, C_1 \quad \text{...(i)}$$

$$\text{Now,} \quad \frac{V'_A}{V'_B} = \frac{C'_B}{C'_A}$$

$$\text{or} \quad \frac{10}{90} = \frac{C_2}{(C_1 + 2)}$$

$$\text{or} \quad C_1 + 2 = 9C_2 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$C_1 = 0.16 \text{ μF}$$

$$\text{and} \quad C_2 = 0.24 \text{ μF}$$

22. (a) $q = CV$

(b) $C = \frac{\varepsilon_0 A}{d}$ or $C \propto \frac{1}{d}$

If d is doubled, C will remain half. Hence, q will also remain half.

(c) $q = CV = \left(\frac{\varepsilon_0 A}{d}\right)V = \frac{\varepsilon_0 (\pi R^3)V}{d}$

$$\text{or} \quad q \propto R^2$$

R is doubled. Hence, q will become four times or 480 μC.

23. Energy lost = energy stored = $\frac{1}{2}CV^2$

24. (a) $C = \frac{\varepsilon_0 A}{d}$

(b) $q = CV$

(c) $E = \frac{V}{d}$

25. (a) $\frac{1}{C_{\text{net}}} = \frac{1}{8.4} + \frac{1}{8.2} + \frac{1}{4.2}$

$$\therefore \quad C_{\text{net}} = 2.09 \text{ μF}$$

$$q_{\text{net}} = C_{\text{net}} V$$

$$= (2.09)(36)$$

$$= 75.14 \text{ μC} \approx 76 \text{ μC}$$

In series, charge remains same in all capacitors.

(b) $U_{\text{total}} = \dfrac{1}{2} C_{\text{net}} V^2$

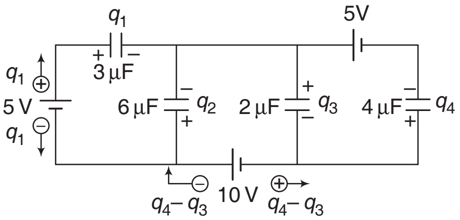
(c) $q_{\text{total}} = (3)\,(76)\,\mu\text{C} = 288\,\mu\text{C}$

Now common potential in parallel,

$$V = \frac{q_{\text{total}}}{C_{\text{total}}} = \frac{228\,\mu\text{C}}{(8.4 + 8.2 + 4.2)\,\mu\text{F}}$$

(d) $U_{\text{total}} = \dfrac{1}{2} C_{\text{net}} V^2$

26.



If we see the charge on positive plate of 6 μF capacitor, then

$$q_2 = -\,q_1 - (q_4 - q_3) \qquad \ldots\text{(i)}$$

Now, applying three loop equations, we have

$$5 - \frac{q_1}{3} + \frac{q_2}{6} = 0 \qquad \ldots\text{(ii)}$$

$$10 - \frac{q_2}{6} - \frac{q_3}{2} = 0 \qquad \ldots\text{(iii)}$$

$$5 - \frac{q_3}{2} - \frac{q_4}{4} = 0 \qquad \ldots\text{(iv)}$$

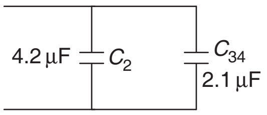
Solving these four equations, we can find q_1, q_2, q_3 and q_4.

27. (a) Simple series and parallel grouping of capacitors.

(b)

$$\begin{aligned} q_{\text{net}} &= C_{\text{net}} V \\ &= (2.5 \times 10^{-6})\,(220) \\ &= 5.5 \times 10^{-4}\,\text{C} \end{aligned}$$

C_1, C_5 and equivalent of other three capacitors are in series. Hence, charges across them are same.



q_{total} will distribute between C_2 and C_{34} in direct ratio of capacitance.

$$\therefore \quad \frac{q_2}{q_{34}} = \frac{4.2}{2.1} = \frac{2}{1}$$

$$\therefore \quad q_2 = \frac{2}{3}\,(5.5 \times 10^{-4}) = 3.7 \times 10^{-4}\,\text{C}$$

$$q_{34} = \frac{1}{3}\,(5.5 \times 10^{-4}) = 1.8 \times 10^{-4}\,\text{C}$$

For finding PD across any capacitor, use the equation

$$C = \frac{q}{V}$$

28. In series, potential difference distributes in inverse ratio of capacitance.

$$\therefore \quad \frac{V_A}{V_B} = \frac{C_2}{C_1} \quad \text{or} \quad \frac{130}{100} = \frac{C_2}{C_1}$$

$$\text{or} \quad C_1 = \frac{C_2}{1.3} = C_A$$

K is made 2.5 times. Therefore, C_1 will also become 2.5 times.

$$C_1' = 2.5\,C_1 = \frac{2.5\,C_2}{1.3}$$

$$\text{or} \quad \frac{C_1'}{C_2} = \frac{25}{13}$$

$$\text{Now,} \quad \frac{V_A'}{V_B'} = \frac{C_2}{C_1'} = \frac{13}{25}$$

$$\text{or} \quad V_A' = \left(\frac{13}{13+25}\right)(230) = 78.68\ \text{V}$$

$$V_B' = 230 - V_B' = 151.32\ \text{V}$$

29. $C_{23} = \dfrac{2 \times 3}{2 + 3} = 1.2\,\mu\text{F}$

$$q_{\text{total}} = C_1 V = 110\,\mu\text{C}$$

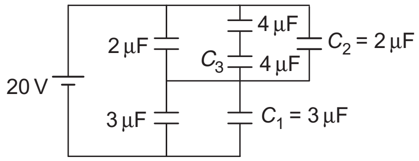
Common potential in parallel is given by

$$V = \frac{\text{Total charge}}{\text{Total capacity}} = \frac{110}{1 + 1.2} = 50\,\text{V}$$

$$q_{23} = (C_{23})V = 60\,\mu\text{C}$$

So, this much charge flows through the switch.

30. (a) Simple circuit is as shown below.



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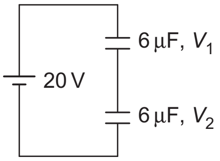


(b) $q_{\text{net}} = (C_{\text{net}})V = (34\,\mu\text{F})\,(20\,\text{V}) = 60\,\mu\text{C}$

(c) Upper network and lower network both have same capacitance = 6 μF

$$V_1 - V_2 = \frac{20}{2} = 10\,\text{V}$$

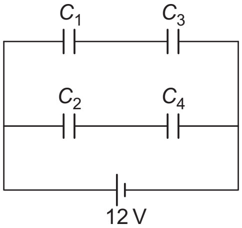
$$V_{C_1} = 10\,\text{V}$$

$$\therefore \qquad q_{C_1} = (C_1)(V_{C_1}) = 30\,\mu\text{C}$$

(d) $V_{C_2} = 10\,\text{V}, \quad \therefore \; q_{C_2} = (C_2)(V_{C_2}) = 20\,\mu\text{C}$

(e) $V_{C_3} = 5\,\text{V}, \quad \therefore \; q_{C_3} = (C_3)(V_{C_3}) = 20\,\mu\text{C}$

31. (a)



$$C_{13} = \frac{1 \times 3}{1 + 3} = \frac{3}{4}\,\mu\text{F}$$

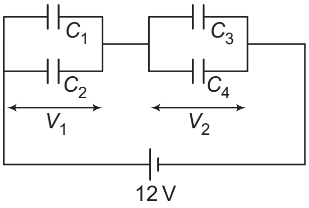
$$C_{24} = \frac{2 \times 4}{2 + 4} = \frac{4}{3}\,\mu\text{F}$$

$$V_{C_1 C_3} = V_{C_2 C_4} = 12\,\text{V}$$

$$\therefore \qquad q_1 = q_3 = (C_{13})\,(V_{C_1 C_3}) = \frac{3}{4} \times 12 = 9\,\mu\text{C}$$

$$q_2 = q_4 = (C_{24})\,(V_{C_2 C_4}) = \frac{4}{3} \times 12 = 16\,\mu\text{C}$$

(b)



$$C_{12} = \frac{1 \times 2}{1 + 2} = \frac{2}{3}\,\mu\text{F}$$

$$C_{34} = \frac{3 \times 4}{3 + 4} = \frac{12}{7}\,\mu\text{F}$$

$$\frac{V_1}{V_2} = \frac{C_{34}}{C_{12}} = \frac{12/7}{2/3} = \frac{18}{7}$$

$$\therefore \qquad V_1 = \left(\frac{18}{25}\right)(12) = 8.64\ \text{V}$$

$$V_2 = 12 - 8.64 = 3.36\ \text{V}$$

Now, we can apply $q = CV$ for finding charge on different capacitors.

32. $q_{\text{total}} = C_1 V_0$

After switch is thrown towards right, C_{23} and C_1 are in parallel. The common potential is

$$V = \frac{\text{Total charge}}{\text{Total capacity}} = \frac{C_1 V_0}{C_1 + \left(\dfrac{C_2 C_3}{C_2 + C_3}\right)}$$

Now, $q_{C_1} = C_1 V = \dfrac{C_1^2 V_0}{C_1 + \left(\dfrac{C_2 C_3}{C_2 + C_3}\right)}$

This is the same result as given in the answer.

$$q_{C_2} = q_{C_3} = C_{23} V$$

$$= \left(\frac{C_2 C_3}{C_2 + C_3}\right) \left[\frac{C_1 V_0}{C_1 + \dfrac{C_2 C_3}{C_2 + C_3}}\right]$$

33. (a) $q = C_i V = \left(\dfrac{\varepsilon_0 A}{d}\right) V$

$$V_f = \frac{q}{C_f} = \frac{(\varepsilon_0 A V/d)}{(\varepsilon_0 A/2d)} = 2V$$

(b) $U_i = \dfrac{1}{2} C_i V^2 = \dfrac{1}{2}\left(\dfrac{\varepsilon_0 A}{d}\right) V^2$

$$U_f = \frac{1}{2} C_f V_f^2 = \frac{1}{2}\left(\frac{\varepsilon_0 A}{2d}\right)(2V)^2$$

$$= \left(\frac{\varepsilon_0 A}{d}\right) V^2$$

(c) $W = U_f - U_i = \dfrac{1}{2}\left(\dfrac{\varepsilon_0 A}{d}\right) V^2$

34. (a) After long time, capacitor gets fully charged by E_1.

$$\therefore \qquad i_C = 0$$

and $\quad i_{R_1} = i_{R_2} = \dfrac{E_1}{R_1 + R_2}$

$$= \frac{20}{20 \times 10^3}$$

$$= 10^{-3}\ \text{A} = 1\ \text{mA}$$

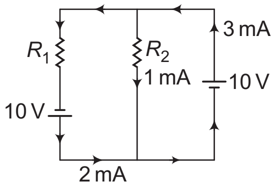
(b) In steady state (with E_1).

$$V_C = V_{R_2} = V_{R_1}$$

$$= \frac{E_1}{2} = 10\ \text{V}$$

Now, when the switch is shifted to position B, capacitor (at $t = 0$) behaves like a battery of 10 V.

The circuit in that case is as shown below.



Now, with the help of Kirchhoff's laws we can find different currents. Final currents are shown in the diagram.

35. (a) $V_a = 18$ V and $V_b = 0$ as no current flow through the resistors.

$$\therefore \quad V_a - V_b = 10 \text{ V}$$

(b) $V_a - V_b = +$ ve.

Hence, $V_a > V_b$

(c) Current flows through two resistors,

$$i = \frac{18 - 0}{6 + 3} = 2 \text{A}$$

$$\therefore \quad V_b - 0 = iR = 2 \times 3$$

$$\text{or} \quad V_b = 6 \text{V}$$

(d) Initially, $V_{3\mu\text{F}} = V_{6\mu\text{F}} = 18 \text{V}$

$$\therefore \quad q_{3\mu\text{F}} = 54\ \mu\text{C} \qquad (q = CV)$$

$$\text{and} \quad q_{6\mu\text{F}} = 108\ \mu\text{C}$$

Finally,

$$V_{6\mu\text{F}} = V_{6\Omega} = iR$$
$$= 2 \times 6 = 12 \text{ V}$$

$$\therefore \quad q_{6\mu\text{F}} = 72\ \mu\text{C}$$
$$V_{3\mu\text{F}} = V_{3\Omega} = 6\text{V}$$
$$\therefore \quad q_{3\mu\text{F}} = 18\ \mu\text{C}$$

$\Delta q = q_f - q_i = -36\ \mu\text{C}$ on both capacitors.

36. (a) In resistors (in series) potential drops in direct ratio of resistance and in capacitors (in series) potential drops in inverse ratio of capacitance.

$$18 - V_a = \left(\frac{6}{6+3}\right)(18)$$

$$\therefore \quad V_a = 6 \text{ V}$$

$$18 - V_b = \left(\frac{3}{6+3}\right)(18)$$

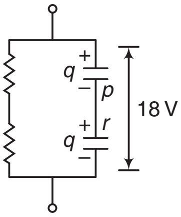
$$V_b = 12 \text{ V}$$

(c) $V_{3\mu\text{F}} = V_{3\Omega} = iR = \left(\dfrac{18-0}{6+3}\right)(3) = 6\text{V}$

$$\therefore \quad V_b - 0 = 6 \text{ V}$$

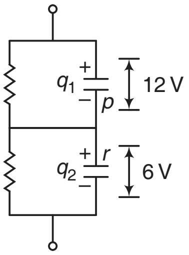
$$\therefore \quad V_b = 6 \text{ V}$$

(d) Initially,



$$q = C_{\text{net}} V = (2\ \mu\text{F})(18)$$
$$= 36\ \mu\text{C}$$

Finally,



$$q_1 = 72\ \mu\text{C}$$
$$q_2 = 18\ \mu\text{C}$$

Charge flow from S = (Final charge on plates p and r) – (Initial charges on plates p and r)

$$= (-72 + 18) - (-36 + 36)$$
$$= -54\ \mu\text{C}$$

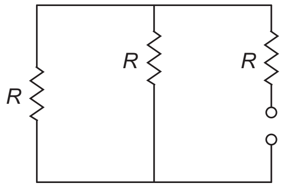
37. (a) In steady state,

$$V_C = \frac{V}{2}$$

∴ Steady state charge,

$$q_0 = CV_C = \frac{CV}{2}$$

For equivalent value of τ_C : We short circuit the battery and find the value of R_{net} across capacitors and then



$$R_{\text{net}} = \frac{3R}{2}$$

$$\therefore \quad \tau_C = CR_{\text{net}} = \frac{3RC}{2}$$

Now, $q = q_0 (1 - e^{-t/\tau_C})$

(b) **At $t = 0$,** capacitor offers zero resistance.

$$R_{\text{net}} = \frac{3R}{2}$$

$$\therefore \quad i = \frac{V}{3R/2} = \frac{2V}{3R}$$

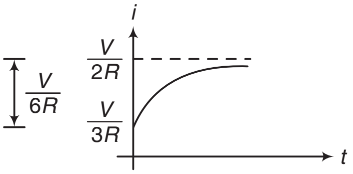
$$i_C = i_{AB} = \frac{1}{2} = \frac{V}{3R}$$

At t $= \infty$, capacitor offers infinite resistance. So, $i_C = 0$.

$$\therefore \quad i_{\text{battery}} = i_{AB} = \frac{V}{2R}$$

Now, current through AB increases exponentially from $\frac{V}{3R}$ to $\frac{V}{2R}$ with same time constant.

$(i\text{-}t)$ graph is as shown below.



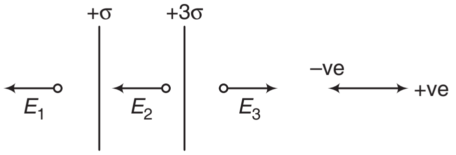
$(i\text{-}t)$ equation corresponding to this graph is

$$i = \frac{V}{3R} + \frac{V}{6R}(1 - e^{-t/\tau_C})$$

LEVEL 2

Single Correct Option

1.



$$E_1 = \frac{\sigma}{2\varepsilon_0} + \frac{3\sigma}{2\varepsilon_0} = \frac{2\sigma}{\varepsilon_0} = E_3$$

$$E_2 = \frac{3\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

E_1 and E_2 are in the negative direction and E_3 in positive direction.

2. Let E be the external field (toward right). Then,

$$E - \frac{\sigma}{2\varepsilon_0} = 8 \qquad \ldots\text{(i)}$$

$$E + \frac{\sigma}{2\varepsilon_0} = 12 \qquad \ldots\text{(ii)}$$

Solving these equations, we get $\sigma = 4\varepsilon_0$

3. At $t = 0$ when capacitors are initially uncharged, their equivalent resistance is zero. Hence, whole current passes through these capacitors.

4. Changing current is given by

$$i = i_0 e^{-t/\tau_C}$$

or

$$i = \frac{V}{R} e^{-t/CR}$$

If we have take log on both sides, we have

$$\ln(i) = \ln\left(\frac{V}{R}\right) - \left(\frac{1}{CR}\right)t$$

Hence, $\ln(i)$ *versus* t graph is a straight line with slope $\left(-\frac{1}{CR}\right)$ and intercept $+\ln\left(\frac{V}{R}\right)$.

Intercepts are same, but $|\text{slope}|_1 > |\text{slope}|_2$.

5. During charging of a capacitor 50% of the energy supplied by the battery is lost and only 50% is stored.

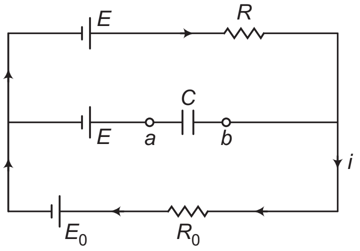
$$\therefore \quad \text{Total energy lost} = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{(EC/2)^2}{C} = \frac{E^2C}{8}$$

Now, this total loss is in direct ratio $r : 2r$ or $1 : 2$

$$\therefore \quad \text{Energy lost in battery is } \frac{1}{3} \text{ rd of } \frac{E^2C}{8}.$$

6. Equal and opposite charges should transfer from two terminals of a battery. For charging of a capacitor, it should lie on a closed loop.

7.



$$i = \frac{[E - E_0]}{R + R_0}$$

Now, $V_a - E + E_0 + iR_0 = V_b$

$$\therefore \quad V_a - V_b = (E - E_0) - iR_0$$

$$= (E - E_0)\left[1 - \frac{R_0}{R + R_0}\right]$$

$$= \frac{R\,(E - E)}{R + R_0}$$

$$\therefore \quad q = C\,(V_a - V_b) = \frac{CR\,(E - E_0)}{R + R_0}$$

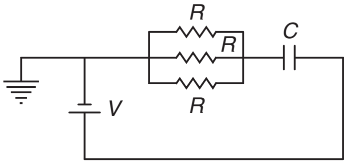
8. **Initially**

$$C_{\text{net}} = \frac{(C_0)(C_0)}{C_0 + C_0} = \frac{C_0}{2}$$
$$= 0.5\,C_0$$

Finally

$$C_{\text{net}} = \frac{(C_0/2)(2C_0)}{(C_0/2) + 2C_0}$$
$$= 0.4\,C_0$$

9. The simple circuit is as shown below.



$$R_{\text{net}} = \frac{R}{3}$$

$$\therefore \qquad \tau_C = CR_{\text{net}} = \frac{CR}{3}$$

$$q = q_0\,(1 - e^{-t/\tau_C}),$$

where, $q_0 = CV$

10. Common potential in parallel grouping,

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$
$$= \frac{(2\times 100) + (4\times 50)}{2+4}$$
$$= \frac{200}{3}\ \text{V}$$

$$\text{Loss} = U_i - U_f$$

$$= 10^{-6}\left[\left(\frac{1}{2}\times 2\times 100\times 100 + \frac{1}{2}\times 4\times 50\times 50\right) - \left(\frac{1}{2}\times 6\times\frac{200}{3}\times\frac{200}{3}\right)\right]$$

$$= 1.7\times 10^{-3}\,\text{J}$$

11. $V_0 = i_0 R = (10)(10) = 100$ V

After 2 s, current becomes $\frac{1}{4}$th. Therefore, after 1 s, current will remain half also called half-life.

$$t_{1/2} = (\ln 2)\,\tau_C = (\ln 2)\,CR$$

$$\therefore \qquad C = \frac{(t_{1/2})}{(\ln 2)R} = \frac{1}{10\ln 2}\ \text{F}$$

$$\text{Total heat} = \frac{1}{2}CV_0^2$$
$$= \frac{1}{2}\times\frac{1}{10\ln 2}\,(100)^2$$
$$= \frac{500}{\ln 2}\,\text{J}$$

12. Net capacitance between points *A* and *P* will be equal to the net capacitance between points *P* and *B*.

13. Total charge $= (2C)(4V) - CV$

$$= 7CV$$

Common potential after they are connected is

$$V_C = \frac{\text{Total charge}}{\text{Total capacitance}}$$
$$= \frac{7\,CV}{2C + C} = \frac{7}{3}V$$

$$\text{Heat} = U_i - U_f$$
$$= \frac{1}{2}CV^2 + \frac{1}{2}(2C)(4V)^2 - \frac{1}{2}\times 3C\times\left(\frac{7}{3}V\right)^2$$
$$= \frac{25}{3}CV^2$$

14. Total heat produced $= \frac{1}{2}CV^2$

$$= \frac{1}{2}(2\,\mu\text{F})(5)^2$$
$$= 25\ \mu\text{J}$$

Now, this should distribute in inverse ratio of resistors, as they are in parallel.

$$\therefore \qquad \frac{H_{5\Omega}}{H_R} = \frac{R}{5}$$

$$\text{or} \qquad H_{5\Omega} = \left(\frac{R}{R+5}\right) \quad (\text{Total heat})$$

$$\text{or} \qquad 10 = \left(\frac{R}{R+5}\right)(25)$$

Solving this equation, we get

$$R = \left(\frac{10}{3}\right)\Omega$$

15. In position-1, initial maximum current is

$$i_0 = \frac{V}{R} = \frac{10}{5} = 2\,\text{A}$$

At the given time, given current is 1A or half of the above value. Hence, at this is instant capacitor is also charged to half of the final value of 5 V.

Now, it is shifted to position-2 wherein steady state it is again charged to 5 V but with opposite polarity.

$$U_i = U_f = \frac{1}{2}CV^2 \qquad (\because\ V = 5\text{ V})$$

∴ Total energy supplied by the lower battery is converted into heat. But double charge transfer (from the normal) takes place from this battery.

∴ Heat produced = Energy supplied by the battery

$$= (\Delta q)V = (2CV)(V) = 2CV^2$$

$$= 2 \times 2 \times 10^{-6} \times (5)^2$$

$$= 100 \times 10^{-6}\text{J}$$

$$= 100\,\mu\text{J}$$

16. Equivalent capacitance of 6 μF and 3 μF is also 2 μF and charge across it is also q or circuit is balanced. Hence, there is no flow of charge.

17. Two capacitors are in parallel.

$$\therefore \quad U = \frac{1}{2}C_{\text{net}}V^2$$

$$= \frac{1}{2}(2C)V^2 = CV^2$$

$$= \left(\frac{\varepsilon_0 A}{d}\right)V^2$$

18. Initially, the rate of charging is fast.

19. $V_{1\Omega} = 5 + 2 = 7$ V

$$\therefore \quad i_{1\Omega} = \frac{V}{R} = 7\text{A}$$

$$V_{2\mu\text{F}} = 6\text{ V}$$

$$\therefore \quad q_{2\mu\text{F}} = CV = 12\,\mu\text{C}$$

20. During charging capacitor and resistance of its wire are independently connected with the battery. Hence,

$$\tau_C = CR$$

During discharging capacitor is discharged through both resistors (in series). Hence,

$$\tau_C = C\,(2R) = 2CR$$

21. Total charge = 3 × 100 − 1 × 100 = 200 μC

Common potential (in parallel) after S is closed, is

$$V = \frac{\text{Total charge}}{\text{Total capacity}}$$

$$= \frac{200\,\mu\text{C}}{4\,\mu\text{F}}$$

$$= 50\text{ V}$$

22. $\dfrac{V_1}{V_{1.5}} = \dfrac{1.5}{1}$

$$\therefore \quad V_1 = \left(\frac{1.5}{1.5+1}\right)(30)$$

$$= 18\text{ V}$$

$$\frac{V_{2.5}}{V_{0.5}} = \frac{0.5}{2.5} = \frac{1}{5}$$

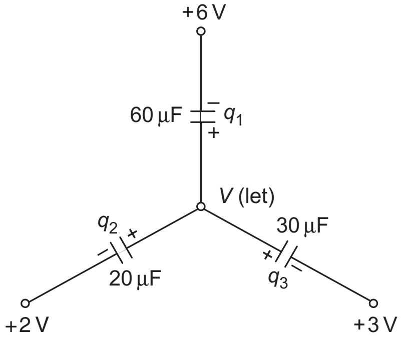
$$\therefore \quad V_{2.5} = \left(\frac{1}{1+5}\right)(30)$$

$$= 5\text{ V}$$

Now, $|V_{ab}| = V_1 - V_{2.5}$

$$= 13\text{ V}$$

23. In the figure,



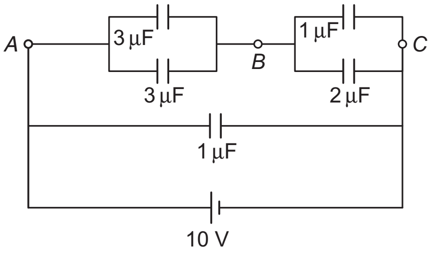
$$q_1 + q_2 + q_3 = 0$$

$$60\,(V-6) + 20\,(V-2) + 30\,(V-3) = 0$$

Solving this equation, we get

$$V = \frac{49}{11}\text{ V}$$

24.

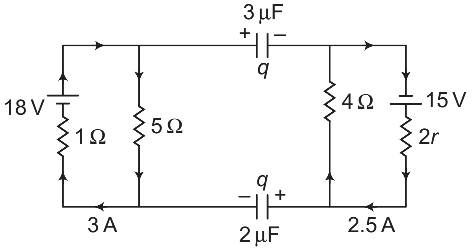


$$\frac{V_{AB}}{V_{BC}} = \frac{(C)_{BC}}{(C)_{AB}} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \quad V_{AB} = \left(\frac{1}{1+2}\right)(10)\text{ V}$$

$$= \frac{10}{3}\text{ V}$$

25. Applying Kirchhoff's loop law in outermost loop, we have



$$-\frac{q}{3}+15-2\times 2.5-\frac{q}{2}-3\times 1+18=0$$

Solving this equation, we get

$$q = 30\,\mu\text{C}$$

26. $\tau_C = CR = 6\text{ s}$

$$q_0 = CV = 10\,\mu\text{C}$$

Now, $q = q_0 e^{-t/\tau_C} = (10\,\mu\text{C})\,e^{-12/6}$

$$= \left(\frac{1}{e}\right)^2 (10\,\mu\text{C})$$

$$= (0.37)^2\,(10\,\mu\text{C})$$

27. $q = (E_1+E_2)\,C_{\text{net}}$

$$= (E_1+E_2)\frac{C_1C_2}{C_2+C_2}$$

$$V_{ab} = \frac{q}{C_2} = \left(\frac{E_1+E_2}{C_1+C_2}\right)C_1$$

28. $H_1 = H_2 = U_i - U_f$

The only change is by increasing the resistance τ_C increase. Hence, process of redistribution of charge slows down.

29. Just after the switch is closed C_1 is short-circuited and current passes through R_1 and C_1 only.

30.
$$i_1 = \left(\frac{V}{2R}\right)e^{-\frac{t}{6\,CR}}$$

$\therefore$
$$i_2 = \left(\frac{V}{R}\right)e^{-\frac{t}{CR}}$$

$\therefore$
$$\frac{i_1}{i_2} = \frac{e^{\frac{5t}{6\,CR}}}{2}$$

We can see that this ratio is increasing with time.

31. $\tau_C = CR$

$$= \left(\frac{K\,\varepsilon_0 A}{d}\right)\left(\frac{d}{A\sigma}\right) \qquad \left(R = \frac{l}{\sigma A}\right)$$

$$= \frac{K\varepsilon_0}{\sigma}$$

$$= \frac{5\times 8.86\times 10^{-12}}{7.4\times 10^{-12}}$$

$$= 6\text{ s}$$

32. The given time is the half-life time of the exponentially decreasing equation.

$\therefore \quad t = t_{1/2} = (\ln 2)\,\tau_C = (\ln 2)\,CR_{\text{net}}$

$\therefore \quad R_{\text{net}} = \dfrac{t}{(\ln 2)\,C}$

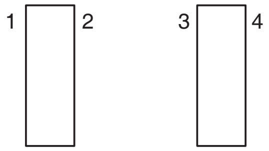
$$= \frac{2\,(\ln 2)\,\mu\text{s}}{(\ln 2)\,(0.5\,\mu\text{F})} = 4\Omega$$

$\therefore$ Resistance of ammeter $= 2\Omega$

33. Four capacitors are in parallel charge across each is $q = CV$. Two surfaces of plate C marks two capacitors, one with B and other with D and C is connected to positive terminal of the battery. Hence,

$$q_C = 2CV = +\,40\,\mu\text{C}$$

34.



$$q_1 = q_4 = \frac{q_{\text{total}}}{2} = \frac{CV - CV + Q}{2} = \frac{Q}{2}$$

$$q_2 = (Q + CV) - \frac{Q}{2} = \left(\frac{Q}{2} + CV\right)$$

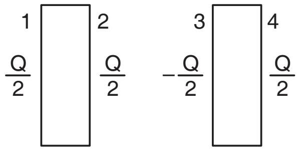
$$q_3 = -\,q_2 = -\left(\frac{Q}{2} + CV\right)$$

Electric field between two plates and hence the potential difference is due to q_2 and q_3 only.

$$\text{PD} = \frac{q_2}{C} = V + \frac{Q}{2C}$$

More than One Correct Options

1.



$$q_1 = q_4 = \frac{q_{\text{total}}}{2} = \frac{Q}{2}$$

$$q_2 = Q - q_1 = \frac{Q}{2}$$

$$q_3 = -q_2 = \frac{-Q}{2}$$

$$E_A = E_1 + E_4 \qquad \text{(towards left)}$$

$$= \frac{Q}{4A\varepsilon_0} + \frac{Q}{4A\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

$$E_C = -E_A$$

$$= \frac{Q}{2A\varepsilon_0} \qquad \text{(towards right)}$$

$$E_B = E_2 + E_3 \qquad \text{(towards right)}$$

$$= \frac{Q}{4A\varepsilon_0} + \frac{Q}{4A\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

2. In steady state,

$$q_C = EC \text{ and } q_{2C} = 2EC$$

$$\tau_C = 2CR \text{ of both circuits}$$

At time t,

$$q_C = EC\ (1 - e^{t/\tau_C})$$

$$q_{2C} = 2EC\ (1 - e^{-t/\tau_C})$$

$$\therefore \qquad \frac{q_C}{q_{2C}} = \frac{1}{2}$$

3. In steady state, current through capacitor wire is zero. Current flows through 200 Ω, 900 Ω and A_2.

$$V_C = \frac{q}{C} = \frac{4 \times 10^{-3}}{100 \times 10^{-6}}$$

$$= 40 \text{ V}$$

This is also potential drop across 900 Ω resistance and 100 Ω ammeter A_2 (Total resistance = 1000 Ω). Now, this 1000 Ω and 200 Ω are in series. Therefore,

$$V_2 = V_{200\,\Omega} = \frac{V_{1000\,\Omega}}{5}$$

$$= \frac{40}{5} = 8 \text{ V}$$

$$\text{Emf} = V_{1000\,\Omega} + V_{200\,\Omega} = 48 \text{ V}$$

$$i = \frac{\text{Emf}}{\text{Net resistance}}$$

$$= \frac{48}{1200} = \frac{1}{25} \text{ A}$$

4. Current through A is the main current passing through the battery. So, this current is more than the current passing through B. Hence, during charging more heat is produced in A.

In steady state,

$$i_C = 0$$

$$\text{and} \qquad i_A = i_B$$

Hence, heat is produced at the same rate in A and B.

Further, in steady state

$$V_C = V_B = \frac{\varepsilon}{2}$$

$$\therefore \qquad U = \frac{1}{2}CV_C^2 = \frac{1}{8}C\varepsilon^2$$

5. $F = qE = (q)\left(\dfrac{\sigma}{2\varepsilon_0}\right) = \dfrac{q^2}{2\varepsilon_0 A}$

q remains unchanged. Hence, F remains unchanged.

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$$

q remains unchanged. Hence, E also remains unchanged.

$$U = \frac{q^2}{2C} \quad \text{or} \quad U \propto \frac{1}{C}$$

C will decrease. Hence, U will increase.

$$V = Ed \quad \text{or} \quad V \propto d$$

d is increasing. Hence, V will increase.

6. $C_i = \dfrac{(C)\,(2C)}{C + 2C} = \dfrac{2}{3}C$

$$q_i = C_iE = \frac{2}{3}EC$$

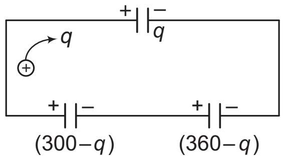
$$C_f = 2C$$

$$\therefore \qquad q_f = 2EC$$

$$\Delta q = q_f - q_i$$

$$= \frac{4}{3}CE$$

7. Let $(+\,q)\,\mu$C charge flows in the closed loop in clockwise direction. Then, final charges on different capacitors are as shown in figure.



Now, applying Kirchhoff's loop law

$$\frac{360 - q}{3} + \frac{300 - q}{2} = \frac{q}{1.5}$$

Solving the above equation, we get

$$q = 180\,\mu\text{C}$$

8. If the battery is disconnected, then q = constant

$$C = \frac{\varepsilon_0 A}{d} \quad \text{or} \quad C \propto \frac{1}{d}$$

d is decreased. Hence, C will increase.

$$U=\frac{1}{2}\frac{q^2}{C} \quad \text{or} \quad U \propto \frac{1}{C}$$

C is increasing. Hence, U will decrease.

$$V=\frac{q}{C} \quad \text{or} \quad V \infty \frac{1}{C}$$

C is increasing. Hence, V will decrease.

9. (a) At $t=0$, emf of the circuit = PD across the capacitor = 6 V.

$$\therefore \qquad i=\frac{6}{1+2}=2\,\text{A}$$

Half-life of the circuit

$= (\ln 2)\,\tau_C\,(\ln 2)\,CR = (6 \ln 2)$ s.

In half-life time, all values get halved.

For example

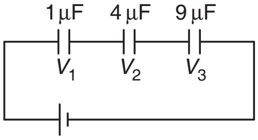
$$V_C=\frac{6}{2}=3\,\text{V}$$

$$i=\frac{2}{1}=1\,\text{A}$$

$$\therefore \qquad V_{1\Omega}=iR=1\,\text{V}$$

$$V_{2\Omega}=iR=2\,\text{V}$$

10.



In series, $\qquad V \propto \frac{1}{C} \qquad$ (as q = constant)

$$\therefore \qquad \frac{V_2}{V_1}=\frac{1}{4}$$

$$\text{or} \qquad V_2=\frac{V_1}{4}=\frac{10}{4}=2.5\ \text{V}$$

$$\frac{V_3}{V_1}=\frac{1}{9}$$

$$\therefore \qquad V_3=\frac{V_1}{9}=\frac{10}{9}\ \text{V}$$

$$\text{Now,} \qquad E=\left(10+2.5+\frac{10}{9}\right)\text{V}$$

Comprehension Based Questions

1. Finally, the capacitors are in parallel and total charge ($=q_0$) distributes between them in direct ratio of capacity.

$$\therefore \qquad q_{C_2}=\left(\frac{C_2}{C_1+C_2}\right)q_0 \rightarrow \text{in steady state.}$$

But this charge increases exponentially.

Hence, charge on C_2 at any time t is

$$q_{C_2}=\left(\frac{C_2\,q_0}{C_1+C_2}\right)(1-e^{-t/\tau_C})$$

Initially, C_2 is uncharged so, whatever is the charge on C_2, it is charge flown through switches.

2. Common potential in steady state when they finally come in parallel is

$$V=\frac{\text{Total charge}}{\text{Total capacity}}=\frac{q_0}{C_1+C_2}$$

Total heat dissipated $=U_i-U_f$

$$=\frac{q_0^2}{2C_1}-\frac{1}{2}(C_1+C_2)\left(\frac{q_0}{C_1+C_2}\right)^2$$

$$=\left(\frac{q_0^2}{2C_1}\right)\left(\frac{C_1C_2}{C_1+C_2}\right)$$

3. $E_{\text{air}}=E_0=\frac{V}{d}$

4. $E_{\text{dielectric}}=\frac{E_0}{K}=\frac{V}{Kd}$

Match the Columns

1. (a) $C_i=\frac{4\times 4}{4+4}=2\,\mu\text{F}$

$$C_f=\frac{C_1C_2}{C_1+C_2}=\frac{8\times 2}{8+2}$$

$$=1.6\,\mu\text{F}$$

$$q=CV$$

Since, total capacity is decreasing. Hence, charge on both capacitors will decrease.

(b) $U_2=\frac{1}{2}\frac{q^2}{C}$

q has become $\frac{1.6}{2}$ or 0.8 times but C is halved.

Hence, U_2 will increase.

(c) $V_2=\frac{q}{C}$

q has become 0.8 times and C is halved.

Hence, V_2 will increase.

(d) $E_2=\frac{V_2}{d}$ or $E_2 \propto V_2$

2. (a) $C_i=2\,\mu\text{F}$

$$\therefore \qquad q_i=60\,\mu\text{C}$$

$$C_f=6\,\mu\text{F}$$

$$\therefore \qquad q_f=180\,\mu\text{C}$$

$\therefore$ Δq from the battery $=q_f-q_i=120\,\mu\text{C}$.

(b) Between 4 μF and 2 μF charge distributes indirect ratio of capacity. Hence, on 2 μF

$$q_i = \left(\frac{2}{2+4}\right)(60\,\mu\text{C}) = 20\,\mu\text{C}$$

$$q_f = \left(\frac{2}{2+4}\right)(180)\;60\,\mu\text{C}$$

$$\therefore \quad \Delta q = \Delta f - q_i = 40\,\mu\text{C}$$

(c) On 3 μF, initial charge is 60 μF and final charge is zero.

$$\therefore \quad \Delta q = 60\,\mu\text{C}$$

(d) On 4 μF

$$q_i = \left(\frac{4}{2+4}\right)(60\,\mu\text{C}) = 40\,\mu\text{C}$$

$$q_f = \left(\frac{4}{2+4}\right)(180\,\mu\text{C}) = 120\,\mu\text{C}$$

$$\therefore \quad \Delta q = q_f - q_i = 80\,\mu\text{C}$$

3. (a) In second figure, $V_{C_1} = V =$ maximum

Hence, q_{C_1} is maximum.

(b) In first figure, $V_{C_2} = \dfrac{V}{3}$

In second figure, $V_{C_2} = V$

In third figure, $V_{C_2} = \left(\dfrac{C}{2C + C}\right) V = \dfrac{V}{3}$

In fourth figure, $V_{C_2} = \left(\dfrac{2C}{C + 2C}\right) V = \dfrac{2V}{3}$

Now, $q = CV$

Hence, q_{C_2} is minimum in first and third figures.

(c) In second figure, $V_{C_1} = V =$ maximum

(d) Similar to option (b)

4. After closing the switch, the common potential is parallel.

$$V = \frac{\text{Total charge}}{\text{Total capacity}} = \frac{CV}{3C} = \frac{V}{3}$$

$$U_C = \frac{1}{2} C \left(\frac{V}{3}\right)^2 = \frac{1}{18} CV^2$$

$$U_{2C} = \frac{1}{2}(2C)\left(\frac{V}{3}\right)^2 = \frac{1}{9} CV^2$$

$$\text{Loss of energy} = U_i - U_f$$

$$= \frac{1}{2} CV^2 - \frac{1}{2} \times 3C \times \left(\frac{V}{3}\right)^2$$

$$= \frac{1}{3} CV^2$$

5. Let $C_0 = \dfrac{\varepsilon_0 A}{d}$

$$C_1 = \frac{\varepsilon_0 (A/2)}{d} + \frac{K\,\varepsilon_0 (A/2)}{d}$$

$$= \frac{C_0}{2} + C_0 = \frac{3C_0}{2}$$

$$C_2 = \frac{\varepsilon_0 A}{d - d/2 + \dfrac{d/2}{K}} = \frac{2\,\varepsilon_0 A}{d\left(1 + \dfrac{1}{K}\right)}$$

$$= \frac{4}{3}\frac{\varepsilon_0 A}{d} = \frac{4}{3} C_0$$

$$\therefore \quad \frac{C_1}{C_2} = \frac{9}{8}$$

Capacitors are in series.

Hence, $q_1 = q_2$

or $\dfrac{q_1}{q_2} = 1$

$$U = \frac{1}{2}\frac{q^2}{C}$$

or $U \propto \dfrac{1}{C}$ (as q is same)

$$\therefore \quad \frac{U_1}{U_2} = \frac{C_2}{C_1} = \frac{8}{9}$$

6.

$$q_1 = q_8 = \frac{q_{\text{total}}}{2} = 7Q$$

$$q_2 = 4Q - q_1 = -3Q$$

$$q_3 = -q_2 = +3Q$$

$$q_4 = Q - q_3 = -2Q$$

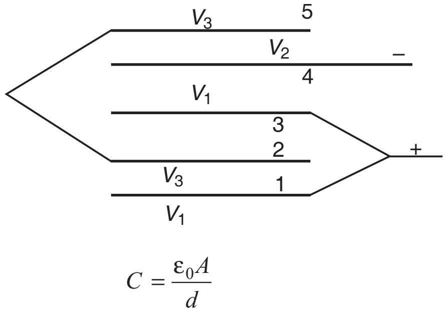
$$q_5 = -q_4 = +2Q$$

$$q_6 = 2Q - q_5 = 0$$

$$q_7 = -q_6 = 0$$

Subjective Questions

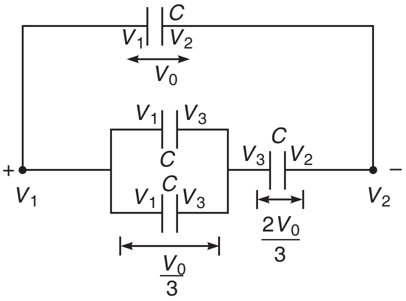
1.



$$C = \frac{\varepsilon_0 A}{d}$$

(between two successive plates).

The effective capacity has to be found between V_1 and V_2.



$$C_{\text{net}} = C + \frac{(2C)(C)}{2C + C} = \frac{5}{3}C = \frac{5}{3}\frac{\varepsilon_0 A}{d}$$

$$q_3 = +[C(V_1 - V_2) + C(V_1 - V_3)]$$

$$= \frac{\varepsilon_0 A}{d}\left[V_0 + \frac{V_0}{3}\right] = \frac{4\varepsilon_0 A V_0}{3d}$$

$$q_5 = C(V_3 - V_2) = \left(\frac{\varepsilon_0 A}{d}\right)\left(\frac{2V_0}{3}\right)$$

$$= \frac{2\varepsilon_0 A V_0}{3d}$$

2. (a) $V = \dfrac{q_{\text{net}}}{C_{\text{net}}} = \dfrac{C_1 V_0}{C_1 + C_2} = \dfrac{V_0}{1 + C_2/C_1}$

$$= \frac{120}{1 + 4/8} = 80 \text{ V}$$

(b) $U_i = \dfrac{1}{2}C_1 V_0^2 = \dfrac{1}{2} \times 8 \times 10^{-6} \times (120)^2$

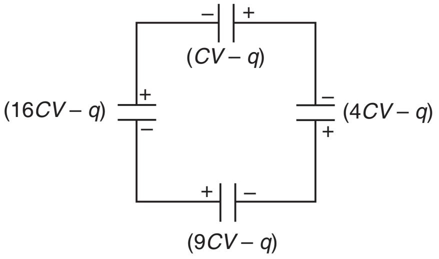
$$= 5.76 \times 10^{-2} \text{ J}$$

$$U_f = \frac{1}{2}(C_1 + C_2)V^2$$

$$= \frac{1}{2} \times 12 \times 10^{-6} \times (80)^2$$

$$= 3.84 \times 10^{-2} \text{ J}$$

3. Let $+q$ charge rotates in the loop in clockwise direction for achieving equilibrium state. In final steady state, charges on the capacitors will be as shown below



Now, applying Kirchhoff's loop law we have

$$\left(\frac{CV - q}{C}\right) + \left(\frac{4CV - q}{2C}\right) + \left(\frac{9CV - q}{3C}\right) + \left(\frac{16CV - q}{4C}\right) = 0$$

or $\qquad q = 4.8\,CV = \dfrac{24}{5}CV$

Now, $V_1 = V - \dfrac{q}{C} = V - \dfrac{24}{5}V = -\dfrac{19}{5}$ V

$$V_2 = 2V - \frac{q}{2C} = 2V - \frac{24}{10}V = -\frac{2}{5}V$$

$$V_3 = 3V - \frac{q}{3C} = 3V - \frac{24}{15}V = \frac{7}{5}V \text{ and}$$

$$V_4 = 4V - \frac{q}{4C} = 4V - \frac{6}{5}V = \frac{14}{5}V$$

4. At $t = 5$ ms, $V = 10$ V

∴ $\qquad i_R = \dfrac{V}{R} = \dfrac{10}{4} = 2.5$ A **Ans.**

Further, $q = CV = (300 \times 10^{-6})(2000t) = 0.6t$

$$i_C = \frac{dq}{dt} = 0.6 \text{ A} = \text{constant}$$ **Ans.**

5. Potential energy stored in the capacitor,

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times 5 \times 10^{-6} \times (200)^2 = 0.1 \text{ J}$$

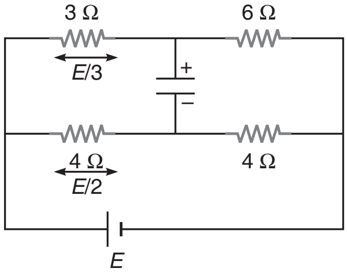
During discharging this 0.1 J will distribute in direct ratio of resistance,

∴ $\qquad H_{400} = \dfrac{400}{400 + 500} \times 0.1$

$$= 44.4 \times 10^{-3} \text{ J}$$

$$= 44.4 \text{ mJ}$$ **Ans.**

6.



(a) Current in lower branch = $E/8 = 3$ A
Current in upper branch = $E/9 = 24/9 = 2.67$ A

(b) PD across the capacitor = $E/2 - E/3 = E/6$

From $q = CV$, we have $16 = (4)\dfrac{E}{6}$

∴ $\qquad E = 24$ V

(c) After short-circuiting the battery, we will have to find net resistance across capacitor to calculate equivalent value of τ_C in discharging. $3\,\Omega$ and $6\,\Omega$ are in parallel. Similarly, $4\,\Omega$ and $4\,\Omega$ are in parallel. They are then in series.

$$\therefore \quad R_{\text{net}} = 4\,\Omega,$$

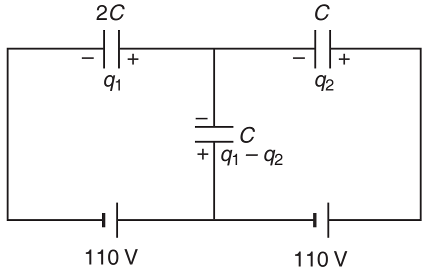
$$\tau_C = CR_{\text{net}} = (4 \times 4)\,\mu\text{s} = 16\,\mu\text{s}$$

During discharging $q = q_0 e^{-t/\tau_C}$

or $\quad 8 = 16 e^{-t/16}$

Solving this equation, we get $t = 11.1\,\mu\text{s}$ **Ans.**

7.



Applying loop law in two closed loops, we have

$$110 - \frac{q_2}{C} + \frac{q_1 - q_2}{C} = 0 \quad \text{or} \quad q_2 = (110C)$$

and $$-110 + \frac{q_1}{2C} + \frac{q_1 - q_2}{C} = 0$$

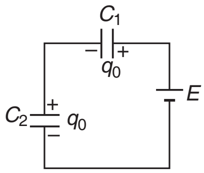
or $$q_1 = \left(\frac{440C}{3}\right)$$

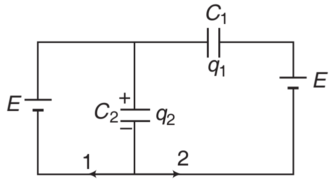
Potential difference between points M and N is

$$V_N - V_M = \frac{q_1 - q_2}{C}$$

$$= \frac{110}{3} \text{ volt}$$ **Ans.**

8. Let us first find charges on both capacitors before and after closing the switch.



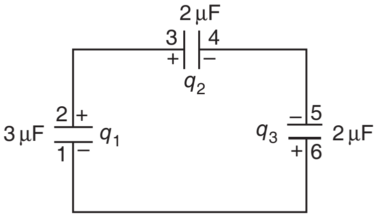


$$q_2 = EC_2 \quad \text{and} \quad q_1 = 0$$

$$q_0 = E\left(\frac{C_1 C_2}{C_1 + C_2}\right)$$

From 2, $-q_0$ charge will flow, so that charge on right hand side plate of C_1 becomes zero. From 1, q_2 charge will flow.

9. (i) Charge on capacitor A, before joining with an uncharged capacitor.



$$q_A = CV = (100)(3)\,\mu\text{C} = 300\,\mu\text{C}$$

Similarly, charge on capacitor B

$$q_B = (180)(2)\,\mu\text{C} = 360\,\mu\text{C}$$

Let q_1, q_2 and q_3 be the charges on the three capacitors after joining them as shown in figure. (q_1, q_2 and q_3 are in microcoulombs)

From conservation of charge

Net charge on plates 2 and 3 before joining = net charge after joining

$$\therefore \quad 300 = q_1 + q_2 \quad \ldots\text{(i)}$$

Similarly, net charge on plates 4 and 5 before joining = net charge after joining

$$-360 = -q_2 - q_3$$

or $$360 = q_2 + q_3 \quad \ldots\text{(ii)}$$

Applying Kirchhoff's second law in closed loop

$$\frac{q_1}{3} - \frac{q_2}{2} + \frac{q_3}{2} = 0$$

or $$2q_1 - 3q_2 + 3q_3 = 0 \quad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$q_1 = 90\,\mu\text{C}$$

$$q_2 = 210\,\mu\text{C}$$

and $$q_3 = 150\,\mu\text{C}$$

(ii) (a) Electrostatic energy stored before completing the circuit,

$$U_i = \frac{1}{2}(3 \times 10^{-6})(100)^2 + \frac{1}{2}(2 \times 10^{-6})(180)^2 \qquad \left(U = \frac{1}{2}CV^2\right)$$

$$= 4.74 \times 10^{-2} \text{ J} \quad \text{or} \quad U_i = 47.4 \text{ mJ}$$

(b) Electrostatic energy stored after completing the circuit,

$$U_f = \frac{1}{2}\frac{(90 \times 10^{-6})^2}{(3 \times 10^{-6})} + \frac{1}{2}\frac{(210 \times 10^{-6})^2}{(2 \times 10^{-6})} + \frac{1}{2}\frac{(150 \times 10^{-6})^2}{(2 \times 10^{-6})} \qquad \left[U = \frac{1}{2}\frac{q^2}{C}\right]$$

$$= 1.8 \times 10^{-2} \text{ J} \quad \text{or} \quad U_f = 18 \text{ mJ}$$ **Ans.**

10. (a) In steady state, capacitors will be in parallel. Charge will distribute in direct ratio of their capacity.

$$\therefore \qquad q_1 = \left(\frac{C_1}{C_1 + C_2}\right) q_0$$

$$\text{and} \qquad q_2 = \left(\frac{C_2}{C_1 + C_2}\right) q_0$$

Initial emf in the circuit is potential difference across capacitor C_1 or q_0/C_1.

Therefore, initial current would be

$$i_0 = \frac{q_0/C_1}{R} = \frac{q_0}{C_1 R}$$

Current as function of time will be $i = i_0 e^{-t/\tau_C}$

Here, $\tau_C = \left(\frac{C_1 C_2}{C_1 + C_2}\right) R$ **Ans.**

(b) $U_i = \frac{1}{2}\frac{q_0^2}{C_1}$ and $U_f = \frac{1}{2}\frac{q_0^2}{C_1 + C_2}$

Heat lost in the resistor

$$= U_i - U_f = \frac{q_0^2}{2}\left[\frac{C_2}{C_1(C_1 + C_2)}\right]$$ **Ans.**

11. $\tau_C = CR = \left(\frac{K\varepsilon_0 A}{d}\right)\left(\frac{\rho d}{A}\right) = K\varepsilon_0 \rho$

$$= 5 \times 8.86 \times \frac{10^{-12}}{7.4 \times 10^{-12}} \approx 6\text{ s}$$

Initial current,

$$i_0 = \frac{q_0/C}{R} = \frac{q_0}{CR} = \frac{q_0}{\tau_C} = \frac{8.55}{6} = 1.425\ \mu\text{A}$$

Now, current as function of time $i = i_0 e^{-t/\tau_C}$

or $i = (1.425)e^{-12/6} = 0.193\ \mu\text{A}$ **Ans.**

12. (a) $C = \frac{\varepsilon_0 A}{x}$, $U = \frac{Q^2}{2C} = \frac{Q^2 x}{2\varepsilon_0 A}$

(b) $\frac{dU}{dx} = \frac{Q^2}{2\varepsilon_0 A}$

$$\therefore \qquad dU = \left(\frac{Q^2}{2\varepsilon_0 A}\right) dx$$

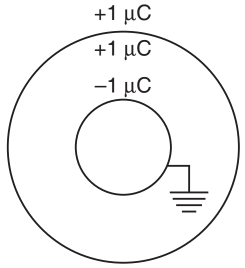
(c) $\left(\frac{Q^2}{2\varepsilon_0 A}\right) dx = dW = F dx$

$$\therefore \qquad F = \frac{Q^2}{2\varepsilon_0 A}$$

(d) Because E between the plates is due to both the plates.

While $F = (Q)$ (field due to other plate) **Ans.**

13. (a) Let q be the charge on smaller sphere. Then,



$$V_{\text{inner}} = 0$$

$$\therefore \qquad \frac{Kq}{2} + \frac{K(2)}{4} = 0 \quad \text{or} \quad q = -1\ \mu\text{C}$$

$$\text{Now,} \quad V_{\text{outer}} = \frac{K(2-1) \times 10^{-6}}{4 \times 10^{-2}}$$

$$= \frac{9 \times 10^{9} \times 10^{-6}}{4 \times 10^{-2}}$$

$$= 2.25 \times 10^{5}\text{ V}$$ **Ans.**

(b) Charge distribution is as shown in above figure.

14. (a) **Fig. (a)** $V_6 = \frac{90}{3} = 30\text{ V}, q_6 = 6 \times 30 = 180\ \mu\text{C}$

$$V_3 = \frac{90}{3} = 30\text{ V}$$

$$q_3 = 30 \times 3 = 90\ \mu\text{C}$$ **Ans.**

Fig. (b) Capacitor $1\ \mu$F is short-circuited.

Therefore, $q_1 = 0$.

$$V_{26} = \left(\frac{20}{20 + 20 + 10}\right) \times 100 = 40\text{ V}$$

This 40 V will distribute in inverse ratio of capacity.

$$\therefore \qquad V_6 = \frac{2}{8} \times 40 = 10\text{ V}$$

$$V_2 = \frac{6}{8} \times 40 = 30\text{ V}$$

$$\therefore \qquad q_6 = 60\ \mu\text{C}, q_2 = 60\ \mu\text{C}$$ **Ans.**

(b) **Fig. (a)** When S is open, $6\ \mu$F is short-circuited

or $V_6 = 0, q_6 = 0$

and $V_3 = 90\text{ V}, \ q_3 = 270\ \mu\text{C}$ **Ans.**

Fig. (b) When S is open, $V_1 = 100$ V

$$q_1 = 100\ \mu\text{C}$$

$$V_{26} = 100\text{ V}$$

$$\therefore \qquad V_6 = \frac{2}{8} \times 100 = 25\text{ V}$$

and $$V_2 = \frac{6}{8} \times 100 = 75 \text{ V}$$

$$\therefore \quad q_6 = 150\,\mu\text{C},\; q_2 = 150\,\mu\text{C}$$

Further, $V_A - 0 = V_2$

$$\therefore \quad V_A = V_2 = 75 \text{ V}$$ **Ans.**

15. Current in the circuit, $i = \dfrac{10}{4+1+2+3} = 1 \text{ A}$

Now, $$V_{5\,\mu\text{F}} = V_{1,2\Omega} = 3 \text{ V}$$

$$\therefore \quad q_5 = 15\,\mu\text{C}$$

Further, $$V_{3\mu\text{F}} = V_{2,\,3\Omega} = 5 \text{ V}$$

$$\therefore \quad q_3 = 15\,\mu\text{C}$$ **Ans.**

16. (a) At $t = 0$, when capacitor is uncharged, its equivalent resistance is zero.

$$\therefore \quad R_{\text{net}} = 4 + \frac{6 \times 3}{6 + 3} = 6 \text{ M}\Omega$$

or $$i_1 = \frac{18 \times 10^3}{6 \times 10^6} \text{ A} = 3 \text{ mA}$$

This will distribute in inverse ratio of resistances.

$$\therefore \quad i_2 = \frac{3}{6+3} i_1 = 1 \text{ mA} \;\text{ and }\; i_3 = 2 \text{ mA}$$

At $t = \infty$, when capacitor is completely charged, equivalent resistance of capacitor is infinite.

$$i_3 = 0, i_1 = i_2 = \frac{18 \times 10^3}{(4 + 6) \times 10^6} = 1.8 \text{ mA}$$ **Ans.**

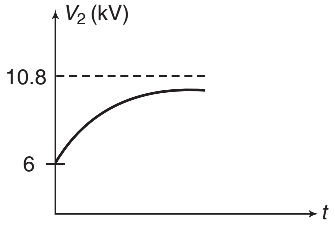
(b) At $t = 0$,

$$V_2 = i_2 R_2 = (1 \times 10^{-3})(6 \times 10^6) \text{ V}$$
$$= 6 \text{ kV}$$

At $t = \infty$,

$$V_2 = i_2 R_2 = (1.8 \times 10^{-3})(6 \times 10^6) \text{ V}$$
$$= 10.8 \text{ kV}$$ **Ans.**

(c) To find time constant of the circuit we will have to short-circuit the battery and find resistance across capacitor. In that case, R_1 and R_2 are in parallel and they are in series with R_3.



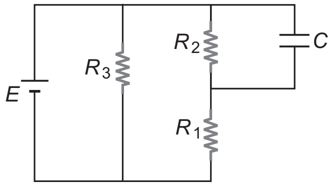
$$\therefore \quad R_{\text{net}} = 3 + \frac{4 \times 6}{4 + 6} = 5.4 \text{ M}\Omega$$

$$\therefore \quad \tau_C = CR_{\text{net}} = (10 \times 10^{-6})(5.4 \times 10^6)$$
$$= 54 \text{ s}$$

$$\therefore \quad V_2 = 6 + (10.8 - 6)(1 - e^{-t/C})$$
$$= 6 + 4.8(1 - e^{t/54})$$

Here, V_2 is in kV and t is second.

17. Circuit can be drawn as shown in figure.

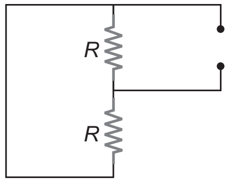


In charging of capacitor, R_3 has no role.
In steady state, potential difference across capacitor = potential difference across $R_2 = E/2$
Therefore, steady state charge across capacitor

$$q_0 = \frac{CE}{2}$$

To find time constant of circuit we will have to short circuit the battery, then we will find net resistance across capacitor.

$$R_{\text{net}} = \frac{R}{2} \Rightarrow \tau_C = CR_{\text{net}} = \frac{CR}{2}$$



$\therefore$ Charge in the capacitor at time t would be

$$q = q_0(1 - e^{-t/\tau_C}) = \frac{CE}{2}(1 - e^{-\frac{2t}{CR}})$$ **Ans.**

18. $q_2 = 20\,\mu\text{C}$

$\therefore \quad q_1 = 10\,\mu\text{C}$ (as they are in parallel)

Energy stored at this instant,

$$U = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2}$$
$$= \frac{1}{2} \times \frac{(10^{-5})^2}{10^{-6}} + \frac{1}{2} \times \frac{(2 \times 10^{-5})^2}{2 \times 10^{-6}}$$
$$= 1.5 \times 10^{-4} \text{ J}$$
$$= 0.15 \text{ mJ}$$

In charging of a capacitor 50% of the energy is stored and rest 50% is dissipated in the form of heat.

Therefore, 0.15 mJ will be dissipated in the form of heat across all the resistors. In series in direct ratio of resistance ($H = i^2Rt$) and in parallel in inverse ratio of resistance.

$\therefore \qquad H_2 = 0.075 \text{ mJ}, H_3 = 0.05 \text{ mJ}$

and $\qquad H_6 = 0.025 \text{ mJ}$ **Ans.**

19. (a) At $t = 0$, capacitor is equivalent to a battery of emf $\frac{E}{2}$.

Net emf of the circuit $= E - E/2 = E/2$

Total resistance is R.

Therefore, current in the circuit at $t = 0$ would be

$$i = \frac{E/2}{R} = \frac{E}{2R}$$ **Ans.**

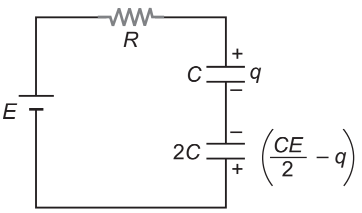
(b) Let in steady state there is total q charge on C.

Initial charge on C was $CE/2$. Therefore, charge on $2C$ in steady state would be $\left(\frac{CE}{2} - q\right)$ with polarities as shown. This is because net charge on lower plate of C and of upper plate on $2C$ should remain constant.

Applying loop law in the circuit in steady state, we have

$$E - \frac{q}{C} + \frac{CE/2 - q}{2C} = 0$$

$$\therefore \qquad q = \frac{5}{6}CE$$



Therefore, charge on C increases from $q_i = \frac{CE}{2}$ to $q_f = \frac{5CE}{6}$ exponentially.

Equivalent time constant would be

$$\tau_C = \left(\frac{C \times 2C}{C + 2C}\right)R = \frac{2}{3}CR$$

Therefore, charge as function of time would be

$$q = q_i + (q_f - q_i)(1 - e^{-t/\tau_C})$$

$$= \frac{CE}{2} + \frac{CE}{3}\left(1 - e^{-\frac{3t}{2CR}}\right)$$ **Ans.**

20. When S_1 is closed and S_2 open, capacitor will discharge. At time $t = R_1C$, one time constant, charge will remain $q_1 = \left(\frac{1}{e}\right)$ times of CV or

$$q_1 = \frac{CV}{e}$$

When S_1 is open and S_2 closed, charge will increase (or may decrease also) from $\frac{CV}{e}$ to CE exponentially. Time constant for this would be $(R_1C + R_2C)$. Charge as function of time would be

$$q = q_i + (q_f - q_i)(1 - e^{-t/\tau_C})$$

$$q = \frac{CV}{e} + \left(CE - \frac{CV}{e}\right)(1 - e^{-t/\tau_C})$$

After total time $2R_1C + R_2C$ or $t = R_1C + R_2C$, one time constant in above equation, charge will remain

$$q = \frac{CV}{e} + \left(CE - \frac{CV}{e}\right)\left(1 - \frac{1}{e}\right)$$

$$= EC\left(1 - \frac{1}{e}\right) + \frac{VC}{e^2}$$

21. At $t = 0$, capacitor C_0 is like a battery of

$$\text{emf} = \frac{Q_0}{C_0} = 1 \text{ V}$$

Net emf of the circuit $= 4 - 1 = 3$ V

Total resistance is $R = 100\ \Omega$

$\therefore$ Initial current $= \frac{3}{100} = 0.03$ A

This current will decrease exponentially to zero.

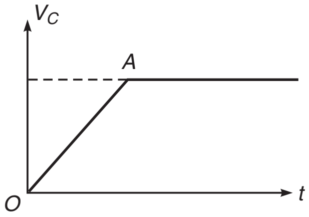
$$\therefore \qquad i = 0.03\, e^{-t/\tau_C}$$

Here, $\qquad \tau_C = C_{\text{net}}R = (1 \times 10^{-6})(100)$

$$= 10^{-4} \text{ s}$$

$$\therefore \qquad i = 0.03\, e^{-10^4 t}$$ **Ans.**

22. **From *O* to *A***



$$V_C = at \qquad (a = \text{constant})$$

$$\therefore \qquad q_C = CV_C = Cat$$

∴ $$i = \frac{dq_C}{dt} = aC$$

$$V_R = iR = aCR = \text{constant}$$

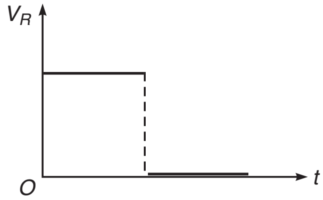
From *A* onwards $V_C =$ constant

∴ $$q_C = \text{constant}$$

∴ $$i = \frac{dq_C}{dt} = 0$$

or $$V_R = 0$$

Therefore, V_R *versus* t graph is as shown in figure.



23. **From *O* to *A*** $V = at$

Here, a is a positive constant.

∴ $$at = \frac{q}{C} + iR$$

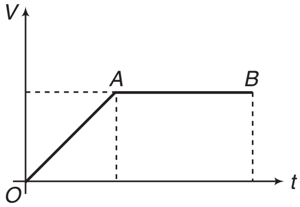
Differentiating w.r.t. time, we have

$$a = \frac{1}{C}\left(\frac{dq}{dt}\right) + \left(\frac{di}{dt}\right)R$$

or $$\left(\frac{di}{dt}\right)R = a - \frac{i}{C} \qquad \left(\text{as } i = \frac{dq}{dt}\right)$$

∴ $$\int_0^i \frac{di}{a - i/C} = \int_0^t \frac{dt}{R}$$

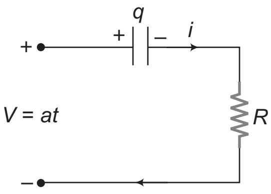
∴ $$i = aC(1 - e^{-t/CR})$$



i.e. current in the circuit increases exponentially

∴ $$V_{ED} = iR = aCR\,(1 - e^{-t/CR})$$

or V_{ED} also increases exponentially.



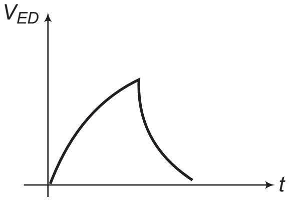
From *A* onwards When $V =$ constant (say V_0)

$$V_0 = at \quad \text{or} \quad t = \frac{V_0}{a}$$

∴ $$V_{ED} = aCR\,(1 - e^{-V_0/aCR})$$

After this V_{ED} will decrease exponentially.

Hence, a rough graph is as shown in figure.



24. $q_i = CV_i = 100\,\mu\text{C}, q_f = CV_f = -50\,\mu\text{C}$

Therefore, charge will vary from 100 μC to − 50 μC exponentially.

∴ $q = -50 + 150\,e^{-t/\tau_C}$, Here q is in μC

$$\tau_C = CR = (10^{-6})(5 \times 10^3) = 5 \times 10^{-3}$$

∴ $$q = -50 + 150\,e^{-t/\tau_C}$$

$$V_C = \frac{q}{C} = (-50 + 150e^{-t/\tau_C})\text{ V}$$

or $$V_C = 50\,(3e^{-200t} - 1)$$

$$i = -\frac{dq}{dt} = \frac{150 \times 10^{-6}}{\tau_C}\,e^{-200t}$$

$$= \frac{150 \times 10^{-6}}{5 \times 10^{-3}}\,e^{-200t} = 30 \times 10^{-3} e^{-200t}$$

$$V_R = iR = 150e^{-200t}$$ **Ans.**

25. At $t = 0$, equivalent resistance of an uncharged capacitor is zero and a charged capacitor is like a battery of emf = potential difference across the capacitor.

(a) $V_{C_1} = q_1/C_1 = 2$ V

∴ Net emf of the circuit = 9 − 2 = 7 V

or 9 + 2 = 11 V

Net resistance $= 30 + \frac{30 \times 60}{30 + 60} = 50\,\Omega$

∴ Current at $t = 0$ would be $i_0 = \frac{7}{50}$ A

or $\frac{11}{50}$ A **Ans.**

(b) In steady state, no current will flow through the circuit. C_2 will therefore be short-circuited, while PD across C_1 will be 9 V.

∴ $Q_2 = 0$ and $Q_1 = 9\,\mu\text{C}$ **Ans.**

26. Time constant of the circuit is

$$\tau_C = CR = (0.5 \times 10^{-6})\,(500) = 2.5 \times 10^{-4}\ \text{s}$$

For $t \le 250\ \mu\text{s}$ $\quad i = i_0 e^{-t/\tau_C}$

Here, $\quad i_0 = \dfrac{20}{500} = 0.04\ \text{A}$

∴ $\quad i = (0.04\, e^{-4000t})$ amp

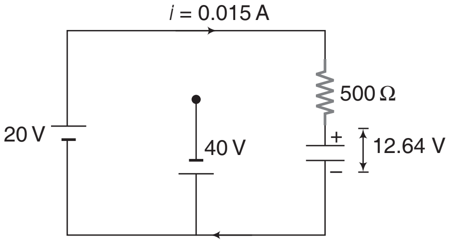
At $\quad t = 250\ \mu\text{s} = 2.5 \times 10^{-4}\ \text{s}$

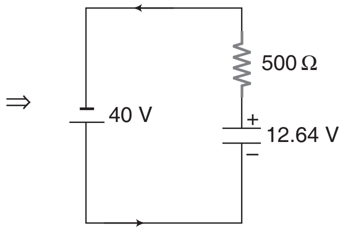
$$i = 0.04\, e^{-1} = 0.015\ \text{amp}$$

At this moment PD across the capacitor,

$$V_C = 20\,(1 - e^{-1}) = 12.64\ \text{V}$$

So when the switch is shifted to position 2, the current in the circuit is 0.015 A (clockwise) and PD across capacitor is 12.64 V





As soon as the switch is shifted to position 2 current will reverse its direction with maximum current.

$$i_0' = \frac{40 + 12.64}{500}$$

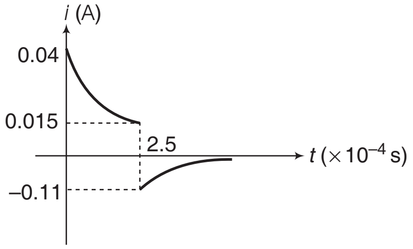
$$= 0.11\ \text{A}$$

Now, it will decrease exponentially to zero.

For $t \ge 250\ \mu\text{s}$

$$i = -\,i_0'\, e^{-t/\tau_C} = -\,0.11\, e^{-4000\,t}$$

The $(i\text{ - }t)$ graph is as shown in figure.



27. Capacitor C_1 will discharge according to the equation,

$$q = q_0 e^{-t/\tau_C} \qquad \text{…(i)}$$

Here, $\quad \tau_C = C_1 R$

and discharging current :

$$i = \frac{-\,dq}{dt} = \frac{q_0}{\tau_C}.e^{-t/\tau_C} = \frac{q_0 e^{-t/\tau_C}}{C_1 R} \qquad \text{…(ii)}$$

At the given instant $i = i_0$

Therefore, from Eq. (ii)

$$q_0 e^{-t/\tau_C} = i_0 C_1\, R \text{ at this instant}$$

or charge C_1 at this instant will be

$$q = (i_0 C_1 R) \qquad \text{[From Eq. (ii)]}$$

Now, this charge q will later on distribute in C_1 and C_2.

∴ $\quad U_i = \dfrac{q^2}{2C_1}$

and $\quad \dfrac{q^2}{2(C_1 + C_2)} = U_f$

∴ Heat generated in resistance,

$$H = U_i - U_f$$

Substituting values of $q : U_i$ and U_f, we get

$$H = \frac{(I_0 R)^2 C_1 C_2}{2(C_1 + C_2)} \qquad \textbf{Ans.}$$

26. Magnetics

INTRODUCTORY EXERCISE 26.1

1. $\because \quad qE = Bqv \sin\theta$

$\therefore \quad [E/B] = [v] = [\mathrm{LT}^{-1}]$

2. From the property of cross product, **F** is always perpendicular to both **v** and **B**.

3. May be possible that $\theta = 0°$ or $180°$ between **v** and **B**, so that $\mathbf{F}_m = 0$

4. $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

Here, q has to be substituted with sign.

5. Apply Fleming's left hand rule.

6. $\because \quad F = Bqv\sin\theta$

$$\therefore \quad v = \frac{F}{Bq\sin\theta}$$

$$= \frac{4.6 \times 10^{-15}}{3.5 \times 10^{-3} \times 1.6 \times 10^{-19} \times \sin 60°}$$

$$= 9.47 \times 10^{6} \text{ m/s}$$

7. $\because \quad F = Bqv\sin 90°$

$$= 0.8 \times 2 \times 1.6 \times 10^{-19} \times 10^{5}$$

$$= 2.56 \times 10^{-14} \text{ N}$$

INTRODUCTORY EXERCISE 26.2

1. Path C is undeviated. Therefore, it is of neutron's path. From Fleming's left hand rule magnetic force on positive charge will be leftwards and on negative charge is rightwards. Therefore, track D is of electron. Among A and B one is of proton and other of α-particle.

Further, $r = \dfrac{mv}{Bq}$ or $r \propto \dfrac{m}{q}$

Since, $\left(\dfrac{m}{q}\right)_\alpha > \left(\dfrac{m}{q}\right)_P$

$\therefore \quad r_\alpha > r_P$

or track B is of α-particle.

2. $r = \dfrac{\sqrt{2km}}{Bq}$ or $r \propto \sqrt{m}$ (k, q and B are same)

$$m_p > m_e \Rightarrow r_p > r_e$$

3. The path will be a helix. Path is circle when it enters normal to the magnetic field.

4. Magnetic force may be non-zero. Hence, acceleration due to magnetic force may be non-zero.

Magnetic force is always perpendicular to velocity. Hence, its power is always zero or work done by magnetic force is always zero. Hence, it can be change the speed of charged particle.

5. $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

F is along position y-direction. q is negative and **v** is along positive x-direction. Therefore, **B** should be along positive z-direction.

6. (a) $r = \dfrac{mv}{Bq}$

or $r \propto m$ as other factors are same.

(b) $f = \dfrac{Bq}{2\pi m}$

or $f \propto \dfrac{1}{m}$

7. $\because \quad r = \dfrac{\sqrt{2qVm}}{Bq}$

INTRODUCTORY EXERCISE 26.3

1. $\because \quad \mathbf{l} = l\hat{\mathbf{i}}$

Now, $\mathbf{F} = i(\mathbf{l} \times \mathbf{B})$

$$= i[(l\hat{\mathbf{i}}) \times (B_0\hat{\mathbf{j}} + B_0\hat{\mathbf{k}})]$$

$$= ilB_0(\hat{\mathbf{i}} - \hat{\mathbf{j}}) = |\mathbf{F}| = \sqrt{2}ilB_0$$

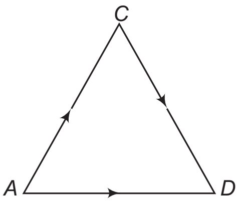
2. No, it will not change, as the new $\hat{\mathbf{i}}$ component of **B** is in the direction of **l**.

3. $\because \quad i = 3.5 \text{ A}$

$$\mathbf{l} = (-10^{-2}\hat{\mathbf{j}})$$

Now, apply $\mathbf{F} = i(\mathbf{l} \times \mathbf{B})$ in all parts.

4.



$\mathbf{F}_{ACD} = \mathbf{F}_{AD}$

$\therefore \quad |\mathbf{F}_{\text{net}}| = 2|\mathbf{F}_{AD}| = 2ilB$

$$= 2 \times 2 \times 4 \times 2 = 32 \text{ N}$$

INTRODUCTORY EXERCISE 26.4

1. ∵ $$\frac{M}{L} = \frac{q}{2m}$$

∴ $$M = \left(\frac{q}{2m}\right)L = \left(\frac{q}{2m}\right)(I\omega)$$
$$= \left(\frac{q}{2m}\right)\left(\frac{1}{2}mR^2\right)(\omega) = \frac{qR^2\omega}{4}$$

2. ∵ $$M = iA = i(\pi R^2)$$
$$= (0.2)(\pi)(8 \times 10^{-2})^2$$
$$= (4.0 \times 10^{-3})\ \text{A-m}^2$$

Now, $\mathbf{M} = M\hat{\mathbf{M}} = (4.0 \times 10^{-3})(0.6\hat{\mathbf{i}} - 0.8\hat{\mathbf{j}})$

(a) $\tau = \mathbf{M} \times \mathbf{B}$

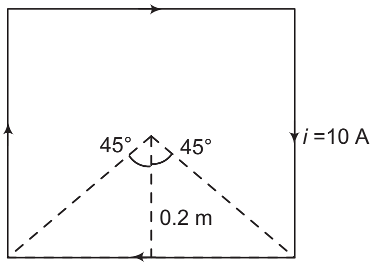
(b) $U = -\mathbf{M} \cdot \mathbf{B}$

3. ∵ $$L = 2\pi R \Rightarrow R = \frac{L}{2\pi}$$
$$M = iA = i(\pi R^2)$$
$$= (i)(\pi)\left(\frac{L}{2\pi}\right)^2 = \frac{iL^2}{4\pi}$$
$$\tau_{\max} = MB \sin 90° = \frac{iBL^2}{4\pi}$$

4. ∵ $$\Delta U = U_{0°} - U_{180°}$$
$$= -MB\cos 0° + MB\cos 180° = -2MB$$

INTRODUCTORY EXERCISE 26.5

1. (a) From screw law, we can see that direction of magnetic field at centre of the square is inwards as the current is clockwise.



$$B = 4\left(\frac{\mu_0}{4\pi}\frac{i}{r}\right)(\sin\alpha + \sin\beta)$$
$$= \frac{4 \times 10^{-7} \times 10}{0.2}(\sin 45° + \sin 45°)$$
$$= 2.83 \times 10^{-5}\ \text{T}$$

(b) $2\pi R = 4(0.4)$
$$2R = \frac{1.6}{\pi}\text{m}$$
$$B = \frac{\mu_0 i}{2R} = \frac{(4\pi \times 10^{-7})(10)}{(1.6/\pi)}$$
$$= 24.7 \times 10^{-6}\ \text{T}$$

2. Magnetic field due to horizontal wire is zero. Magnetic field due to vertical is
$$B = \frac{\mu_0}{4\pi}\frac{i}{x}(\sin 90° + \sin 0°)$$
$$= \frac{\mu_0}{4\pi}\frac{i}{x} \qquad \text{(inwards)}$$

3. Both straight and circular wires will produce magnetic fields inwards.

∴ $$B = \frac{\mu_0}{2\pi}\frac{i}{R} + \frac{\mu_0 i}{2R}$$
$$= \frac{(2 \times 10^{-7})(7)}{0.1} + \frac{(4\pi \times 10^{-7})(7)}{2 \times 0.1}$$
$$= 5.8 \times 10^{-5}\ \text{T}$$

4. Magnetic field at O due to two straight wires = 0

Magnetic field due to circular wire,
$$B = \frac{1}{4}\ \text{(due to whole circle)}$$
$$= \frac{1}{4}\left(\frac{\mu_0 i}{2R}\right) \qquad \text{(inwards)}$$
$$= \frac{1}{4} \times \frac{(4\pi \times 10^{-7})(5)}{2 \times 0.03}$$
$$= 2.62 \times 10^{-5}\ \text{T}$$

5. Magnetic field at P due to straight wires = 0

Due to circular wires one is outwards (of radius a) and other is inwards. 60° means $\frac{1}{6}$ th of whole circle.

∴ $$B = \frac{1}{6}\left[\frac{\mu_0 i}{2a} - \frac{\mu_0 i}{2b}\right] \qquad \text{(outwards)}$$

INTRODUCTORY EXERCISE 26.6

1. Applying Ampere's circuital law,
$$B_A = \frac{\mu_0}{2\pi}\frac{i_{\text{in}}}{r}$$
$$= \frac{(2 \times 10^{-7})(1)}{10^{-3}}$$
$$= 2 \times 10^{-4}\ \text{T}$$

This is due to (·) current of 1 A. Hence, magnetic lines are circular and anti-clockwise. Hence, magnetic field is upwards.

$$B_b = \frac{\mu_0}{2\pi}\frac{i_{\text{in}}}{r}$$

$$= \frac{(2\times10^{-7})(3-1)}{3\times10^{-3}}$$

$$= 1.33\times10^{-4}\ \text{T}$$

This is due to net ⊗ current. Hence, magnetic lines are clockwise.

So, magnetic field at B is downwards.

2. $\int \mathbf{B}\cdot d\mathbf{l} = \mu_0(i_{\text{net}})$

Along path (a), net current enclosed by this path is zero.

Hence, line integral = 0

Along path (b), i_{net} is ⊗. So, magnetic lines along this current is clockwise. But, we have to take line integral in counter clockwise direction. Hence, line integral will be negative.

3. Using Ampere's circuital law over a circular loop of any radius less than the radius of the pipe, we can see that net current inside the loop is zero. Hence, magnetic field at every point inside the loop will be zero.

INTRODUCTORY EXERCISE 26.7

1. $$i = \left(\frac{k}{NBA}\right)\phi$$

$$\therefore\quad B = \frac{k\phi}{NiA} = \frac{(10^{-8})(90)}{100\times10^{-6}\times10^{-4}}$$

$$= 90\ \text{T}$$

2. $$i = \left(\frac{k}{NBA}\right)\phi$$

$$= \frac{(0.125\times10^{-7})(6)(\pi/180)}{200\times5\times10^{-2}\times5\times2\times10^{-4}}$$

$$= 1.3\times10^{-7}\ \text{A}$$

Exercises

LEVEL 1

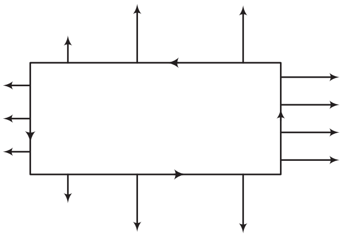
Assertion and Reason

2. By changing the direction of velocity direction of magnetic force will change. So, it is not a constant force.

3. To balance the weight, force on upper wire should be upwards (repulsion). Further equilibrium can be checked by displacing the wire from equilibrium position.

4. $\tau = MB\sin 90° \neq 0$

5. Magnetic field is outwards and increasing with x. So, magnetic force will also increase with x. The force on different sections are as shown in figure.



Force will act in positive x-direction. But, no torque will act.

6. $\because\quad r = \dfrac{\sqrt{2qVm}}{Bq} \Rightarrow r \propto \sqrt{\dfrac{m}{q}}$

m and q both are different. Ratio $\dfrac{m}{q}$ is not same for both.

7. $\because\quad \mathbf{F}_e + \mathbf{F}_m = 0$

$\therefore\quad q\mathbf{E} + q(\mathbf{v}\times\mathbf{B}) = 0$

$\therefore\quad \mathbf{E} = -(\mathbf{v}\times\mathbf{B}) = (\mathbf{B}\times\mathbf{v})$

8. $\because\quad \mathbf{F}_m = q(\mathbf{v}\times\mathbf{B})$

$\mathbf{F}_m \perp \mathbf{v}$ (always)

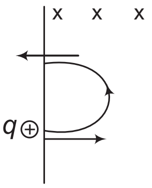
and $P = \mathbf{F}\cdot\mathbf{v}$

$\therefore$ Power of magnetic force is always zero.

$\mathbf{F}_e = q\mathbf{E}$

If $\mathbf{F}_e$ is also perpendicular to $\mathbf{v}$, then its power is also zero.

9.



10. $|\mathbf{v}| = \sqrt{2}v_0$ = speed, which always remains constant.

11. $R \propto v$, by increasing the speed two times radius also becomes two times. Hence, acceleration $(= v^2/R)$ will also become only two times.

Objective Questions

2. $T = \dfrac{2\pi m}{Bq}$, independent of v.

3. $B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$, independent of diameter of wire.

8. In uniform **B** is force on any current carrying loop is always zero.

9. $M = NiA$

10. $\mathbf{B} \cdot \mathbf{F} = 0$ as $\mathbf{B} \perp \mathbf{F}$

$$\therefore \quad \mathbf{B} \cdot \alpha = 0$$
$$\text{or} \quad 2x + 3 - 4 = 0$$
$$\therefore \quad x = 0.5$$

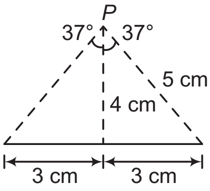
12. In uniform magnetic field, force on any current carrying loop is always zero.

13. $r = \dfrac{P}{Bq}$ or $r \propto \dfrac{1}{q}$ (as $P =$ constant)

14. $\theta_i = 180^\circ,\ \theta_f = 180^\circ - \theta$

$$W = U_f - U_i$$
$$= -MB\cos(180^\circ - \theta) - (-MB\cos 180^\circ)$$
$$= MB\cos\theta - MB$$

15.



$r = 5$ cm

$$B = \frac{\mu_0}{4\pi}\frac{i}{r}(\sin 37^\circ + \sin 37^\circ)$$

16. $B_x = \dfrac{\mu_0 NiR^2}{2(R^2 + x^2)}$ and $B_c = \dfrac{\mu_0 Ni}{2R}$ $c \rightarrow$ centre.

17. $\mathbf{F} = I\,(\mathbf{l} \times \mathbf{B}) = I\,(\mathbf{ba} \times \mathbf{B})$

We can see that all (a), (b) and (c) options are same.

18. $F = Bqv$ or $F \propto v$

$$\text{Now,} \quad v = \sqrt{\frac{2qV}{m}}$$
$$\therefore \quad F \propto \sqrt{V}$$

19. Two fields are additive.

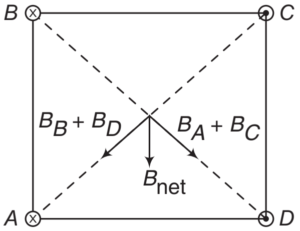
$$\therefore \quad B_{\text{net}} = 2\left[\frac{\mu_0 I}{2\pi\,(R/2)}\right] = \frac{2\mu_0 I}{\pi R}$$

20.
$$B_x = \frac{1}{8}B_c$$
$$\therefore \quad \frac{\mu_0 NiR^2}{2(R^2 + x^2)^{3/2}} = \frac{1}{8}\left[\frac{\mu_0 Ni}{2R}\right]$$

21. Electric field (acting along $\hat{\mathbf{j}}$ direction) will change the velocity component which is parallel to **B** (which is also along $\hat{\mathbf{j}}$ direction).
$\mathbf{B} = B_0\hat{\mathbf{j}}$ and $\mathbf{v} = v_0\hat{\mathbf{j}}$ will rotate the particle in a circle. Hence, the net path is helical with variable pitch.

22. $r = \sqrt{\dfrac{2Km}{Bq}} \Rightarrow r \propto \dfrac{\sqrt{K}}{B}$

23.



24. $r = \sqrt{\dfrac{2qVm}{Bq}} = \left(\sqrt{\dfrac{2Vm}{q}}\right)\left(\dfrac{1}{B}\right)$

26. In (c), two wires are producing ⊙ magnetic field and two wires are producing ⊗ magnetic field.

27. I_1 produces circular magnetic lines current I_2 is each small circular element is parallel to $(\theta = 0^\circ)$ magnetic field. Hence, force is zero.

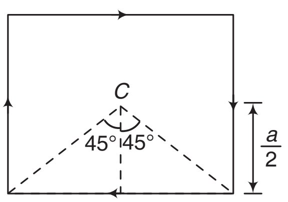
28. Arc of radius a $\left(\dfrac{1}{4}\text{th of circle}\right)$ produces magnetic field in $\hat{\mathbf{k}}$ direction or outwards, while arc of radius b produces magnetic field in $-\hat{\mathbf{k}}$ direction.

$$\therefore \quad \mathbf{B} = \frac{1}{4}\left(\frac{\mu_0 I}{2a}\right)\hat{\mathbf{k}} + \frac{1}{4}\left(\frac{\mu_0 I}{2a}\right)(-\hat{\mathbf{k}})$$
$$= \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{k}}$$

29. Equivalent current,

$$i = qf = ef$$
$$B = \frac{\mu_0 i}{2R} = \frac{\mu_0 ef}{2R}$$

30.



$$B_C = 4\left[\frac{\mu_0}{4\pi}\,\frac{I}{a/2}\,(\sin 45° + \sin 45°)\right]$$

$$= \frac{2\sqrt{2}\mu_0 I}{\pi a}$$

31. At distance r from centre,

$$B = \frac{\mu_0}{2\pi}\,\frac{(i_{\text{in}})}{r} \qquad \text{(From Ampere's circuital law)}$$

For path-1, $\quad i_{\text{in}} \neq 0$

$\therefore \quad B_1 \neq 0$

For path-2, $\quad i_{\text{in}} = 0$

$\therefore \quad B_2 = 0$

32. $d\mathbf{B} = \dfrac{\mu_0}{4\pi}\,\dfrac{i}{r^3}\,(d\mathbf{l} \times \mathbf{r})$

$\therefore$ $d\mathbf{B}$ is in the direction $d\mathbf{l} \times \mathbf{r}$. Hence, $d\mathbf{l}$ is outwards and $\mathbf{r}$ is from $d\mathbf{l}$ towards P.

33. Magnetic field due the straight portions is zero. It is only due to arc of circle.

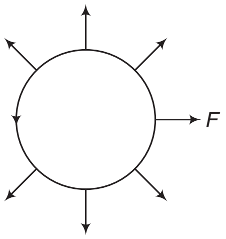
$$\therefore \quad B = \frac{\phi}{2\pi}\left(\frac{\mu_0 I}{2x}\right) \qquad \text{(Radius} = x\text{)}$$

$$= \frac{\mu_0 I \phi}{4\pi x}$$

34. From centre, $r = (R - x)$

$$B = \frac{\mu_0}{2\pi}\,\frac{I}{R^2}\,r = \frac{\mu_0}{2\pi}\,\frac{I}{R^2}\,(R - x)$$

35. From Fleming's left hand rule, we can see that magnetic force is outwards on the loop.



So, it tends to expand.

Subjective Questions

1. $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, where $q = -1.6 \times 10^{-19}$ C for an electron and $q = +1.6 \times 10^{-19}$ C for a proton

2. $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

3. (a) $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$

or $\quad [(7.6 \times 10^{-3})\,\hat{\mathbf{i}} - (5.2 \times 10^{-3})\hat{\mathbf{k}}]$

$= (7.8 \times 10^{-6})\,[(-3.8 \times 10^3)\hat{\mathbf{j}}\,(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})]$

$\therefore \quad (7.8 \times 10^{-6})\,(3.8 \times 10^{-3})\,(B_x)$

$= (-5.2 \times 10^{-3})$

$\therefore \quad B_x = -0.175$ T

Similarly,

$(7.8 \times 10^{-6})\,(-3.8 \times 10^3)\,(B_z)$

$= 7.6 \times 10^{-3}$

or $\quad B_z = -0.256$ T

(c) From the property of cross product.

$\mathbf{F}$ is always perpendicular to $\mathbf{B}$.

Hence, $\quad \mathbf{F} \cdot \mathbf{B} = 0$

4. Apply $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$

For example, let us apply for charged particle at e.

$$\mathbf{F}_e = q\left[\left(\frac{v}{\sqrt{2}}\,\hat{\mathbf{j}} - \frac{v}{\sqrt{2}}\,\hat{\mathbf{k}}\right) \times (B\hat{\mathbf{i}})\right]$$

$$= \frac{qvB}{\sqrt{2}}(-\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

5. $r = \dfrac{\sqrt{2qVm}}{Bq}$

$$\therefore \quad B = \frac{1}{r}\sqrt{\frac{2Vm}{q}}$$

$$= \frac{1}{0.18}\sqrt{\frac{2 \times 2 \times 10^3 \times 9.1 \times 10^{-31}}{1.6 \times 10^{-19}}}$$

$$= 8.38 \times 10^{-4} \text{ T}$$

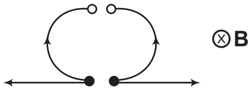
6. (a) $r = \dfrac{mv}{Bq} \Rightarrow v = \dfrac{Bqr}{m}$

(b) $t = \dfrac{T}{2} = \dfrac{\pi m}{Bq}$

(c) $r = \dfrac{\sqrt{2qVm}}{Bq}$

$$\therefore \quad V = \frac{r^2B^2q^2}{2qm} = \frac{r^2B^2q}{2m}$$

7. (a) Conservation of charge



They will collide after time,

$$t = \frac{T}{2} = \frac{\pi m}{Bq}$$

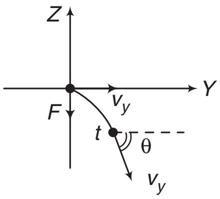
8. (a) At A, magnetic force should be towards right. From Fleming's left hand rule, magnetic field should be inwards.

Further, $$r = \frac{mv}{Bq}$$

$$\therefore \quad B = \frac{mv}{qr} \qquad (r = 5\text{ cm})$$

(b) $t_{AB} = \dfrac{T}{2} = \dfrac{\pi m}{Bq}$

9. Component of velocity parallel to $\mathbf{B}$, i.e. v_x will remain unchanged $v_y\hat{\mathbf{j}}$ and $B\hat{\mathbf{j}}$ will rotate the particle in yz-plane ($\perp$ to $\mathbf{B}$).



At the beginning direction to magnetic force is $-\hat{\mathbf{k}}$ [from the relation, $\mathbf{F} = q\,(v_y\hat{\mathbf{j}} \times B\hat{\mathbf{i}})$

In the time t, particle rotates an angle $\theta = \omega t = \left(\dfrac{Bq}{m}\right)t$ from its original path.

In the figure, we can set that, y- component of velocity at time t is $v_y \cos\theta$ and z - component is $-v_y \sin\theta$.

10. $\mathbf{F}_e + \mathbf{F}_m = 0$

or $\quad q\mathbf{E} + q\,(\mathbf{v} \times \mathbf{B}) = 0$

or $\quad \mathbf{E} = -\,(\mathbf{v} \times \mathbf{B})$

or $\quad \mathbf{E} = (\mathbf{B} \times \mathbf{v})$

11. Work is done only by electrostatic force. Hence, from work-energy theorem

$= \dfrac{1}{2}mv^2 =$ work done by electrostatic force only

$= (qE_0)z$ or Speed $v = \sqrt{\dfrac{2qE_0z}{m}}$

Particle rotates in a plane perpendicular to $\mathbf{B}$, i.e. in xz-plane only. Hence, $v_y = 0$

12. When they are moving rectilinearly, net force is zero.

$$\therefore \quad qE = Bqv \sin(90^\circ - \theta)$$

$$\therefore \quad v = \frac{E}{B\cos\theta}$$

When electric field is switched off,

$$p = \left(\frac{2\pi m}{Bq}\right) v \cos(90^\circ - \theta)$$

$$= \frac{2\pi m\, E \tan\theta}{qB^2}$$

13. $W = F_m$ or $mg = ilB \sin 90^\circ$

$$\therefore \quad i = \frac{mg}{Bl} = \frac{(13 \times 10^{-3})\,(10)}{0.44 \times 0.62}$$

$$= 0.47\text{ A}$$

Magnetic force should be upwards to balance the weight. Hence, from Fleming's left hand rule we can see that direction of current should be from left to right.

14. (a) $ilB = mg$

$$\text{or} \quad \frac{V}{R}\,lB = mg \quad \text{or} \quad V = \frac{mgR}{lB}$$

$$= \frac{(0.75)\,(9.8)\,(25)}{0.5 \times 0.45}$$

$$\approx 817\text{ V}$$

(b) $a = \dfrac{ilB - mg}{m} = \dfrac{VlB}{mR} - g \qquad \left(\text{as } i = \dfrac{V}{R}\right)$

$$= \frac{(817)\,(0.5)\,(0.45)}{(0.75)\,(2.0)} - 9.8$$

$$\approx 112.8\text{ m/s}^2$$

15. $i = 5\text{ A} \;\Rightarrow\; \mathbf{B} = (0.02\hat{\mathbf{j}})\text{ T}$

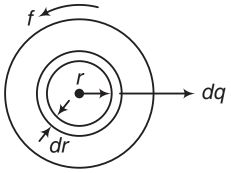
Now, applying $\mathbf{F} = i\,(\mathbf{l} \times \mathbf{B})$ in all parts. Let us find $\mathbf{l}$ for anyone parts.

$$\mathbf{l}_{cd} = \mathbf{r}_d - \mathbf{r}_c$$

$$= (0.4\,\hat{\mathbf{j}} + 0.4\hat{\mathbf{k}}) - (0.4\,\hat{\mathbf{i}} + 0.4\hat{\mathbf{k}})$$

$$= (0.4\,\hat{\mathbf{j}} - 0.4\,\hat{\mathbf{i}})$$

16. Let surface charge density is σ.



$$dq = [(2\pi r)dr]\sigma$$

Equivalent current,

$$i = (dq)f$$

$$dM = iA = [(dq)f][\pi r^2]$$

$$\therefore \quad M = \int_0^R dM$$

$$dB = \frac{\mu_0 i}{2r} = \frac{\mu_0 (dq) f}{2r}$$

$$\therefore \quad B = \int_0^R dB$$

Now, we can find the ratio $\dfrac{M}{B}$.

17. (a) From energy conservation,

$$U_\theta = K_\theta = U_{0^\circ} + K_{0^\circ}$$

or $(-MB\cos\theta) + 0 = (-MB\cos 0^\circ) + K_{0^\circ}$

Substituting the given values, we can calculate.

(b) To other side also it rotates upto the same angle.

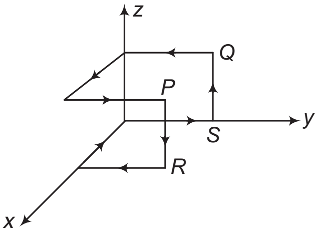
18. (a) $T = \dfrac{2\pi r}{v}$

(b) $I = qf = (e)\left(\dfrac{v}{2\pi r}\right)$

(c) $M = IA = \left(\dfrac{ev}{2\pi r}\right)(\pi r^2) = \dfrac{evr}{2}$

19. Assume equal and opposite currents in wires *cf* and *eh*.

20. Assume equal and opposite currents in wires *PQ* and *RS*, then find **M**.

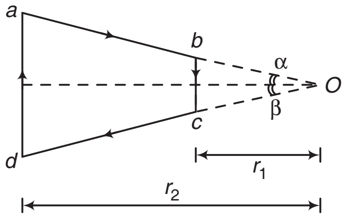


Now, $\mathbf{B}\,(2\hat{\mathbf{j}}) \Rightarrow \tau = \mathbf{M} \times \mathbf{B}$

21. $B_{\text{net}} = \sqrt{B^2 + B^2} = \sqrt{2}B$

where, $B = \dfrac{\mu_0}{4\pi}\dfrac{i}{r}(\sin 0^\circ + \sin 90^\circ)$

22. Magnetic fields at *O* due to currents in wires *ab* and *cd* are zero.



Magnetic field due to current in wire *da* (say B_2) is inwards due to current in wire *bc* (say B_1) is outwards.

$$B_1 = \frac{\mu_0}{4\pi}\frac{i}{r_1}(\sin\alpha + \sin\beta)$$

$$B_2 = \frac{\mu_0}{4\pi}\frac{i}{r_2}(\sin\alpha + \sin\beta)$$

$$B_1 > B_2 \quad \text{as} \quad r_1 < r_2$$

$\therefore \quad B_{\text{net}} = B_1 - B_2$ (outwards)

23. In first quadrant magnetic field due to I_1 is outwards and due to I_2 is inwards. So, net magnetic field may be zero.

Similarly, in third quadrant magnetic field due to I_1 is inwards and due to I_2 magnetic field is outwards. Hence, only in first and third quadrants magnetic field may be zero.

Let magnetic field is zero at point *P*(*xy*), then

$$B_{I_1} = B_{I_2}$$

$$\therefore \quad \frac{\mu_0}{2\pi}\frac{I_1}{y} = \frac{\mu_0}{2\pi}\frac{I_2}{x}$$

$$\therefore \quad y = \frac{I_1}{I_2}x$$

24. Two straight wires produces outward magnetic field by arc of circle produces inward magnetic field.

Due to straight wires,

$$B_1 = 2\left[\frac{\mu_0}{4\pi}\frac{i}{R}(\sin\theta + \sin 90^\circ)\right]$$

$$= \frac{\mu_0 i}{2\pi R} \quad \text{(outwards)}$$

Due to circular arc, $B_2 = \dfrac{\theta}{2\pi}\left(\dfrac{\mu_0 i}{2R}\right)$ (inwards)

For net field to be zero,

$$B_1 = B_2$$

or $\theta = 2\,\text{rad}$

25. (a) If currents are in the same direction, then above and below the wires magnetic fields are in the same direction. Hence, they can't produce zero magnetic field.

In between the wires, let $B = 0$ at a distance (r) cm from the wire carrying 75 A current. Then,

$$\frac{\mu_0}{2\pi}\left(\frac{75}{r}\right) = \left(\frac{\mu_0}{2\pi}\right)\left(\frac{25}{40-\pi}\right)$$

Solving, we get $r = 30$ cm.

(b) If currents are in opposite direction, then in between the wire magnetic field are in the same direction. So, they cannot produce zero magnetic field. The points should be above or below the wires, nearer to wire having smaller current. Let it is at a distance r from the wire having 25 A current. Then,

$$\frac{\mu_0}{2\pi}\left(\frac{25}{r}\right) = \left(\frac{\mu_0}{2\pi}\right)\left(\frac{75}{40+r}\right)$$

Solving this equation, we get $r = 20$ cm

26. Apply $B = \dfrac{\mu_0 N i R^2}{2(R^2 + x^2)^{3/2}}$

27. I_2 produces inwards magnetic field at centre. Hence, I_1 should produce outward magnetic field. Or current should be towards right. Further,

$$\frac{\mu_0 I_2}{2R} = \left(\frac{\mu_0}{2\pi}\right)\left(\frac{I_1}{D}\right)$$

$$\therefore \qquad I_1 = \left(\frac{\pi D}{R}\right) I_2$$

28. (a) $B_1 = \frac{\mu_0 Ni}{2R}$

(b) $B_2 = \frac{\mu_0 NiR^2}{2(R^2 + x^2)^{3/2}}$, given $B_2 = \frac{B_1}{2}$

29. 10 A and 8 A current produce inward magnetic field. While 20 A current produces outward magnetic field. Hence, current in fourth wire should be (20 – 10 – 8) A or 2 A and it should produce inward magnetic field. So, it should be downwards toward the bottom.

30. (a) **B** at origin $\mathbf{B}_{KLM} = \mathbf{B}_{KNM}$

$$= \frac{\mu_0 I}{4R}(-\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

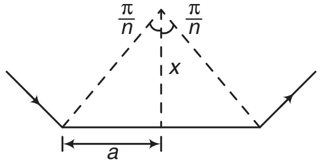
Now, we can apply $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ for finding force on it.

(b) In uniform magnetic field,

$$\begin{aligned}\mathbf{F}_{KLM} = \mathbf{F}_{KNM} &= \mathbf{F}_{KM}\\ &= I(\mathbf{l} \times \mathbf{B})\\ &= I[\{2R(-\hat{\mathbf{k}})\} \times \{B_0 \hat{\mathbf{j}}\}]\\ &= (2B_0 IR)\,\hat{\mathbf{i}}\end{aligned}$$

$\therefore$ Net force is two times of the above value.

31. (a)



$$a = \frac{1}{2}\left(\frac{2\pi r}{n}\right) = \frac{\pi r}{n}$$

$$x = a\cot\frac{\pi}{n} = \left(\frac{\pi r}{n}\right)\cot\left(\frac{\pi}{n}\right)$$

$$B = n\left[\frac{\mu_0}{4\pi}\,\frac{i}{x}\left(\sin\frac{\pi}{n} + \sin\frac{\pi}{n}\right)\right]$$

$$= n\left[\frac{\mu_0}{4\pi}\,\frac{i}{(\pi r/n)\cot\pi/n}(2\sin\pi/n)\right]$$

$$= \frac{\mu_0 i n^2 \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)}{2\pi^2 r}$$

(b) The above calculated magnetic field can be written as

$$B = \frac{\mu_0 i}{2r}\,\frac{\left(\dfrac{\sin\pi/n}{\pi/n}\right)^2}{\cos\pi/n}$$

As, $\qquad n \to \propto, \ \frac{\pi}{n} \to 0$

Hence, $\displaystyle\lim_{\pi/n\to 0}\left(\frac{\sin\pi/n}{\pi/n}\right) \to 1$

and $\qquad \displaystyle\lim_{\pi/n\to 0}(\cot\pi/n) \to 1$

or $\qquad B \to \frac{\mu_0 i}{2r}$

32. Current per unit area,

$$\sigma = \frac{I}{\pi a^2 - \pi(a/2)^2 - (a/2)^2}$$

$$= \frac{2I}{\pi a^2}$$

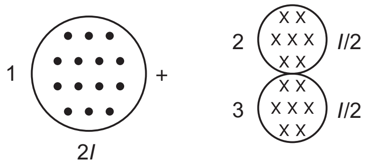
Total area is (πa^2). Therefore, the total current is

$$I_1 = (\sigma)(\pi a^2) = 2I$$

Cavity area is $\pi(a/2)^2$. Therefore, cavity current is

$$I_2 = (\sigma)(\pi a^2/4) = \frac{I}{2}$$

Now, the given current system can be assumed as shown below.



(a) **At P_1**, $\quad B_1 = \frac{\mu_0}{2\pi}\,\frac{2I}{5} = \frac{\mu_0 I}{\pi r}$ (towards left)

$B_2 = \frac{\mu_0}{2\pi}\,\frac{I/2}{(r - a/2)}$ (towards right)

and $\quad B_2 = \frac{\mu_0}{2\pi}\,\frac{I/2}{(r + a/2)}$ (towards right)

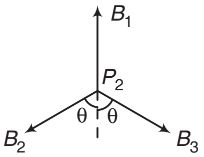
$\therefore \quad B_{net} = B_1 - B_2 - B_2$ (towards left)

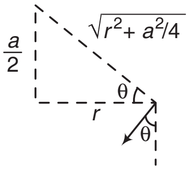
$$= \frac{\mu_0 I}{\pi}\left[\frac{1}{r} - \frac{1}{4r - 2a} - \frac{1}{4r + 2a}\right]$$

$$= \frac{\mu_0 I}{\pi}\left[\frac{16r^3 - 4a^2 - 4r^2 - 2ar - 4r^2 + 2ar}{r(16r^2 - 4a^2)}\right]$$

$$= \frac{\mu_0 I}{\pi r}\left[\frac{2r^2 - a^2}{4r^2 - a^2}\right]$$ (towards left)

(b)



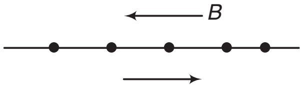


$$B_{\text{net}} = B_1 - 2B_2 \cos\theta] \qquad \text{(towards the top)}$$

$$= \frac{\mu_0}{2\pi}\frac{2I}{R} - 2\left[\frac{\mu_0}{2\pi}\frac{I/2}{\sqrt{r^2+a^2/4}}\right]\frac{r}{\sqrt{r^2+a^2/4}}$$

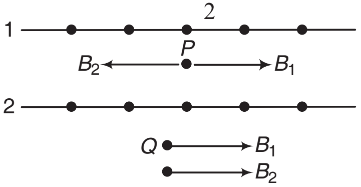
$$= \frac{\mu_0 I}{\pi r}\left[\frac{2r^2+a^2}{4r^2+a^2}\right] \qquad \text{(towards the top)}$$

33.



B in the above situation is given by

$$B = \frac{\mu_0 \lambda}{2}$$



At point P, B_1 and B_2 are in opposite directions.

Hence, $B_P = 0$

At point, Q, B_1 and B_2 are in same direction.

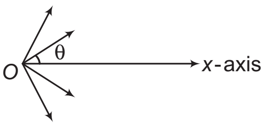
Hence, $B_Q = 2\left(\frac{\mu_0 \lambda}{2}\right) = \mu_0 \lambda$

34. $\frac{F}{L} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{a}$

$\therefore$ $F = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{a} L$ (Repulsion or upwards)

M of the loop is inwards and magnetic field to I_1 on the plane of loop is outwards. Hence, $\tau = 0$, as $\tau = \mathbf{M} \times \mathbf{B}$ and angle between two is 180°.

35.



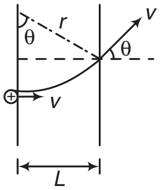
Electrons touch the x-axis again after every pitch. Therefore, the asked distance is

$$d = p = v_{11}T = (v\cos\theta)\left(\frac{2\pi m}{Bq}\right)$$

For paraxial electron $\theta \approx 0°$ and $q = e$,

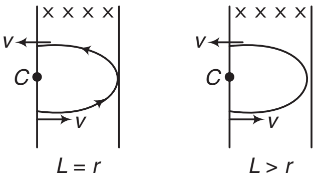
$\therefore$ $$d = \frac{2\pi m v}{Be}$$

36.



Deviation, $\theta = \sin^{-1}\left(\frac{L}{r}\right)$ for $L < r$

where, $r = \frac{mv}{Bq}$



and $\theta = \pi$ if $L \le r$

37. $a_E = \frac{qE_0}{m}$ (along negative z-direction)

Electric field will make z-component of velocity zero. At that time speed of the particle will be minimum and that minimum speed is the other component, i.e. v_0.

This is minimum when,

$$v_z = u_z + a_z t$$

or $$0 = v_0 - \frac{qE_0}{m}t$$

or $$t = \frac{mv_0}{qE_0}$$

38. Path is helix and after one rotation only x-coordinate will change by a distance equal to pitch.

$\therefore$ $$x = p = (v_0 \cos\theta)\left(\frac{2\pi m}{Bq}\right)$$

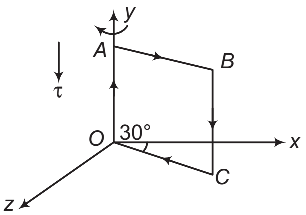
39. $\mathbf{M} = i(\mathbf{CO} \times \mathbf{OA})$

$= i(\mathbf{CO} \times \mathbf{CB})$

$= 4[(-0.1\,\hat{\mathbf{i}}) \times (0.2\cos 30°\,\hat{\mathbf{j}} + 0.2\sin 30°\,\hat{\mathbf{k}})$

$= (0.04\,\hat{\mathbf{j}} - 0.07\,\hat{\mathbf{k}})$ A-m^2

40.



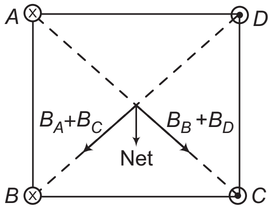
$$\mathbf{M} = Ni\,(\mathbf{OA} \times \mathbf{AB})$$
$$= Ni\,(\mathbf{OA} \times \mathbf{OC})$$
$$= (100)\,(1.2)\,[(0.4\,\hat{\mathbf{j}}) \times (0.03\cos 30^\circ\,\hat{\mathbf{i}} + 0.3\sin 30^\circ\,\hat{\mathbf{k}})]$$
$$= (7.2\hat{\mathbf{i}} - 12.47\hat{\mathbf{k}})\ \text{A-m}^2$$
$$\tau = \mathbf{M} \times \mathbf{B}$$
$$= [(7.2\,\hat{\mathbf{i}} - 12.47\,\hat{\mathbf{k}}) \times (0.8\,\hat{\mathbf{i}})]$$
$$= (-\,9.98\hat{\mathbf{j}})\ \ \text{N-m}$$

$\therefore \quad |\tau| = 9.98$ N-m

Torque vector and expected direction of rotation is shown in figure.

41. $B_A = B_B = B_C = B_D$

$$= B = \frac{\mu_0}{2\pi}\,\frac{i}{r} = \frac{(2 \times 10^{-7})\,(5)}{(0.2/\sqrt{2})}$$
$$= (5.0\sqrt{2}) \times 10^{-6}\ \text{T}$$



Net magnetic field

$$= \sqrt{(B_A + B_C)^2 + (B_B + B_D)^2}$$
$$= 2\sqrt{2}B$$
$$= 2.0 \times 10^{-5}\ \text{T} \quad \text{(towards bottom as shown)}$$

42. $i = \int_0^r (2\pi r\,dr)\,j = \int_0^r (2\pi r\,dr)\,(br) = \dfrac{2\pi\,br^3}{3}$

(a) **For $r_1 < R$**

$$B = \frac{\mu_0}{2\pi}\,\frac{i_{\text{in}}}{r_1}$$
$$= \frac{\mu_0}{2\pi}\left(\frac{2\pi b r_1^3/3}{r_1}\right)$$
$$= \frac{\mu_0 b r_1^2}{3}$$

(b) **For $r_2 > R$**

$$B = \frac{\mu_0}{2\pi}\,\frac{i_{\text{in}}}{r_2}$$
$$= \left(\frac{\mu_0}{2\pi}\right)\left(\frac{2\pi b R^3/b}{r_2}\right) = \frac{\mu_0 b R^3}{3r_2}$$

LEVEL 2

Single Correct Option

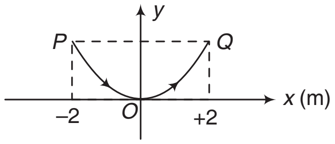
1. τ_{mg} about the left end (from where string is connected)

$$= |\mathbf{M} \times \mathbf{B}| = MB\sin 90^\circ$$

or $\quad (mgR) = (NiA)B_0 = i\,(\pi R^2)B_0$

or $\quad i = \dfrac{mg}{\pi R B_0}$

2. In uniform field,



Magnetic force on POQ = magnetic force on straight wire PQ having the same current. Hence,

$$\mathbf{F} = i\,(\mathbf{l} \times \mathbf{B}) = i\,(\mathbf{PQ} \times \mathbf{B})$$
$$= 2\,[(4\hat{\mathbf{i}}) \times (-\,0.02\,\hat{\mathbf{k}})]$$
$$= (0.16\,\hat{\mathbf{j}})$$

$\therefore \quad \mathbf{a} = \dfrac{\mathbf{F}}{m} = \dfrac{(0.16\,\hat{\mathbf{j}})}{0.1} = (1.6\,\hat{\mathbf{j}})\ \text{m/s}^2$

3. Linear impulse $= mv$

or $\quad F\Delta t = m\sqrt{2gh} \quad$ or $\quad (ilB)\,\Delta t = m\sqrt{2gh}$

But, $\quad i\,\Delta t = \Delta q$

$\therefore \quad (\Delta q)\,(lB) = m\sqrt{2gh}$

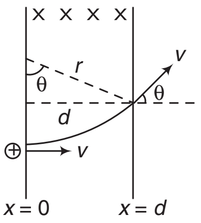
Hence, $\quad \Delta q = \dfrac{m\sqrt{2gh}}{Bl}$

4. $\dfrac{M}{L} = \dfrac{q}{2m}$

$\therefore \quad M = \left(\dfrac{q}{2m}\right)L = \left(\dfrac{q}{2m}\right)(I\omega)$

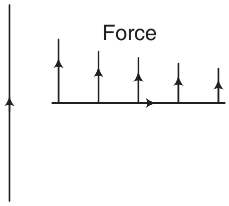
$$= \left(\frac{q}{2m}\right)\left(\frac{2}{5}\,mR^2\omega\right) = \frac{1}{5}\,qR^2\omega$$

5.



$$\sin\theta = \frac{d}{r} = \frac{d}{(mv/Bq)} \quad \Rightarrow \quad \frac{q}{m} = \frac{v\sin\theta}{Bd}$$

6.



Variation of magnetic force on wire *ACB* is as shown in figure. Point of application of net force lies some where between *A* and *C*.

7. At a distance X from current I_2,

$$B = \frac{\mu_0}{2\pi}\frac{I_2}{X}$$

Magnetic force of small element dX of wire AB

$dF = I_2\,(dX)B \sin 90°$

$$\therefore \qquad F = \int_{x=a}^{x=2a} dF$$

8. Apply screw law for finding magnetic field around a straight current carrying wire.

9. $B_1 = \dfrac{\mu_0 IR^3}{2\,(R^2 + x^2)^{3/2}}$

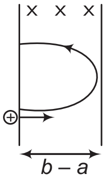
$B_2 = \dfrac{\mu_0 I(2R)^2}{2\,[(2R)^2 + (2x)^2]^{3/2}}$

$\dfrac{B_1}{B_2} = 2$

10. $(b - a) =$ radius of circular part

$$= \frac{mv}{Bq}$$

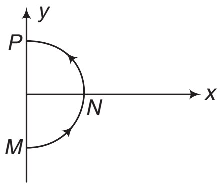
$$\therefore \qquad V = \frac{Bq\,(b-a)}{m}$$



11. $\dfrac{M}{L} = \dfrac{q}{2m}$

$$M = \left(\frac{q}{2m}\right)(I\omega) = \left(\frac{q}{2m}\right)\left(\frac{ml^2}{3}\right)(2\pi f) = \frac{\pi q f l^2}{3}$$

12. At $x = 0$,



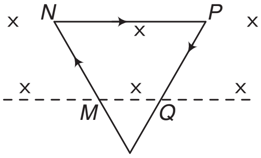
$y = \pm\, 2m$

$\mathbf{F}_{\text{MNP}} = \mathbf{F}_{\text{MP}} = i\,[\mathbf{MP} \times \mathbf{B}\,]$

$= 3\,[(4\,\hat{\mathbf{j}}) \times (5\hat{\mathbf{k}})]$

$= 60\,\hat{\mathbf{i}}$

13. $\mathbf{F}_{\text{MNPQ}} = \mathbf{F}_{\text{MQ}}$ and this force should be upwards to balance the weight.



$\therefore \qquad ilB = mg,$ where

$l = MQ = \dfrac{a}{2}$

$\therefore \qquad i\,(a/2)\,B = mg$ or $i = \dfrac{2Mg}{aB}$

Force is upwards if current is clockwise or current in *MQ* is towards right.

14. $\dfrac{r}{2} =$ radius of circular path

$$= \frac{mv}{Bq} = \frac{mv}{(\mu_0 ni)\,(q)}$$

$$\therefore \qquad v = \frac{\mu_0 qr\,ni}{2m}$$

15. (a) $\mathbf{B} \perp \mathbf{v}$, so it may along y- axis

(b) $\mathbf{F} \perp \mathbf{v}$,

$\therefore \qquad \mathbf{a} \perp \mathbf{v} = 0$

or $\qquad \mathbf{a} \cdot \mathbf{v} = 0$

or $\qquad a_1 b_1 = a_2 b_2$

(c) See the logic of option (a).

(d) Magnetic force cannot change the kinetic energy of a particle.

16. Magnetic force is always perpendicular to velocity. So, it will always act in radial direction which will change tension at different points. But, time period and θ will remain unchanged.

17. Force on the wires parallel to x-axis will be obtained by integration (as $B \propto x$ and x coordinates vary along these wires). But on a loop there are two such wires. Force on them will be equal and opposite.

Forces on two wires parallel to y-axis can be obtained directly (without integration) as value of B is same along these wires. But their values will be different (as x-coordinate and therefore B is different).

$F_{\text{net}} = \Delta F \qquad$ (on two wires)

$= Ia\,(\Delta B) = Ia\,(B_0)\,(\Delta x)$

$= Ia\,B_0(a) = IB_0 a^2$

This is indecent of x

$\therefore \qquad F_1 = F_2 = IBla^2 \neq 0$

18. $B_x = \dfrac{\mu_0}{2\pi}\dfrac{I_{\text{in}}}{x}$

where, $I_{\text{in}} = I - \left[\dfrac{I}{\pi (c^2 - b^2)}\right][\pi (x^2 - b^2)]$

$$= \frac{I (c^2 - x^2)}{(c^2 - b^2)}$$

19. $\mathbf{F}_e = q\mathbf{E} = (1.6 \times 10^{-19}) (-102.4 \times 10^3)\, \hat{\mathbf{k}}$

$$= (-1.6384 \times 10^{-14}\, \hat{\mathbf{k}})\text{N}$$

$$\mathbf{F}_m = q\, (\mathbf{v} \times \mathbf{B})$$

$$= (1.6 \times 10^{-19})[(1.28 \times 10^6 \hat{\mathbf{i}}) \times (8 \times 10^{-2} \hat{\mathbf{j}})]$$

$$= (1.6384 \times 10^{-14} \hat{\mathbf{k}})\text{N}$$

Now, we can see that

$$\mathbf{F}_e + \mathbf{F}_m = 0$$

20. Due to $\hat{\mathbf{j}}$ component of **B**, magnetic force is zero. It is only due to $\hat{\mathbf{i}}$ component.

l is towards $-y$ direction, **B** is towards $\hat{\mathbf{i}}$ direction. Hence, $\mathbf{l} \times \mathbf{B}$ (the direction of magnetic force is $+\hat{\mathbf{k}}$)

$$F = \int dF = \int_0^1 (i)(dy)\,(B)$$

$$= \int_0^1 (2 \times 10^{-3})\,(dy)\,(0.3\, y)$$

$$= 3 \times 10^{-4}\ \text{N}$$

21. $r = \dfrac{mv}{Bq}$ (during circular path)

$$\therefore \quad v = \frac{Bqr}{m}$$

Now, $qE = Bqv$

$$\therefore \quad E = Bv = \frac{B^2 qr}{m}$$

$$\therefore \quad E = \frac{(0.1)^2\, (20 \times 10^{-6})\, (5 \times 10^{-2})}{(20 \times 10^{-9})}$$

$$= 0.5\ \text{V/m}$$

22. $B_1 = \dfrac{\mu_0 i_1}{2R_1} = \dfrac{(4\pi \times 10^{-7})\,(5)}{(2)\left(\dfrac{5}{\sqrt{2}} \times 10^{-2}\right)}$

$$= 2\sqrt{2}\pi \times 10^{-5}\,\text{T}$$

$$B_2 = \frac{\mu_0 i_2}{2R_2} = \frac{(4\pi \times 10^{-7})\,(5\sqrt{2})}{(2)\,(5 \times 10^{-2})}$$

$$= 2\sqrt{2}\pi \times 10^{-5}\,\text{T}$$

$$B_{\text{net}} = \sqrt{B_1^2 + B_2^2}$$

23. $qE = Bqv$

$$\therefore \quad v = \frac{E}{B}$$

$$r = \frac{mv}{Bq} = \frac{m\,(E/B)}{B(q)}$$

$$= \frac{E}{B^2 S} \qquad \text{(where, } S = q/m\text{)}$$

24. $\dfrac{F}{l} =$ weight per unit length

$$\therefore \quad \frac{\mu_0}{2\pi}\frac{i_1 i_2}{r} = 0.01 \times 10$$

or $\dfrac{(2 \times 10^{-7})\,(100 \times 50)}{r} = 0.01 \times 10$

or $r = 0.1$ m

When B wire is displaced downwards from equilibrium position, magnetic attraction from A wire will decrease (which is upwards). But, weight (which is downwards). So, net force is downwards, in the direction of displacement from the mean position or away from the mean position. Hence, equilibrium is unstable.

25. Magnetic field due to current in the wire along z-axis is zero. Magnetic field due to wire along x-axis is along $\hat{\mathbf{j}}$ direction and magnetic field due to wire along y-axis is along $-\hat{\mathbf{i}}$ direction. Both wires produce

$$B = \frac{\mu_0}{2\pi}\frac{i}{a}$$

26. $|\,\mathbf{F}_{\text{ABC}}| + = |\mathbf{F}_{\text{AC}}| = ilB$

$$= (2)\,(5)\,(2) = 20\ \text{N}$$

27. $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qx}{(R^2 + x^2)^{3/2}}$

$$B = \frac{\mu_0 i R^2}{2\,(R^2 + x^2)^{3/2}}$$

where, $i = qf = q\left(\dfrac{V}{2\pi R}\right)$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

More than One Correct Options

1. $B_1 = \dfrac{\mu_0 N_1 i_1}{2R_1}$

$$= \frac{(4\pi \times 10^{-7})\,(50)\,(2)}{2\,(5 \times 10^{-2})}$$

$$= 4\pi \times 10^{-4}\ \text{T}$$

$$B_2 = \frac{\mu_0 N_2 i_2}{2R_2}$$

$$= \frac{(4\pi \times 10^{-7})\,(100)\,(2)}{(2)\,(10 \times 10^{-2})}$$

$$= 4\pi \times 10^{-4}\,\text{T}$$

When currents are in the same direction, then

$$B_{\text{net}} = B_1 + B_2$$

When currents are in the opposite directions, then

$$B_{\text{net}} = B_1 - B_2$$

2. (a) $\mathbf{v}$ is parallel on anti-parallel to $\mathbf{B}$.

(c) $q\mathbf{E} + q\,(\mathbf{v} \times \mathbf{B}) = 0$

or $\quad \mathbf{E} = -\,(\mathbf{v} \times \mathbf{B}) = (\mathbf{B} \times \mathbf{v})$

3. $r = \dfrac{mV}{Bq} = \dfrac{(1)\,(10)}{(2)\,(1)} = 5\text{ m}$

$T = \dfrac{2\pi m}{Bq} = \dfrac{(2)\,(\pi)\,(1)}{(2)\,(1)} = \pi$

$= 3.14\text{ s}$

Plane of circle is perpendicular to $\mathbf{B}$, i.e. xy-plane.

4. $\theta = 180°$

$\tau = MB \sin\theta = 0$

$U = -\,MB \cos\theta = +\,MB = \text{maximum}$

5. If current flows in a conductor, then

$E \neq 0 \qquad$ (for inside points)

$E = 0 \qquad$ (for outside points)

$B = \dfrac{\mu_0}{2\pi}\dfrac{i}{R^2}\,r \qquad$ (for inside points)

$B = 0$ at $r = 0$, i.e. at centre

$B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$ for outside points.

6. $\mathbf{F}_{ab}$ = upwards

$\mathbf{F}_{abc}$ = leftwards

∴ Net force on loop is neither purely leftwards or rightwards or upwards or downwards.

7. $\mathbf{F}_m = q\,(\mathbf{v} \times \mathbf{B})$

Depending on sign of q, $\mathbf{F}_m$ may be along positive z-axis or along negative z-axis.

$$\mathbf{F}_e = q\mathbf{E}$$

Again, depending on the value of q it may be along positive z-axis or along negative z-axis.

If q is positive, $\mathbf{v} \times \mathbf{B}$ and $\mathbf{F}_m$ comes along negative z-axis also. But, $\mathbf{F}_e$ comes along positive z-axis. So, it may also pass undeflected.

8. $\text{KE} = qV \quad$ or $\quad \text{KE} \propto V$

$r = \dfrac{\sqrt{2qVm}}{Bq} \quad$ or $\quad r \propto \sqrt{V}$

$T = \dfrac{2\pi m}{Bq} \quad$ or T is independent of V.

9. (a) Point a lies to the right hand side of ef and fg. Hence, both wires produce inward magnetic field. Hence, net magnetic field is inwards. Same logic can be applied for other points also.

10. See the hint of Q.No-3 of Assertion & Reason section for Level 1

Match the Columns

1. (a) $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

q is negative, $\mathbf{v}$ is along $+\,\hat{\mathbf{i}}$ and $\mathbf{B}$ along $+\,\hat{\mathbf{j}}$.

Therefore, $\mathbf{F}$ is along negative z.

(b) Same logic is given in (a).

(c) $\mathbf{B}$ is parallel to $\mathbf{v}$. So, magnetic force is zero. Charge is negative so, electric force is opposite to $\mathbf{E}$.

(d) Charge is negative. So, electrostate force is in opposite direction of $\mathbf{E}$.

2. For direction of magnetic force apply Fleming's left hand rule. According to that w and x are positively charged particles and y and z negatively charged particle.

Secondly,

$$r = \frac{\sqrt{2Km}}{Bq}$$

$$\therefore \qquad r \propto \frac{\sqrt{m}}{q} \qquad (K \rightarrow \text{same})$$

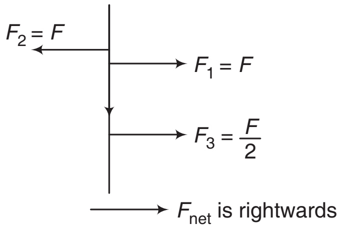
3. Force on a current carrying loop is zero for all angles. τ is maximum when Q, then angle between $\mathbf{M}$ and $\mathbf{B}$ is 90° minimum potential energy is at $\theta = 0°$. Positive potential energy is for obtuse angle. Direction of $\mathbf{M}$ is obtained by screw law.

4. Two currents are lying in the plane of paper. Its point is lying to the right hand side of the current carrying wire, magnetic field is inward ($-\,\hat{\mathbf{k}}$ direction).

If point lies to left hand side, field is outward ($\hat{\mathbf{k}}$ direction).

5. Let us take $F = \dfrac{\mu_0}{2\pi}\cdot\dfrac{i.\,i}{r}$

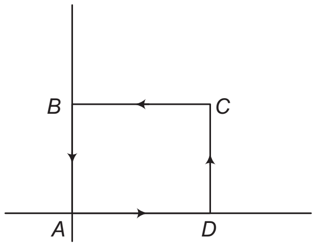
Current in same direction means attraction and current in opposite direction means repulsion. Let us find force on wire-2 by other three wires 1,3 and 4.



6. $\mathbf{F}_{ABC} = \mathbf{F}_{ADC} = \mathbf{F}_{AC}$

$$\therefore \quad \mathbf{F}_{\text{loop}} = 2\,\mathbf{F}_{AC} = \left[\frac{I}{2}\,(\mathbf{l} \times \mathbf{B})\right] = I\,(\mathbf{AC} \times \mathbf{B})$$

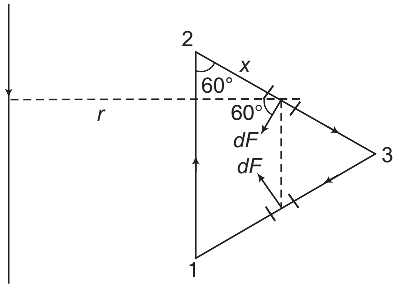
Hence, $\quad \mathbf{AC} = (l\hat{\mathbf{i}} + l\hat{\mathbf{j}})$



Now, putting value of **B** we can find net force in different cases.

Subjective Questions

1. Force on wire 12 will be $\left(\frac{\mu_0}{2\pi}\frac{I}{a}\right)(Ia)$ in positive x-direction.



Force on 2-3 and 3-1 :

$$r = a + x\sin 60° = a + \sqrt{3}\frac{x}{2}$$

$$B_r = \frac{\mu_0}{2\pi}\frac{I}{r} = \frac{\mu_0}{2\pi}\frac{I}{a + \sqrt{3}\frac{x}{2}}$$

$$= \frac{\mu_0}{\pi}\frac{I}{2a + \sqrt{3}x}$$

$$\therefore \quad \mathbf{F}_{231} = \left[2\int_{x=0}^{x=a}(dF\,\cos 60°)\right](-\hat{\mathbf{i}})$$

$$= \left[2\int_0^a \frac{\mu_0}{\pi}\cdot\frac{I}{2a+\sqrt{3}x}.I.(dx)\frac{1}{2}\right](-\hat{\mathbf{i}})$$

$$= \left(\frac{\mu_0 I^2}{\pi}\right).\frac{1}{\sqrt{3}}[\ln\,(2a + \sqrt{3}x)]_0^a\,(-\hat{\mathbf{i}})$$

$$= \frac{\mu_0 I^2}{\sqrt{3}\pi}\ln\left(\frac{2+\sqrt{3}}{2}\right)(-\hat{\mathbf{i}})$$

and $\quad \mathbf{F}_{12} = \frac{\mu_0 I^2}{2\pi}\,(\hat{\mathbf{i}})$

$$\therefore \quad \mathbf{F}_{\text{Total}} = \frac{\mu_0 I^2}{\pi}\left[\frac{1}{2} - \frac{1}{\sqrt{3}}\ln\left(\frac{2+\sqrt{3}}{2}\right)\right](\hat{\mathbf{i}}) \qquad \textbf{Ans.}$$

2. **Magnetic moment** According to Bohr's hypothesis, angular momentum in nth orbit is $L = n\left(\frac{h}{2\pi}\right)$.

Further, $\quad \frac{M}{L} = \frac{q}{2m} \quad$ or $\quad \frac{e}{2m}$

$\therefore$ Magnetic moment,

$$M = \left(\frac{e}{2m}\right)(L) = \left(\frac{e}{2m}\right)\left(n\frac{h}{2\pi}\right) = \left(\frac{neh}{4\pi m}\right)$$

Magnetic field induction

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(e)(e)}{r^2} \qquad \ldots\text{(i)}$$

From Bohr's hypothesis :

$$mvr = \frac{nh}{2\pi} \qquad \ldots\text{(ii)}$$

Solving these two equations, we find

$$v = \frac{e^2}{2\varepsilon_0 nh} \quad \text{and} \quad r = \frac{\varepsilon_0 n^2 h^2}{\pi m e^2}$$

Now, magnetic field induction at centre

$$B = \frac{\mu_0 i}{2r}$$

Here, $\quad i = qf = (e)\left(\frac{v}{2\pi r}\right)$

$$\therefore \quad B = \left(\frac{\mu_0}{2r}\right)\left(\frac{ev}{2\pi r}\right) = \frac{\mu_0 e}{4\pi}\left(\frac{v}{r^2}\right)$$

$$= \left(\frac{\mu_0 e}{4\pi}\right)\left(\frac{e^2}{2\varepsilon_0 nh}\right)\left(\frac{\pi^2 m^2 e^4}{\varepsilon_0^2 n^4 h^4}\right)$$

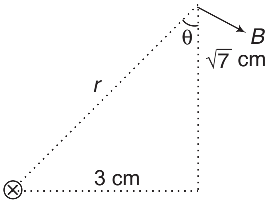
$$= \frac{\mu_0 \pi m^2 e^7}{8\varepsilon_0^3 h^5 n^5} \qquad \textbf{Ans.}$$

3. $r = \sqrt{9+7} = 4$ cm

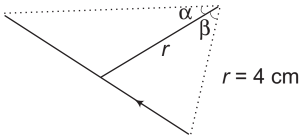
$$B = \frac{\mu_0}{4\pi}\frac{i}{r}(2\sin\alpha)$$

$$= (10^{-7})\left(\frac{30}{4\times10^{-2}}\right)\left(2\times\frac{3}{5}\right)$$

$$= 9.0\times10^{-5}\text{ T}$$



(a)



(b)

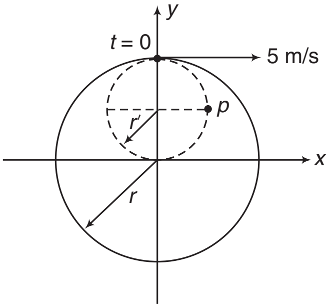
$$\text{Net field} = 4B\sin\theta$$

$$= 4\times9.0\times10^{-5}\times\frac{3}{4}$$

$$= 2.7\times10^{-4}\text{ T} \qquad \textbf{Ans.}$$

4. ***xy*-plane** $T = \dfrac{2\pi r}{v} = \dfrac{2\pi(0.4)}{5} = \dfrac{8}{50}\pi$

Since, $T = \dfrac{2\pi m}{Bq}$ or $T \propto \dfrac{m}{q}$



After collision mass has become $\dfrac{5}{4}$ times and charge two times.

$$\therefore \quad T' = \left(\frac{5}{4}\times\frac{1}{2}\right)T = \frac{5}{8}\times\frac{8}{50}\pi = \frac{\pi}{10}\text{ s}$$

Given time $t = \dfrac{T'}{4}$, i.e. combined mass will complete one-quarter circle.

Further $\qquad r = \dfrac{P}{Bq}$

or $\qquad r \propto \dfrac{1}{q} \qquad$ (as P = constant)

Since, charge has become two times

$$\therefore \quad r' = \frac{r}{2} = 0.2\text{ m}$$

At $t = (\pi/40)$ second, particle will be at P in xy-plane.

$$\therefore \quad x = r' = 0.2\text{ m}$$

$$y = r' = 0.2\text{ m} \qquad \textbf{Ans.}$$

***z*-coordinate** Mass of combined body has become 5 times of the colliding particle. Therefore, from conservation of linear momentum, velocity component in z-direction will become $\dfrac{1}{5}$ times. Or

$$v_z = \frac{1}{5}\times\frac{40}{\pi}\text{ m/s} = \frac{8}{\pi}\text{ m/s}$$

$$\therefore \quad z = v_z t = \frac{8}{\pi}\times\frac{\pi}{40} = 0.2\text{ m} \qquad \textbf{Ans.}$$

5. $F_e = F_m$ or $eE = eBv$

$$\therefore \quad v = \frac{E}{B} = \frac{120\times10^3}{50\times10^{-3}}$$

$$= 2.4\times10^6\text{ m/s}$$

Let n be the number of protons striking per second. Then,

$$ne = 0.8\times10^{-3}$$

or $$n = \frac{0.8\times10^{-3}}{1.6\times10^{-19}}$$

$$= 5\times10^{15}\text{ m/s}$$

Force imparted = Rate of change of momentum

$$= nmv$$

$$= 5\times10^{15}\times1.67\times10^{-27}\times2.4\times10^6$$

$$= 2.0\times10^{-5}\text{ N} \qquad \textbf{Ans.}$$

6. (a) Speed of particle at origin, $v = \sqrt{\dfrac{2qV}{m}} = v_x$

$$t = \frac{x}{v_x} = \frac{a}{\sqrt{\frac{2qV}{m}}} = a\sqrt{\frac{m}{2qV}}$$

$$y = \frac{1}{2}a_y t^2 = \frac{1}{2}\left[\frac{qE}{m}\right]\left[a^2\times\frac{m}{2qV}\right] = \frac{a^2E}{4V} \qquad \textbf{Ans.}$$

(b) Component parallel to **B** is

$$v_y = a_y t = \left(\frac{qE}{m}\right)\left(a\sqrt{\frac{m}{2qV}}\right) = Ea\sqrt{\frac{q}{2mV}}$$

Now, pitch = component parallel to **B** × time period

$$= v_y T = \left(Ea\sqrt{\frac{q}{2mV}}\right)\left(\frac{2\pi m}{Bq}\right)$$

$$= \frac{\pi Ea}{B}\sqrt{\frac{2m}{qV}} \qquad \textbf{Ans.}$$

7. **To graze at *C*** Using equation of trajectory of parabola,

$$y = x \tan\theta - \frac{ax^2}{2v^2\cos^2\theta} \qquad \ldots\text{(i)}$$

Her , $a = \frac{qE}{m} = \frac{10^{-6}\times 10^{-3}}{10^{-10}}$

$$= 10 \text{ m/s}^2$$

Substituting in Eq. (i), we have

$$0.05 = 0.17 \tan 30° - \frac{10\times(0.17)^2}{2v^2\times(\sqrt{3}/2)^2}$$

Solving this equation, we have

$$v = 2 \text{ m/s}$$

In magnetic field, $AC = 2r$

or $0.1 = 2r$

or $r = 0.05 \text{ m} = \frac{mv\cos 30°}{Bq}$

$\therefore$ $B = \frac{mv\cos 30°}{(0.05)q}$

$$= \frac{(10^{-10})(2)(\sqrt{3}/2)}{(0.05)(10^{-6})}$$

$$= 3.46\times 10^{-3} \text{ T}$$

$$= 3.46 \text{ mT} \qquad \textbf{Ans.}$$

8. Magnetic moment of the loop, $\mathbf{M} = (iA)\hat{\mathbf{j}}$

$$= (I_0L^2)\hat{\mathbf{k}}$$

Magnetic field, $\mathbf{B} = (B\cos 45°)\,\hat{\mathbf{i}} + (B\sin 45°)\hat{\mathbf{j}}$

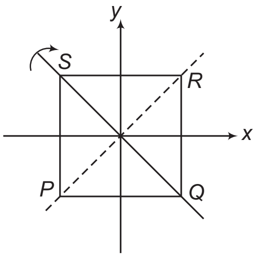
$$= \frac{B}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$$

(a) Torque acting on the loop, $\tau = \mathbf{M}\times\mathbf{B}$

$$= (I_0L^2\hat{\mathbf{k}})\times\left[\frac{B}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})\right]$$

$$\therefore \quad \tau = \frac{I_0L^2B}{\sqrt{2}}(\hat{\mathbf{j}}-\hat{\mathbf{i}}) \quad \text{or} \quad |\tau| = I_0L^2B \qquad \textbf{Ans.}$$

(b) Axis of rotation coincides with the torque and since torque is along $\hat{\mathbf{j}}-\hat{\mathbf{i}}$ direction or parallel to QS. Therefore, the loop will rotate about an axis passing through Q and S as shown in the figure.



Angular acceleration, $\alpha = \frac{|\tau|}{I}$

where, I = moment of inertia of loop about QS.

$$I_{QS} = I_{PR} = I_{ZZ}$$

(From the theorem of perpendicular axis)

But, $I_{QS} = I_{PR}$

$\therefore$ $2I_{QS} = I_{ZZ} = \frac{4}{3}ML^2$

$$I_{QS} = \frac{2}{3}ML^2$$

$$\therefore \quad \alpha = \frac{|\tau|}{I} = \frac{I_0L^2B}{2/3ML^2} = \frac{3}{2}\frac{I_0B}{M}$$

$\therefore$ Angle by which the frame rotates in time Δt is

$$\theta = \frac{1}{2}\alpha(\Delta t)^2$$

or $\theta = \frac{3}{4}\frac{I_0B}{M}\,.\,(\Delta t)^2$ **Ans.**

9. In equilibrium,

$$2T_0 = mg \quad \text{or} \quad T_0 = \frac{mg}{2} \qquad \ldots\text{(i)}$$

Magnetic moment, $M = iA = \left(\frac{\omega}{2\pi}Q\right)(\pi R^2)$

$$\tau = MB\sin 90° = \frac{\omega BQR^2}{2}$$

Let T_1 and T_2 be the tensions in the two strings when magnetic field is switched on ($T_1 > T_2$).

For translational equilibrium of ring in vertical direction,

$$T_1 + T_2 = mg \qquad \ldots\text{(ii)}$$

For rotational equilibrium,

$$(T_1 - T_2)\frac{D}{2} = \tau = \frac{\omega BQR^2}{2}$$

or $T_1 - T_2 = \frac{\omega BQR^2}{2}$ …(iii)

Solving Eqs. (ii) and (iii), we have

$$T_1 = \frac{mg}{2} + \frac{\omega BQR^2}{2D}$$

As $T_1 > T_2$ and maximum values of T_1 can be $\frac{3T_0}{2}$,

We have

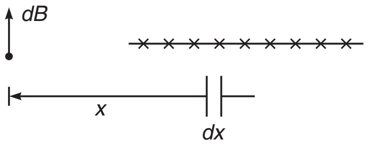
$$\frac{3T_0}{2} = T_0 + \frac{\omega_{\max} BQR^2}{2D}$$

$$\therefore \qquad \omega_{\max} = \frac{DT_0}{BQR^2} \qquad \textbf{Ans.}$$

10. $dB = \frac{\mu_0}{2\pi}\frac{(i/\omega).\,dx}{x}$

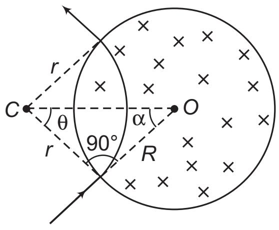
$$\therefore \qquad B = \frac{\mu_0 i}{2\pi\omega}\int_d^{d+\omega}\frac{dx}{x}$$

$$B = \frac{\mu_0 i}{2\pi\omega}\ln\left(\frac{d+\omega}{d}\right) \quad \text{(upwards)} \quad \textbf{Ans.}$$



11. $\theta = \tan^{-1}\left(\frac{R}{r}\right) = \tan^{-1}\left(\frac{R}{mv/Bq}\right) = \tan^{-1}\left(\frac{BqR}{mv}\right)$

Deviation $= 2\theta = 2\tan^{-1}\left(\frac{BqR}{mv}\right)$



12. (a) Yes, magnetic force for calculation of torque can be assumed at centre. Since, variation of torque about P from one end of the rod to the other end comes out to be linear.

$$\therefore \qquad \tau = (IlB)\left(\frac{l}{2}\right) = \frac{Il^2B}{2}$$

$$= \frac{(6.5)(0.2)^2(0.34)}{2}$$

$$= 0.0442 \text{ N-m} \qquad \textbf{Ans.}$$

(b) Magnetic torque on rod will come out to be clockwise. Therefore, torque of spring force should be anti-clockwise or spring should be stretched.

(c) In equilibrium,

Clockwise torque of magnetic force

= anti-clockwise torque of spring force

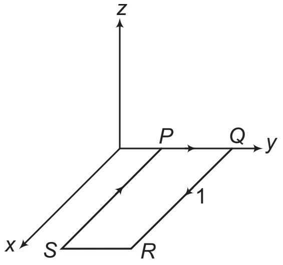
$$\therefore\ 0.0442 = (kx)(l\sin 53°) = (4.8)(x)(0.2)\left(\frac{4}{5}\right)$$

or $\qquad x = 0.057$ m

$$U = \frac{1}{2}kx^2 = \frac{1}{2}\times(4.8)(0.057)^2$$

$$= 7.8\times10^{-3}\text{J} \qquad \textbf{Ans.}$$

13. Let the direction of current in wire PQ is from P to Q and its magnitude be I.



The magnetic moment of the given loop is

$$\mathbf{M} = -Iab\hat{\mathbf{k}}$$

Torque on the loop due to magnetic force is

$$\tau_1 = \mathbf{M}\times\mathbf{B}$$

$$= (-Iab\hat{\mathbf{k}})\times(3\hat{\mathbf{i}} + 4\hat{\mathbf{k}})B_0\hat{\mathbf{i}}$$

$$= -3IabB_0\hat{\mathbf{j}}$$

Torque on weight of the loop about axis PQ is

$$\tau_2 = \mathbf{r}\times\mathbf{F} = \left(\frac{a}{2}\hat{\mathbf{i}}\right)\times(-mg\hat{\mathbf{k}})$$

$$= \frac{mga}{2}\hat{\mathbf{j}}$$

We see that when the current in the wire PQ is from P to Q, τ_1 and τ_2 are in opposite directions, so they can cancel each other and the loop may remain in equilibrium. So, the direction of current I in wire PQ is from P to Q. Further for equilibrium of the loop

$$|\tau_1| = |\tau_2|$$

or $$3IabB_0 = \frac{mga}{2}$$

$$I = \frac{mg}{6bB_0} \qquad \textbf{Ans.}$$

Magnetic force on wire RS is

$$\mathbf{F} = I(\mathbf{l}\times\mathbf{B})$$

$$= I[(-b\hat{\mathbf{j}})\times\{(3\hat{\mathbf{i}} + 4\hat{\mathbf{k}})B_0\}]$$

$$\mathbf{F} = IbB_0(3\hat{\mathbf{k}} - 4\hat{\mathbf{i}}) \qquad \textbf{Ans.}$$

27. Electromagnetic Induction

INTRODUCTORY EXERCISE 27.1

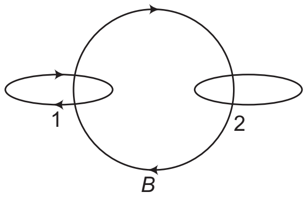
1. ⊗ magnetic field passing through loop is increasing. Hence, induced current will produce magnetic field. So, induced current should be anti-clockwise.

2. It is true that magnetic flux passing through the loop is calculated by integration. But, it remains constant.

3. $$\left|\frac{d\phi_B}{dt}\right| = [\text{Potential or EMF}]$$
$$= [\text{ML}^2\text{A}^{-1}\ \text{T}^{-3}]$$

4. ⊙ is increasing. Hence, ⊗ is produced by the induced current. So, it is clockwise.

5. By increasing the current in loop-1, magnetic field in ring-2 in downward direction will increase. Hence, induced current in ring-2 should produce upward magnetic field. Or current in ring should be in the same direction.



6. $2\pi R = 4L$

$$\therefore \quad L = \frac{\pi R}{2} = \frac{(\pi)(10)}{2}$$
$$= (5\pi)\ \text{cm}$$
$$\Delta S = S_i - S_f = (\pi R^2) - L^2$$
$$= \pi\ (0.1)^2 - (5\pi)^2 \times 10^{-4}$$
$$= 0.0067\ \text{m}^2$$
$$e = \frac{\Delta\phi}{\Delta t} = B\left(\frac{\Delta S}{\Delta t}\right)$$
$$= \frac{100 \times 0.0067}{0.1}$$
$$= 6.7\ \text{V}$$

7. $\Delta\phi = 2\,(NBS)$

$$\Delta q = \frac{\Delta\phi}{R} = \frac{2NBS}{R}$$
$$= \frac{2 \times 500 \times 0.2 \times 4 \times 10^{-4}}{50}$$
$$= 1.6 \times 10^{-3}\text{C}$$

8. $\mathbf{S} = [(5 \times 10^{-4})\,\hat{\mathbf{k}}]\ \text{m}^2$

$$\phi = |\mathbf{B} \cdot \mathbf{S}| = 9 \times 10^{-7}\ \text{Wb}$$

INTRODUCTORY EXERCISE 27.2

1. $\because \quad e = Bvl = 1.1 \times 5 \times 0.8$
$$= 4.4\ \text{V}$$

Apply right hand rule for polarity of this emf.

2. $\because \quad e = Bvl$
$$i = \frac{e}{R} = \frac{Bvl}{R}$$
$$F = ilB = \frac{B^2l^2v}{R}$$
$$= \frac{(0.15)^2\ (0.5)^2\ (2)}{3}$$
$$= 0.00375\ \text{N}$$

3. $$V_A - V_C = \frac{B\omega l^2}{2}$$
$$V_D - V_C = \frac{B\omega(2l)^2}{2}$$

From these two equations, we find
$$V_A - V_D = -3B\omega l^2/2$$

4. Circuit is not closed. So, current is zero or magnetic force is zero.

INTRODUCTORY EXERCISE 27.3

1. $|e| = \left|L\frac{\Delta i}{\Delta t}\right| \quad \text{or} \quad \left|L\frac{di}{dt}\right|$

Here, $\quad L = 1\text{H}$

and $\quad \frac{di}{dt} = 3\,[\sin t + t\cos t]$

$\therefore \quad |e| = 3\,(t\cos t + \sin t)$

2. $V_L = +L\frac{di}{dt} = (2)\frac{d}{dt}(10e^{-4t})$
$$= -80\,e^{-4t}$$

Further, $V_a - iR - L\frac{di}{dt} = V_b$

$$\therefore \quad V_a - V_b = iR + L\frac{di}{dt}$$

or $$V_{ab} = (10e^{-4t})\,(4) - 80e^{-4t}$$
$$= -40e^{-4t}$$

3. (a) $dI/dt = 16\,\text{A/s}$

$$\therefore \quad L = \left|\frac{e}{dI/dt}\right| = \frac{10\times10^{-3}}{16}$$

$$= 0.625\times10^{-3}\ \text{H} = 0.625\ \text{mH}$$

(b) At $t = 1\,\text{s}$, $I = 21$ A

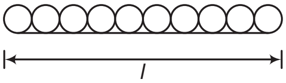
$$U = \frac{1}{2}LI^2 = \frac{1}{2}\times(0.625\times10^{-3})(21)^2$$

$$= 0.137\ \text{J}$$

$$P = Ei = (10\times10^{-3})\,(21)$$

$$= 0.21\ \text{J/s}$$

4. (a)



$$L = \frac{\mu_0 N^2 S}{l}$$

$$N = \frac{l}{d}$$

$$\therefore \quad L = \frac{\mu_0 l S}{d^2} = \frac{(4\pi\times10^{-7})\,(0.4)\,(0.9\times10^{-4})}{(0.1\times10^{-2})^2}$$

$$= 4.5\times10^{-5}\ \text{H}$$

(b) $e = \left|L\dfrac{\Delta i}{\Delta t}\right| = \dfrac{(4.5\times10^{-5})\,(10)}{0.1} = 4.5\times10^{-3}$ V

INTRODUCTORY EXERCISE 27.4

1. $\because \quad M = \dfrac{e_2}{di_1/dt} = \dfrac{(50\times10^{-3})}{(8/0.5)}$

$$= 3.125\times10^{-3}\ \text{H} = 3.125\ \text{mH}$$

$$e_1 = M\left(\frac{\Delta i_2}{\Delta t}\right) = \frac{(3.125\times10^{-3})\,(6)}{0.02}$$

$$= 0.9375\ \text{V}$$

2. (a) $M = \dfrac{N_2\phi_2}{i_1} = \dfrac{(1000)\,(6\times10^{-23})}{3} = 2$ H

(b) $|e_2| = M\left(\dfrac{\Delta i_1}{\Delta t}\right) = \dfrac{(2)\,(3)}{0.2} = 30$ V

(c) $L_1 = \dfrac{N_1\phi_1}{i_1} = \dfrac{(600)\,(5\times10^{-3})}{3} = 1$H

3. (a) $|e_2| = M\left(\dfrac{di_1}{dt}\right)$

$$= (3.24\times10^{-4})\,(830)$$

$$= 0.27\ \text{V}$$

(b) Result will remain same.

INTRODUCTORY EXERCISE 27.5

1. $\because \quad E = \dfrac{1}{2}Li^2 = i^2Rt$

$\therefore \quad \dfrac{L}{R}$ has the units of time.

2. (a) $\tau_L = \dfrac{L}{R} = \dfrac{2}{10} = 0.2$ s

(b) $i_0 = \dfrac{E}{R} = \dfrac{100}{10} = 10$ A

(c) $i = i_0\,(1 - e^{-t/\tau_L})$

$$= 10\,(1 - e^{-1/0.2})$$

$$= 9.93\ \text{A}$$

3. $E = V_R + V_L$

INTRODUCTORY EXERCISE 27.6

1. $U = \dfrac{1}{2}Li^2 = \dfrac{1}{2}\dfrac{q^2}{C}$

$\therefore \quad \sqrt{LC} = \dfrac{q}{i} = \dfrac{it}{i} = t$

3. (a) $V_L = V_C$

$$L\frac{di}{dt} = \frac{q}{C}$$

$$q = (LC)\,\frac{di}{dt}$$

$$= (0.75\times18\times10^{-6})\,(3.4)$$

$$= 45.9\times10^{-6}\ \text{C}$$

$$= 45.9\,\mu\text{C}$$

(b) $V_L = V_C = \dfrac{q}{C}$

$$= \frac{4.2\times10^{-4}}{18\times10^{-6}}$$

$$= 23.3\ \text{V}$$

4. $\dfrac{1}{2}\,Li_{\max}^2 = \dfrac{1}{2}\,CV_{\max}^2$

$\therefore \quad V_{\max} = \left(\sqrt{\dfrac{L}{C}}\right) i_{\max}$

$$= \left(\sqrt{\frac{20\times10^{-3}}{0.5\times10^{-6}}}\right)(0.1)$$

$$= 20\ \text{V}$$

INTRODUCTORY EXERCISE 27.7

1. In the theory we have already derived mutual inductance between solenoid and coil,

$$M = \frac{\mu_0 N_1 N_2 (\pi R_1^2)}{l_1}$$

$$= \frac{\mu_0 N_1 N_2 S_1}{l_1}$$

$$|e_2| = \left| M \frac{di}{dt} \right| = \left| \frac{\mu_0 N_1 N_2 S_1}{l_1} \frac{di_1}{dt} \right|$$

$$= \frac{(4\pi \times 10^{-7})(25)(10)(5 \times 10^{-4})(0.2)}{10^{-2}}$$

$$= 3.14 \times 10^{-6} \text{ V}$$

(b)
$$El = \left| \frac{d\phi}{dt} \right| = e$$

$$\therefore \quad E = \frac{e}{l} = \frac{e}{2\pi R_2}$$

$$= \frac{3.14 \times 10^{-6}}{(2\pi)(0.25)}$$

$$= 2 \times 10^{-6} \text{ V/m}$$

2. (a) **At P_2**

$$El = \frac{d\phi}{dt}$$

$$E(2\pi r_2) = \pi R^2 \cdot \frac{dB}{dt}$$

$$\therefore \quad E = \frac{R^2}{2r_2}\left(\frac{dB}{dt}\right)$$

$$F = qE = \frac{eR^2}{2r_2}(6t^2 - 8t)$$

Substituting the values, we have

$$F = \frac{(1.6 \times 10^{-19})(2.5 \times 10^{-2})}{2 \times 5 \times 10^{-2}} [6(2)^2 - 8(2)]$$

$$= 8.0 \times 10^{-21} \text{N}$$

⊗ magnetic field at the given instant is increasing. Hence, induced current in an imaginary circular loop passing through P_2 should produced ⊙ magnetic field. Or, current in this should be anti-clockwise. Hence, electrons should move in clockwise direction. Or electron at P_2 should experience force in downward direction (perpendicular to r_2).

(b) **At P_1**

$$El = \frac{d\phi}{dt}$$

$$\therefore \quad E(2\pi r_1) = s\frac{dB}{dt} = (\pi r_1^2)\left(\frac{dB}{dt}\right)$$

$$\therefore \quad E = \frac{r_1}{2}\left(\frac{dB}{dt}\right)$$

$$= \frac{r_1}{2}[6t^2 - 8t]$$

$$= \frac{0.02}{2}[6(3)^3 - 8(3)]$$

$$= 0.3 \text{ V/m}$$

As discussed in the above part, direction of electric field is in the direction of induced current (anti-clockwise) in an imaginary circular conducting loop passing through P_1.

Exercises

LEVEL 1

Assertion and Reason

1. Due to non-uniform magnetic field (a function of x) magnetic flux passing through the loop obtained by integration. But that remains constant with time.

Hence, $\dfrac{d\phi}{dt} = 0$

or $e = 0$

Magnetic field is along $-\hat{\mathbf{k}}$ direction or in ⊗ magnetic is increasing. Hence, induced current should produce ⊙ magnetic field. Or induced current should be anti-clockwise.

2. At time t_0, magnetic field is negative or and increasing.

Hence, induced current will produce ⊗ magnetic field. Or induced current should be clockwise.

If $\dfrac{d\phi}{dt}$ = constant. Then, e = constant

∴ i or rate of flow of charge is constant.

3. It can exert a force on charged particle.

4. $\dfrac{di}{dt} = 2$ A/s

$$V_a - V_b = L\frac{di}{dt}$$

$$= (2)(2) = 4 \text{ V}$$

5. Comparing with spring-block system

$$\left(\frac{dI}{dt}\right) \text{ is acceleration.}$$

$$a_{\max} = \omega^2 A = \omega\,(\omega A)$$

$$= \omega\,(v_{\max})$$

$$\therefore \quad \left(\frac{dI}{dt}\right)_{\max} = \omega\, I_{\max}$$

6. Applying RHR, we can find that

$$V_a > V_b$$

8. Ferromagnetic substance will attack more number of magnetic lines through it. So, flux passing through it will increase. Hence, coefficient of self-inductance will increase.

L depends on number of turns in the coil's radius of coil etc. It does not depend on the current passing through it.

9. Current developed in the inductor wire will decrease exponentially through wire ab.

10. $V_{L_1} = V_{L_2}$

$$\therefore \quad L\frac{di_1}{dt} = L_2\frac{di_2}{dt}$$

$$\text{or} \quad L_1 di_1 = L_2 di_2$$

$$\text{or} \quad L_1 i_1 = L_2 i_2$$

$$\text{or} \quad i \propto \frac{1}{L}$$

Objective Questions

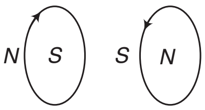
1. $U = \frac{1}{2}Li^2$

$$\therefore \quad [L] = \left[\frac{U}{i^2}\right] = \left[\frac{\text{ML}^2\text{T}^{-2}}{\text{A}^2}\right]$$

$$= [\text{ML}^2\text{T}^{-2}\text{A}^{-2}]$$

2. $M \propto N_1 N_2$

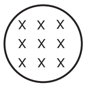
3.



When brought closer induced effects should produced repulsion. So, currents should increase, so that pole strength increases. Hence, repulsion increases.

4. Magnetic field of ring is also along its axis, or in the direction of velocity of charged particle. Hence, no magnetic force will act on charged particle. But, due to g velocity of charged particle will increase.

5.



If magnetic field in the shown cylindrical region is changing, then induced electric field exists even outside the cylindrical regions also where magnetic field does not exist.

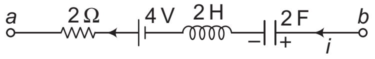
6. $i = \frac{dq}{dt} = (8t)\text{A}$

$$\frac{di}{dt} = 8 \text{ A/s}$$

At $t = 1$ s, $q = 4$ C, $i = 8$ A

and $\frac{di}{dt} = 8$ A/s

Charge on capacitor is increasing. So, charge on positive plate is also increasing. Hence, direction of current is towards left.



$$\text{Now,} \quad V_a + 2\times 8 - 4 + 2\times 8 + \frac{4}{2} = V_b$$

$$\therefore \quad V_a - V_b = -30 \text{ V}$$

7. $\frac{dI}{dt} = I_0\omega \cos\omega t$

$$e = M\frac{dI}{dt} = MI_0\omega\cos\omega t$$

$$\therefore \quad e_{\max} = MI_0\omega$$

$$= 0.005 \times 10 \times 100\pi$$

$$= (5\pi) \text{ V}$$

8. $\frac{1}{2}Li^2 = \frac{1}{2}CV^2$

$$\therefore \quad V = \sqrt{\frac{L}{C}} \Rightarrow i = \sqrt{\frac{2}{4\times10^{-6}}}\,(2)$$

$$= \sqrt{2}\times10^3 \text{ V}$$

9. $e = \frac{B\omega l^2}{2} = \text{constant}$

10. $\Delta q = \frac{\Delta\phi}{R}$

$$\text{or} \quad i\Delta t = \frac{\Delta\phi}{R}$$

$$\therefore \quad \Delta\phi = i(\Delta t)R = (10\times10^{-3})\,(5)\,(0.5)$$

$$= 25\times10^{-3} \text{ Wb}$$

11. ⊗ magnetic field is increasing. Therefore, induced electric lines are circular and anti-clockwise. Force on negative charge is opposite to electric field.

12. $e = \left|\dfrac{d\phi}{dt}\right| = (a\tau - 2at)$

$$i = \frac{e}{R} = \frac{a\tau - 2at}{R}$$

$$H = \int_0^{\tau} i^2 R\, dt$$

13. $e = L\left|\dfrac{di}{dt}\right| = L$ (Slope of i-t graph)

Initially, slope = 0 $\Rightarrow$ $e = 0$

Then in remaining two regions slopes are constants but of opposite signs. Hence, induced emfs are constants but of opposite signs.

14. $V_A - 1\times 5 + 15 + (5\times 10^{-3})(10^3) = V_B$

$\therefore$ $V_B - V_A = 15\text{ V}$

15. $I = (10t + 5)\text{A}$

$$\frac{dI}{dt} = 10\text{ A/s} = \text{constant}$$

At, $t = 0, I = 5\text{ A}$

Now, $V_A - 3\times 5 - 1\times 10 + 10 = V_B$

$\therefore$ $V_A - V_B = 15\text{ V}$

16.

$$(V_C)_{\max} = (V_L)_{\max}$$

$$\frac{q_0}{C} = L\left(\frac{dI}{dt}\right)_{\max}$$

$$\therefore \quad \left(\frac{dI}{dt}\right)_{\max} = \frac{q_0}{LC}$$

17. $V_L = L\dfrac{di}{dt}$

18. $\phi_i = BS\cos 0° = 2\text{ Wb}$

$\phi_f = BS\cos 180° = -2\text{ Wb}$

$$|\Delta\phi| = 4\text{Wb}$$

$$|\Delta q| = \frac{|\Delta\phi|}{R}$$

$$= \frac{4}{10} = 0.4\text{ C}$$

19. $\mathbf{S} = (ab)\,\hat{\mathbf{k}} \rightarrow$ perpendicular to xy-plane

$\phi = \mathbf{B}\cdot\mathbf{S} = (50)(ab) = \text{constant}$

$$\frac{d\phi}{dt} = 0$$

$\therefore$ $e = 0$

20. Back emf = Applied voltage potential drop across armature coil

$$= 200 - iR$$

$$= 200 - 1.5\times 20$$

$$= 170\text{ V}$$

21. $\dfrac{N_S}{N_P} = \dfrac{I_P}{I_S} = \dfrac{V_S}{V_P}$

$$V_P = V_i = \left(\frac{N_P}{N_S}\right)V_0$$

$$= \frac{2}{1}\times 20 = 40\text{ V}$$

$$I_P = \left(\frac{N_S}{N_P}\right)I_S$$

$$= \left(\frac{1}{2}\right)(4) = 2\text{ A}$$

22. Relative velocity = 0

$\therefore$ Charge in flux = 0

23. In case of free fall,

$$d = \frac{1}{2}gt^2$$

$$= \frac{1}{2}(10)(1)^2 = 5\text{m}$$

Here due to repulsion from induced effects

$$a < g$$

$\therefore$ $d < 5\text{ m}$

24. $V_A - V_B = L\dfrac{di}{dt}$

$$= L(-\alpha) = -\alpha L$$

25. $i_0 = \dfrac{E}{R} = \dfrac{12}{0.3} = 40\text{ A}$

$$U = \frac{1}{2}Li_0^2 = \frac{1}{2}\times 50\times 10^{-3}(40)^2$$

$$= 40\text{ J}$$

26. Value remains $\dfrac{1}{4}$th in 20 ms times. Hence, two half-lives are equal to 20 ms. So, one half-life is 10 ms.

$$t_{1/2} = (\ln 2)\,\tau_C = (\ln 2)\frac{L}{R}$$

$$\therefore \quad R = \frac{(\ln 2)\,L}{t_{1/2}}$$

$$= \frac{(\ln 2)(2)}{10\times 10^{-3}} = (100\ln 4)\,\Omega$$

27. $\because$ $i = \dfrac{e}{R}$

$$= \frac{N(\Delta\phi/\Delta t)}{R} = \frac{NS(\Delta B/\Delta t)}{R}$$

$$= \frac{10(10\times 10^{-4})(10^4)}{20}$$

$$= 5\text{ A}$$

28. In steady state, whole current passes through the inductor.

29. If current is passed through the straight wire, magnetic lines are circular and tangential to the loop. So, no flux is linked with the loop.

30. In second position, $\Delta\phi = 0$

$$\therefore \qquad |Q_2| = \frac{\Delta\phi}{R} = 0$$

31. From Lenz's law, induced effects always oppose the cause due to which they are produced. So, when the first loop is moved towards the smaller loop, it will face repulsion.

32. $\tau_L = \dfrac{L}{R} = 2\text{ s}$

$$i_0 = \frac{E}{R} = 3\text{A}\,,\ t = 2\text{ s}$$

$$i = i_0\,(1 - e^{-t/\tau_L})$$

Substituting the given values, we can find i.

33. In AB, $\mathbf{l}$ is parallel to its $\mathbf{v}$. Hence, PD = 0

34. $\mathbf{v}$ is parallel to $\mathbf{l}$.

35. For wire ab, velocity vector is parallel to $\mathbf{l}$.

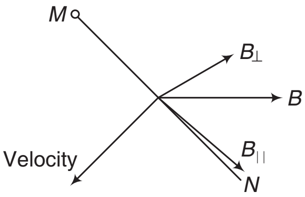
36. Current increases with time. So, flux passing through B will increase with time. From Lenz's law, it should have a tendency to move away from the coil to decrease flux.

37. For $E \neq 0$, ϕ must change

$$\text{or} \qquad \frac{d\phi}{dt} \neq 0$$

38. Even if radius is doubled, flux is not going to change.

39.



$B_{||}$ is parallel to MN (or $\mathbf{l}$) and $B_\perp$ is parallel or antiparallel to velocity.

Subjective Questions

1. When switch is opened current suddenly decreasing from steady state value to zero. When switch is closed, it takes time to increase from 0 to steady state value.

$$e = \left| L\frac{\Delta i}{\Delta t} \right|$$

Δt in second case is large. Hence, induced emf is less.

2. $e = \dfrac{N\Delta\phi}{\Delta t} = \left(\dfrac{NS\Delta B}{\Delta t}\right)\cos 30°$

$$\therefore \qquad S = \left(\frac{e\Delta t}{N\Delta B}\right)\sec 30°$$

$$= \left[\frac{(80\times10^{-3})\,(0.4)}{(50)\,(400\times10^{-6})}\right]\left[\frac{2}{\sqrt{3}}\right] = 1.85\text{ m}$$

$\therefore$ Side of square = 1.36 m

Total length of wire = 50 (4 × 1.36)

= 272 m

3. $\phi = BS = B_0 S\, e^{-at}$

$$\text{Induced emf} = \left|\frac{d\phi}{dt}\right| = aB_0 S e^{-at}$$

4. (a) At a distance x from the wire, magnetic field over the wire ab is

$$B = \frac{\mu_0}{2\pi}\frac{i}{x}$$

$$dV = Bvdx = \left(\frac{\mu_0 i}{2\pi x}\right)vdx$$

$$\therefore \qquad \text{Total emf} = \int_{x=d}^{x=d+l} dV$$

(b) Magnetic field due to current i over the wire ab is inwards. Velocity of wire ab is towards right. Applying right hand rule, we can see that a point is at higher potential.

(c) Net change in flux through the loop $abcd$ is zero. Hence, induced emf is zero. So, induced current is zero.

5. At $t = 0$, inductor offers infinite resistance. Hence, current through inductor wire is zero. Whole current passes through two resistors of 4Ω each.

$$i_1 = \frac{10}{4+4} = 1.25\text{ A}$$

At $t = \infty$, inductor offers zero resistance.

$$R_{\text{net}} = 4 + \frac{8\times4}{8+4}$$

$$= 6.67\ \Omega$$

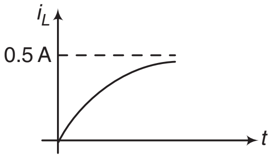
So, main current

$$i_2 = \frac{10}{R_{\text{net}}} = 1.5\text{A}$$

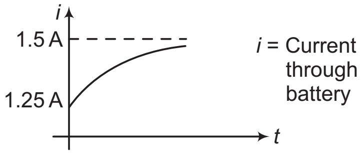
This distributes in 4Ω and 3Ω in inverse ratio of resistance. Hence, current through 4Ω is 1A and through 8Ω is 0.5A.

For equivalent τ_L of the circuit R_{net} across inductor after short-circuiting, the battery is 10 Ω.

$$\therefore \qquad \tau_L = \frac{1}{R_{\text{net}}} = \frac{1}{10} = 0.1 \text{ s}$$



$$i_L = 0.5\,(1 - e^{-\frac{t}{0.1}})$$
$$= 0.5\,(1 - e^{-10t})$$



$$i = 1.25 + 0.25\,(1 - e^{-t/0.1})$$
$$= 1.5 - 0.25\, e^{-10t}$$

6. Similar to above problem

$$\Delta q = \frac{2NBS}{R}$$

$$\therefore \qquad B = \frac{(\Delta q)R}{2NS}$$

$$= \frac{(4.5 \times 10^{-6})\,(40)}{(2)\,(60)\,(3 \times 10^{-6})}$$

$$= 0.5 \text{ T}$$

7. $e = \dfrac{d\phi}{dt} = s\left(\dfrac{dB}{dt}\right) = \pi R^2$ (Slope of B - t graph)

(a) $e = (\pi)\,(0.12)^2 \left(\dfrac{0.5}{2}\right) = 0.011$ V/m

(b) Slope of B-t graph is zero. Hence,

$$e = 0$$

(c) Slope is just opposite to the slope of part (a).

8. Induced emf ($e = BvL$) and therefore induced current is developed only during entering and during existing from the magnetic field.

$$i = \frac{e}{R} = \frac{BvL}{R}$$

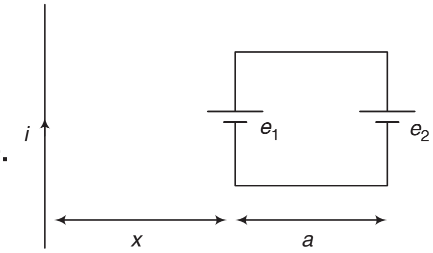
$$F = iLB = \frac{B^2L^2v}{R}$$

Further, magnetic force always opposes the change. Hence, external force is always positive.

During entering into the field, ⊗ magnetic field increases. Hence, induced current should produce ⊙ magnetic field. Or it should be anti-clockwise.

During existing from the magnetic field case is just opposite.

9.



$$\begin{aligned} e_{\text{net}} &= e_1 - e_2 \\ &= B_1va - B_2va \\ &= (B_1 - B_2)\,va \\ &= \left[\frac{\mu_0}{2\pi}\frac{i}{x} - \frac{\mu_0}{2\pi}\frac{i}{x+a}\right] va \\ &= \frac{\mu_0}{4\pi}\frac{2ia^2v}{x\,(x+a)} \end{aligned}$$

10. $B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$

$$dV = Bv\,dr = \frac{\mu_0 iv}{2\pi}\int \frac{dr}{r}$$

$$V = \int_{r_1}^{r_2} dV = \frac{\mu_0 iv}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$$

11. $V_L = L\dfrac{di}{dt}$

$$\therefore \quad di = \frac{1}{L}\,(V_L dt)$$

$$\therefore \quad \int di = i = \frac{1}{L}\int V_L dt$$

or $\quad i = \dfrac{1}{L}$ (area under V_L *versus* t graph)

(a) **At $t = 2$ ms**

$$i = (150 \times 10^{-3})^{-1}\left(\frac{1}{2} \times 2 \times 10^{-3} \times 5\right)$$

$$= 3.33 \times 10^{-2} \text{A}$$

(b) **At $t = 4$** ms

Area is just double. Hence, current is also double.

12. (a) $L = \left|\dfrac{e}{di/dt}\right| = \dfrac{0.016}{0.064} = 0.25$ H

(b) $L = \dfrac{N\phi}{i}$

$$\therefore \qquad \phi = \frac{Li}{N} = \frac{(0.25)\,(0.72)}{400}$$

$$= 4.5 \times 10^{-4} \text{ Wb}$$

13. (a) $M = \frac{N_2\phi_2}{i_1} = \frac{(400)\,(0.032)}{6.52} = 1.96\text{ H}$

(b) $M = \frac{N_1\phi_1}{i_2}$

$$\phi_2 = \frac{Mi_2}{N_1} = \frac{(1.96)\,(2.54)}{700}$$

$$= 7.12 \times 10^{-3}\text{ Wb}$$

14. $\tau_L = \frac{L}{R} = 0.1\text{ s}$

The given time $t = 0.1$ s is one time constant.

The desired ratio is $\frac{iV_L}{iE}$ (∵ $P = Vi$)

$$= \frac{V_L}{E}$$

After on time constant $V_L = \frac{E}{e}$ as

$$V_L = Ee^{-t/\tau_L}$$

Hence, the desired ratio is $\frac{1}{e} \approx 0.37$.

15. $i_0 = \frac{V}{R} = \frac{3.24}{12.8} = 0.253\text{ A}$

$\tau_L = \frac{L}{R} = \frac{3.56}{12.8} = 0.278\text{ s}$

(a) After one time constant ($t = 0.278\text{ s} = \tau_C$)

$$i = \left(1 - \frac{1}{e}\right) i_0$$

$$\approx 0.63\, i_0$$

$$= 0.16\text{ A}$$

Power supplied by battery $= Ei$

$$P = (3.24)\,(0.16) = 0.518\text{ W}$$

(b) $P_R = i^2R$

$$= (0.16)^2\,(12.8) = 0.328\text{ W}$$

(c) $P_L = P - P_R = 0.191\text{ W}$

16. (a) After one half-life,

$$t = t_{1/2} = (\ln 2)\,\tau_L$$

$$= 0.693\,\frac{L}{R}$$

$$= \frac{(0.693)\,(1.25 \times 10^{-3})}{50}$$

$$= 1.73 \times 10^{-5}\text{ s}$$

(b) $\left(\frac{1}{2}Li^2\right) = \left(\frac{1}{2}Li_0^2\right)/2$

∴ $i = \frac{i_0}{\sqrt{2}}$

Now, apply

$$i = i_0\,(1 - e^{-t/\tau_L})$$

where, $\tau_L = \frac{L}{R}$

17. Steady state current developed in the inductor

$$= \frac{E}{r} = i_0 \text{ (say)}$$

(a) Now this current decreases to zero exponentially through r and R.

∴ $i = i_0e^{-t/\tau_L}$

where, $\tau_L = \frac{L}{R + r}$

Energy stored in inductor,

$$U_0 = \frac{1}{2}Li_0^2 = \left(\frac{1}{2}L\right)\left(\frac{E}{r}\right)^2$$

Now, this energy dissipates in r and R in direct ratio of resistances.

∴ $H_r = \left(\frac{r}{R + r}\right)U_0 = \frac{E^2L}{2r\,(R + r)}$

18. In steady state, main current from the battery is

$$i_0 = \frac{E}{R} = \frac{20}{5} = 4\text{ A}$$

Now, this current distributes in inverse ratio of inductor.

∴ $i_5 = \left(\frac{10}{10 + 5}\right)(4\text{A}) = \frac{8}{3}\text{ A}$

19. (a) $\frac{1}{2}Li_0^2 = \frac{1}{2}CV_0^2$

∴ $L = \frac{CV_0^2}{i_0^2} = \frac{(4 \times 10^{-6})\,(1.5)}{(50 \times 10^{-3})^2}$

$$= 3.6 \times 10^{-3}\text{ H}$$

(b) $f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(3.6 \times 10^{-3})\,(4 \times 10^{-6})}}$

$$= 0.133 \times 10^4\text{ Hz}$$

$$= 1.33\text{ kHz}$$

(c) $t = \frac{T}{4} = \frac{1}{4f} = \frac{1}{4 \times 1.33 \times 10^3}\text{ s}$

$$= 0.188 \times 10^{-3}\text{ s}$$

$$= 0.188\text{ ms}$$

20. (a) $\omega = \frac{2\pi}{f},\ T = \frac{1}{f}$

(b) At $t = 0$, $q = q_0 = CV_0 = (100\,\mu\text{C})$

Now, $q = q_0 \cos \omega t$

(c) $\omega = \dfrac{1}{\sqrt{LC}} \quad \Rightarrow \quad L = \dfrac{1}{\omega^2 C}$

(d) $|i| = \left| \dfrac{dq}{dt} \right| = q_0 \omega \sin \omega t$

Average value of current in first quarter cycle

$$= \frac{\int_0^{T/4} dt}{T/4}$$

21. (a) $\dfrac{1}{2}\dfrac{q_i^2}{C} + \dfrac{1}{2} L I_i^2 = \dfrac{1}{2} C V_0^2$

$\therefore \qquad V_0 = \dfrac{q_i}{C} \qquad$ (as $I_i = 0$)

(b) $\dfrac{1}{2} C V_0^2 = \dfrac{1}{2} L I_0^2$

$\therefore \qquad I_0 = \sqrt{\dfrac{C}{L}} V_0$

(c) $U_{\max} = \dfrac{1}{2} L I_0^2$

(d) $U_L = \dfrac{1}{2} L \left(\dfrac{I_0}{2} \right)^2$

$$U_C = \frac{1}{2}\frac{q^2}{C} = U_{\max} - U_L$$

LEVEL 2

Single Correct Option

1. $\dfrac{1}{2} m v_0^2 = \dfrac{1}{2} L i_{\max}^2$

$\therefore \qquad i_{\max} = \sqrt{\dfrac{m}{L}}\, v_0$

2. $V_C = Bvl$

$\therefore \qquad q = C V_C = BvlC = \text{constant}$

$\therefore \qquad I_C = \dfrac{dq}{dt} = 0$

$$U_C = \frac{1}{2} C V^2 = \frac{1}{2} C B^2 L^2 v$$

3. From right hand rule, we can see that P and Q points are at higher potential than O.

4. At mean position, velocity is maximum. Hence, motional emf Bvl is also maximum. v oscillates simple harmonically. Hence, motional emf will also move simple harmonically. Further, polarity of induced emf will keep on changing.

5. At $t = t$ side of square,

$$l = (a + 2v_0 t)$$

Area, $\qquad S = l^2 = (a + 2v_0 t)^2$

$$\phi = BS = B\,(a + 2v_0 t)^2$$

$$e = \frac{d\phi}{dt} = 4 B v_0\,(a + 2v_0 t)$$

$$R = \lambda\,[4l] = 4\lambda\,(a + 2v_0 t)$$

$\therefore \qquad i = \dfrac{e}{R} = \dfrac{Bv_0}{\lambda}$

6. **At time *t***

Side of square $l = l_i - dt$

$$S = l^2 = (l_i - \alpha t)^2$$

At given time

$$l = l_i - \alpha t = a$$

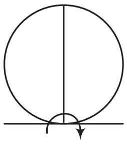
$$\phi = BS = B\,(l_i - \alpha t)^2$$

$$e = \left| \frac{d\phi}{dt} \right| = 2B\alpha\,(l_i - \alpha t)$$

But, $\qquad (l_i - \alpha t) = a$

$\therefore \qquad e = 2a\alpha\, B$

7.



$$\omega = \frac{v}{R} = \frac{v}{l/2} = \frac{2v}{l}$$

$$e = \frac{B\omega l^2}{2} = \frac{B\left(\frac{2v}{l}\right) l^2}{2} = Bvl$$

8. From right hand rule, we can see that

$$V_A > V_B$$

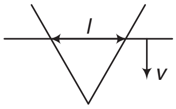
$\therefore \quad q_A$ is positive and q_B is negative.

$$q = CV = C\,(Bvl)$$

$$= (20 \times 10^{-6})\,(0.5)\,(0.2)\,(0.1)$$

$$= 0.2 \times 10^{-6}\,\text{C} = 0.2\,\mu\text{C}$$

9. $i = \dfrac{Bvl}{R}$



Let λ = resistance per unit length of conducting rod, then

$$i = \frac{Bvl}{\lambda l} = \frac{Bv}{\lambda} = \text{constant}$$

10. At time t, angle rotated by loop is $\theta = \omega t$. This is also the angle between **B** and **S**. Then,

$$\phi = BS \cos\theta$$
$$= Bb^2 \cos\omega t$$
$$e = \left|\frac{d\phi}{dt}\right| = b^2 B\omega \sin\omega t$$

11. $El = \frac{d\phi}{dt} = S\frac{dB}{dt}$

$\therefore \quad E(2\pi r) = (\pi r^2)\frac{dB}{dt}$

or $\quad E = \frac{r}{2}\frac{dB}{dt}$

$$F = qE = \frac{qr}{2}\frac{dB}{dt}$$
$$W = Fd = (2\pi r)$$
$$= \pi r^2 q\left(\frac{dB}{dt}\right)$$
$$= \left(\frac{22}{7}\right)(1)^2(10^{-6})(2\times 10^{-3})$$
$$= 2\pi\times 10^{-9}\text{ J}$$

12. Initial current $= \frac{10}{10} = 1$ A

$\therefore \quad \phi_i = L(I_i) = 500\text{ mWb} = 0.5\text{ Wb}$

Final current $= \frac{20}{5} = 4$A

$$\phi_f = L(I_f) = (0.5)\times 4 = 2\text{ Wb}$$

$\therefore \quad \Delta\phi = 1.5$ Wb

13. $\frac{1}{2}Li^2 = \frac{1}{2}\left[\frac{1}{2}Li_0^2\right]$

$\therefore \quad i = \frac{i_0}{\sqrt{2}} = i_0(1 - e^{-t/\tau_L})$

$$e^{-t/\tau_L} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$\therefore \quad \frac{t}{\tau_L} = \ln\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)$

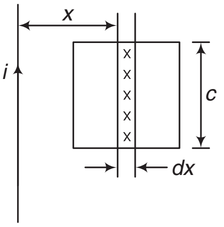
or $\quad t = \tau_L \ln\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)$

$$= \frac{L}{R}\ln\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)$$

14. $B = \frac{\mu_0}{2\pi}\frac{i}{a}$

$$F = Bqv\sin 90° = \frac{\mu_0}{2\pi}\frac{i}{a}(qv)$$

15.



$$B = \frac{\mu_0}{2\pi}\frac{i}{x} \quad \Rightarrow \quad dS = c\,dx$$
$$d\phi = BdS = \frac{\mu_0}{2\pi}\frac{ic}{x}dx$$
$$\phi = \int_a^b d\phi = \frac{\mu_0 ic}{2\pi}\ln\left(\frac{b}{a}\right)$$
$$M = \frac{\phi}{i} = \frac{\mu_0 c}{2\pi}\ln\left(\frac{b}{a}\right)$$

16. $B_x = \frac{\mu_0}{2\pi}\frac{I}{x}$

$$de = B_x\,v dx = \frac{\mu_0}{2\pi}\frac{I}{x}v\,dx$$
$$e = \int_a^b de = \frac{\mu_0 Iv}{2\pi}\ln\left(\frac{b}{a}\right)$$
$$i = \frac{e}{R} = \frac{\mu_0 Iv}{2\pi R}\ln\left(\frac{b}{a}\right) = \text{Induced current}$$
$$dF = (i)(dx)B_x$$
$$= \left[\frac{\mu_0 Iv}{2\pi R}\ln\left(\frac{b}{a}\right)\right]\left[\frac{\mu_0}{2\pi}\frac{I}{x}\right]dx$$
$$F = \int_a^b dF$$

17. $El = \frac{d\phi}{dt} = S\frac{dB}{dt}$

$\therefore \quad E(2\pi r) = \pi r^2\frac{dB}{dt}$

or $\quad E = \frac{r}{2}\frac{dB}{dt}$ or $E \propto r$

18. Magnetic field through Q (by I_2) is downwards. By decreasing I_1, downward magnetic field through Q will decrease. Hence, induced current in Q should produce magnetic field in same direction.

19. $i = 0(1 - e^{-t./\tau_L}) = \frac{E}{R}(1 - e^{-t/\tau_L})$

$$= \frac{E}{R} - \frac{Ee^{-t/\tau_L}}{R} = i_0 - \left(\frac{V_L}{R}\right) \quad (\text{as } V_L = Ee^{-t/\tau_L})$$

$\therefore \quad V_L = (i_0 R) - (R)\,i$

i.e. V_L *versus* i graph is a straight line with positive intercept and negative slope.

20. $e = Bvl = 0.5 \times 4 \times 0.25 = 0.5\ \text{V}$

12 Ω and 4Ω are parallel. Hence, their net resistance $R = 3\ \Omega$.

$$i = \frac{e}{R + r} = \frac{0.5}{3 + 2} = 0.1\ \text{A}$$

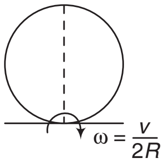
21. $El = \frac{d\phi}{dt} = S\,\frac{dB}{dt}$

$$E\,(2\pi R) = (\pi r^2)\,(\beta)$$

$$\therefore \quad E = \frac{r^2}{2R}\beta \quad \Rightarrow \quad F = qE$$

and $$\tau = FR = qER = \frac{1}{2}qr^2\beta$$

22.



$$\text{PD} = \frac{B\omega l^2}{2} = \frac{(B)\,(v/2R)\,(4R)^2}{2} = 4\,BvR$$

23. $L_1 = \left(\frac{\eta}{\eta + 1}\right) L,\ R_1 = \left(\frac{\eta}{\eta + 1}\right) R$

$$L_2 = \left(\frac{1}{\eta + 1}\right) L,\ R_2 = \left(\frac{1}{\eta + 1}\right) R$$

$$L_{\text{net}} = \frac{L_1 L_2}{L_1 + L_2}$$

Similarly, $$R_{\text{net}} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\tau_L = \frac{L_{\text{net}}}{R_{\text{net}}} = \frac{L}{R}$$

24. $i = i_0 e^{-t/\tau_L}$

$$\beta\, i_0 = i_0 e^{-T/\tau_L}$$

$$\therefore \quad \tau_L = \frac{T}{\ln\,(1/\beta)}$$

25. $P = i_0^2 R \quad \Rightarrow \quad i_0^2 = \frac{P}{R} \quad \Rightarrow \quad \tau = \frac{L}{R}$

$$\therefore \quad L = \tau R$$

Heat dissipated $= \frac{1}{2}Li_0^2 = \frac{1}{2}\,(\tau R)\left(\frac{P}{R}\right) = \frac{1}{2}P\tau$

26. In decay of current through *L-R* circuit, current can not remain constant.

27. By short-circuiting the battery, net resistance across inductor is $\frac{R}{2}$ (*R* and *R* in parallel).

$$\therefore \quad \tau_{\text{net}} = \frac{L}{R_{\text{net}}} = \frac{2L}{R}$$

28. At $t = 0$, $i = E/R$

Now, this current will decay in closed loop in anti-clockwise direction. So, $|i_2| = i_2 = E/R$ in upward or opposite direction.

Hence, $$i_2 = -\frac{E}{R}$$

29. $\frac{1}{2}Li^2 = \frac{1}{4}\left[\frac{1}{4}Li_0^2\right]$

So, $i = \frac{i_0}{2}$, half value

$$\therefore \quad t = t_{1/2} = (\ln 2)\,\tau_L = (\ln 2)\left(\frac{L}{R}\right)$$

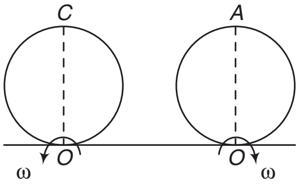
30. Steady state current through inductor in $\frac{E}{R}$.

So, at $t = 0$, current in closed loop (confiding of capacitor) will remain same.

31. At $t = 0$, $V_L = -E$

32. $V_A - V_0 = \frac{B\omega\,(2R)^2}{2} = 2B\omega R^2$ …(i)

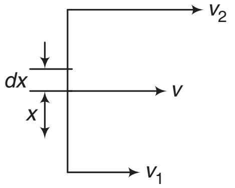
$V_0 - V_C = \frac{B\omega\,(2R)^2}{2} = 2B\omega R^2$ …(ii)



Adding these two equations, we get

$$V_A - V_C = 4\,B\omega R^2$$

33.



$$v = v_1 + \left(\frac{v_2 - v_1}{l}\right)x$$

Small potential difference $= Bv\,(dx)$

$\therefore$ Total potential difference $= \int_0^l Bv\,dx$

$$= \frac{1}{2}B\,(v_1 + v_2)\,l$$

34. At time $t = 0$, resistance offered by a capacitor = 0 and resistance offered by an inductor = α.

$$R_{\text{net}} = \frac{R}{2} + \frac{R}{3} = \frac{5R}{6} = 5\ \Omega$$

$\therefore$ Current from the battery,

$$i = \frac{E}{R_{\text{net}}} = \frac{5}{5} = 1\ \text{A}$$

35. $\tau_L = \dfrac{L}{R} = \dfrac{0.01}{10} = 10^{-3}\ \text{s}$

$$\tau_C = CR = (0.1 \times 10^{-3})\ (10) = 10^{-3}\text{s}$$

$$(i_0)_L = \frac{20}{10} = 2\ \text{A}$$

$$(i_0)_C = \frac{20}{10} = 2\ \text{A}$$

The given time is the half-life time of both the circuits.

$$\therefore \qquad i_L = i_C = \frac{2}{2} = 1\ \text{A}$$

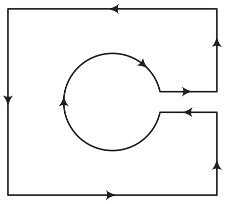
or total current is 2 A.

36. $|e| = \dfrac{d\phi}{dt} = S\dfrac{dB}{dt} = (4b^2 - \pi a^2)\ B_0$

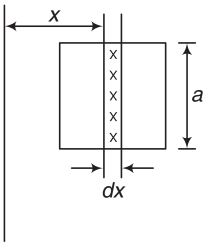
$$i = \frac{|e|}{R} = \frac{(4b^2 - \pi a^2)\ B_0}{R}$$

⊗ magnetic field is increasing. So,

⊙ magnetic field is produced.



37. $\phi_i = \displaystyle\int_b^{b+a} \frac{\mu_0}{2\pi}\frac{i}{x}\,(adx) = \frac{\mu_0 ia}{2\pi}\ln\left(\frac{b+a}{b}\right)$



Similarly,

$$\phi_f = \frac{\mu_0 ia}{2\pi}\ln\left(\frac{b-a}{b}\right)$$

$$\Delta\phi = |Q_i - Q_f| = \frac{\mu_0 ia}{2\pi}\ln\left(\frac{b+a}{b-a}\right)$$

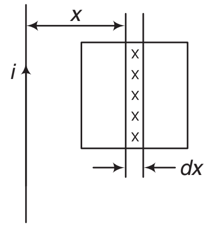
$$\Delta q = \frac{\Delta\phi}{R} = \frac{\mu_0 ia}{2\pi R}\ln\left(\frac{b+a}{b-a}\right)$$

More than One Correct Options

1. $e = Bvl, \quad \text{where} \quad l = \dfrac{L}{2}$

For polarity of this motional emf, we can use right hand rule.

2. (a)



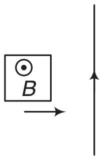
$$B_x = \frac{\mu_0}{2\pi}\frac{i}{x}$$

$$d\phi = (B_x)\ dS = \left(\frac{\mu_0}{2\pi}\frac{i}{x}\right)(adx)$$

$$\phi = \int_a^{2a} d\phi = \frac{\mu_0 ia}{2\pi}\ln 2$$

$$M = \frac{\phi}{i} = \frac{\mu_0 a}{2\pi}\ln 2$$

(c)



Wire produces ⊙ magnetic field over the loop. If the loop is brought closer to the wire, ⊙ magnetic field passing through the loop increases. Hence, induced current produces ⊗ magnetic field so, induced current is clockwise.

3. (a) $L = \dfrac{N\phi}{i} \quad \Rightarrow \quad \phi = \dfrac{Li}{N}$

So, SI unit of flux is Henry-ampere.

(c) $L = \dfrac{-e}{\Delta i/\Delta t} = -\dfrac{-e\Delta t}{\Delta i}$

Hence, SI unit of L is $\dfrac{\text{V-s}}{\text{ampere}}$.

4. $\tau_L = \dfrac{L}{R} = \dfrac{2}{2} = 1\,\text{s}$

$$t_{\frac{1}{2}} = (\ln 2)\ \tau_L = (\ln 2)\ \ \text{s}$$

Hence, the given time is half-life time.

$$\therefore \qquad i = \frac{i_0}{2} = \frac{8/2}{2} = 2\text{ A}$$

Rate of energy supplied by battery

$$= Ei = 8 \times 2 = 16\text{ J/s}$$

$$P_R = i^2R = (2)^2(2) = 8\text{ J/s}$$

$$V_a - V_b = E - iR = 8 - 2 \times 2 = 4\text{ V}$$

5. According to Lenz's law, induced effects always oppose the change i_1 and i_2 both are in same direction. Hence, magnetic lines from B due to both currents are from right to left. By bringing A closer to B or increasing i_1 right to left magnetic field from B will increase. So, i_2 should decrease.

6. $\phi_i = BS\cos 0° = (4)(2) = 8\text{ Wb}$

$\phi_f = BS\cos 90° = 0$

$$\Delta\phi = 8\text{ Wb}$$

$$|e| = \frac{\Delta\phi}{\Delta t} = \frac{8}{0.1} = 80\text{ V}$$

$$i = \frac{|e|}{R} = 20\text{ A}$$

$$\Delta q = \frac{\Delta\phi}{R} = 2\text{ C}$$

This current is not constant. So, we cannot find the heat generated unless current function with time is not known.

7. $i_{\max} = \omega q_0 = \left(\dfrac{1}{\sqrt{LC}}\right)q_0$

$$\left(\frac{di}{dt}\right)_{\max} = \omega^2 q_0 = \left(\frac{1}{LC}\right)q_0$$

8. If ⊗ magnetic field increases, then induced electric lines are anti-clockwise. If ⊗ magnetic field decreases, then induced electric lines are clockwise (both inside and outside the cylindrical region).

On positive charge, force is in the direction of **E**. On negative charge, force is in the opposite direction of **E**.

9. $q = 2t^2$

$$i = \frac{dq}{dt} = 4t$$

$$\frac{di}{dt} = 4\text{ A/s}$$

At $t = 1$ s, $q = 2$C, $i = 4$A

and $\qquad \dfrac{di}{dt} = 4\text{ A/s}$

$$V_a - V_b = L\frac{di}{dt} = 1 \times 4 = 4\text{ V}$$

$$V_b - V_c = \frac{q}{C} = \frac{2}{2} = 1\text{ V}$$

$$V_c - V_d = iR = 4 \times 4 = 16\text{ V}$$

$V_a - V_d$ is summation of above three, i.e. 21 V.

10. $V_a - V_c = 0$ as **l** is parallel to **v**.

$$V_a - V_b = V_c - V_b = \frac{B\omega l^2}{2}$$

Comprehension Based Questions

1. $\dfrac{dB}{dt} = (6t^2 + 24)\text{ T/s}$

At $t = 2$ s, $\dfrac{dB}{dt} = 48\text{ T/s}$

$$El = \frac{d\phi}{dt} = S\left(\frac{dB}{dt}\right)$$

or $\qquad E(2\pi r) = \pi r^2\left(\dfrac{dB}{dt}\right)$

$$\therefore \qquad E = \frac{r}{2}\cdot\frac{dB}{dt} \qquad \text{...(i)}$$

$$F = qE = \frac{qr}{2}\frac{dB}{dt}$$

$$= \frac{(1.6 \times 10^{-19})(1.25 \times 10^{-2})}{2}(48)$$

$$= 48 \times 10^{-21}\text{N}$$

2. From Eq. (i) of above problem, we can see that

$$E \propto r$$

i.e. E - r graph is a straight line passing through origin.

3. ⊗ Magnetic field is increasing. Hence, ⊙ magnetic field is produced by a conducting circular loop placed there. For producing, magnetic field induced current should be anti-clockwise. Direction of induced circular electric lines are also anti-clockwise.

4. $|e| = \left|\dfrac{d\phi}{dt}\right| = S\dfrac{dB}{dt} = (\pi a^2)B_0$

5. $El = \dfrac{d\phi}{dt}$

$$\therefore \qquad E(2\pi a) = (\pi a^2)B_0$$

or $\qquad E = \dfrac{1}{2}aB_0$

6. $\qquad F = qE = \dfrac{1}{2}qaB_0$

$$\tau = Fa = \frac{1}{2}qa^2B_0$$

$$\alpha = \frac{\tau}{I} = \frac{\left(\frac{1}{2}qa^2B_0\right)}{ma^2} = \frac{qB_0}{2m}$$

7. $\omega = \alpha t = \left(\frac{qB_0}{2m}\right)t$

$$P = \tau\,\omega = \left(\frac{1}{2}qa^2B_0\right)\left(\frac{qB_0}{2m}\right)$$

$$= \frac{q^2B_0^2a^2}{4m}$$

8. $|e| = \frac{d\phi}{dt} = S\frac{dB}{dt}$

$$= (0.2 \times 0.4)\,(2) = 0.16 \text{ V}$$

$$i = \frac{|e|}{R} = \frac{0.16}{(1)\,(40 + 40 + 20) \times 10^{-2}}$$

$$= 0.16 \text{ A}$$

⊙ Magnetic field passing through the loop is increasing. So, induced current should produce ⊗ magnetic field. Hence, induced current is clockwise.

9. At $t = 2$ s, rod will move 10 cm. Hence, 40 cm side will become 30 cm.

$$|e| = e_1 \text{ (say)} = S\left(\frac{dB}{dt}\right)$$

$$(0.2 \times 0.3)\,(2) = 0.12 \text{ V}$$

At $t = 2 \text{ s}, B = 4 \text{ T}$

$\therefore \quad e_2 = Bvl$

$$= (4)\,(5 \times 10^{-2})\,(0.2)$$

$$= 0.04 \text{ V}$$

$\therefore \quad e_{\text{net}} = e_1 - e_2 = 0.08 \text{ V}$

10. $i = \frac{e_{\text{net}}}{R} = \frac{0.08}{(1)\,(30 + 30 + 20) \times 10^{-2}}$

$$= 0.1 \text{ A}$$

$$F = ilB$$

$$= (0.1)\,(0.2)\,(4) = 0.08 \text{ N}$$

11 to **13**

At terminal velocity,

$$iLB = mg$$

$\therefore \quad i = \frac{mb}{LB} = \frac{0.2 \times 98}{1 \times 0.6}$

$$i = 3.27 \text{ A} \qquad \text{...(i)}$$

$$e = BvL \qquad (v = \text{terminal velocity})$$

$$= (0.6)\,(v)\,(1)$$

$$e = 0.6v$$

$$P_{R_1} = \frac{e^2}{R_1}$$

$\therefore \quad 0.76 = \frac{0.36\,v^2}{R_1} \qquad \text{...(ii)}$

$$P_{R_2} = \frac{e^2}{R_2}$$

$\therefore \quad 1.2 = \frac{0.36\,v^2}{R_2} \qquad \text{...(iii)}$

R_1 and R_2 are in parallel.

$\therefore \quad R_{\text{net}} = \frac{R_1R_2}{R_1 + R_2} \qquad \text{...(iv)}$

$$i = \frac{e}{R_{\text{net}}} \qquad \text{...(v)}$$

Solving these five equations, we can get the results.

Match the Columns

1. (a) $B = \frac{F}{il}$

$$[B] = \left[\frac{\text{MLT}^{-2}}{\text{AL}}\right] = [\text{MT}^{-2}\text{A}^{-1}]$$

(b) $U = \frac{1}{2}Li^2$

$$\therefore \; [L] = \left[\frac{U}{i^2}\right] = \left[\frac{\text{ML}^2\text{T}^{-2}}{\text{A}^2}\right] = [\text{ML}^2\text{T}^{-2}\text{A}^{-2}]$$

(c) $\omega = \frac{1}{\sqrt{LC}} \;\Rightarrow\; [LC] = \left[\frac{L}{\omega^2}\right] = [\text{T}^2]$

(d) $[\phi] = [BS] = [\text{MT}^{-2}\text{A}^{-1}\text{L}^2]$

2. $V_L = Ee^{-t/\tau_L} = 10e^{-t/\tau_L}$

$$\tau_L = \frac{L}{R} = 1 \text{ s}$$

$\therefore \quad V_L = 10e^{-t}$

$$V_R = E - V_L = 10\,(1 - e^{-t})$$

Now, we can put $t = 0$ and $t = 1$ second.

3. $\omega = \frac{1}{\sqrt{LC}} = 2 \text{ rad/s}$

(a) $i_{\max} = \omega q_0 = 8 \text{ A}$

(b) $\left|\frac{di}{dt}\right|_{\max} = \omega^2 q_0 = 16 \text{ A/s}$

(c) $V_L = V_C = \frac{q}{C} = \frac{2}{1/4} = 8 \text{ V}$

(d) $V_C = V_L = L\frac{di}{dt} = (1)\left(\frac{16}{2}\right) = 8 \text{ V}$

4. Steady state current through inductor,

$$i_0 = \frac{9}{3} = 3\ \text{A}$$

Now, this current decays exponentially across inductor and two resistors.

$$\tau_L = \frac{L}{R} = \frac{9}{6+3} = 1\ \text{s}$$

$$t_{1/2} = (\ln 2)\,\tau_L = (\ln 2)\ \text{s}$$

Given time is half-life time. Hence, current will remain 1.5 A.

$$i = i_0 e^{-t/\tau_L} = 3e^{-t}$$

$$\left(-\frac{di}{dt}\right) = 3e^{-t}$$

In the beginning $\left(\frac{-di}{dt}\right) = 3\ \text{A/s}$

After one half-life time $\left(\frac{-di}{dt}\right) = 1.5\ \text{A/s}$

(a) $V_L = L\left(\frac{-di}{dt}\right) = 9 \times 1.5 = 13.5\ \text{V}$

(b) $V_{3\Omega} = iR = 1.5 \times 3 = 4.5\ \text{V}$

(c) $V_{6\Omega} = iR = 1.5 \times 6 = 9\ \text{V}$

(d) $V_{bc} = V_L - V_{3\Omega} = 9\ \text{V}$

5. $\phi = 2t$

(a) $e = \frac{d\phi}{dt} = 2\ \text{V}$

(b) $i = \frac{e}{R} = 1\ \text{A} = \text{constant}$

(c) $\Delta q = i\Delta t = 1 \times 2 = 2\ \text{C}$

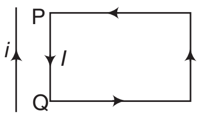
(d) $H = i^2 R\Delta t = (1)^2(2)\,(2) = 4\text{J}$

6. (a) If current is increased, ⊗ magnetic field passing through loop will increase. So, induced current will produce ⊙ magnetic field. Hence, induced current is anti-clockwise.

Now, i and I currents in PQ are in opposite directions. Hence, they will repel each other.

Same logic can be applied for (b) part.

(c) situation is similar to (b) situation and
(d) situation is similar to (a) situation.



Subjective Questions

1. $\tau_L = \frac{L}{R}$

and $\tau_C = CR$, $\frac{\tau_C}{\tau_L} = \frac{CR^2}{L} = \frac{C}{L}\cdot\frac{L}{C} = 1$

$\therefore \quad \tau_L = \tau_C$

$\therefore$ For the given condition $\tau_L = \tau_C = \tau$ (say)

Now, in L-R circuit

$$I_1 = \frac{V}{R}(1 - e^{-t/\tau})$$

In CR circuit, $I_2 = \frac{V}{R}e^{-t/\tau}$

$$\therefore \qquad I = I_1 + I_2 = \frac{V}{R} = \text{constant} \qquad \textbf{Ans.}$$

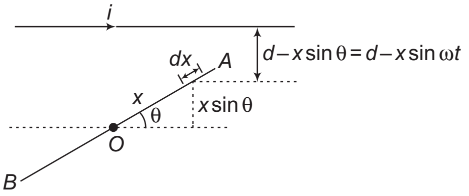
2. Motional emf, $V = Bvl$

Net resistance of the circuit $= R + \frac{R_1R_2}{R_1 + R_2}$

$\therefore$ Current through the connector,

$$i = \frac{Bvl}{R + \frac{R_1R_2}{R_1 + R_2}} \qquad \textbf{Ans.}$$

3. $\theta = \omega t$



$$de = B\,(\omega x)\,dx$$

Here, $B = \frac{\mu_0}{2\pi}\,\frac{i}{d - x\sin\omega t}$

$$\therefore \qquad de = \frac{\mu_0 i\omega}{2\pi}\,\frac{x}{d - x\sin\omega t}\cdot dx$$

$$V_{OA} = V_0 - V_A = \int_0^a de = \frac{\mu_0 i\omega}{2\pi}\int_0^a \frac{x}{d - x\sin\omega t}\,dx$$

$$= -\frac{\mu_0 i\omega}{2\pi\sin\omega t}\left[\frac{d}{\sin\omega t}\ln\left(\frac{d - a\sin\omega t}{d}\right) + a\right]$$

Similarly,

$$V_{OB} = V_O - V_B = \frac{\mu_0 i\omega}{2\pi}\int_0^a \frac{x}{d + x\sin\omega t}\,dx$$

$$= \frac{\mu_0 i\omega}{2\pi\sin\omega t}\left[a - \frac{d}{\sin\omega t}\ln\left(\frac{d + a\sin\omega t}{d}\right)\right]$$

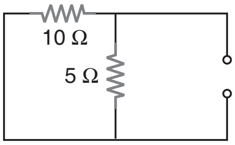
$$\therefore \quad V_{AB} = V_{OB} - V_{OA}$$

$$= \frac{\mu_0 i\omega}{2\pi\sin\omega t}\left[2a + \frac{d}{\sin\omega t}\ln\left(\frac{d - a\sin\omega t}{d + a\sin\omega t}\right)\right]$$

Ans.

Note *This function is discontinuous at* $\omega t = n\pi$.

4. At $t = 0$, equivalent resistance of an inductor in infinite and at $t = \infty$, equivalent resistance is zero.



$\therefore$ Initial current through inductor = 0 and

Final current through inductor $= \frac{36}{10} = 3.6$ A

To find equivalent time constant, we will have to short circuit the battery and find net resistance across inductor.

$$R_{\text{net}} = \frac{10 \times 5}{10 + 5} = \frac{10}{3}\ \Omega$$

$$\tau_L = \frac{L}{R_{\text{net}}} = \frac{3}{10}\ \text{ms}$$

Current through inductor will increase exponentially from 0 to 3.6 A.

$\therefore\quad i = 3.6(1 - e^{-t/\tau_L})$, where $\tau_L = \frac{3}{10}$ ms = 300 µs

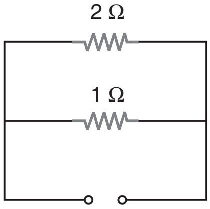
Current through 10 Ω will vary with time. **Ans.**

5. At $t = 0$, Current through inductor will be zero.

At $t = \infty$, net emf $= \frac{2/2 + 4/1}{1/2 + 1/1} = \frac{10}{3}$ V

Net resistance $= \frac{2 \times 1}{2 + 1} = \frac{2}{3}\ \Omega$

$$\therefore\quad i = \frac{10/3}{2/3} = 5\ \text{A}$$



To find equivalent time constant short circuit, both the batteries and find net resistance across inductor.

$$R_{\text{net}} = \frac{2 \times 1}{2 + 1} = \frac{2}{3}\ \Omega$$

$$\therefore\quad \tau_L = \frac{L}{R_{\text{net}}} = \frac{1 \times 10^{-3}}{2/3} = \frac{3}{2000}\ \text{s}$$

Current through inductor will increase exponentially from 0 to 5 A.

$$\therefore\quad i = 5\left(1 - e^{-\frac{2000t}{3}}\right)$$

6. These are two independent parallel circuits across the battery.

(a) $V_{ab} = E = 120$ volt (at all instants)

(b) a is at higher potential.

(c) V_{cd} will decrease exponentially from 120 V to zero.

$\therefore V_{cd} = 120$ volt, just after the switch is closed.

(d) c will be at higher potential.

(e) When switch is opened, current through R_1 will immediately become zero. While through R_2, will decrease to zero from the value $\frac{E}{R_2} = 2.4$ A $= i_0$ (say), exponentially. Path of this decay of current will be *cdbac*.

$\therefore$ Just after the switch is opened,

$V_{ab} = -i_0R_1 = -2.4 \times 30 = -72$ volt

(f) Point b is at higher potential.

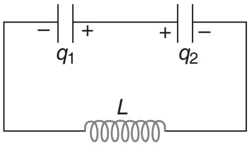
(g) $V_{cd} = -i_0(R_1 + R_2) = -2.4(80) = -192$ volt

(h) This time point d will be at higher potential.

7. $q_1 = 8CV_0 \quad : \quad q_2 = CV_0$

$$q_1 + q_2 = 9CV_0$$

In the absence of inductor, this $9C_0V$ will distribute as $6CV_0$ in $2C$ and $3CV_0$ in C. Thus, mean position of q_1 is $6CV_0$ and mean position of q_2 is $3CV_0$.



At $t = 0$, q_1 is $2CV_0$ more than its mean position and q_2 is $2CV_0$ less.

Thus, $\quad q_0 = 2CV_0$

$$C_{\text{net}} = \frac{2C}{3}$$

$$\therefore\quad \omega = \frac{1}{\sqrt{LC_{\text{net}}}} = \sqrt{\frac{3}{2LC}}$$

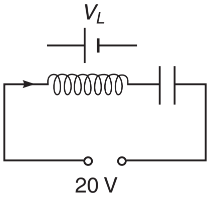
(a) $I_{\max} = q_0\omega$

(b) $V_1 = \frac{6CV_0}{2C} = 3V_0$

and $$V_2 = \frac{3CV_0}{C} = 3V_0$$

(c) $i = q_0 \sin\omega t$

8.



(a) $V_L = L\dfrac{di}{dt} = (1\times10^{-3})\dfrac{d}{dt}(20t) = 0.02$ V

$= 20$ mV

(b) $q = \int_0^t i\,dt = \int_0^t (20t)\,dt = 10t^2$

$$V = \frac{q}{C} = \frac{10t^2}{10^{-6}} = (10^{+7}t^2)\ \text{V}$$

(c) $\dfrac{q^2}{2C} > \dfrac{1}{2}Li^2$

or $$\frac{(10t^2)^2}{2\times10^{-6}} > \frac{1}{2}\times10^{-3}\times(20t)^2$$

or $$t > 63.2\times10^{-6}\ \text{s}$$

or $$t > 63.2\ \mu\text{s}$$ **Ans.**

9. In steady state when switch was closed,

$$i_0 = E/R = (1/5)\text{A} = 0.2\ \text{A}$$

After switch is opened, it becomes *L-C* circuit in which peak value current is 0.2 A.

$\therefore$ $$\frac{1}{2}Li_0^2 = \frac{1}{2}CV_0^2$$

or $$L = \frac{V_0^2}{i_0^2}.C = \frac{(150)^2}{(0.2)^2}\times0.5\times10^{-6} = 0.28\ \text{H}$$ **Ans.**

10. (a) $e = Bvl = 0.8\times7.5\times0.5 = 3$ V

(b) Current will flow in anti-clockwise direction, as magnetic field in ⊗ direction passing through the closed loop is increasing. Therefore, induced current will produce magnetic field in ⊙ direction.

(c) $F = F_m = ilB = \dfrac{e}{R}lB = \left(\dfrac{Bvl}{R}\right)Bl = \dfrac{B^2l^2}{R}v$

$$= \frac{(0.8)^2(0.5)^2}{1.5}\times7.5 = 0.8\ \text{N}$$

(d) $Fv = 0.8\times7.5 = 6$ W **Ans.**

$$i^2R = \left(\frac{Bvl}{R}\right)^2 R = \frac{B^2l^2}{R}.v^2$$

$$= \frac{(0.8)^2(0.5)^2}{1.5}\times(7.5)^2 = 6\ \text{W}$$ **Ans.**

So, we can see that both rates are equal.

11. (a) Magnitude of induced electric field due to change in magnetic flux is given by

$$\oint \mathbf{E}\cdot d\mathbf{l} = \frac{d\phi}{dt} = S.\frac{dB}{dt}$$

or $$El = \pi R^2(2B_0t) \qquad \left(\frac{dB}{dt} = 2B_0t\right)$$

Here, E = induced electric field due to change in magnetic flux

or $$E(2\pi R) = 2\pi R^2B_0t$$

or $$E = B_0Rt$$

Hence, $$F = QE = B_0QRt$$

This force is tangential to ring. Ring starts rotating when torque of this force is greater than the torque due to maximum friction ($f_{\max} = \mu mg$) or when

$$\tau_F \geq \tau_{f_{\max}}$$

Taking the limiting case

$$\tau_F = \tau_{f_{\max}} \text{ or } FR = (\mu mg)R$$

or $$F = \mu mg$$

or $$B_0QRt = \mu mg$$

It is given that ring starts rotating after 2. So, putting $t = 2$, we get

$$\mu = \frac{2B_0RQ}{mg}$$ **Ans.**

(b) After 2

$$\tau_F > \tau_{f_{\max}}$$

Therefore, net torque is

$$\tau = \tau_F - \tan_{f_{\max}} = B_0QR^2t - \mu mgR$$

Substituting $\mu = \dfrac{2B_0QR}{mg}$, we get

$$\tau = B_0QR^2(t-2)$$

or $$I\left(\frac{d\omega}{dt}\right) = B_0QR^2(t-2)$$

or $$mR^2\left(\frac{d\omega}{dt}\right) = B_0QR^2(t-2)$$

or $$\int_0^\omega d\omega = \frac{B_0Q}{m}\int_2^4 (t-2)\,dt$$

or $$\omega = \frac{2B_0Q}{m} \qquad \text{...(i)}$$

Now, magnetic field is switched off, i.e only retarding torque is present due to friction. So, angular retardation will be

$$\alpha = \frac{\tau_{f_{\max}}}{I} = \frac{\mu mgR}{mR^2} = \frac{\mu g}{R}$$

Therefore, applying

$$\omega^2 = \omega_0^2 - 2\alpha\theta$$

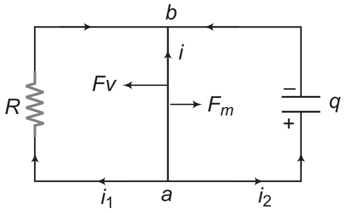
or $$0 = \left(\frac{2B_0Q}{m}\right)^2 - 2\left(\frac{\mu g}{R}\right)\theta$$

or $$\theta = \frac{2B_0^2Q^2R}{\mu m^2 g}$$

Substituting $\mu = \frac{2B_0RQ}{mg}$

We get $\theta = \frac{B_0Q}{m}$ **Ans.**

12. Let v be the velocity of connector at some instant of time. Then,



$$V_{ab} = Bvl,\ i_1 = \frac{Bvl}{R},\ q = C(Bvl)$$

$$\therefore \quad i_2 = \frac{dq}{dt} = CBl\frac{dv}{dt}$$

Now, $i = i_1 + i_2 = \frac{Bvl}{R} + CBl\frac{dv}{dt}$

Magnetic force, $F_m = ilB = \frac{B^2l^2}{R}\cdot v + B^2l^2C\cdot\frac{dv}{dt}$

Further, $F_{\text{net}} = F - F_m$

or $$m\frac{dv}{dt} = F - \frac{B^2l^2}{R}v - B^2l^2C\frac{dv}{dt}$$

$$\therefore \quad \int_0^v \frac{dv}{F - \frac{B^2l^2}{R}v} = \int_0^t \frac{dt}{m + B^2l^2C}$$

Integrating we get,

$$v = \frac{FR}{B^2l^2}\left[1 - e^{-\left(\frac{B^2l^2}{mR + RB^2l^2C}\right)t}\right]$$

Terminal velocity in this case is : $v_T = \frac{FR}{B^2l^2}$ **Ans.**

13. With key K_1 closed, C_1 and C_2 are in series with the battery in steady state.

$\therefore \quad C_{\text{net}} = 1\,\mu\text{F}$ or $q_0 = C_{\text{net}}\ V = 20\,\mu\text{C}$

(a) With K_1 opened and K_2 closed, charge on C_2 will remain as it is, while charge on C_1 will oscillate in L-C_1 circuit.

$$\omega = \frac{1}{\sqrt{LC_1}}$$

$$= \frac{1}{\sqrt{0.2\times10^{-3}\times2\times10^{-6}}}$$

$$= 5\times10^4 \text{ rad/s}$$ **Ans.**

(b) Since, at $t = 0$, charge is maximum $(= q_0)$. Therefore, current will be zero.

$$\frac{1}{2}Li^2 = \frac{1}{3}\left(\frac{1}{2}\frac{q^2}{C}\right)$$

or $$i = \frac{q}{\sqrt{3LC}} = \frac{q\omega}{\sqrt{3}}$$

From the expression,

$$i = \omega\sqrt{q_0^2 - q^2}$$

We have, $$\frac{q\omega}{\sqrt{3}} = \omega\sqrt{q_0^2 - q^2}$$

or $$q = \frac{\sqrt{3}}{2}q_0$$

Since at $t = 0$, charge is maximum or q_0, so we can write

$$q = q_0\cos\omega t \text{ or } \frac{\sqrt{3}q_0}{2} = q_0\cos\omega t$$

or $$\omega t = \frac{\pi}{6} \text{ or } t = \frac{\pi}{6\omega} = \frac{\pi}{6\times5\times10^4}$$

$$= 1.05\times10^{-5}\text{ s}$$

(c) $q = \frac{\sqrt{3}}{2}q_0 = \frac{\sqrt{3}}{2}\times20 = 10\sqrt{3}\,\mu\text{C}$ **Ans.**

14. In the capacitor,

$$q_i = CV_i = (10\times10^{-3})(5) = 0.05\text{ C}$$

$$q_f = CV_f = (10\times10^{-3})(10) = 0.1\text{ C}$$

$\therefore$ Charge in capacitor will increase from 0.05 C to 0.1 C exponentially.

Time constant for this increase would be

$$\tau_C = CR = 1\text{ s.}$$

$\therefore$ Charge at time t will be

$$q = 0.05 + (0.1 - 0.05)(1 - e^{-t/\tau_C})$$

$$= 0.1 - 0.05\,e^{-t/\tau_C}$$

(a) At $t = 1\text{ s}, q = 0.1 - 0.05\, e^{-1} = 0.0816\text{ C}$

$$V = \frac{q}{C} = \frac{0.0816}{10 \times 10^{-3}} = 8.16\text{ V}$$ **Ans.**

(b) This charge 0.0816 C is also the maximum charge q_0 of L-C oscillations.

From energy conservation equation,

$\frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}Li_0^2$ we have,

$$i_0 = \frac{q_0}{\sqrt{LC}} = \frac{0.0816}{\sqrt{25 \times 10^{-3} \times 10 \times 10^{-3}}}$$

$$= 5.16\text{ A}$$ **Ans.**

Further, $\omega = \frac{1}{\sqrt{LC}}$ or $f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{25 \times 10^{-3} \times 10 \times 10^{-3}}}$$

$$= 10\text{ Hz}$$ **Ans.**

15. (a) Let at time t velocity of rod be v (towards right) and current in the circuit is i (from a to b). The magnetic force on it is ilB (towards right). Writing the equation of motion of the rod,

$$m \cdot \frac{dv}{dt} = i\,l\,B = \left(\frac{E_0 - Blv}{R}\right) lB$$

$$\int_0^v \frac{dv}{\frac{E_0Bl}{mR} - \frac{B^2l^2}{mR}v} = \int_0^t dt$$

$$\therefore \quad v = \frac{E_0}{Bl}\left(1 - e^{-\frac{B^2l^2}{mR}t}\right)$$ **Ans.**

(b) $i = \frac{E_0 - Blv}{R}$

16. Let v be the velocity at some instant. Then, motional emf, $V = Bvl$

Charge stored in capacitor $q = CV = (CBl)v$

Current in the wire $= \frac{dq}{dt} = (CBl)\frac{dv}{dt}$

Magnetic force, $F_m = ilB = (CB^2l^2)\frac{dv}{dt}$ (upwards)

$\therefore$ Net force, $F_{\text{net}} = mg - F_m$

or $$m\frac{dv}{dt} = mg - (CB^2l^2)\frac{dv}{dt}$$

$\therefore$ $\frac{dv}{dt}$ = acceleration, $a = \frac{mg}{m + CB^2l^2}$

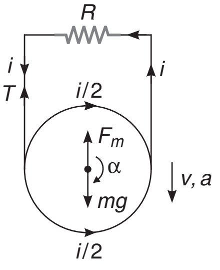
Since, a = constant

$$\therefore \quad x = \frac{1}{2}at^2 = \frac{mgt^2}{2(m + CB^2l^2)}$$

17. Let at time t velocity of ring be v (downwards)

$$e = Bv\,(2r) = 2Bvr$$

(Two batteries of emf $2Bvr$ are connected in parallel)



$$\therefore \quad i = \frac{e}{R} = \frac{2Bvr}{R}$$

Now, $$a = \frac{mg - F_m - T}{m}$$

Here, $$F_m = 2\left[\left(\frac{i}{2}\right)(2r)B\right] = 2irB = \frac{4B^2r^2v}{R}$$

$$\therefore \quad a = g - \frac{4B^2r^2v}{mR} - \frac{T}{m} \quad \text{...(i)}$$

$$\alpha = \frac{Tr}{mr^2} = \frac{T}{mr} \quad \text{...(ii)}$$

$$a = r\alpha = \frac{T}{m} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$a = \frac{g}{2} - \frac{2B^2r^2v}{mR}$$

or $$\int_0^v \frac{dv}{\frac{g}{2} - \frac{2B^2r^2v}{mR}} = \int_0^t dt$$

or $$v = \frac{mgR}{4B^2r^2}\left(1 - e^{-\frac{2B^2r^2}{mR}t}\right)$$ **Ans.**

$$i = \frac{2Bvr}{R} = \frac{mg}{2Br}\left(1 - e^{-\frac{2B^2r^2}{mR}t}\right)$$

and $$v_T = \frac{mgR}{4B^2r^2}$$ **Ans.**

18. (a) Suppose v be the velocity of rod ef when it has fallen a distance, x. Then,

$V_{fe} = V_{cb}$ or $Bvl = L(di/dt)$

or $B(dx/dt)l = L(di/dt)$ or $Bl(dx) = L(di)$

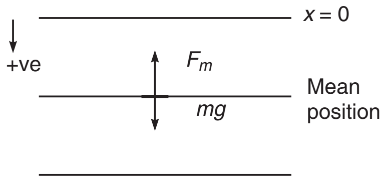
Integrating, we get $Li = Blx$

or $$i = \left(\frac{Bl}{L}\right)x \quad \text{...(i)}$$

Now, magnetic force opposite to displacement x will be

$$F = F_m = ilB = \left(\frac{B^2l^2}{L}\right)x$$

A constant downward force is mg.



So, this is similar situation like spring-block system in vertical position. In which a force $F = kx$ acts upwards and a constant force mg acts downwards.

Hence, the wire will execute SHM, where

$$k = \frac{B^2l^2}{L}$$

Amplitude will be at $F_m = mg$

$$\text{or} \qquad \left(\frac{B^2l^2}{L}\right)A = mg \Rightarrow A = \frac{mgL}{B^2l^2}$$

At $t = 0$, rod is in its extreme position. Therefore, if we write the equation from mean position we will write,

$$X = -A\cos\omega t$$

But, $x = X + A = A - A\cos\omega t = A(1-\cos\omega t)$

$$\text{where,} \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{B^2l^2}{mL}} \qquad \textbf{Ans.}$$

(b) From Eq. (i),

$$i_{\max} = \left(\frac{Bl}{L}\right)x_{\max}$$

$$\text{Here,} \quad x_{\max} = 2A = \frac{2mgL}{B^2l^2}$$

$$\therefore \quad i_{\max} = \left(\frac{Bl}{L}\right)\left(\frac{2mgL}{B^2l^2}\right) = \frac{2mg}{Bl} \qquad \textbf{Ans.}$$

(c) Maximum velocity,

$$v_0 = \omega A = \left(\sqrt{\frac{B^2l^2}{mL}}\right)\left(\frac{mgL}{B^2l^2}\right) = g\sqrt{\frac{mL}{B^2l^2}} = \frac{g\sqrt{mL}}{Bl}$$

Ans.

19. (a) At time t, $v = a_0t$

Motional emf, $V = Bvl = Ba_0lt$

Total resistance $= \dfrac{R_1R_2}{R_1+R_2}$

$$\therefore \quad i = \frac{(Ba_0lt)(R_1+R_2)}{R_1R_2} \qquad \textbf{Ans.}$$

(b) From right hand rule, we can see that points a and b will be at higher potential and c and d at lower potentials.

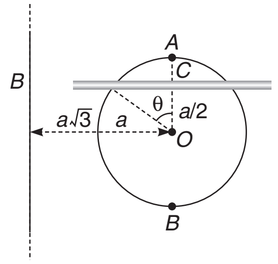
$$F_m = ilB = \frac{B^2l^2a_0t}{R_1R_2}(R_1+R_2)$$

Let F be the external force applied, then,

$$F - F_m = ma_0$$

$$\therefore \quad F = F_m + ma_0 = \frac{B^2l^2a_0t}{R_1R_2}(R_1+R_2) + ma_0$$

20. (a) At the given instant,



$$AC = \frac{a}{2}, \quad OC = \frac{a}{2}$$

$$\text{and} \qquad \cos\theta = \frac{a/2}{a} = \frac{1}{2}, \quad \theta = \frac{\pi}{3}$$

$\therefore$ Velocity of rod

$= \left(+\dfrac{v_0}{2}\right)$ along the direction of current.

Emf induced across the ends M and N

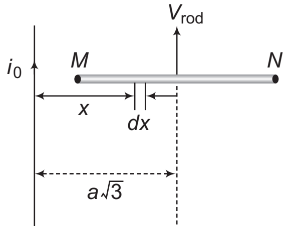
$$E_{\text{rod}} = \int_{a\sqrt{3}-\frac{a\sqrt{3}}{2}}^{a\sqrt{3}+\frac{a\sqrt{3}}{2}} v_{\text{rod}}B\,dx$$

$$= v_{\text{rod}}\int_{a\frac{\sqrt{3}}{2}}^{3a\frac{\sqrt{3}}{2}} \frac{\mu_0 i_0}{2\pi x}\,dx$$

$$= \left(\frac{v_0}{2}\right)\frac{\mu_0 i_0}{2\pi}\ln\left(\frac{3}{1}\right)$$

with end M at higher potential.

Since, the effective length of both the arcs MAN and MBN is MN.



$$\therefore \quad E_{MAN} = E_{MBN} = v_{\text{loop}} \frac{\mu_0 i_0}{2\pi} \ln 3$$

$$= v_0 \frac{\mu_0 i_0}{2\pi} \ln 3$$

with point M at higher potential.

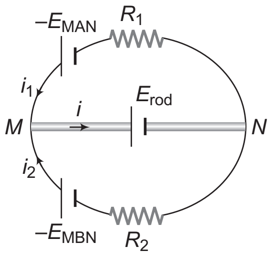
Resistance of arc MAN

$$\Rightarrow \quad R_1 = (R)2(a\theta) = 2aR\,\frac{\pi}{3}$$

$\Rightarrow$ Resistance of arc MBN

$$\Rightarrow \quad R_2 = (R)a(2\pi - 2\theta) = 4aR\,\frac{\pi}{3}$$

Equivalent circuit at the given instant is shown in the figure.



Current through the rod MN,

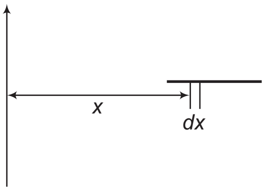
$$i = (i_1 + i_2) = \left(\frac{E_{MAN} - E_{\text{rod}}}{R_1}\right) + \left(\frac{E_{MBN} - R_{\text{rod}}}{R_2}\right)$$

$$i = (E_{MAN} - E_{\text{rod}})\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$= \frac{v_0 \mu_0 i_0 (\ln 3)}{4\pi}\left[\left(\frac{1}{2} + \frac{1}{4}\right)\frac{3}{aR\pi}\right]$$

$$= \frac{9 v_0 i_0 \mu_0}{16 aR\pi^2} \ln (3)$$ **Ans.**

(b) Force on the rod



$$F_{\text{rod}} = \int_{\frac{a\sqrt{3}}{2}}^{\frac{3a\sqrt{3}}{2}} i\,dxB$$

$$= \frac{i\mu_0 i_0}{2\pi} \ln 3 = \frac{9\,\mu_0^2 i_0^2 v_0}{32 aR\pi^3} (\ln 3)^2$$

21. Since, PQ and DC both cut the lines of field.

$\therefore$ Motional emf will be induced across both of them.

Integrating, potential difference across

$$dx \;\Rightarrow\; \int de = \int_a^{2a} v\left(\frac{\mu_0 i_0}{2\pi x}\right) dx$$

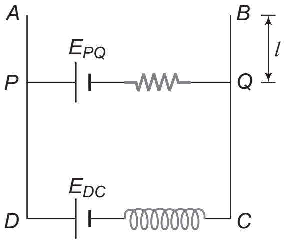
$$e_{DC} = \frac{v\mu_0 i_0}{2\pi} \ln 2 \text{ with } D \text{ at higher potential}$$

$$e_{PQ} = \frac{2v\mu_0 i_0}{2\pi} \ln 2 \text{ with } P \text{ at higher potential}$$

The relative velocity of the rod PQ w.r.t. U frame

$$v_{\text{rel}} = 2v - v = v$$

Now, time taken by it to loose the contact $t = \dfrac{l}{v}$



From equivalent electrical network

Net emf in the closed loop $QPDC$.

$$e = e_{PQ} - e_{DC} = \frac{v\mu_0 i_0}{2\pi} \ln 2$$

Growth of current in the L-R circuit is given by

$$i = i_0(1 - e^{-tR/L}) = \left(\frac{e}{R}\right)(1 - e^{-tR/L})$$

At time $t = \dfrac{l}{v}$

$$\Rightarrow \qquad i = \left(\frac{e}{R}\right)(1 - e^{-\frac{Rl}{vL}})$$

28. Alternating Current

INTRODUCTORY EXERCISE 28.1

1. (a) $X_L = 2\pi fL$

(b) $L = \dfrac{X_L}{2\pi f}$

(c) $X_C = \dfrac{1}{2\pi fC}$

(d) $C = \dfrac{1}{2\pi f X_C}$

2.

$$V_L = \sqrt{V^2 - V_R^2}$$

$$= \sqrt{(150)^2 - (100)^2}$$

$$= 111.8 \text{ V}$$

$$V_L = IX_L = I\,(2\pi fL)$$

$$\therefore \quad L = \frac{V_L}{2\pi fI} = \frac{111.8}{2\pi \times 50 \times 10}$$

$$= 0.036 \text{ H}$$

3. $\phi = 0$, if

$$X_L = X_C$$

$$\text{or} \quad 2\pi f\,L = \frac{1}{2\pi\, fC}$$

$$\therefore \quad L = \frac{1}{(2\pi f)^2\, C}$$

$$= \frac{1}{(360)^2 \times 10^{-6}}$$

$$= 7.7 \text{ H}$$

$$Z = R \qquad (\text{When } X_L = X_C)$$

$$\therefore \quad I = \frac{V}{Z} = \frac{V}{R} = \frac{120}{20}$$

$$= 6 \text{ A}$$

INTRODUCTORY EXERCISE 28.2

1.

$$f = \frac{1}{2\pi\sqrt{2C}} = \text{resonance frequency}$$

$$= \frac{1}{2\pi\sqrt{0.03 \times 2 \times 10^{-6}}}$$

$$= 650 \text{ Hz}$$

At resonance, $X_L = X_C$ and $Z = R$

$$\therefore \quad \cos\phi = \frac{R}{Z} = 1 \quad \text{or} \quad \phi = 0°$$

2. $R = \dfrac{V}{I} = \dfrac{40}{10} = 4\Omega$

$Z = \dfrac{V}{I} = \dfrac{200}{10} = 20\ \Omega$

Power factor, $\cos\phi = \dfrac{R}{Z} = \dfrac{4}{20}$

$$= 0.2$$

Exercises

LEVEL 1

Assertion and Reason

1. $Z = \sqrt{R^2 + (X_C - X_L)^2}$

From this expression, we can see that X_C may be greater than Z also.

2. At resonance frequency f_r,

$$X_C = X_L$$

Now, $X_L = 2\pi fL$ or $X_L \propto f$

For $f > f_r$, $X_L > X_C$

At resonance, $X_L = X_C \Rightarrow Z = R$

$$\therefore \quad \cos\phi = \frac{R}{Z} = 1$$

$$\text{or} \quad \phi = 0°$$

3. $\omega = \dfrac{1}{\sqrt{LC}}$

By inserting a slab, C will increase. So, ω will decrease.

4. Average value $= \dfrac{\text{total area under } i\text{-}t \text{ graph}}{\text{total time interval}}$

$$= \frac{8 + 2 + 2 + 4 + 2}{6}$$

$$= \frac{18}{6} = 3 \text{ A}$$

5. $Z = \sqrt{R^2 + (X_C \sim X_L)^2}$

X_L will increase. So, Z may increase or decrease, depending on the value of X_C. Therefore, current may decrease or increase.

6. $V_L = V_C \Rightarrow X_L = X_C$

So this resonance condition.

8. $$P = I_{\text{rms}}^2 R$$
$$= \left(\frac{2}{\sqrt{2}}\right)^2 (10)$$
$$= 20 \text{ W}$$

9. $I_{\text{DC}} = \frac{V_{\text{DC}}}{r}$ (r = internal resistance of inductor)

$$I_{\text{AC}} = \frac{V_{\text{AC}}}{Z} = \frac{V_{\text{AC}}}{\sqrt{r^2 + X_L^2}}$$

If $V_{\text{DC}} = V_{\text{AC}}$, then $I_{\text{DC}} > I_{\text{AC}}$

10. $I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_L^2}}$

$$= \frac{V}{\sqrt{R^2 + (2\pi f L)^2}}$$

with increase in frequency, I will decrease.

$$\tan\phi = \frac{X_L}{R} = \frac{2\pi f L}{R}$$

with increase in frequency $\tan\phi$ and therefore ϕ will increase.

11. At resonance, $X_L = X_C$

$\Rightarrow$ $Z = R$

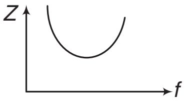
Hence, $I = \frac{V}{Z} = \frac{V}{R}$

So, current at resonance depends on R.

Objective Questions

2. Average value in AC comes out to be zero.

3.



Impedance first decreases, then increases. At resonance frequency Z is minimum.

4. In case of only capacitor and inductor phase difference between current and voltage should be 90°.

5. $I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \approx 0.707\, I_0$

6. $\phi = 90°$ between V and I functions.

$$\therefore \quad P = V_{\text{rms}} I_{\text{rms}} \cos\phi = 0$$

7. $Z = \sqrt{R^2 + (X_L - X_C)^2}$

8. $\because \quad P = V_{\text{rms}} I_{\text{rms}} \cos\theta$

$$= \left(\frac{V_0}{\sqrt{2}}\right)\left(\frac{I_0}{\sqrt{2}}\right)\cos\theta$$
$$= \frac{V_0 I_0}{2}\cos\theta$$

9. $V_0 = 240$ V

$$\therefore \quad V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = \frac{240}{\sqrt{2}} = 170 \text{ V}$$
$$\omega = 120 \text{ rad/s}$$
$$f = \frac{\omega}{2\pi} = \frac{120}{2 \times 3.14} = 19 \text{ Hz}$$

10. $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 8 \times 10^{-6}}}$

$= 500$ rad/s

11. $P = V_{\text{rms}} I_{\text{rms}} \cos\phi$

$$= \left(\frac{100}{\sqrt{2}}\right)\left(\frac{100 \times 10^{-3}}{\sqrt{2}}\right) \cdot \cos\left(\frac{\pi}{3}\right)$$

$= 2.5$ W

12. $X_C = \frac{1}{\omega C}$

$\omega = 0$ for DC

$\therefore \quad X_C = \infty$

or it becomes a perfect insulator.

13. Substituting $t = \frac{1}{600}$ s in the given equation, we have

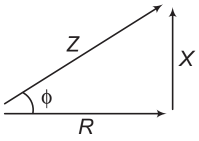
$$V = 10\cos(100\pi)\left(\frac{1}{600}\right)$$
$$= 10\cos\frac{\pi}{6}$$
$$= 5\sqrt{3} \text{ V}$$

15. $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$ or $X_C \propto \frac{1}{f}$

i.e. X_C *versus* f graph is a rectangular hyperbola.

16. $\sin\phi = \frac{X}{Z} = \frac{1}{\sqrt{3}}$

$\therefore \quad \phi = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$



17. $i = \sqrt{2}\sin\left[\omega t + \frac{\pi}{4} + \frac{\pi}{2}\right]$

Hence, phase difference between V and i is $\frac{\pi}{2}$. So, power consumed = 0.

18. $I_{DC} = \dfrac{V_{DC}}{R}$

$$I = \frac{100}{R} \Rightarrow R = 100\ \Omega$$

$$I_{AC} = \frac{V_{AC}}{\sqrt{R^2 + X_L^2}}$$

$$0.5 = \frac{100}{\sqrt{(100)^2 + X_L^2}}$$

or $X_L = 100\sqrt{3}\ \Omega = (2\pi fL)$

$\therefore$ $L = \dfrac{100\sqrt{3}}{2\pi f} = \dfrac{100\sqrt{3}}{2\pi\ (50)}$

$$= \left(\frac{\sqrt{3}}{\pi}\right) \text{H}$$

19. $X_C = \dfrac{1}{\omega C} = \dfrac{1}{100 \times 10^{-6}} = 10^4\ \Omega$

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{(200\sqrt{2})/\sqrt{2}}{10^4}$$

$$= 0.02\text{ A } = 20\text{ mA}$$

20. $V = \sqrt{V_R^2 + V_L^2}$

$$= \sqrt{(20)^2 + (15)^2} = 25\text{ V}$$

But this is the rms value.

$\therefore$ Peak value $= \sqrt{2}\ V_{rms} = 25\sqrt{2}$ V

22. Resistance does not depend on the frequency of AC.

23. An ideal choke coil should have almost zero internal resistance. Otherwise, it will consume some power.

24. 45° phase angle means,

$$X_L = R$$

$\therefore$ $(2\pi fL) = R$

$\therefore$ $L = \dfrac{R}{2\pi f}$

$$= \frac{100}{(2\pi)\ (10^3)}$$

$$= 0.0159\text{ H}$$

$$\approx 16\text{ mH}$$

25. $T = \dfrac{1}{f} = \dfrac{1}{50}$ s

$$t = \frac{T}{4} = \frac{1}{200}\text{ s}$$

$$= 5 \times 10^{-3}\text{ s}$$

$$= 5\text{ ms}$$

26. $P = V_{rms}\, I_{rms} \cos\phi$

$$= \left(\frac{V_0}{\sqrt{2}}\right)\left(\frac{I_0}{\sqrt{2}}\right)\cos 60°$$

$$= \left(\frac{220}{\sqrt{2}}\right)\left(\frac{4}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= 220\text{ W}$$

27. I_C is 90° ahead of the applied voltage and I_L lags behind the applied voltage by 90°. So, there is a phase difference of 180° between I_L and I_C.

$\therefore$ $I = I_C - I_L = 0.2$ A

28. V_L function is cos function, which is 90° ahead of the current function. Hence, current function should be sin function.

29. $\text{H}_{DC} = I^2Rt$

$$\text{H}_{AC} = I_{rms}^2\ Rt = \left(\frac{I}{\sqrt{2}}\right)^2 Rt = \frac{I^2\ Rt}{2}$$

$\therefore$ $\dfrac{\text{H}_{DC}}{\text{H}_{AC}} = \dfrac{2}{1}$

30. $V_C = \sqrt{V^2 - V_R^2} = \sqrt{(20)^2 - (12)^2}$

$$= 16\text{ V}$$

Subjective Questions

1. $X_L = \omega L = 100\ \Omega$

$$X_C = \frac{1}{\omega C} = 312.5\ \Omega$$

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$

$$= \sqrt{(300)^2 + (312.5 - 100)^2}$$

$$= 368\ \Omega$$

(a) $I_0 = \dfrac{V_0}{Z} = \dfrac{120}{368} = 0.326$ A

(b) Since, $X_C > X_L$, voltage lags the current by an angle given by

$$\phi = \cos^{-1}\left(\frac{R}{Z}\right) = \cos^{-1}\left(\frac{300}{368}\right) = 35.3°$$

(c) $(V_0)_R = I_0R = (0.326)\ 300 = 97.8$ V

$(V_0)_L = I_0X_L = (0.326)\ (100) = 32.6$ V

$(V_0)_C = I_0X_C = (0.326)\ (312.5) = 102$ V

2. (a) Voltage lags

$\therefore$ $X_C > X_L$

Power factor, $\cos\phi = \dfrac{R}{Z}$

$$= \frac{R}{\sqrt{R^2 + (X_C - X_L)^2}}$$

To increase the power factor denominator should decrease. Hence, X_L should increase. Therefore, an inductor is required to be connected.

(b) $\cos\phi = \frac{R}{Z} = 0.72$

$$\therefore \quad R = 0.72\,Z = 0.72 \times 60 = 43.2\ \Omega$$

$$(X_C - X_L) = \sqrt{(60)^2 - (43.2)^2} = 41.64\ \Omega$$

New inductor of inductance 41.64 Ω should be added in the circuit.

$$L = \frac{X_L}{2\pi f} = \frac{41.64}{2\pi\ (50)} = 0.133\ \text{H}$$

3. (b) $f = \frac{\omega}{2\pi} = \frac{6280}{2 \times 3.14} = 1000\ \text{Hz}$

(c) $\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ or 30°

Power factor $= \cos\phi = \cos 30° = \frac{\sqrt{3}}{2}$

From the given functions of V and i, we can see that current function leads the voltage function.

(d) $Z = \frac{170}{8.5} = 20\ \Omega$ ...(i)

$$\cos\phi = \frac{\sqrt{3}}{2} = \frac{R}{Z} = \frac{R}{20}$$

$$\therefore \quad R = 10\sqrt{3}\ \Omega = 17.32\ \Omega$$

$$X_C = \sqrt{Z^2 - R^2} = 10\ \Omega$$

$$C = \frac{1}{\omega X_C} = \frac{1}{6280 \times 10}\ \text{F} = 15.92 \times 10^{-6}\ \text{F}$$

4. $I_0 = \frac{V_0}{X_L} = \frac{V_0}{\omega L}$

5. (a) $I_0 = \frac{(V_0)_R}{R} = \frac{2.5}{300} = 8.33 \times 10^{-3}\ \text{A} = 8.33\ \text{mA}$

Current function and V_R function are in phase. Hence,

$$I = (8.33\ \text{mA}) \cos [(950\ \text{rad/s})\ t]$$

(b) $X_L = \omega L = 950 \times 0.8 = 760\ \Omega$

(c) $(V_0)_L = I_0 X_L = (8.33 \times 10^{-3})\ (760) = 6.33\ \text{V}$

Now, V_L function leads the current (or V_R) function by 90°.

$$\therefore \quad V_L = 6.33 \cos (950\,t + 90°) = -6.33 \sin (950\,t)$$

6. $X_L = 2\pi f L = 301\ \Omega$

$$X_C = \frac{1}{2\pi f C} = 55\ \Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(240)^2 + (301 - 55)^2} = 343\ \Omega$$

(a) $\phi = \cos^{-1}\left(\frac{R}{Z}\right) = \cos^{-1}\left(\frac{240}{343}\right) = 45.8°$

$$\cos\phi = \frac{R}{Z} = \frac{240}{343} = 0.697$$

Since, $X_L > X_C$, voltage leads the current.

(b) Impedance $= Z = 343\ \Omega$

(c) $V_{\text{rms}} = I_{\text{rms}} Z = 0.45 \times 343 = 155\ \text{V}$

(d) $P = I_{\text{rms}}^2 R = (0.45)^2\ (240) = 48.6\ \text{W}$

(e) $P = P_R = 48.6\ \text{W}$

(f) $P_C = 0$

(g) $P_L = 0$

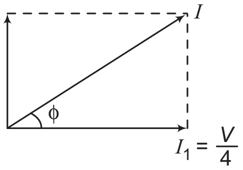
LEVEL 2

Single Correct Option

1. $I_2 = \frac{V}{X_C} = \frac{V}{3}$ (here V = rms value)

$$I_1 = \frac{V}{R} = \frac{V}{4}$$

I_2 is 90° ahead of applied voltage function and I_1 is in phase with it.



$$\tan\phi = \frac{V/3}{V/4} = \frac{4}{3}$$

$$\therefore \qquad \phi = 53°$$

2. I_R and I_L are in same phase and phase difference between them and applied voltage lies between 0° and 90°.

3. $X_L = \omega L = (5 \times 10^{-3})\,(2000) = 10\ \Omega$

$$X_C = \frac{1}{\omega C} = \frac{1}{(2000)\,(50 \times 10^{-6})} = 10\ \Omega$$

Since, $X_L = X_C$ circuit is in resonance.

$$Z = R = (6 + 4) = 10\ \Omega$$

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{(20/\sqrt{2})}{10} = 1.414\ \text{A}$$

This is also the reading of ammeter.

$$V = 4I_{\text{rms}}$$
$$\approx 5.6\ \text{volt}$$

4. $$I_R = \frac{V_{\text{rms}}}{R} = \frac{200}{100} = 0.2\ \text{A}$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{(2\pi)\,(5 \times 10^3)\left(\frac{1}{\pi} \times 10^{-6}\right)}$$

$$= 100\ \Omega$$

$$\therefore \quad I_C = \frac{V_{\text{rms}}}{X_C} = \frac{200}{100} = 2\ \text{A}$$

I_C is 90° ahead of the applied voltage and I_R is in phase with the applied voltage. Hence, there is a phase difference of 90° between I_R and I_C too.

$$\therefore \qquad I = \sqrt{I_R^2 + I_C^2}$$
$$= \sqrt{(2)^2 + (2)^2}$$
$$= 283\ \text{A}$$

5. Average value of 5 sin 100 ωt is zero. But average value of 5 A (= constant current) is 5 A. Hence, average value of total given function is 5 A.

6. V function is sin function. I function is ahead of V function. Hence, the circuit should be capacitive in nature.

Further, $\qquad \phi = 45°$

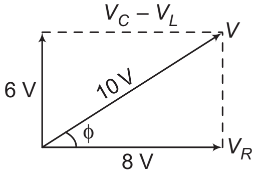
$$\therefore \qquad X_C = R \quad \text{or} \quad \omega C = R$$

$$\text{or} \qquad C = \frac{R}{\omega} = \frac{R}{100} = 0.01\,R$$

In option (b), this condition is satisfied.

7. $V = \sqrt{V_R^2 + (V_C - V_L)^2} = 10\ \text{V}$

$V_C > V_L$, hence current leads the voltage.



$$\text{Power factor} = \cos\phi = \frac{8}{10} = 0.8$$

8. See the hint of miscellaneous example numbers 6 and 7 of solved examples.

9. $$V_S = \sqrt{V_R^2 + V_L^2}$$
$$= \sqrt{(70)^2 + (20)^2} = 72.8\ \text{V}$$

$$\tan\phi = \frac{X_L}{R} = \frac{V_L}{V_R} = \frac{20}{70} = \frac{2}{7}$$

10. In first case, $X_C = \frac{V}{I} = \frac{220}{0.25} = 880\ \Omega$

In the second case, $R = \frac{V}{I} = \frac{220}{0.25} = 880\ \Omega$

In the combination of P and Q,

$$\tan\phi = \frac{X_C}{R} = 1$$

$$\therefore \qquad \phi = 45°$$

Since the circuit is capacitive, current leads the voltage. Further,

$$Z = \sqrt{R^2 + X_C^2} = 880\sqrt{2}\ \Omega$$

$$I = \frac{V}{Z} = \frac{220}{880\sqrt{2}} = \frac{1}{4\sqrt{2}}\ \text{A}$$

11. See the hints of miscellaneous example numbers 6 and 7 of solved examples.

12. $i = \frac{V}{R}$, i.e. circuit is in resonance. Hence,

$$V_C = V_L = 200\ \text{V}$$

13. $$P = I_{\text{rms}}^2\, R = \left(\frac{V_{\text{rms}}}{Z}\right)^2 R$$

$$= \left[\frac{(V_0/\sqrt{2})^2}{R^2 + \omega^2 L^2}\right] R$$

$$= \frac{V_0^2 R}{2\,(R^2 + \omega^2 L^2)}$$

14. $X_L = \omega L$

If ω is very low, then $X_L \approx 0$

$\therefore \quad V_L \approx 0$

or $\quad V = V_C = V_0$

15. $I_{\max} = \dfrac{V}{R}$ (at resonance)

$$6 = \frac{24}{R}$$

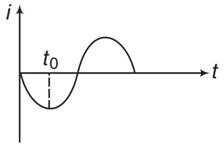
$\therefore \quad R = 4\ \Omega$

$$I_{DC} = \frac{V}{R+r} = \frac{12}{4+4} = 1.5\ \text{A}$$

16. $V_R = \sqrt{V^2 - V_C^2} = \sqrt{(10)^2 - (8)^2} = 6\ \text{V}$

$$\tan\phi = \frac{X_C}{X_R} = \frac{V_C}{V_R} = \frac{8}{6} = \frac{4}{3}$$

17. Current will lead the voltage function by 90° voltage function is cos function. Therefore, current function will be – sin function.



$$t_0 = \frac{T}{4} = \frac{(2\pi/\omega)}{4} = \frac{\pi}{2\omega}$$

$$= \frac{\pi}{2\,(\pi/2)} = 1\ \text{s}$$

18. $R = \dfrac{1}{\omega C} = X_C$

$\therefore \quad Z = \sqrt{R^2 + X_C^2} = \sqrt{2}\,R \quad$ (as $X_C = R$)

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{2}\,R} \quad \text{...(i)}$$

When ω becomes $\dfrac{1}{\sqrt{3}}$ times, X_C will become $\sqrt{3}$ times or $\sqrt{3}\,R$.

$$Z' = \sqrt{(R^2) + (\sqrt{3}R)^2} = 2R$$

$$I_0' = \frac{V_0}{Z'} = \frac{V_0}{2R} = \frac{I_0}{\sqrt{2}}$$

More than One Correct Options

1. $\sqrt{V_R^2 + V_L^2} = 100 \quad$ ...(i)

$V_L \sim V_C = 120 \quad$ ...(ii)

$\sqrt{V_R^2 + (V_L \sim V_C)^2} = 130 \quad$ ...(iii)

Solving these three equations, we get

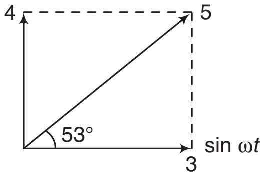
$V_R = 50$ V, $V_L = 86.6$ V and

$V_C = 206.6$ V

$$\text{Power factor} = \cos\phi = \frac{R}{Z} = \frac{V_R}{V} = \frac{50}{130} = \frac{5}{13}$$

Since $V_C > V_L$, circuit is capacitive in nature.

2. $i = 5 \sin(\omega t + 53°)$



$i_0 = 5\,\text{A}$

$$\therefore \quad i_{\text{rms}} = \frac{i_0}{\sqrt{2}} = \frac{5}{\sqrt{2}}\ \text{A}$$

Mean value of current in positive half cycle is

$$\frac{2}{\pi}\,i_0 = \left(\frac{2}{\pi}\right)(5) = \left(\frac{10}{\pi}\right)\ \text{A}$$

In $V = V_m \sin\omega t$, current $i = 5\sin(\omega t + 53°)$ leads the voltage function. Hence, circuit is capacitive in nature. Same is the case with part (*d*).

3. $P_R = V_R\, i$

$$\therefore \quad i = \frac{P_R}{V_R} = \frac{60}{60} = 1\ \text{A}$$

Now, $\quad V_L = \sqrt{V^2 - V_R^2}$

$= \sqrt{(100)^2 - (60)^2}$

$= 80\ \text{V} = i X_L = i\,(2\pi f L)$

$$L = \frac{80}{2\pi f i}$$

$$= \frac{80}{(2\pi)(50)(1)} = \frac{4}{5\pi}\ \text{H}$$

If we connect another resistance R in series, then it should consume 40 V, so that remaining 60 V is used by the tube light.

$$R = \frac{V}{i} = \frac{40}{1} = 40\ \Omega$$

4. Power factor, $\cos\phi = \dfrac{R}{Z}$

When circuit contains only resistance, then

$$Z = R \quad \Rightarrow \quad \cos\phi = 1$$

When circuit contains only inductance, then

$$R = 0$$

$\therefore \quad \cos\phi = 0$

5. (a) $X_L > X_C$, hence voltage function will lead the current function.

(b) $Z = \sqrt{R^2 + (X_L - X_C)^2}$

$= \sqrt{(10)^2 + (20 - 10)^2}$

$= 10\sqrt{2}\ \Omega$

(c) $\cos\phi = \frac{R}{Z} = \frac{1}{\sqrt{2}}$

Hence, $\phi = 45°$

(d) Power factor $= \cos\phi = \frac{R}{Z} = \frac{1}{\sqrt{2}}$

6. (b) At resonance frequency (ω_r)

$$X_L > X_C$$

In the given values, $X_L > X_C$. Hence,

$$\omega > \omega_r$$

As, $X_L = \omega L \Rightarrow X_L \propto \omega$

and $X_C = \frac{1}{\omega C} \Rightarrow X_C \propto \frac{1}{\omega}$

(c) If frequency is increased from the given value, X_L will further increase. So, $X_L - X_C$ will increase. Hence, net impedance will increase.

(d) If frequency is decreased from the given value, then X_C will increase and X_L will decrease. So, $X_L - X_C$ may be less than the previous value or $X_C - X_L$ may be greater than the previous. So, Z may either increase or decrease. Hence, current may decrease or increase.

7. (a) $V_R = IR = 80$ V

(b) $X_C = \frac{V_C}{I} = \frac{100}{2} = 50\ \Omega$

(c) $V_L = IX_L = 40$ V

(d) $V = V_{\text{rms}} = \sqrt{V_R^2 + (V_C - V_L)^2}$

$= \sqrt{(80)^2 + (100 - 40)^2}$

$= 60$ V

$\therefore\ V_0 = \sqrt{2}\, V_{\text{rms}} = 60\sqrt{2}$ V

8. $I = \frac{V}{Z}$

$$= \frac{V}{\sqrt{R^2 + \left(\omega L \sim \frac{1}{\omega C}\right)^2}}$$

By increasing R, current will definitely decrease by change in L or C, current may increase or decrease.

Comprehension Based Questions

1 to **3.**

$$V_{\text{DC}} = I_{\text{DC}}\, R$$

$$\therefore \quad R = \frac{V_{\text{DC}}}{I_{\text{DC}}} = \frac{12}{4} = 3\ \Omega$$

$$I_{\text{AC}} = \frac{V_{\text{AC}}}{Z} = \frac{V_{\text{AC}}}{\sqrt{R^2 + X_L^2}}$$

$$2.4 = \frac{12}{\sqrt{(3)^2 + X_L^2}}$$

Solving this equation, we get

$$X_L = 4\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{50 \times 2500 \times 10^{-6}}$$

$$= 8\ \Omega$$

$$Z = \sqrt{R^2 + (X_C - X_L)^2} = 5\Omega$$

$$\therefore \quad I = \frac{V_{\text{DC}}}{Z} = \frac{12}{5} = 2.4\text{ A} = I_{\text{rms}}$$

$$P = I_{\text{rms}}^2\, R = (2.4)^2\,(3)$$

$$= 17.28\text{ W}$$

At given frequency, $X_C > X_L$. If ω is further decreased, X_C will increase $\left(\text{as } X_C \propto \frac{1}{\omega}\right)$ and X_L will increase (as $X_L \propto \omega$).

Therefore, $X_C - X_L$ and hence Z will increase. So, current will decrease.

4.

$$\omega = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{4.9 \times 10^{-3} \times 10^{-6}}}$$

$$= \frac{10^5}{7}\text{ rad/s}$$

5. $X_C = \frac{1}{\omega C} = \frac{1}{\left(\frac{10^5}{7}\right)(10^{-6})} = 70\ \ \Omega$

$$Z_P = \sqrt{R_P^2 + X_C^2}$$

$$= \sqrt{(32)^2 + (70)^2}$$

$$\approx 77\ \Omega$$

6. At maximum current means at resonance,

$$X_L = X_C,\ Z = R$$

$$\therefore \quad \text{Power factor} = \cos\phi = \frac{R}{Z} = 1$$

Match the Columns

2. (a) $\phi = 0°$ between voltage function and current function.

(b) $I = I_0 \sin(\omega t - 90°)$

i.e. $\phi = 90°$ and voltage function leads the current function.

(c) Current function leads the voltage function. So,

$$X_C > X_L$$

(d) Voltage function leads the current function. So,

$$X_L > X_C$$

3. See the hint of Q.No. 6 and 8 of section more than one correct options. Then,

$$P = I_{\text{rms}}^2 R$$

By increasing R, current i_{rms} will decrease but the power, $P = I_{\text{rms}}^2 R$ may increase or decrease.

4. (a) $R = \dfrac{V_R}{I} = \dfrac{40}{2} = 20\ \Omega$

(b) $V_C = IX_C = 2 \times 30 = 60$ V

(c) $V_L = I\ X_L = 2 \times 15 = 30$ V

(d) $V = \sqrt{V_R^2 + (V_C - V_L)^2} = 50$ V

5. (a) Resistance does not depend on the value of ω.

(b) $X_C = \dfrac{1}{\omega C}$ or $X_C \propto \dfrac{1}{\omega}$

(c) $X_L = \omega L$ or $X_L \propto \omega$

(d) Z is minimum at $\omega = \omega_r$ and $Z_{\text{min}} = R$

Below or above ω_r

$$Z = \sqrt{R^2 + (X_L \sim X_C)^2}$$

or $\quad Z > R$

Subjective Questions

1. $P = V_{\text{rms}} i_{\text{rms}} \cos\phi$

or $\quad 200 = 230 \times 8 \times \cos\phi$

$\therefore \quad \cos\phi = 0.108$

or $\quad \phi = 83.8°$

Further, $\quad P = i_{\text{rms}}^2 R$

$\therefore \quad R = \dfrac{P}{i_{\text{rms}}^2} = \dfrac{200}{(8)^2} = 3.125\ \Omega$

(a) $\tan\phi = \dfrac{X_C - X_L}{R}$

$\therefore \quad \dfrac{1}{2\pi\ fC} - (2\pi fL) = R\tan\phi$

$\therefore \quad L = \dfrac{1}{(2\pi f)^2 C} - \dfrac{R\tan\phi}{2\pi f}$

$= \dfrac{1}{(2\pi \times 50)^2 \times 20 \times 10^{-6}} - \dfrac{3.125 \tan 83.8°}{2\pi \times 50}$

$= 0.416$ H

(b) $\tan\phi = \dfrac{X_L - X_C}{R}$

or $\quad 2\pi fL - \dfrac{1}{(2\pi fC)} = R\tan\phi$ **Ans.**

$$L = \frac{1}{(2\pi f)^2 C} + \frac{R\tan\phi}{(2\pi f)}$$

$= \dfrac{1}{(2 \times \pi \times 50)^2 \times 20 \times 10^{-6}} + \dfrac{3.125 \tan 83.8°}{(2\pi \times 50)}$

$= 0.597$ H **Ans.**

2. Average current will be zero as positive and negative half cycles are symmetrical RMS current can also be obtained from 0 to τ/2.

$$I = \left(\frac{I_0}{\tau/2}\right)t = \left(\frac{2I_0}{\tau}\right)t$$

$\Rightarrow \quad I^2 = \left(\dfrac{4I_0^2}{\tau^2}\right)t^2$

$\Rightarrow \quad \langle I^2 \rangle_{0-\tau/2} = \dfrac{\displaystyle\int_0^{\tau/2} \frac{4I_0^2}{\tau^2} t^2 dt}{\tau/2}$

or $\quad \langle I^2 \rangle_{0-\tau/2} = \dfrac{I_0^2}{3}$

$\Rightarrow \quad I_{\text{rms}} = \sqrt{\dfrac{I_0^2}{3}} = \dfrac{I_0}{\sqrt{3}}$

3. (a) $0.5 = \dfrac{R_1}{Z_1}$

Further, $\quad P = V_{\text{rms}} i_{\text{rms}} \cos\phi$

or $\quad 100 = 230 \times \dfrac{230}{Z_1} \times 0.5$

$\therefore \quad Z_1 = 264.5\ \Omega$

and $\quad R_1 = 132.25\ \Omega$

Further, $X_L = \sqrt{Z_1^2 - R_1^2} = \dfrac{\sqrt{3}}{2} Z_1$

$= 229\ \Omega$

In second case, $0.6 = \dfrac{R_2}{Z_2}$

and $\quad 60 = \dfrac{230 \times 230}{Z_2} \times 0.6$

$\therefore \quad Z_2 = 529\ \Omega$

and $\quad R_2 = 317.4\ \Omega$

Further, $\quad X_C = \sqrt{Z_2^2 - R_2^2}$

$= 423.2\ \Omega$

When connected in series,

$R = R_1 + R_2 = 449.65\ \Omega$

$X_C - X_L = 194.2$

$$\therefore \quad Z = \sqrt{(449.65)^2 + (194.2)^2}$$

$$= 489.79\ \Omega$$

Power factor, $\cos\phi = \dfrac{R}{Z} = 0.92$ (leading)

$$P = V_{\text{rms}}\, i_{\text{rms}} \cos\phi$$

$$= (230)\left(\frac{230}{489.79}\right)(0.92)$$

$$= 99\ \text{W}$$ **Ans.**

(b) Since, $X_C - X_L = 194.2\ \Omega$

Therefore, if 194.2 Ω inductive reactance is to be added in series, then it will become only R circuit and power factor will become unity.

4. (a) $i_{\text{rms}} = \dfrac{V_1}{R} = \dfrac{40}{4} = 10$ A

$$i_0 = \sqrt{2}\, i_{\text{rms}} = 10\sqrt{2}\ \text{A}$$ **Ans.**

(b) $\dfrac{E_0}{\sqrt{2}} = \sqrt{V_1^2 + (V_2 - V_1)^2} = 50$ V

$$\therefore \quad E_0 = 50\sqrt{2}\ \text{V}$$

(c) $X_L = (\omega L) = \dfrac{V_2}{i_{\text{rms}}} = \dfrac{40}{10} = 4\ \Omega$

$$\therefore \quad L = \frac{4}{\omega} = \frac{4}{100\pi}\ \text{H} = \frac{1}{25\pi}\ \text{H}$$ **Ans.**

$$X_C = \frac{1}{\omega C} = \frac{V_1}{i_{\text{rms}}} = \frac{10}{10} = 1\ \Omega$$

$$\therefore \quad C = \frac{1}{\omega} = \frac{1}{100\pi}\ \text{F}$$ **Ans.**

5. $\cos\phi_1 = 0.5$

$$\therefore \quad \phi_1 = 60°$$

$$\cos\phi_2 = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \phi_2 = 30°$$

Let R be the effective resistance of the box. Then,

$$\tan\phi_1 = \frac{X_C}{R} \quad \text{or} \quad \sqrt{3} = \frac{X_C}{R} \qquad \ldots\text{(i)}$$

$$\tan\phi_2 = \frac{X_C}{R+10} \quad \text{or} \quad \frac{1}{\sqrt{3}} = \frac{X_C}{R+10} \qquad \ldots\text{(ii)}$$

From these two equations, we get $R = 5\ \Omega$

6. (a) $V_R = IR = 80$ V, $V_C = 100$ V

and $V_L = IX_L = 160$ V

$$\therefore \quad V = \sqrt{V_R^2 + (V_L - V_C)^2} = 100\ \text{V}$$ **Ans.**

Note *Value of X_L have been taken from part (b).*

(b) Since the current is lagging behind, there should be an inductor in the box.

$$X_C = \frac{V_C}{I} = 100\ \Omega$$

Now, $0.8 = \dfrac{R}{Z} = \dfrac{80}{\sqrt{(80)^2 + (X_L - 100)^2}}$

Solving, we get

$$X_L = 160\ \Omega$$

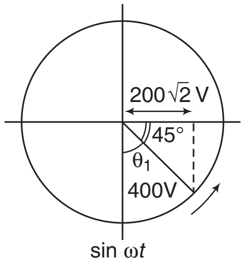
or $\quad \omega L = 160$

$\therefore \quad (2\pi f L) = 160$

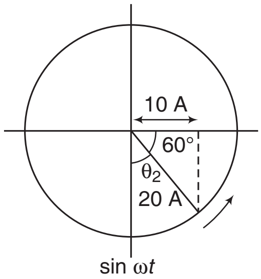
$$\therefore \quad L = \frac{160}{2\pi f} = \frac{160}{(2\pi)\times 50}$$

$$= \frac{1.6}{\pi}\ \text{H}$$

7. (a)



Reference circle for voltage



Reference circle for current

$$\omega = 2\pi f = (100\pi)\ \text{rad/s}$$

From the above two figures, we can write

$$V = 400 \sin(\omega t + \theta_1) = 400 \sin\left[100\pi t + \frac{\pi}{4}\right]$$ **Ans.**

$$i = 20 \sin(\omega t + \theta_2) = 20 \sin\left[100\pi t + \frac{\pi}{6}\right]$$ **Ans.**

(b) Phase difference between V and i

$$\phi = (\pi/4 - \pi/6) = \frac{\pi}{12} \text{ or } 15°$$

$$P = V_{\text{rms}} i_{\text{rms}} \cos\phi = \left(\frac{400}{\sqrt{2}}\right)\left(\frac{20}{\sqrt{2}}\right)\cos 15°$$

$$= 3864 \text{ W}$$ **Ans.**

8. $\omega = \dfrac{2}{\sqrt{LC}} = \dfrac{2}{\sqrt{5\times10^{-3}\times20\times10^{-6}}}$

$$= 6324.5 \text{ rad/s}$$

$$X_L = \omega L = (6324.5)(5\times10^{-3}) = 31.62\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{6324.5\times20\times10^{-6}} = 7.9\ \Omega$$

$\therefore \quad Z = X_L - X_C = 23.72\ \Omega$

(a) Maximum voltage across capacitor

$= i_0 X_C = (0.211)(7.9) = 1.67$ mV

$\therefore$ Maximum charge

$q_0 = (20\times10^{-6})(1.67\times10^{-3}) = 33.4$ nC

(b) $i_0 = \dfrac{V_0}{Z} = \dfrac{5}{23.72}$ mA $= 0.211$ mA

(c) Since $X_L > X_C$, current in the circuit will lag behind the applied voltage by $\pi/2$.

Further voltage across the inductor will lead this current by $\pi/2$.

Therefore, applied voltage and voltage across inductor are in phase.

Voltage across the capacitor will lag the circuit current by $\pi/2$.

Therefore, phase difference between V_L and V_C will be 180°.

9. $X_{L_1} = \omega L_1 = (2\pi\times50)(0.02) = 6.28\ \Omega$

$\therefore \quad Z_1 = \sqrt{R_1^2 + X_{L_1}^2}$

$$= \sqrt{(5)^2 + (6.28)^2} = 8.0\ \Omega$$

$$P_1 = (I_{\text{rms}})_1 V_{\text{rms}} \cos\phi_1 = \left(\frac{100}{8}\right)(100)\left(\frac{5}{8}\right)$$

$$= 781.25 \text{ W}$$

$$X_{L_2} = \omega L_2 = (2\pi\times50)(0.08)$$

$$= 25.13\ \Omega$$

$\therefore \quad Z_2 = \sqrt{R_2^2 + X_{L_2}^2}$

$$= 25.15\ \Omega$$

$\therefore \quad P_2 = (i_{\text{rms}})_2 V_{\text{rms}} \cos\phi_2$

$$= \left(\frac{100}{25.15}\right)(100)\left(\frac{1}{25.15}\right)$$

$$= 15.8 \text{ W}$$

$\therefore \quad P_{\text{Total}} = P_1 + P_2 = 797$ W

10. $Z_1 = \dfrac{115}{3} = 38.33\ \Omega$

$$\cos\phi_1 = \frac{R_1}{Z_1}$$

$\Rightarrow \quad R_1 = Z_1 \cos\phi_1$

$$= (38.33)(0.6) = 23\ \Omega$$

$$X_L = \sqrt{Z_1^2 - R_1^2} = 30.67\ \Omega$$

$$Z_2 = \frac{115}{5} = 23\ \Omega$$

$$R_2 = Z_2\cos\phi_2 = (23)(0.707)$$

$$= 16.26$$

$$X_C = \sqrt{Z_2^2 - R_2^2}$$

$$= \sqrt{(23)^2 - (16.26)^2}$$

$$= 16.26\ \Omega$$

When connected in series,

$$R = R_1 + R_2 = 39.26\ \Omega$$

$$X_L - X_C = 14.41\ \Omega$$

$\therefore \quad Z = \sqrt{R^2 + (X_L - X_C)^2}$

$$= 41.82\ \Omega$$

(a) $i = \dfrac{V}{Z} = \dfrac{230}{41.82} = 5.5$ A

(b) $P = i^2 R = (5.5)^2(39.26)$

$$= 1187.6 \text{ W} \approx 1.188 \text{ kW}$$

(c) Power factor $= \cos\phi = \dfrac{R}{Z} = \dfrac{39.26}{41.92} = 0.939$

Since $X_L > X_C$, this power factor is lagging.

JEE Main and Advanced

Previous Years' Questions (2018-13)

JEE Main

1. Three concentric metal shells A, B and C of respective radii a, b and c $(a < b < c)$ have surface charge densities $+\sigma, -\sigma$ and $+\sigma$, respectively. The potential of shell B is **(2018)**

(a) $\frac{\sigma}{\varepsilon_0}\left[\frac{a^2-b^2}{a}+c\right]$ (b) $\frac{\sigma}{\varepsilon_0}\left[\frac{a^2-b^2}{b}+c\right]$

(c) $\frac{\sigma}{\varepsilon_0}\left[\frac{b^2-c^2}{b}+a\right]$ (d) $\frac{\sigma}{\varepsilon_0}\left[\frac{b^2-c^2}{c}+a\right]$

2. A parallel plate capacitor of capacitance 90 pF is connected to a battery of emf 20 V. If a dielectric material of dielectric constant $K = \frac{5}{3}$ is inserted between the plates, the magnitude of the induced charge will be **(2018)**

(a) 1.2 nC (b) 0.3 nC (c) 2.4 nC (d) 0.9 nC

3. In an AC circuit, the instantaneous emf and current are given by

$e = 100 \sin 30t$, $i = 20 \sin\left(30t - \frac{\pi}{4}\right)$

In one cycle of AC, the average power consumed by the circuit and the wattless current are, respectively

(a) 50 , 10 (b) $\frac{1000}{\sqrt{2}}$, 10

(c) $\frac{50}{\sqrt{2}}$, 0 (d) 50 , 0

4. Two batteries with emf 12 V and 13 V are connected in parallel across a load resistor of 10 Ω. The internal resistances of the two batteries are 1 Ω and 2 Ω, respectively. The voltage across the load lies between **(2018)**

(a) 11.6 V and 11.7 V (b) 11.5 V and 11.6 V

(c) 11.4 V and 11.5 V (d) 11.7 V and 11.8 V

5. An electron, a proton and an alpha particle having the same kinetic energy are moving in circular orbits of radii r_e, r_p, r_α respectively, in a uniform magnetic field B. The relation between r_e, r_p, r_α is **(2018)**

(a) $r_e > r_p = r_\alpha$ (b) $r_e < r_p = r_\alpha$

(c) $r_e < r_p < r_\alpha$ (d) $r_e < r_\alpha < r_p$

6. The dipole moment of a circular loop carrying a current I, is m and the magnetic field at the centre of the loop is B_1. When the dipole moment is doubled by keeping the current constant, the magnetic field at the centre of the loop is B_2. The ratio $\frac{B_1}{B_2}$ is **(2018)**

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

7. For an R-L-C circuit driven with voltage of amplitude v_m and frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, the current exhibits resonance. The quality factor, Q is given by **(2018)**

(a) $\frac{\omega_0 L}{R}$ (b) $\frac{\omega_0 R}{L}$

(c) $\frac{R}{\omega_0 C}$ (d) $\frac{CR}{\omega_0}$

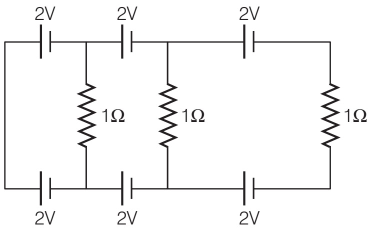
8. In a potentiometer experiment, it is found that no current passes through the galvanometer when the terminals of the cell are connected across 52 cm of the potentiometer wire. If the cell is shunted by a resistance of 5 Ω, a balance is found when the cell is connected across 40 cm of the wire. Find the internal resistance of the cell. **(2018)**

(a) 1 Ω (b) 1.5 Ω (c) 2 Ω (d) 2.5 Ω

9. On interchanging the resistances, the balance point of a meter bridge shifts to the left by 10 cm. The resistance of their series combination is 1 kΩ. How much was the resistance on the left slot before interchanging the resistances? **(2018)**

(a) 990 Ω (b) 505 Ω (c) 550 Ω (d) 910 Ω

10. In the below circuit, the current in each resistance is **(2017)**



(a) 0.25 A (b) 0.5 A (c) 0 A (d) 1 A

11. When a current of 5 mA is passed through a galvanometer having a coil of resistance 15 Ω, it shows full scale deflection. The value of the resistance to be put in series with the galvanometer to convert it into a voltmeter of range 0-10 V is **(2017)**

(a) 2.045×10^3 Ω (b) 2.535×10^3 Ω
(c) 4.005×10^3 Ω (d) 1.985×10^3 Ω

12. Which of the following statements is false? **(2017)**

(a) In a balanced Wheatstone bridge, if the cell and the galvanometer are exchanged, the null point is disturbed
(b) A rheostat can be used as a potential divider
(c) Kirchhoff's second law represents energy conservation
(d) Wheatstone bridge is the most sensitive when all the four resistances are of the same order of magnitude

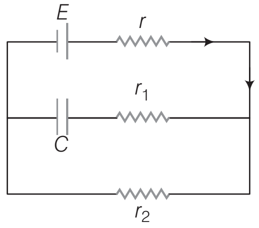
13. An electric dipole has a fixed dipole moment **p**, which makes angle θ with respect to X-axis. When subjected to an electric field $\mathbf{E}_1 = E\hat{\mathbf{i}}$, it experiences a torque $\mathbf{T}_1 = \tau\hat{\mathbf{k}}$. When subjected to another electric field $\mathbf{E}_2 = \sqrt{3}E_1\hat{\mathbf{j}}$, it experiences a torque $\mathbf{T}_2 = -\mathbf{T}_1$. The angle θ is **(2017)**

(a) 45° (b) 60° (c) 90° (d) 30°

14. A capacitance of 2 μF is required in an electrical circuit across a potential difference of 1kV. A large number of 1 μF capacitors are available which can withstand a potential difference of not more than 300 V. The minimum number of capacitors required to achieve this is **(2017)**

(a) 16 (b) 24 (c) 32 (d) 2

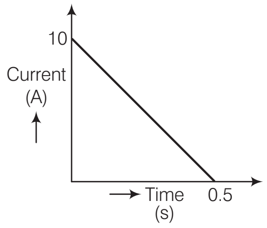
15. In the given circuit diagram, when the current reaches steady state in the circuit, the charge on the capacitor of capacitance C will be **(2017)**



(a) $CE \dfrac{r_1}{(r_2 + r)}$ (b) $CE \dfrac{r_2}{(r + r_2)}$

(c) $CE \dfrac{r_1}{(r_1 + r)}$ (d) CE

16. In a coil of resistance 100 Ω, a current is induced by changing the magnetic flux through it as shown in the figure. The magnitude of change in flux through the coil is **(2017)**

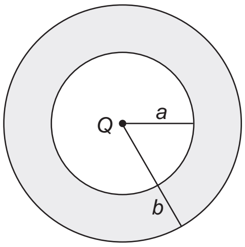


(a) 225 Wb (b) 250 Wb (c) 275 Wb (d) 200 Wb

17. A galvanometer having a coil resistance of 100 Ω gives a full scale deflection when a current of 1 mA is passed through it. The value of the resistance which can convert this galvanometer into ammeter giving a full scale deflection for a current of 10 A, is **(2016)**

(a) 0.01 Ω (b) 2 Ω (c) 0.1 Ω (d) 3 Ω

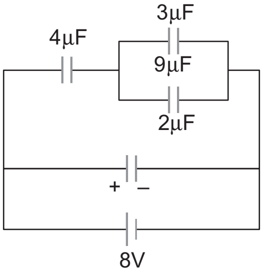
18. The region between two concentric spheres of radii a and b, respectively (see the figure), has volume charge density $\rho = \frac{A}{r}$, where, A is a constant and r is the distance from the centre. At the centre of the spheres is a point charge Q. The value of A, such that the electric field in the region between the spheres will be constant, is **(2016)**



(a) $\frac{Q}{2\pi a^2}$ (b) $\frac{Q}{2\pi(b^2 - a^2)}$

(c) $\frac{2Q}{\pi(a^2 - b^2)}$ (d) $\frac{2Q}{\pi a^2}$

19. A combination of capacitors is set-up as shown in the figure. The magnitude of the electric field, due to a point charge Q (having a charge equal to the sum of the charges on the $4\mu F$ and $9\mu F$ capacitors), at a point distant 30 m from it, would equal to **(2016)**



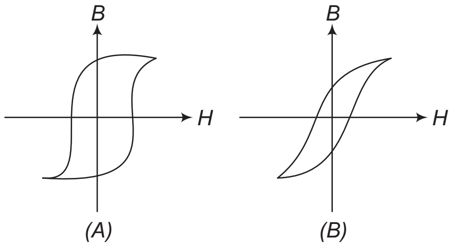
(a) 240 N/C (b) 360 N/C

(c) 420 N/C (d) 480 N/C

20. Two identical wires A and B, each of length l, carry the same current I. Wire A is bent into a circle of radius R and wire B is bent to form a square of side a. If B_A and B_B are the values of magnetic field at the centres of the circle and square respectively, then the ratio $\frac{B_A}{B_B}$ is **(2016)**

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{16\sqrt{2}}$ (c) $\frac{\pi^2}{16}$ (d) $\frac{\pi^2}{8\sqrt{2}}$

21. Hysteresis loops for two magnetic materials A and B are as given below: **(2016)**



These materials are used to make magnets for electric generators, transformer core and electromagnet core. Then, it is proper to use

(a) A for electric generators and transformers

(b) A for electromagnets and B for electric generators

(c) A for transformers and B for electric generators

(d) B for electromagnets and transformers

22. An arc lamp requires a direct current of 10 A at 80 V to function. If it is connected to a 220 V (rms), 50 Hz AC supply, the series inductor needed for it to work is close to **(2016)**

(a) 80 H (b) 0.08 H

(c) 0.044 H (d) 0.065 H

23. Arrange the following electromagnetic radiations in the order of increasing energy. **(2016)**

A. Blue light B. Yellow light

C. X-ray D. Radio wave

(a) D, B, A, C (b) A, B, D, C

(c) C, A, B, D (d) B, A, D, C

24. When 5V potential difference is applied across a wire of length 0.1m, the drift speed of electrons is $2.5 \times 10^{-4} \text{ ms}^{-1}$. If the electron density in the wire is $8 \times 10^{28} \text{ m}^{-3}$ the resistivity of the material is close to **(2015)**

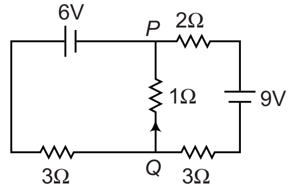
(a) $1.6 \times 10^{-8}\ \Omega\text{-m}$

(b) $1.6 \times 10^{-7}\ \Omega\text{-m}$

(c) $1.6 \times 10^{-5}\ \Omega\text{-m}$

(d) $1.6 \times 10^{-6}\ \Omega\text{-m}$

25. In the circuit shown below, the current in the 1Ω resistor is



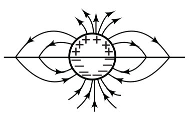
(2015)

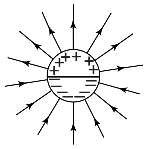
(a) 1.3 A, from P to Q
(b) 0.13 A, from Q to P
(c) 0 A
(d) 0.13 A, from P to Q

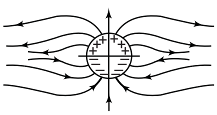
26. A uniformly charged solid sphere of radius R has potential V_0 (measured with respect to ∞) on its surface. For this sphere, the equipotential surfaces with potentials $\frac{3V_0}{2}, \frac{5V_0}{4}, \frac{3V_0}{4}$ and $\frac{V_0}{4}$ have radius R_1, R_2, R_3, and R_4 respectively. Then, **(2015)**

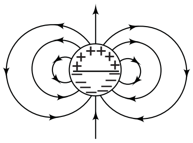
(a) $R_1 \neq 0$ and $(R_2 - R_1) > (R_4 - R_3)$
(b) $R_1 = 0$ and $R_2 > (R_4 - R_3)$
(c) $2R < R_4$
(d) $R_1 = 0$ and $R_2 < (R_4 - R_3)$

27. A long cylindrical shell carries positive surface charge σ in the upper half and negative surface charge $-\sigma$ in the lower half. The electric field lines around the cylinder will look like figure given in (figures are schematic and not drawn to scale) **(2015)**

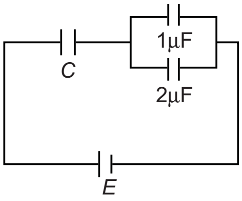
(a)


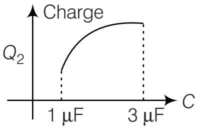
(b)


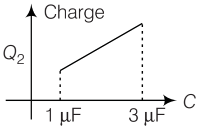
(c)


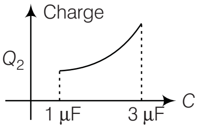
(d)


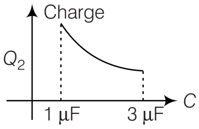
28. In the given circuit, charge Q_2 on the $2\mu F$ capacitor changes as C is varied from $1\,\mu F$ to $3\,\mu F$. Q_2 as a function of C is given properly by (figures are drawn schematically and are not to scale) **(2015)**



(a)


(b)


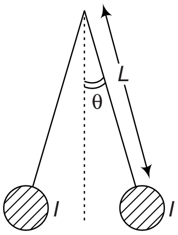
(c)


(d)


29. Two coaxial solenoids of different radii carry current I in the same direction. Let $\mathbf{F}_1$ be the magnetic force on the inner solenoid due to the outer one and $\mathbf{F}_2$ be the magnetic force on the outer solenoid due to the inner one. Then, **(2015)**

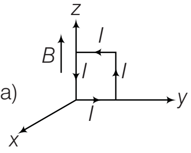
(a) $\mathbf{F}_1$ is radially outwards and $\mathbf{F}_2 = 0$
(b) $\mathbf{F}_1$ is radially inwards and $\mathbf{F}_2$ is radially outwards
(c) $\mathbf{F}_1$ is radially inwards and $\mathbf{F}_2 = 0$
(d) $\mathbf{F}_{21} = \mathbf{F}_2 = 0$

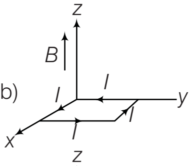
30. Two long current carrying thin wires, both with current I, are held by insulating threads of length L and are in equilibrium as shown in the figure, with threads making an angle θ with the vertical. If wires have mass λ per unit length then, the value of I is (g = gravitational acceleration) **(2015)**

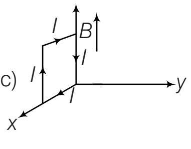


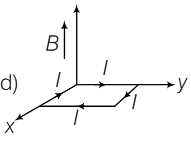
(a) $2\sin\theta\sqrt{\frac{\pi\lambda gL}{\mu_0\cos\theta}}$
(b) $\sin\theta\sqrt{\frac{\pi\lambda gL}{\mu_0\cos\theta}}$
(c) $2\sqrt{\frac{\pi gL}{\mu_0}\tan\theta}$
(d) $\sqrt{\frac{\pi\lambda gL}{\mu_0}\tan\theta}$

31. A rectangular loop of sides 10 cm and 5 cm carrying a current I of 12 A is placed in different orientations as shown in the figures below. **(2015)**

(a)


(b)


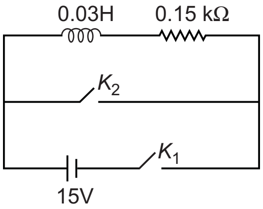
(c)


(d)


If there is a uniform magnetic field of 0.3 T in the positive z-direction, in which orientations the loop would be in (i) stable equilibrium and (ii) unstable equilibrium?

(a) (a) and (b) respectively
(b) (b) and (d) respectively
(c) (a) and (c) respectively
(d) (b) and (c) respectively

32. An inductor ($L = 0.03\,\text{H}$) and a resistor ($R = 0.15\,\text{k}\Omega$) are connected in series to a battery of 15V EMF in a circuit shown below. The key K_1 has been kept closed for a long time. Then at $t = 0$, K_1 is opened and key K_2 is closed simultaneously. At $t = 1$ ms, the current in the circuit will be ($e^5 \cong 150$) **(2015)**



(a) 100 mA (b) 67 mA (c) 0.67mA (d) 6.7 mA

33. In a large building, there are 15 bulbs of 40 W, 5 bulbs of 100 W, 5 fans of 80 W and 1 heater of 1 kW. The voltage of the electric mains is 220 V. The minimum capacity of the main fuse of the building will be **(2014)**

(a) 8 A (b) 10 A (c) 12 A (d) 14 A

34. Assume that an electric field $\mathbf{E} = 30x^2\,\hat{\mathbf{i}}$ exists in space. Then, the potential difference $V_A - V_O$, where V_O is the potential at the origin and V_A the potential at $x = 2$ m is **(2014)**

(a) 120 J (b) −120 J (c) −80 J (d) 80 J

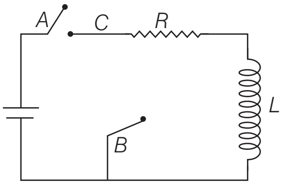
35. A parallel plate capacitor is made of two circular plates separated by a distance of 5 mm and with a dielectric of dielectric constant 2.2 between them. When the electric field in the dielectric is 3×10^4 V/m, the charge density of the positive plate will be close to **(2014)**

(a) 6×10^{-7} C/m^2 (b) 3×10^{-7} C/m^2
(c) 3×10^4 C/m^2 (d) 6×10^4 C/m^2

36. The coercivity of a small magnet where the ferromagnet gets demagnetised is $3 \times 10^3\ \text{Am}^{-1}$. The current required to be passed in a solenoid of length 10 cm and number of turns 100, so that the magnet gets demagnetised when inside the solenoid is **(2014)**

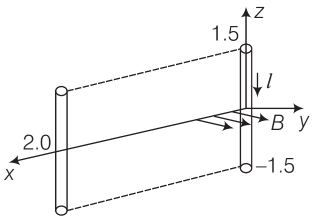
(a) 30 mA (b) 60 mA
(c) 3 A (d) 6 A

37. In the circuit shown here, the point C is kept connected to point A till the current flowing through the circuit becomes constant. Afterward, suddenly point C is disconnected from point A and connected to point B at time $t = 0$. Ratio of the voltage across resistance and the inductor at $t = L / R$ will be equal to **(2014)**



(a) $\dfrac{e}{1-e}$ (b) 1 (c) −1 (d) $\dfrac{1-e}{e}$

38. A conductor lies along the z-axis at $-1.5 \le z < 1.5$ m and carries a fixed current of 10.0 A in $-a_z$ direction (see figure).
For a field $\mathbf{B} = 3.0 \times 10^{-4}\, e^{-0.2x} a_y\ \text{T}$, find the power required to move the conductor at constant speed to $x = 2.0$ m, $y = 0$ in 5×10^{-3}s.
Assume parallel motion along the x-axis. **(2014)**



(a) 1.57 W (b) 2.97 W
(c) 14.85 W (d) 29.7 W

39. This question has Statement I and Statement II. Of the four choices given after the statements, choose the one that best describes the two statements. **(2013)**

Statement I Higher the range, greater is the resistance of ammeter.

Statement II To increase the range of ammeter, additional shunt needs to be used across it.

(a) If Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I

(b) If Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) If Statement I is true; Statement II is false

(d) If Statement I is false; Statement II is true

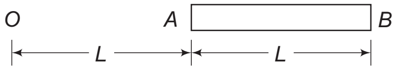
40. The supply voltage in a room is 120 V. The resistance of the lead wires is 6 Ω. A 60 W bulb is already switched on. What is the decrease of voltage across the bulb, when a 240 W heater is switched on in parallel to the bulb? **(2013)**

(a) zero (b) 2.9 V (c) 13.3 V (d) 10.4V

41. Two charges, each equal to q, are kept at $x = -a$ and $x = a$ on the x-axis. A particle of mass m and charge $q_0 = \frac{q}{2}$ is placed at the origin. If charge q_0 is given a small displacement $y\,(y << a)$ along the y-axis, the net force acting on the particle is proportional to **(2013)**

(a) y (b) $-y$ (c) $\frac{1}{y}$ (d) $-\frac{1}{y}$

42. A charge Q is uniformly distributed over a long rod AB of length L as shown in the figure. The electric potential at the point O lying at distance L from the end A is **(2013)**



(a) $\frac{Q}{8\pi\,\varepsilon_0 L}$ (b) $\frac{3\,Q}{4\pi\,\varepsilon_0 L}$

(c) $\frac{Q}{4\pi\,\varepsilon_0 L \ln(2)}$ (d) $\frac{Q\ln(2)}{4\pi\,\varepsilon_0 L}$

43. Two capacitors C_1 and C_2 are charged to 120 V and 200 V respectively. It is found that by connecting them together the potential on each one can be made zero. Then **(2013)**

(a) $5C_1 = 3C_2$ (b) $3C_1 = 5C_2$

(c) $3C_1 + 5C_2 = 0$ (d) $9C_1 = 4C_2$

44. Two short bar magnets of length 1 cm each have magnetic moments 1.20 Am^2 and 1.00 Am^2, respectively. They are placed on a horizontal table parallel to each other with their N poles pointing towards the South. They have a common magnetic equator and are separated by a distance of 20.0 cm. The value of the resultant horizontal magnetic induction at the mid-point O of the line joining their centres is close to (Horizontal component of the earth's magnetic induction is 3.6×10^{-5} Wb/m^2) **(2013)**

(a) 3.6×10^{-5} Wb /m^2 (b) 2.56×10^{-4} Wb /m^2

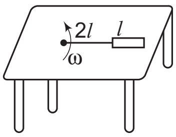
(c) 3.50×10^{-4} Wb /m^2 (d) 5.80×10^{-4} Wb /m^2

45. A circular loop of radius 0.3 cm lies parallel to a much bigger circular loop of radius 20 cm. The centre of the smaller loop is on the axis of the bigger loop. The distance between their centres is 15 cm. If a current of 2.0 A flows through the bigger loop, then the flux linked with smaller loop is **(2013)**

(a) 9.1×10^{-11} Wb (b) 6×10^{-11} Wb

(c) 3.3×10^{-11} Wb (d) 6.6×10^{-9} Wb

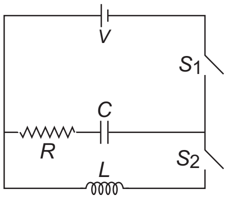
46. A metallic rod of length l is tied to a string of length $2l$ and made to rotate with angular speed ω on a horizontal table with one end of the string fixed. If there is a vertical magnetic field B in the region, the emf induced across the ends of the rod is **(2013)**



(a) $\frac{2B\omega l^3}{2}$ (b) $\frac{3B\omega l^3}{2}$

(c) $\frac{4B\omega l^2}{2}$ (d) $\frac{5B\omega l^2}{2}$

47. In a L-C-R circuit as shown below, both switches are open initially. Now, switch S_1 and S_2, are closed. (q is charge on the capacitor and $\tau = RC$ is capacitance time constant). Which of the following statement is correct? **(2013)**



(a) Work done by the battery is half of the energy dissipated in the resistor

(b) At $t = \tau$, $q = CV/2$

(c) At $t = 2\tau$, $q = CV(1 - e^{-2})$

(d) At $t = \tau/2$, $q = CV(1 - e^{-1})$

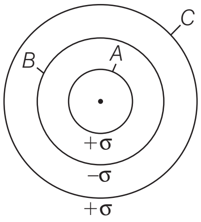
48. The amplitude of a damped oscillator decreases to 0.9 times its original magnitude is 5 s. In another 10 s, it will decrease to α times its original magnitude, where α equals **(2013)**

(a) 0.7 (b) 0.81

(c) 0.729 (d) 0.6

Answer *with* Explanations

1. (b) Potential of B = Potential due to charge on A + Potential due to charge on B + Potential due to charge on C.



$$\therefore \quad V_B = \frac{k(Q_A + Q_B)}{b} + \frac{kQ_C}{c}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{\sigma 4\pi a^2}{b} - \frac{\sigma 4\pi b^2}{b} + \frac{\sigma 4\pi c^2}{c}\right]$$

$$= \frac{\sigma\varepsilon}{\varepsilon_0}\left(\frac{a^2 - b^2}{b} + \frac{c^2}{c}\right) = \frac{\sigma}{\varepsilon_0}\left(\frac{a^2 - b^2}{b} + c\right)$$

$$V_B = \frac{\sigma}{\varepsilon_0}\left(\frac{a^2 - b^2}{b} + c\right)$$

2. (a) Magnitude of induced charge is given by

$$Q' = (K - 1)CV_0$$

$$= \left(\frac{5}{3} - 1\right) 90 \times 10^{-12} \times 20 = 1.2 \times 10^{-9}\text{C}$$

$$\Rightarrow \quad Q' = 1.2 \text{ nC}$$

3. (b) Given, $e = 100\sin 30\,t$

and $\quad i = 20\sin\left(30\,t - \frac{\pi}{4}\right)$

$\therefore$ Average power ,

$$P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos\phi$$

$$= \frac{100}{\sqrt{2}} \times \frac{20}{\sqrt{2}} \times \cos\frac{\pi}{4} = \frac{1000}{\sqrt{2}} \text{ watt}$$

Wattless current is,

$$I = I_{\text{rms}} \sin\phi$$

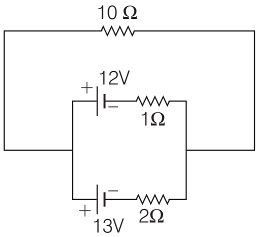
$$= \frac{20}{\sqrt{2}} \times \sin\frac{\pi}{4}$$

$$= \frac{20}{2} = 10\text{A}$$

$$\therefore \quad P_{\text{av}} = \frac{1000}{\sqrt{2}} \text{ watt}$$

and $\quad I_{\text{wattless}} = 10\text{A}$

4. (b)



For parallel combination of cells,

$$E_{\text{eq}} = \frac{\dfrac{E_1}{r_1} + \dfrac{E_2}{r_2}}{\dfrac{1}{r_1} + \dfrac{1}{r_2}}$$

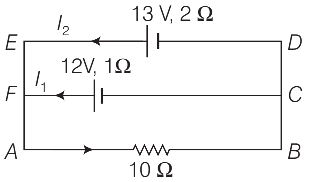
$$\therefore \quad E_{\text{eq}} = \frac{\dfrac{12}{1} + \dfrac{13}{2}}{\dfrac{1}{1} + \dfrac{1}{2}} = \frac{37}{3}\text{ V}$$

Potential drop across 10 Ω resistance,

$$V = \left(\frac{E}{R_{\text{total}}}\right) \times 10 = \frac{\dfrac{37}{3}}{\left(10 + \dfrac{2}{3}\right)} \times 10 = 11.56\text{V}$$

$$\therefore \quad V = 11.56\text{ V}$$

Alternative Method



Applying KVL, in loop *ABCFA*,

$$-12 + 10(I_1 + I_2) + 1 \times I_1 = 0$$

$$\Rightarrow \quad 12 = 11I_1 + 10I_2 \quad \ldots\text{(i)}$$

Similarly,

in loop *ABDEA*,

$$-13 + 10(I_1 + I_2) + 2 \times I_2 = 0$$

$$\Rightarrow \quad 13 = 10I_1 + 12I_2 \quad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$I_1 = \frac{7}{16}\text{A}, \; I_2 = \frac{23}{32}\text{A}$$

∴ Voltage drop across 10 Ω resistance is,

$$V = 10\left(\frac{7}{16} + \frac{23}{32}\right) = 11.56\text{ V}$$

5. **(b)** From $Bqv = \frac{mv^2}{r}$, we have

$$r = \frac{mv}{Bq} = \frac{\sqrt{2mK}}{Bq}$$

where, K is the kinetic energy.

As, kinetic energies of particles are same;

$$r \propto \frac{\sqrt{m}}{q} \quad \Rightarrow r_e : r_p : r_\alpha = \frac{\sqrt{m_e}}{e} : \frac{\sqrt{m_p}}{e} : \frac{\sqrt{4m_p}}{2e}$$

Clearly, $r_p = r_\alpha$ and r_e is least $\quad [\because m_e < m_p]$

So, $r_p = r_\alpha > r_e$

6. **(c)** As $m = IA$, so to change dipole moment (current is kept constant), we have to change radius of loop.

Initially, $\quad m = I\pi R^2$ and $B_1 = \frac{\mu_0 I}{2R_1}$

Finally, $\quad m' = 2m = I\pi R_2^2$

$$\Rightarrow \quad 2I\pi R_1^2 = I\pi R_2^2$$

or $\quad R_2 = \sqrt{2}R_1$

So, $\quad B_2 = \frac{\mu_0 I}{2(R_2)} = \frac{\mu_0 I}{2\sqrt{2}R_1}$

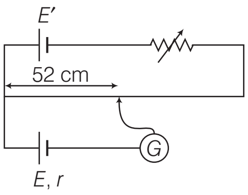
Hence, $\quad$ ratio $\frac{B_1}{B_2} = \frac{\left(\frac{\mu_0 I}{2R_1}\right)}{\left(\frac{\mu_0 I}{2\sqrt{2}R_1}\right)} = \sqrt{2}$

∴ Ratio $\frac{B_1}{B_2} = \sqrt{2}$

7. **(a)** Sharpness of resonance of a resonant *L-C-R* circuit is determined by the ratio of resonant frequency with the selectivity of circuit. This ratio is also called ''Quality Factor'' or *Q*-factor.

$$Q\text{-factor} = \frac{\omega_0}{2\Delta\omega} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

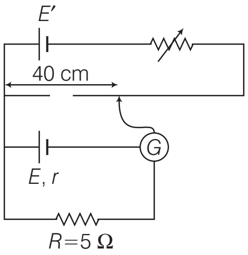
8. **(b)** With only the cell,



On balancing, $\quad E = 52 \times x \quad \ldots$(i)

where, x is the potential gradient of the wire.

When the cell is shunted,



Similarly, on balancing,

$$V = E - \frac{Er}{(R+r)} = 40 \times x \quad \ldots\text{(ii)}$$

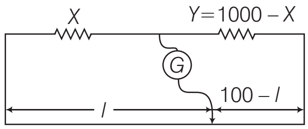
Solving Eqs. (i) and (ii), we get

$$\frac{E}{V} = \frac{1}{1 - \frac{r}{R+r}} = \frac{52}{40}$$

$$\Rightarrow \quad \frac{E}{V} = \frac{R+r}{R} = \frac{52}{40} \quad \Rightarrow \quad \frac{5+r}{5} = \frac{52}{40}$$

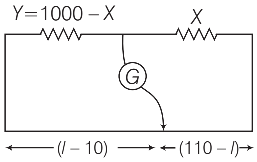
$$\Rightarrow \quad r = \frac{3}{2}\,\Omega \quad \Rightarrow \quad r = 1.5\,\Omega$$

9. **(c)** We have, $\quad X + Y = 1000\,\Omega$



Initially, $\quad \frac{X}{l} = \frac{1000 - X}{100 - l} \quad \ldots$(i)

When X and Y are interchanged, then



$$\frac{1000-X}{l-10}=\frac{X}{100-(l-10)}$$

or $$\frac{1000-X}{l-10}=\frac{X}{110-l} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{100-l}{l}=\frac{l-10}{110-l}$$

$$(100-l)\,(110-l)=(l-10)l$$

$$11000-100l-110l+l^2=l^2-10l$$

$\Rightarrow$ $$11000=200l$$

$\therefore$ $$l=55\text{ cm}$$

Substituting the value of l in Eq. (i), we get

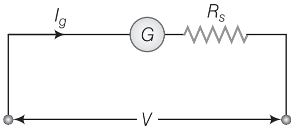
$$\frac{X}{55}=\frac{1000-55}{100-55}$$

$\Rightarrow$ $$20X=11000$$

$\therefore$ $$X=550\ \Omega$$

10. **(c)** A potential drop across each resistor is zero, so the current through each of resistor is zero.

11. **(d)** For a voltmeter,



$I_g\,(G+R_s)=V$

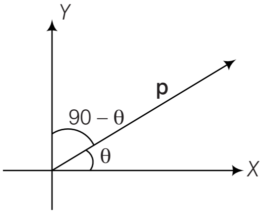
$\Rightarrow$ $R=\dfrac{V}{I_g}-G$

$\Rightarrow R=1985=1.985\text{ k}\Omega$ or $R=1.985\times 10^3\ \Omega$

12. **(a)** In a balanced Wheatstone bridge, there is no effect on position of null point, if we exchange the battery and galvanometer. So, option (a) is incorrect.

13. **(b)** Torque applied on a dipole $\tau = pE\sin\theta$

where θ = angle between axis of dipole and electric field.



For electric field $E_1 = E\hat{\mathbf{i}}$ it means field is directed along positive X direction, so angle between dipole and field will remain θ, therefore torque in this direction

$$E_1 = pE_1\sin\theta$$

In electric field $E_2=\sqrt{3}\,E\hat{\mathbf{j}}$, it means field is directed along positive Y-axis, so angle between dipole and field will be $90-\theta$

Torque in this direction $T_2 = pE\sin(90-\theta)$.

$$= p\sqrt{3}\,E_1\cos\theta$$

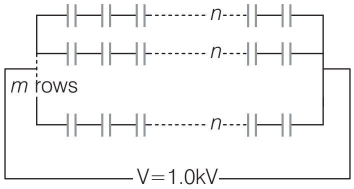
According to question $\tau_2=-\tau_1 \Rightarrow |\tau_2|=|\tau_1|$

$\therefore$ $$pE_1\sin\theta = p\sqrt{3}\,E_1\cos\theta$$

$\tan\theta=\sqrt{3}\Rightarrow \tan\theta=\tan 60^\circ$

$\therefore$ $$\theta=60^\circ$$

14. **(c)** Let there are n capacitors in a row with m such rows in parallel.



As voltage not to exceed 300 V

$\therefore$ $n\times 300>1000$

[a voltage greater than 1 kV to be withstand]

$\Rightarrow$ $$n>\frac{10}{3} \Rightarrow n=4\text{ (or 3.33)}$$

Also, $$C_{\text{Eq}}=\frac{mC}{n}=2\,\mu\text{F}$$

$\Rightarrow$ $$\frac{m}{n}=2 \Rightarrow m=8 \quad [\because C=1\,\mu\text{F}]$$

So, total number of capacitors required

$$= m\times n = 8\times 4 = 32$$

15. **(b)** In steady state no current flows through the capacitor.

So, the current in circuit $I=\dfrac{E}{r+r_2}$

$\because$ Potential drop across capacitor = Potential drop across r_2 $$= Ir_2 = \frac{Er_2}{r+r_2}$$

$\therefore$ Stored charge of capacitor, $Q = CV = \dfrac{CEr_2}{r+r_2}$

16. **(b)** Induced constant, $I=\dfrac{e}{R}$

Here, e = induced emf $=\dfrac{d\phi}{dt}$

$$I=\frac{1}{R}=\left(\frac{d\phi}{dt}\right)\cdot\frac{1}{R}$$

$\Rightarrow$ $d\phi = IRdt$

$\Rightarrow$ $\phi=\int IRdt$

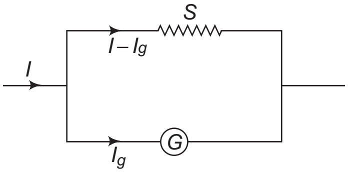
$\therefore$ Here, R is constant $\Rightarrow \phi = R\int Idt$

$\int I\cdot dt$ = Area under $I-t$ graph

$$=\frac{1}{2}\times 10\times 0.5 = 2.5$$

$\therefore$ $$\phi = R\times 2.5 = 100\times 2.5 = 250\text{ Wb.}$$

17. **(a)**



In parallel, current distributes in inverse ratio of resistance. Hence,

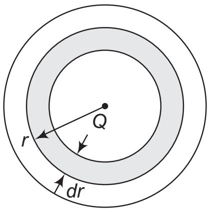
$$\frac{I-I_g}{I_g}=\frac{G}{S} \quad \Rightarrow \quad S=\frac{GI_g}{I-I_g}$$

As I_g is very small, hence

$$S=\frac{GI_g}{I} \Rightarrow b=\frac{(100)(1\times10^{-3})}{10}=0.01\,\Omega$$

18. **(a)** As E is constant,

Hence, $E_a=E_b$



As per Guass theorem, only Q_{in} contributes in electric field.

$$\therefore \qquad \frac{kQ}{a^2}=\frac{k\left[Q+\int_a^b 4\pi r^2 dr\cdot\frac{A}{r}\right]}{b^2}$$

Here, $k=\frac{1}{4\pi\varepsilon_0}$

$$\Rightarrow \quad Q\frac{b^2}{a^2}=Q+4\pi A\left[\frac{r^2}{2}\bigg|_a^b\right]=Q+4\pi A\cdot\left(\frac{b^2-a^2}{2}\right)$$

$$\Rightarrow \quad Q\left(\frac{b^2}{a^2}\right)=Q+2\pi A(b^2-a^2)$$

$$\Rightarrow \quad Q\left(\frac{b^2-a^2}{a^2}\right)=2\pi A(b^2-a^2) \Rightarrow A=\frac{Q}{2\pi a^2}$$

19. **(c)** $3\,\mu\text{F}$ and $9\mu\text{F}=12\mu\text{F}$

$$4\mu\text{F and }12\mu\text{F}=\frac{4\times12}{4+12}=3\mu\text{F}$$

$$Q=CV=3\times8=24\mu\text{C (on }4\mu\text{F and }3\mu\text{F)}$$

Now, this 24μC distributes in direct ratio of capacity between

3μF and 9μF. Therefore,

$$Q_{9\,\mu\text{F}}=18\mu\text{C}$$

$$\therefore Q_{4\mu\text{F}}+Q_{9\,\mu\text{F}}=24+18=42\mu\text{C}=Q \text{ (say)}$$

$$E=\frac{kQ}{R^2}=\frac{9\times10^9\times42\times10^{-6}}{30^2}=420\text{ N/C}$$

20. **(d)** B at centre of a circle $=\frac{\mu_0 I}{2R}$

B at centre of a square

$$=4\times\frac{\mu I}{4\pi\cdot\frac{l}{2}}[\sin45°+\sin45°]=4\sqrt{2}\,\frac{\mu_0 I}{2\pi l}$$

Now, $R=\frac{L}{2\pi}$ and $l=\frac{L}{4}$ (as $L=2\pi R=4l$)

where, L = length of wire.

$$\therefore \qquad B_A=\frac{\mu_0 I}{2\cdot\frac{L}{2\pi}}=\frac{\pi\mu_0 I}{L}=\pi\left[\frac{\mu_0 I}{L}\right]$$

$$B_B=4\sqrt{2}\,\frac{\mu_0 I}{2\pi\left(\frac{L}{4}\right)}=\frac{8\sqrt{2}\mu_0 I}{\pi L}=\frac{8\sqrt{2}}{\pi}\left[\frac{\mu_0 I}{L}\right]$$

$$\therefore \qquad \frac{B_A}{B_B}=\frac{\pi^2}{8\sqrt{2}}$$

21. **(d)** We need high retentivity and high coercivity for electromagnets and small area of hysteresis loop for transformers.

22. **(d)** $V^2=V_R^2+V_L^2 \Rightarrow 220^2=80^2+V_L^2$

Solving, we get

$$V_L\approx205\text{ V}$$

$$X_L=\frac{V_L}{I}=\frac{205}{10}=20.5\,\Omega=\omega L$$

$$\therefore \qquad L=\frac{20.5}{2\pi\times50}=0.065\text{ H}$$

23. **(a)** Theoretical question. Therefore, no solution is required.

24. **(c)** $i=neAv_d$ or $\frac{V}{R}=neAv_d \Rightarrow \frac{V}{\left(\frac{\rho l}{A}\right)}=neAv_d$

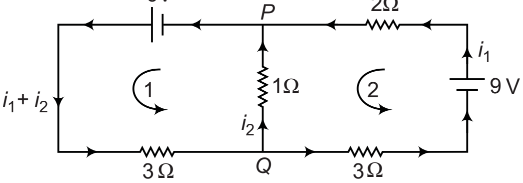
$$\therefore \qquad \rho=\frac{V}{nelv_d}=\text{resistivity of wire}$$

Substituting the given values we have

$$\rho=\frac{5}{(8\times10^{28})(1.6\times10^{-19})(0.1)(2.5\times10^{-4})}$$

$$\approx 1.6\times10^{-5}\,\Omega\text{-m}$$

25. **(b)**



Applying Kirchhoff's loop law in loops 1 and 2 in the directions shown in figure we have

$$6 - 3(i_1 + i_2) - i_2 = 0 \quad \text{...(i)}$$
$$9 - 2i_1 + i_2 - 3i_1 = 0 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii) we get,

$$i_2 = 0.13\text{ A}$$

Hence, the current in 1 Ω resister is 0.13 A from Q to P.

26. **(b, c)** V_0 = potential on the surface $= \frac{Kq}{R}$

where, $K = \frac{1}{4\pi\varepsilon_0}$ and q is total charge on sphere.

Potential at centre $= \frac{3}{2}\frac{Kq}{R} = \frac{3}{2}V_0$

Hence, $R_1 = 0$

From centre to surface potential varies between $\frac{3}{2}V_0$ and V_0 From surface to infinity, it varies between V_0 and 0, $\frac{5V_0}{4}$ will be potential at a point between centre and surface. At any point, at a distance $r (r \le R)$ from centre potential is given by

$$V = \frac{Kq}{R^3}\left(\frac{3}{2}R^2 - \frac{1}{2}r^2\right)$$
$$= \frac{V_0}{R^2}\left(\frac{3}{2}R^2 - \frac{1}{2}r^2\right) \quad \left(\text{as } V_0 = \frac{Kq}{R}\right)$$

Putting $V = \frac{5}{4}V_0$ and $r = R_2$ in this equation, we get

$$R_2 = \frac{R}{\sqrt{2}}$$

$\frac{3V_0}{4}$ and $\frac{V_0}{4}$ are the potentials lying between V_0 and zero hence these potentials lie outside the sphere. At a distance $r (\ge R)$ from centre potential is given by

$$V = \frac{Kq}{r} = \frac{V_0 R}{r}$$

Putting $V = \frac{3}{4}V_0$ and $r = R_3$ in this equation we get,

$$R_3 = \frac{4}{3}R$$

Further putting $V = \frac{V_0}{4}$ and $r = R_4$ in above equation,

we get $R_4 = 4R$

Thus, $R_1 = 0, R_2 = \frac{R}{\sqrt{2}}, R_3 = \frac{4R}{3}$ and $R_4 = 4R$ with these values, option (b) and (c) are correct.

27. **(d)** Electric field lines originate from position charge and termination negative charge. They cannot form closed loops and they are smooth curves. Hence the most appropriate answer is (d).

28. **(a)** Resultant of 1 μF and 2 μF is 3 μF. Now in series, potential difference distributes in inverse ratio of capacity.

$$\therefore \quad \frac{V_{3\mu\text{F}}}{V_c} = \frac{C}{3} \quad \text{or} \quad V_{3\mu\text{F}} = \left(\frac{C}{C+3}\right)E$$

This is also the potential difference across 2 μF.

$$\therefore \quad Q_2 = (2\ \mu\text{F})(V_{2\mu\text{F}})$$

$$\text{or} \quad Q_2 = \left(\frac{2CE}{C+3}\right) = \left(\frac{2}{1+\frac{3}{C}}\right)E$$

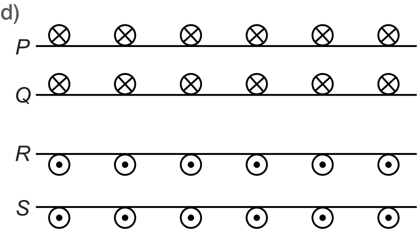
From this expression of Q_2, we can see that Q_2 will increase with increase in the value of c (but not linearly). Therefore, only options (a) and (b) may be correct.

Further, $\frac{d}{dc}(Q_2) = 2E\left[\frac{(C+3) - C}{(C+3)^2}\right] = \frac{6E}{(C+3)^2}$

= Slope of Q_2 *versus* C graph.

i.e. slope of Q_2 *versus* C graph decreases with increase in the value of C. Hence, the correct graph is (a).

29. **(d)**

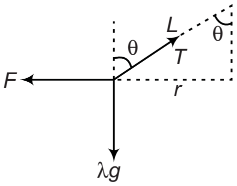


If we calculate the force on inner solenoid. Force on Q due to P is outwards (attraction between currents in same direction. Similarly, force on R due to S is also outwards. Hence, net force $\mathbf{F}_1$ is zero)

Force on P due to Q and force on S due to R is inwards. Hence, net force $\mathbf{F}_2$ is also zero.

Alternate Thought Field of one solenoid is uniform and other solenoid may be assumed a combination of circular closed loops. In uniform magnetic field, net force on a closed current carrying loop is zero.

30. **(a)**



$r = L\sin\theta$

F = Magnetic force (repulsion) per unit length

$$= \frac{\mu_0}{2\pi}\frac{I^2}{2r} = \frac{\mu_0}{4\pi}\frac{I^2}{L\sin\theta}$$

λg = weight per unit length

Each wire is in equilibrium under three concurrent forces as shown in figure. Therefore, applying Lami's theorem.

$$\frac{F}{\sin(180 - \theta)} = \frac{\lambda g}{\sin(90 + \theta)} \quad \text{or} \quad \frac{\frac{\mu_0}{4\pi}\frac{I^2}{L\sin\theta}}{\sin\theta} = \frac{\lambda g}{\cos\theta}$$

$$\therefore \quad I = 2\sin\theta\sqrt{\frac{\pi\lambda g L}{\mu_0\cos\theta}}$$

31. **(b)** Direction of magnetic dipole moment **M** is given by screw law and this is perpendicular to plane of loop.

In stable equilibrium position, angle between **M** and **B** is 0° and in unstable equilibrium this angle is 180°.

32. **(c)** Steady state current i_0 was already flowing in the L-R circuit when K_1 was closed for a long time. Here,

$$i_0 = \frac{V}{R} = \frac{15\ \text{V}}{150\ \Omega} = 0.1\ \text{A}$$

Now, K_1 is opened and K_2 is closed. Therefore, this i_0 will decrease exponentially in the L-R circuit. Current i at time t will be given by $i = i_0 e^{\frac{-t}{\tau_L}}$

where, $\tau_L = \frac{L}{R} \Rightarrow \therefore \quad i = i_0 e^{\frac{-Rt}{L}}$

Substituting the values, we have

$$i = (0.1)\, e^{\frac{-(0.15\times 10^3)(10^{-3})}{(0.03)}} = (0.1)(e^{-5})$$

$$= \frac{0.1}{150} = 6.67 \times 10^{-4}\ \text{A}$$

$$= 0.67\ \text{mA}$$

33. **(c)** Total power (P) consumed

$$= (15 \times 40) + (5 \times 100) + (5 \times 80) + (1 \times 1000)$$

$$= 2500\ \text{W}$$

As we know,

Power i.e. $P = VI \Rightarrow I = \frac{2500}{220}\ \text{A} \quad = \frac{125}{11} = 11.3\ \text{A}$

Minimum capacity should be 12 A.

34. **(c)** As we know, potential difference $V_A - V_O$ is

$$dV = -E dx$$

$$\Rightarrow \int_{V_O}^{V_A} dV = -\int_0^2 30x^2 dx \quad \Rightarrow \quad V_A - V_O = -30 \times \left[\frac{x^3}{3}\right]_0^2$$

$$= -10 \times [2^3 - (0)^3] = -10 \times 8 = -80 \quad \text{J}$$

35. **(a)** When free space between parallel plates of capacitor, $E = \frac{\sigma}{\varepsilon_0}$

When dielectric is introduced between parallel plates of capacitor, $E' = \frac{\sigma}{K\varepsilon_0}$

Electric field inside dielectric, $\frac{\sigma}{K\varepsilon_0} = 3 \times 10^4$

where, K = dielectric constant of medium = 2.2

ε_0 = permitivity of free space = 8.85×10^{-12}

$\Rightarrow \sigma = 2.2 \times 8.85 \times 10^{-12} \times 3 \times 10^4$

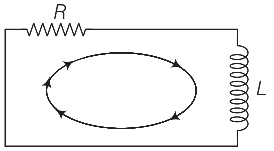
$= 6.6 \times 8.85 \times 10^{-8} = 5.841 \times 10^{-7} = 6 \times 10^{-7}\ \text{C/m}^2$

36. For solenoid, the magnetic field needed to be magnetised the magnet. $B = \mu_0 nI$

where, $n = 100$, $l = 10\text{cm} = \frac{10}{100}\text{m} = 0.1\ \text{m}$

$$\Rightarrow \quad 3 \times 10^3 = \frac{100}{0.1} \times I \Rightarrow I = 3\ \text{A}$$

37. **(c)** After connecting C to B hanging the switch, the circuit will act like an L-R discharging circuit.



Applying Kirchhoff's loop equation,

$$V_R + V_L = 0 \quad \Rightarrow \quad V_R = -V_L \quad \Rightarrow \quad \frac{V_R}{V_L} = -1$$

38. **(b)** When force exerted on a current carrying conductor $F_{\text{ext}} = BIL$

Average power = $\frac{\text{Work done}}{\text{Time taken}}$

$$P = \frac{1}{t}\int_0^2 F_{\text{ext.}}\ .\ dx = \frac{1}{t}\int_0^2 B(x)IL\, dx$$

$$= \frac{1}{5 \times 10^{-3}} \int_0^2 3 \times 10^{-4} e^{-0.2x} \times 10 \times 3\, dx$$

$$= 9\,[1 - e^{-0.4}] = 9\left[1 - \frac{1}{e^{0.4}}\right] = 2.967 \approx 2.97\ \text{W}$$

39. **(d)** Statement I is false and Statement II is true.

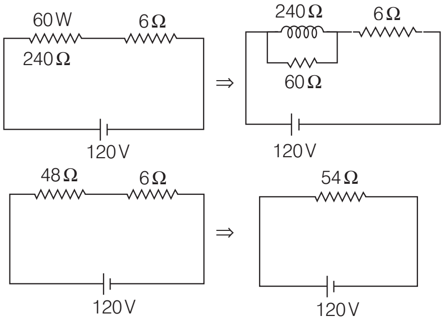
40. **(d)** As, $P = \frac{V^2}{R}$

where, P = power dissipates in the circuit,

V = applied voltage,

R = net resistance of the circuit

$$R = \frac{120 \times 120}{60} = 240\ \Omega \quad \text{[resistance of bulb]}$$



$$R_{\text{eq}} = 240 + 6 = 246\ \Omega \quad \Rightarrow \quad i_1 = \frac{V}{R_{\text{eq}}} = \frac{120}{246}$$

[before connecting heater]

$$R = \frac{V^2}{R} = \frac{120 \times 120}{240}$$

$\Rightarrow \quad R = 60\ \Omega$ [resistance of heater]

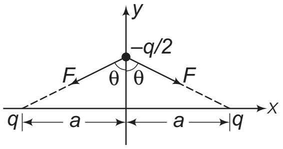
So, from figure,

$$V_1 = \frac{240}{246} \times 120 = 117.073\ \text{V} \qquad [\because V = IR]$$

$$\Rightarrow \quad i_2 = \frac{V}{R_2} = \frac{120}{54} \quad \Rightarrow V_2 = \frac{48}{54} \times 120 = 106.66 \text{ V}$$

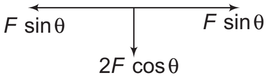
$$V_1 - V_2 = 10.04 \text{ V}$$

41. (a) $F_{\text{net}} = 2F\cos\theta$



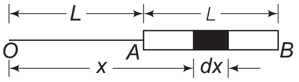
$$F_{\text{net}} = \frac{2kq\,(q/2)}{(\sqrt{y^2+a^2})^2} \cdot \frac{y}{\sqrt{y^2+a^2}}$$

$$F_{\text{net}} = \frac{2kq\,(q/2)\,y}{(y^2+a^2)^{3/2}}$$



$$\Rightarrow \frac{kq^2y}{a^3} \propto y$$

42. (d)



$$V = \int_L^{2L} \frac{kdQ}{x} = \int_L^{2L} \frac{k\left(\frac{Q}{L}\right)dx}{x} = \frac{Q}{4\pi\varepsilon_0 L}\int_L^{2L}\left(\frac{1}{x}\right)dx$$

$$= \frac{Q}{4\pi\varepsilon_0 L}[\log_e x]_L^{2L} = \frac{Q}{4\pi\varepsilon_0 L}[\log_e 2L - \log_e L]$$

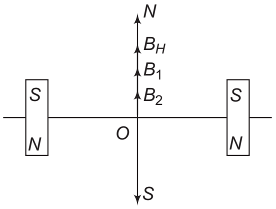
$$= \frac{Q}{4\pi\varepsilon_0 L}\ln(2)$$

43. (b, c) Polarity should be mentioned in the question. Potential on each of them can be zero if, $q_{\text{net}} = 0$

or $q_1 \pm q_2 = 0$

or $120C_1 \pm 200C_2 = 0$ or $3C_1 \pm 5C_2 = 0$

44. (b) $B_{\text{net}} = B_1 + B_2 + B_H$



$$B_{\text{net}} = \frac{\mu_0}{4\pi}\frac{(M_1+M_2)}{r^3} + B_H$$

$$= \frac{10^{-7}\,(1.2+1)}{(0.1)^3} + 3.6\times 10^{-5} = 2.56\times 10^{-4} \text{ Wb/m}^2$$

45. (a) Magnetic field at the centre of smaller loop

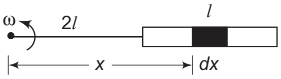
$$B = \frac{\mu_0 i R_2^2}{2(R_2^2+x^2)^{3/2}}$$

Area of smaller loop $S = \pi R_1^2$

∴ Flux through smaller loop $\phi = BS$

Substituting the values, we get, $\phi \approx 9.1\times 10^{-11}$ Wb

46. (d) $e = \int_{2l}^{3l}(\omega x)\,Bdx = B\omega\,\frac{[(3l)^2-(2l)^2]}{2} = \frac{5B\omega l^2}{2}$



47. (c) For charging of capacitor $q = CV\,(1 - e^{t/\tau})$

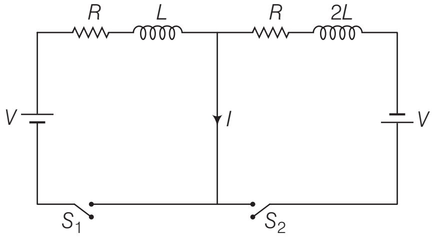
At $t = 2\tau$; $q = CV\,(1 - e^{-2})$

48. (c) Amplitude decreases exponentially. In 5 s, it remains 0.9 times. Therefore, in total 15 s it will remains (0.9) (0.9) (0.9) = 0.729 times its original value.

JEE Advanced

1. In the figure below, the switches S_1 and S_2 are closed simultaneously at $t = 0$ and a current starts to flow in the circuit. Both the batteries have the same magnitude of the electromotive force (emf) and the polarities are as indicated in the figure. Ignore mutual inductance between the inductors. The current I in the middle wire reaches its maximum magnitude $I_{\max}$ at time $t = \tau$. Which of the following statements is (are) true?

(More than One Correct Option, 2018)



(a) $I_{\max} = \frac{V}{2R}$

(b) $I_{\max} = \frac{V}{4R}$

(c) $\tau = \frac{L}{R}\ln 2$

(d) $\tau = \frac{2L}{R}\ln 2$

2. Two infinitely long straight wires lie in the xy-plane along the lines $x = \pm R$. The wire located at $x = +R$ carries a constant current I_1 and the wire located at $x = -R$ carries a constant current I_2. A circular loop of radius R is suspended with its centre at $(0, 0, \sqrt{3}R)$ and in a plane parallel to the xy-plane. This loop carries a constant current I in the clockwise direction as seen from above the loop. The current in the wire is taken to be positive, if it is in the $+\hat{\mathbf{j}}$-direction. Which of the following statements regarding the magnetic field $\mathbf{B}$ is (are) true? **(More than One Correct Option, 2018)**

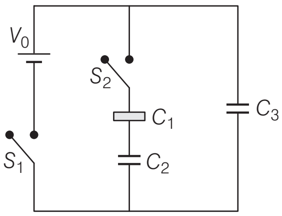
(a) If $I_1 = I_2$, then $\mathbf{B}$ cannot be equal to zero at the origin (0,0,0)

(b) If $I_1 > 0$ and $I_2 < 0$, then $\mathbf{B}$ can be equal to zero at the origin (0,0,0)

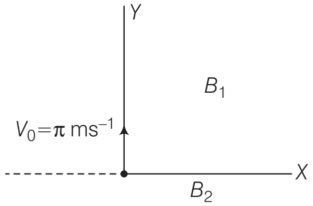
(c) If $I_1 < 0$ and $I_2 > 0$, then $\mathbf{B}$ can be equal to zero at the origin (0,0,0)

(d) If $I_1 = I_2$, then the z-component of the magnetic field at the centre of the loop is $\left(-\frac{\mu_o I}{2R}\right)$

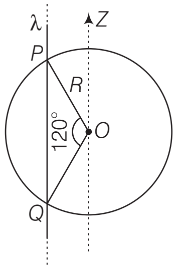
3. Three identical capacitors C_1, C_2 and C_3 have a capacitance of 1.0μF each and they are uncharged initially. They are connected in a circuit as shown in the figure and C_1 is then filled completely with a dielectric material of relative permittivity ε_r. The cell electromotive force (emf) $V_0 = 8$V. First the switch S_1 is closed while the switch S_2 is kept open. When the capacitor C_3 is fully charged, S_1 is opened and S_2 is closed simultaneously. When all the capacitors reach equilibrium, the charge on C_3 is found to be 5μC. The value of ε_r =........... **(Numerical Value, 2018)**



4. In the xy-plane, the region $y > 0$ has a uniform magnetic field $B_1\hat{\mathbf{k}}$ and the region $y < 0$ has another uniform magnetic field $B_2\hat{\mathbf{k}}$. A positively charged particle is projected from the origin along the positive Y-axis with speed $v_0 = \pi$ ms^{-1} at $t = 0$, as shown in figure. Neglect gravity in this problem. Let $t = T$ be the time when the particle crosses the X-axis from below for the first time. If $B_2 = 4B_1$, the average speed of the particle, in ms^{-1}, along the X-axis in the time interval T is........... . **(Numerical Value, 2018)**



5. An infinitely long thin non-conducting wire is parallel to the Z-axis and carries a uniform line charge density λ. It pierces a thin non-conducting spherical shell of radius R in such a way that the arc PQ subtends an angle 120° at the centre O of the spherical shell, as shown in the figure. The permittivity of free space is ε_0. Which of the following statements is (are) true? **(More than One Correct Option, 2018)**



(a) The electric flux through the shell is $\sqrt{3}\,R\lambda/\varepsilon_0$.

(b) The z-component of the electric field is zero. at all the points on the surface of the shell.

(c) The electric flux through the shell is $\sqrt{2}\,R\lambda/\varepsilon_0$.

(d) The electric field is normal to the surface of the shell at all points.

6. A particle of mass 10^{-3}kg and charge 1.0 C is initially at rest. At time $t = 0$, the particle comes under the influence of an electric field $\mathbf{E}(t) = E_0 \sin \omega t\,\hat{\mathbf{i}}$, where $E_0 = 1.0\,\text{NC}^{-1}$ and $\omega = 10^3$ rad s^{-1}. Consider the effect of only the electrical force on the particle. Then, the maximum speed in m s^{-1}, attained by the particle at subsequent times is .... **(Numerical Value, 2018)**

7. A moving coil galvanometer has 50 turns and each turn has an area $2 \times 10^{-4}\,\text{m}^2$. The magnetic field produced by the magnet inside the galvanometer is 0.02 T. The torsional constant of the suspension wire is 10^{-4} N - m rad^{-1}. When a current flows through the galvanometer, a full scale deflection occurs, if the coil rotates by 0.2 rad. The resistance of the coil of the galvanometer is 50 Ω. This galvanometer is to be converted into an ammeter capable of measuring current in the range 0 – 1.0 A. For this purpose, a shunt resistance is to be added in parallel to the galvanometer. The value of this shunt resistance in ohms, is ....

(Numerical Value, 2018)

8. The electric field E is measured at a point $P\,(0, 0, d)$ generated due to various charge distributions and the dependence of E on d is found to be different for different charge distributions. List-I contains different relations between E and d. List-II describes different electric charge distributions, along with their locations. Match the functions in List-I with the related charge distributions in List-II.

(Matching Type, 2018)

	List-I		List-II
P.	E is independent of d	1.	A point charge Q at the origin
Q.	$E \propto \frac{1}{d}$	2.	A small dipole with point charges Q at (0, 0, l) and – Q at (0, 0, – 1). (Take, $2l << d$)
R.	$E \propto \frac{1}{d^2}$	3.	An infinite line charge coincident with the X-axis, with uniform linear charge density λ.
S.	$E \propto \frac{1}{d^3}$	4.	Two infinite wires carrying a uniform linear charge density parallel to the X- axis. The one along (y = 0, z = l) has a charge density + λ and the one along (y = 0, z = – l) has a charge density – λ. (Take, $2l << d$).
		5.	Infinite plane charge coincident with the xy-plane with uniform surface charge density.

(a) P → 5; Q → 3,4; R → 1; S → 2
(b) P → 5; Q → 3; R → 1, 4; S → 2
(c) P → 5; Q → 3; R → 1, 2; S → 4
(d) P → 4; Q → 2, 3; R → 1; S → 5

Passage (Q. Nos. 9-10)

Consider a simple RC circuit as shown in Figure 1.

Process 1 In the circuit the switch S is closed at $t = 0$ and the capacitor is fully charged to voltage V_0 (i.e. charging continues for time $T >> RC$). In the process some dissipation (E_D) occurs across the resistance R. The amount of energy finally stored in the fully charged capacitor is E_c.

Process 2 In a different process the voltage is first set to $\frac{V_0}{3}$ and maintained for a charging time $T >> RC$. Then, the voltage is raised to $\frac{2V_0}{3}$ without discharging the capacitor and again maintained for a time $T >> RC$. The process is repeated one more time by raising the voltage to V_0 and the capacitor is charged to the same final voltage V_0 as in Process 1. These two processes are depicted in Figure 2.

(Passage Type, 2017)

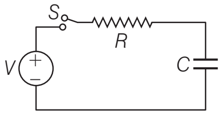


Figure 1

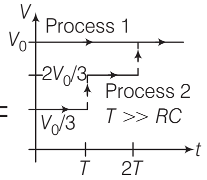


Figure 2

9. In Process 1, the energy stored in the capacitor E_C and heat dissipated across resistance E_D are related by

(a) $E_C = E_D \ln 2$
(b) $E_C = E_D$
(c) $E_C = 2E_D$
(d) $E_C = \frac{1}{2}E_D$

10. In Process 2, total energy dissipated across the resistance E_D is

(a) $E_D = \frac{1}{3}\left(\frac{1}{2}CV_0^2\right)$
(b) $E_D = 3\left(\frac{1}{2}CV_0^2\right)$
(c) $E_D = 3CV_0^2$
(d) $E_D = \frac{1}{2}CV_0^2$

Directions (Q.Nos. 11 to 13) Matching the information given in the three columns of the following table.

A charged particle (electron or proton) is introduced at the origin ($x=0,\ y=0,\ z=0$) with a given initial velocity **v**. A uniform electric field **E** and a uniform magnetic field **B** exist everywhere. The velocity **v**, electric field **E** and magnetic field **B** are given in columns 1, 2 and 3, respectively. The quantities E_0, B_0 are positive in magnitude. **(Matching Type, 2017)**

Column 1	Column 2	Column 3
(I) Electron with $\mathbf{v} = 2\frac{E_0}{B_0}\hat{x}$	(i) $\mathbf{E} = E_0\hat{z}$	(P) $\mathbf{B} = -B_0\hat{x}$
(II) Election with $\mathbf{v} = \frac{E_0}{B_0}\hat{y}$	(ii) $\mathbf{E} = -E_0\hat{y}$	(Q) $\mathbf{B} = B_0\hat{x}$
(III) Proton with $\mathbf{v} = 0$	(iii) $\mathbf{E} = -E_0\hat{x}$	(R) $\mathbf{B} = B_0\hat{y}$
(IV) Proton with $\mathbf{v} = 2\frac{E_0}{B_0}\hat{x}$	(iv) $\mathbf{E} = E_0\hat{x}$	(S) $\mathbf{B} = B_0\hat{z}$

11. In which case would the particle move in a straight line along the negative direction of Y-axis?

(a) (IV) (ii) (S)
(b) (II) (iii) (Q)
(c) (III), (ii) (R)
(d) (III) (ii) (P)

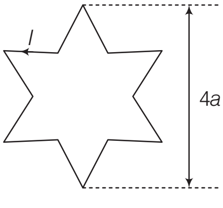
12. In which case will the particle move in a straight line with constant velocity?

(a) (II) (iii) (S)
(b) (III) (iii) (P)
(c) (IV) (i) (S)
(d) (III) (ii) (R)

13. In which case will the particle describe a helical path with axis along the positive z-direction?

(a) (II) (ii) (R)
(b) (III) (iii) (P)
(c) (IV) (i) (S)
(d) (IV) (ii) (R)

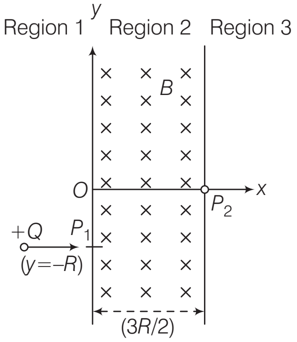
14. A symmetric star shaped conducting wire loop is carrying a steady state current I as shown in the figure. The distance between the diametrically opposite vertices of the star is $4a$. The magnitude of the magnetic field at the center of the loop is **(Single Correct Option, 2017)**



(a) $\frac{\mu_0 I}{4\pi a}6[\sqrt{3}-1]$
(b) $\frac{\mu_0 I}{4\pi a}6[\sqrt{3}+1]$
(c) $\frac{\mu_0 I}{4\pi a}3[\sqrt{3}-1]$
(d) $\frac{\mu_0 I}{4\pi a}3[2-\sqrt{3}]$

15. A uniform magnetic field B exists in the region between $x=0$ and $x=\frac{3R}{2}$ (region 2 in the figure) pointing normally into the plane of the paper. A particle with charge $+Q$ and momentum p directed along X-axis enters region 2 from region 1 at point P_1 ($y=-R$).

Which of the following option(s) is/are correct? **(More than One Correct Option, 2017)**



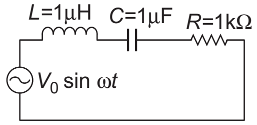
(a) When the particle re-enters region 1 through the longest possible path in region 2, the magnitude of the change in its linear momentum between point P_1 and the farthest point from Y-axis is $\frac{p}{\sqrt{2}}$

(b) For $B = \frac{8}{13}\frac{p}{QR}$, the particle will enter region 3 through the point P_2 on X-axis

(c) For $B > \frac{2}{3}\frac{p}{QR}$, the particle will re-enter region 1

(d) For a fixed B, particles of same charge Q and same velocity v, the distance between the point P_1 and the point of re-entry into region 1 is inversely proportional to the mass of the particle

16. In the circuit shown, $L = 1\,\mu\text{H}$, $C = 1\,\mu\text{F}$ and $R = 1\,\text{k}\Omega$. They are connected in series with an AC source $V = V_0 \sin\omega t$ as shown. Which of the following options is/are correct? **(More than One Correct Option, 2017)**



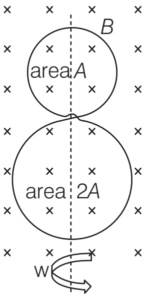
(a) At $\omega \sim 0$ the current flowing through the circuit becomes nearly zero

(b) The frequency at which the current will be in phase with the voltage is independent of R

(c) The current will be in phase with the voltage if $\omega = 10^4\ \text{rads}^{-1}$

(d) At $\omega >> 10^6\ \text{rads}^{-1}$, the circuit behaves like a capacitor

17. A circular insulated copper wire loop is twisted to form two loops of area A and $2A$ as shown in the figure. At the point of crossing the wires remain electrically insulated from each other. The entire loop lies in the plane (of the paper). A uniform magnetic field **B** points into the plane of the paper. At $t = 0$, the loop starts rotating about the common diameter as axis with a constant angular velocity ω in the magnetic field. Which of the following options is/are correct? **(More than One Correct Option, 2017)**



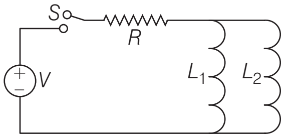
(a) the emf induced in the loop is proportional to the sum of the areas of the two loops

(b) The rate of change of the flux is maximum when the plane of the loops is perpendicular to plane of the paper

(c) The net emf induced due to both the loops is proportional to $\cos\omega t$

(d) The amplitude of the maximum net emf induced due to both the loops is equal to the amplitude of maximum emf induced in the smaller loop alone

18. A source of constant voltage V is connected to a resistance R and two ideal inductors L_1 and L_2 through a switch S as shown. There is no mutual inductance between the two inductors. The switch S is initially open. At $t = 0$, the switch is closed and current begins to flow. Which of the following options is/are correct? **(More than One Correct Option, 2017)**



(a) After a long time, the current through L_1 will be $\frac{V}{R}\left(\frac{L_2}{L_1 + L_2}\right)$

(b) After a long time, the current through L_2 will be $\frac{V}{R}\left(\frac{L_1}{L_1 + L_2}\right)$

(c) The ratio of the currents through L_1 and L_2 is fixed at all times ($t > 0$)

(d) At $t = 0$, the current through the resistance R is $\frac{V}{R}$

19. The instantaneous voltages at three terminals marked X, Y and Z are given by $V_X = V_0 \sin\omega t, V_Y = V_0 \sin\left(\omega t + \frac{2\pi}{3}\right)$ and

$$V_Z = V_0 \sin\left(\omega t + \frac{4\pi}{3}\right).$$

An ideal voltmeter is configured to read rms value of the potential difference between its terminals. It is connected between points X and Y and then between Y and Z. The reading(s) of the voltmeter will be **(More than One Correct Option, 2017)**

(a) $V_{YZ}^{\text{rms}} = V_0\sqrt{\frac{1}{2}}$

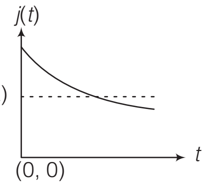
(b) $V_{XY}^{\text{rms}} = V_0\sqrt{\frac{3}{2}}$

(c) independent of the choice of the two terminals

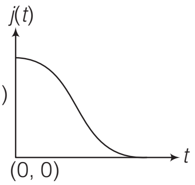
(d) $V_{XY}^{\text{rms}} = V_0$

20. An infinite line charge of uniform electric charge density λ lies along the axis of an electrically conducting infinite cylindrical shell of radius R. At time $t = 0$, the space inside the cylinder is filled with a material of permittivity ε and electrical conductivity σ. The electrical conduction in the material follows Ohm's law. Which one of the following graphs best describes the subsequent variation of the magnitude of current density $j(t)$ at any point in the material? **(Single Correct Option, 2016)**

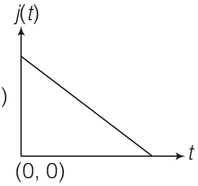
(a)



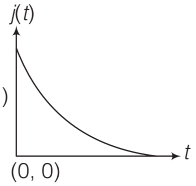
(b)



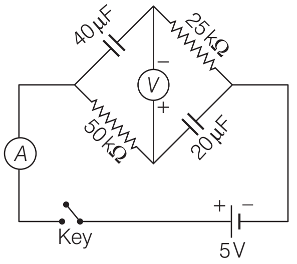
(a)



(d)



21. In the circuit shown below, the key is pressed at time $t = 0$. Which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**



(a) The voltmeter display −5 V as soon as the key is pressed and displays +5 V after a long time

(b) The voltmeter will display 0 V at time $t = \ln 2$ seconds

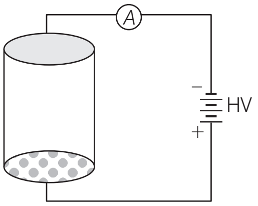
(c) The current in the ammeter becomes $1/e$ of the initial value after 1 second

(d) The current in the ammeter becomes zero after a long time

Passage (Q. Nos. 22-23)

Consider an evacuated cylindrical chamber of height h having rigid conducting plates at the ends and an insulating curved surface as shown in the figure. A number of spherical balls made of a light weight and soft material and coated with a conducting material are placed on the bottom plate. The balls have a radius $r << h$. Now, a high voltage source (HV) connected across the conducting plates such that the bottom plate is at $+V_0$ and the top plate at $-V_0$. Due to their conducting surface, the balls will get charge, will become equipotential with the plate and are repelled by it.

The balls will eventually collide with the top plate, where the coefficient of restitution can be taken to be zero due to te soft nature of the material of the balls. The electric field in the chamber can be considered to be that of a parallel plate capacitor. Assume that there are no collisions between the balls and the interaction between them is negligible. (Ignore gravity) **(Passage Type, 2016)**



22. Which one of the following statements is correct?

(a) The balls will execute simple harmonic motion between the two plates

(b) The balls will bounce back to the bottom plate carrying the same charge they went up with

(c) The balls will stick to the top plate and remain there

(d) The balls will bounce back to the bottom plate carrying the opposite charge they went up with

23. The average current in the steady state registered by the ammeter in the circuit will be

(a) proportional to V_0^2

(b) proportional to the potential V_0

(c) zero

(d) proportions to $V_0^{1/2}$

24. An incandescent bulb has a thin filament of tungsten that is heated to high temperature by passing an electric current. The hot filament emits black-body radiation. The filament is observed to break up at random locations after a sufficiently long time of operation due to non-uniform evaporation of tungsten from the filament. If the bulb is powered at constant voltage, which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**

(a) The temperature distribution over the filament is uniform

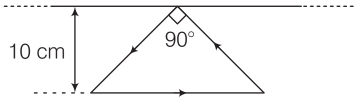
(b) The resistance over small sections of the filament decreases with time

(c) The filament emits more light at higher band of requencies before it breaks up

(d) The filament consumes less electrical power towards the end of the life of the bulb

25. A conducting loop in the shape of a right angled isosceles triangle of height 10 cm is kept such that the 90° vertex is very close to an infinitely long conducting wire (see the figure). The wire is electrically insulated from the loop. The hypotenuse of the triangle is parallel to the wire. The current in the triangular loop is in counter- clockwise direction and increased at a constant rate of $10\,\text{As}^{-1}$. Which of the following statement(s) is (are) true?

(More than One Correct Option, 2016)



(a) There is a repulsive force between the wire and the loop

(b) If the loop is rotated at a constant angular speed about the wire, an additional emf of (μ_0/π) volt is induced in the wire

(c) The magnitude of induced emf in the wire is $\left(\frac{\mu_0}{\pi}\right)$ volt

(d) The induced current in the wire is in opposite direction to the current along the hypotenuse

26. Consider two identical galvanometers and two identical resistors with resistance R. If the internal resistance of the galvanometers $R_c < R/2$, which of the following statement(s) about anyone of the galvanometers is (are) true?

(More than One Correct Option, 2016)

(a) The maximum voltage range is obtained when all the components are connected in series

(b) The maximum voltage range is obtained when the two resistors and one galvanometer are connected in series, and the second galvanometer is connected in parallel to the first galvanometer

(c) The maximum current range is obtained when all the components are connected in parallel

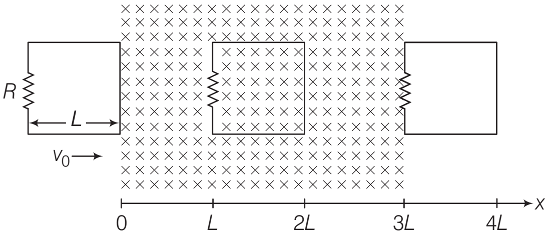
(d) The maximum current range is obtained when the two galvanometers are connected in series, and the combination is connected in parallel with both the resistors

27. Two inductors L_1 (inductance 1mH, internal resistance $3\,\Omega$) and L_2 (inductance 2 mH, internal resistance $4\,\Omega$), and a resistor R (resistance $12\,\Omega$) are all connected in parallel across a 5V battery. The circuit is switched on at time $t = 0$. The ratio of the maximum to the minimum current $(I_{\max}/I_{\min})$ drawn from the battery is **(Single Integer Type, 2016)**

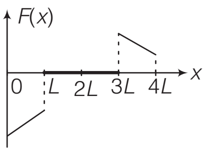
28. A rigid wire loop of square shape having side of length L and resistance R is moving along the x-axis with a constant velocity v_0 in the plane of the paper .

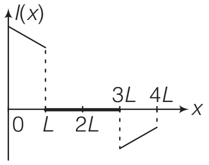
At $t = 0$, the right edge of the loop enters a region of length $3L$ where there is a uniform magnetic field B_0 into the plane of the paper, as shown in the figure. For sufficiently large v_0, the loop eventually crosses the region. Let x be the location of the right edge of the loop. Let $v(x)$, $I(x)$ and $F(x)$ represent the velocity of the loop, current in the loop, and force on the loop, respectively, as a function of x. Counter-clockwise current is taken as positive.

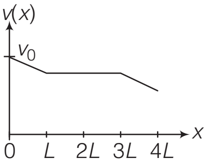
(More than One Correct Option, 2016)

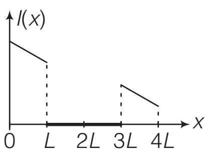


Which of the following schematic plot(s) is (are) correct? (Ignore gravity)

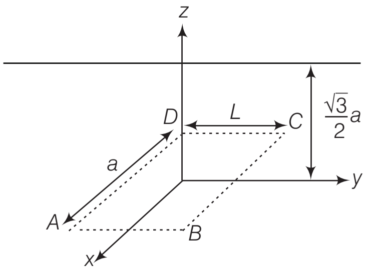
(a)


(b)


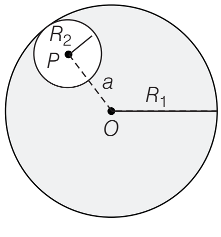
(c)


(d)


29. An infinitely long uniform line charge distribution of charge per unit length λ lies parallel to the y-axis in the y-z plane at $z = \frac{\sqrt{3}}{2} a$ (see figure). If the magnitude of the flux of the electric field through the rectangular surface $ABCD$ lying in the x-y plane with its centre at the origin is $\frac{\lambda L}{n\varepsilon_0}$ (ε_0 = permittivity of free space), then the value of n is = 6) **(Single Integer Type, 2015)**

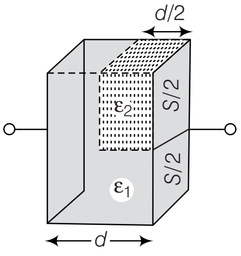


30. Consider a uniform spherical charge distribution of radius R_1 centred at the origin O. In this distribution, a spherical cavity of radius R_2, centred at P with distance $OP = a = R_1 - R_2$ (see figure) is made. If the electric field inside the cavity at position **r** is $\mathbf{E}(\mathbf{r})$, then the correct statement(s) is/are **(Single Correct Option, 2015)**



(a) **E** is uniform, its magnitude is independent of R_2 but its direction depends on **r**
(b) **E** is uniform, its magnitude depends on R_2 and its direction depends on **r**
(c) **E** is uniform, its magnitude is independent of 'a' but its direction depends on **a**
(d) **E**is uniform and both its magnitude and direction depend on **a**

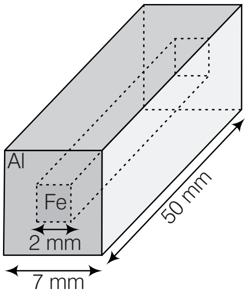
31. A parallel plate capacitor having plates of area S and plate separation d, has capacitance C_1 in air. When two dielectrics of different relative permittivities ($\varepsilon_1 = 2$ and $\varepsilon_2 = 4$) are introduced between the two plates as shown in the figure, the capacitance becomes C_2. The ratio $\frac{C_2}{C_1}$ is



(a) $\frac{6}{5}$ (b) $\frac{5}{3}$
(c) $\frac{7}{5}$ (d) $\frac{7}{3}$

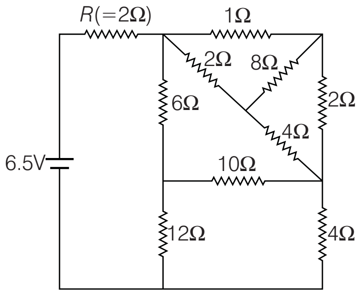
(Single Correct Option, 2015)

32. In an aluminium (Al) bar of square cross section, a square hole is drilled and is filled with iron (Fe) as shown in the figure. The electrical resistivities of Al and Fe are 2.7×10^{-8} Ωm and 1.0×10^{-7} Ωm, respectively. The electrical resistance between the two faces P and Q of the composite bar is **(Single Correct Option, 2015)**

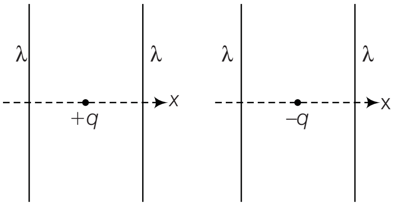


(a) $\frac{2475}{64} \mu\Omega$ (b) $\frac{1875}{64} \mu\Omega$
(c) $\frac{1875}{49} \mu\Omega$ (d) $\frac{2475}{132} \mu\Omega$

33. In the following circuit, the current through the resistor $R(= 2\,\Omega)$ is I amperes. The value of I is **(Single Integer Type, 2015)**



34. The figures below depict two situations in which two infinitely long static line charges of constant positive line charge density λ are kept parallel to each other.



In their resulting electric field, point charges q and $-q$ are kept in equilibrium between them. The point charges are confined to move in the x direction only. If they are given a small displacement about their equilibrium positions, then the correct statements is/are

(Single Correct Option, 2015)

(a) both charges execute simple harmonic motion

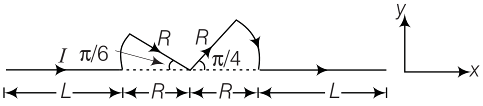
(b) both charges will continue moving in the direction of their displacement

(c) charge $+q$ executes simple harmonic motion while charge $-q$ continues moving in the direction of its displacement

(d) charge $-q$ executes simple harmonic motion while charge $+q$ continues moving in the direction of its displacement

35. A conductor (shown in the figure) carrying constant current I is kept in the x-y plane in a uniform magnetic field $\mathbf{B}$. If F is the magnitude of the total magnetic force acting on the conductor, then the correct statements is/are

(More than One Correct Option, 2015)



(a) if $\mathbf{B}$ is along $\hat{\mathbf{z}}$, $F \propto (L+R)$

(b) if $\mathbf{B}$ is along $\hat{\mathbf{x}}$, $F = 0$

(c) if $\mathbf{B}$ is along $\hat{\mathbf{y}}$, $F \propto (L+R)$

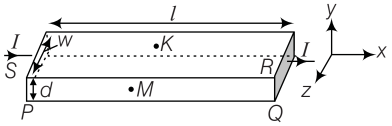
(d) if $\mathbf{B}$ is along $\hat{\mathbf{z}}$, $F = 0$

Passage (Q. Nos. 36-37)

In a thin rectangular metallic strip a constant current I flows along the positive x-direction, as shown in the figure. The length, width and thickness of the strip are l, w and d, respectively. A uniform magnetic field $\mathbf{B}$ is applied on the strip along the positive y-direction. Due to this, the charge carriers experience a net deflection along the z-direction.

This results in accumulation of charge carriers on the surface $PQRS$ and appearance of equal and opposite charges on the face opposite to $PQRS$. A potential difference along the z-direction is thus developed. Charge accumulation continues until the magnetic force is balanced by the electric force. The current is assumed to be uniformly distributed on the cross section of the strip and carried by electrons.

(Passage Type, 2015)



36. Consider two different metallic strips (1 and 2) of the same material. Their lengths are the same, widths are w_1 and w_2 and thicknesses are d_1 and d_2, respectively. Two points K and M are symmetrically located on the opposite faces parallel to the x-y plane (see figure). V_1 and V_2 are the potential differences between K and M in strips 1 and 2, respectively. Then, for a given current I flowing through them in a given magnetic field strength B, the correct statements is/are

(a) If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = 2V_1$

(b) If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = V_1$

(c) If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = 2V_1$

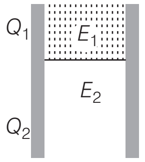
(d) If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = V_1$

37. Consider two different metallic strips (1 and 2) of same dimensions (length l, width w and thickness d) with carrier densities n_1 and n_2, respectively. Strip 1 is placed in magnetic field B_1 and strip 2 is placed in magnetic field B_2, both along positive y-directions. Then V_1 and V_2 are the potential differences developed between K and M in strips 1 and 2, respectively. Assuming that the current I is the same for both the strips, the correct options is/are

(a) If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = 2V_1$
(b) If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = V_1$
(c) If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = 0.5V_1$
(d) If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = V_1$

38. A parallel plate capacitor has a dielectric slab of dielectric constant K between its plates that covers 1/3 of the area of its plates, as shown in the figure. The total capacitance of the capacitor is C while that of the portion with dielectric in between is C_1. When the capacitor is charged, the plate area covered by the dielectric gets charge Q_1 and the rest of the area gets charge Q_2. The electric field in the dielectric is E_1 and that in the other portion is E_2. Choose the correct option/options, ignoring edge effects.

(More than One Correct Option, 2014)



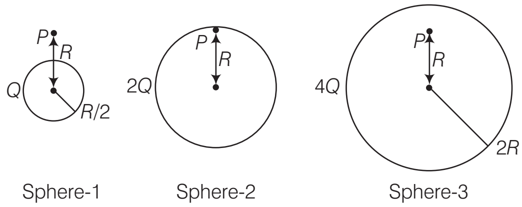
(a) $\dfrac{E_1}{E_2} = 1$ (b) $\dfrac{E_1}{E_2} = \dfrac{1}{K}$
(c) $\dfrac{Q_1}{Q_2} = \dfrac{3}{K}$ (d) $\dfrac{C}{C_1} = \dfrac{2+K}{K}$

39. Let $E_1(r)$, $E_2(r)$ and $E_3(r)$ be the respective electric fields at a distance r from a point charge Q, an infinitely long wire with constant linear charge density λ, and an infinite plane with uniform surface charge density σ. If $E_1(r_0) = E_2(r_0) = E_3(r_0)$ at a given distance r_0, then

(More than One Correct Option, 2014)

(a) $Q = 4\sigma\pi r_0^2$ (b) $r_0 = \dfrac{\lambda}{2\pi\sigma}$
(c) $E_1\left(\dfrac{r_0}{2}\right) = 2E_2\left(\dfrac{r_0}{2}\right)$ (d) $E_2\left(\dfrac{r_0}{2}\right) = 4E_3\left(\dfrac{r_0}{2}\right)$

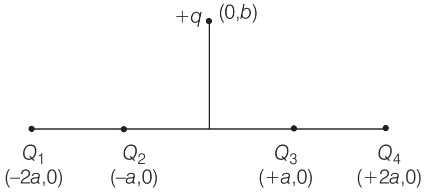
40. Charges Q, $2Q$ and $4Q$ are uniformly distributed in three dielectric solid spheres 1, 2 and 3 of radii $R/2$, R and $2R$ respectively, as shown in figure. If magnitudes of the electric fields at point P at a distance R from the centre of spheres 1, 2 and 3 are E_1, E_2 and E_3 respectively, then **(Single Correct Option, 2014)**



(a) $E_1 > E_2 > E_3$ (b) $E_3 > E_1 > E_2$
(c) $E_2 > E_1 > E_3$ (d) $E_3 > E_2 > E_1$

41. Four charges Q_1, Q_2, Q_3 and Q_4 of same magnitude are fixed along the x-axis at $x = -2a, -a, +a$ and $+2a$ respectively. A positive charge q is placed on the positive y-axis at a distance $b > 0$. Four options of the signs of these charges are given in Column I. The direction of the forces on the charge q is given in Column II. Match Column I with Column II and select the correct answer using the code given below the lists.

(Matching Type, 2014)



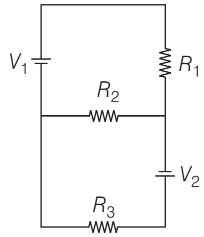
Column I	Column II
P. Q_1, Q_2, Q_3, Q_4 all positive	1. $+x$
Q. Q_1, Q_2 positive; Q_3, Q_4 negative	2. $-x$
R. Q_1, Q_4 positive; Q_2, Q_3 negative	3. $+y$
S. Q_1, Q_3 positive; Q_2, Q_4 negative	4. $-y$

Codes

	P	Q	R	S
(a)	3,	1,	4,	2
(b)	4,	2,	3,	1
(c)	3,	1,	2,	4
(d)	4,	2,	1,	3

42. Two ideal batteries of emf V_1 and V_2 and three resistances R_1, R_2 and R_3 are connected as shown in the figure. The current in resistance R_2 would be zero if

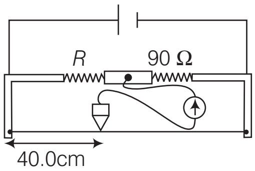
(More than One Correct Option, 2014)



(a) $V_1 = V_2$ and $R_1 = R_2 = R_3$
(b) $V_1 = V_2$ and $R_1 = 2R_2 = R_3$
(c) $V_1 = 2V_2$ and $2R_1 = 2R_2 = R_3$
(d) $2V_1 = V_2$ and $2R_1 = R_2 = R_3$

43. During an experiment with a metre bridge, the galvanometer shows a null point when the jockey is pressed at 40.0 cm using a standard resistance of 90 Ω, as shown in the figure. The least count of the scale used in the metre bridge is 1 mm. The unknown resistance is

(Single Correct Option, 2014)



(a) $60 \pm 0.15\,\Omega$
(b) $135 \pm 0.56\,\Omega$
(c) $60 \pm 0.25\,\Omega$
(d) $135 \pm 0.23\,\Omega$

44. Two parallel wires in the plane of the paper are distance X_0 apart. A point charge is moving with speed u between the wires in the same plane at a distance X_1 from one of the wires. When the wires carry current of magnitude I in the same direction, the radius of curvature of the path of the point charge is R_1. In contrast, if the currents I in the two wires have directions opposite to each other, the radius of curvature of the path is R_2. If $\frac{X_0}{X_1} = 3$, and value of $\frac{R_1}{R_2}$ is

(Single Integer Type, 2014)

45. A galvanometer gives full scale deflection with 0.006 A current. By connecting it to a 4990 Ω resistance, it can be converted into a voltmeter of range 0-30 V. If connected to a $\frac{2n}{249}\,\Omega$ resistance, it becomes an ammeter of range 0-1.5 A. The value of n is **(Single Integer Type, 2014)**

Passage (Q. Nos. 46-47)

The figure shows a circular loop of radius a with two long parallel wires (numbered 1 and 2) all in the plane of the paper. The distance of each wire from the centre of the loop is d. The loop and the wires are carrying the same current I. The current in the loop is in the counter-clockwise direction if seen from above. **(Passage Type, 2014)**

46. When $d \approx a$ but wires are not touching the loop, it is found that the net magnetic field on the axis of the loop is zero at a height h above the loop. In that case

(a) current in wire 1 and wire 2 is the direction PQ and RS, respectively and $h \approx a$
(b) current in wire 1 and wire 2 is the direction PQ and SR, respectively and $h \approx a$
(c) current in wire 1 and wire 2 is the direction PQ and SR, respectively and $h \approx 1.2a$
(d) current in wire 1 and wire 2 is the direction PQ and RS, respectively and $h \approx 1.2a$

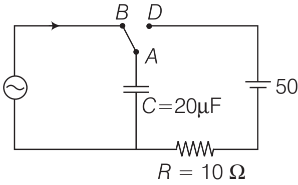
47. Consider $d >> a$, and the loop is rotated about its diameter parallel to the wires by 30° from the position shown in the figure. If the currents in the wires are in the opposite directions, the torque on the loop at its new position will be (assume that the net field due to the wires is constant over the loop)

(a) $\frac{\mu_0 I^2 a^2}{d}$
(b) $\frac{\mu_0 I^2 a^2}{2d}$
(c) $\frac{\sqrt{3}\,\mu_0 I^2 a^2}{d}$
(d) $\frac{\sqrt{3}\,\mu_0 I^2 a^2}{2d}$

48. At time $t = 0$, terminal A in the circuit shown in the figure is connected to B by a key and an alternating current $I(t) = I_0 \cos(\omega t)$, with $I_0 = 1$ A and

$\omega = 500\ \text{rad s}^{-1}$ starts flowing in it with the initial direction shown in the figure. At $t = \frac{7\pi}{6\omega}$, the key is switched from B to D. Now onwards only A and D are connected. A total charge Q flows from the battery to charge the capacitor fully. If $C = 20\ \mu\text{F}$, $R = 10\ \Omega$ and the battery is ideal with emf of 50 V, identify the correct statement(s).

(More than One Correct Option, 2014)



(a) Magnitude of the maximum charge on the capacitor before $t = \frac{7\pi}{6\omega}$ is 1×10^{-3} C

(b) The current in the left part of the circuit just before $t = \frac{7\pi}{6\omega}$ is clockwise

(c) Immediately after A is connected to D, the current in R is 10 A

(d) $Q = 2 \times 10^{-3}$ C

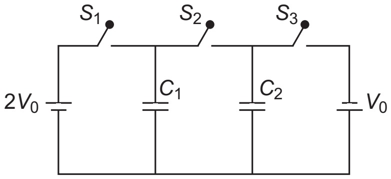
49. Two non-conducting solid spheres of radii R and $2R$, having uniform volume charge densities ρ_1 and ρ_2 respectively, touch each other. The net electric field at a distance $2R$ from the centre of the smaller sphere, along the line joining the centre of the spheres, is zero. The ratio $\frac{\rho_1}{\rho_2}$ can be **(More than One Correct Option, 2013)**

(a) -4 (b) $-\frac{32}{25}$

(c) $\frac{32}{25}$ (d) 4

50. In the circuit shown in the figure, there are two parallel plate capacitors each of capacitance C. The switch S_1 is pressed first to fully charge the capacitor C_1 and then released. The switch S_2 is then pressed to charge the capacitor C_2. After some time, S_2 is released and then S_3 is pressed. After some time

(More than One Correct Option, 2013)



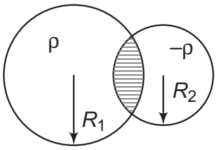
(a) the charge on the upper plate of C_1 is $2\,CV_0$

(b) the charge on the upper plate of C_1 is CV_0

(c) the charge on the upper plate of C_2 is 0

(d) the charge on the upper plate of C_2 is $-CV_0$

51. Two non-conducting spheres of radii R_1 and R_2 and carrying uniform volume charge densities $+\rho$ and $-\rho$, respectively, are placed such that they partially overlap, as shown in the figure. At all points in the overlapping region **(More than One Correct Option, 2013)**



(a) the electrostatic field is zero

(b) the electrostatic potential is constant

(c) the electrostatic field is constant in magnitude

(d) the electrostatic field has same direction

52. A particle of mass M and positive charge Q, moving with a constant velocity $\mathbf{u}_1 = 4\mathbf{i}\text{ms}^{-1}$, enters a region of uniform static magnetic field normal to the x-y plane. The region of the magnetic field extends from $x = 0$ to $x = L$ for all values of y. After passing through this region, the particle emerges on the other side after 10 milliseconds with a velocity $\mathbf{u}_2 = 2(\sqrt{3}\mathbf{i} + \mathbf{j})\ \text{ms}^{-1}$. The correct statement(s) is (are)

(More than One Correct Option, 2013)

(a) the direction of the magnetic field is $-z$ direction.

(b) the direction of the magnetic field is $+z$ direction

(c) the magnitude of the magnetic field is $\frac{50\pi M}{3Q}$ units.

(d) the magnitude of the magnetic field is $\frac{100\pi M}{3Q}$ units.

53. A steady current I flows along an infinitely long hollow cylindrical conductor of radius R. This cylinder is placed coaxially inside an infinite solenoid of radius $2R$. The solenoid has n turns per unit length and carries a steady current I. Consider a point P at a distance r from the common axis. The correct statement(s) is (are)

(More than One Correct Option, 2013)

(a) In the region $0 < r < R$, the magnetic field is non-zero
(b) In the region $R < r < 2R$, the magnetic field is along the common axis
(c) In the region $R < r < 2R$, the magnetic field is tangential to the circle of radius r, centered on the axis
(d) In the region $r > 2R$, the magnetic field is non-zero

Passage (Q. Nos. 54-55)

A point charge Q is moving in a circular orbit of radius R in the x-y plane with an angular velocity ω. This can be considered as equivalent to a loop carrying a steady current $\frac{Q\omega}{2\pi}$. A uniform magnetic field along the positive z-axis is now switched on, which increases at a constant rate from 0 to B in one second. Assume that the radius of the orbit remains constant. The applications of the magnetic field induces an emf in the orbit.

The induced emf is defined as the work done by an induced electric field in moving a unit positive charge around a closed loop. It is known that, for an orbiting charge, the magnetic dipole moment is proportional to the angular momentum with a proportionality constant γ.

(Passage Type, 2013)

54. The change in the magnetic dipole moment associated with the orbit, at the end of the time interval of the magnetic field change, is

(a) γBQR^2 (b) $-\gamma \frac{BQR^2}{2}$
(c) $\gamma \frac{BQR^2}{2}$ (d) γBQR^2

55. The magnitude of the induced electric field in the orbit at any instant of time during the time interval of the magnetic field change is

(a) $\frac{BR}{4}$ (b) $\frac{-BR}{2}$ (c) BR (d) $2BR$

Passage (Q. Nos. 56-57)

A thermal power plant produces electric power of 600 kW at 4000 V, which is to be transported to a place 20 km away from the power plant for consumers' usage. It can be transported either directly with a cable of large current carrying capacity or by using a combination of step-up and step-down transformers at the two ends. The drawback of the direct transmission is the large energy dissipation. In the method using transformers, the dissipation is much smaller. In this method, a step-up transformer is used at the plant side so that the current is reduced to a smaller value. At the consumers' end a step-down transformer is used to supply power to the consumers at the specified lower voltage. It is reasonable to assume that the power cable is purely resistive and the transformers are ideal with a power factor unity. All the current and voltages mentioned are rms values.

(Passage Type, 2013)

56. If the direct transmission method with a cable of resistance $0.4\ \Omega\ \text{km}^{-1}$ is used, the power dissipation (in %) during transmission is

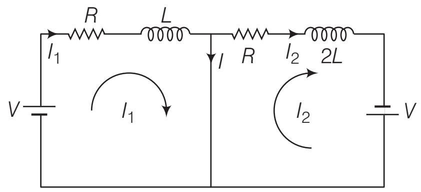
(a) 20 (b) 30 (c) 40 (d) 50

57. In the method using the transformers, assume that the ratio of the number of turns in the primary to that in the secondary in the step-up transformer is $1:10$. If the power to the consumers has to be supplied at 200V, the ratio of the number of turns in the primary to that in the secondary in the step-down transformer is

(a) $200:1$ (b) $150:1$
(c) $100:1$ (d) $50:1$

Answer *with* Explanations

1. **(b, d)** $I_1 = \frac{V}{R}\left(1-e^{-\frac{tR}{L}}\right)$



$$I_2 = \frac{V}{R}\left(1-e^{-\frac{tR}{2L}}\right)$$

From principle of superposition,

$$I = I_1 - I_2$$

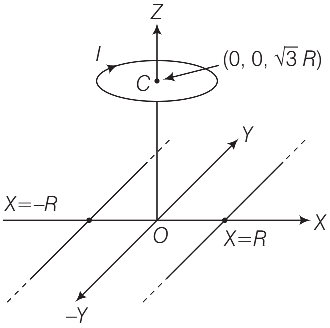
$$\Rightarrow \quad I = \frac{V}{R}e^{-\frac{tR}{2L}}\left(1-e^{-\frac{tR}{2L}}\right) \qquad \dots\text{(i)}$$

I is maximum when $\frac{dI}{dt}=0$, which gives $e^{-\frac{tR}{2L}} = \frac{1}{2}$ or $t = \frac{2L}{R}\ln 2$

Substituting this time in Eq. (i), we get

$$I_{\max} = \frac{V}{4R}$$

2. **(a,b,d)**

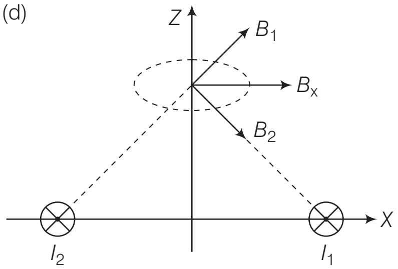


(a) At origin, $\mathbf{B}=0$ due to two wires if $I_1 = I_2$, hence ($\mathbf{B}_{\text{net}}$) at origin is equal to $\mathbf{B}$ due to ring. which is non-zero.

(b) If $I_1 > 0$ and $I_2 < 0$, $\mathbf{B}$ at origin due to wires will be along $+\hat{\mathbf{k}}$. Direction of $\mathbf{B}$ due to ring is along $-\hat{\mathbf{k}}$ direction and hence $\mathbf{B}$ can be zero at origin.

(c) If $I_1 < 0$ and $I_2 > 0$, $\mathbf{B}$ at origin due to wires is along $-\hat{\mathbf{k}}$ and also along $-\hat{\mathbf{k}}$ due to ring, hence $\mathbf{B}$ cannot be zero.

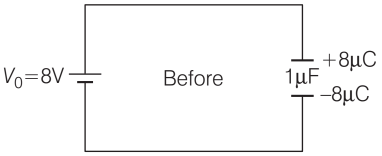
(d)

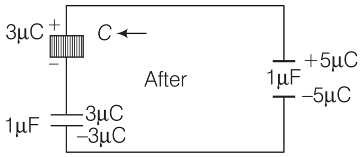


At centre of ring, **B** due to wires is along *X*-axis.

Hence *Z*-component is only because of ring which $\mathbf{B} = \frac{\mu_0 i}{2R}(-\hat{\mathbf{k}})$.

3. **(1.50)**



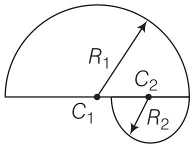


Applying loop rule,

$$\frac{5}{1} - \frac{3}{\varepsilon_r} - \frac{3}{1} = 0 \quad \Rightarrow \quad \frac{3}{\varepsilon_r} = 2 \quad \Rightarrow \varepsilon_r = 1.50$$

4. (2 m/s) If average speed is considered along *X*-axis,

$$R_1 = \frac{mv_0}{qB_1},\; R_2 = \frac{mv_0}{qB_2} = \frac{mv_0}{4qB_1} \quad \Rightarrow \quad R_1 > R_2$$



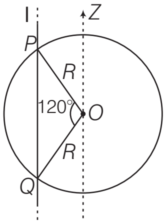
Distance travelled along *x*-axis, $\Delta x = 2(R_1 + R_2) = \frac{5mv_0}{2qB_1}$

$$\text{Total time} = \frac{T_1}{2} + \frac{T_2}{2} = \frac{\pi m}{qB_1} + \frac{\pi m}{qB_2}$$

$$= \frac{\pi m}{qB_1} + \frac{\pi m}{4qB_1} = \frac{5\pi m}{4qB_1}$$

$$\text{Magnitude of average speed} = \frac{\frac{5mv_0}{2qB_1}}{\frac{5\pi m}{4qB_1}} = 2 \text{ m/s}$$

5. **(a, b)** $PQ = (2)\,R\sin 60°$



$$= (2R)\frac{\sqrt{3}}{2} = (\sqrt{3}R)$$

$$q_{\text{enclosed}} = \lambda\,(\sqrt{3}R)$$

We have, $\phi = \dfrac{q_{\text{enclosed}}}{\varepsilon_0} \Rightarrow \phi = \left(\dfrac{\sqrt{3}\lambda R}{\varepsilon_0}\right)$

Also, electric field is perpendicular to wire, so Z-component will be zero.

6. **(2)** $a = \dfrac{F}{m} = \dfrac{qE}{m} = 10^3\sin(10^3 t)$

$$\frac{dv}{dt} = 10^3\sin(10^3 t) \Rightarrow \int_0^v dv = \int_0^t 10^3\sin(10^3 t)\,dt$$

$$\therefore \quad v = \frac{10^3}{10^3}[1-\cos(10^3 t)]$$

Velocity will be maximum when $\cos(10^3 t) = -1$

$$v_{\max} = 2\text{ m/s}$$

7. **(5.55)** Given, $N = 50$,

$$A = 2\times 10^{-4}\text{ m}^2, C = 10^{-4}, R = 50\,\Omega,$$

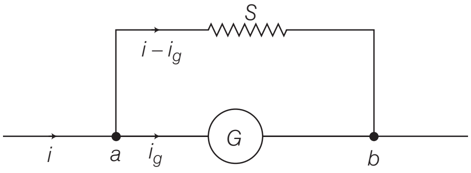
$$B = 0.02\text{ T}, \theta = 0.2\text{ rad}$$

$$\therefore \quad Ni_g AB = C\theta$$

$$\Rightarrow \quad i_g = \frac{C\,\theta}{N\,AB} = \frac{10^{-4}\times 0.2}{50\times 2\times 10^{-4}\times 0.02} = 0.1\text{ A}$$

$$\therefore \quad V_{ab} = i_g \times G = (i - i_g)S$$

$$0.1\times 50 = (1-0.1)\times S$$



$$5 = 0.9\times S$$

$$\therefore \quad S = \frac{50}{9}\,\Omega = 5.55\,\Omega$$

8. **(b)** **List-II**

(1) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{d^2} \Rightarrow E \propto \dfrac{1}{d^2}$

(2) $E_{\text{axis}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q(2l)}{d^3} \Rightarrow E \propto \dfrac{1}{d^3}$

(3) $E = \dfrac{\lambda}{2\pi\varepsilon_0 d} \Rightarrow E \propto \dfrac{1}{d}$

(4) $E = \dfrac{\lambda}{2\pi\varepsilon_0(d-l)} - \dfrac{\lambda}{2\pi\varepsilon_0(d+l)} = \dfrac{\lambda(2l)}{2\pi\varepsilon_0 d^2}$

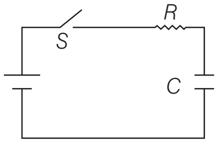
$\Rightarrow E \propto \dfrac{1}{d^2}$

(5) $E = \dfrac{\sigma}{2\varepsilon_0} \Rightarrow E$ is independent of d

9. **(b)** When switch is closed for a very long time capacitor will get fully charged and charge on capacitor will be $q = CV$

Energy stored in capacitor,

$$E_C = \frac{1}{2}CV^2 \qquad \text{...(i)}$$



Work done by a battery,

$$W = Vq = VCV = CV^2$$

Energy dissipated across resistance

E_D = (work done by eq. battery) – (energy stored)

$$E_D = CV^2 - \frac{1}{2}CV^2 = \frac{1}{2}CV^2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$E_D = E_C$$

10. **(a)** **For process (1)**

Charge on capacitor $= \dfrac{CV_0}{3}$

Energy stored in capacitor $= \dfrac{1}{2}C\dfrac{V_0^2}{9} = \dfrac{CV_0^2}{18}$

Work done by battery $= \dfrac{CV_0}{3}\times\dfrac{V}{3} = \dfrac{CV_0^2}{9}$

Heat loss $= \dfrac{CV_0^2}{9} - \dfrac{CV_0^2}{18} = \dfrac{CV_0^2}{18}$

For process (2)

Charge on capacitor $= \dfrac{2CV_0}{3}$

Extra charge flow through battery $= \dfrac{CV_0}{3}$

Work done by battery $= \dfrac{CV_0}{3}\cdot\dfrac{2V_0}{3} = \dfrac{2CV_0^2}{9}$

Final energy stored in capacitor $= \dfrac{1}{2}C\left(\dfrac{2V_0}{3}\right)^2 = \dfrac{4CV_0^2}{18}$

Energy stored in process 2 $= \dfrac{4CV_0^2}{18} - \dfrac{CV_0^2}{18} = \dfrac{3CV_0^2}{18}$

Heat loss in process (2) = work done by battery in process (2) – energy store in capacitor process (2)

$$= \frac{2CV_0^2}{9} - \frac{3CV_0^2}{18} = \frac{CV_0^2}{18}$$

For process (3)

Charge on capacitor $= CV_0$

Extra charge flown through battery

$$= CV_0 - \frac{2CV_0}{3} = \frac{CV_0}{3}$$

Work done by battery in this process

$$= \left(\frac{CV_0}{3}\right)(V_0) = \frac{CV_0^2}{3}$$

Final energy stored in capacitor $= \frac{1}{2}CV_0^2$

Energy stored in this process

$$= \frac{1}{2}CV_0^2 - \frac{4CV_0^2}{18} = \frac{5CV_0^2}{18}$$

Heat loss in process (3)

$$= \frac{CV_0^2}{3} - \frac{5CV_0^2}{18} = \frac{CV_0^2}{18}$$

Now, total heat loss (E_D)

$$= \frac{CV_0^2}{18} + \frac{CV_0^2}{18} + \frac{CV_0^2}{18} = \frac{CV_0^2}{6}$$

Final energy stored in capacitor $= \frac{1}{2}CV_0^2$

So we can say that $E_D = \frac{1}{3}\left(\frac{1}{2}CV_0^2\right)$

11. **(c)** For particle to move in negative y-direction, either its velocity must be in negative y-direction (if initial velocity $\neq 0$) and force should be parallel to velocity or it must experience a net force in negative y-direction only (if initial velocity $= 0$)

12. **(a)** $\mathbf{F}_{\text{net}} = \mathbf{F}_e + \mathbf{F}_B = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

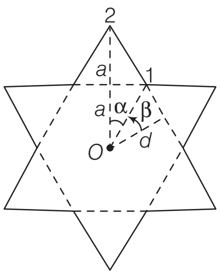
For particle to move in straight line with constant velocity, $\mathbf{F}_{\text{net}} = 0$

$\therefore \quad q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = 0$

13. **(c)** For path to be helix with axis along positive z-direction, particle should experience a centripetal acceleration in xy-plane.

For the given set of options only option (c) satisfy the condition. Path is helical with increasing pitch.

14. **(a)**



$$B_{12} = \frac{\mu_0 I}{4\pi d}[\sin\alpha + \sin\beta]$$

$\alpha = 60°$ and $\beta = -30°$

$$= \frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right]$$

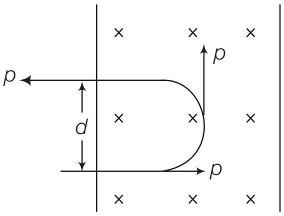
$$B_{12} = \frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{3}-1}{2}\right]$$

$d = a$

$B_0 = 12\ B_{12}$

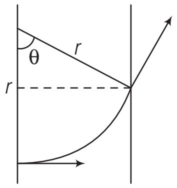
$$= 12 \times \frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{3}-1}{2}\right] = \frac{\mu_0 I}{4\pi a} 6\,[\sqrt{3}-1]$$

15. **(b,c)** (a) $|\Delta\mathbf{P}| = \sqrt{2}p$



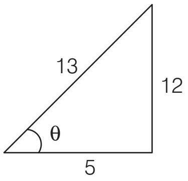
(b) $r(1 - \cos\theta) = R \quad \Rightarrow \quad r\sin\theta = \frac{3R}{2}$

$$\frac{\sin\theta}{1-\cos\theta} = \frac{3}{2}\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{3}{2}$$



$$\cot\frac{\theta}{2} = \frac{3}{2} \quad \Rightarrow \quad \tan\frac{\theta}{2} = \frac{2}{3}$$

$$\Rightarrow \qquad \tan\theta = \frac{2\left(\frac{2}{3}\right)}{1-\frac{4}{9}} = \frac{\frac{4}{3}}{\frac{5}{9}} = \frac{12}{5}$$



$$\sin\theta = \frac{12}{13}\ r\left(\frac{12}{13}\right) = \frac{3R}{2};\ r = \frac{13R}{8} = \frac{P}{QB};\ B = \frac{8P}{13QR}$$

(c) $\frac{P}{QB} < \frac{3R}{2}, B > \frac{2P}{3QR}$

(d) $r = \frac{mv}{QB}, d = 2r = \frac{2mv}{QB} \quad \Rightarrow \quad d \propto m$

16. **(a,b)** At $\omega \approx 0$, $X_C = \dfrac{1}{\omega C} = \infty$. Therefore, current is nearly zero.

Further at resonance frequency, current and voltage are in phase. This resonance frequency is given by,

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-6} \times 10^{-6}}} = 10^6 \text{ rad/s}$$

We can see that this frequency is independent of R.

Further, $X_L = \omega L$, $X_C = \dfrac{1}{\omega C}$

$$\text{At, } \omega = \omega_r = 10^6 \text{ rad/s}, X_L = X_C.$$

For $\omega > \omega_r$, $X_L > X_C$. So, circuit is inductive.

17. **(b,d)** The net magnetic flux through the loops at time t is

$$\phi = B(2A - A)\cos \omega t = BA \cos \omega t$$

so, $\left|\dfrac{d\phi}{dt}\right| = B\omega A \sin\omega t$

$\therefore \left|\dfrac{d\phi}{dt}\right|$ is maximum when $\phi = \omega t = \pi/2$

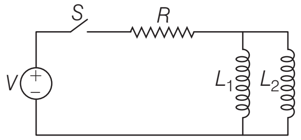
The emf induced in the smaller loop,

$$\varepsilon_{\text{smaller}} = -\frac{d}{dt}(BA \cos \omega t) = B\omega A \sin\omega t$$

$\therefore$ Amplitude of maximum net emf induced in both the loops

= Amplitude of maximum emf induced in the smaller loop alone.

18. **(a,b,c)**



Since inductors are connected in parallel

$$V_{L_1} = V_{L_2}; \quad L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt}$$

$$L_1 I_1 = L_2 I_2; \quad \frac{I_1}{I_2} = \frac{L_2}{L_1}$$

Current through resistor at any time t is given by

$$I = \frac{V}{R}(1 - e^{-\frac{RT}{L}}), \quad \text{where} \quad L = \frac{L_1 L_2}{L_1 + L_2}$$

After long time $I = \dfrac{V}{R}$

$$I_1 + I_2 = I \qquad \text{...(i)}$$

$$L_1 I_1 = L_2 I_2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$I_1 = \frac{V}{R}\frac{L_2}{L_1 + L_2} \Rightarrow I_2 = \frac{V}{R}\frac{L_1}{L_1 + L_2}$$

(d) Value of current is zero at $t = 0$

Value of current is V/R at $t = \infty$

Hence option (d) is incorrect.

19. **(b,c)** $V_{XY} = V_0 \sin\left(\omega t + \dfrac{2\pi}{3}\right) - V_0 \sin \omega t$

$$= V_0 \sin\left(\omega t + \frac{2\pi}{3}\right) + V_0 \sin(\omega t + \pi)$$

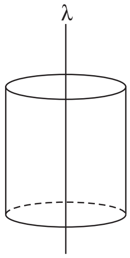
$$\Rightarrow \quad \phi = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

$$\Rightarrow \quad V_0' = 2V_0 \cos\left(\frac{\pi}{6}\right) = \sqrt{3}\, V_0$$

$$\Rightarrow \quad V_{XY} = \sqrt{3} V_0 \sin(\omega t + \phi)$$

$$\Rightarrow \quad (V_{XY})_{\text{rms}} = (V_{YZ})_{\text{rms}} = \sqrt{3}\frac{V_0}{\sqrt{2}}$$

20. **(d)** Suppose charger per unit length at any instant is λ.



Initial value of λ is suppose λ_0.

Electric field s at a distance r at any instant is

$$E = \frac{\lambda}{2\pi\varepsilon r}$$

$$J = \sigma E = \sigma \frac{\lambda}{2\pi\varepsilon r}$$

$$i = \frac{dq}{dt} = J(A) = -J\sigma 2\pi rl$$

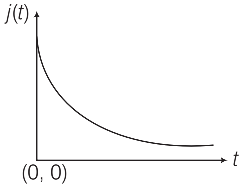
$$\frac{d\lambda l}{dt} = -\frac{\lambda}{2\pi\varepsilon r} \times \sigma 2\pi rl \qquad (q = \lambda l)$$

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\lambda} = -\frac{\sigma}{\varepsilon}\int_0^t dt \quad \Rightarrow \lambda = \lambda_0 e^{-\frac{\sigma}{\varepsilon}t}$$

$$J = \frac{\sigma}{2\pi\varepsilon r}\lambda = \frac{\sigma\lambda_0}{2\pi\varepsilon r} e^{-\frac{\sigma}{\varepsilon}t} = J_0 e^{-\frac{\sigma}{\varepsilon}t}$$

Here, $J_0 = \dfrac{\sigma\lambda_0}{2\pi\varepsilon r}$

$\therefore J(t)$ decreases exponentially as shown in figure below.



21. **(a,b,c,d)** Just after pressing key,

$5 - 25000 i_1 = 0$

$5 - 50000 i_2 = 0$ (As charge in both capacitors = 0)

$\Rightarrow i_1 = 0.2$ mA $\Rightarrow i_2 = 0.1$ mA

and $V_B + 25000 i_1 = V_A \Rightarrow V_B - V_A = -5$ V

After a long time, i_1 and $i_2 = 0$ (steady state)

$$\Rightarrow \quad 5 - \frac{q_1}{40} = 0 \Rightarrow q_1 = 200\mu\text{C}$$

$$\text{and} \quad 5 - \frac{q_2}{20} = 0 \Rightarrow q_2 = 100\mu\text{C}$$

$$V_B - \frac{q_2}{20} = V_A \Rightarrow V_B - V_A = +5 \text{ V}$$

⇒ (a) is correct.

For capacitor 1, $q_1 = 200[1 - e^{-t/1}]\,\mu\text{C}$

$$i_1 = \frac{1}{5}e^{-t/1}\ \text{mA}$$

For capacitor 2, $q_2 = 100[1 - e^{-t/1}]\,\mu\text{C}$

$$i_2 = \frac{1}{10}e^{-t/1}\ \text{mA}$$

$$\Rightarrow \quad V_B - \frac{q_2}{20} + i_1 \times 25 = V_A$$

$$\Rightarrow \quad V_B - V_A = 5[1 - e^{-t}] - 5e^{-t}$$

$$= -5[1 - 2e^{-t}]$$

At $t = \ln 2$, $V_B - V_A = 5[1 - 1] = 0$

⇒ (b) is correct.

At $t = 1$, $i = i_1 + i_2 = \frac{1}{5}e^{-1} + \frac{1}{10}e^{-1} = \frac{3}{10}\cdot\frac{1}{e}$

At $t = 0$, $i = i_1 + i_2 = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$

⇒ (c) is correct.

After a long time, $i_1 = i_2 = 0 \Rightarrow$ (d) is correct.

22. **(d)** Balls will gain positive charge and hence move towards negative plate.

On reaching negative plate, balls will attain negative charge and come back to positive plate.

and so on, balls will keep oscillating.

But oscillation is not S.H.M.,

As force on balls is not $\propto x$.

⇒ option (d) is correct.

23. **(a)** As the balls keep on carrying charge form one plate to another, current will keep on flowing even in steady state. When at bottom plate, if all balls attain charge q,

$$\frac{kq}{r} = V_0 \left(k = \frac{1}{4\pi\varepsilon_0}\right) \Rightarrow q = \frac{V_0 r}{k}$$

Inside cylinder, electric field

$$E = [V_0 - (-V_0)]h = 2V_0 h.$$

⇒ Acceleration of each ball,

$$a = \frac{qE}{m} = \frac{2hr}{km}\cdot V_0^2$$

⇒ Time taken by balls to reach other plate,

$$t = \sqrt{\frac{2h}{a}} = \sqrt{\frac{2h.km}{2hrV_0^2}} = \frac{1}{V_0}\sqrt{\frac{km}{r}}$$

If there are n balls, then

Average current,

$$i_{\text{av}} = \frac{nq}{t} = n \times \frac{V_0 r}{k} \times V_0\sqrt{\frac{r}{km}} \quad \Rightarrow i_{\text{av}} \propto V_0^2$$

24. **(c,d)** Because of non-uniform evaporation at different section, area of cross-section would be different at different sections.

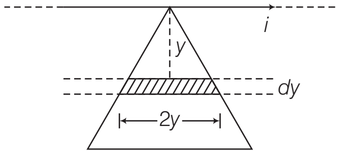
Region of highest evaporation rate would have rapidly reduced area and would become break up cross-section.

Resistance of the wire as whole increases with time.

Overall resistance increases hence power decreases.

$\left(P = \frac{V^2}{R} \text{ or } P \propto \frac{1}{R} \text{ as } V \text{ is constant}\right)$. At break up junction temperature would be highest, thus light of highest band frequency would be emitted at those cross-section.

25. **(a,c)** By reciprocity theorem of mutual induction, it can be assumed that current in infinite wire is varying at 10A/s and EMF is induced in triangular loop.



Flux of magnetic field through triangle loop, if current in infinite wire is ϕ, can be calculated as follows:

$$d\phi = \frac{\mu_0 i}{2\pi y}\cdot 2y\,dy \quad \Rightarrow \quad d\phi = \frac{\mu_0 i}{\pi}dy$$

$$\Rightarrow \quad \phi = \frac{\mu_0 i}{\pi}\left(\frac{l}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \text{EMF} = \left|\frac{d\phi}{dt}\right| = \frac{\mu_0}{\pi}\left(\frac{l}{\sqrt{2}}\right)\cdot\frac{di}{dt}$$

$$= \frac{\mu_0}{\pi}(10\text{ cm})\left(10\frac{\text{A}}{\text{s}}\right) = \frac{\mu_0}{\pi}\text{ volt}$$

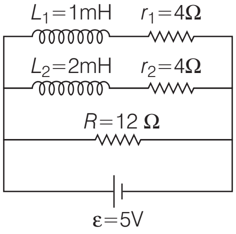
If we assume the current in the wire towards right then as the flux in the loop increases we know that the induced current in the wire is counter clockwise. Hence, the current in the wire is towards right.

Field due to triangular loop at the location of infinite wire is into the paper. Hence, force on infinite wire is away from the loop.

By cylindrical symmetry about infinite wire, rotation of triangular loop will not cause any additional EMF.

26. **(a,c)** For maximum range of voltage resistance should be maximum. So, all four should be connected in series. For maximum range of current, net resistance should be least. Therefore, all four should be connected in parallel.

27. **(8)**



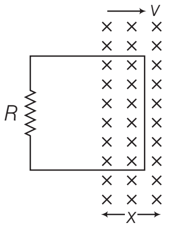
$$I_{\max} = \frac{\varepsilon}{R} = \frac{5}{12}\text{A} \qquad \text{(Initially at } t = 0)$$

$$I_{\min} = \frac{\varepsilon}{R_{\text{eq}}} = \varepsilon\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{R}\right) \qquad \text{(finally in steady state)}$$

$$= 5\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{12}\right) = \frac{10}{3}\text{A}$$

$$\frac{I_{\max}}{I_{\min}} = 8$$

28. **(b,c)**



When loop was entering (x < L)

$$\phi = BLx$$

$$e = -\frac{d\phi}{dt} = -BL\frac{dx}{dt}$$

$$|e| = BLv$$

$$i = \frac{e}{R} = \frac{BLv}{R} \qquad \text{(anticlockwise)}$$

$$F = ilB \text{ (Left direction)} = \frac{B^2L^2v}{R} \qquad \text{(in left direction)}$$

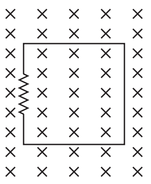
$$\Rightarrow \quad a = \frac{F}{m} = -\frac{B^2L^2v}{mR} \quad \Rightarrow \quad a = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = -\frac{B^2L^2v}{mR} \quad \Rightarrow \quad \int_{v_0}^{v} dv = -\frac{B^2L^2}{mR}\int_0^x dx$$

$$\Rightarrow v = v_0 - \frac{B^2L^2v}{mR}x$$

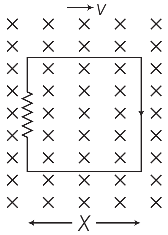
(straight line of negative slope for $x < L$)

$I = \frac{BL}{R}v \Rightarrow$ (I vs x will also be straight line of negative slope for x < L) $L \leq x \leq 3L$



$$\frac{d\phi}{dt} = 0 \quad e = 0, i = 0$$

$$F = 0 \Rightarrow x > 4L \Rightarrow e = Blv$$



Force also will be in left direction.

$$i = \frac{BLv}{R} \text{ (clockwise)}$$

$$a = -\frac{B^2L^2v}{mR} = v\frac{dv}{dx}$$

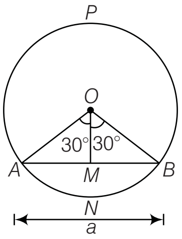
$$F = \frac{B^2L^2v}{R}\int_L^x -\frac{B^2L^2}{mR}dx = \int_{v_i}^{v_f} dv$$

$$\Rightarrow \quad -\frac{B^2L^2}{mR}(x - L) = v_f - v_i$$

$$v_f = v_i - \frac{B^2L^2}{mR}(x - L) \qquad \text{(straight line of negative slope)}$$

$$I = \frac{BLv}{R} \rightarrow \text{(Clockwise)} \qquad \text{(straight line of negative slope)}$$

29. **(6)** *ANBP* is cross-section of a cylinder of length *L*. The line charge passes through the centre *O* and perpendicular to paper.



$$AM = \frac{a}{2}, MO = \frac{\sqrt{3}a}{2}$$

$$\therefore \quad \angle AOM = \tan^{-1}\left(\frac{AM}{OM}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

Electric flux passing from the whole cylinder

$$\phi_1 = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

∴ Electric flux passing through *ABCD* plane surface (shown only *AB*) = Electric flux passing through cylindrical surface *ANB*

$$= \left(\frac{60^\circ}{360^\circ}\right)(\phi_1) = \frac{\lambda L}{6\varepsilon_0}$$

$$\therefore \qquad n = 6$$

30. (d) The sphere with cavity can be assumed as a complete sphere with positive charge of radius R_1 + another complete sphere with negative charge and radius R_2.

$\mathbf{E}_+ \to \mathbf{E}$ due to total positive charge

$\mathbf{E}_- \to \mathbf{E}$ due to total negative charge.

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$$

If we calculate it at P, then $\mathbf{E}_-$ comes out to be zero.

$\therefore \quad \mathbf{E} = \mathbf{E}_+$

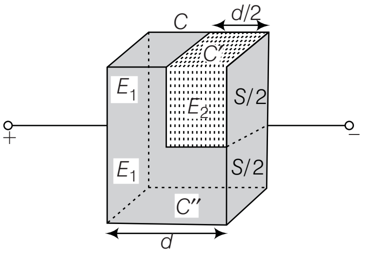
and $\mathbf{E}_+ = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{R_1^3}(OP)$, in the direction of OP.

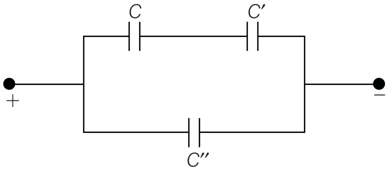
Here, q is total positive charge on whole sphere.

It is in the direction of OP or $\mathbf{a}$.

Now, inside the cavity electric field comes out to be uniform at any point. This is a standard result.

31. (d)





$$C_1 = \frac{\varepsilon_0 S}{d}, C = \frac{2\varepsilon_0 \frac{S}{2}}{d/2} = \frac{2\varepsilon_0 S}{d}$$

$$C' = \frac{4\varepsilon_0 \frac{S}{2}}{d/2} = \frac{4\varepsilon_0 S}{d}$$

and $$C'' = \frac{2\varepsilon_0 \frac{S}{2}}{d} = \frac{\varepsilon_0 S}{d}$$

$$C_2 = \frac{CC'}{C + C'} + C'' = \frac{4}{3}\frac{\varepsilon_0 S}{d} + \frac{\varepsilon_0 S}{d}$$

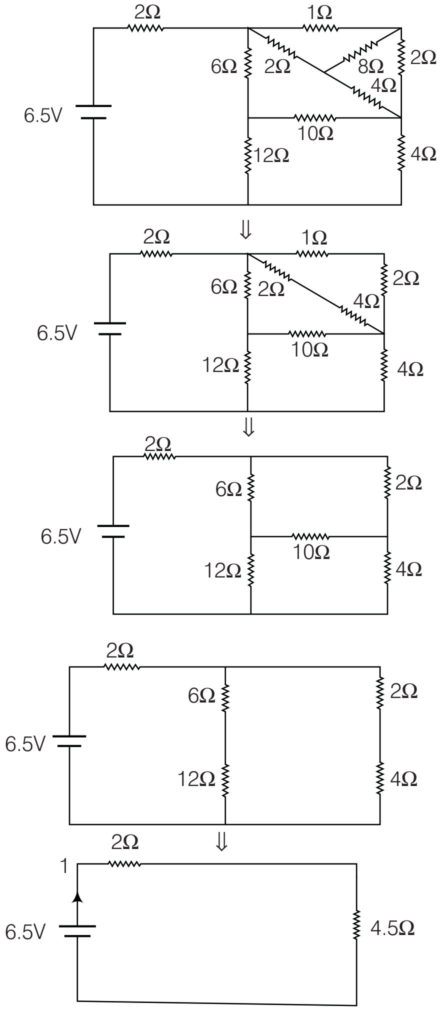
$$= \frac{7}{3}\frac{\varepsilon_0 S}{d} \quad \frac{C_2}{C_1} = \frac{7}{3}$$

32. (b) $\dfrac{1}{R} = \dfrac{1}{R_{\text{Al}}} + \dfrac{1}{R_{\text{Fe}}} = \left(\dfrac{A_{\text{Al}}}{\rho_{\text{Al}}} + \dfrac{A_{\text{Fe}}}{\rho_{\text{Fe}}}\right)\dfrac{1}{\ell}$

$$= \left[\frac{(7^2 - 2^2)}{2.7} + \frac{2^2}{10}\right]\frac{10^{-6}}{10^{-8}} \times \frac{1}{50 \times 10^{-3}}$$

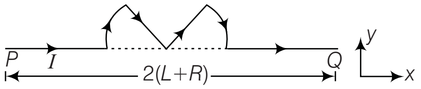
Solving we get, $R = \dfrac{1875}{64} \times 10^{-6}\ \Omega = \dfrac{1875}{64}\ \mu\Omega$

33. (1)



34. (c) At the shown position, net force on both charges is zero. Hence they are in equilibrium. But equilibrium of $+q$ is stable equilibrium. So, it will start oscillations when displaced from this position. These small oscillations are simple harmonic in nature. While equilibrium of $-q$ is unstable. So, it continues to move in the direction of its displacement.

35. (a,b,c)



Force on the complete wire = force on straight wire PQ carrying a current I.

$$\mathbf{F} = I(\mathbf{PQ} \times \mathbf{B})$$

$$= I[\{2(L + R)\hat{\mathbf{i}}\} \times \mathbf{B}]$$

This force is zero if $\mathbf{B}$ is along $\hat{\mathbf{i}}$ direction or x-direction. If magnetic field is along $\hat{\mathbf{j}}$ direction or $\hat{\mathbf{k}}$ direction,

$$|\mathbf{F}| = F = (I)(2)(L + R)B\sin 90°$$

or $\quad F = 2I(L + R)B$

or $\quad F \propto (L + R)$

$\therefore$ Options (a), (b) and (c) are correct.

36. **(a,d)** $F_B = Bev = Be\dfrac{I}{nAe} = \dfrac{BI}{nA}$

$$F_e = eE \quad \Rightarrow \quad F_e = F_B$$

$$eE = \frac{BI}{nA} \quad \Rightarrow \quad E = \frac{B}{nAe}$$

$$V = Ed = \frac{BI}{nAe}\cdot w = \frac{BIw}{n(wd)e} = \frac{BI}{ned}$$

$$\frac{V_1}{V_2} = \frac{d_2}{d_1}$$

$\Rightarrow$ if $w_1 = 2w_2$ and $d_1 = d_2$

$$V_1 = V_2$$

$\therefore$ Correct answers are (a) and (d).

37. **(a,c)** $V = \dfrac{BI}{ned} \quad \Rightarrow \quad \dfrac{V_1}{V_2} = \dfrac{B_1}{B_2} \times \dfrac{n_2}{n_1}$

If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = 2V_1$

If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = 0.5V_1$

$\therefore$ Correct answers are (a) and (c).

38. **(a,d)** $C = C_1 + C_2$

$$C_1 = \frac{K\varepsilon_0 A/3}{d}, C_2 = \frac{\varepsilon_0 2A/3}{d}$$

$$\Rightarrow \quad C = \frac{(K+2)\varepsilon_0 A}{3d} \quad \Rightarrow \quad \frac{C}{C_1} = \frac{K+2}{K}$$

Also, $E_1 = E_2 = \dfrac{V}{d}$, where V is potential difference between the plates.

39. **(c, d)** $\dfrac{Q}{4\pi\varepsilon_0 r_0^2} = \dfrac{\lambda}{2\pi\varepsilon_0 r_0} = \dfrac{\sigma}{2\varepsilon_0}$

$$Q = 2\pi\sigma r_0^2$$

(a) is incorrect, $r_0 = \dfrac{\lambda}{\pi\sigma}$

(b) is incorrect, $E_1\left(\dfrac{r_0}{2}\right) = 4E_1(r_0)$

As $\quad E_1 \propto \dfrac{1}{r^2}$

$$E_2\left(\frac{r_0}{2}\right) = 2E_2(r_0) \text{ as } E_2 \propto \frac{1}{r}$$

$\Rightarrow$ (c) is correct

$$E_3\left(\frac{r_0}{2}\right) = E_3(r_0) = E_2(r_0)$$

as $\quad E_3 \propto r^0$

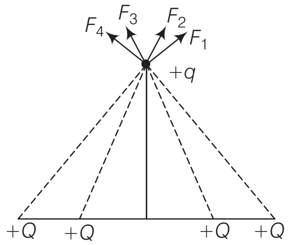
$\Rightarrow$ (d) is incorrect

40. **(c)** $E_1 = \dfrac{kQ}{R^2}$, where $k = \dfrac{1}{4\pi\varepsilon_0}$

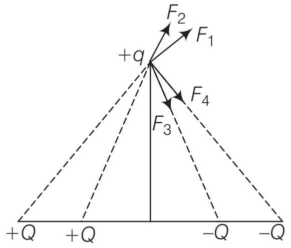
$$E_2 = \frac{k(2Q)}{R^2} \Rightarrow E_2 = \frac{2kQ}{R^2}$$

$$E_3 = \frac{k(4Q)\,R}{(2R)^3} \Rightarrow E_3 = \frac{kQ}{2R^2} \quad E_3 < E_1 < E_2$$

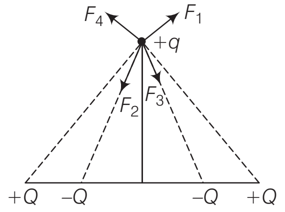
41. **(c)** (P) Component of forces along x-axis will vanish. Net force along positive y-axis



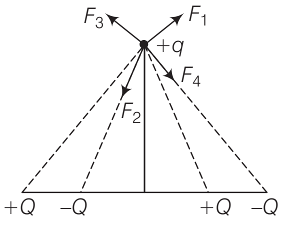
(Q) Component of forces along y-axis will vanish. Net force along positive x-axis



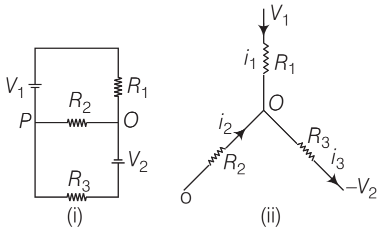
(R) Component of forces along x-axis will vanish. Net force along negative y-axis



(S) Component of forces along y-axis will vanish. Net force along negative x-axis



42. **(a,b,d)** Let us take $V_P = 0$. Then potentials across R_1, R_2 and R_3 are as shown in figure (ii)
In the same figure



(i) (ii)

$$i_1 + i_2 = i_3$$

$$\therefore \quad \frac{V_1 - V_0}{R_1} + \frac{0 - V_0}{R_2} = \frac{V_0 - (-V_2)}{R_3}$$

Solving this equation we get

$$V_0 = \frac{\frac{V_1}{R_1} + 0 - \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Current through R_2 will be zero if

$$V_0 = 0 \quad \Rightarrow \quad \frac{V_1}{V_2} = \frac{R_1}{R_3}$$

In options (a), (b) and (d) this relation is satisfied.

43. **(c)** For balanced meter bridge

$$\frac{X}{R} = \frac{l}{(100 - l)} \qquad \text{(where, } R = 90\ \Omega\text{)}$$

$$\therefore \quad \frac{X}{90} = \frac{40}{100 - 40}$$

$$\therefore \quad X = 60\ \Omega$$

$$X = R\,\frac{l}{(100 - l)}$$

$$\frac{\Delta X}{X} = \frac{\Delta l}{l} + \frac{\Delta l}{100 - l} = \frac{0.1}{40} + \frac{0.1}{60}$$

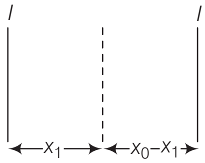
$$\Delta X = 0.25$$

So, $X = (60 \pm 0.25)\ \Omega$

44. **(3)** $B_2 = \dfrac{\mu_0 I}{2\pi x_1} + \dfrac{\mu_0 I}{2\pi (x_0 - x_1)}$

(when currents are in opposite directions)

$$B_1 = \frac{\mu_0 I}{2\pi x_1} - \frac{\mu_0 I}{2\pi (x_0 - x_1)}$$



(when currents are in same direction)

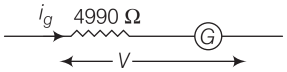
Substituting $x_1 = \dfrac{x_0}{3}$ (as $\dfrac{x_0}{x_1} = 3$)

$$B_1 = \frac{3\mu_0 I}{2\pi x_0} - \frac{3\mu_0 I}{4\pi x_0} = \frac{3\mu_0 I}{4\pi x_0}$$

$$R_1 = \frac{mv}{qB_1} \quad \text{and} \quad B_2 = \frac{9\mu_0 I}{4\pi x_0}$$

$$R_2 = \frac{mv}{qB_2} \quad \Rightarrow \quad \frac{R_1}{R_2} = \frac{B_2}{B_1} = \frac{9}{3} = 3$$

45. **(5)**



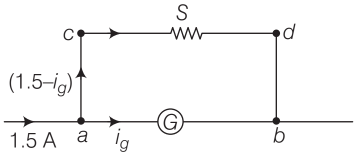
$$i_g (G + 4990) = V$$

$$\Rightarrow \quad \frac{6}{1000}(G + 4990) = 30$$

$$\Rightarrow \quad G + 4990 = \frac{30{,}000}{6} = 5000$$

$$\Rightarrow \quad G = 10\ \Omega$$

$$V_{ab} = V_{cd} \quad \Rightarrow \iota_\gamma \Gamma = (1.5 - \iota_\gamma)\,\Sigma$$



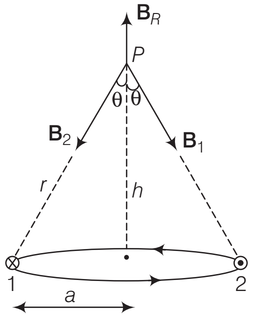
$$\Rightarrow \quad \frac{6}{1000} \times 10 = \left(1.5 - \frac{6}{1000}\right) S$$

$$\Rightarrow \quad S = \frac{60}{1494} = \frac{2n}{249} \quad \Rightarrow n = \frac{249 \times 30}{1494} = \frac{2490}{498} = 5$$

46. **(c)** $\mathbf{B}_R = \mathbf{B}$ due to ring

$\mathbf{B}_1 = \mathbf{B}$ due to wire-1

$\mathbf{B}_2 = \mathbf{B}$ due to wire-2



In magnitudes $\mathbf{B}_1 = \mathbf{B}_2 = \dfrac{\mu_0 I}{2\pi r}$

Resultant of $\mathbf{B}_1$ and $\mathbf{B}_2$

$$= 2\mathbf{B}_1 \cos\theta = 2\left(\frac{\mu_0 I}{2\pi r}\right)\left(\frac{h}{r}\right) = \frac{\mu_0 I h}{\pi r^2}$$

$$B_R = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}} = \frac{2\mu_0 I \pi a^2}{4\pi r^3}$$

As, $R = a$, $x = h$ and $a^2 + h^2 = r^2$

For zero magnetic field at P,

$$\frac{\mu_0 I h}{\pi r^2} = \frac{2\mu_0 I \pi a^2}{4\pi r^3} \Rightarrow \pi a^2 = 2rh \quad \Rightarrow \eta \approx 1.2\alpha$$

47. (b) Magnetic field at mid-point of two wires

$= 2$ (magnetic field due to one wire) $= 2\left[\frac{\mu_0}{2\pi}\frac{I}{d}\right]$

$= \frac{\mu_0 I}{\pi d} \otimes$

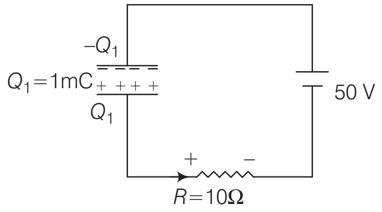
Magnetic moment of loop $M = IA = I\pi a^2$

Torque on loop $= MB \sin 150° = \frac{\mu_0 I^2 a^2}{2d}$

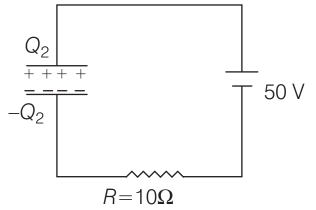
48. (c,d) $\frac{dQ}{dt} = I \quad \Rightarrow \quad Q = \int I\, dt = \int (I_0 \cos \omega t)\, dt$

$$\therefore \qquad Q_{\max} = \frac{I_0}{\omega} = \frac{1}{500} = 2 \times 10^{-3}\ \text{C}$$

Just after switching



In steady state



At $\qquad t = \frac{7\pi}{6\omega}$ or $\omega t = \frac{7\pi}{6}$

Current comes out to be negative from the given expression. So, current is anti-clockwise.

Charge supplied by source from $t = 0$ to

$$t = \frac{7\pi}{6\omega} \quad \Rightarrow \quad Q = \int_0^{\frac{7\pi}{6\omega}} \cos(500t)\, dt$$

$$= \left[\frac{\sin 500t}{500}\right]_0^{\frac{7\pi}{6\omega}} = \frac{\sin \frac{7\pi}{6}}{500} = -1\,\text{mC}$$

Apply Kirchhoff's loop law just after changing the switch to position D

$$50 + \frac{Q_1}{C} - IR = 0$$

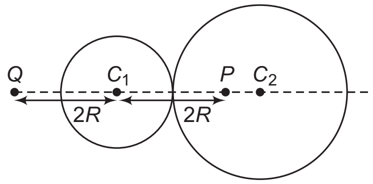
Substituting the values of Q_1, C and R we get

$$I = 10\ \text{A}$$

In steady state $Q_2 = CV = 1\,\text{mC}$

$\therefore$ Net charge flown from battery = 2 mC

49. (b,d) **At point** P



If resultant electric field is zero then

$$\frac{KQ_1}{4R^2} = \frac{KQ_2}{8R^3}R \quad \Rightarrow \quad \frac{\rho_1}{\rho_2} = 4$$

At point Q

If resultant electric field is zero then

$$\frac{KQ_1}{4R^2} + \frac{KQ_2}{25R^2} = 0$$

$$\frac{\rho_1}{\rho_2} = -\frac{32}{25} \quad (\rho_1 \text{ must be negative})$$

50. (b, d) After pressing S_1 charge on upper plate of C_1 is $+\,2CV_0$.

After pressing S_2 this charge equally distributes in two capacitors. Therefore charge an upper plates of both capacitors will be $+\,CV_0$.

When S_2 is released and S_3 is pressed, charge on upper plate of C_1 remains unchanged ($= +\,CV_0$) but charge on upper plate of C_2 is according to new battery ($= -\,CV_0$).

51. (c,d) For electrostatic field,

Field at P

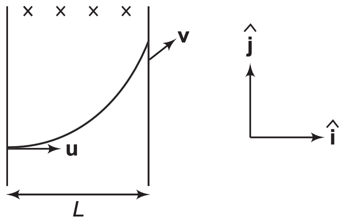
$$\mathbf{E}_P = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{C_1P} + \frac{(-\rho)}{3\varepsilon_0}\mathbf{C_2P}$$

$$= \frac{\rho}{3\varepsilon_0}(\mathbf{C_1P} + \mathbf{PC_2})$$

$$\mathbf{E}_P = \frac{\rho}{3\varepsilon_0}\mathbf{C_1C_2}$$

For electrostatic. Since, electric field is non-zero so it is not equipotential.

52. (a,c) $\mathbf{u} = 4\mathbf{i}$; $\mathbf{v} = 2(\sqrt{3}\mathbf{i} + \mathbf{j})$



According to the figure, magnetic field should be in ⊗ direction, or along $-z$ direction.

Further, $\tan\theta = \frac{v_y}{v_x} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\therefore \quad \theta = 30° \quad \text{or} \quad \frac{\pi}{6}$

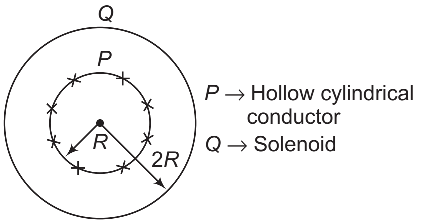
= angle of $\mathbf{v}$ with x-axis

= angle rotated by the particle

$= Wt = \left(\frac{BQ}{M}\right)t$

$\therefore \quad B = \frac{\pi M}{6Qt} = \frac{50\pi M}{3Q}$ units (as $t = 10^{-3}$ second)

53. (a,d)



In the region, $0 < r < R$

$B_P = 0,$

$B_Q \neq 0$, along the axis

$\therefore \quad B_{\text{net}} \neq 0$

In the region, $R < r < 2R$

$B_P \neq 0$, tangential to the circle of radius r, centred on the axis.

$B_Q \neq 0$, along the axis.

$\therefore B_{\text{net}} \neq 0$ neither in the directions mentioned in options (b) or (c).

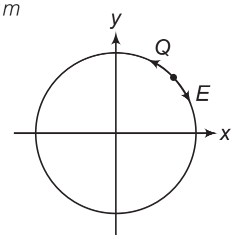
In region, $r > 2R$

$B_P \neq 0$

$B_Q \neq 0$

$\therefore \quad B_{\text{net}} \neq 0$

54. (b) $\frac{M}{L} = \frac{Q}{2m}$



$\therefore \quad M = \left(\frac{Q}{2m}\right)L \quad \Rightarrow \quad M \propto L,$

where $\gamma = \frac{Q}{2m} = \left(\frac{Q}{2m}\right)(I\omega)$

$= \left(\frac{Q}{2m}\right)(mR^2\omega) = \frac{Q\omega R^2}{2}$

Induced electric field is opposite. Therefore,

$\omega' = \omega - \alpha t$

$$\alpha = \frac{\tau}{I} = \frac{(QE)R}{mR^2} = \frac{(Q)\left(\frac{BR}{2}\right)R}{mR^2} = \frac{QB}{2m}$$

$$\therefore \quad \omega' = \omega - \frac{QB}{2m}\cdot 1 = \omega - \frac{QB}{2m}$$

$$M_f = \frac{Q\omega' R^2}{2} = Q\left(\omega - \frac{QB}{2m}\right)\frac{R^2}{2}$$

$$\therefore \quad \Delta M = M_f - M_i = -\frac{Q^2BR^2}{4m}$$

$$M = -\gamma\frac{QBR^2}{2} \qquad \left(\text{as } \gamma = \frac{Q}{2m}\right)$$

55. (b) The induced electric field is given by,

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{d\phi}{dt} \quad \text{or} \quad El = -s\left(\frac{dB}{dt}\right)$$

$$\therefore \; E(2\pi R) = -(\pi R^2)(B)$$

or $\quad E = -\frac{BR}{2}$

56. (b) $\quad P = Vi$

$\therefore \quad i = \frac{P}{V} = \frac{600 \times 10^3}{4000} = 150$ A

Total resistance of cables,

$R = 0.4 \times 20 = 8\Omega$

$\therefore$ Power loss in cables

$= i^2R = (150)^2(8)$

$= 180000$ W $= 180$ kW

This loss is 30% of 600 kW.

57. (a) During step-up,

$$\frac{N_p}{N_s} = \frac{V_p}{V_s}$$

or $\quad \frac{1}{10} = \frac{4000}{V_s}$

or $\quad V_s = 40{,}000$ V

In step, down transformer,

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{40000}{200} = \frac{200}{1}$$

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Understanding Physics

JEE Main & Advanced

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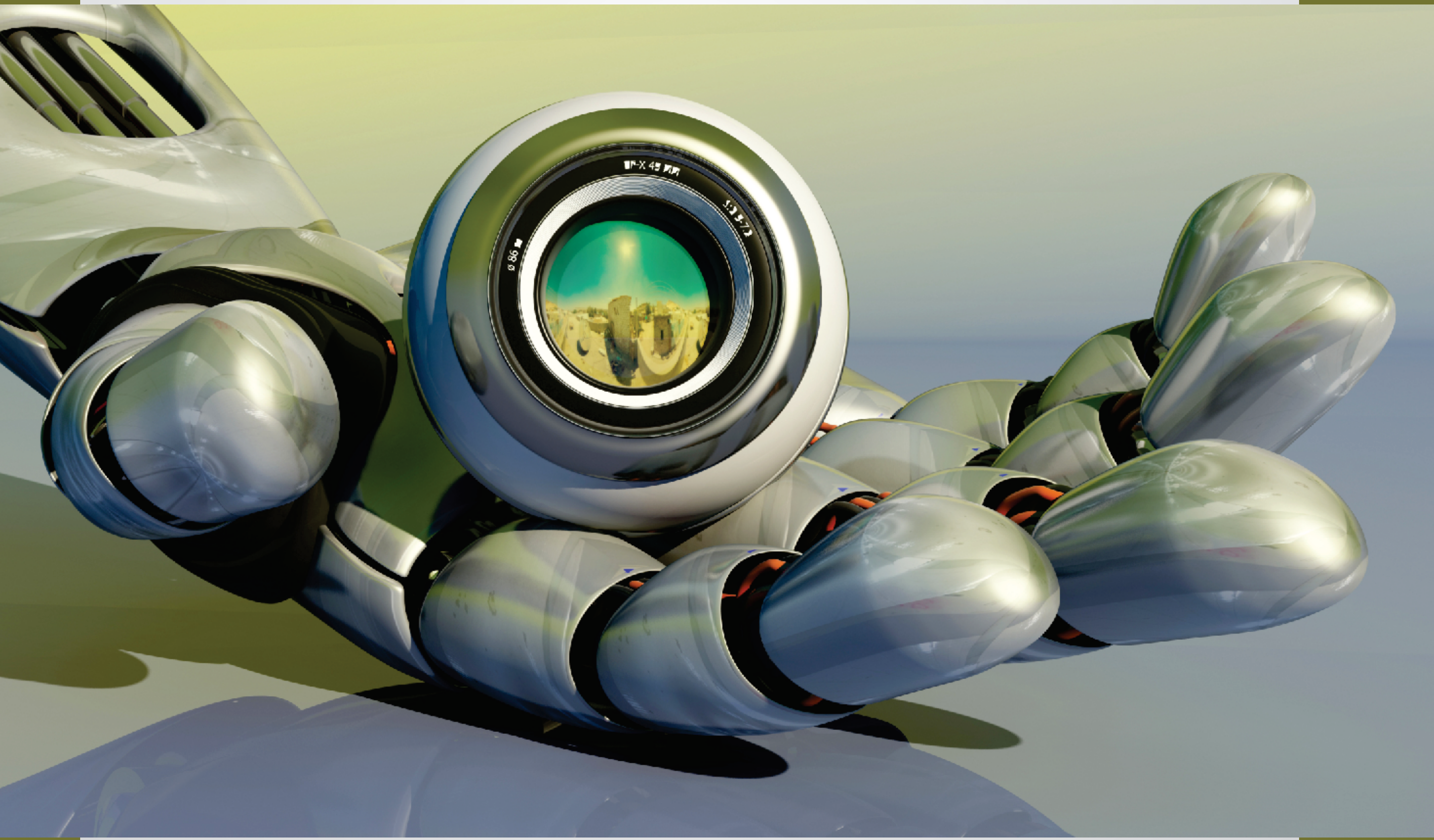

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced, the NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. The exercises in this book have been divided into two sections viz., JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions.

I am extremely thankful to (Dr.) Mrs. Sarita Pandey, Mr. Anoop Dhyani and Nisar Ahmad for their endless efforts during the project.

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

ELECTROMAGNETIC WAVES

Electromagnetic waves and their characteristics. Transverse nature of electromagnetic waves.

Electromagnetic spectrum (radio waves, microwaves, infrared, visible, ultraviolet, X-rays, g- rays). Applications of e.m. waves.

OPTICS

Reflection and refraction of light at plane and spherical surfaces, Mirror formula, Total internal reflection and its applications, Deviation and Dispersion of light by a prism, Lens formula, Magnification, Power of a lens, Combination of thin lenses in contact, Microscope and astronomical telescope (reflecting and refracting) and their magnifying powers.

WAVE OPTICS

Wave front and Huygens' principle, Laws of reflection and refraction using Huygen's principle. Interference, Young's double slit experiment and expression for fringe width. Diffraction due to a single slit, width of central maximum. Resolving power of microscopes and astronomical telescopes, Polarisation, Plane polarized light; Brewster's law, uses of plane polarized light and polaroids.

DUAL NATURE OF MATTER AND RADIATION

Dual nature of radiation. Photoelectric effect, Hertz and Lenard's observations; Einstein's photoelectric equation; Particle nature of light. Matter waves-wave nature of particle, de-Broglie relation. Davisson-Germer experiment.

ATOMS AND NUCLEI

a-particle scattering experiment; Rutherford's model of atom; Bohr model, Energy levels, Hydrogen spectrum. Composition and size of nucleus, Atomic masses, Isotopes, Isobars; Isotones. Radioactivity - a, b and g particles/rays and their properties; radioactive decay law. Mass-energy relation, Mass defect; Binding energy per nucleon and its variation with mass number, Nuclear fission and fusion.

ELECTRONIC DEVICES

Semiconductors; semiconductor diode: I-V characteristics in forward and reverse bias; Diode as a rectifier; I-V characteristics of LED, photodiode, solar cell and Zener diode; Zener diode as a voltage regulator. Junction Transistor, transistor action, Characteristics of a transistor; transistor as an amplifier (common emitter configuration) and oscillator. Logic gates (OR, AND, NOT, NAND and NOR). Transistor as a switch.

COMMUNICATION SYSTEMS

Propagation of electromagnetic waves in the atmosphere; Sky and space wave propagation, Need for modulation, Amplitude and frequency modulation, Bandwidth of signals, Bandwidth of transmission medium, Basic elements of a communication system (block diagram only).

JEE Advanced

GENERAL

Focal length of a concave mirror and a convex lens using U V method.

OPTICS

Rectilinear propagation of light, Reflection and refraction at plane and spherical surfaces, Total internal reflection, Deviation and dispersion of light by a prism, Thin lenses, Combinations of mirrors and thin lenses, Magnification.

WAVE NATURE OF LIGHT

Huygens principle, Interference limited to Young's double-slit experiment.

MODERN PHYSICS

Atomic nucleus, a, b and g radiations, Law of radioactive decay, Decay constant, Half-life and mean life, Binding energy and its calculation, Fission and fusion processes, Energy calculation in these processes.

Photoelectric effect, Bohr's theory of hydrogen-like atoms, Characteristic and continuous X-rays, Moseley's law, de-Broglie wavelength of matter waves.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

29

Electromagnetic Waves

Chapter Contents

29.1 Introduction
29.2 Displacement Current
29.3 Electromagnetic Waves
29.4 Electromagnetic Spectrum

29.1 Introduction

Earlier we have learned that a time varying magnetic field produces an electric field. Is the converse also true? Does a time varying electric field can produce a magnetic field? James Clerk Maxwell argued that not only an electric current but also a time varying electric field generates magnetic field.

Maxwell formulated a set of equations (known as Maxwell's equations) involving electric and magnetic fields. Maxwell's equations and Lorentz force formula make all the basic laws of electromagnetism.

The most important outcome of Maxwell's equations is the existence of electromagnetic waves.

The changing electric and magnetic fields form the basis of electromagnetic waves. A combination of time varying electric and magnetic fields (referred as electromagnetic wave) propagate in space very close to the speed of light ($= 3 \times 10^8$ m/s) obtained from optical measurements. We shall take a brief discussion of electromagnetic waves mainly developed by Maxwell around 1864.

29.2 Displacement Current

An electric current produces magnetic field. Value of magnetic field (due to an electric current) at some point can be obtained by Biot-Savart law or Ampere's circuital law.

We have stated Ampere's law as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i \qquad \text{...(i)}$$

where left hand side of this equation is the line integral of magnetic field over a closed path and i is the electric current crossing the surface bounded by that closed path.

Ampere's law in this form is not valid if the electric field at the surface varies with time. For an example if we place a magnetic needle in between the plates of a capacitor during its charging or discharging then it deflects. Although, there is no current between the plates, so magnetic field should be zero. Hence, the needle should not show any deflection. But deflection of needle shows that there is a magnetic field in the region between plates of capacitor during charging or discharging. So, there must be some other source (other than current) of magnetic field. This other source is nothing but the changing electric field. Because at the time of charging or discharging of capacitor electric field between the plates changes.

The relation between the changing electric field and the magnetic field resulting from it is given by

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}} \qquad \text{...(ii)}$$

Here, ϕ_E is the flux of the electric field through the area bounded by the closed path along which line integral of **B** is calculated.

Combining Eqs. (i) and (ii), we can make a general expression of Ampere's circuital law and that is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} = \mu_0 \left(i + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

or

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 (i + i_d)} \qquad \text{...(iii)}$$

Here,
$$\boxed{i_d = \varepsilon_0 \frac{d\phi_E}{dt}} \qquad \text{...(iv)}$$

is called the **displacement current** and which is produced by the change in electric field. The current due to the flow of charge is often called **conduction current** and is denoted by i_c. Thus, Eq. (iii) can also be written as

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 (i_c + i_d)} \qquad \text{...(v)}$$

Example

In the figure, a capacitor is charged by a battery through a resistance R. Charging of capacitor will be exponential. A time varying charging current i flows in the circuit (due to flow of charge) till charging continues. A time varying electric field is also produced between the plates. This causes a displacement current i_d between the plates. There is no current between the plates due to flow of charge, as a medium between the plates is insulator.

Consider two closed paths a and b as shown in figure. Ampere's circuital law in these two paths is

Along path *a*

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i \quad \text{or} \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_c$$

Along path *b*

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \, i_d$$

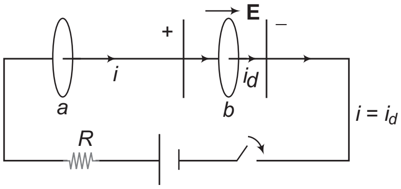


Fig. 29.1

Here, $i_d = \varepsilon_0 \frac{d\phi_E}{dt}$ is in the direction shown in figure. In sample example 29.1, we have shown that

$$\boxed{i_c = i_d}$$

✓ Extra Points to Remember

Faraday's law of electromagnetic induction says that changing magnetic field gives rise to an electric field and its line integral (= *emf*) is given by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi_B}{dt}$$

A changing electric field gives rise to a magnetic field is the symmetrical counterpart of Faraday's law. This is a consequence of the displacement current and given by

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} = \mu_0 i_d$$

Thus, time dependent electric and magnetic fields give rise to each other.

Maxwell's Equations

- $\oint \mathbf{E} \cdot d\mathbf{s} = q_{\text{in}}/\varepsilon_0$ (Gauss's law for electricity)
- $\oint \mathbf{B} \cdot d\mathbf{s} = 0$ (Gauss's law for magnetism)
- $\oint \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d\phi_B}{dt}$ (Faraday's law)
- $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(i_c + i_d) = \mu_0 i_c + \mu_0 \varepsilon_0 \dfrac{d\phi_E}{dt}$ (Ampere-Maxwell's law)

Example 29.1 *During charging of a capacitor show that the displacement current between the plates is equal to the conduction current in the connecting wire.*

Solution Let A is the area of plates, q is the charge on capacitor at some instant and d the separation between the plates.

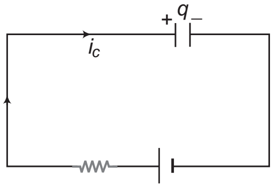


Fig. 29.2

$$\text{Conduction current, } i_c = \frac{dq}{dt} \quad \text{...(i)}$$

Electric field between the plates,

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q/A}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$$

The flux of the electric field through the given area is

$$\phi_E = EA = \left(\frac{q}{A\,\varepsilon_0}\right) A = \frac{q}{\varepsilon_0}$$

$$\therefore \quad \frac{d\phi_E}{dt} = \frac{1}{\varepsilon_0}\left(\frac{dq}{dt}\right)$$

Displacement current,

$$i_d = \varepsilon_0 \frac{d\phi_E}{dt}$$

$$= \varepsilon_0 \left[\frac{1}{\varepsilon_0} \cdot \frac{dq}{dt}\right]$$

or

$$i_d = \frac{dq}{dt} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$i_c = i_d$$

Hence Proved.

INTRODUCTORY EXERCISE 29.1

1. A parallel-plate capacitor with plate area A and separation between the plates d, is charged by a constant current i. Consider a plane surface of area $A/2$ parallel to the plates and drawn symmetrically between the plates. Find the displacement current through this area.

29.3 Electromagnetic Waves

Stationary charges produce only electric field. Charges in uniform motion (or steady currents) produce both electric and magnetic fields. **Accelerated charges radiate electromagnetic waves.** It is an important result of Maxwell's theory. Thus, an accelerated charge produces all three electric field, magnetic field and electromagnetic waves.

Consider an oscillating charged particle. Let f is the frequency of its oscillations. This oscillating charged particle produces an oscillating electric field (of same frequency f). Now, this oscillating electric field becomes a source of oscillating magnetic field (Ampere-Maxwell's law). This oscillating magnetic field again becomes a source of oscillating electric field (Faraday's law) and so on.

The oscillating electric and magnetic fields regenerate each other and electromagnetic wave propagates through the space. The frequency of the electromagnetic wave is equal to the frequency of oscillation of the charge.

Frequency of visible light is of the order of 10^{14} Hz, while the maximum frequency that we can get with modern electronic circuits is of the order of 10^{11} Hz. Therefore, it is difficult to experimentally demonstrate the production of visible light. **Hertz's** experiment (in 1887) demonstrated the production of electromagnetic waves of low frequency (in radio wave region). **Jagdish Chandra Bose** succeeded in producing the electromagnetic waves of much higher frequency in the laboratory.

✓ *Extra Points to Remember*

- When electromagnetic waves propagate in space then electric and magnetic fields oscillate in mutually perpendicular directions. Further, they are perpendicular to the direction of propagation of electromagnetic wave also.
- Consider a plane electromagnetic wave propagating along the z-direction. The electric field E_x is along the x-axis and varies sinusoidally. The magnetic field B_y is along the y-axis and again varies sinusoidally. We can write E_x and B_y as

$$\boxed{E_x = E_0 \sin(\omega t - kz) \quad \text{and} \quad B_y = B_0 \sin(\omega t - kz)}$$

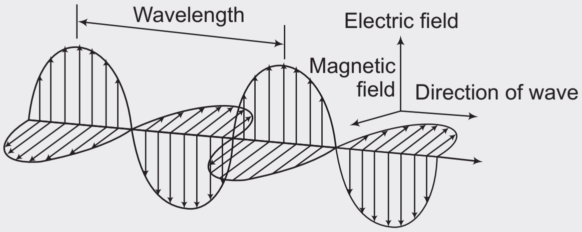


Fig. 29.3

Thus, electromagnetic wave travels in the direction of $\mathbf{E} \times \mathbf{B}$.

- From Maxwell's equations and the knowledge of waves we can write the following expressions,

$$\boxed{\begin{array}{c} k = 2\pi/\lambda \quad \text{and} \quad \omega = 2\pi f \\ c = \dfrac{\omega}{k} = f\lambda = \dfrac{E_0}{B_0} = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}} \end{array}} \qquad \text{Speed of light (in vacuum)}$$

where, f is the frequency of electromagnetic wave and λ its wavelength.

- Unlike a mechanical wave (like sound wave) an electromagnetic wave does not require any material medium for the oscillations of electric and magnetic fields. They can travel in vacuum also. Oscillations of electric and magnetic fields are self sustaining in free space or vacuum.
- In a material medium (like glass, water etc.), electric and magnetic fields are different from the external fields. They are described in terms of permittivity ε and magnetic permeability μ. In Maxwell's equations, ε_0 and μ_0 are thus replaced by ε and μ and the velocity of light becomes,

  $$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

  Thus, the velocity of light depends on electric and magnetic properties of the medium.
- Like other waves, electromagnetic waves also carry energy and momentum.In previous chapters, we have studied that, energy density in electric field $= \frac{1}{2}\varepsilon_0 E^2$ and energy density in magnetic field $= \frac{B^2}{2\mu_0}$.

  An electromagnetic wave contains both electric and magnetic field. Therefore, energy density is associated with both the fields.
- Consider a plane perpendicular to the direction of propagation of the electromagnetic wave. If the total energy transferred to a surface in time t is E, then total momentum delivered to this surface for complete absorption is

  $$\Delta p = \frac{E}{c} \qquad \text{(complete absorption)}$$

  If the wave is totally reflected, the momentum delivered is

  $$\Delta p = \frac{2E}{c} \qquad \text{(completely reflected)}$$
- The energy transferred per unit area per unit time perpendicular to the direction of propagation of electromagnetic wave is called the intensity of wave. It is given by

  $$I = \frac{1}{2}\varepsilon_0 E^2 c$$

  Here, E is the rms value of electric field or E_{rms}.

Example 29.2 *A plane electromagnetic wave of frequency 25 MHz travels in free space along the x-direction. At a particular point in space and time, $E = (6.3\,\hat{\mathbf{j}})$ V/m. What is **B** at this point?*

Solution

$$c = \frac{E_0}{B_0} \quad \text{or} \quad = \frac{E}{B}$$

$$\therefore \qquad B = \frac{E}{c}$$

Substituting the values in SI units,

$$B = \frac{6.3}{3 \times 10^8}$$

$$= 2.1 \times 10^{-8}\ \text{T}$$

From the relation $\mathbf{c} = \mathbf{E} \times \mathbf{B}$

We can see that **B** is along positive **z**-direction. Because, **E** is along $\hat{\mathbf{j}}$ direction and **c** along $\hat{\mathbf{i}}$ direction.

$\therefore$ $$\mathbf{B} = (2.1 \times 10^{-8}\ \text{T})\,\hat{\mathbf{k}}$$ **Ans.**

Example 29.3 *The magnetic field in a plane electromagnetic wave is given by* $B_y = 2 \times 10^{-7} T \sin(0.5 \times 10^3 x + 1.5 \times 10^{11}\ t)\ T$

(a) What is the wavelength and frequency of the wave?

(b) Write an expression for the electric field.

Solution (a) **Wavelength**

From the given equation, we can see that

$$k = 0.5 \times 10^3\ \text{m}^{-1}$$

But, $$k = \frac{2\pi}{\lambda}$$

$\therefore$ $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.5 \times 10^3}$$

$$= 1.25 \times 10^{-2}\ \text{m}$$ **Ans.**

Frequency

Angular frequency,

$$\omega = 1.5 \times 10^{11}\ \text{rad/s}$$

But, $$\omega = 2\pi f$$

$\therefore$ $$f = \frac{\omega}{2\pi} = \frac{1.5 \times 10^{11}}{2\pi}$$ \

$$= 2.39 \times 10^{10}\ \text{Hz}$$ **Ans.**

(b) $c = \dfrac{E_0}{B_0}$

$\therefore$ $$E_0 = cB_0$$

$$= (3.0 \times 10^8)(2 \times 10^{-7}) = 60\ \text{V/m}$$

From the relation,

$$\mathbf{c} = \mathbf{E} \times \mathbf{B}$$

We can see that **E** is along z-direction.

$\therefore$ $$E_z = (60\ \text{V/m}) \sin(0.5 \times 10^3\ x + 1.5 \times 10^{11}\ t)\ \text{V/m}$$ **Ans.**

Example 29.4 *Light with an energy flux of 18 W/cm² falls on a non-reflecting surface at normal incidence. If the surface has an area of 20 cm², find the average force exerted on the surface during a 30 minute time span.*

Solution Total energy incident on the given surface in the given time interval is

$$E = (18 \times 10^4\ \text{W/m}^2)(20 \times 10^{-4}\ \text{m}^2)(30 \times 60\,\text{s})$$

$$= 6.48 \times 10^5\ \text{J}$$

Therefore, the total momentum transferred to the given surface for complete absorption is

$$\Delta p = \frac{E}{c} = \frac{6.48 \times 10^5}{3.0 \times 10^8}$$

$$= 2.16 \times 10^{-3} \text{ kg-m/s}$$

$\therefore$

$$F_{\text{av}} = \frac{\Delta p}{\Delta t} = \frac{2.16 \times 10^{-3}}{30 \times 60}$$

$$= 1.2 \times 10^{-6} \text{ N}$$ **Ans.**

Example 29.5 *In the above example what is the average force if surface is perfectly reflecting?*

Solution (a) If the surface is perfectly reflecting, then

$$\Delta p = \frac{2E}{c}$$

Therefore, average force is doubled or

$$F_{\text{av}} = 2.4 \times 10^{-6} \text{ N}$$ **Ans.**

Example 29.6 *Calculate the electric and magnetic fields produced by the radiation coming from a* 100 *W bulb at a distance of* 3 *m. Assume that the efficiency of the bulb is* 2.5 % *and it is a point source.*

Solution Intensity at a distance r from a point source of power P is given by

$$I = \frac{P}{4\pi r^2}$$

So, intensity at a distance of 3 m from the bulb with 2.5% efficiency will be

$$I = \frac{100}{4\pi (3)^2} \times \frac{2.5}{100}$$

$$= 0.022 \text{ W/m}^2$$

Half of the intensity is provided by electric field and half by magnetic field.

$\therefore$

$$I_E = \frac{I}{2} = 0.011 \text{ W/m}^2$$

But, I_E is given by $\frac{1}{2} \varepsilon_0 E^2 c$

$\therefore$

$$I_E = \frac{1}{2} \varepsilon_0 E^2 c \text{ or } E = \sqrt{\frac{2 I_E}{\varepsilon_0 c}}$$

Substituting the values, we have

$$E = \sqrt{\frac{2 \times 0.011}{(8.85 \times 10^{-12})(3 \times 10^8)}}$$

$$= 2.9 \text{ V/m}$$ **Ans.**

Note *that this is actually the rms value of electric field.*

From the equation, $c = E/B$

$$B = \frac{E}{c} = \frac{2.9}{3.0 \times 10^8}$$

$$= 9.6 \times 10^{-9} \text{ T}$$ **Ans.**

This is again the rms value of magnetic field.

INTRODUCTORY EXERCISE 29.2

1. Show that the unit of $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is m/s.
2. A capacitor is connected to an alternating current source. Is there a magnetic field between the plates?
3. The sunlight reaching the earth has maximum electric field of 810 Vm^{-1}. What is the maximum magnetic field in this light?
4. The electric field in an electromagnetic wave is given by

$$E = (50 \text{ NC}^{-1}) \sin \omega (t - x/c) .$$

Find the energy contained in a cylinder of cross-section 10 cm^2 and length 50 cm along the x-axis.

29.4 Electromagnetic Spectrum

The basic source of electromagnetic wave is an accelerated charge. This produces the changing electric and magnetic fields which constitute an electromagnetic wave. An electromagnetic wave may have its wavelength varying from zero to infinity. Not all of them are known till date. Today, we are familiar with electromagnetic waves having wavelengths as small as 30 fm ($1 \text{ fm} = 10^{-15}$ m) to as large as 30 km. The boundaries separating different regions of spectrum are not sharply defined, with visible light ranging from 4000 Å to 7000 Å. An approximate range of wavelengths associated with each colour are violet (4000 Å – 4500 Å), blue (4500 Å – 5200 Å), green (5200 Å – 5600 Å), yellow (5600 Å – 6000 Å), orange (6000 Å – 6250 Å) and red (6250 Å – 7000 Å).

The classification of electromagnetic waves according to frequency or wavelength is called electromagnetic spectrum. Table below gives range of wavelengths and frequencies for different waves.

Table 29.1

S.No.	Type	Wavelength range	Frequency range
1.	Radio waves	> 0.1 m	$< 3 \times 10^9$ Hz
2.	Micro waves	0.1 m to 1 mm	3×10^9 Hz to 3×10^{11} Hz
3.	Infrared	1 mm to 7000 Å	3×10^{11} Hz to 4.3×10^{14} Hz

S.No.	Type	Wavelength range	Frequency range
4.	Visible light	7000 Å to 4000 Å	4.3×10^{14} Hz to 7.5×10^{14} Hz
5.	Ultraviolet	4000 Å to 10 Å	7.5×10^{14} Hz to 3×10^{17} Hz
6.	X-rays	10 Å to 0.01 Å	3×10^{17} Hz to 3×10^{20} Hz
7.	Gamma rays	< 0.01 Å	$> 3 \times 10^{20}$ Hz

Note *In the above table, wavelength is decreasing from top to bottom. But, frequency is increasing. Now, let us discuss them in brief in the order of increasing wavelength.*

1. **Gamma Rays** These high frequency radiations are usually produced in nuclear reactions and also emitted by radioactive nuclei. They are used in medicines to destroy cancer cells.
2. **X-Rays** X-rays were discovered in 1895 by W.Roentgen. These are produced by the rapid deceleration of electrons that bombard a heavy metal target. These are also produced by electronic transitions between the energy levels in an atom. X-rays are used as a diagnostic tool in medicine and as a treatment for certain forms of cancer.
3. **Ultraviolet Rays** Ultraviolet radiation is produced by special lamps and very hot bodies. The sun is an important source of ultraviolet light. It plays an important role in the production of vitamin-D. But prolonged doses of UV radiation can induce skin cancers in human beings. Ozone layer in atmosphere at an altitude of about 40-50 km plays a protective role in this regarding. Depletion of this layer by chlorofluorocarbon (CFC) gas (such as Freon) is a matter of international concern now a days.
4. **Visible Light** It is most familiar form of electromagnetic waves.Our eye is sensitive to visible light. Visible light emitted or reflected from objects around us provides information about world. Process of photosynthesis in plants needs visible light. Visible light is produced by the transition of electrons in an atom from one energy level to other.
5. **Infrared Radiation** Infrared rays also sometimes referred as heat waves are produced by hot bodies. They are perceived by us as heat. In most of the materials, water molecules are present. These molecules readily absorb infrared rays. After absorption, their thermal motion increases, i.e. they heat up and heat their surroundings. Infrared rays are used for early detection of tumors. Infrared detectors are also used to observe growth of crops and for military purposes.
6. **Microwaves** Microwaves may be generated by the oscillations of electrons in a device called klystron. Microwave ovens are used in kitchens. In microwave ovens frequency of the microwaves is selected to match the resonant frequency of water molecules so that energy from the waves is transferred to the kinetic energy of the molecules. This raises the temperature of any food containing water.
7. **Radio Waves** Radio waves are generated when charges are accelerating through conducting wires. They are generated in L - C oscillators and are used in radio and television communication systems.

✓ *Extra Points to Remember*

- Our eyes are sensitive to visible light (λ between 4000 Å to 7000 Å). Similarly, different animals are sensitive to different ranges of wavelengths. For example, snakes can detect infrared waves.
- The basic difference between different types of electromagnetic waves lies in their frequencies and wavelengths. All of them travel with same speed in vacuum. Further they differ in their mode of interaction with matter.

  For example infrared waves vibrate the electrons, atoms and molecules of a substance. These vibrations increase the internal energy and temperature of the substance. This is why infrared waves are also called heat waves.
- Electromagnetic waves interact with matter *via* their electric and magnetic fields, which set in oscillation with the charges present in all matter. This interaction depends on the wavelength of the electromagnetic wave and the nature of atoms or molecules in the medium.
- **Microwave Oven** In electromagnetic spectrum frequency and energy of microwaves is smaller than the visible light. Water content is required for cooking food in microwave oven. Almost all food items contain some water content. Microwaves interact with water molecules and atoms *via* their electric and magnetic fields. Temperature of water molecules increases by this. These water molecules share this energy with neighboring food molecules, heating up the food.

  Porcelain vessels are used for cooking food in microwave oven. Because its large molecules vibrate and rotate with much smaller frequencies and do not get heated up. We cannot use metal vessels. Metal vessels interact with microwaves. These vessels may melt due to heating.

Solved Examples

Example 1 *Long distance radio broadcasts use short wave bands. Explain why?*

Solution Short radio waves are reflected by ionosphere.

Example 2 *It is necessary to use satellites for long distance TV transmission. Explain why?*

Solution TV waves (part of radio waves) range from 54 MHz to 890 MHz. Unlike short wave bands (used in radio broadcasts) which are reflected by ionosphere, TV waves are not properly reflected by ionosphere. This is why, satellites are used for long distance TV transmission.

Example 3 *The ozone layer on the top of the stratosphere is crucial for human survival. Explain why?*

Solution Ozone layer protects ourselves from ultraviolet radiations. Over exposure to UV radiation can cause skin cancer in human beings. Ozone layer absorbs UV radiations. But unfortunately over use of Chlorofluoro Carbon Gases (CFCs) is depleting this ozone layer and it is a matter of international concern now a days.

Example 4 *Optical and radio telescopes are built on ground but X-ray astronomy is possible only from satellites orbiting the earth. Explain why?*

Solution Visible and radio waves can penetrate the atmosphere, while X-rays are absorbed by the atmosphere. This is why X-ray telescopes are installed in satellites orbiting the earth.

Example 5 *If the earth did not have an atmosphere, would its average surface temperature be higher or lower than what it is now?*

Solution Due to presence of atmosphere green house effect takes place. Heat radiated by earth is trapped due to green house effect. In the absence of atmosphere, temperature of the earth would be lower because the green house effect of the atmosphere would be absent.

Example 6 *Some scientists have predicted the global nuclear war on the earth would be followed by a severe nuclear winter with a devastating effect on life on earth. What might be the basis of this prediction?*

Solution After nuclear war, clouds would perhaps cover the atmosphere of earth preventing solar light from reaching many parts of earth. This would cause a winter.

Example 7 *Why is the orientation of the portable radio with respect to broadcasting station important?*

Solution Electromagnetic waves are plane polarised, so the receiving antenna should be parallel to electric and magnetic part of wave.

Example 8 *A plane electromagnetic wave propagating in the x-direction has a wavelength of* 5.0 *mm. The electric field is in the y-direction and its maximum magnitude is* $30\ Vm^{-1}$. *Write suitable equations for the electric and magnetic fields as a function of x and t.*

Solution Given, $\lambda = 5\text{ mm} = 5 \times 10^{-3}\text{ m}$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{5 \times 10^{-3}}\text{ m}^{-1}$$

$$= 1257\text{ m}^{-1}$$

From the equation, $c = \dfrac{\omega}{k}$

$$\omega = c\,k = (3 \times 10^{8})\ (1257)$$

$$= 3.77 \times 10^{11}\text{ rad/s}$$

$$E_0 = 30\text{ V/m}$$

$$c = \frac{E_0}{B_0}$$

$$\therefore \qquad B_0 = \frac{E_0}{c} = \frac{30}{3 \times 10^{8}}$$

$$= 10^{-7}\text{ T}$$

Now, $E_y = E_0 \sin(\omega t - kx)$

$$= (30\text{ V/m}) \sin\left[(3.77 \times 10^{11}\text{ s}^{-1})\ t - (1257\text{ m}^{-1})\ x\right]$$

and $B_z = (10^{-7}\text{ T}) \sin\left[(3.77 \times 10^{11}\text{ s}^{-1})\ t - (1257\text{ m}^{-1})\ x\right]$ **Ans.**

Example 9 *A light beam travelling in the x-direction is described by the electric field* $E_y = (300\ Vm^{-1}) \sin \omega (t - x/c)$. *An electron is constrained to move along the y-direction with a speed of* $2.0 \times 10^{7}\ ms^{-1}$. *Find the maximum electric force and the maximum magnetic force on the electron.*

Solution **Maximum Electric Force**

Maximum electric field, $E_0 = 300\text{ V/m}$

$\therefore$ Maximum electric force $F = qE_0$

$$= (1.6 \times 10^{-19})\ (300)$$

$$= 4.8 \times 10^{-17}\text{ N}$$ **Ans.**

Maximum Magnetic Force

From the equation, $c = \dfrac{E_0}{B_0}$

Maximum magnetic field, $B_0 = \dfrac{E_0}{c}$

or $B_0 = \dfrac{300}{3.0 \times 10^{8}} = 10^{-6}\text{ T}$

$\therefore$ Maximum magnetic force $= B_0 qv \sin 90^\circ = B_0 qv$

Substituting the values, we have

Maximum magnetic force $= (10^{-6})(1.6 \times 10^{-19})(2.0 \times 10^{7})$

$$= 3.2 \times 10^{-18}\text{ N}$$ **Ans.**

Example 10 *A parallel plate capacitor having plate area A and plate separation d is joined to a battery of emf V and internal resistance R at $t = 0$. Consider a plane surface of area A/2, parallel to the plates and situated symmetrically between them. Find the displacement current through this surface as a function of time. [The charge on the capacitor at time t is given by $q = CV(1 - e^{-t/\tau})$, where $\tau = CR$]*

Solution Given, $$q = CV(1 - e^{-t/\tau})$$

∴ Surface charge density, $$\sigma = \frac{q}{A} = \frac{CV}{A}(1 - e^{-t/\tau})$$

Electric field between the plates of capacitor,

$$E = \frac{\sigma}{\varepsilon_0} = \frac{CV}{\varepsilon_0 A}(1 - e^{-t/\tau})$$

Electric flux from the given area,

$$\phi_E = \frac{EA}{2} = \frac{CV}{2\varepsilon_0}(1 - e^{-t/\tau})$$

Displacement current, $$i_d = \varepsilon_0 \frac{d\phi_E}{dt}$$

or $$i_d = \varepsilon_0 \frac{d}{dt}\left[\frac{CV}{2\varepsilon_0}(1 - e^{-t/\tau})\right] = \frac{CV}{2\tau} e^{-t/\tau}$$

Substituting, $$\tau = CR$$

We have, $$i_d = \frac{V}{2R} e^{-t/CR}$$

Again substituting, $$C = \frac{\varepsilon_0 A}{d}$$

$$i_d = \frac{V}{2R} e^{-\frac{td}{\varepsilon_0 AR}}$$ **Ans.**

Example 11 *About 5% of the power of a 100 W light bulb is converted to visible radiation. What is the average intensity of visible radiation*

(a) at a distance of 1 m from the bulb?

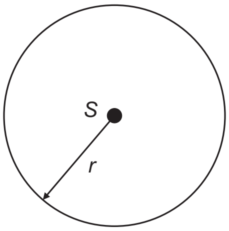
(b) at a distance of 10 m?

Assume that the radiation is emitted isotropically and neglect reflection.

Solution Effective power (energy radiated per second)

$$= 5\% \text{ of } 100 \text{ W}$$

$$P = 5 \text{ W}$$



This energy will distribute on a sphere. At a distance r from the point source, area on which light is incident is

$$S = 4\pi r^2$$

$\therefore$ Intensity at distance r from the point source,

$I = \frac{P}{S} = \frac{5}{4\pi r^2}$ = Energy incident per unit area per unit time

(a) At $r = 1\text{ m}$,

$$I = \frac{5}{4\pi (1)^2}$$

$$= 0.4\text{ W/m}^2 \qquad \textbf{Ans.}$$

(b) At $r = 10\text{ m}$,

$$I = \frac{5}{4\pi (10)^2}$$

$$= 0.004\text{ W/m}^2 \qquad \textbf{Ans.}$$

Example 12 *Suppose that the electric field of an electromagnetic wave in vacuum is* $\mathbf{E} = \{(3.0\ N/C)\cos[(1.8\ rad/m)\ y + (5.4\times 10^6 rad/s)\ t]\}\ \hat{\mathbf{i}}$.

(a) What is the direction of propagation of wave?

(b) What is the wavelength λ?

(c) What is the frequency f?

(d) What is the amplitude of the magnetic field of the wave?

(e) Write an expression for the magnetic field of the wave.

Solution (a) From the knowledge of wave we can see that electromagnetic wave is travelling along negative y-direction, as ωt and ky both are positive.

(b) $k = 1.8\text{ rad/m}$

$$k = \frac{2\pi}{\lambda}$$

$$\therefore \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi}{1.8} = 3.5\text{ m} \qquad \textbf{Ans.}$$

(c) $\omega = 5.4\times 10^6\text{ rad/s}$

$$\omega = 2\pi f$$

$$\therefore \qquad f = \frac{\omega}{2\pi} = \frac{5.4\times 10^6}{2\pi}$$

$$= 8.6\times 10^5\text{ Hz} \qquad \textbf{Ans.}$$

(d) $$E_0 = 3.0\text{ N/C}$$

From the relation, $$c = \frac{E_0}{B_0}$$

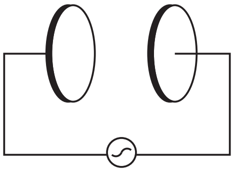
We have, $$B_0 = \frac{E_0}{c} = \frac{3.0}{3.0\times 10^8}$$

$$= 10^{-8}\text{ T} \qquad \textbf{Ans.}$$

(e) $\mathbf{E}$ is along $\hat{\mathbf{i}}$ direction, wave is travelling along negative y-direction. Therefore, oscillations of $\mathbf{B}$ are along z-direction or

$$\mathbf{B} = (10^{-8}\text{ T})\cos[(1.8\text{ rad/m})\ y + (5.4\times 10^6\text{ rad/s})\ t]\ \hat{\mathbf{k}} \qquad \textbf{Ans.}$$

Example 13 *A parallel plate capacitor made of circular plates each of radius $R = 6.0$ cm has a capacitance $C = 100$ pF. The capacitor is connected to a 230 V AC supply with a (angular) frequency of 300 rad/s.*



(a) What is the rms value of the conduction current?

(b) Is the conduction current equal to the displacement current?

(c) Determine the amplitude of B at a point 3.0 cm from the axis between the plates.

Solution (a) Capacitive reactance,

$$X_C = \frac{1}{\omega C} = \frac{1}{300 \times 100 \times 10^{-12}} = \frac{10^8}{3}\ \Omega$$

There is only capacitance in the circuit.

$$\therefore \quad i_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{230}{(10^8/3)} = 6.9 \times 10^{-6}\ \text{A}$$ **Ans.**

(b) Yes, the derivation in example 29.1 is true even if current is alternating.

(c) Here, i_d is displacement current and i the conduction current. Magnetic field at a distance r from the axis,

$$B = \frac{\mu_0}{2\pi}\frac{i_d}{R^2} r$$

$$\therefore \quad B_{\text{rms}} = \frac{\mu_0}{2\pi}\frac{i_{\text{rms}}}{R^2} r \qquad (i_d = i = i_{\text{rms}})$$

Substituting the values, we have

$$B_{\text{rms}} = \frac{(2 \times 10^{-7})(6.9 \times 10^{-6})}{(6 \times 10^{-2})^2}(3 \times 10^{-2}) = 1.15 \times 10^{-11}\ \text{T}$$

$$\therefore \quad B_0 = \sqrt{2}\ B_{\text{rms}} = (\sqrt{2})(1.15 \times 10^{-11})\ \text{T} = 1.63 \times 10^{-11}\ \text{T}$$ **Ans.**

Exercises

Single Correct Option

1. One requires 11eV of energy to dissociate a carbon monoxide molecule into carbon and oxygen atoms. The minimum frequency of the appropriate electromagnetic radiation to achieve the dissociation lies in

(a) visible region
(b) infrared region
(c) ultraviolet region
(d) microwave region

2. If **E** and **B** represent electric and magnetic field vectors of the electromagnetic wave, the direction of propagation of electromagnetic wave is along

(a) $\mathbf{E}$
(b) $\mathbf{B}$
(c) $\mathbf{B} \times \mathbf{E}$
(d) $\mathbf{E} \times \mathbf{B}$

3. The ratio of contributions made by the electric field and magnetic field components to the intensity of an EM wave is

(a) $c : 1$
(b) $c^2 : 1$
(c) $1 : 1$
(d) $\sqrt{c} : 1$

4. Light with an energy flux of 20 W/cm^2 falls on a non-reflecting surface at normal incidence. If the surface has an area of 30 cm^2. The total momentum delivered (for complete absorption) during 30 minutes is

(a) 36×10^{-5} kg-m/s
(b) 36×10^{-4} kg-m/s
(c) 1.08×10^{4} kg-m/s
(d) 1.08×10^{7} kg-m/s

More than One Correct Options

5. A plane electromagnetic wave propagating along x-direction can have the following pairs of **E** and **B**

(a) E_x, B_y
(b) E_y, B_z
(c) B_x, E_y
(d) E_z, B_y

6. The source of electromagnetic waves can be a charge

(a) moving with a constant velocity
(b) moving in a circular orbit
(c) at rest
(d) falling in an electric field

7. An electromagnetic wave of intensity I falls on a surface kept in vacuum and exerts radiation pressure p on it. Which of the following are true?

(a) Radiation pressure is I/c if the wave is totally absorbed
(b) Radiation pressure is I/c if the wave is totally reflected
(c) Radiation pressure is $2\,I/c$ if the wave is totally reflected
(d) Radiation pressure is in the range $I/c < p < 2\,I/c$ for real surfaces

8. A charged particle oscillates about its mean equilibrium position with a frequency of 10^9 Hz. The electromagnetic waves produced

(a) will have frequency of 10^9 Hz
(b) will have frequency of 2×10^9 Hz
(c) will have a wavelength of 0.3 m
(d) fall in the region of radio waves

Subjective Questions

9. Can an electromagnetic wave be deflected by an electric field? By a magnetic field?

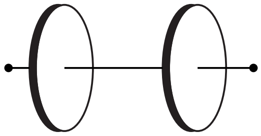
10. What physical quantity is the same for X-rays of wavelength 10^{-10}m, red light of wavelength 6800 Å and radio waves of wavelength 500 m?

11. A plane electromagnetic wave travels in vacuum along z-direction. What can you say about the directions of its electric and magnetic field vectors. If the frequency of the wave is 30 MHz, what is its wavelength?

12. A radio can tune into any station in the 7.5 MHz to 12 MHz band. What is the corresponding wavelength of band?

13. The amplitude of the magnetic field part of a harmonic electromagnetic wave in vacuum is $B_0 = 510$ nT. What is the amplitude of the electric field part of the wave?

14. Figure shows a capacitor made of two circular plates each of radius 12 cm, and separated by 5.0 cm. The capacitor is being charged by an external source (not shown in the figure). The charging current is constant and equal to 0.15 A.



(a) Calculate the capacitance and the rate of change of potential difference between the plates.

(b) Obtain the displacement current across the plates.

(c) Is Kirchhoff's first rule (junction rule) valid at each plate of the capacitor? Explain.

15. Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\,\text{N/C}$ and that its frequency is 50.0 MHz. (a) Determine B_0, ω, k and λ, (b) Find expressions for **E** and **B**.

16. A variable frequency AC source is connected to a capacitor. How will the displacement current change with decrease in frequency?

17. A laser beam has intensity 2.5×10^{14} Wm^{-2}. Find the amplitudes of electric and magnetic fields in the beam.

18. In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of 2.0×10^{10} Hz and amplitude 48 Vm^{-1}.

(a) What is the wavelength of the wave?

(b) What is the amplitude of the oscillating magnetic field?

(c) Show that the average energy density of the field **E** equals the average energy density of the field **B**. [$c = 3 \times 10^8$ ms^{-1}].

19. The charge on a parallel plate capacitor varies as $q = q_0 \cos 2\pi ft$. The plates are very large and close together (area = A, separation = d). Neglecting the edge effects, find the displacement current through the capacitor.

Answers

Introductory Exercise 29.1

1. $i/2$

Introductory Exercise 29.2

2. Yes **3.** $2.7\ \mu\text{T}$ **4.** 5.55×10^{-12} J

Exercises

1. (c) **2.** (d) **3.** (c) **4.** (b) **5.** (b,d)
6. (b,d) **7.** (a,c,d) **8.** (a,c,d) **9.** No, No
10. The speed in vacuum is the same for all
11. **E** and **B** lie in x-y plane and are mutually perpendicular, 10 m
12. Wavelength band from 40 m to 25 m
13. 153 N/C
14. (a) 8.0 pF, 1.87×10^{10} Vs^{-1} (b) 0.15 A

(c) Yes, provided by current we mean the sum of conduction and displacement currents.

15. (a) 400 nT, 3.14×10^{8} rad/s, 1.05 rad/m, 6.00 m

(b) **E** = (120 N/C) sin [(1.05 rad/m)] x – $(3.14 \times 10^{8}$ rad/s)t]

B = (400 nT) sin [(1.05 rad/m)] x – $(3.14 \times 10^{8}$ rad/s) t]

16. Displacement current will decrease
17. 4.3×10^{8} N/C , 1.44 T
18. (a) 1.5×10^{-2} m (b) 1.6×10^{-7} T
19. $-2\pi q_0 f \sin 2\pi ft$

30

Reflection of Light

Chapter Contents

30.1 Introduction
30.2 General concepts used in Geometrical Optics
30.3 Reflection of light
30.4 Reflection from a spherical surface

30.1 Introduction

The branch of physics called **optics** deals with the behaviour of light and other electromagnetic waves. Light is the principal means by which we gain knowledge of the world. Consequently, the nature of light has been the source of one of the longest debates in the history of science.

Electromagnetic radiation with wavelengths in the range of about 4000 Å to 7000 Å, to which eye is sensitive is called light.

In the present and next two chapters we investigate the behaviour of a beam of light when it encounters simple optical devices like mirrors, lenses and apertures. Under many circumstances, the wavelength of light is negligible compared with the dimensions of the device as in the case of ordinary mirrors and lenses. A light beam can then be treated as a ray whose propagation is governed by simple geometric rules. The part of optics that deals with such phenomena is known as **geometric optics.** However, if the wavelength is not negligible compared with the dimensions of the device (for example a very narrow slit), the ray approximation becomes invalid and we have to examine the behaviour of light in terms of its wave properties. This study is known as **physical optics.**

30.2 General Concepts used in Geometrical Optics

Some general concepts which are used in whole geometrical optics are given below.

1. Normal incidence means angle of incidence (with normal) is 0°. If angle of incidence is 90°, it is called grazing incidence.

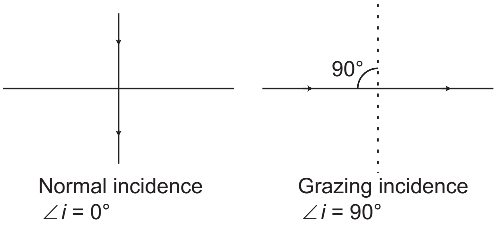


Fig. 30.1

2. An image is formed either by reflection or refraction. Minimum two (reflected or refracted) rays are required for image formation. More the number of rays, more will be the intensity of image.
3. A light is reflected only from a silvered surface. Without any reflecting surface on the path of ray of light it keeps on moving ahead.

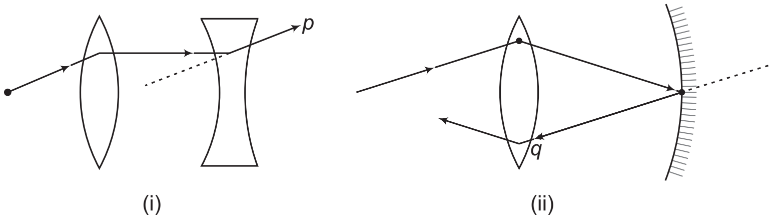


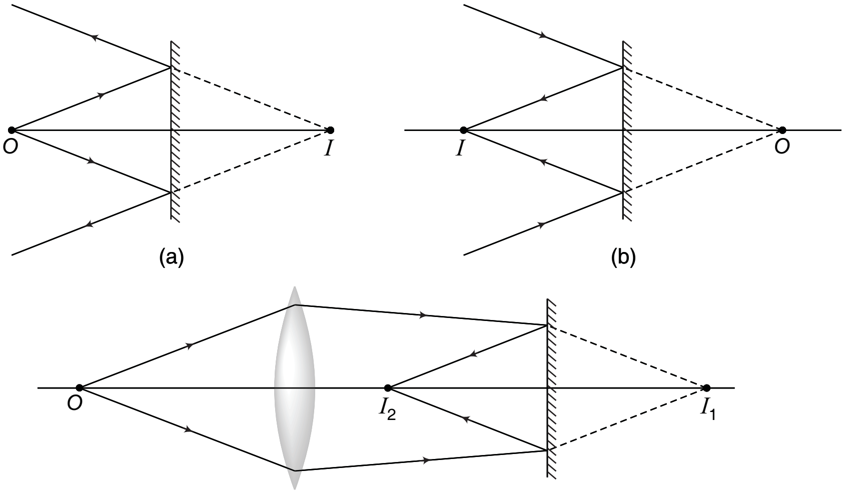
Fig. 30.2

In figure (i) For image formation, if reflected ray-*p* is required to the left of concave lens, then we will take it as dotted line.

In figure (ii) For image formation, if reflected ray-q is required to the right of concave mirror, then we will take it as dotted line.

4. **Real object, virtual object, real image, virtual image** The point where the rays meet (or appear to meet) before refraction or reflection is called object and the point where the rays meet (or appear to meet) after refraction or reflection is called image. Further, object (or image) is real if dark lines meet and virtual if dotted lines meet.

   In figure (a), object is real, while image is virtual. In figure (b), object is virtual while its image is real.

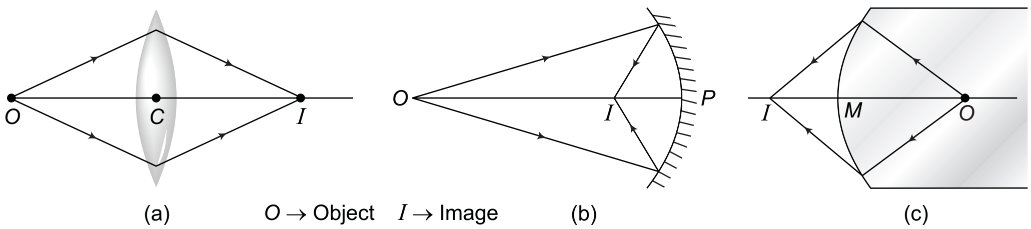


(c)

Fig. 30.3

In figure (c), the object O is real. Its image formed by the lens (i.e. I_1) is real. But, it acts as a virtual object for mirror which forms its real image I_2.

5. The virtual images cannot be taken on screen. But, they can be seen by our eye. Because our eye lens forms their real image on our retina. Thus, if we put a screen at I in the above figure (a) no image will be formed on it. At the same time if we put the screen at I in figure (b), image will be formed.
6. Normally, the object is kept on the left hand side of the optical instrument (mirror, lens etc.), i.e. the ray of light travels from left to right. Sometimes, it may happen that the light is travelling in opposite direction. See the figure.



(a) $O \rightarrow$ Object $I \rightarrow$ Image (b) (c)

Fig. 30.4

In figures (a) and (b), light is travelling from left to right and in figure (c) it is travelling from right to left.

7. **Sign convention** The distances measured along the incident light are taken as positive while the distances against incident light are taken as negative. For example, in figures (a) and (b) the incident light travels from left to right. So, the distances measured in this direction are positive. While in figure (c) the incident light travels from right to left. So, in this case right to left direction will be positive. Distances are measured from pole of the mirror [point *P* in figure (b)], optical centre of the lens [point *C* in figure (a)] and the centre of the refracting surface [point *M* in figure (c)].

   It may happen in some problem that sign convention does not remain same for the whole problem. For example, in the Fig. 30.5 shown, the ray of light starting from *O* first undergoes refraction at *A*, then reflection at *B* and then finally refraction at *C*. For refraction and reflection at *A* and *B* the incident light is travelling from left to right, so distances measured along this direction are positive. For final refraction at *C* the incident light travels from right to left, so now the sign convention will change or right to left is positive.

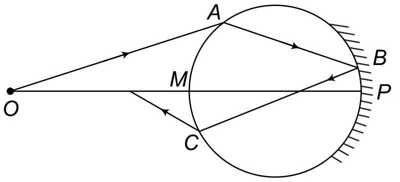


Fig. 30.5

8. Object distance (from P, C or M along the optic axis) in Fig. 30.4 is shown by u and image distance by v.
9. In front of mirror, object (or image) is always real, lines are always dark and u (or v) are always negative.

   Behind the mirror object (or image) is always virtual, lines are always dotted and u (or v) are always positive.

   This is because light always falls from front side of the mirror.

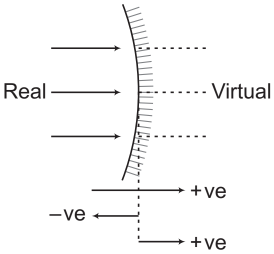


Fig. 30.6

10. In most of the cases, objects are real (whether refraction or reflection) and u for them is negative.

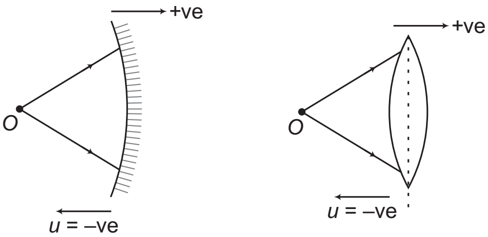


Fig. 30.7

11. In case of mirror light always falls from front of the mirror. But in a lens (or some other refracting surface like slab) light can fall from both sides.

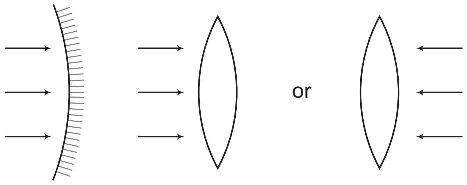


Fig. 30.8

12. **Total steps and reduced steps**

In Fig. 30.9 total steps are five. Four of them are refraction. Only third is reflection. But we have made a lens formula for steps 1 and 2 or 4 and 5. So, the reduced steps are three,

lens → mirror → lens.

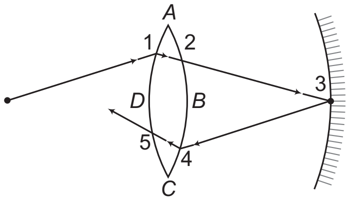


Fig. 30.9

13. **Image real or virtual** In Fig.30.9, if we wish to find the nature of I_2 (after 2nd refraction), then it is real if it is formed to the right of *ABC* because ray of light has moved to this side and it is virtual to the left of *ABC*. Similarly, I_5 is real to the left of *ADC* and virtual to the right of *ADC*.

14. **Final image coincides with the object** In most of these cases there will be one mirror, plane or spherical (convex or concave) and light will be falling normal ($\angle i = 0°$) to this mirror. In case of spherical mirror it is normal if ray of light passes through centre of curvature. In case of normal incidence, ray of light retraces its path and final image coincides with the object.

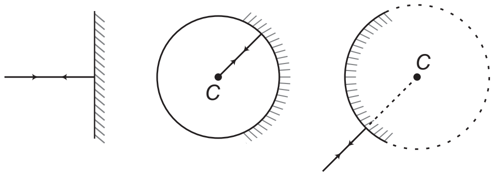


Fig. 30.10

In all above figures, $\angle i = 0°$ (normal incidence). Ray of light retraces its path.

15. Image at infinity means rays after refraction or reflection have become parallel to the optic axis. If a screen is placed directly in between these parallel rays no image will be formed on the screen. But if a lens (or a mirror) is placed on the path of these parallel rays, then image is formed at focus. Sometimes, our eye plays the role of this lens and the image is formed on our retina.

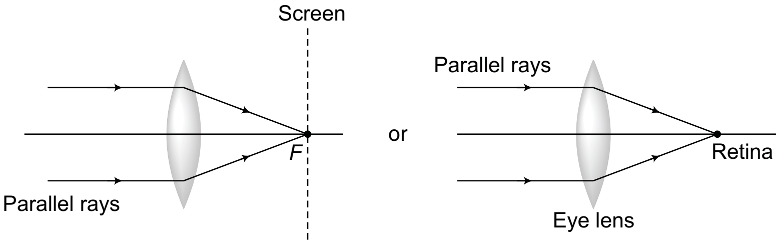


Fig. 30.11

16. **Visual angle** θ Angle subtended by an object on our eye is called the visual angle. The apparent size depends on the visual angle. As the object moves away from the eye, actual size remains the same but visual angle decreases. Therefore, apparent size decreases.

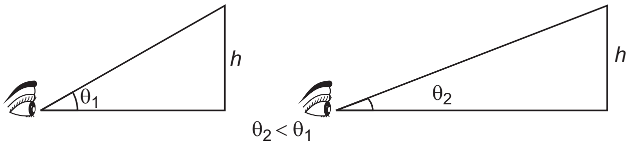


Fig. 30.12

30.3 Reflection of Light

When waves of any type strike the interface between two different materials, new waves are generated which move away from the interface. Experimentally, it is found that the rays corresponding to the incident and reflected waves make equal angles with the normal to the interface and that the reflected ray lies in the plane of incidence formed by the incident ray and the normal. Thus, the two laws of reflection can be summarised as under:

(i) $\angle i = \angle r$

(ii) Incident ray, reflected ray and normal lie on the same plane.

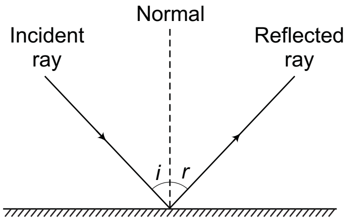
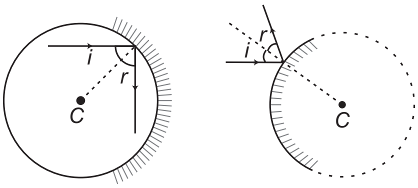


Fig. 30.13

Two Important Points in Reflection Laws

1. The first law $\angle i = \angle r$ can be applied for any type of surface. The main point is, normal at point of incidence. In spherical surface (convex mirror or concave mirror) normal at any point passes through centre of curvature.



Concave mirror Convex mirror

Fig. 30.14

2. Incident ray, reflected ray and normal are sometimes represented in the form of three vectors. Then, these three vectors should be coplanar.

Reflection from a Plane Surface (or Plane Mirror)

Almost everybody is familiar with the image formed by a plane mirror. If the object is real, the image formed by a plane mirror is virtual, erect, of same size and at the same distance from the mirror. The ray diagram of the image of a point object and of an extended object is as shown below.

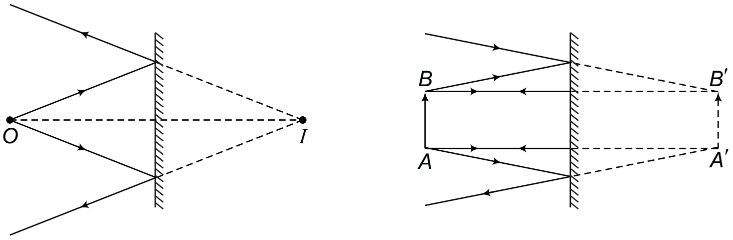


Fig. 30.15

Important Points in Reflection from Plane Mirror

1. Relation between object distance (u) and the image distance (v) in case of plane mirror is

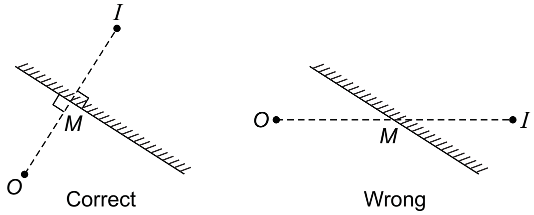
$$\boxed{v = -u}$$

Here, v and u are measured from the plane surface. Two conclusions can be drawn from this equation.

(i) Negative sign implies that object and image are on opposite sides of the mirror. So, if object is real then image is virtual and *vice-versa*.

(ii) $|v| = |u|$

And this implies that perpendicular distance of the object from the mirror is equal to the perpendicular distance of image from the mirror.

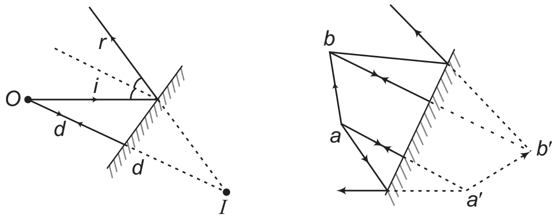


Correct
$OM = MI$

Wrong
$OM = MI$

Fig. 30.16

2. **Ray Diagram** Let us draw the ray diagram of a point object and an extended object.



(i)
$O \to$ Real
$I \to$ Virtual

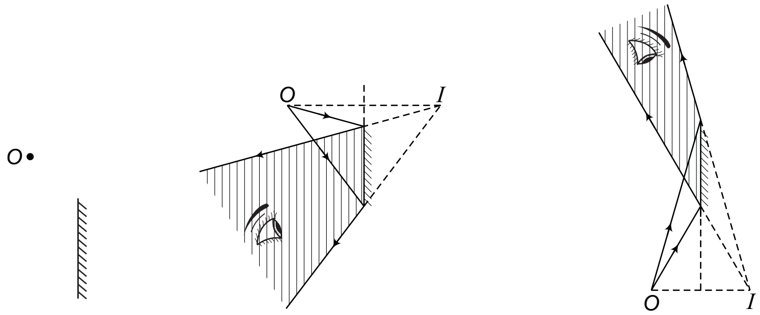
(ii)
$ab \to$ Real
$a'b' \to$ Virtual

Fig. 30.17

Just as we have drawn the ray diagram of point object O in figure (i), we can also draw the ray diagrams of points a and b in figure (ii).

3. **Field of view of an object for a given mirror** Suppose a point object O is placed in front of a small mirror as shown in Fig. 30.18 (a), then a question arises in mind whether this mirror will make the image of this object or not. Or suppose an elephant is standing in front of a small mirror, will the mirror form the image of the elephant or not. The answer is yes, it will form. A mirror whatever may be the size of it forms the images of all objects lying in front of it. But every object has its own field of view for the given mirror. The field of view is the region between the extreme reflected rays and depends on the location of the object in front of the mirror. If our eye lies in the field of view then only we can see the image of the object otherwise not. The field of view of an object placed at different locations in front of a plane mirror are shown in Fig. 30.18 (b) and (c).

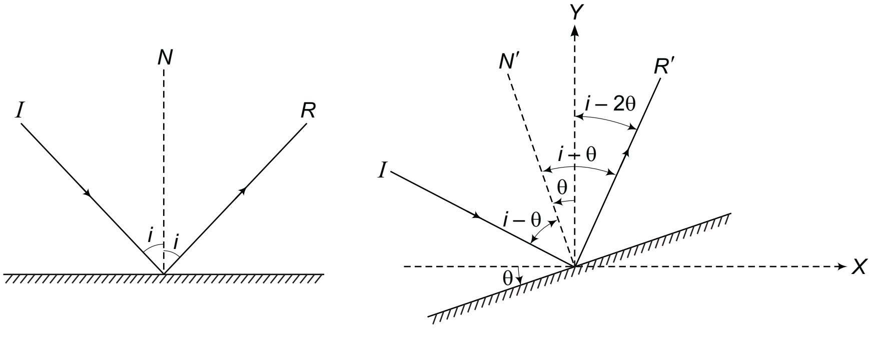
   The region between extreme reflected rays (reflected from the end points of the mirror) is called the field of view. To see the image of object eye should lie in this region, as all reflected rays lie in this region.



(a) (b) (c)

Fig. 30.18

4. Suppose a mirror is rotated by an angle θ (say anti-clockwise), keeping the incident ray fixed then the reflected ray rotates by 2θ along the same direction, i.e. anti-clockwise.



(a) (b)

Fig. 30.19

In figure (a), I is the incident ray, N the normal and R the reflected ray.

In figure (b), I remains as it is N and R shift to N' and R'.

From the two figures, we can see that earlier the reflected ray makes an angle i with y-axis while after rotating the mirror it makes an angle $i - 2\theta$. Thus, we may conclude that the reflected ray has been rotated by an angle 2θ.

5. The minimum length of a plane mirror to see one's full height is $\frac{H}{2}$, where H is the height of person. But, the mirror should be placed in a fixed position which is shown in Fig. 30.20.

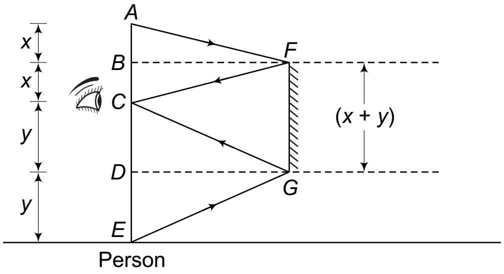


Fig. 30.20

A ray starting from head (A) after reflecting from upper end of the mirror (F) reaches the eye at C. Similarly, the ray starting from the foot (E) after reflecting from the lower end (G) also reaches the eye at C. In two similar triangles ABF and BFC, $AB = BC = x$ (say), Similarly in triangles CDG and DGE,

$$CD = DE = y \text{ (say)}$$

Now, we can see that height of the person is $2(x + y)$ and that of mirror is $(x + y)$, i.e. height of the mirror is half the height of the person.

Note *The mirror can be placed anywhere between the centre lines BF (of AC) and DG (of CE). As the mirror is moved away on this line, image also moves away from the person. So, apparent size keeps on decreasing.*

6. A person is standing exactly at midway between a wall and a mirror and he wants to see the full height of the wall (behind him) in a plane mirror (in front of him). The minimum length of mirror in this case should be $\frac{H}{3}$, where H is the height of wall. The ray diagram in this case is shown in Fig. 30.21

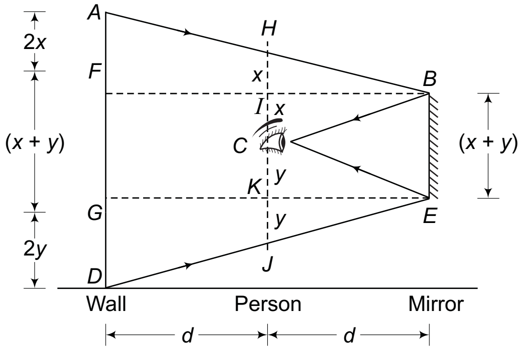


Fig. 30.21

In triangles HBI and IBC, $HI = IC = x$ (say). Now, in triangles HBI and ABF,

$$\frac{AF}{HI} = \frac{FB}{BI} \quad \text{or} \quad \frac{AF}{x} = \frac{2d}{d} \quad \text{or} \quad AF = 2x$$

Similarly, we can prove that $DG = 2y$ if, $CK = KJ = y$

Now, we can see that height of the wall is $3(x + y)$ while that of the mirror is $(x + y)$.

7. **Object and image velocity** There are four important points related to object and image velocity.
   (i) Image speed is equal to the object speed.
   (ii) Image velocity and object velocity make same angles from the plane mirror on two opposite sides of the mirror.
   (iii) Components of velocities which are along the mirror are equal.
   (iv) Components of velocities which are perpendicular to the mirror are equal and opposite.
   The following four figures demonstrate the above four points.

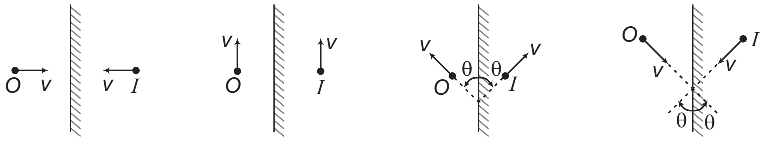


Fig. 30.22

Three Types of Problems in Reflection from Plane Mirror

Type 1. *Based on law of reflection $\angle i = \angle r$*

Concept

These problems are purely based on geometry. Proper normal at point of incidence is very important.

Example 30.1 *Two plane mirrors M_1 and M_2 are inclined at angle θ as shown. A ray of light 1, which is parallel to M_1 strikes M_2 and after two reflections, the ray 2 becomes parallel to M_2. Find the angle θ.*

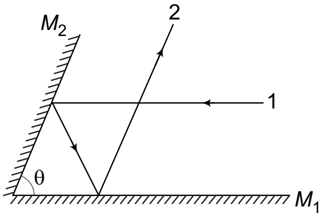


Fig. 30.23

Solution Different angles are as shown in Fig. 30.24. In triangle *ABC*,

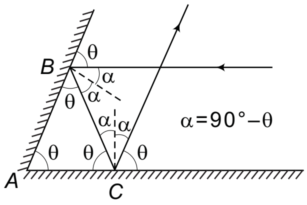


Fig. 30.24

$$\theta + \theta + \theta = 180^\circ$$

$\therefore$ $$\theta = 60^\circ$$ **Ans.**

Example 30.2 *Prove that for any value of angle i, rays 1 and 2 are parallel.*

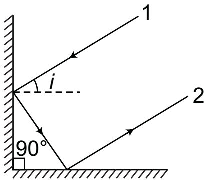


Fig. 30.25

Solution *PQ* and *MN* are mutually parallel. Rays 1 and 2 are making equal angles $(= \angle i)$ from *PQ* and *MN*. So, they are mutually parallel.

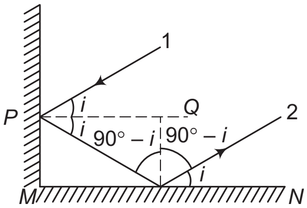


Fig. 30.26

Important Result Two plane mirrors kept at 90° deviate each ray of light by 180° from its original path.

Type 2. *Based on field of view*

Concept

The region between extreme reflected rays is called field of view. To see the image of object our eye should lie in the field of view.

Example 30.3 *A point source of light S, placed at a distance L in front of the centre of a mirror of width d, hangs vertically on a wall. A man walks in front of the mirror along a line parallel to the mirror at a distance 2L from it as shown. The greatest distance over which he can see the image of the light source in the mirror is*

(a) $d/2$ (b) d

(c) $2\,d$ (d) $3\,d$ **(JEE 2000)**

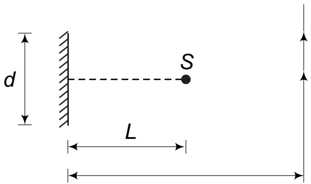


Fig. 30.27

Solution (d) The ray diagram will be as shown in Fig. 30.28.

$$HI = AB = d$$

$$DS = CD = \frac{d}{2}$$

Since, $AH = 2AD$

$\therefore$ $GH = 2\,CD = 2\frac{d}{2} = d$

Similarly, $IJ = d$

$\therefore$ $GJ = GH + HI + IJ = d + d + d = 3d$

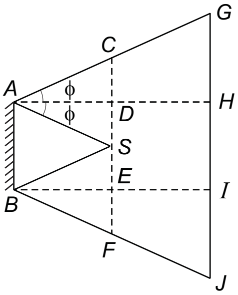


Fig. 30.28

Example 30.4 *A pole of height 4 m is kept in front of a vertical plane mirror of length 2 m. The lower end of the mirror is at a height of 6 m from the ground. The horizontal distance between the mirror and the pole is 2 m. Upto what minimum and maximum heights a man can see the image of top of the pole at a horizontal distance of 4 m (from the mirror) standing on the same horizontal line which is passing through the pole and the horizontal point below the mirror?*

Solution PQ = Pole, MN = Image of pole

$$\frac{HG}{GN} = \frac{BD}{BN}$$

$$\therefore \qquad BD = \frac{(HG)(BN)}{GN} = \frac{(2)(6)}{2}$$

$$= 6 \text{ m}$$

Minimum height required $= AD = BD + AB = 10$ m

Further, $\dfrac{IG}{GN} = \dfrac{BE}{BN}$

$$\therefore \qquad BE = \frac{(IG)(BN)}{GN} = \frac{(4)(6)}{2} = 12 \text{ m}$$

$\therefore$ Maximum height required $= AE$

$$= BE + AB = 16 \text{ m}$$

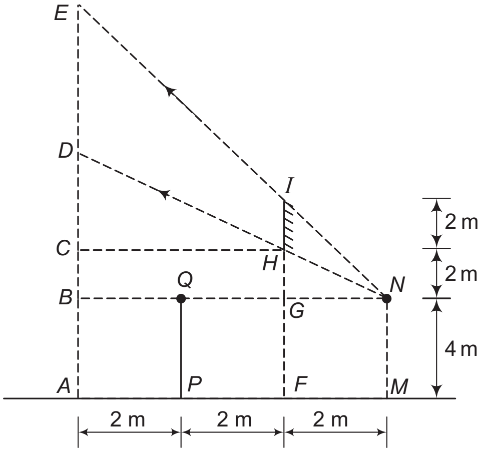


Fig. 30.29

Type 3. *Based on object and image velocity*

Concept

We have already discussed four points on this topic.

Example 30.5 *A plane mirror is lying in x-y plane. Object velocity is* $\mathbf{v}_0 = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ *m/s. Find the image velocity.*

Solution Components of object velocity parallel to plane mirror (or lying in x - y plane) remain unchanged. But, component perpendicular to plane mirror changes its direction but magnitude remains the same. Hence, the image velocity is

$$\mathbf{v}_I = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \text{ m/s}$$ **Ans.**

Example 30.6 *An object is falling vertically downwards with velocity 10 m/s. In terms of* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$*, find the image velocity.*

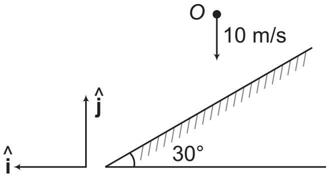


Fig. 30.30

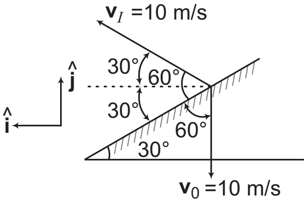


Fig. 30.31

Solution Image speed will also be 10 m/s. Further, image velocity will make same angle ($= 60°$) from the mirror on opposite side of it.

In terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$,

$$\mathbf{v}_I = 10 \cos 30° \,\hat{\mathbf{i}} + 10 \sin 30° \,\hat{\mathbf{j}}$$

or $$\mathbf{v}_I = (5\sqrt{3}\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}) \text{ m/s} \qquad \textbf{Ans.}$$

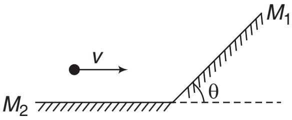


Fig. 30.32

Example 30.7 *A point object is moving with a speed v before an arrangement of two mirrors as shown in figure. Find the magnitude of velocity of image in mirror M_1 with respect to image in mirror M_2.*

Solution Using the same concept used in above problem, we have to find magnitude of relative velocity between $\mathbf{v}_{I_1}$ and $\mathbf{v}_{I_2}$. The angle between these two vectors is 2θ.

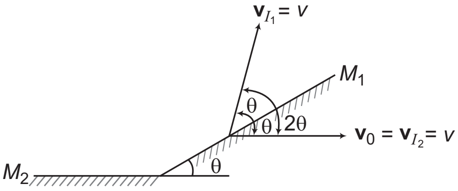


Fig. 30.33

Hence, $$\mathbf{v}_r = \mathbf{v}_{I_1} - \mathbf{v}_{I_2}$$

$\therefore$ $$|\mathbf{v}_r| = |\mathbf{v}_{I_1} - \mathbf{v}_{I_2}|$$

This is nothing but magnitude of subtraction of two velocity vectors of equal magnitudes v each and angle between them equal to 2θ.

Hence, $$|\mathbf{v}_r| = \sqrt{v^2 + v^2 - 2(v)(v)\cos 2\theta}$$

Solving these two equations, we get

$$|\mathbf{v}_2| = 2\, v \sin \theta \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 30.1

1. A man approaches a vertical plane mirror at speed of 2 m/s. At what rate does he approach his image?
2. An object M is placed at a distance of 3 m from a mirror with its lower end at 2 m from ground as shown in Fig. 30.34. There is a person at a distance of 4 m from object. Find minimum and maximum height of person to see the image of object.

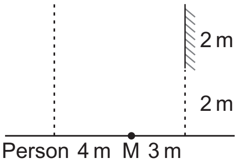


Fig. 30.34

3. In terms of θ find the value of i, so that ray of light retraces its path after third reflection.

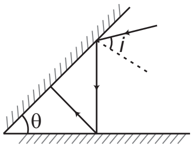


Fig. 30.35

30.4 Reflection from a Spherical Surface

We shall mainly consider the spherical mirrors, i.e. those which are part of a spherical surface.

Terms and Definitions

There are two types of spherical mirrors, concave and convex.

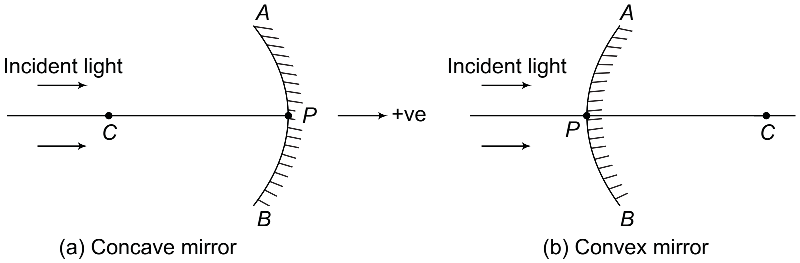


Fig. 30.36

Centre *C* of the sphere of which the mirror is a part is called the **centre of curvature** of the mirror and *P* the centre of the mirror surface, is called the **pole.** The line *CP* produced is the **principal axis** and *AB* is the **aperture** of the mirror. The distance *CP* is called the **radius of curvature (*R*).** All distances are measured from point *P*. We can see from the two figures that *R* is positive for convex mirror and negative for concave mirror.

Principal Focus

Observation shows that a narrow beam of rays, parallel and near to the principal axis, is reflected from a concave mirror so that all rays converge to a point *F* on the principal axis. *F* is called the **principal focus** of the mirror and it is a **real** focus, since, light actually passes through it. Concave mirrors are also known as converging mirrors because of their action on a parallel beam of light. They are used in car head-lights, search-lights and telescopes.

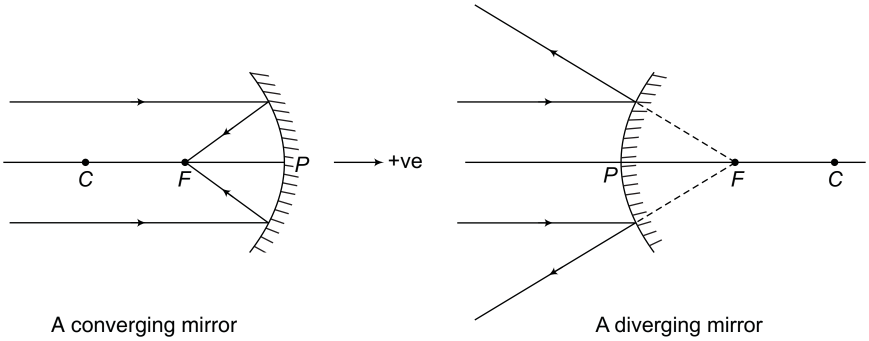


Fig. 30.37

A narrow beam of rays, parallel and near to the principal axis, falling on a convex mirror is reflected to form a divergent beam which appears to come from a point *F* behind the mirror. A convex mirror thus has a virtual principal focus. It is also called a diverging mirror. The distance *FP* is called the **focal length (*f*)** of the mirror. Further, we can see that *f* is negative for a concave mirror and positive for convex mirror. Later, we will see that $f = R/2$.

Paraxial rays Rays which are close to the principal axis and make small angles with it, i.e. they are nearly parallel to the axis, are called paraxial rays. Our treatment of spherical mirrors will be restricted to such rays which means we shall consider only mirrors of small aperture. In diagrams, however, they will be made larger for clarity.

Images formed by Spherical Mirrors

In general position of image and its nature (i.e. whether it is real or virtual, erect or inverted, magnified or diminished) depend on the distance of object from the mirror.

Case 1 **Concave mirror**

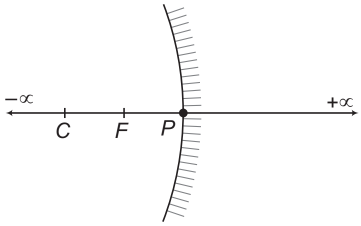


Fig. 30.38

Table 30.1

Object position	Image position	Image nature	Object speed and image speed
P	P	–	–
F	$\pm\infty$	–	–
C	C	Real, inverted and same size	$v_I = v_O$
Between P and F	Between P and $+\infty$	Virtual, erect and magnified	$v_I > v_O$
Between F and C	Between $-\infty$ and C	Real, inverted and magnified	$v_I > v_O$
Between C and $-\infty$	Between C and F	Real, inverted and diminished	$v_O > v_I$

Note *(i) The above table is only for real objects lying in front of the mirror for which u is negative.*
(ii) v_I and v_O are image and object speeds.
(iii) From the above table we can see that image and object always travel in opposite directions as long as they move along the principal axis.

Case 2 **Convex mirror** For real objects there is only one case

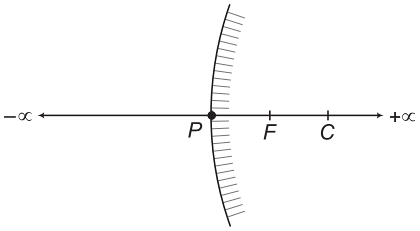


Fig. 30.39

Object lies between P and $-\infty$, then image lies between P and F. Image is virtual, erect and diminished. Object speed is greater than the image speed and they travel in opposite directions (along the principal axis).

Ray diagrams We shall consider the small objects and mirrors of small aperture so that all rays are paraxial. To construct the image of a point object two of the following four rays are drawn passing through the object. To construct the image of an extended object, the image of two end points is only drawn. The image of a point object lying on principal axis is formed on the principal axis itself. The four rays are as under:

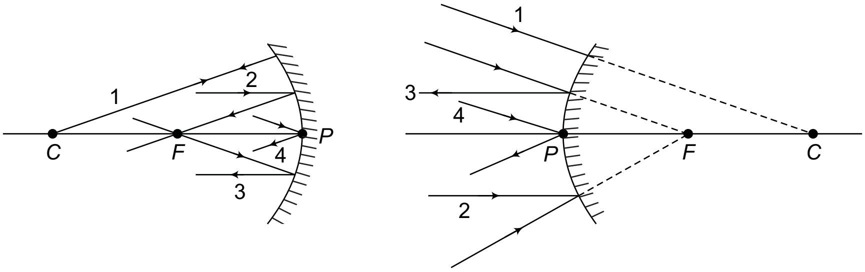


Fig. 30.40

Ray 1. A ray through the centre of curvature which strikes the mirror normally and is reflected back along the same path.

Ray 2. A ray parallel to principal axis after reflection either actually passes through the principal focus F or appears to diverge from it.

Ray 3. A ray passing through the principal focus F or a ray which appears to converge at F is reflected parallel to the principal axis.

Ray 4. A ray striking at pole P is reflected symmetrically back in the opposite side.

Convex Mirror

Image formed by convex mirror is always virtual, erect and diminished, no matter where the object is.

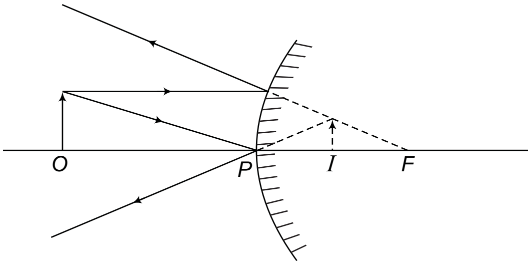
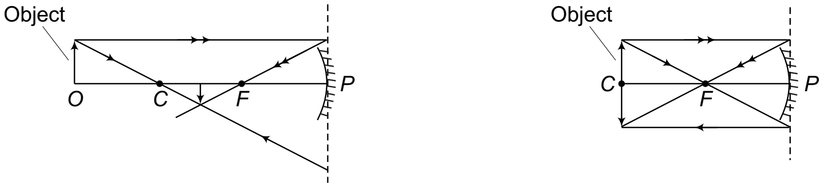


Fig. 30.41

Figure shows that convex mirror gives a wider **field of view** than a plane mirror, convex mirrors are therefore, used as rear view mirrors in cars or scooters. Although, they make the estimation of distances more difficult but still they are preferred because there is only a small movement of the image for a large movement of the object.

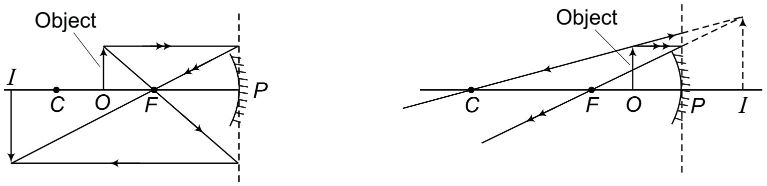
Concave Mirror

In case of a concave mirror the image is erect and virtual when the object is placed between F and P. In all other positions of object the image is real.



(a) Object beyond C
Image between C and F, real, inverted, diminished

(b) Object at C
Image at C, real, inverted, same size



(c) Object between C and F
Image beyond C, real inverted, magnified

(d) Object between F and P
Image behind mirror, virtual upright, magnified

Fig. 30.42

List of Formulae

(i) $f = \dfrac{R}{2}$

(ii) $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f} = \dfrac{2}{R}$

(iii) Lateral, transverse or linear magnification,

$$m = \frac{\text{Image height}}{\text{Object height}} = \frac{I}{O} = \frac{-v}{u}$$

(iv) Power of a mirror (in dioptre) $= \dfrac{1}{\text{Focal length (in metre)}}$

(v) Image velocity

Case 1 Along the principal axis, $\mathbf{v}_I = -m^2 \mathbf{v}_O$

Here, negative sign implies that object and image always travel in opposite directions.

Case 2 Perpendicular to principal axis,

$$\mathbf{v}_I = m \mathbf{v}_O$$

Here, m has to be substituted with sign. If m is positive, then $\mathbf{v}_I$ and $\mathbf{v}_O$ travel in same direction. If m is negative, then they travel in opposite directions.

Important Points in Formulae

(i) For plane mirror, $R = \infty$

$$\therefore \quad f = \frac{R}{2} \quad \text{or} \quad f = \infty \quad \text{and} \quad \frac{1}{v} + \frac{1}{u} = \frac{2}{R} = \frac{2}{\infty} = 0 \quad \text{or} \quad v = -u$$

(ii) From the value of m, we can know nature of image, type of mirror and an approximate location of object. But, always remind that real (and inverted) image is formed only by a concave mirror (for real objects) but virtual image is formed by all three mirrors. The only difference is, in their sizes. Magnified image is obtained from concave mirror, same size from plane mirror and diminished size from convex mirror. Let us make a table :

Table 30.2

Value of m	Nature of image	Type of mirror	Object position
-4	Inverted, real and magnified	Concave	Between F and C
-1	Inverted, real and same size	Concave	At C
$-\frac{1}{2}$	Inverted, real and diminished	Concave	Between C and $-\infty$
$+3$	Erect, virtual and magnified	Concave	Between P and F
$+1$	Erect, virtual and same size	Plane	For all positions
$+\frac{1}{2}$	Erect, virtual and diminished	Convex	Between P and $-\infty$

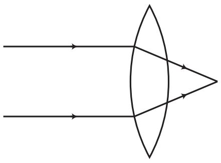
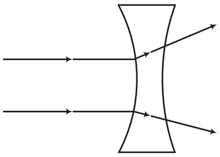
(iii) **Power** Optical power means power of bending of light. By convention, converging nature is taken as the positive power and diverging nature as negative power.

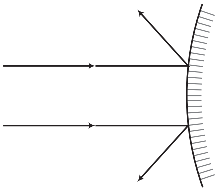
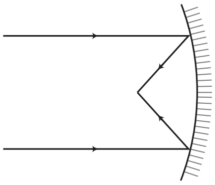
$$\text{Power of a lens (in dioptre)} = +\frac{1}{f\,(\text{in metre})}$$

$$\text{Power of a mirror (in dioptre)} = -\frac{1}{f\,(\text{in metre})}$$

Now, let us make a table :

Table 30.3

Lens/Mirror	f	P	Converging/Diverging	Diagram
Convex lens	+ ve	+ ve	Converging	
Concave lens	– ve	– ve	Diverging	

Lens/Mirror	f	P	Converging/Diverging	Diagram
Convex mirror	+ ve	– ve	Diverging	
Concave mirror	– ve	+ ve	Converging	

(iv) **Image velocity**

Case 1 **Along the principal axis**

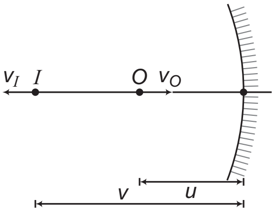


Fig. 30.43

By the motion of object and image v and u will change but focal length will remain unchanged. If we differentiate the mirror formula

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

with respect to time, we get

$$-v^{-2}.\frac{dv}{dt}-u^{-2}\frac{du}{dt}=0 \qquad \text{(as } f = \text{constant)}$$

or

$$\boxed{\frac{dv}{dt}=-\left(\frac{v^2}{u^2}\right)\frac{du}{dt}}$$

Here, $\frac{du}{dt}$ is the rate by which u is changing. Or it is the object speed if mirror is stationary. Similarly, $\frac{dv}{dt}$ is the rate by which v (distance between image and mirror) is changing. Or it is image speed if mirror is stationary.

$$\frac{v^2}{u^2}=m^2$$

So, the above relation becomes :

$$\text{image speed} = m^2 \times \text{object speed}$$

As, object and image travel in opposite directions. So, in terms of velocity, the correct relation is

$$\boxed{\mathbf{v}_I = -m^2 \mathbf{v}_O}$$

Case 2 **Perpendicular to axis**

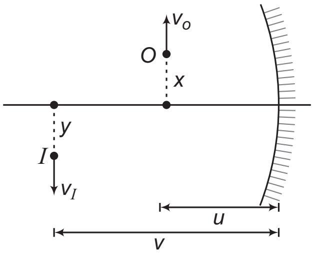


Fig. 30.44

This time v and u are constants. Therefore, $m = -\frac{v}{u}$ is also constant.

But, x and y are variables

$$m = \frac{I}{O} = \frac{y}{x}$$

$\Rightarrow$ $$y = mx$$

If we differentiate with respect to time, we get,

$$\frac{dy}{dt} = m\frac{dx}{dt}$$

$\Rightarrow$ $$\boxed{\mathbf{v}_I = m\mathbf{v}_O}$$

Proofs of Different Formulae Discussed in Theory

(i) **Relation between f and R** A ray AM parallel to the principal axis of a concave mirror of small aperture is reflected through the principal focus F. If C is the centre of curvature, CM is the normal to the mirror at M because the radius of a spherical surface is perpendicular to the surface. From first law of reflection,

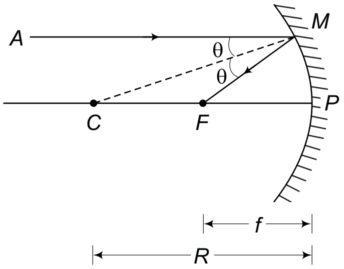


Fig. 30.45

$$\angle i = \angle r$$

or $$\angle AMC = \angle CMF = \theta \qquad \text{(say)}$$

But, $$\angle AMC = \angle MCF \qquad \text{(alternate angles)}$$

$\therefore$ $$\angle CMF = \angle MCF$$

Therefore, ΔFCM is thus isosceles with $FC = FM$.

The rays are paraxial and so M is very close to P. Therefore,

$$FM \approx FP$$

$\therefore$ $$FC = FP$$

or $$FP = \frac{1}{2} CP$$

or $$\boxed{f = \frac{R}{2}}$$

EXERCISE Prove the above relation for convex mirror.

(ii) **The mirror formula** In Fig. 30.46 (a) and (b), a ray OM from a point object O on the principal axis is reflected at M so that the angle θ, made by the incident and reflected rays with the normal CM are equal. A ray OP strikes the mirror normally and is reflected back along PO. The intersection I of the reflected rays MI and PO in figure (a) gives a **real** point image of O and in figure (b) gives a **virtual** point image of O. Let α, β and γ be the angles as shown. As the rays are paraxial, these angles are small, we can take

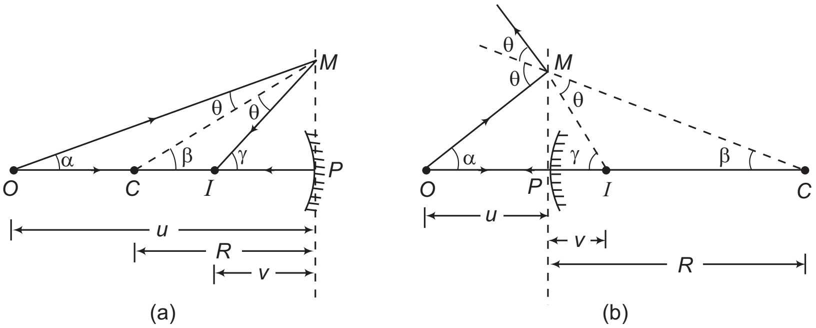


Fig. 30.46

$$\alpha \approx \tan\alpha = \frac{MP}{OP},$$

$$\beta = \frac{MP}{CP}$$

and $$\gamma = \frac{MP}{IP}$$

Now, let us take the two figures simultaneously

Table 30.4

Concave	Convex
In triangle CMO, $\beta = \alpha + \theta$ (the exterior angle) or $\theta = \beta - \alpha$ ...(i)	In triangle CMO, $\theta = \alpha + \beta$ ...(iv) (the exterior angle)
In $\Delta\, CMI$, $\gamma = \beta + \theta$ $\therefore$ $\theta = \gamma - \beta$ ...(ii)	In ΔCMI $\gamma = \theta + \beta$ or $\theta = \gamma - \beta$ ...(v)
From Eqs. (i) and (ii), we get $2\beta = \gamma + \alpha$ ...(iii)	From Eqs. (iv) and (v), we get $2\beta = \gamma - \alpha$ ...(vi)
Substituting the values of α, β and γ, we get $\frac{2}{CP} = \frac{1}{IP} + \frac{1}{OP}$ ...(A)	Substituting the values of α,β and γ, we get $\frac{2}{CP} = \frac{1}{IP} - \frac{1}{OP}$ ...(B)
If we now substitute the values with sign, i.e. $CP = -R$, $IP = -v$ and $OP = -u$ we get, $\frac{2}{R} = \frac{1}{v} + \frac{1}{u}$	If we now substitute the values with sign, i.e. $CP = +R$, $IP = +v$ and $OP = -u$, we get $\frac{2}{R} = \frac{1}{v} + \frac{1}{u}$
or $\boxed{\frac{1}{v} + \frac{1}{u} = \frac{1}{f}}$ $\left(\text{as } f = \frac{R}{2}\right)$	or $\boxed{\frac{1}{v} + \frac{1}{u} = \frac{1}{f}}$ $\left(\text{as } f = \frac{R}{2}\right)$

(iii) **Magnification** The lateral, transverse or linear magnification m is defined as

$$m = \frac{\text{Height of image}}{\text{Height of object}} = \frac{I'I}{O'O} = \frac{IP}{OP} \quad \text{...(i)}$$

(From similar triangles)

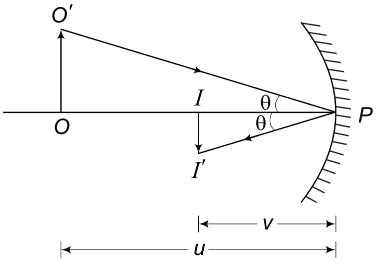


Fig. 30.47

Here, $IP = -v$ and $OP = -u$, further object is erect and image is inverted so we can take $I'I$ as negative and $O'O$ as positive and Eq. (i) will then become

$$\frac{I'I}{O'O} = -\frac{v}{u}$$

or
$$\boxed{m = -\frac{v}{u}}$$

Note *We have derived* $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ *and* $m = -\frac{v}{u}$ *for special cases of the position of object but the same result can be derived for other cases also.*

Types of Problems in Spherical Mirror

Type 1. *To find image distance and its magnification corresponding to given object distance and focal length of mirror*

How to Solve?

- In the mirror formula substitute signs of only u and f. Sign of v automatically comes after calculations. For real objects sign of u is always negative, sign of f is positive for convex mirror and negative for concave mirror.

Example 1 *An object is placed at a distance of 30 cm from a concave mirror of focal length 20 cm. Find image distance and its magnification. Also, draw the ray diagram.*

Solution Substituting $u = -30$ cm and $f = -20$ cm in the mirror formula $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$, we have

$$\frac{1}{v} + \frac{1}{-30} = \frac{1}{-20}$$

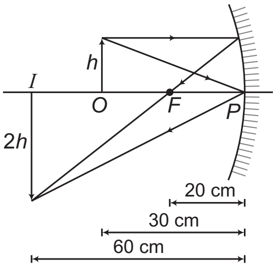
Solving, we get $v = -60$ cm **Ans.**

Magnification, $m = \frac{-v}{u} = -\frac{(-60)}{(-30)}$

or $m = -2$ **Ans.**

Magnification is –2, which implies that image is inverted, real and two times magnified.

Ray diagram is as shown below.



Example 2 *An object is placed at a distance of 40 cm from a convex mirror of focal length 40 cm. Find image position and its magnification. Also, draw its ray diagram.*

Solution Substituting, $u = -40$ cm and $f = +40$ cm in the mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}, \text{ we have}$$

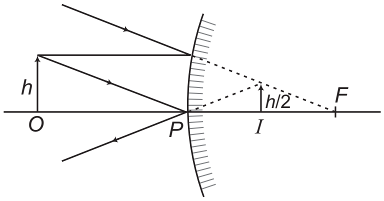
$$\frac{1}{v} + \frac{1}{-40} = \frac{1}{+40}$$

We get, $v = +20$ cm **Ans.**

Magnification, $m = -\frac{v}{u} = -\frac{(+20)}{(-40)} = +\frac{1}{2}$

Magnification is $+\frac{1}{2}$, which implies that image is erect, virtual and half in size.

Ray diagram is as shown below.



$$PO = PF = 40 \text{ cm}$$
$$PI = 20 \text{ cm}$$

Note *Here, PO = PF = 40 cm or object is placed at a distance of its focal length, but object is not actually kept at F. Otherwise, image would be formed at infinity.*

Type 2. *To find object/image distance corresponding to given magnification of image if focal length of mirror is also given*

How to Solve?

- In this type, substitute all three signs of u, v and f.
- Signs of u and f have been discussed in Type 1. Sign of v will be positive for virtual image and negative for real image.
- $m = -\frac{v}{u} \quad \Rightarrow \quad |v| = |mu|$

Example 3 *Find the distance of object from a concave mirror of focal length 10 cm so that image size is four times the size of the object.*

Solution Concave mirror can form real as well as virtual image. Here, nature of image is not given in the question. So, we will consider two possible cases.

***Case* 1 (When image is real)** Real image is formed on the same side of the object, i.e. u, v and f all are negative. So let,

$$u = -x$$

then $$v = -4x \quad \text{as} \quad \left|\frac{v}{u}\right| = |m| = 4$$

and $$f = -10 \text{ cm}$$

Substituting in $$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have $$\frac{1}{-4x} - \frac{1}{x} = \frac{1}{-10} \quad \text{or} \quad \frac{5}{4x} = \frac{1}{10}$$

∴ $$x = 12.5 \text{ cm}$$ **Ans.**

Note *$|f| < |x| < |2f|$ and we know that in case of a concave mirror, image is real and erect when object lies between F and C.*

***Case* 2 (When image is virtual)** In case of a mirror, image is virtual, when it is formed behind the mirror, i.e. u and f are negative, while v is positive. So let,

$$u = -y$$

then $v = +4y$ and $f = -10\text{ cm}$

Substituting in $$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

We have $$\frac{1}{4y}-\frac{1}{y}=\frac{1}{-10}$$

or $$\frac{3}{4y}=\frac{1}{10}$$

or $$y = 7.5\text{ cm}$$ **Ans.**

Note *Here, $|y| < |f|$, as we know that image is virtual and erect when the object lies between F and P.*

Type 3. *Based on making some condition*

How to Solve?

- Initially, substitute sign of f only.
- Make an equation of v.
- Now, for real image v should be negative and for virtual object v should be positive. With these concepts we can make the necessary condition.

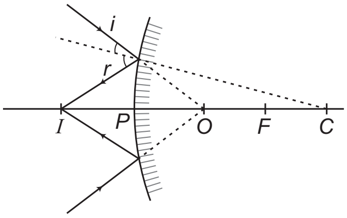
Example 4 *Find the condition under which a convex mirror can make a real image.*

Solution Substituting the sign of f only in the mirror formula, we have

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{+f} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

For real image v should be negative and for this u should be positive and less than f. Object distance u is positive means object should be virtual and lying between P and F.

The ray diagram is as shown below.



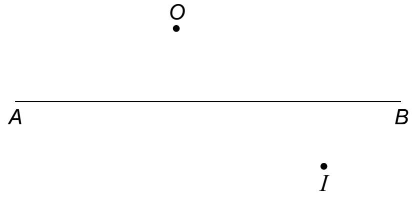
Here, O = virtual between P and F

I = real

Note *Under normal conditions a concave mirror makes a real image. But, it makes a virtual image if a real object is kept between P and F. On the other hand a convex mirror makes a virtual image. But it makes a real image if a virtual object is kept between P and F.*

Type 4. *To find nature of image and type of mirror corresponding to given optic axis of mirror a point object and a point image. With the help of ray diagram we have to find focus and pole of the mirror also.*

Example 5 *An image I is formed of a point object O by a mirror whose principal axis is AB as shown in figure.*



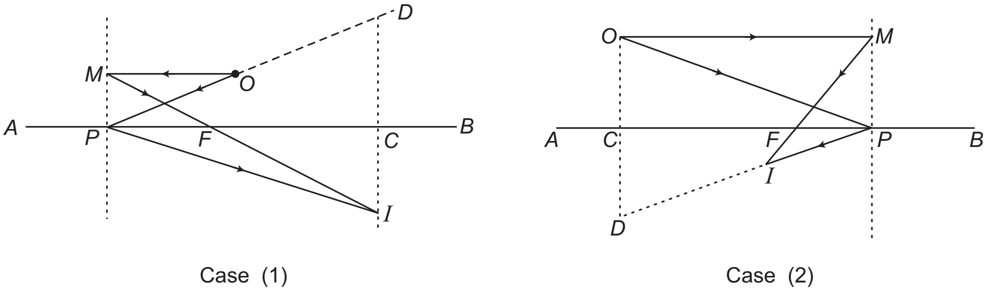
(a) *State whether it is a convex mirror or a concave mirror.*

(b) *Draw a ray diagram to locate the mirror and its focus. Write down the steps of construction of the ray diagram. Consider the possible two cases:*

(1) *When distance of I from AB is more than the distance of O from AB and*

(2) *When distance of O from AB is more than the distance of I from AB*

Solution



Case (1) Case (2)

(a) As the image is on the opposite side of the principal axis, the mirror is concave. Because convex mirror always forms an erect image.

(b) Two different cases are shown in figure. Steps are as under :

(i) From I or O drop a perpendicular on principal axis, such that $CI = CD$ or $OC = CD$.

(ii) Draw a line joining D and O or D and I so that it meets the principal axis at P. The point P will be the pole of the mirror as a ray reflected from the pole is always symmetrical about principal axis.

(iii) From O draw a line parallel to principal axis towards the mirror so that it meets the mirror at M. Join M to I, so that it intersects the principal axis at F. F is the focus of the mirror as any ray parallel to principal axis after reflection from the mirror intersects the principal axis at the focus.

Note *In both figures, mirror should face towards the object.*

Exercise In the above problem, find centre of curvature of the mirror with the help of only ray diagram.

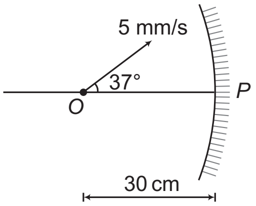
Type 5. *Based on velocity of image*

How to Solve?

- Using the steps discussed in Type 1, find v and then m
- Along the axis $\mathbf{v}_I = -m^2 \mathbf{v}_O$
- Perpendicular to axis $\mathbf{v}_I = m\mathbf{v}_O$

Here, m has to be substituted with sign.

Example 6 *Focal length of the mirror shown in figure is 20 cm. Find the image position and its velocity.*



Solution Substituting the values,

$$u = -30 \text{ cm and } f = -20 \text{ cm}$$

in the mirror formula, we have

$$\frac{1}{v} + \frac{1}{-30} = \frac{1}{-20}$$

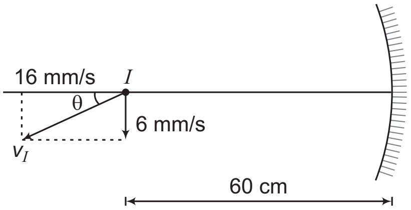
Solving this equation, we get

$$v = -60 \text{ cm}$$

Further, $$m = -\frac{v}{u} = -\frac{(-60)}{(-30)} = -2$$

and $$m^2 = 4$$

Object velocity along the axis is 5 cos 37° = 4 mm/s (towards OP). Therefore, image velocity along the axis should be m^2 times or 16 mm/s in the opposite direction of object velocity. Object velocity perpendicular to axis is 5 sin 37° = 3 mm/s (upwards). Therefore, image velocity will be m times or –6 mm/s downwards. The position and velocity of image is shown below.



$$v_I = \sqrt{(16)^2 + (6)^2} = \sqrt{292} \text{ mm/s}$$

$$\tan\theta = \frac{6}{16} = \frac{3}{8} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{3}{8}\right)$$

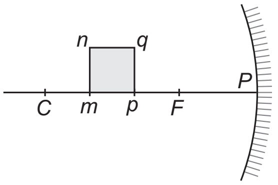
Type 6. *To find a rough image of a square or rectangular type of object kept along the axis*

Concept

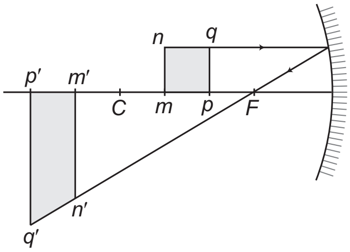
(i) If object is towards P or C, then its image is also towards P or C.

(ii) If object is towards F, then its image is towards infinity and it is more magnified.

Example 7 *A square mnqp is kept between F and C on the principal axis of a concave mirror as shown in figure. Find a rough image of this object.*



Solution Object is placed between F and C, therefore, image is real, inverted magnified and beyond C. Further pq is towards F. Therefore, its image $p'q'$ is towards infinity and it is more magnified. A rough image is as shown alongside.



From the ray shown in figure, we can see that $n'q'$ will be a straight line.

Exercise In the given problem, focal length of mirror is 30 cm and side of square is 10 cm with $pP = 40$ cm. Find perimeter of the image

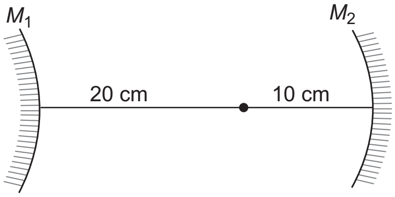
Ans $(90 + 15\sqrt{10})$ cm

Type 7. *Two mirror problems*

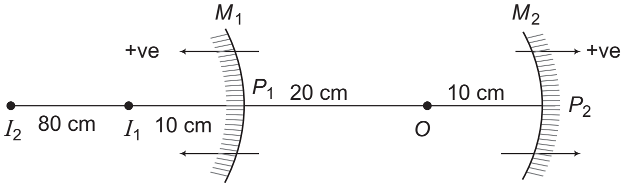
Concept

If an object is placed between two mirrors, then infinite reflections will take place. Therefore, infinite images are formed. But normally position of second image is asked. So, we have to apply mirror formula two times. Image from first mirror acts as an object for the second mirror. Sign convention for second reflection will change but sign of focal length will not change.

Example 8 *Focal length of convex mirror M_1 is 20 cm and that of concave mirror M_2 is 30 cm. Find position of second image I_2. Take first reflection from M_1.*



Solution



For M_1

$$u_1 = -20 \text{ cm}$$
$$f_1 = +20 \text{ cm}$$

Using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}, \text{ we have}$$

$$\frac{1}{v_1} + \frac{1}{-20} = \frac{1}{+20}$$

Solving, we get $v_1 = +10$ cm

For M_2 $u_2 = -40$ cm ($PI_1 = 40$ cm)

$$f_2 = -30 \text{ cm}$$

Using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{v_2} + \frac{1}{-40} = \frac{1}{-30}$$

$\therefore$ $v_2 = P_2I_2 = -120$ cm

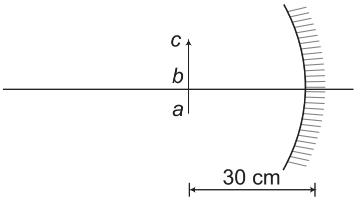
Note *I_1 is virtual from M_1 point of view (behind M_1). But it behaves like a real object for M_2 (in front of M_2).*

Type 8. *An extended object is kept perpendicular to principal axis and we have to make its image*

How to Solve?

With the help of type 1 first find v and then m. Now, suppose $m = -2$ and object is 2 mm above the principal axis, then its image will be formed 4 mm below its principal axis.

Example 9



Focal length of the concave mirror shown in figure is 20 cm.

$$ab = 1 \text{ mm}$$

and $$bc = 2 \text{ mm}$$

For the given situation, make its image

Solution $u = -30$ cm and $f = -20$ cm

Using the mirror formula, $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$, we have

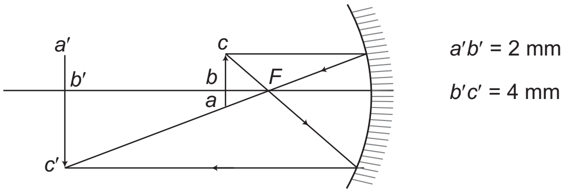
$$\frac{1}{v} + \frac{1}{-30} = \frac{1}{-20}$$

Solving this equation, we get

$$v = -60 \text{ cm}$$

Now, $m = -\frac{v}{u} = -\frac{(-60)}{(-30)} = -2$

Now c point is 2 mm above the principal axis and magnification is –2. Hence, image c' will be formed 4 mm below the principal axis. Similarly a point is 1 mm below the principal axis and value of m is –2. Hence, image a' is formed 2 mm above the principal axis. Image with ray diagram of c is as shown below.



Type 9. *An extended object is kept along the principal axis. Now, in this type further two cases are possible.*

Case 1 **When object size is very small.**

In this case image length,

$$L_I = m^2 (L_o)$$

Here, L_o is the object length. So, using type 1, we have to find v and then m.

Proof We have already proved that image speed.

$$v_I = m^2 v_O \qquad \text{(Along the axis)}$$

or

$$\frac{dv}{dt} = m^2 \left(\frac{du}{dt}\right)$$

For small change in the values of v and u, we can write

$$|\Delta v| = |m^2 \times \Delta u|$$

Δv is nothing but difference in two values of v or image length L_I. Similarly, Δu is object length L_o.

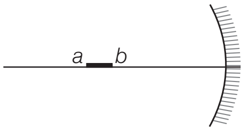
Case 2 **When object size is large.**

If an extended object is lying along the principal axis, then we will get two values of u corresponding to its two ends. Now, apply mirror formula two times and find two values of v. The image length now becomes,

$$L_I = |v_1 \sim v_2|$$

Note *If one end of the object is placed either at C or P, then its image will also be formed at C or P. So, we will have to apply the mirror formula only for the other end.*

Example 10



A small object ab of size 1 mm is kept at a distance of 40 cm from a concave mirror of focal length 30 cm. Make image of this object.

Solution Here, object size is very small. So, this is case 1.

$$u = -40 \text{ cm}$$

and $$f = -30 \text{ cm}$$

Applying the mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}, \text{ we have}$$

$$\frac{1}{v} + \frac{1}{-40} = \frac{1}{-30}$$

Solving this equation, we get

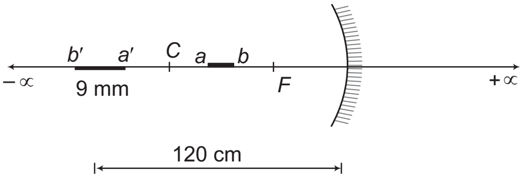
$$v = -120 \text{ cm}$$

Further, $$m = -\frac{v}{u} = -\frac{(-120)}{(-40)} = -3$$

$$m^2 = 9$$

Now, $$L_I = m^2 L_o = (9)\,(1 \text{ mm}) = 9 \text{ mm}$$

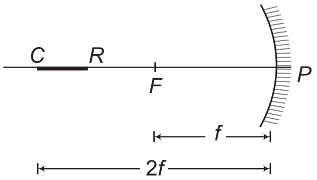
Image diagram is as shown below.



Note *Point b is towards F, therefore its image b′ should be towards − ∝.*

Example 11 *A thin rod of length $\frac{f}{3}$ is lying along the principal axis of a concave mirror of focal length f. Image is real, magnified and inverted and one of the end of rod coincides with its image itself. Find length of the image.*

Solution Image is real, magnified and inverted. So, the given rod lies between F and C. Further, one end of the rod is coinciding with its image itself. Therefore, it is lying at C. So, the thin rod CR is kept as shown below.



$$CR = \frac{f}{3}$$

$$PR = 2f - \frac{f}{3}$$

$$= \frac{5f}{3}$$

We have to apply mirror formula only for point R.

$$u = -\frac{5f}{3}, \text{ focal length} = -f$$

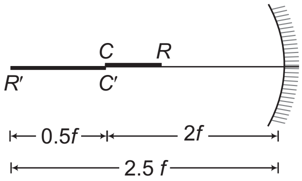
Using the mirror formula $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$,

We have, $\frac{1}{v} + \frac{1}{-\frac{5f}{3}} = \frac{1}{-f}$

Solving this equation, we get

$$v = \frac{-5f}{2} \quad \text{or} \quad -2.5f$$

So, the image of rod CR is $C'R'$ as shown below.



So, image length $= C'R' = 0.5\,f$ or $\frac{f}{2}$.

Note *In this problem if magnification of rod is asked then we can write*

$$m = -\frac{L_I}{L_O} = -\frac{f/2}{f/3} = -\frac{3}{2}$$

Negative sign has been used for inverted image.

Type 10. *Based on u versus v graph or 1/u versus 1/v graph (only for real objects)*

Example 12 *Draw u versus v graph or* $\frac{1}{u}$ *versus* $\frac{1}{v}$ *graph for a concave mirror of focal length f.*

Solution The mirror formula is

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

If we take $\frac{1}{v}$ along y-axis and $\frac{1}{u}$ along x-axis, then the above equation becomes

$$y + x = c \qquad \left(\text{as } \frac{1}{f} = \text{constant}\right)$$

Therefore, $\frac{1}{v}$ *versus* $\frac{1}{u}$ graph will be a straight line. Let us take origin at pole.

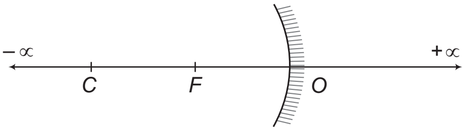
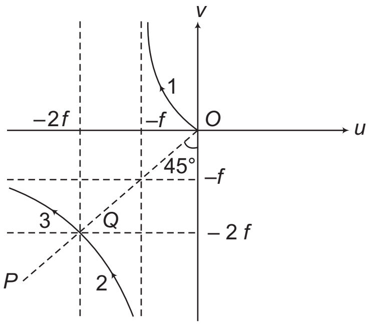


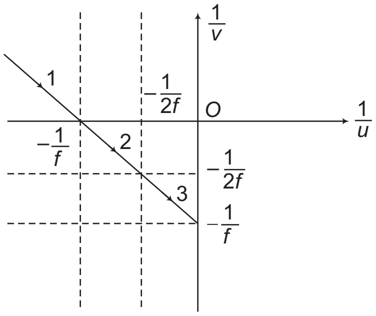
Table 30.5

S.No.	u	v	$\frac{1}{u}$	$\frac{1}{v}$
1.	0 to $-f$	-0 to $+\infty$	$-\infty$ to $-\frac{1}{f}$	$+\infty$ to 0
2.	$-f$ to $-2f$	$-\infty$ to $-2f$	$-\frac{1}{f}$ to $-\frac{1}{2f}$	0 to $-\frac{1}{2f}$
3.	$-2f$ to $-\infty$	$-2f$ to $-f$	$-\frac{1}{2f}$ to 0	$-\frac{1}{2f}$ to $-\frac{1}{f}$

u *versus* v and $\frac{1}{u}$ *versus* $\frac{1}{v}$ graphs are as shown below.



Note *OP line cuts the u-v graph at Q (−2f, −2f)*



Exercise Draw above two graphs for convex mirror.

Miscellaneous Examples

Example 13 *An object is 30.0 cm from a spherical mirror, along the central axis. The absolute value of lateral magnification is* $\frac{1}{2}$*. The image produced is inverted. What is the focal length of the mirror?*

Solution Image is inverted, so it is real. u and v both are negative. Magnification is $\frac{1}{2}$, therefore, $v = \frac{u}{2}$.

Given, $u = -30$ cm, $v = -15$ cm

Using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{f} = \frac{1}{-15} - \frac{1}{30} = -\frac{1}{10}$$

$\therefore$ $f = -10$ cm **Ans.**

Example 14 *A concave mirror has a radius of curvature of 24 cm. How far is an object from the mirror if an image is formed that is :*

(a) virtual and 3.0 times the size of the object,

(b) real and 3.0 times the size of the object and

(c) real and 1/3 the size of the object?

Solution Given, $R = -24$ cm (concave mirror)

Hence,

$$f = \frac{R}{2} = -12 \text{ cm}$$

(a) Image is virtual and 3 times larger. Hence, u is negative and v is positive. Simultaneously, $|v| = 3|u|$. So let,

$$u = -x$$

then

$$v = +3x$$

Substituting in the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{3x} - \frac{1}{x} = \frac{1}{-12}$$

$\therefore$ $x = 8$ cm

Therefore, object distance is 8 cm. **Ans.**

(b) Image is real and three times larger. Hence, u and v both are negative and $|v| = 3|u|$.

So let,

$$u = -x$$

then $v = -3x$

Substituting in mirror formula, we have

$$\frac{1}{-3x} - \frac{1}{x} = -\frac{1}{12}$$

or $x = 16$ cm

$\therefore$ Object distance should be 16 cm. **Ans.**

(c) Image is real and $\frac{1}{3}$rd the size of object. Hence, both u and v are negative and $|v| = \frac{|u|}{3}$.

So let, $u = -x$

then $v = -\frac{x}{3}$

Substituting in the mirror formula, we have

$$-\frac{3}{x} - \frac{1}{x} = -\frac{1}{12}$$

$\therefore$ $x = 48$ cm

$\therefore$ Object distance should be 48 cm. **Ans.**

Example 15 *A ray of light is incident on a plane mirror along a vector* $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$. *The normal on incidence point is along* $\hat{\mathbf{i}} + \hat{\mathbf{j}}$. *Find a unit vector along the reflected ray.*

Solution Reflection of a ray of light is just like an elastic collision of a ball with a horizontal ground. Component of incident ray along the inside normal gets reversed while the component perpendicular to it remains unchanged. Thus, the component of incident ray vector $\mathbf{A} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ parallel to normal, i.e. $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ gets reversed while perpendicular to it, i.e. $-\hat{\mathbf{k}}$ remains unchanged. Thus, the reflected ray can be written as

$$\mathbf{R} = -\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$\therefore$ A unit vector along the reflected ray will be

$$\hat{\mathbf{r}} = \frac{\mathbf{R}}{R} = \frac{-\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{3}}$$

or $$\hat{\mathbf{r}} = -\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$ **Ans.**

Note *In this problem, given normal is inside the mirror surface. Think why?*

Example 16 *A gun of mass* m_1 *fires a bullet of mass* m_2 *with a horizontal speed* v_0. *The gun is fitted with a concave mirror of focal length* f *facing towards a receding bullet. Find the speed of separations of the bullet and the image just after the gun was fired.*

Solution Let v_1 be the speed of gun (or mirror) just after the firing of bullet. From conservation of linear momentum,

$$m_2 v_0 = m_1 v_1$$

or $$v_1 = \frac{m_2 v_0}{m_1} \quad \text{...(i)}$$

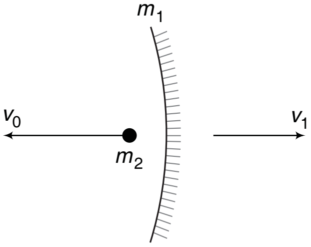
Now, $\frac{du}{dt}$ is the rate at which distance between mirror and bullet is increasing $= v_1 + v_0$ ...(ii)

We have already read in extra points that :

$$\therefore \qquad \frac{dv}{dt} = \left(\frac{v^2}{u^2}\right)\frac{du}{dt}$$

Here, $\frac{v^2}{u^2} = m^2 = 1$ (as at the time of firing, bullet is at pole).

$$\therefore \qquad \frac{dv}{dt} = \frac{du}{dt} = v_1 + v_0 \qquad \text{...(iii)}$$



Here, $\frac{dv}{dt}$ is the rate at which distance between image (of bullet) and mirror is increasing. So, if v_2 is the absolute velocity of image (towards right), then

$$v_2 - v_1 = \frac{dv}{dt}$$

$$= v_1 + v_0$$

or $$v_2 = 2v_1 + v_0 \qquad \text{...(iv)}$$

Therefore, speed of separation of bullet and image will be

$$v_r = v_2 + v_0$$

$$= 2v_1 + v_0 + v_0$$

or $$v_r = 2\,(v_1 + v_0)$$

Substituting value of v_1 from Eq. (i), we have

$$v_r = 2\left(1 + \frac{m_2}{m_1}\right)v_0$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. **Assertion :** A convex mirror can never make a real image.

Reason : For all real objects image formed by a convex mirror is virtual.

2. **Assertion :** Focal length of a convex mirror is 20 cm. If a real object is placed at distance 20 cm from the mirror, its virtual erect and diminished image will be formed.

Reason : If a virtual object is placed at 20 cm distance, its image is formed at infinity.

3. **Assertion :** In case of a concave mirror if a point object is moving towards the mirror along its principal axis, then its image will move away from the mirror.

Reason : In case of reflection (along the principal axis of mirror) object and image always travel in opposite directions.

4. **Assertion :** Real view mirror of vehicles is a convex mirror.

Reason : It never makes real image of real objects.

5. **Assertion :** If magnification of a real object is -2. Then, it is definitely a concave mirror.

Reason : Only concave mirror can make real images of real objects.

6. **Assertion :** Any ray of light suffers a deviation of $(180°-2i)$ after one reflection.

Reason : For normal incidence of light deviation is zero.

7. **Assertion :** Two plane mirrors kept at right angles deviate any ray of light by 180° after two reflections.

Reason : The above condition is satisfied only for angle of incidence $i = 45°$.

8. **Assertion :** In reflection from a denser medium, any ray of light suffers a phase difference of π.

Reason : Denser medium is that medium in which speed of wave is less.

9. **Assertion :** For real objects, image formed by a convex mirror always lies between pole and focus.

Reason : When object moves from pole to infinity, its image will move from pole to focus.

10. **Assertion :** Light converges on a virtual object.

Reason : Virtual object is always behind a mirror.

Objective Questions

1. A plane mirror reflects a beam of light to form a real image. The incident beam should be

(a) parallel (b) convergent

(c) divergent (d) not possible

2. When an object lies at the focus of a concave mirror, then the position of the image formed and its magnification are

(a) pole and unity (b) infinity and unity

(c) infinity and infinity (d) centre of curvature and unity

3. Two plane mirrors are inclined to each other at 90°. A ray of light is incident on one mirror. The ray will undergo a total deviation of

(a) 180° (b) 90°

(c) 45° (d) Data insufficient

4. A concave mirror cannot form

(a) virtual image of virtual object

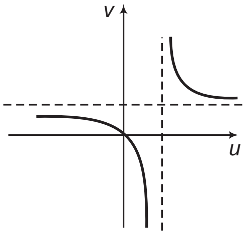
(b) virtual image of real object

(c) real image of real object

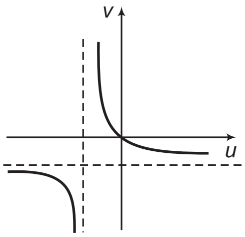
(d) real image of virtual object

5. Which of the following is correct graph between *u* and *v* for a concave mirror for normal sign convention?

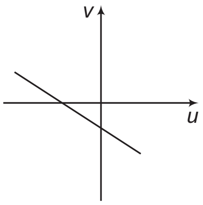
(a)



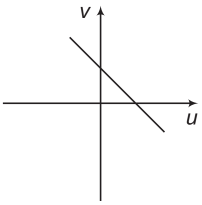
(b)



(c)



(d)



6. Two plane mirrors are inclined at 70°. A ray incident on one mirror at incidence angle θ, after reflection falls on the second mirror and is reflected from there parallel to the first mirror. The value of θ is

(a) 50° (b) 45°

(c) 30° (d) 25°

7. The radius of curvature of a convex mirror is 60 cm. When an object is placed at *A*, its image is formed at *B*. If the size of image is half that of the object, then the distance between *A* and *B* is

(a) 30 cm (b) 60 cm

(c) 45 cm (d) 90 cm

8. A boy of height 1.5 m with his eye level at 1.4 m stands before a plane mirror of length 0.75 m fixed on the wall. The height of the lower edge of the mirror above the floor is 0.8 m. Then,

(a) the boy will see his full image
(b) the boy cannot see his hair
(c) the boy cannot see his feet
(d) the boy can see neither his hair nor his feet

9. A spherical mirror forms an erect image three times the size of the object. If the distance between the object and the image is 80 cm, the nature and the focal length of the mirror are

(a) concave, 30 cm
(b) convex, 30 cm
(c) concave, 15 cm
(d) convex, 15 cm

10. A convex mirror of focal length f produces an image $(1/n)^{\text{th}}$ of the size of the object. The distance of the object from the mirror is

(a) nf
(b) f/n
(c) $(n+1)f$
(d) $(n-1)f$

11. An object is moving towards a concave mirror of focal length 24 cm. When it is at a distance of 60 cm from the mirror, its speed is 9 cm/s. The speed of its image at that instant, is

(a) 4 cm/s towards the mirror
(b) 6 cm/s towards the mirror
(c) 4 cm/s away from the mirror
(d) 6 cm/s away from the mirror

12. All the following statements are correct except (for real objects)

(a) the magnification produced by a convex mirror is always less than one
(b) a virtual, erect and same sized image can be obtained using a plane mirror
(c) a virtual, erect, magnified image can be formed using a concave mirror
(d) a real, inverted same sized image can be formed using a convex mirror

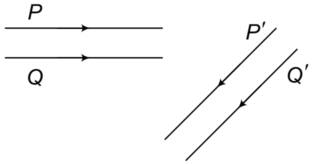
13. A particle moves perpendicularly towards a plane mirror with a constant speed of 4 cm/s. What is the speed of the image observed by an observer moving with 2 cm/s along the same direction? Mirror is also moving with a speed of 10 cm/s in the opposite direction.
(All speeds are with respect to ground frame of reference)

(a) 4 cm/s
(b) 12 cm/s
(c) 14 cm/s
(d) 26 cm/s

Subjective Questions

Note *You can take approximations in the answers.*

1. Figure shows two rays P and Q being reflected by a mirror and going as P' and Q'. State which type of mirror is this?



2. A candle 4.85 cm tall is 39.2 cm to the left of a plane mirror. Where does the mirror form the image, and what is the height of this image?

3. A plane mirror lies face up, making an angle of 15° with the horizontal. A ray of light shines down vertically on the mirror. What is the angle of incidence? What will be the angle between the reflected ray and the horizontal?

4. Two plane mirrors are placed parallel to each other and 40 cm apart. An object is placed 10 cm from one mirror. What is the distance from the object to the image for each of the five images that are closest to the object?

5. If an object is placed between two parallel mirrors, an infinite number of images result. Suppose that the mirrors are a distance $2b$ apart and the object is put at the mid-point between the mirrors. Find the distances of the images from the object.

6. Show that a ray of light reflected from a plane mirror rotates through an angle 2θ when the mirror is rotated through an angle θ about its axis perpendicular to both the incident ray and the normal to the surface.

7. Two plane mirrors each 1.6 m long, are facing each other. The distance between the mirrors is 20 cm. A light incident on one end of one of the mirrors at an angle of incidence of 30°. How many times is the ray reflected before it reaches the other end?

8. Two plane mirrors are inclined to each other at an angle θ. A ray of light is reflected first at one mirror and then at the other. Find the total deviation of the ray.

9. Assume that a certain spherical mirror has a focal length of – 10.0 cm. Locate and describe the image for object distances of (a) 25.0 cm (b) 10.0 cm (c) 5.0 cm.

10. A ball is dropped from rest 3.0 m directly above the vertex of a concave mirror that has a radius of 1.0 m and lies in a horizontal plane.

(a) Describe the motion of ball's image in the mirror.

(b) At what time do the ball and its image coincide?

11. An object 6.0 mm is placed 16.5 cm to the left of the vertex of a concave spherical mirror having a radius of curvature of 22.0 cm.

(a) Draw principal ray diagram showing formation of the image.

(b) Determine the position, size, orientation, and nature (real or virtual) of the image.

12. An object 9.0 mm tall is placed 12.0 cm to the left of the vertex of a convex spherical mirror whose radius of curvature has a magnitude of 20.0 cm.

(a) Draw a principal ray diagram showing formation of the image.

(b) Determine the position, size, orientation, and nature (real or virtual) of the image.

13. How far should an object be from a concave spherical mirror of radius 36 cm to form a real image one-ninth its size?

14. As the position of an object in front of a concave spherical mirror of 0.25 m focal length is varied, the position of the image varies. Plot the image distance as a function of the object distance letting the later change from 0 to $+\infty$. Where is the image real? Where virtual?

15. An object is placed 42 cm, in front of a concave mirror of focal length 21 cm. Light from the concave mirror is reflected onto a small plane mirror 21 cm in front of the concave mirror. Where is the final image?

16. Prove that for spherical mirrors the product of the distance of the object and the image to the principal focus is always equal to the square of the principal focal length.

17. Convex and concave mirrors have the same radii of curvature R. The distance between the mirrors is $2R$. At what point on the common optical axis of the mirrors should a point source of light A be placed for the rays to coverage at the point A after being reflected first on the convex and then on the concave mirror?

18. A spherical mirror is to be used to form on a screen 5.0 m from the object an image five times the size of the object.

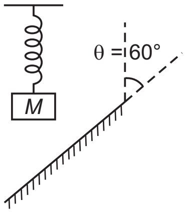
(a) Describe the type of mirror required.

(b) Where should the mirror be positioned relative to the object?

LEVEL 2

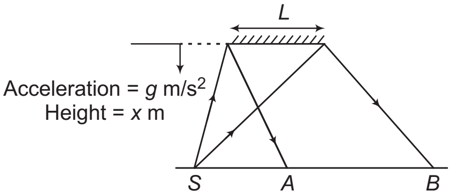
Single Correct Option

1. An insect of negligible mass is sitting on a block of mass M, tied with a spring of force constant k. The block performs simple harmonic motion with amplitude A in front of a plane mirror as shown. The maximum speed of insect relative to its image will be



(a) $A\sqrt{\frac{k}{M}}$ (b) $\frac{A\sqrt{3}}{2}\sqrt{\frac{k}{M}}$ (c) $A\sqrt{3}\sqrt{\frac{k}{M}}$ (d) $2A\sqrt{\frac{M}{k}}$

2. A plane mirror is falling vertically as shown in the figure. If S is a point source of light, the rate of increase of the length AB is

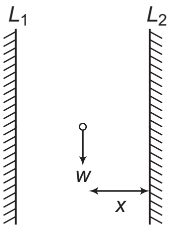


(a) directly proportional to x
(b) constant but not zero
(c) inversely proportional to x
(d) zero

3. A point object is placed at a distance of 10 cm and its real image is formed at a distance of 20 cm from a concave mirror. If the object is moved by 0.1 cm towards the mirror. The image will shift by about

(a) 0.4 cm away from the mirror
(b) 0.4 cm towards the mirror
(c) 0.8 cm away from the mirror
(d) 0.8 cm towards the mirror

4. Two plane mirrors L_1 and L_2 are parallel to each other and 3 m apart. A person standing x m from the right mirror L_2 looks into this mirror and sees a series of images. The distance between the first and second image is 4 m. Then, the value of x is



(a) 2 m (b) 1.5 m (c) 1 m (d) 2.5 m

5. A piece of wire bent into an L shape with upright and horizontal portion of equal lengths 10 cm each is placed with the horizontal portion along the axis of the concave mirror towards pole of mirror whose radius of curvature is 10 cm. If the bend is 20 cm from the pole of the mirror, then the ratio of the lengths of the images of the upright and horizontal portion of the wire is

(a) 1 : 2
(b) 1 : 3
(c) 1 : 1
(d) 2 : 1

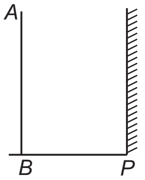
6. A point object at 15 cm from a concave mirror of radius of curvature 20 cm is made to oscillate along the principal axis with amplitude 2 mm. The amplitude of its image will be

(a) 2 mm
(b) 4 mm
(c) 8 mm
(d) None of these

7. A ray of light falls on a plane mirror. When the mirror is turned, about an axis at right angles to the plane of mirror by 20° the angle between the incident ray and new reflected ray is 45°. The angle between the incident ray and original reflected ray was therefore

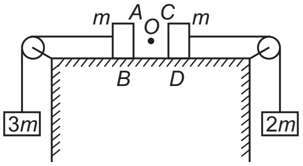
(a) 35° or 50°
(b) 25° or 65°
(c) 45° or 5°
(d) None of these

8. A person AB of height 170 cm is standing in front of a plane mirror. His eyes are at height 164 cm. At what distance from P should a hole be made in mirror so that he cannot see his hair?



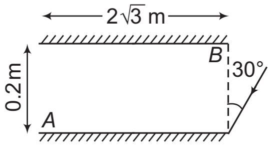
(a) 167 cm
(b) 161 cm
(c) 163 cm
(d) 165 cm

9. Two blocks each of masses m lie on a smooth table. They are attached to two other masses as shown in figure. The pulleys and strings are light. An object O is kept at rest on the table. The sides AB and CD of the two blocks are made reflecting. The acceleration of two images formed in those two reflecting surfaces with respect to each other is



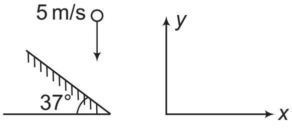
(a) $\frac{5g}{6}$
(b) $\frac{5g}{3}$
(c) $\frac{17g}{12}$
(d) $\frac{17g}{6}$

10. Two plane mirrors A and B are aligned parallel to each other as shown in the figure. A light ray is incident at an angle of 30° at a point just inside one end of A. The number of times the ray undergoes reflections (including the first one) before it emerges out is

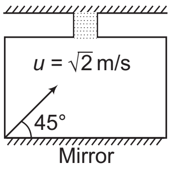


(a) 29
(b) 30
(c) 31
(d) 32

11. An object O is just about to strike a perfectly reflecting inclined plane of inclination 37°. Its velocity is 5 m/s. Find the velocity of its image.



(a) $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$
(b) $4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
(c) $4.8\,\hat{\mathbf{i}} + 1.4\hat{\mathbf{j}}$
(d) $1.4\,\hat{\mathbf{i}} + 4.8\hat{\mathbf{j}}$



12. An elevator at rest which is at 10th floor of a building is having a plane mirror fixed to its floor. A particle is projected with a speed $\sqrt{2}$ m/s and at 45° with the horizontal as shown in the figure. At the very instant of projection, the cable of the elevator breaks and the elevator starts falling freely. What will be the separation between the particles and is image 0.5 s after the instant of projection?

(a) 0.5 m (b) 1 m

(c) 2 m (d) 1.5 m

13. A plane mirror is moving with velocity $4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$. A point object in front of the mirror moves with a velocity $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$. Here, $\hat{\mathbf{k}}$ is along the normal to the plane mirror and facing towards the object. The velocity of the image is

(a) $-3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$

(c) $-4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$ (d) $7\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

14. Point A (0, 1 cm) and B (12 cm, 5 cm) are the coordinates of object and image. x-axis is the principal axis of the mirror. Then, this object image pair is

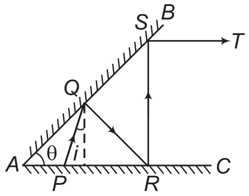
(a) due to a convex mirror of focal length 2.5 cm

(b) due to a concave mirror having its pole at (2 cm, 0)

(c) due to a concave mirror having its pole at (– 2 cm, 0)

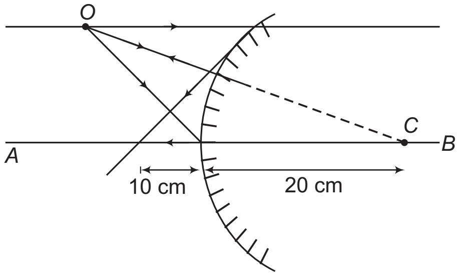
(d) Data is insufficient

15. Two plane mirrors AB and AC are inclined at an angle $\theta = 20°$. A ray of light starting from point P is incident at point Q on the mirror AB, then at R on mirror AC and again on S on AB. Finally, the ray ST goes parallel to mirror AC. The angle which the ray makes with the normal at point Q on mirror AB is



(a) 20° (b) 30° (c) 40° (d) 60°

16. A convex mirror of radius of curvature 20 cm is shown in figure. An object O is placed in front of this mirror. Its ray diagram is shown. How many mistakes are there in the ray diagram (AB is principal axis)



(a) 3 (b) 2 (c) 1 (d) 0

More than One Correct Options

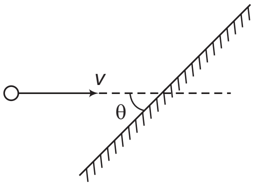
1. The image formed by a concave mirror is twice the size of the object. The focal length of the mirror is 20 cm. The distance of the object from the mirror is/are

(a) 10 cm
(b) 30 cm
(c) 25 cm
(d) 15 cm

2. Magnitude of focal length of a spherical mirror is f and magnitude of linear magnification is $\frac{1}{2}$.

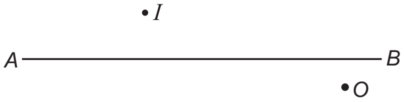
(a) If image is inverted, it is a concave mirror
(b) If image is erect, it is a convex mirror
(c) Object distance from the mirror may be $3f$
(d) Object distance from the mirror may be f

3. A point object is moving towards a plane mirror as shown in figure. Choose the correct options.



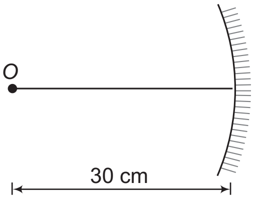
(a) Speed of image is also v
(b) Image velocity will also make an angle θ with mirror
(c) Relative velocity between object and image is $2v$
(d) Relative velocity between object and image is $2v \sin \theta$

4. AB is the principal axis of a spherical mirror. I is the point image corresponding to a point object O. Choose the correct options.



(a) Mirror is lying to the right hand side of O
(b) Focus of mirror is lying to the right hand side of O
(c) Centre of curvature of mirror is lying to the right hand side of O
(d) Centre of curvature of mirror is lying between I and O

5. A point object is placed on the principal axis of a concave mirror of focal length 20 cm. At this instant object is given a velocity v towards the axis (event-1) or perpendicular to axis (event-2). Then, speed of image



(a) In event-1 is $2v$
(b) In event-1 is $4v$
(c) In event-2 is $2v$
(d) In event-2 is $4v$

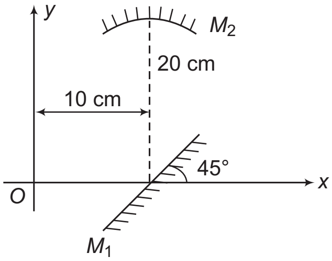
6. A point object is placed at equal distance $3f$ in front of a concave mirror, a convex mirror and a plane mirror separately (event-1). Now, the distance is decreased to $1.5\,f$ from all the three mirrors (event-2). Magnitude of focal length of convex mirror and concave mirror is f. Then, choose the correct options.

(a) Maximum distance of object in event-1 from the mirror is from plane mirror
(b) Minimum distance of object in event-1 from the mirror is from convex mirror
(c) Maximum distance of object in event-2 from the mirror is from concave mirror
(d) Minimum distance of object in event-2 from the mirror is from plane mirror

Comprehension Based Questions

Passage : (Q. No. 1 to 4)

A plane mirror (M_1) and a concave mirror (M_2) of focal length 10 cm *are arranged as shown in figure. An object is kept at origin. Answer the following questions. (consider image formed by single reflection in all cases)*



1. The coordinates of image formed by plane mirror are

(a) (– 20 cm, 0) (b) (10 cm, – 60 cm) (c) (10 cm, – 10 cm) (d) (10 cm, 10 cm)

2. The coordinates of image formed by concave mirror are

(a) (10 cm, – 40 cm) (b) (10 cm, – 60 cm) (c) (10 cm, 8 cm) (d) None of these

3. If concave mirror is replaced by convex mirror of same focal length, then coordinates of image formed by M_2 will be

(a) (10 cm, 12 cm) (b) (10 cm, 22 cm) (c) (10 cm, 8 cm) (d) None of these

4. If concave mirror is replaced by another plane mirror parallel to x-axis, then coordinates of image formed by M_2 are

(a) (40 cm, 20 cm) (b) (20 cm, 40 cm) (c) (– 20 cm, 20 cm) (d) None of these

Match the Columns

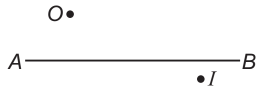
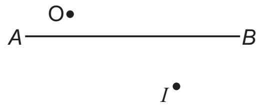
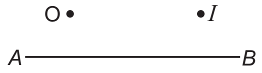
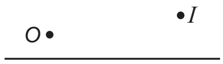
1. For real objects, match the following two columns corresponding to linear magnification m given in Column I.

Column I	Column II
(a) $m = -2$	(p) convex mirror
(b) $m = -\frac{1}{2}$	(q) concave mirror
(c) $m = +2$	(r) real image
(d) $m = +\frac{1}{2}$	(s) virtual image

2. For virtual objects, match the following two columns.

Column I	Column II
(a) Plane mirror	(p) only real image
(b) Convex mirror	(q) only virtual image
(c) Concave mirror	(r) may be real or virtual image

3. Principal axis of a mirror (AB), a point object O and its image I are shown in Column I, match it with Column II.

Column I	Column II
(a) 	(p) plane mirror
(b) 	(q) convex mirror
(c) 	(r) concave mirror
(d) 	(s) Not possible

4. Focal length of a concave mirror M_1 is – 20 cm and focal length of a convex mirror M_2 is + 20 cm. A point object is placed at a distance X in front of M_1 or M_2. Match the following two columns.

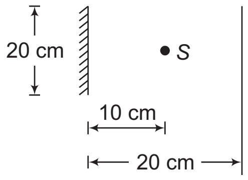
Column I	Column II
(a) $X = 20$, mirror is M_1	(p) image is at infinity
(b) $X = 20$, mirror is M_2	(q) image is real
(c) $X = 30$, mirror is M_1	(r) image is virtual
(d) $X = 30$, mirror is M_2	(s) image is magnified

5. Focal length of a concave mirror is –20 cm. Match the object distance given in Column II corresponding to magnification (only magnitude) given is Column I.

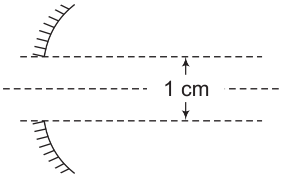
Column I	Column II
(a) 2	(p) 10 cm
(b) 1/2	(q) 30 cm
(c) 1	(r) 20 cm
(d) 1/4	(s) None of these

Subjective Questions

1. A point source of light S is placed at a distance 10 cm in front of the centre of a mirror of width 20 cm suspended vertically on a wall. An insect walks with a speed 10 cm/s in front of the mirror along a line parallel to the mirror at a distance 20 cm from it as shown in figure. Find the maximum time during which the insect can see the image of the source S in the mirror.

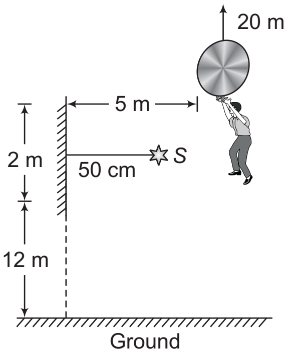


2. A concave mirror forms the real image of a point source lying on the optical axis at a distance of 50 cm from the mirror. The focal length of the mirror is 25 cm. The mirror is cut into and its halves are drawn a distance of 1 cm apart in a direction perpendicular to the optical axis. How will the image formed by the halves of the mirror be arranged?

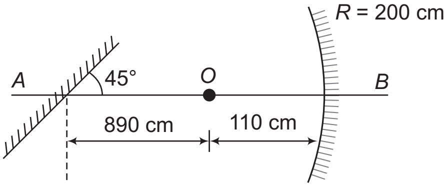


3. A point source of light S is placed on the major optical axis of the concave mirror at a distance of 60 cm. At what distance from the concave mirror should a flat mirror be placed for the rays to converge again at the point S having been reflected from the concave mirror and then from the flat one? Will the position of the point where the rays meet change if they are first reflected from the flat mirror? The radius of the concave mirror is 80 cm.

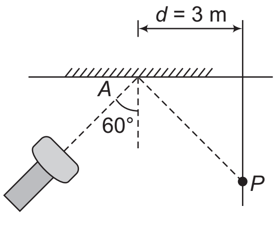
4. A balloon is moving upwards with a speed of 20 m/ s. When it is at a height of 14 m from ground in front of a plane mirror in situation as shown in figure, a boy drops himself from the balloon. Find the time duration for which he will see the image of source S placed symmetrically before plane mirror during free fall.



5. A plane mirror and a concave mirror are arranged as shown in figure and O is a point object. Find the position of image formed by two reflections, first one taking place at concave mirror.



6. Figure shows a torch producing a straight light beam falling on a plane mirror at an angle 60°. The reflected beam makes a spot P on the screen along y-axis. If at $t = 0$, mirror starts rotating about the hinge A with an angular velocity $\omega = 1°$ per second clockwise. Find the speed of the spot on screen after time $t = 15$ s.



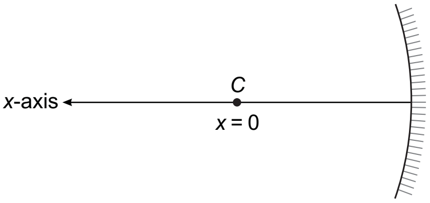
7. A thief is running away in a car with velocity of 20 m/s. A police jeep is following him, which is sighted by thief in his rear view mirror, which is a convex mirror of focal length 10 m. He observes that the image of jeep is moving towards him with a velocity of 1 cm/s. If the magnification of mirror for the jeep at that time is $\frac{1}{10}$. Find

(a) the actual speed of jeep,

(b) rate at which magnification is changing.

Assume the police's jeep is on the axis of the mirror.

8. A ball swings back and forth in front of a concave mirror. The motion of the ball is described approximately by the equation $x = f \cos \omega t$, where f is the focal length of the mirror and x is measured along the axis of mirror. The origin is taken at the centre of curvature of the mirror.

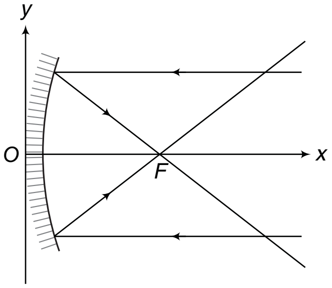


(a) Derive an expression for the distance from the mirror of the image of the swinging ball.

(b) At what point does the ball appear to coincide with its image?

(c) What will be the lateral magnification of the image of the ball at time $t = \frac{T}{2}$, where T is time period of oscillation?

9. Show that a parallel bundle of light rays parallel to the x-axis and incident on a parabolic reflecting surface given by $x = 2\,by^2$, will pass through a single point called focus of the reflecting surface. Also, find the focal length.



Answers

Introductory Exercise 30.1

1. 4 m/s **2.** $\frac{20}{3}$ m, $\frac{40}{3}$ m **3.** 2θ

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (a) **4.** (b) **5.** (a or b) **6.** (c) **7.** (c) **8.** (b) **9.** (a,b) **10.** (b)

Objective Questions

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (b) **6.** (a) **7.** (c) **8.** (c) **9.** (a) **10.** (d)
11. (c) **12.** (d) **13.** (d)

Subjective Questions

1. Plane mirror

2. 39.2 cm to the right of mirror, 4.85 cm
3. 15°, 60°

4. 20 cm, 60 cm, 80 cm, 100 cm, 140 cm
5. The images are at $2nb$ from the object with n as integer.
7. 14
8. $360^\circ - 2\theta$

9. (a) – 16.7 cm, real (b) ∞ (c) + 10.0 cm, virtual
10. (a) A real image moves from – 0.6 m to $-\infty$, then a virtual image moves from $+\infty$ to 0.

(b) 0.639 s and 0.782 s.
11. (b) 33.0 cm to the left of vertex 1.20 cm tall, inverted, real
12. (b) 5.46 cm to the right of vertex, 4.09 mm tall, erect, virtual
13. 180 cm

14.

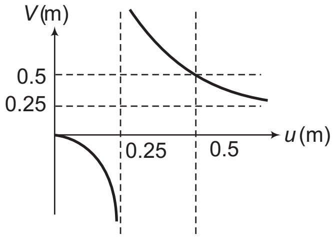


Image is virtual when object distance is from 0 to 0.25 m
15. 21 cm in front of plane mirror
17. At a distance $\left(\frac{\sqrt{3}+1}{2}\right)R$ from convex mirror

18. (a) A concave mirror with radius of curvature 2.08 m (b) 1.25 m from the object

LEVEL 2

Single Correct Option

1. (c) **2.** (d) **3.** (a) **4.** (c) **5.** (c) **6.** (c) **7.** (d) **8.** (a) **9.** (c) **10.** (c)
11. (c) **12.** (b) **13.** (b) **14.** (b) **15.** (b) **16.** (b)

More than One Correct Options

1. (a,b) **2.** (a,b,c,d) **3.** (a,b,d) **4.** (a,b,d) **5.** (b,c) **6.** (a,b,c)

Comprehension Based Questions

1.(c) **2.**(d) **3.**(d) **4.**(d)

Match the Columns

1. (a) → q,r (b) → q,r (c) → q,s (d) → p,s

2. (a) → p (b) → r (c) → p

3. (a) → r (b) → r (c) → p (d) → r

4. (a) → p,s (b) → r (c) → q,s (d) → r

5. (a) → p,q (b) → s (c) → s (d) → s

Subjective Questions

1. 6 s **2.** At a distance of 50 cm from mirror and 2 cm from each other

3. 90 cm, Yes **4.** 1.7 s **5.** 100 cm vertically below *A* **6.** $\frac{2\pi}{15}$ m/s

7. (a) 21 m/s (b) 10^{-3}/s **8.** (a) Distance = $\left(\frac{2+\cos\omega t}{1+\cos\omega t}\right)f$ (b) At $x = 0$ (c) $m = \infty$

9. $f = \frac{1}{8b}$

31

Refraction of Light

Chapter Contents

31.1 Refraction of light and Refractive Index of a medium
31.2 Law of refraction (Snell's law)
31.3 Single refraction from plane surface
31.4 Shift due to a glass slab
31.5 Refraction from spherical surface
31.6 Lens Theory
31.7 Total Internal Reflection (TIR)
31.8 Refraction through prism
31.9 Deviation
31.10 Optical Instruments

31.1 Refraction of Light and Refractive Index of a Medium

When a ray of light travels from one medium to other medium with or without bending, the phenomenon is called refraction of light. Under following two conditions the ray of light does not bend in refraction.

(i) For normal incidence (i 0)

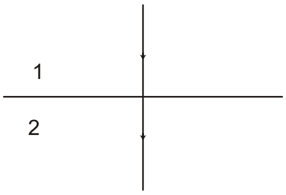


Fig. 31.1

(ii) If refractive index of both media is same, angle of incidence does not matter in this case.

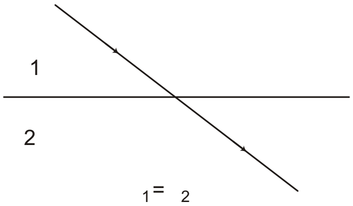


Fig. 31.2

Here, refractive index of medium

Refractive Index

(i) In general speed of light in any medium is less than its speed in vacuum. It is convenient to define refractive index of a medium as,

$$\boxed{\frac{\text{Speed of light in vacuum}}{\text{Speed of light in medium}} \quad \frac{c}{v}}$$

(ii) As a ray of light travels from medium 1 to medium 2, its wavelength changes but its frequency remains unchanged.

2 1, v_1 v_2, 1 2

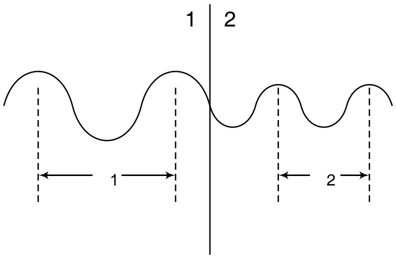


Fig. 31.3

(iii) $_{1\;2}\quad\frac{_2}{_1}$ Refractive index of 2 w.r.t. 1

and $\quad{}_{2\;1}\quad\frac{_1}{_2}$ Refractive index of 1 w.r.t. 2

$$_{1\;2}\quad\frac{1}{_{2\;1}}$$

(iv) $_{1\;2}\quad\frac{_2}{_1},\quad{}_{2\;3}\quad\frac{_3}{_2}$ and $\quad{}_{3\;1}\quad\frac{_1}{_3}$

$$\boxed{_{1\;2}\quad{}_{2\;3}\quad{}_{3\;1}\quad 1}$$

(v) $v\quad\frac{c}{}$ and $\quad\frac{_0}{}$

Here, is the wavelength in a medium and $_0$ the wavelength in vacuum. Thus, in travelling from vacuum to a medium speed and wavelength decrease times but frequency remains unchanged.

(vi) $_{1\;2}\quad\frac{_2}{_1}\quad\frac{c/v_2}{c/v_1}\quad\frac{v_1}{v_2}\quad\frac{f\;_1}{f\;_2}\quad\frac{_1}{_2}$

Here, f frequency of light which remains same in both media.

Thus, $\quad\boxed{_{1\;2}\quad\frac{_2}{_1}\quad\frac{v_1}{v_2}\quad\frac{_1}{_2}}$

Example 31.1 *(a) Find the speed of light of wavelength* 780 *nm (in air) in a medium of refractive index* 1.55.

(b) What is the wavelength of this light in the given medium?

Solution (a) $v\quad\frac{c}{}\quad\frac{3.0\quad 10^8}{1.55}\quad 1.94\quad 10^8$ m/s **Ans.**

(b) $_{\text{medium}}\quad\frac{_{\text{air}}}{}\quad\frac{780}{1.55}$ 503 nm **Ans.**

Example 31.2 *Refractive index of glass with respect to water is* (9/8). *Refractive index of glass with respect to air is* (3/2). *Find the refractive index of water with respect to air.*

Solution Given, $_{w\;g}\quad 9/8$ and $\quad{}_{a\;g}\quad 3/2$

As, $\quad{}_{a\;g}\quad{}_{g\;w}\quad{}_{w\;a}\quad 1$

$$\frac{1}{_{w\;a}}\quad{}_{a\;w}\quad{}_{a\;g}\quad{}_{g\;w}\quad\frac{_{a\;g}}{_{w\;g}}$$

$$_{a\;w}\quad\frac{3/2}{9/8}\quad\frac{4}{3}$$ **Ans.**

Example 31.3 *A ray of light passes through two slabs of same thickness. In the first slab n_1 waves are formed and in the second slab n_2. Find refractive index of second medium with respect to first.*

Solution One wave means one wavelength. So, if t is the thickness of slab, the wavelength and n the number of waves, then

$$n \qquad t \qquad\qquad \frac{t}{n}$$

or
$$\frac{1}{n} \quad \text{(as } t \text{ is same)} \quad \text{or} \quad \frac{_1}{_2} \quad \frac{n_2}{n_1}$$

Now, refractive index of second medium w.r.t. first medium is

$$_1 \;\; _2 \quad \frac{_1}{_2} \quad \frac{n_2}{n_1}$$ **Ans.**

INTRODUCTORY EXERCISE 31.1

1. Given that $_1 \quad _2$ 4/3, $_2 \quad _3$ 3/2. Find $_1 \quad _3$.
2. What happens to the frequency, wavelengths and speed of light that crosses from a medium with index of refraction $_1$ to one with index of refraction $_2$?
3. A monochromatic light beam of frequency 6.0 10^{14} Hz crosses from air into a transparent material where its wavelength is measured to be 300 nm. What is the index of refraction of the material?

31.2 Law of Refraction (Snell's Law)

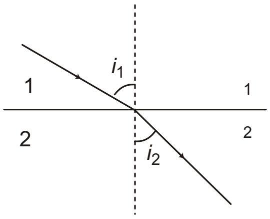


Fig. 31.4

If a ray of light passes through one medium to other medium, then according to Snell's law,

$$\boxed{\sin i \quad \text{constant}} \qquad \text{...(i)}$$

For two media,
$$_1 \sin i_1 \qquad _2 \sin i_2$$

or
$$\boxed{\frac{_2}{_1} \quad \frac{\sin i_1}{\sin i_2} \quad _1 \; _2} \qquad \text{...(ii)}$$

From Eq. (i) we can see that $i_1 \quad i_2$ if $_2 \quad _1$, i.e. if a ray of light passes from a rarer to a denser medium, it bends towards normal.

Eq. (ii) can be written as,

$$\boxed{{}_1 \quad {}_2 \qquad \frac{\sin i_1}{\sin i_2} \qquad \frac{v_1}{v_2} \qquad \frac{{}_1}{{}_2} \qquad \frac{{}_2}{{}_1}} \qquad \ldots\text{(iii)}$$

Here, v_1 is the speed of light in medium 1 and v_2 in medium 2. Similarly, ${}_1$ and ${}_2$ are the corresponding wavelengths.

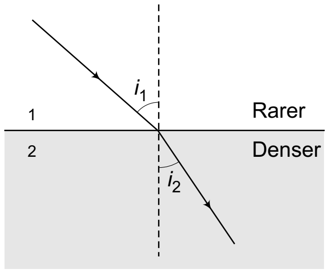


Fig. 31.5

If ${}_2 \quad {}_1$ then $v_1 \quad v_2$ and ${}_1 \quad {}_2$, i.e. in a rarer medium speed and hence, wavelength of light is more.

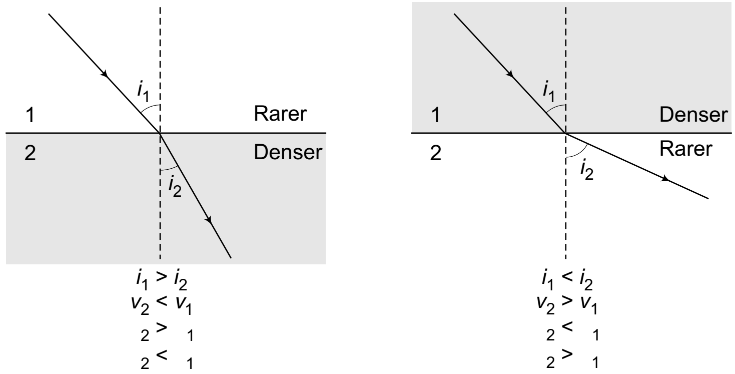


Fig. 31.6

Experiments show that if the boundaries of the media are **parallel**, the emergent ray *CD* although laterally displaced, is parallel to the incident ray *AB* if ${}_1 \quad {}_5$. We can also directly apply the Snell's law ($\sin i$ constant) in medium 1 and 5, i.e.

$$\boxed{{}_1 \sin i_1 \quad {}_5 \sin i_5}$$

So, $i_1 \quad i_5$ if ${}_1 \quad {}_5$

If any of the boundary is not parallel we cannot use this law directly by jumping the intervening media.

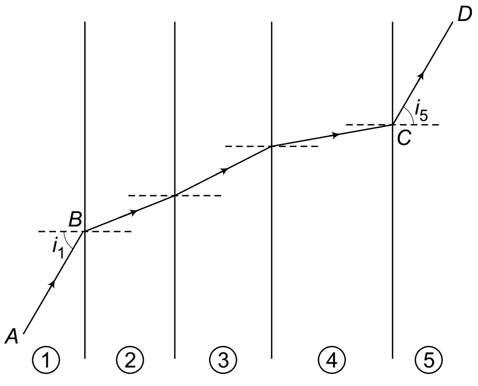


Fig. 31.7

Extra Points to Remember

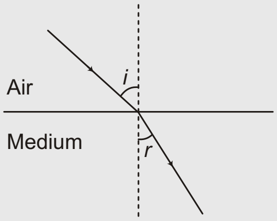


Fig. 31.8

- $\dfrac{\sin i}{\sin r}$ is a special case of Snell's law when one medium is air.

  In Fig. 31.8, if we apply the Snell's law in original form then it is

$$_{\text{air}} \sin i_{\text{air}} \qquad _{\text{medium}} \sin i_{\text{medium}}$$

or $$(1) \sin i \quad (\ ) \sin r$$

$$\frac{\sin i}{\sin r}$$

- In $\dfrac{\sin i}{\sin r}$, angle i is not always the angle of incidence but it is the angle of ray of light in air (with normal).

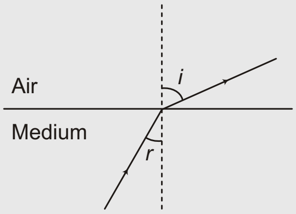


Fig. 31.9

  In the above figure, ray of light is travelling from medium to air. So, angle of incidence is actually r. But we have to take i angle in air and now we can apply $\dfrac{\sin i}{\sin r}$.

- In $\dfrac{\sin i}{\sin r}$, if i is changed, then r angle also changes. But $\dfrac{\sin i}{\sin r}$ remains constant and this constant is called refractive index of that medium.

- $\dfrac{\sin i}{\sin r}$ can be applied for any pair of angles i and r except the normal incidence for which $i \quad r \quad 0$ and $\dfrac{\sin i}{\sin r}$ is an indeterminant form.

Example 31.4 *A light beam passes from medium* 1 *to medium* 2. *Show that the emerging beam is parallel to the incident beam.*

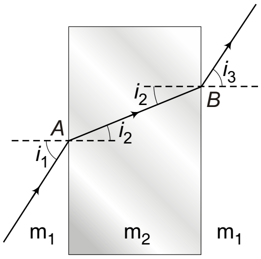


Fig. 31.10

Solution Applying Snell's law at A,

$$_1 \sin i_1 \qquad _2 \sin i_2$$

or $$\frac{1}{2} \quad \frac{\sin i_2}{\sin i_1} \qquad \text{...(i)}$$

Similarly at B, $$_2 \sin i_2 \qquad _1 \sin i_3$$

$$\frac{1}{2} \quad \frac{\sin i_2}{\sin i_3} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have $$i_3 \quad i_1$$

i.e. the emergent ray is parallel to incident ray. **Proved.**

Problems Based on $\dfrac{\sin i}{\sin r}$

Example 31.5 *A ray of light falls on a glass plate of refractive index* 1.5. *What is the angle of incidence of the ray if the angle between the reflected and refracted rays is* 90 ?

Solution In the figure, $r \quad 90 - i$

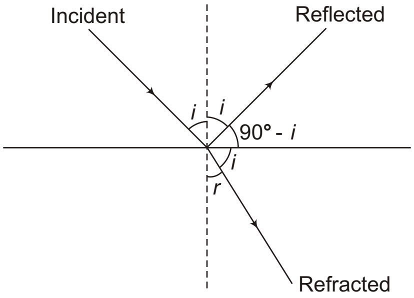


Fig. 31.11

From Snell's law,
$$1.5 \quad \frac{\sin i}{\sin r} \quad \frac{\sin i}{\sin(90 - i)} \quad \tan i$$

$$i \quad \tan^{-1}(1.5) \quad 56.3$$ **Ans.**

Example 31.6 *A pile* 4 *m high driven into the bottom of a lake is* 1 *m above the water. Determine the length of the shadow of the pile on the bottom of the lake if the sun rays make an angle of* 45 *with the water surface. The refractive index of water is* 4/3.

Solution From Snell's law, $\dfrac{4}{3} \quad \dfrac{\sin 45}{\sin r}$

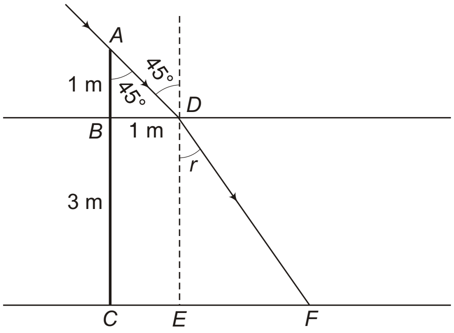


Fig. 31.12

Solving this equation, we get $r \quad 32$

Further, $EF \quad (DE)\tan r \quad (3)\tan 32 \quad 1.88$ m

Total length of shadow, $L \quad CF$ or $L \quad (1 \quad 1.88)$ m

2.88 m **Ans.**

Example 31.7 *An observer can see through a pin-hole the top end of a thin rod of height h, placed as shown in the figure. The beaker height is 3h and its radius h. When the beaker is filled with a liquid up to a height 2h, he can see the lower end of the rod. Then, the refractive index of the liquid is* **(JEE 2002)**

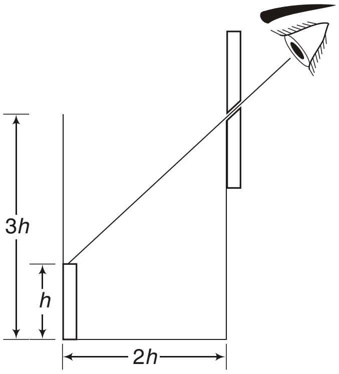


Fig. 31.13

(a) $\frac{5}{2}$ (b) $\sqrt{\frac{5}{2}}$ (c) $\sqrt{\frac{3}{2}}$ (d) $\frac{3}{2}$

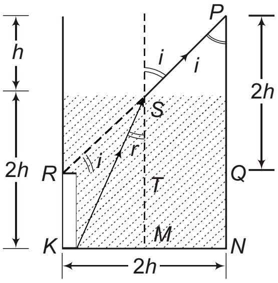


Fig. 31.14

Solution $PQ \quad QR \quad 2h$

$$i \quad 45$$

$$ST \quad RT \quad h \quad KM \quad MN$$

So, $$KS \quad \sqrt{h^2 \quad (2h)^2} \quad h\sqrt{5}$$

$$\sin r \quad \frac{h}{h\sqrt{5}} \quad \frac{1}{\sqrt{5}}$$

$$\frac{\sin i}{\sin r} \quad \frac{\sin 45}{1/\sqrt{5}} \quad \sqrt{\frac{5}{2}}$$

The correct answer is (b).

INTRODUCTORY EXERCISE 31.2

1. In the figure shown, find ${}_1 \quad {}_2$.

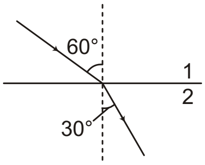


Fig. 31.15

2. If ${}_1 \quad {}_2$ is 1.5, then find the value of $\frac{{}_1}{{}_2}$.

31.3 Single Refraction from Plane Surface

(i) If ray of light travels from first medium (refractive index $_1$) to second medium (refractive index $_2$), then image distance (or v) from the plane surface is given by

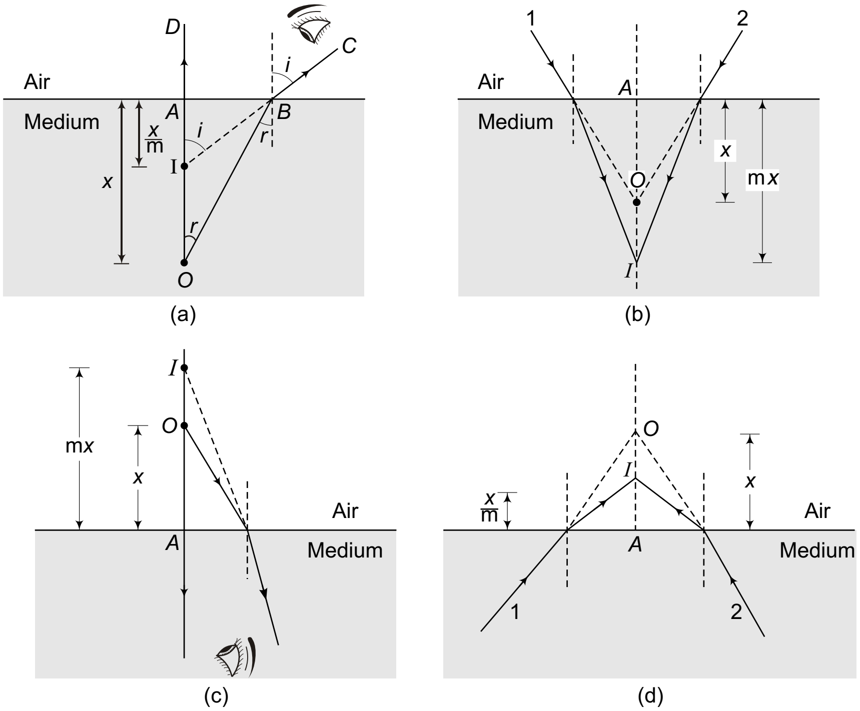
$$\boxed{v \quad \frac{{}_{2}}{{}_{1}}\, u}$$

(ii) From this equation, we can see that v and u are of same sign. This implies that object and image lie on same side of the plane surface. If one is real, then the other is virtual.

(iii) If ray of light travels from denser medium to rarer medium (or $_1$ $_2$), then we can see that v u) or the image distance is less than the object distance. If the light travels from rarer to denser medium ($_1$ $_2$), then v u or the image distance is greater than the object distance.

Further, if rarer medium is air (or vacuum), then this decrease or increase in image distance will be times.

(iv) In all the four figures, single refraction is taking place through a plane surface. Refractive index of medium (may be glass, water etc.) is . In figures (a) and (d), the ray of light is travelling from denser to rarer medium and hence, it bends away from the normal. In figures (b) and (c), the ray of light is travelling from a rarer to a denser medium and hence, it bends towards the normal. Now, let us take the four figures individually.



(a) (b) (c) (d)

Fig. 31.16

Refer figure (a) Object O is placed at a distance x from A. Ray OA, which falls normally on the plane surface, passes undeviated as AD. Ray OB, which falls at angle r (with the normal) on the

plane surface, bends away from the normal and passes as BC in air. Rays AD and BC meet at I after extending these two rays backwards. Let BC makes an angle i (> r) with normal.
In the figure, AOB will be r and AIB is i. For normal incidence (i.e. small angles of i and r)

$$\sin i \quad \tan i \quad \frac{AB}{AI} \qquad \ldots\text{(i)}$$

and

$$\sin r \quad \tan r \quad \frac{AB}{AO} \qquad \ldots\text{(ii)}$$

Dividing Eq. (i) by Eq. (ii), we have

$$\frac{\sin i}{\sin r} \quad \frac{AO}{AI} \quad \text{or} \quad \frac{AO}{AI} \qquad \text{as} \quad \frac{\sin i}{\sin r}$$

$$AI \quad \frac{AO}{\ } \quad \frac{x}{\ }$$

If point O is at a depth of d from a water surface, then the above result is also sometimes written as,

$$\boxed{d_{\text{apparent}} \quad \frac{d_{\text{actual}}}{\ }}$$

or the apparent depth is times less than the actual depth.

Refer figure (b) In the absence of the plane refracting surface, the two rays 1 and 2 would have met at O. Proceeding in the similar manner we can prove that after refraction from the plane surface, they will now meet at a point I, where

$$AI \quad x \qquad \text{(if } AO \quad x\text{)}$$

Refer figure (c) In this case object is at O, a distance $AO \quad x$ from the plane surface. When seen from inside the medium, it will appear at I, where

$$AI \quad x$$

If point O is at a height of h from the water surface, then the above relation is also written as

$$\boxed{h_{\text{app}} \quad h}$$

Refer figure (d) The two rays 1 and 2 meeting at O will now meet at I after refraction from the plane surface, where $AI \quad \frac{AO}{\ } \quad \frac{x}{\ }$.

Note *In all the four cases, the change in the value of x is times whether it is increasing or decreasing. All the relations can be derived for small angles of incidence as done in part (a).*

Exercise Three immiscible liquids of refractive indices $_1$, $_2$ and $_3$ (with $_3$ $_2$ $_1$) are filled in a vessel. Their depths are d_1, d_2 and d_3 respectively. Prove that the apparent depth (for almost normal incidence) when seen from top of the first liquid will be

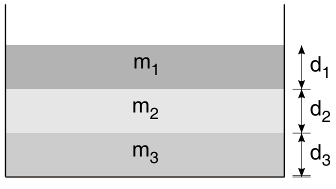


Fig. 31.17

$$\boxed{d_{\text{app}} \quad \frac{d_1}{_1} \quad \frac{d_2}{_2} \quad \frac{d_3}{_3}}$$

Example 31.8 *In Fig.* 31.16, *find position of second image* I_2 *formed after two times refraction from two plane surfaces AB and CD.*

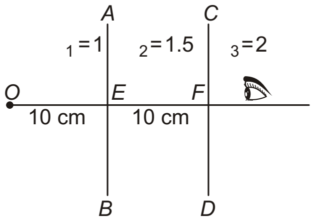


Fig. 31.18

Solution We will apply $v \quad \frac{2}{1} u$, two times with using the fact that object and image are on same side of the surface.

Refraction from *AB*

$$_1 \quad 1$$

$$_2 \quad 1.5$$

$$u \quad EO \quad 10\,\text{cm} \qquad \text{(towards left of } AB\text{)}$$

$$v \quad \frac{2}{1} \quad u \quad \frac{1.5}{1}(10)$$

or

$$v \quad EI_1 \quad 15\,\text{cm} \qquad \text{(towards left of } AB\text{)}$$

Refraction from *CD* I_1 will act as an object for refraction from *CD*

$$_1 \quad 1.5 \quad \text{and} \quad _2 \quad 2$$

$$u \quad FI_1 \quad FE \quad EI_1 \quad (10 \quad 15)\,\text{cm}$$

$$25\,\text{cm} \qquad \text{(towards left of } CD\text{)}$$

$$v \quad \frac{2}{1} \quad (u) \quad \frac{2}{1.5} \quad (25)$$

$$\frac{100}{3}\,\text{cm} \qquad \text{(towards left of } CD\text{)}$$

$$FI_2$$

The correct figure is as shown below

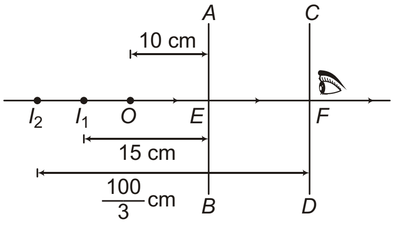


Fig. 31.19

Note *I_1 and I_2 both are virtual as the light has moved towards right of AB and CD (because it is a refraction) but I_1 and I_2 are towards left of AB or CD.*

Example 31.9 *Refractive index of the glass slab is* 1.5. *There is a point object* O *inside the slab as shown. To eye* E_1 *object appears at a distance of* 6 *cm (from the top surface) and to eye* E_2 *it appears at a distance of* 8 *cm (from the bottom surface). Find thickness of the glass slab.*

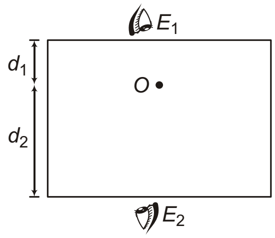


Fig. 31.20

Solution Applying $d_{\text{app}} \quad \frac{d}{}$

$$d \quad (\ )\, d_{\text{app}}$$
$$d_1 \quad (1.5)(6) \quad 9\,\text{cm}$$
$$d_2 \quad (1.5)(8) \quad 12\,\text{cm}$$

Therefore, actual thickness of the glass slab is $d_1 \quad d_2 \quad 21\,\text{cm}$. **Ans.**

INTRODUCTORY EXERCISE 31.3

1. In the figure shown, at what distance

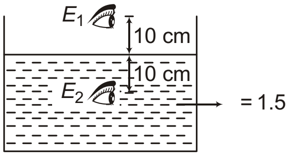


Fig. 31.21

(a) E_2 will appear to E_1
(b) E_1 will appear to E_2

31.4 Shift due to a Glass Slab

(i) It is a case of double refraction from two plane surfaces. So, we are talking about the second (or final) image and let us call it I.

(ii) If O is real then second image I is virtual and *vice-versa*.

(iii) I is shifted (w.r.t.) O by a distance OI shift $1 \quad \frac{1}{} \quad t$ in the direction of ray of light.

(iv) If E_1 observes E_2, then E_2 is object. So, light travels from E_2 towards E_1. So, shift is also in the same direction or E_1 will observe second image of E_2 at a distance

$$d_1 \quad d \quad \text{shift}$$

Same is the case when E_2 observes E_1.

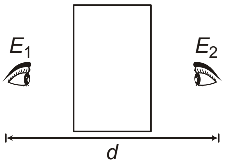
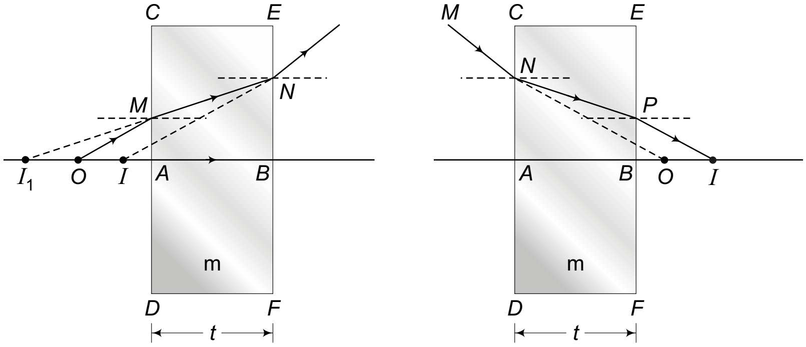


Fig. 31.22

(v) If two or more than two slabs are kept jointly or separately, then total shift is added.

$$S_{\text{Total}} \quad S_1 \quad S_2 \quad 1 \quad \frac{1}{{}_1} \quad t_1 \quad 1 \quad \frac{1}{{}_2} \quad t_2$$

(vi)



(a) (b)

Fig. 31.23

Refer figure (a) An object is placed at O. Plane surface CD forms its image (virtual) at I_1. This image acts as an object for EF which finally forms the image (virtual) at I. Distance OI is called the normal shift and its value is

$$\boxed{OI \quad 1 - \frac{1}{\ } \quad t}$$

This can be proved as under

Let $\quad OA \quad x$

then $\quad AI_1 \quad x \qquad$ (Refraction from CD)

$$BI_1 \quad x \quad t$$

$$BI \quad \frac{BI_1}{\ } \quad x \quad \frac{t}{\ } \qquad \text{(Refraction from } EF\text{)}$$

$$OI \quad (AB \quad OA) - BI$$

$$(t \quad x) - \quad x \quad \frac{t}{\ } \qquad 1 - \frac{1}{\ } \quad t$$ **Hence Proved.**

Note *For two refractions (at CD and EF) we have used,*

$$v \quad \frac{{}_2}{{}_1}$$

Refer figure (b) The ray of light which would have met line *AB* at *O* will now meet this line at *I* after two times refraction from the slab. Here,

$$OI \quad 1-\frac{1}{} \quad t$$

Example 31.10 *Refractive index of glass slab shown in figure is* 1.5. *Focal length of mirror is* 20 *cm. Find*

(a) total number of refractions and reflections before final image is formed.

(b) reduced steps.

(c) value of x, so that final image coincides with the object.

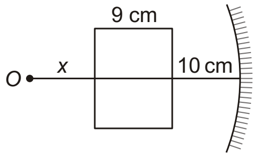


Fig. 31.24

Solution (a) There are total four refractions and one reflection.

(b) Reduced steps are three, first slab, then mirror and then again slab.

(c) Shift due to the slab,

$$s \quad 1 \quad \frac{1}{} \quad t \quad 1 \quad \frac{2}{3} \quad (9) \quad 3\,\text{cm}$$

Actual distance from mirror to object is $(19 \quad x)$ cm. Slab will reduce this distance by 3 cm. So, apparent distance will be $(16 \quad x)$ cm.

Now, if $\quad 16 \quad x \quad R \quad 2f \quad 40\,\text{cm} \quad \text{or} \quad x \quad 24\,\text{cm}$

then ray of light will fall normal to the concave mirror. It will retrace its path and final image will coincide with the object

$$x \quad 24\,\text{cm}$$ **Ans.**

Note *If ray of light falls normal to a mirror, then there is no need of applying the slab formula in return journey of ray of light. Path is retracing means, slab formula is automatically applied in return journey. But if it is not normal, then we will have to apply the slab formula in return journey too.*

Example 31.11 *A point object O is placed in front of a concave mirror of focal length* 10 *cm. A glass slab of refractive index* 3/2 *and thickness* 6 cm *is inserted between object and mirror. Find the position of final image when the distance x shown in figure is*

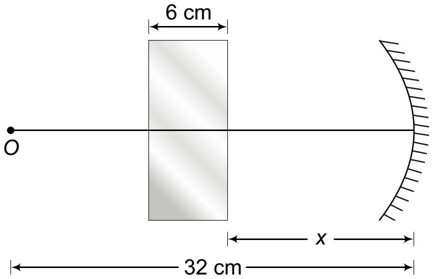


Fig. 31.25

(a) 5 *cm* *(b)* 20 *cm*

Solution As we have read in the above article, the normal shift produced by a glass slab is

$$x \quad 1-\frac{1}{\ } \; t \quad 1-\frac{2}{3} \;(6) \quad 2\ \text{cm}$$

i.e. for the mirror the object is placed at a distance (32 – x) 30 cm from it. Applying mirror formula

$$\frac{1}{v} \quad \frac{1}{u} \quad \frac{1}{f} \quad \text{or} \quad \frac{1}{v}-\frac{1}{30} \quad -\frac{1}{10} \quad \text{or} \quad v \quad -15\,\text{cm}$$

(a) **When x 5 cm** The light falls on the slab on its return journey as shown. But the slab will again shift it by a distance x 2 cm. Hence, the final **real** image is formed at a distance (15 2) 17 cm from the mirror.

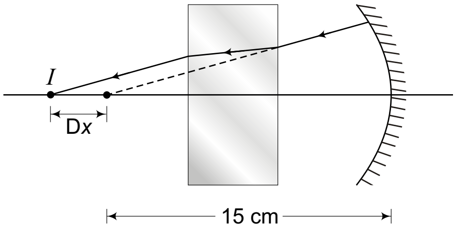


Fig. 31.26

(b) **When x 20 cm** This time also the final image is at a distance 17 cm from the mirror but it is **virtual** as shown.

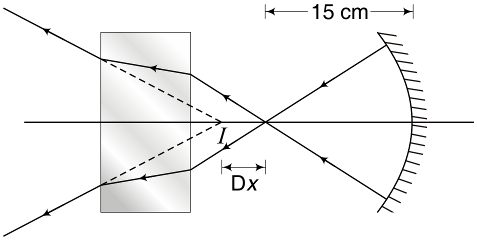


Fig. 31.27

INTRODUCTORY EXERCISE 31.4

1. At what distance eye *E* will observe the fourth image (after four refractions from plane surfaces) of object *O* from itself.

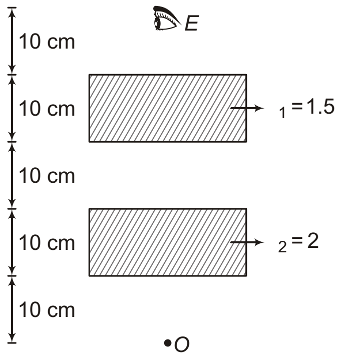


Fig. 31.28

31.5 Refraction from Spherical Surface

(i) There are two types of spherical surfaces, concave and convex.

(ii)

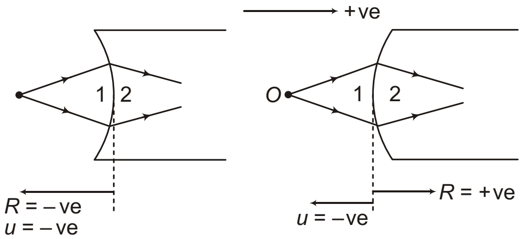


Fig. 31.29

If the ray of light is travelling from first medium to second medium, then for image distance *v*, we have the formula

$$\boxed{\frac{_2}{v} \quad \frac{_1}{u} \quad \frac{_2 \qquad _1}{R}}$$

(iii) For plane surface R . Putting this value in the above formula, we get

$$v \quad \frac{_2}{_1} u$$

and this formula, we have already discussed in article 31.3.

(iv) Ray of light has moved in medium-2, so image formed in medium-2 will be real and v in this medium will be positive.

Proof

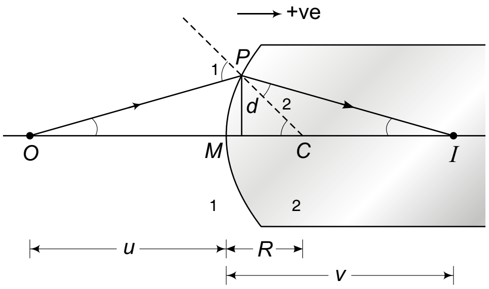


Fig. 31.30

Consider two transparent media having indices of refraction $_1$ and $_2$, where the boundary between the two media is a spherical surface of radius R. We assume that $_1 \quad _2$. Let us consider a single ray leaving point O and focusing at point I. Snell's law applied to this refracted ray gives,

$$_1 \sin \ _1 \qquad _2 \sin \ _2$$

Because $_1$ and $_2$ are assumed to be small, we can use the small angle approximation

$$\sin \qquad \text{(angles in radians)}$$

and say that $$_1 \ _1 \qquad _2 \ _2 \qquad \text{...(i)}$$

From the geometry shown in the figure,

$$_1 \qquad \text{...(ii)}$$

and $$_2 \qquad \text{...(iii)}$$

The above three equations can be rearranged as,

$$\frac{_1}{_2}(\qquad)$$

So, $$_1 \qquad _2 \qquad (_2 - \ _1) \qquad \text{...(iv)}$$

Since, the arc *PM* (of length *s*) subtends an angle at the centre of curvature,

$$\frac{s}{R}$$

Also in the paraxial approximation, $\frac{s}{u}$ and $\frac{s}{v}$

Using these expressions in Eq. (iv) with proper signs, we are left with,

$$\frac{1}{-u} \quad \frac{2}{v} \quad \frac{2 - 1}{R} \quad \text{or} \quad \boxed{\frac{2}{v} - \frac{1}{u} \quad \frac{2 - 1}{R}} \qquad \ldots(v)$$

Although the formula (v) is derived for a particular situation, it is valid for all other situations of refraction at a single spherical surface.

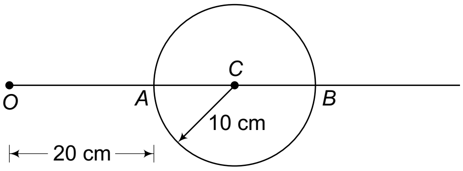


Fig. 31.31

Example 31.12 *A glass sphere of radius R 10 cm is kept inside water. A point object O is placed at 20 cm from A as shown in figure. Find the position and nature of the image when seen from other side of the sphere. Also draw the ray diagram. Given,* $_g$ *3/2 and* $_w$ *4/3.*

Solution A ray of light starting from *O* gets refracted twice. The ray of light is travelling in a direction from left to right. Hence, the distances measured in this direction are taken positive. Applying $\frac{2}{v} - \frac{1}{u} \quad \frac{2 - 1}{R}$, twice with proper signs. We have,

$$\frac{3/2}{AI_1} - \frac{4/3}{-20} \quad \frac{3/2 - 4/3}{10} \quad \text{or} \quad AI_1 \quad -30\,\text{cm}$$

Now, the first image I_1 acts as an object for the second surface, where

$$BI_1 \quad u \quad -(30 \quad 20) \quad -50\,\text{cm}$$

$$\frac{4/3}{BI_2} - \frac{3/2}{-50} \quad \frac{4/3 - 3/2}{-10}$$

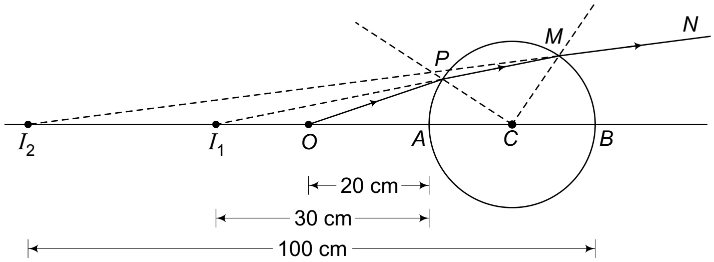


Fig. 31.32

$BI_2 \quad -100$ cm,

i.e. the final image I_2 is **virtual** and is formed at a distance 100 cm (towards left) from *B*.

The ray diagram is as shown in Fig. 31.32

Following points should be noted while drawing the ray diagram.

(i) At P the ray travels from rarer to a denser medium. Hence, it will bend towards normal PC. At M, it travels from a denser to a rarer medium, hence, it moves away from the normal MC.

(ii) PM ray when extended backwards meets at I_1 and MN ray when extended meets at I_2.

INTRODUCTORY EXERCISE 31.5

1. If an object is placed at the centre of a glass sphere and it is seen from outside, then prove that its virtual image is also formed at centre.
2. A glass sphere (1.5) with a radius of 15.0 cm has a tiny air bubble 5 cm above its centre. The sphere is viewed looking down along the extended radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?
3. One end of a long glass rod (1.5) is formed into a convex surface of radius 6.0 cm. An object is positioned in air along the axis of the rod. Find the image positions corresponding to object distances of (a) 20.0 cm, (b) 10.0 cm, (c) 3.0 cm from the end of the rod.
4. A dust particle is inside a sphere of refractive index $\frac{4}{3}$. If the dust particle is 10.0 cm from the wall of the 15.0 cm radius bowl, where does it appear to an observer outside the bowl.
5. A parallel beam of light enters a clear plastic bead 2.50 cm in diameter and index 1.44. At what point beyond the bead are these rays brought to a focus?

31.6 Lens Theory

(i) A lens is one of the most familiar optical devices for a human being. A lens is an optical system with two refracting surfaces. The simplest lens has two spherical surfaces close enough together that we can neglect the distance between them (the thickness of the lens). We call this a **thin lens.**

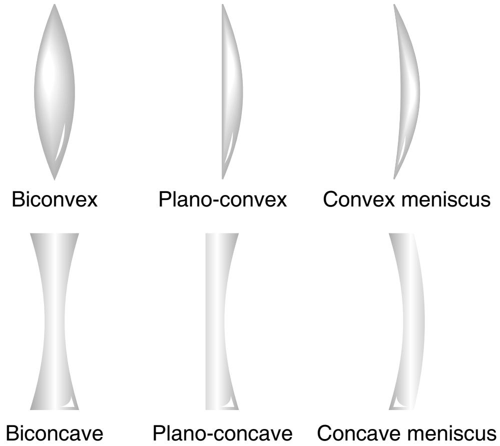


Fig. 31.33 Types of lens.

Lenses are of two basic types **convex** (converging) which are thicker in the middle than at the edges and **concave** (diverging) for which the reverse holds.

Figure shows examples of both types bounded by spherical or plane surfaces.

As there are two spherical surfaces, there are two centres of curvature C_1 and C_2 correspondingly two radii of curvature R_1 and R_2.

The line joining C_1 and C_2 is called the **principal axis** of the lens. The centre P of the thin lens which lies on the principal axis, is called the **optical centre.**

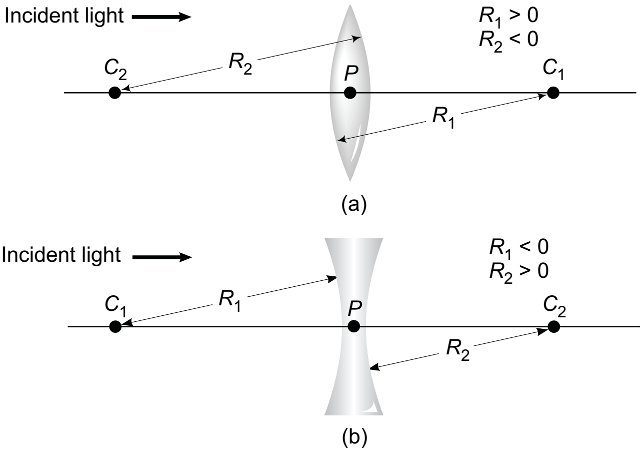


Fig. 31.34 (a) A converging thin lens and (b) a diverging thin lens

(ii) All lens formulae (which we will use in this chapter) can be applied directly under following two conditions.

Condition 1 Lens should be thin or its thickness should be negligible.

Condition 2 On both sides of the lens, medium should be same (not necessarily air)

(iii) If either of the above two conditions are not satisfied, then apply refraction formulae $\left(\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}\right.$ for spherical surface or $v = \frac{\mu_2}{\mu_1} u$ for plane surface) two times.

(iv) In a biconvex (or equiconvex) or biconcave (or equiconcave) lens,

$$|R_1| = |R_2|$$

(v) **Use of thin lens** If the lens is thin, then the first image distance v_1 is exactly equal to the second object distance u_2.

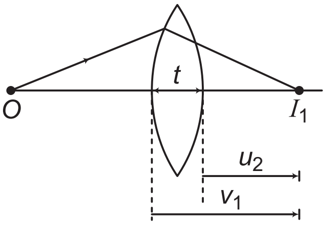


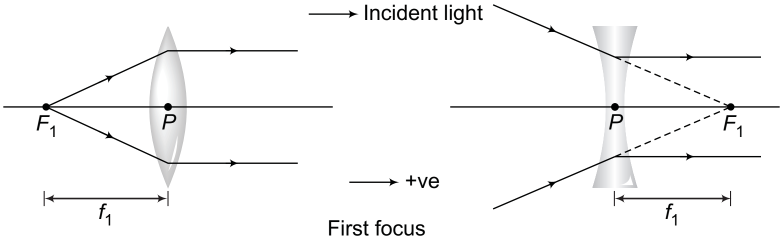
Fig. 31.35

In the figure, we can see that $u_2 = v_1 - t$

But, $u_2 = v_1$ if $t = 0$

(vi) Unlike a mirror, a lens has two foci :

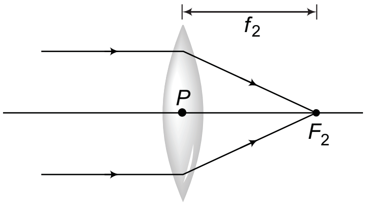
First focus (F_1) It is defined as a point at which if an object (real in case of a convex lens and virtual for concave) is placed, the image of this object is formed at infinity. Or we can say that rays passing through F_1 become parallel to the principal axis after refraction from the lens. The distance PF_1 is the first focal length f_1.



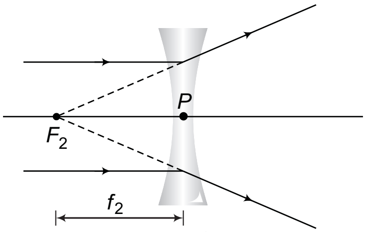
First focus

Fig. 31.36

Second focus or principal focus (F_2) A narrow beam of light travelling parallel to the principal axis either converge (in case of a convex lens) or diverge (in case of a concave lens) at a point F_2 after refraction from the lens. This point F_2 is called the second or principal focus. If the rays converge at F_2, the lens is said a converging lens and if they diverge, they are called diverging lens. Distance PF_2 is the second focal length f_2.



Second focus



Principal focus

Fig. 31.37

From the figure, we can see that f_1 is negative for a convex lens and positive for a concave lens. But, f_2 is positive for convex lens and negative for concave lens.

(vii) We are mainly concerned with the second focus f_2. Thus, wherever we write the focal length f, it means the second or principal focal length. Thus, $f = f_2$ and hence, f is positive for a convex lens and negative for a concave lens.

(viii) If the two conditions mentioned in point number (ii) are satisfied, then

$$|f_1| = |f_2|$$

although their signs are different.

(ix) If those two conditions are satisfied, then object can be placed on either side of the lens or light can fall from both sides of the lens. Following figures will help you to clear this concept.

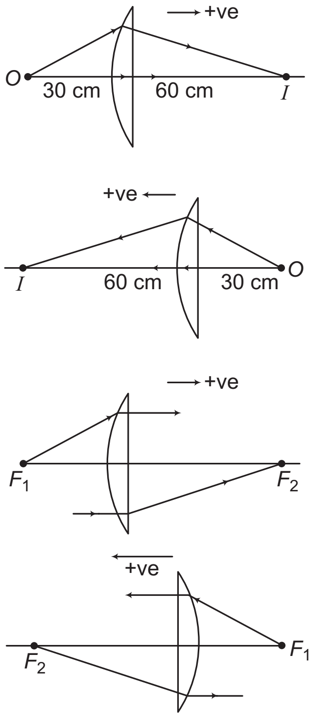


Fig. 31.38

Image Position, its Nature and Speed

Case 1 Convex lens

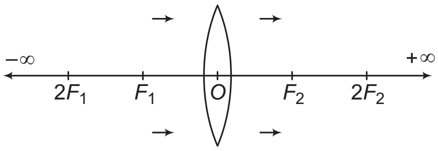


Fig. 31.39

Table 31.1

Object	Image	Nature of image	Speed
At F_1	$\pm\infty$	–	–
At $2F_1$	At $2F_2$	Real, Inverted and same size	$v_I = v_O$
At $-\infty$	At F_2	–	–
Between O and F_1	Between O and $-\infty$	Virtual, Erect and magnified	$v_O > v_I$
Between F_1 and $2F_1$	Between $+\infty$ and $2F_2$	Real, Inverted and magnified	$v_I > v_O$
Between $2F_1$ and $-\infty$	Between $2F_2$ and F_2	Real, Inverted and diminished	$v_O > v_I$

Note *(i) The above table has been made only for real objects (lying between O and $-\infty$), object distance u for them is negative.*

(ii) Since $|f_1| = |f_2|$ (when two conditions discussed earlier are satisfied). Therefore, F_1 and F_2 are sometimes denoted by F and $2F_1$ (and $2F_2$) by 2F.

(iii) If object is travelling along the principal axis, then image also travels along the principal axis in the same direction.

Case 2 **Concave lens** In case of concave lens there is only one case for real objects.

Object lies between O and $-\infty$, then image lies between O and F_2. Nature of image is virtual, erect and diminished. Object speed is always greater than image speed ($v_O > v_I$). Both travel in the same direction.

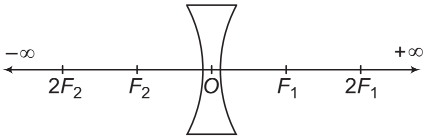


Fig. 31.40

Three Standard Rays for making Ray Diagrams

1. A ray parallel to the principal axis after refraction passes through the principal focus or appears to diverge from it.

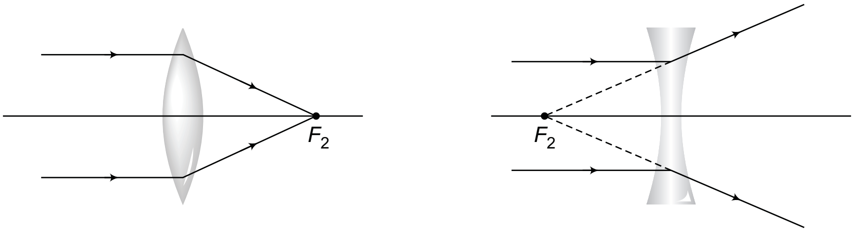


Fig. 31.41

2. A ray through the optical centre P passes undeviated because the middle of the lens acts like a thin parallel-sided slab.

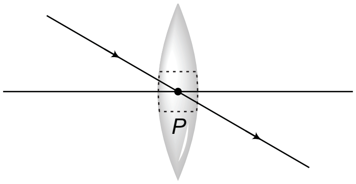


Fig. 31.42

3. A ray passing through the first focus F_1 becomes parallel to the principal axis after refraction.

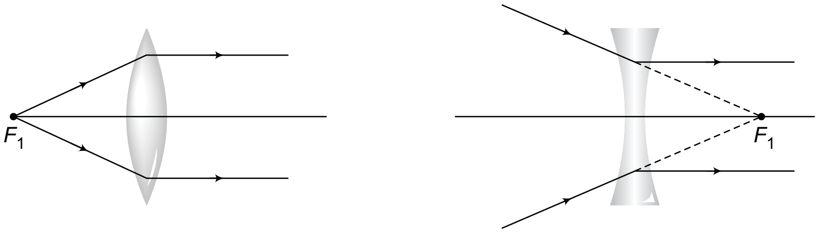


Fig. 31.43

Ray Diagrams

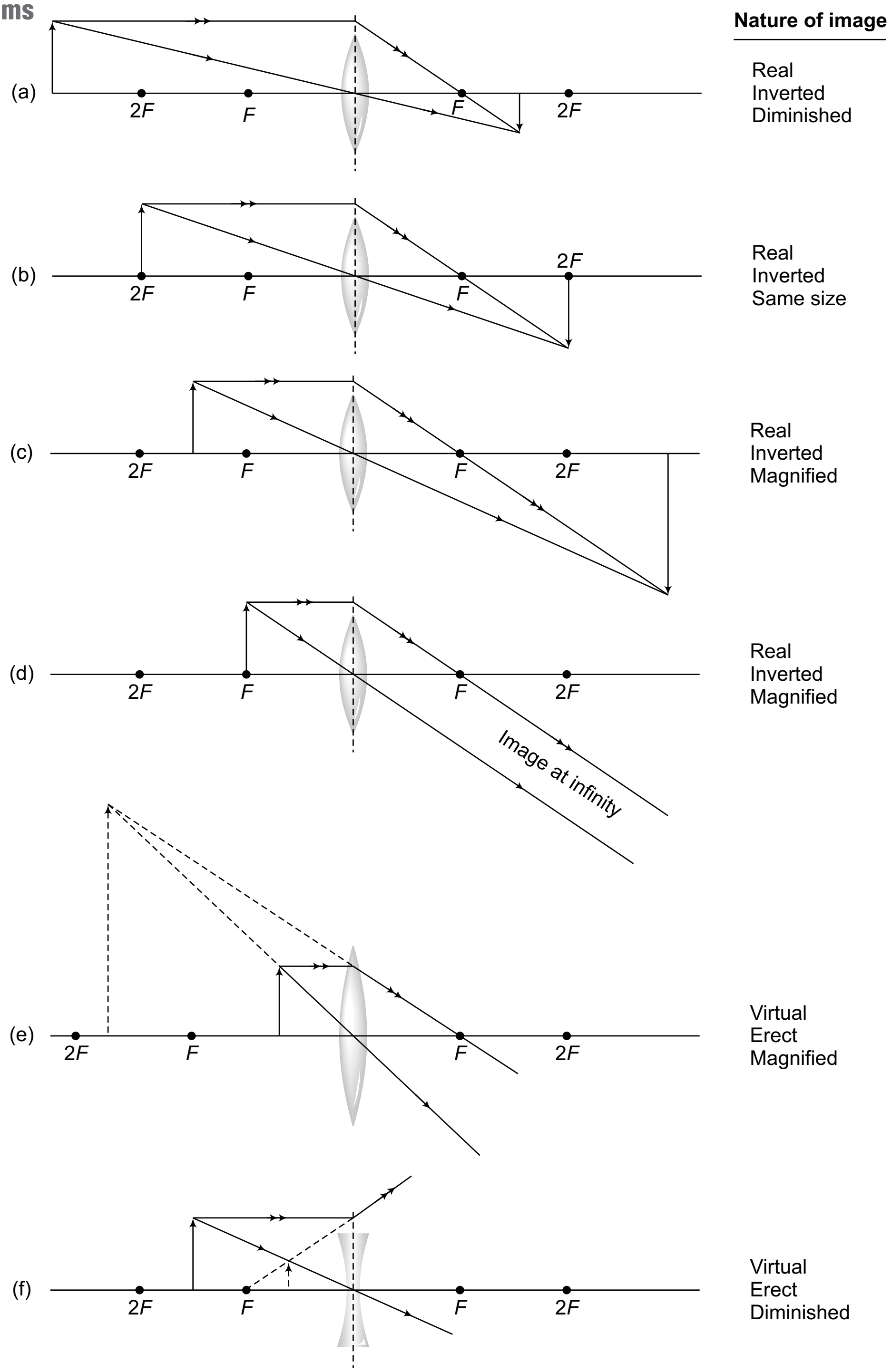


Fig. 31.44 Ray diagrams for a convex lens (a–e) and a concave lens (f).

List of Formulae

(i) $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$

(ii) Linear magnification, $m=\frac{\text{Image height}}{\text{Object height}}=\frac{I}{O}=\frac{v}{u}$

(iii) **Lens maker's formula**

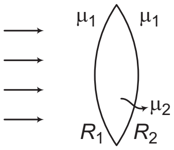


Fig. 31.45

$$\frac{1}{f}=\left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

In air, $\mu_1=1$ and $\mu_2=\mu$. Therefore, $\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$

(iv) Power of a lens (in dioptre) $=\frac{1}{\text{Focal length (in metre)}}$

(v) Two or more than two thin lenses in contact.

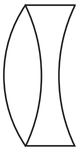


Fig. 31.46

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}\quad\text{or}\quad P=P_1+P_2$$

(vi) Two or more than two thin lenses at some distance

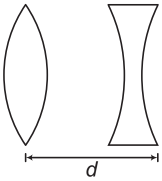


Fig. 31.47

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}$$

or $$P=P_1+P_2-dP_1P_2$$

Note *In the above two equations if* $d=0$, *then*

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}\quad\textit{and}\quad P=P_1+P_2$$

(vii) **Image velocity**

(a) Along the principal axis,

$$\mathbf{v}_I = m^2 \mathbf{v}_0$$

$\mathbf{v}_I$ and $\mathbf{v}_0$ are in the same direction.

(b) Perpendicular to principal axis,

$$\mathbf{v}_I = m\mathbf{v}_0$$

If m is positive then $\mathbf{v}_I$ and $\mathbf{v}_0$ are in the same direction and if m is negative, then $\mathbf{v}_I$ and $\mathbf{v}_0$ are in opposite directions.

Note *The above two formulae have been derived in the previous chapter of reflection. So, same method can be applied here.*

Important points in formulae

(i) **On linear magnification *m*** From the value of *m*, we can determine nature of image, type of lens and an approximate position of object. Following table illustrates this point.

Table 31.2

Value of *m*	Nature of image	Type of lens	Object position
-3	Inverted, real and magnified	convex	Between F_1 and $2F_1$
-1	Inverted, real and same size	convex	at $2F_1$
$-\frac{1}{2}$	Inverted, real and diminished	convex	Between $2F_1$ and $-\infty$
$+2$	Erect, virtual and magnified	convex	Between O and F_1
$+\frac{1}{4}$	Erect, virtual and diminished	concave	Between O and $-\infty$

Note *For real objects, real image is formed only by convex lens. But virtual image is formed by both types of lenses. Their sizes are different. Magnified virtual image is formed by convex lens. Diminished virtual image is formed by concave lens.*

(ii) **On lens maker's formula**

$$\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \ldots \text{(i)}$$

$$= (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \text{ in air} \qquad \ldots \text{(ii)}$$

For a converging lens, R_1 is positive and R_2 is negative. Therefore, $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ in Eq. (ii) comes out a positive quantity and if the lens is placed in air, $(\mu - 1)$ is also a positive quantity. Hence, the focal length f of a converging lens turns out to be positive. For a diverging lens however, R_1 is negative and R_2 is positive and the focal length f becomes negative.

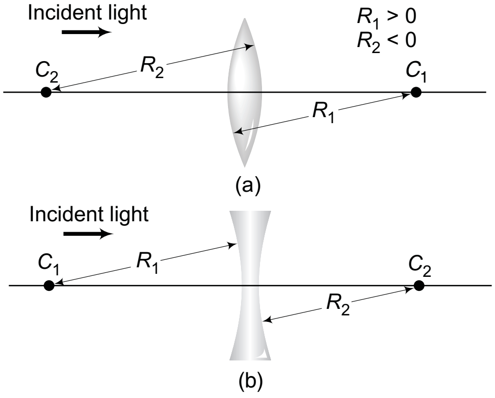


Fig. 31.48

Focal length of a mirror $\left(f_M = \frac{R}{2}\right)$ depends only upon the radius of curvature R while that of a lens [Eq. (i)] depends on μ_1, μ_2, R_1 and R_2. Thus, if a lens and a mirror are immersed in some liquid, the focal length of lens would change while that of the mirror will remain unchanged.

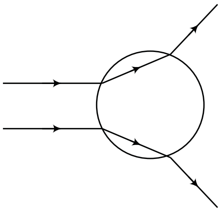


Fig. 31.49 Air bubble in water diverges the parallel beam of light incident on it.

Suppose $\mu_2 < \mu_1$ in Eq. (i), i.e. refractive index of the medium (in which lens is placed) is more than the refractive index of the material of the lens, then $\left(\frac{\mu_2}{\mu_1} - 1\right)$ becomes a negative quantity, i.e. the lens changes its behaviour. A converging lens behaves as a diverging lens and *vice-versa*. An air bubble in water seems as a convex lens but behaves as a concave (diverging) lens.

(iii) **Power of lens** By optical power of an instrument (whether it is a lens, mirror or a refractive surface) we mean the ability of the instrument to deviate the path of rays passing through it. If the instrument converges the rays parallel to the principal axis its power is said to be positive and if it diverges the rays it is said a negative power.

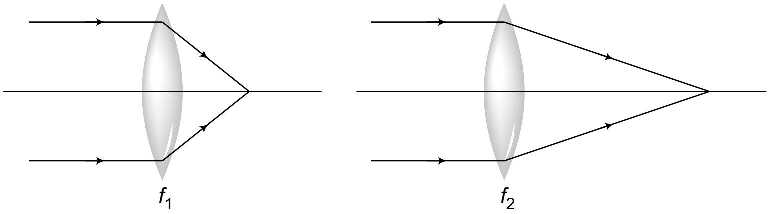


Fig. 31.50

The shorter the focal length of a lens (or a mirror) the more it converges or diverges the light. As shown in the figure,

$$f_1 < f_2$$

and hence the power $P_1 > P_2$, as bending of light in case 1 is more than that of case 2. For a lens,

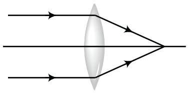
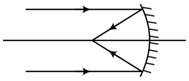
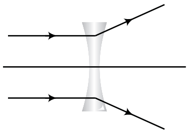
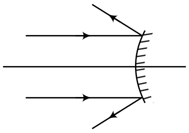
$$P \text{ (in dioptre)} = \frac{1}{f \text{ (in metre)}}$$

and for a mirror,

$$P \text{ (in dioptre)} = \frac{-1}{f \text{ (in metre)}}$$

Following table gives the sign of P and f for different types of lens and mirror.

Table 31.3

Nature of lens/mirror	Focal length (f)	Power $P_L = \frac{1}{f}, P_M = -\frac{1}{f}$	Converging/ diverging	Ray diagram
Convex lens	+ ve	+ ve	converging	
Concave mirror	– ve	+ ve	converging	
Concave lens	– ve	– ve	diverging	
Convex mirror	+ ve	– ve	diverging	

Thus, convex lens and concave mirror have positive power or they are converging in nature. Concave lens and convex mirror have negative power or they are diverging in nature.

(iv) **Based on two or more than two thin lenses in contact (or at some distance)**

If the lenses are kept in contact, then after finding the equivalent focal length F from the equation

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

We can directly apply the formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{F}$$

for finding the image distance v, but we will have to apply lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$ two times if the lenses are kept at some distance.

Proofs of Different Formulae

(i) Consider an object O placed at a distance u from a convex lens as shown in figure. Let its image I after two refractions from spherical surfaces of radii R_1 (positive) and R_2 (negative) be formed at a distance v from the lens. Let v_1 be the distance of image formed by refraction from the refracting surface of radius R_1. This image acts as an object for the second surface.

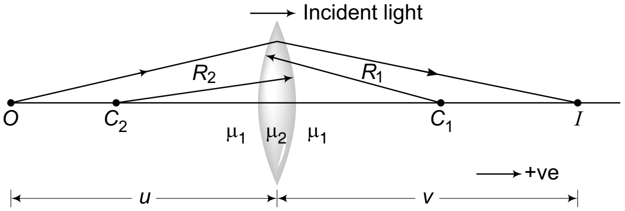


Fig. 31.51

Using,
$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$$

We have,
$$\frac{\mu_2}{v_1}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R_1} \quad \text{...(i)}$$

and
$$\frac{\mu_1}{v}-\frac{\mu_2}{v_1}=\frac{\mu_1-\mu_2}{-R_2} \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii) and then simplifying, we get
$$\frac{1}{v}-\frac{1}{u}=\left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad \text{...(iii)}$$

This expression relates the image distance v of the image formed by a thin lens to the object distance u and to the thin lens properties (index of refraction and radii of curvature). It is valid only for paraxial rays and only when the lens thickness is much less than R_1 and R_2. The **focal length *f*** of a thin lens is the image distance that corresponds to an object at infinity. So, putting $u=\infty$ and $v=f$ in the above equation, we have
$$\boxed{\frac{1}{f}=\left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{...(iv)}$$

If the refractive index of the material of the lens is μ and it is placed in air, $\mu_2=\mu$ and $\mu_1=1$ so that Eq. (iv) becomes
$$\boxed{\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{...(v)}$$

This is called the **lens maker's formula** because it can be used to determine the values of R_1 and R_2 that are needed for a given refractive index and a desired focal length f.

Combining Eqs. (iii) and (v), we get

$$\boxed{\frac{1}{v}-\frac{1}{u}=\frac{1}{f}} \qquad \ldots\text{(vi)}$$

Which is known as the **lens formula.**

(ii) **Magnification** The lateral, transverse or linear magnification m produced by a lens is defined by

$$\boxed{m=\frac{\text{Height of image}}{\text{Height of object}}=\frac{I}{O}}$$

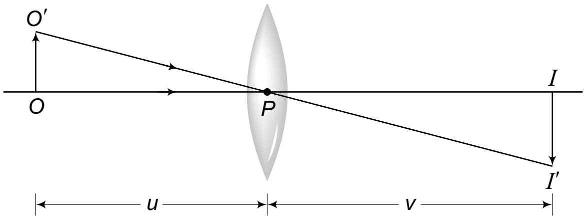


Fig. 31.52

A real image II' of an object OO' formed by a convex lens is shown in figure.

$$\frac{\text{Height of image}}{\text{Height of object}}=\frac{II'}{OO'}=\frac{v}{u}$$

Substituting v and u with proper sign,

$$\frac{II'}{OO'}=\frac{-I}{O}=\frac{v}{-u}$$

or

$$\frac{I}{O}=m=\frac{v}{u}$$

Thus,

$$\boxed{m=\frac{v}{u}}$$

(iii) **Focal length of two or more than two thin lenses in contact**

Combinations of lenses in contact are used in many optical instruments to improve their performance.

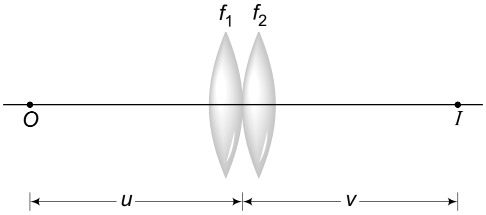


Fig. 31.53

Suppose two lenses of focal lengths f_1 and f_2 are kept in contact and a point object O is placed at a distance u from the combination. The first image (say I_1) after refraction from the first lens is formed at a distance v_1 (whatever may be the sign of v_1) from the combination. This image I_1 acts as an object for the second lens and let v be the distance of the final image from the combination. Applying the lens formula,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

For the two lenses, we have

$$\frac{1}{v_1}-\frac{1}{u}=\frac{1}{f_1} \qquad \text{...(i)}$$

and

$$\frac{1}{v}-\frac{1}{v_1}=\frac{1}{f_2} \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we have

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F} \qquad \text{(say)}$$

Here, F is the equivalent focal length of the combination. Thus,

$$\boxed{\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}}$$

Similarly for more than two lenses in contact, the equivalent focal length is given by the formula,

$$\boxed{\frac{1}{F}=\sum_{i=1}^{n}\frac{1}{f_i}}$$

Note *Here, f_1, f_2 etc., are to be substituted with sign.*

Types of Problems in Lens

Type 1. *Based on two or more than two thin lenses in contact*

How to Solve

Apply, $$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

and $$P=P_1+P_2$$

Example 31.13 *A convex lens of power 2 D and a concave lens of focal length 40 cm are kept in contact, find*

(a) Power of combination *(b) Equivalent focal length*

Solution (a) Applying

$$P=P_1+P_2=P_{\text{convex}}+P_{\text{concave}}$$

$$=2+\frac{1}{(-0.4)} \qquad \left[P(\text{in } D)=\frac{1}{f(\text{in } m)}\right]$$

$$=2-2.5=-0.5\text{ D}$$ **Ans.**

(b)
$$F = \frac{1}{P}$$
$$= \frac{1}{-0.5} = -2\text{ m}$$
$$= -200\text{ cm}$$
Ans.

Note *F and P are negative so, the system behaves like a concave lens.*

Example 31.14 *A converging lens of focal length* 5.0 *cm is placed in contact with a diverging lens of focal length* 10.0 *cm. Find the combined focal length of the system.*

Solution Here, $f_1 = +5.0\text{cm}$ and $f_2 = -10.0\text{cm}$

Therefore, the combined focal length F is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{5.0} - \frac{1}{10.0} = +\frac{1}{10.0}$$

$\therefore$ $F = +10.0\text{cm}$ **Ans.**

i.e. the combination behaves as a converging lens of focal length 10.0 cm.

Type 2. *Based on lens maker's formula*

How to Solve

Apply, $$\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{or} \quad \frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Note *If initial two conditions are satisfied (thin lens and same medium on both sides,) then we can find focal length of the lens (f or f_2) from either side of the lens. Result comes out to be same.*

Example 31.15

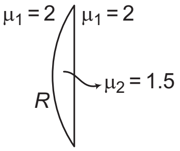


Fig. 31.54

Find focal length of the system shown in figure from left hand side.

Solution $\mu_1 = 2, \mu_2 = 1.5, R_1 = +40$ and $R_2 = \infty$

Using the equation

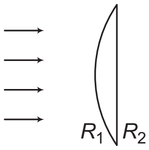


Fig. 31.55

$$\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \text{ we have}$$

$$\frac{1}{f} = \left(\frac{1.5}{2} - 1\right)\left(\frac{1}{+40} - \frac{1}{\infty}\right) \Rightarrow f = -160\text{ cm}$$

Therefore, from left hand side it behaves like a concave lens of focal length 160 cm.

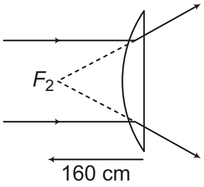


Fig. 31.56

Exercise Find focal length of the above system from right hand side and prove that it is also -160 cm.

Example 31.16 *Focal length of a convex lens in air is 10 cm. Find its focal length in water. Given that* $\mu_g = 3/2$ *and* $\mu_w = 4/3$.

Solution

$$\frac{1}{f_{\text{air}}} = (\mu_g - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \ldots\text{(i)}$$

and

$$\frac{1}{f_{\text{water}}} = \left(\frac{\mu_g}{\mu_w} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \ldots\text{(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{f_{\text{water}}}{f_{\text{air}}} = \frac{(\mu_g - 1)}{(\mu_g/\mu_w - 1)}$$

Substituting the values, we have

$$f_{\text{water}} = \frac{(3/2 - 1)}{\left(\frac{3/2}{4/3} - 1\right)} f_{\text{air}}$$

$$= 4\, f_{\text{air}} = 4 \times 10 = 40\,\text{cm}$$ **Ans.**

Note *(i) Students can remember the result* $f_{water} = 4\, f_{air}$, *if* $\mu_g = 3/2$ *and* $\mu_w = 4/3$.

(ii) In water focal length has become four times and power (power of bending of light) remains $\frac{1}{4}$*th. This is because, difference in refractive index between glass and water has been decreased (compared to glass and air). So, there will be less bending of light.*

Example 31.17 *A biconvex lens* ($\mu = 1.5$) *has radius of curvature* 20 *cm* (*both*). *Find its focal length.*

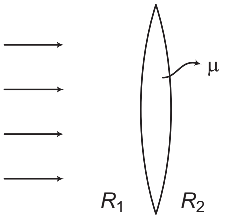


Fig. 31.57

Solution $R_1 = +20\,\text{cm}, R_2 = -20\,\text{cm}, \mu = 1.5$

Substituting the values in the equation

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \text{ we have}$$

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{20} - \frac{1}{-20}\right)$$

or $$f = +20\,\text{cm}$$ **Ans.**

Note *If a biconvex or biconcave lens has refractive index* $\mu = 1.5$, *then*

$$|R_1| = |R_2| = |f|$$

Type 3. *To find image distance and its magnification corresponding to given object distance*

How to Solve?

- Substitute signs of u and f. Sign of v automatically comes after applying the lens formula. Sign of u is negative for real objects. Sign of f is positive for convex lens and negative for concave lens.

Example 31.18 *Find distance of image from a convex lens of focal length* 20 *cm if object is placed at a distance of* 30 *cm from the lens. Also find its magnification.*

Solution $u = -30\,\text{cm}, f = +20\,\text{cm}$

Applying the lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{v} - \frac{1}{-30} = \frac{1}{+20}$$

Solving, we get

$$v = +60\,\text{cm}$$ **Ans.**

$$m = \frac{v}{u} = \frac{+60}{-30} = -2$$ **Ans.**

m is -2, it implies that image is real, inverted and two times magnified. The ray diagram is as shown below.

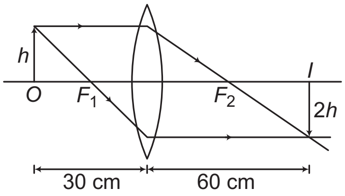


Fig. 31.58

Type 4. *To find object/image distance corresponding to given magnification of image and focal length of lens*

How to Solve?

- Substitute all three signs of u, v and f. Signs of u and f have been discussed in the above type. Sign of v is positive for real image (see the above example) and it is negative for virtual image.
- $m = \frac{v}{u} \quad \Rightarrow \quad |v| = |m \times u|$

Example 31.19 *Find the distance of an object from a convex lens if image is two times magnified. Focal length of the lens is 10 cm.*

Solution Convex lens forms both types of images real as well as virtual. Since, nature of the image is not mentioned in the question, we will have to consider both the cases.

When image is real Means v is positive and u is negative with $|v| = 2|u|$. Thus, if

$$u = -x, \quad \text{then} \quad v = 2x \quad \text{and} \quad f = 10\,\text{cm}$$

Substituting in
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have
$$\frac{1}{2x} + \frac{1}{x} = \frac{1}{10} \quad \text{or} \quad \frac{3}{2x} = \frac{1}{10}$$

$\therefore$
$$x = 15\,\text{cm}$$
Ans.

$x = 15\,\text{cm}$, means object lies between F and $2F$.

When image is virtual Means v and u both are negative. So let,

$$u = -y, \quad \text{then} \quad v = -2y \quad \text{and} \quad f = 10\,\text{cm}$$

Substituting in,
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have,
$$\frac{1}{-2y} + \frac{1}{y} = \frac{1}{10} \quad \text{or} \quad \frac{1}{2y} = \frac{1}{10}$$

$\therefore$
$$y = 5\text{ cm}$$
Ans.

$y = 5\,\text{cm}$, means object lies between F and O.

Type 5. *To make some conditions*

How to Solve?

- Initially, substitute sign of only f, then make equation of v. From this equation of v, find the asked condition.

Example 31.20 *Under what condition, a concave lens can make a real image.*

Solution Substituting sign of f in the lens formula, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{-f} \quad \text{or} \quad \frac{1}{v} = \frac{1}{u} - \frac{1}{f} \qquad \ldots\text{(i)}$$

For real image v should be positive. Therefore, from Eq. (i) we can see that u should be positive and less than f. Further, u is positive and less than f means a virtual object should lie between O and F_1.

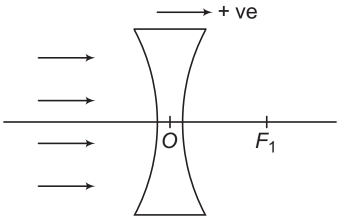


Fig. 31.59

Important Result

Under normal conditions, a convex lens makes a real image. But the image is virtual (and magnified) if a real object is placed between O and F_1. Opposite is the case with concave lens. Under normal conditions it makes a virtual image (for all real objects). But the image is real if a virtual object is placed between O and F_1.

Type 6. *To find image nature, type of lens, its optical centre and focus for given principal axis, point object and its point image.*

Example 31.21 *An image I is formed of point object O by a lens whose optic axis is AB as shown in figure.*

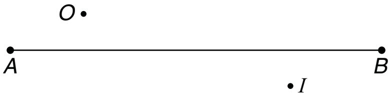


Fig. 31.60

(a) State whether it is a convex lens or concave?

(b) Draw a ray diagram to locate the lens and its focus.

Solution (a) (i) Concave lens always forms an erect image. The given image I is on the other side of the optic axis. Hence, the lens is **convex.**

(ii) Join O with I. Line OI cuts the optic axis AB at optical centre (P) of the lens. The dotted line shows the position of lens.

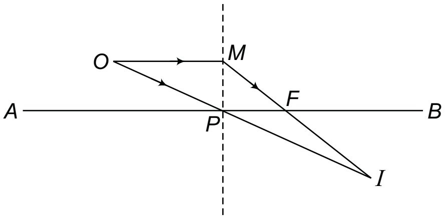


Fig. 31.61

From point O, draw a line parallel to AB. Let it cuts the dotted line at M. Join M with I. Line MI cuts the optic axis at focus (F) of the lens.

Type 7. *Two lens problems*

How to Solve?

- We have to apply lens formula two times. The first image behaves like an object for the second lens.

Example 31.22 *Focal length of convex lens is* 20 *cm and of concave lens* 40 *cm. Find the position of final image.*

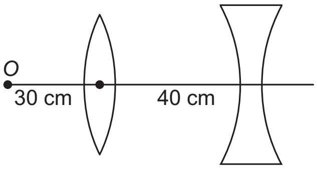


Fig. 31.62

Solution Using the lens formula for convex lens,

$$u = -30\,\text{cm},\ f = +20\,\text{cm}$$

$$\therefore \qquad \frac{1}{v} - \frac{1}{-30} = \frac{1}{+20}$$

Solving this equation, we get $v = +60\,\text{cm}$. Therefore, the first image is 60 cm to the right of convex lens or 20 cm to the right of concave lens. Again applying lens formula for concave lens,

$$u = +20\,\text{cm},\ f = -40\,\text{cm},\ \text{we have}$$

$$\frac{1}{v} - \frac{1}{+20} = \frac{1}{-40}$$

$$v = +40\,\text{cm}$$

So, the final image I_2 is formed at 40 cm to the right of concave lens as shown below.

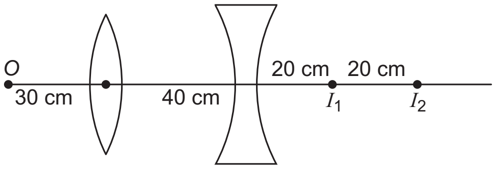


Fig. 31.63

Type 8. *Based on image velocity*

How to Solve?

- Using the methods discussed in Type 3, first find *v* and then *m*. Now,

(i) Along the axis, $\mathbf{v}_I = m^2 \mathbf{v}_0$

(ii) Perpendicular to axis, $\mathbf{v}_I = m\mathbf{v}_0$

Example 31.23 *Focal length of the convex lens shown in figure is* 20 *cm. Find the image position and image velocity.*

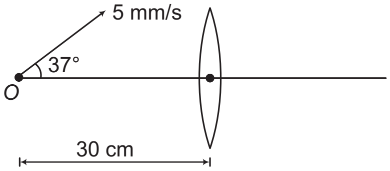


Fig. 31.64

Solution For the given condition,

$$u = -30\,\text{cm},\ f = +20\,\text{cm}$$

Using the lens formula, we have

$$\frac{1}{v} - \frac{1}{-30} = \frac{1}{+20}$$

Solving this equation, we get

$$v = +\,60\,\text{cm and}$$

$$m = \frac{v}{u} = \frac{+60}{-30} = -2$$

$$m^2 = 4$$

Component of velocity of object along the axis = $5\cos 37° = 4$ mm/s (towards the lens) component of velocity of image along the axis = m^2 (4 mm/s) = $4 \times 4 = 16$ mm/s. This component is away from the lens (in the same direction of object velocity component)

Component of velocity of object perpendicular to the axis = $5 \sin 37° = 3$ mm/s (upwards).

∴ Component of velocity of image perpendicular to axis = m(3 mm/s) or (–2)(3 mm/s) = –6 mm/s or this component is 6 mm/s downwards. These all points are shown in the figure given below.

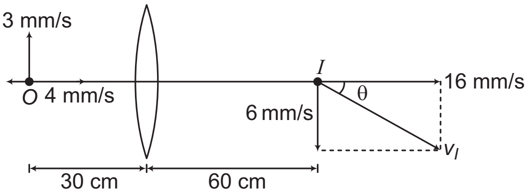


Fig. 31.65

$$v_I = \sqrt{(16)^2 + (6)^2}$$

$$= \sqrt{292}\ \text{mm/s}$$

$$\tan\theta = \frac{6}{16} = \frac{3}{8}$$

or $$\theta = \tan^{-1}(3/8)$$

Type 9. *An extended object kept perpendicular to principal axis*

How to Solve?

- Using the methods discussed in Type 3, first find *v* and then *m*. Now, suppose *m* is –2 and object is 1 mm above the principal axis, then image will be 2 mm below the principal axis.

Example 31.24 *Focal length of concave lens shown in figure is* 60 *cm. Find image position and its magnification.*

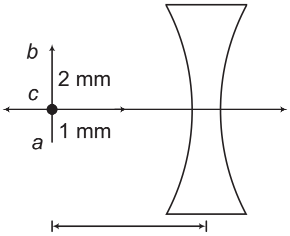


Fig. 31.66

Solution For the given situation,

$$u = -30\,\text{cm},\ f = -60\,\text{cm}$$

Using the lens formula, we have

$$\frac{1}{v} - \frac{1}{-30} = \frac{1}{-60}$$

Solving this equation, we get

$$v = -20\,\text{cm}$$

Further, $m = \frac{v}{u} = \frac{-20}{-30} = +\frac{2}{3}$. Point *b* is 2 mm above the principal axis. Therefore, its image *b*′ will be $(2)\left(\frac{2}{3}\right)$ or $\frac{4}{3}$ mm, above the principal axis.

Similarly, point *a* is 1 mm below the principal axis. Therefore, its image *a*′ will be $(1)\left(\frac{2}{3}\right)$ or $\frac{2}{3}$ mm below the principal axis. The final image is as shown in Fig. 31.67

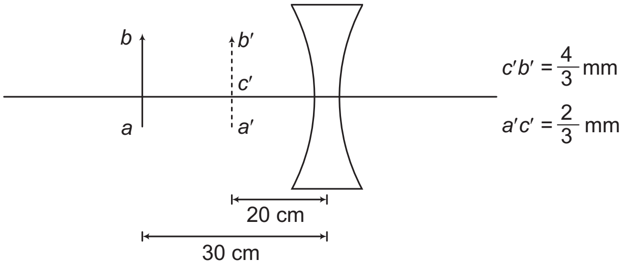


Fig. 31.67

Type 10. *To plot u versus v or $\frac{1}{u}$ versus $\frac{1}{v}$ graph*

Concept

In the previous chapter, we have seen that $\frac{1}{v}$ *versus* $\frac{1}{u}$ graph will be a straight line. Further for real objects, u is always negative. So, u varies from 0 to $-\infty$. Therefore, $\frac{1}{u}$ will vary from $-\infty$ to 0.

Example 31.25 *Plot u versus v and $\frac{1}{u}$ versus $\frac{1}{v}$ graph for convex lens (only for real objects)*

Solution

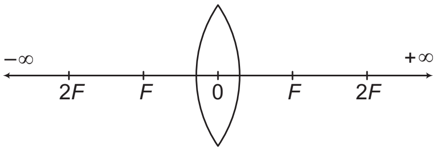


Fig. 31.68

Table 31.4

S.No.	u	v	$\frac{1}{u}$	$\frac{1}{v}$
1.	0 to $-f$	0 to $-\infty$	$-\infty$ to $-\frac{1}{f}$	$-\infty$ to 0
2.	$-f$ to $-2f$	$+\infty$ to $+2f$	$-\frac{1}{f}$ to $-\frac{1}{2f}$	0 to $+\frac{1}{2f}$
3.	$-2f$ to $-\infty$	$+2f$ to $+f$	$-\frac{1}{2f}$ to 0	$+\frac{1}{2f}$ to $+\frac{1}{f}$

***u versus* graph**

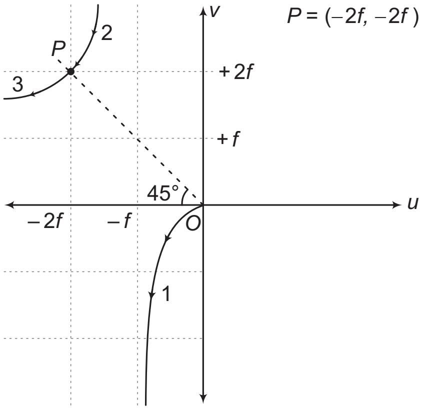


Fig. 31.69

$\frac{1}{u}$ *versus* $\frac{1}{v}$ graph

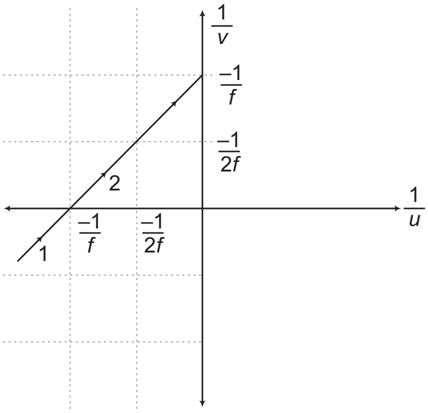


Fig. 31.70

Type 11. *Problems of inclined lenses*

Example 31.26

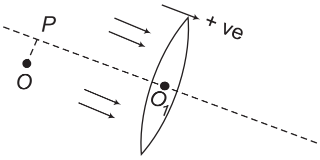


Fig. 31.71

O_1P is the principal axis and O is the point object. Given, $O_1P = 30$ cm, $f = 20$ cm and $OP = 2$ mm. Find the image distance and its position.

Solution For the given situation,

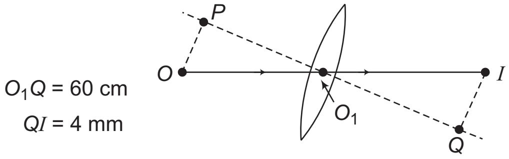


Fig. 31.72

$$u = -30 \text{ cm}$$

and
$$f = +20 \text{ cm}$$

Using the lens formula, we have

$$\frac{1}{v} - \frac{1}{-30} = \frac{1}{+20}$$

Solving this equation, we get

$$v = +60 \text{ cm}$$

Further, $$m = \frac{v}{u} = \frac{+60}{-30} = -2$$

Therefore, image is at a distance of + 60 cm from the lens at a distance of (2 mm) (–2) or 4 mm from the principal axis on other side of the object. The image is as shown above in Fig. 31.72.

Note *Image will always lie on the line joining O and O_1. This is because the ray OO_1 passes undeviated.*

Type 12. *To find focal length of an optical system for which either of the two conditions (thin lens and same medium on both sides) is not satisfied*

Concept

If focal length is asked then we have to find the second focal length f_2. The definition of F_2 is, if object is at infinity ($u_1 = \infty$) then final image after two refractions will be at F_2 ($v_2 = f_2$ or f). The use of thin lens is v_1 is exactly equal to u_2.

Example 31.27 *In the figure, light is incident on a thin lens as shown. The radius of curvature for both the surfaces is R. Determine the focal length of this system.* **(JEE 2003)**

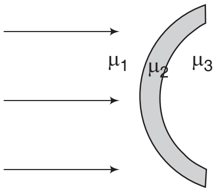


Fig. 31.73

Solution For refraction at first surface,

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{-\infty} = \frac{\mu_2 - \mu_1}{+R} \quad \text{...(i)}$$

For refraction at second surface,

$$\frac{\mu_3}{v_2} - \frac{\mu_2}{v_1} = \frac{\mu_3 - \mu_2}{+R} \quad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get $\frac{\mu_3}{v_2} = \frac{\mu_3 - \mu_1}{R}$ or $v_2 = \frac{\mu_3 R}{\mu_3 - \mu_1}$

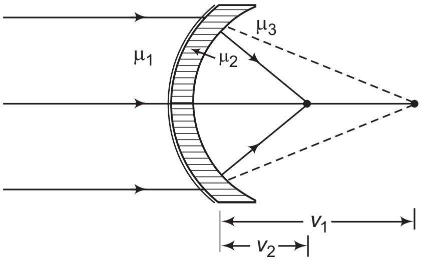


Fig. 31.74

Therefore, focal length of the given lens system is

$$f = \frac{\mu_3 R}{\mu_3 - \mu_1}$$

Note *If we find the focal length of the above system from right hand side, then it will be different because medium on both sides is not same.*

Important Points in Lens Theory

1.

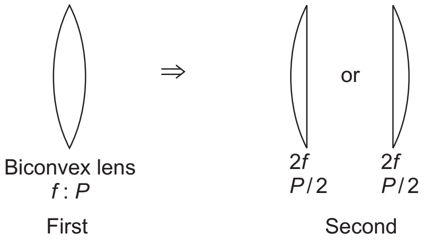


Fig. 31.75

In the first figure,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = (\mu - 1)\left(\frac{2}{R}\right)$$

$$\therefore \qquad f = \frac{R}{2(\mu - 1)}$$

In the second figure,

$$\frac{1}{f'} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{\infty}\right)$$

$$\therefore \qquad f' = \frac{R}{\mu - 1}$$

$$\text{or} \qquad f' = 2f$$

2.

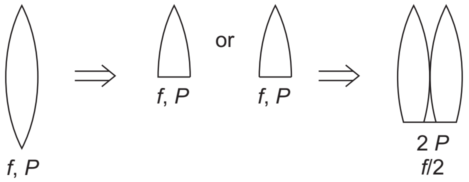


Fig. 31.76

3. The system shown in Fig. 31.77 has single value of u but two different parts will have two focal lengths. Therefore, we get two images I_1 and I_2, horizontally separated from each other. The two focal lengths are

$$\frac{1}{f_1} = (\mu_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and $$\frac{1}{f_2} = (\mu_2 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

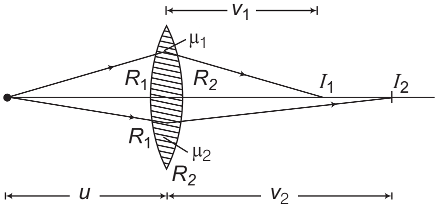


Fig. 31.77

4. This system shown in Fig. 31.78 has single values of u and f. Therefore, we will get single value of v. Still, we will get two images, vertically separated from each other. Let us take an example in support of this.

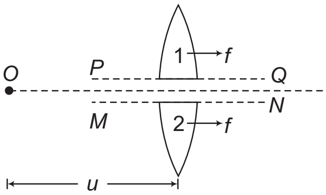


Fig. 31.78

If $u = 30\,\text{cm}$, f is 20 cm, principal axis PQ of upper part 1 is 1 mm above the object and principal axis MN of lower part 2 is 1 mm below the object O. Then, after applying lens formula, we get

$$v = +60\,\text{cm} \text{ and } m = -2$$

Now, O is 1 mm below PQ and $m = -2$. Therefore, upper part will make I_1, 2 mm above PQ. Similarly, O is 1 mm above MN, therefore lower part will make I_2, 2 mm below MN. Ray diagram from the two parts is as shown in Fig. 31.79 below

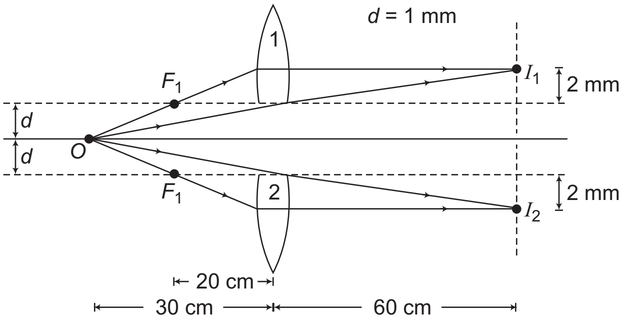


Fig. 31.79

5. If a liquid is filled between two thin convex glass lenses, then it is a group of three lenses as shown in figure.

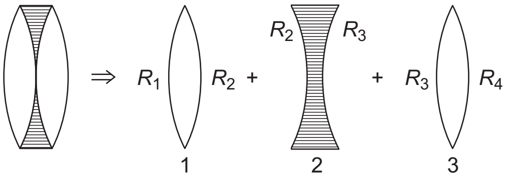


Fig. 31.80

$$\therefore \qquad \frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

where, $$\frac{1}{f_1}=(\mu_g-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{1}{f_2}=(\mu_l-1)\left(\frac{1}{R_2}-\frac{1}{R_3}\right) \quad \text{and}$$

$$\frac{1}{f_3}=(\mu_g-1)\left(\frac{1}{R_3}-\frac{1}{R_4}\right)$$

6. The system shown in figure behaves like a lens of zero power (or $f=\infty$). This is because

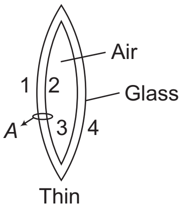


Fig. 31.81

$$R_1 \approx R_2 \quad \text{and} \quad R_3 \approx R_4$$

Now if we find focal length or power of part A, then

$$\frac{1}{f} \quad \text{or} \quad P=(\mu_g-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=0$$

as $$R_1 \approx R_2$$

Similarly, we can prove that power of other part is also zero.

7. Minimum distance between real object and its real image from a convex lens is $4f$.

Exercise Prove the above result.

8. **Silvered lens** A point object O is placed in front of a silvered lens as shown in figure.

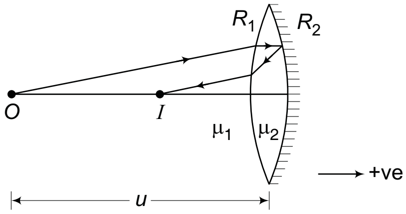


Fig. 31.82

Ray of light is first refracted, then reflected and then again refracted. In first two steps, light is travelling from left to right and in the last one direction of light is reversed. But we will take one sign convention, i.e. left to right as positive and in the last step will take v, u and R as negative.

$$\frac{\mu_2}{v_1}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R_1} \qquad \ldots\text{(i)}$$

$$\frac{1}{v_2}+\frac{1}{v_1}=\frac{1}{f_{\text{mirror}}}=\frac{2}{R_2} \qquad \ldots\text{(ii)}$$

$$\frac{\mu_1}{-v} - \frac{\mu_2}{-v_2} = \frac{\mu_1 - \mu_2}{-R_1} \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\frac{1}{v} + \frac{1}{u} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1} \qquad \text{...(iv)}$$

This is the desired formula for finding position of image for the given situation.

Note *The given system behaves as a mirror because the ray of light finally reflects in the same medium. Whose focal length can be found by comparing Eq. (iv) with mirror formula 1/v + 1/u = 1/f.*

$$\boxed{\frac{1}{f} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1}} \qquad \text{...(v)}$$

Let us take one example in support of this.

Example 31.28

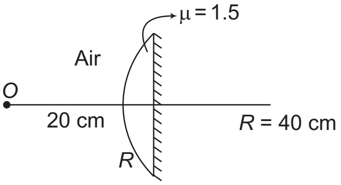


Fig. 31.83

(a) Find focal length of the system as shown in figure.
(b) Find image position.

Solution (a) $\mu_1 = 1, \mu_2 = 1.5, R_1 = R = +40\,\text{cm}$ and $R_2 = \infty$

Using the formula,

$$\frac{1}{f} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1}$$

$$= \frac{2(1.5)}{\infty} - \frac{2(1.5-1)}{40}$$

or $$f = -40\,\text{cm}$$

Thus, the given system behaves like a concave mirror of focal length 40 cm.

(b) Using the mirror formula, we have

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\therefore \quad \frac{1}{v} + \frac{1}{-20} = \frac{1}{-40}$$

$$\therefore \quad v = +40\,\text{cm}$$

Therefore, image will be formed at a distance of 40 cm to the right hand side of the given system.

9. **Displacement method of finding focal length of a convex lens** If the distance d between an object and screen is greater than 4 times the focal length of a convex lens, then there are two

positions of the lens between the object and the screen at which a sharp image of the object is formed on the screen. This method is called **displacement method** and is used in laboratory to determine the focal length of convex lens.

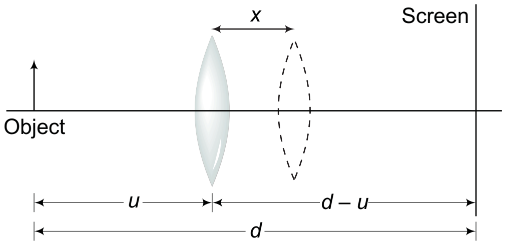


Fig. 31.84

To prove this, let us take an object placed at a distance u from a convex lens of focal length f. The distance of image from the lens $v = (d - u)$. From the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{d-u} - \frac{1}{-u} = \frac{1}{f}$$

or

$$u^2 - du + df = 0$$

$\therefore$

$$u = \frac{d \pm \sqrt{d\,(d-4f)}}{2}$$

Now, there are following possibilities:

(i) If $d < 4f$, then u is imaginary.

So, physically no position of the lens is possible.

(ii) If $d = 4f$, then $u = \frac{d}{2} = 2f$. So, only one position is possible. From here we can see that the **minimum distance between an object and its real image in case of a convex lens is 4 *f*.**

(iii) If $d > 4f$, there are two positions of lens at distances

$$\frac{d + \sqrt{d\,(d-4f)}}{2} \quad \text{and} \quad \frac{d - \sqrt{d\,(d-4f)}}{2}$$

for which real image is formed on the screen.

(iv) Suppose I_1 is the image length in one position of the object and I_2 the image length in second position, then object length O is given by

$$\boxed{O = \sqrt{I_1 I_2}}$$

This can be proved as under

$$|u_1| = \frac{d + \sqrt{d\,(d-4f)}}{2}$$

$\therefore$

$$|v_1| = d - |u_1| = \frac{d - \sqrt{d\,(d-4f)}}{2}$$

$$|u_2| = \frac{d - \sqrt{d(d-4f)}}{2}$$

$\therefore$ $$|v_2| = d - |u_2| = \frac{d + \sqrt{d(d-4f)}}{2}$$

Now, $$|m_1 m_2| = \frac{I_1}{O} \times \frac{I_2}{O} = \frac{|v_1|}{|u_1|} \times \frac{|v_2|}{|u_2|}$$

Substituting the values, we get $$\frac{I_1 I_2}{O^2} = 1$$

or $$O = \sqrt{I_1 I_2}$$ **Hence Proved.**

(v) Focal length of the lens is given by

$$\boxed{f = \frac{d^2 - x^2}{4d}}$$

Proof In the figure, we can see that difference of two values of u is x. Thus,

$$|u_1| - |u_2| = x$$

or $$\left[\frac{d + \sqrt{d(d-4f)}}{2}\right] - \left[\frac{d - \sqrt{d(d-4f)}}{2}\right] = x$$

Solving this equation, we can find that

$$f = \frac{d^2 - x^2}{4d}$$

Example 31.29 *A thin plano-convex lens of focal length f is split into two halves. One of the halves is shifted along the optical axis as shown in figure. The separation between object and image planes is 1.8 m. The magnification of the image formed by one of the half lens is 2. Find the focal length of the lens and separation between the two halves. Draw the ray diagram for image formation.* **(JEE 1996)**

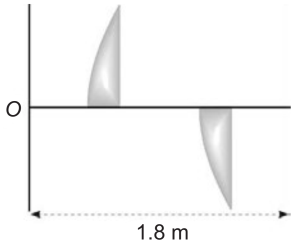


Fig. 31.85

Solution For both the halves, position of object and image is same. Only difference is of magnification. Magnification for one of the halves is given as $2\,(>1)$. This can be for the first one, because for this, $|v| > |u|$. Therefore, magnification, $|m| = |v/u| > 1$.

So, for the first half

$$|v/u| = 2$$

or $$|v| = 2|u|$$

Let $$u = -x,$$

then $$v = +2x$$

and $$|u| + |v| = 1.8 \text{ m}$$

i.e. $$3x = 1.8 \text{ m}$$

or $$x = 0.6 \text{ m}$$

Hence, $$u = -0.6 \text{ m}$$

and $$v = +1.2 \text{ m}$$

Using $$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{1.2} - \frac{1}{-0.6} = \frac{1}{0.4}$$

∴ $$f = 0.4 \text{ m}$$ **Ans.**

For the second half, $$\frac{1}{f} = \frac{1}{1.2 - d} - \frac{1}{-(0.6 + d)}$$

or $$\frac{1}{0.4} = \frac{1}{1.2 - d} + \frac{1}{(0.6 + d)}$$

Solving this, we get $$d = 0.6 \text{ m}$$ **Ans.**

Magnification for the second half will be

$$m_2 = \frac{v}{u} = \frac{0.6}{-(1.2)} = -\frac{1}{2}$$

and magnification for the first half is

$$m_1 = \frac{v}{u} = \frac{1.2}{-(0.6)} = -2$$

The ray diagram is as follows:

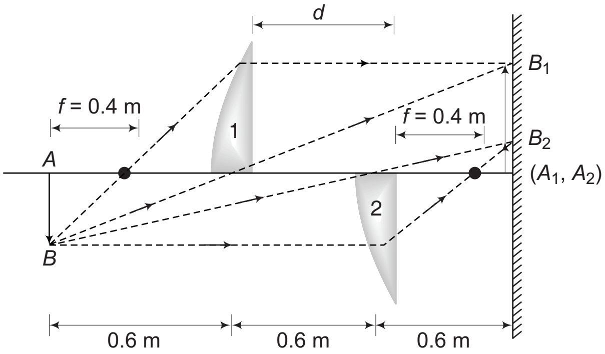


Fig. 31.86

INTRODUCTORY EXERCISE 31.6

1. When an object is placed 60 cm in front of a diverging lens, a virtual image is formed 20 cm from the lens. The lens is made of a material of refractive index $\mu = 1.65$ and its two spherical surfaces have the same radius of curvature. What is the value of this radius?
2. A converging lens has a focal length of 30 cm. Rays from a 2.0 cm high filament that pass through the lens form a virtual image at a distance of 50 cm from the lens. Where is the filament located? What is the height of the image?
3. Show that the focal length of a thin lens is not changed when the lens is rotated so that the left and the right surfaces are interchanged.
4. As an object is moved from the surface of a thin converging lens to a focal point, over what range does the image distance vary?
5. A diverging lens is made of material with refractive index 1.3 and has identical concave surfaces of radius 20 cm. The lens is immersed in a transparent medium with refractive index 1.8.
   (a) What is now the focal length of the lens?
   (b) What is the minimum distance that an immersed object must be from the lens so that a real image is formed?
6. An object is located 20 cm to the left of a converging lens with $f = 10$ cm. A second identical lens is placed to the right of the first lens and then moved until the image it produces is identical in size and orientation to the object. What is the separation between the lenses?
7. Suppose an object has thickness du so that it extends from object distance u to $u + du$. Prove that the thickness dv of its image is given by $\left(-\frac{v^2}{u^2}\right) du$, so the longitudinal magnification $\frac{dv}{du} = -m^2$, where m is the lateral magnification.
8. Two thin similar convex glass pieces are joined together front to front, with its rear portion silvered such that a sharp image is formed 0.2 m for an object at infinity. When the air between the glass pieces is replaced by water $\left(\mu = \frac{4}{3}\right)$, find the position of image.
9. When a pin is moved along the principal axis of a small concave mirror, the image position coincides with the object at a point 0.5 m from the mirror. If the mirror is placed at a depth of 0.2 m in a transparent liquid, the same phenomenon occurs when the pin is placed 0.4 m from the mirror. Find the refractive index of the liquid shown in Fig. 31.87.

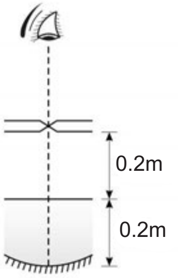


Fig. 31.87

10. When a lens is inserted between an object and a screen which are a fixed distance apart the size of the image is either 6 cm or $\frac{2}{3}$ cm. Find size of the object.
11. A lens of focal length 12 cm forms an upright image three times the size of a real object. Find the distance in cm between the object and image.
12. The distance between an object and its upright image is 20 cm. If the magnification is 0.5, what is the focal length of the lens that is being used to form the image?
13. A thin lens of focal length + 10.0 cm lies on a horizontal plane mirror. How far above the lens should an object be held if its image is to coincide with the object?

31.7 Total Internal Reflection (TIR)

(i) When a ray of light strikes the boundary separating two different media, then part of it is refracted and part is reflected.

(ii) If a ray of light is travelling from a denser to a rarer medium with angle of incidence greater than a critical angle ($i > \theta_C$), then no refraction takes place but ray of light is 100% reflected in the same medium. This phenomenon is called TIR.

(iii) $\sin \theta_C = \dfrac{\mu_R}{\mu_D}$

Here, R stands for rarer medium and D for denser medium. If rarer medium is air, then

$$\sin \theta_C = \frac{1}{\mu}$$

$$\therefore \qquad \boxed{\theta_C = \sin^{-1}\left(\frac{\mu_R}{\mu_D}\right) \quad \text{or} \quad \sin^{-1}\left(\frac{1}{\mu}\right)}$$

(iv) If value of μ increases, then critical angle θ_C decreases. Therefore, chances of TIR increase in travelling from denser to rarer medium.

(v)

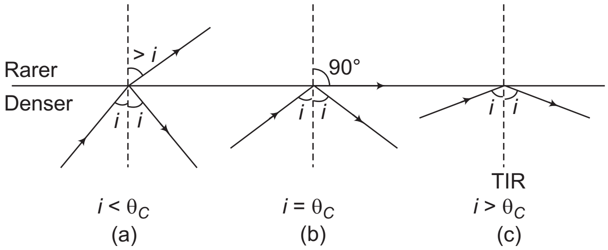


Fig. 31.88

Applying Snell's law of refraction in Fig. (b), we have

$$\mu_R \sin 90° = \mu_D \sin \theta_C$$

or

$$\sin \theta_C = \frac{\mu_R}{\mu_D}$$

or

$$\theta_C = \sin^{-1}\left(\frac{\mu_R}{\mu_D}\right)$$

(vi) In critical case (Fig. b), angle in denser medium is θ_C and angle in rarer medium is 90°.

TIR has following applications

(i) Totally reflecting prisms Refractive index of crown glass is 3/2. Hence,

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{2}{3}\right) \approx 42°$$

A ray OA incident normally on face PQ of a crown glass prism suffers TIR at face PR since, the angle of incidence in the optically denser medium is 45°. A bright ray AB emerges at right angles from face QR. The prism thus, reflects the ray through 90°. Light can be reflected through 180° and an erect image can be obtained of an inverted one if the prism is arranged as shown in figure (b).

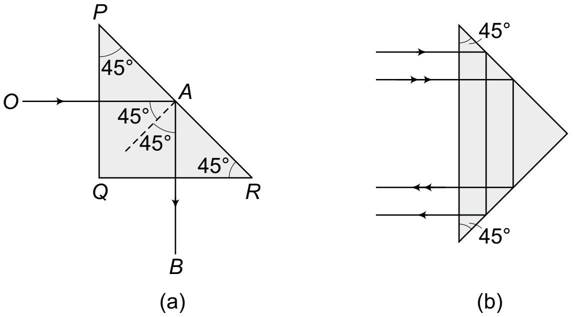


Fig. 31.89 Prism reflectors

(ii) Optical fibres Light can be confined within a bent glass rod by TIR and so 'piped' along a twisted path as in figure. The beam is reflected from side to side practically without loss (except for that due to absorption in the glass) and emerges only at the end of the rod where it strikes the surface almost normally, i.e. at an angle less than the critical angle. A single, very thin, solid glass fibre behaves in the same way and if several thousands are taped together a flexible light pipe is obtained that can be used, for example in medicine and engineering to illuminate an inaccessible spot. Optical fibres are now a days used to carry telephone, television and computer signals from one place to the other.

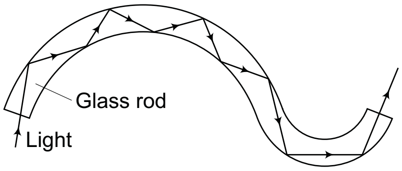


Fig. 31.90 Principle of an optical fibre

Note *As we have seen*

$$\theta_C = \sin^{-1}\left(\frac{\mu_R}{\mu_D}\right)$$

Suppose we have two sets of media 1 *and* 2 *and*

$$\left(\frac{\mu_R}{\mu_D}\right)_1 < \left(\frac{\mu_R}{\mu_D}\right)_2$$

then $(\theta_C)_1 < (\theta_C)_2$

So, a ray of light has more chances to have TIR in case 1.

Examples of TIR

Example 31.30 *An isotropic point source is placed at a depth h below the water surface. A floating opaque disc is placed on the surface of water so that the source is not visible from the surface. What is the minimum radius of the disc? Take refractive index of water* $= \mu$.

Solution

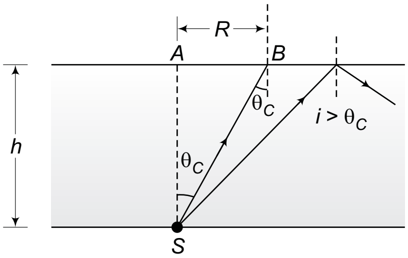


Fig. 31.91

As shown in figure light from the source will not emerge out of water if $i > \theta_C$.

Therefore, minimum radius R corresponds to $i = \theta_C$

In ΔSAB,

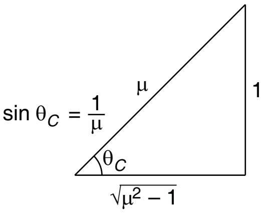


Fig. 31.92

$$\frac{R}{h} = \tan\theta_C$$

$$\therefore \quad R = h\tan\theta_C$$

$$\text{or} \quad R = \frac{h}{\sqrt{\mu^2 - 1}}$$

Ans.

Note *Only that portion of light refracts in air which falls on the circle (on the surface of water) with A as centre and AB as radius.*

Example 31.31 *A point source of light is placed at a distance h below the surface of a large and deep lake. Show that the fraction f of light that escapes directly from water surface is independent of h and is given by*

$$f = \frac{[1 - \sqrt{1 - 1/\mu^2}\,]}{2}$$

Solution Due to TIR, light will be reflected back into the water if $i > \theta_C$. So, only that portion of incident light will escape which passes through the cone of angle $\theta = 2\theta_C$.

So, the fraction of light escaping

$$f = \frac{\text{area } ACB}{\text{Total area of sphere}} = \frac{2\pi R^2 (1 - \cos\theta_C)}{4\pi R^2} = \frac{1 - \cos\theta_C}{2}$$

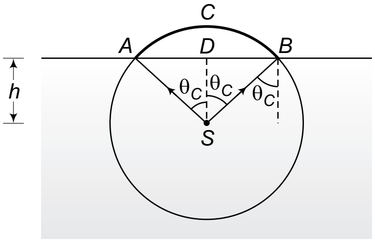


Fig. 31.93

Now, as f depends on θ_C and which depends only on μ, it is independent of h. **Proved.**

Further $$\cos\theta_C = \frac{\sqrt{\mu^2 - 1}}{\mu} = \sqrt{1 - 1/\mu^2} \quad \Rightarrow \quad f = \frac{1 - \sqrt{1 - 1/\mu^2}}{2}$$ **Ans.**

Note *Area of ACB = $2\pi R^2(1 - \cos\theta_C)$ can be obtained by integration. In the above example, we have seen that light falling on the circle with centre at D and radius DB will only refract in air and in the absence of water surface only that light would fall on surface ACB of sphere.*

Example 31.32 *In the figure shown, $\mu_1 > \mu_2$. Find minimum value of i so that TIR never takes place at P.*

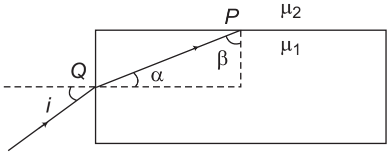


Fig. 31.94

Solution Let us take $\beta = \theta_C$. Then, $\alpha = 90° - \beta$ or $\alpha = 90° - \theta_C$

Here, $$\sin\theta_C = \frac{\mu_R}{\mu_D} = \frac{\mu_2}{\mu_1}$$

$$\therefore \quad \theta_C = \sin^{-1}\left(\frac{\mu_2}{\mu_1}\right)$$

Applying Snell's law at point *Q*,

$$\mu_2 \sin i = \mu_1 \sin\alpha = \mu_1 \sin(90° - \theta_C)$$

$$= \mu_1 \cos\theta_C = \mu_1 \cos\sin^{-1}\left(\frac{\mu_2}{\mu_1}\right)$$

$$\therefore \quad i = \sin^{-1}\left[\frac{\mu_1}{\mu_2}\cos\sin^{-1}\left(\frac{\mu_2}{\mu_1}\right)\right]$$ **Ans.**

Now, we can see that starting from this value of *i*, we get $\beta = \theta_C$.

If *i* is increased from this value, then α will also increase (becomes $> 90° - \theta_C$) and β decreases from θ_C (becomes $< \theta_C$) and no TIR takes place at *P*.

So far no TIR condition *i* has to be increased from the above value. Or this is the minimum value of *i*.

Example 31.33 *Monochromatic light is incident on a plane interface AB between two media of refractive indices μ_1 and μ_2 ($\mu_2 > \mu_1$) at an angle of incidence θ as shown in the figure.*

The angle θ is infinitesimally greater than the critical angle for the two media so that total internal reflection takes place. Now if a transparent slab DEFG of uniform thickness and of refractive index μ_3 is introduced on the interface (as shown in the figure), show that for any value of μ_3 all light will ultimately be reflected back again into medium II. Consider separately the cases. **(1986, 6M)**

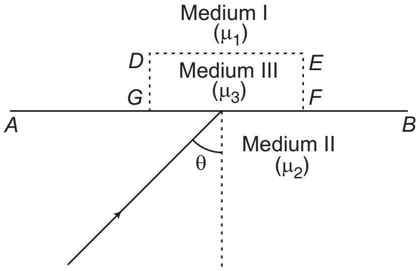


Fig. 31.95

(a) $\mu_3 < \mu_1$ *(b)* $\mu_3 > \mu_1$

Solution Given, θ is slightly greater than $\sin^{-1}\left(\frac{\mu_1}{\mu_2}\right)$.

(a) **When** $\mu_3 < \mu_1$

i.e. $$\mu_3 < \mu_1 < \mu_2$$

or $$\frac{\mu_3}{\mu_2} < \frac{\mu_1}{\mu_2} \quad \text{or} \quad \sin^{-1}\left(\frac{\mu_3}{\mu_2}\right) < \sin^{-1}\left(\frac{\mu_1}{\mu_2}\right)$$

Hence, critical angle for III and II will be less than the critical angle for II and I. So, if TIR is taking place between I and II, then TIR will definitely take place between I and III.

(b) **When** $\mu_3 > \mu_1$ Two cases may arise :

Case 1 $\mu_1 < \mu_3 < \mu_2$

In this case, there will be no TIR between II and III but TIR will take place between III and I.
This is because

Ray of light first enters from II to III. i.e. from denser to rarer.

$\therefore$ $$i > \theta$$

Applying Snell's law at P,

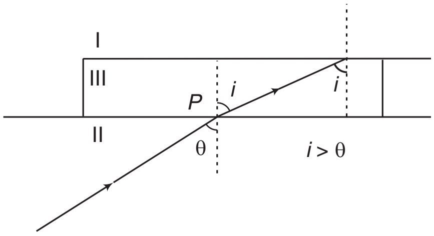


Fig. 31.96

$$\mu_2 \sin\theta = \mu_3 \sin i \quad \text{or} \quad \sin i = \left(\frac{\mu_2}{\mu_3}\right)\sin\theta$$

Since, $\sin\theta$ is slightly greater than $\frac{\mu_1}{\mu_2}$.

$\therefore$ $\sin i$ is slightly greater than $\frac{\mu_2}{\mu_3} \times \frac{\mu_1}{\mu_2}$ or $\frac{\mu_1}{\mu_3}$

but $\frac{\mu_1}{\mu_3}$ is nothing but $\sin(\theta_C)_{\text{I, III}}$

$\therefore$ $\sin(i)$ is slightly greater than $\sin(\theta_C)_{\text{I, III}}$

or TIR will now take place on I and III and the ray of light will be reflected back.

Case 2 $\mu_1 < \mu_2 < \mu_3$

This time while moving from II to III, ray of light will bend towards normal. Again applying Snell's law at P,

$$\mu_2 \sin\theta = \mu_3 \sin i$$

$$\sin i = \frac{\mu_2}{\mu_3}\sin\theta$$

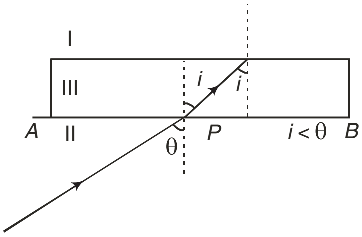


Fig. 31.97

Since, $\sin\theta$ is slightly greater than $\frac{\mu_1}{\mu_2}$.

Therefore, $\sin i$ will be slightly greater than $\frac{\mu_2}{\mu_3} \times \frac{\mu_1}{\mu_2}$ or $\frac{\mu_1}{\mu_3}$

But, $\quad \dfrac{\mu_1}{\mu_3}$ is $\sin(\theta_C)_{\text{I, III}}$

i.e. $\quad \sin i > \sin(\theta_C)_{\text{I, III}}$

or $\quad i > (\theta_C)_{\text{I, III}}$

Therefore, TIR will again take place between I and III and the ray of light will be reflected back.

Note *Two cases of $\mu_3 > \mu_1$ can be explained by one single equation. But two cases are deliberately taken for better understanding of refraction, Snell's law and total internal reflection (TIR).*

Example 31.34 *A right angled prism is to be made by selecting a proper material and the angles A and B ($B \le A$), as shown in figure. It is desired that a ray of light incident on the face AB emerges parallel to the incident direction after two internal reflections.*

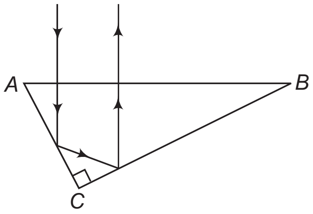


Fig. 31.98

(a) What should be the minimum refractive index n for this to be possible? **(JEE 1987)**

(b) For $n = 5/3$ is it possible to achieve this with the angle B equal to 30 degrees?

Solution (a) At *P*, angle of incidence $i_A = A$ and at *Q*, angle of incidence $i_B = B$

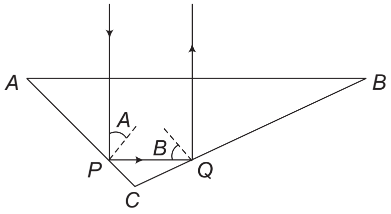


Fig. 31.99

If TIR satisfies for the smaller angle of incidence than for larger angle of incidence is automatically satisfied.

$$B \le A$$

$$\therefore \quad i_B \le i_A$$

Maximum value of *B* can be 45°. Therefore, if condition of TIR is satisfied for 45°, then condition of TIR will be satisfied for all value of i_A and i_B.

Thus, $\quad 45° \ge \theta_C \quad$ or $\quad \sin 45° \ge \sin\theta_C$

or $\quad \dfrac{1}{\sqrt{2}} \ge \dfrac{1}{\mu} \quad$ or $\quad \mu \ge \sqrt{2}$

$\therefore$ Minimum value of μ is $\sqrt{2}$.

(b) For $\mu = \frac{5}{3}$, $\sin \theta_C = \frac{1}{\mu} = \sin^{-1}\left(\frac{3}{5}\right) \approx 37°$

If $B = 30°$, then $i_B = 30°$ and $A = 60°$ or $i_A = 60°$, $i_A > \theta_C$ but $i_B < \theta_C$ i.e. TIR will take place at A but not at B.

INTRODUCTORY EXERCISE 31.7

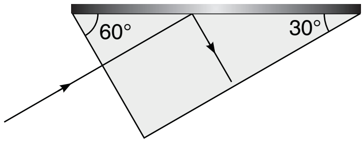


Fig. 31.100

1. Light is incident normally on the short face of a $30° - 60° - 90°$ prism. A liquid is poured on the hypotenuse of the prism. If the refractive index of the prism is $\sqrt{3}$, find the maximum refractive index of the liquid so that light is totally reflected.
2. If the speed of light in ice is 2.3×10^8 m/s, what is its index of refraction? What is the critical angle of incidence for light going from ice to air?

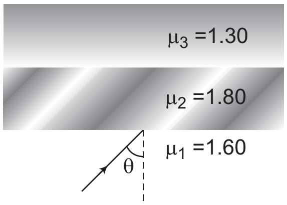


Fig. 31.101

3. In figure, light refracts from material 1 into a thin layer of material 2, crosses that layer, and then is incident at the critical angle on the interface between materials 2 and 3.
   (a) What is the angle θ?
   (b) If θ is decreased, is there refraction of light into material 3?

31.8 Refraction Through Prism

A prism has two plane surfaces AB and AC inclined to each other as shown in figure. $\angle A$ is called the **angle of prism** or **refracting angle of prism.**

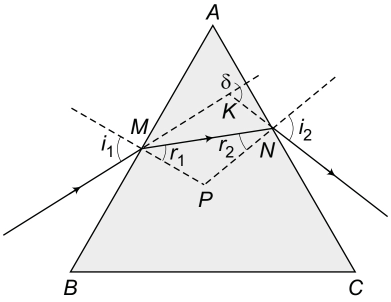


Fig. 31.102

General Formulae

(i) In quadrilateral $AMPN$, $\angle AMP + \angle ANP = 180°$

$\therefore$ $A + \angle MPN = 180°$ ...(i)

In triangle MNP, $r_1 + r_2 + \angle MPN = 180°$ ...(ii)

From Eqs. (ii) and (iii), we have $\boxed{r_1 + r_2 = A}$ ...(iii)

(ii) **Deviation** Deviation δ means angle between incident ray and emergent ray.

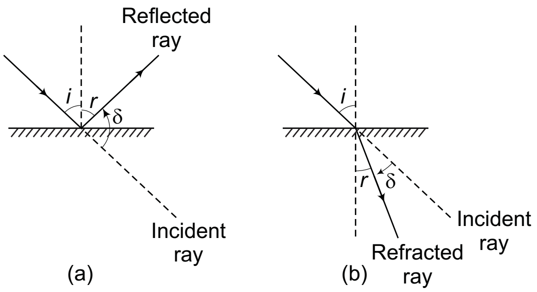


Fig. 31.103

In reflection,

$$\delta = 180° - 2i = 180° - 2r$$

In refraction,

$$\delta = |i - r|$$

In prism, a ray of light gets refracted twice one at M and other at N. At M its deviation is $i_1 - r_1$ and at N it is $i_2 - r_2$. These two deviations are added as both are clockwise. So, the net deviation is

$$\delta = (i_1 - r_1) + (i_2 - r_2) = (i_1 + i_2) - (r_1 + r_2) = (i_1 + i_2) - A$$

Thus, $$\boxed{\delta = (i_1 + i_2) - A} \qquad \text{...(iv)}$$

(iii) **If A and i_1 are small** The expression for the deviation in this case is basically used for developing the lens theory. Consider a ray falling almost normally in air on a prism of small angle A (less than about 6° or 0.1 radian) so that angle i_1 is small. Now, $\mu = \frac{\sin i_1}{\sin r_1}$ or $\sin i_1 = \mu \sin r_1$, therefore, r_1 will also be small. Since, sine of a small angle is nearly equal to the angle in radians, we have

$$i_1 \approx \mu r_1$$

Also, $A = r_1 + r_2$ and so if A and r_1 are small then r_2 and i_2 will also be small. From $\mu = \frac{\sin i_2}{\sin r_2}$, we can say $i_2 \approx \mu r_2$

Substituting these values in Eq. (iv), we have

$$\delta = (\mu r_1 + \mu r_2) - A$$
$$= \mu(r_1 + r_2) - A = \mu A - A$$

or $$\boxed{\delta = (\mu - 1) A} \qquad \text{...(v)}$$

This expression shows that all rays entering a small angle prism at small angles of incidence suffer the same deviation.

(iv) **Minimum deviation** It is found that the angle of deviation δ varies with the angle of incidence i_1 of the ray incident on the first refracting face of the prism. The variation is shown in figure and for one angle of incidence it has a minimum value δ_{min}. At this value, the ray passes symmetrically through the prism (a fact that can be proved theoretically as well as be shown experimentally), i.e. the angle of emergence of the ray from the second face equals the angle of incidence of the ray on the first face.

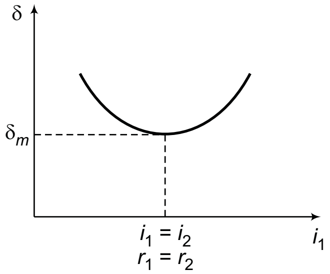


Fig. 31.104

$$i_2 = i_1 = i \qquad \text{...(vi)}$$

It therefore, follows that

$$r_1 = r_2 = r \qquad \text{...(vii)}$$

From Eqs. (iii) and (vii), we get $\quad r = \dfrac{A}{2}$

Further at $\quad \delta = \delta_m = (i + i) - A$

or
$$i = \frac{A + \delta_m}{2} \qquad \text{...(viii)}$$

∴ $\quad \mu = \dfrac{\sin i}{\sin r}$

or
$$\boxed{\mu = \frac{\sin\left(\dfrac{A + \delta_m}{2}\right)}{\sin \dfrac{A}{2}}} \qquad \text{...(ix)}$$

(v) **Condition of no emergence** In this section, we want to find the condition such that a ray of light entering the face AB does not come out of the face AC for any value of angle i_1, i.e. TIR takes place on AC

$$r_1 + r_2 = A$$

∴ $\quad r_2 = A - r_1$

or
$$(r_2)_{\min} = A - (r_1)_{\max} \qquad \text{...(x)}$$

Now, r_1 will be maximum when i_1 is maximum and maximum value of i_1 can be 90°.

Hence, $\quad \mu = \dfrac{\sin(i_1)_{\max}}{\sin(r_1)_{\max}} = \dfrac{\sin 90°}{\sin (r_1)_{\max}}$

∴ $\quad \sin (r_1)_{\max} = \dfrac{1}{\mu} = \sin \theta_C$

∴ $\quad (r_1)_{\max} = \theta_C$

∴ From Eq. (x),
$$(r_2)_{\min} = A - \theta_C \qquad \text{...(xi)}$$

Now, if minimum value of r_2 is greater than θ_C then obviously all values of r_2 will be greater than θ_C and TIR will take place under all conditions. Thus, the condition of no emergence is

$$(r_2)_{\min} > \theta_C \quad \text{or} \quad A - \theta_C > \theta_C$$

or
$$\boxed{A > 2\theta_C} \qquad \ldots\text{(xii)}$$

(vi) **Dispersion and deviation of light by a prism** White light is a superposition of waves with wavelengths extending throughout the visible spectrum. The speed of light in vacuum is the same for all wavelengths, but the speed in a material substance is different for different wavelengths. Therefore, the index of refraction of a material depends on wavelength. In most materials, the value of refractive index μ decreases with increasing wavelength.

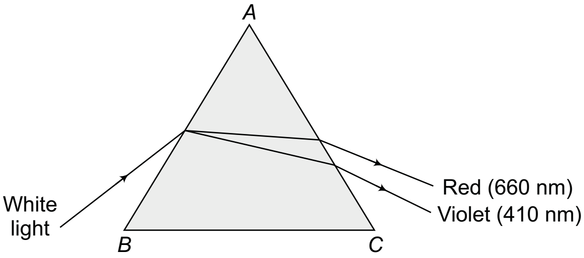


Fig. 31.105

If a beam of white light, which contains all colours, is sent through the prism, it is separated into a spectrum of colours. The spreading of light into its colour components is called **dispersion.**

Dispersive Power

When a beam of white light is passed through a prism of transparent material, light of different wavelengths are deviated by different amounts. If δ_r, δ_y and δ_v are the deviations for red, yellow and violet components then average deviation is measured by δ_y as yellow light falls in between red and violet. $\delta_v - \delta_r$ is called **angular dispersion.** The **dispersive power** of a material is defined as the ratio of angular dispersion to the average deviation when a white beam of light is passed through it. It is denoted by ω. As we know

$$\delta = (\mu - 1) A$$

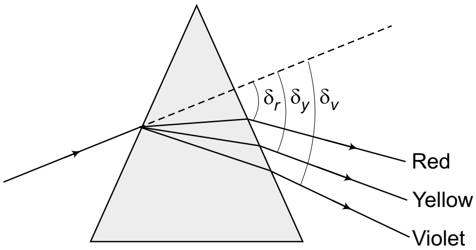


Fig. 31.106

This equation is valid when A and i are small. Suppose, a beam of white light is passed through such a prism, the deviation of red, yellow and violet light are

$$\delta_r = (\mu_r - 1)A, \; \delta_y = (\mu_y - 1)A$$

and
$$\delta_v = (\mu_v - 1)\,A$$

The angular dispersion is $\delta_v - \delta_r = (\mu_v - \mu_r)A$ and the average deviation is $\delta_y = (\mu_y - 1)A$. Thus, the dispersive power of the medium is

$$\boxed{\omega = \frac{\mu_v - \mu_r}{\mu_y - 1}} \qquad \text{...(i)}$$

Dispersion without Average Deviation and Average Deviation without Dispersion

Figure shows two prisms of refracting angles A and A' and dispersive powers ω and ω' respectively. They are placed in contact in such a way that the two refracting angles are reversed with respect to each other. A ray of light passes through the combination as shown. The deviations produced by the two prisms are

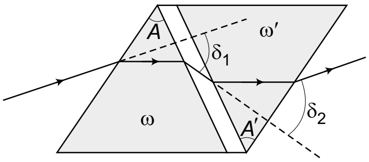


Fig. 31.107

$$\delta_1 = (\mu - 1)A \quad \text{and} \quad \delta_2 = (\mu' - 1)A'$$

As the two deviations are opposite to each other, the net deviation is

$$\delta = \delta_1 - \delta_2 = (\mu - 1)A - (\mu' - 1)A' \qquad \text{...(ii)}$$

Using this equation, the average deviation produced by the combination if white light is passed is

$$\delta_y = (\mu_y - 1)A - (\mu'_y - 1)A' \qquad \text{...(iii)}$$

and the net angular dispersion is

$$\delta_v - \delta_r = (\mu_v - \mu_r)A - (\mu'_v - \mu'_r)A'$$

But, as $\mu_v - \mu_r = \omega(\mu_y - 1)$ from Eq. (i), we have

$$\delta_v - \delta_r = (\mu_y - 1)\omega A - (\mu'_y - 1)\omega' A' \qquad \text{...(iv)}$$

Dispersion without average deviation From Eq. (iii),

$$\delta_y = 0 \quad \text{if}$$

$$\boxed{\frac{A}{A'} = \frac{\mu'_y - 1}{\mu_y - 1}} \qquad \text{...(v)}$$

This is the required condition of dispersion without average deviation. Using this in Eq. (iv), the net angular dispersion produced is

$$\delta_v - \delta_r = (\mu_y - 1)A(\omega - \omega')$$

Average deviation without dispersion

From Eq. (iv),

$$\delta_v - \delta_r = 0 \text{ if}$$

$$\boxed{\frac{A}{A'} = \frac{(\mu'_y - 1)\omega'}{(\mu_y - 1)\omega} = \frac{\mu'_v - \mu'_r}{\mu_v - \mu_r}} \qquad \text{...(vi)}$$

This is the required condition of average deviation without dispersion. Using the above condition in Eq. (iii), the net average deviation is

$$\delta_y = (\mu_y - 1)A\left(1 - \frac{\omega}{\omega'}\right)$$

Important Points in Prism

1. Equation $r_1 + r_2 = A$ can be applied at any of the three vertices. For example in the figure shown, $r_1 + r_2 = B$.

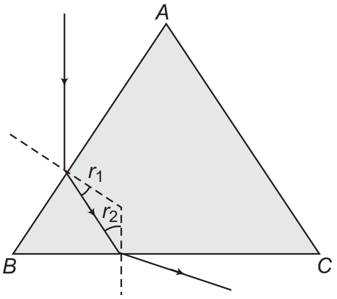
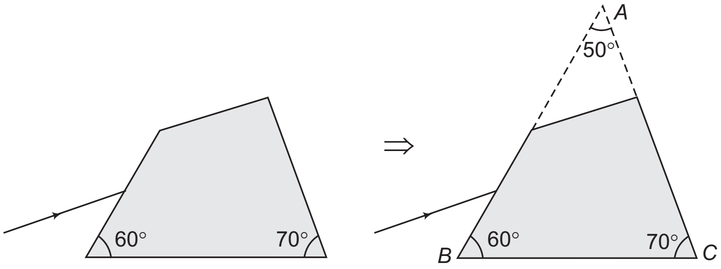


Fig. 31.108

2. Sometimes a part of a prism is given as shown in Fig. 31.109 (a). To solve such problems, first complete the prism then solve as the problems of prism are solved.



(a) (b)

Fig. 31.109

3. For isosceles or equilateral triangle ($\angle B = \angle C$), ray of light is parallel to base of the prism at minimum deviation condition.

   Under minimum deviation condition, we know that $r_1 = r_2 \Rightarrow \alpha = \beta$. Because, $\alpha = 90° + r_1$ and $\beta = 90° + r_2$.

   Further, it is given that $\angle B = \angle C$. Therefore, MN is parallel to BC.

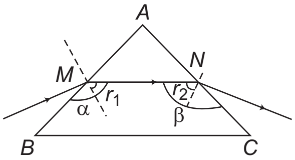


Fig. 31.110

4. In some cases, when deviation at M is clockwise and deviation at N is anti-clockwise, then

$$r_1 - r_2 = A$$

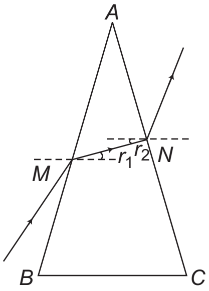


Fig. 31.111

This condition is normally obtained with thin angle prisms as shown above:

At M, deviation is clockwise and at N deviation is anti-clockwise. In triangle AMN,

$$A + (90° - r_1) + (90° + r_2) = 180°$$

$$\therefore \qquad r_1 - r_2 = A$$

5. In the ray diagram shown in Fig. 31.112, we can treat it like a prism ABC of $\angle A = 90°$

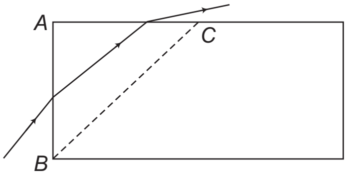


Fig. 31.112

6. Different identical equilateral triangles are arranged as shown in Fig. 31.113. Deviation by prism(s) in each case will be same, if angle of incidence is same.

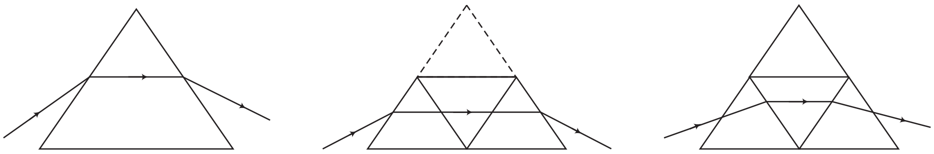


Fig. 31.113

Examples of Prism

Example 31.35 *One face of a prism with a refractive angle of 30° is coated with silver. A ray of light incident on another face at an angle of 45° is refracted and reflected from the silver coated face and retraces its path. What is the refractive index of the prism?*

Solution

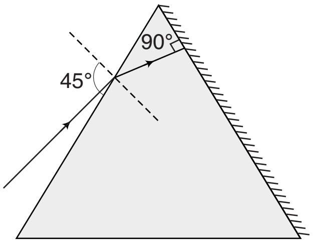


Fig. 31.114

Given, $A = 30°$, $i_1 = 45°$ and $r_2 = 0$

Since, $$r_1 + r_2 = A$$

$$\therefore \qquad r_1 = A = 30°$$

Now, refractive index of the prism,

$$\mu = \frac{\sin i_1}{\sin r_1} = \frac{\sin 45°}{\sin 30°} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} = \sqrt{2} \qquad \textbf{Ans.}$$

Example 31.36 *In the shown figure, mirror is rotated by an angle θ. Find θ if*

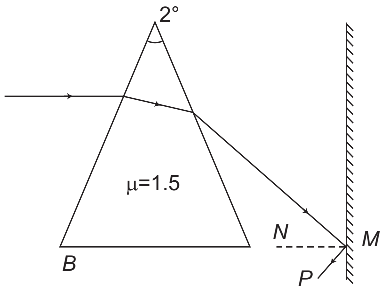


Fig. 31.115

(a) ray of light retraces its path after reflection from M

(b) ray of light MP turns in the direction of MN.

Solution In this case, A and i_1 are small. Therefore, deviation by prism can be obtained by

$$\delta = (\mu - 1)A = (1.5 - 1)(2°) = 1°$$

Hence, the prism will deviate the ray of light by 1° from its original path as shown in Fig. 31.116.

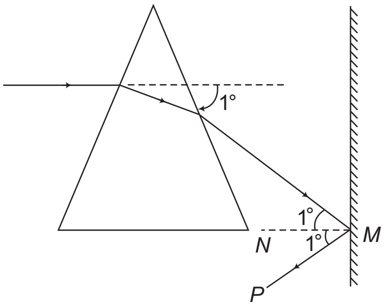


Fig. 31.116

(a) In the absence of prism, ray of light was falling normal to the mirror. So, ray of light was already retracing its path. Prism has rotated the ray by 1° in clockwise direction. So if we rotate the mirror also by 1° in clockwise direction, then ray of light will further fall normal to the mirror and it again retraces its path. Therefore, the correct answer is

$$\theta = 1°, \text{clockwise}$$ **Ans.**

(b) If we rotate a plane mirror by θ in clockwise direction, then reflected ray also rotates in clockwise direction by an angle 2θ.

Here, we have to rotate reflected ray MP by 1° in clockwise direction to make it in the direction of MN. Therefore, we will have to rotate the mirror by 0.5° in clockwise direction. Therefore, the correct answer is

$$\theta = 0.5°, \text{ clockwise}$$ **Ans.**

Example 31.37 ***General method of finding deviation by prism.***

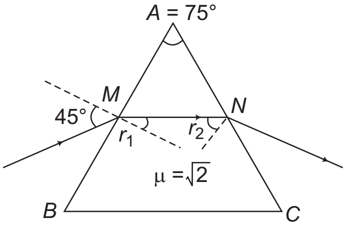


Fig. 31.117

In the ray diagram shown in figure, find total deviation by prism.

Solution Deviation by a prism is given by

$$\delta = (i_1 + i_2) - A \qquad \text{...(i)}$$

Here, $i_1 = 45°$ and $A = 75°$

Therefore, the main objective is to find i_2 and this angle i_2 can be obtained under following three steps.

(i) Applying Snell's law at point M, we have

$$\mu = \frac{\sin i_1}{\sin r_1} \quad \text{or} \quad \sqrt{2} = \frac{\sin 45°}{\sin r_1}$$

Solving this equation, we get $r_1 = 30°$

(ii) $$r_1 + r_2 = A$$

$\therefore$ $$30° + r_2 = 75°$$

$\therefore$ $$r_2 = 45°$$

(iii) Further applying Snell's law at N,

$$\mu = \frac{\sin i_2}{\sin r_2} \Rightarrow \sqrt{2} = \frac{\sin i_2}{\sin 45°}$$

Solving this equation, we get $i_2 = 90°$

Now, substituting the values in Eq. (i), we have

$$\delta = 45° + 90° - 75°$$
$$= 60°$$

Ans.

Example 31.38 *Based on the condition of no emergence from face AC*

In the shown figure,

$$A = 110°, B = 20°, C = 50°, i_1 = 45° \text{ and } \mu = \sqrt{2}$$

Find the total deviation by prism.

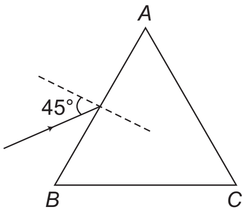


Fig. 31.118

Solution $\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

Here, $A > 2\theta_C$

Therefore, TIR will take place at AC and the ray of light emerges from the prism as shown in figure.

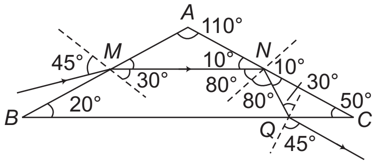


Fig. 31.119

Applying, $\mu = \frac{\sin i}{\sin r}$ at M and Q, we can find the respective angles as shown in figure.

Now, deviation at M is clockwise, deviation at N is clockwise but deviation at Q is anti-clockwise.

$\therefore$
$$\delta_{\text{Total}} = \delta_M + \delta_N - \delta_Q$$
$$= (45° - 30°) + (180° - 2 \times 80°) - (45° - 30°)$$
$$= 20°$$

Ans.

INTRODUCTORY EXERCISE 31.8

1. The prism shown in figure has a refractive index of 1.60 and the angles A are 30°. Two light rays P and Q are parallel as they enter the prism. What is the angle between them after they emerge? $[\sin^{-1}(0.8) = 53°]$

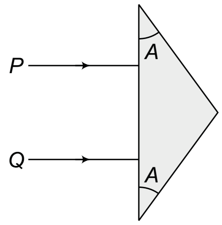


Fig. 31.120

2. A glass vessel in the shape of a triangular prism is filled with water, and light is incident normally on the face xy. If the refractive indices for water and glass are 4/3 and 3/2 respectively, total internal reflection will occur at the glass-air surface xz only for sin θ greater than

   A 1/2 **B** 2/3
   C 3/4 **D** 8/9
   E 16/27.

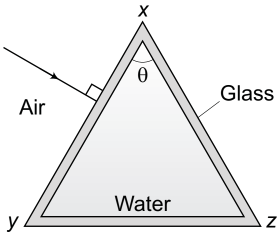


Fig. 31.121

3. A light ray going through a prism with the angle of prism 60°, is found to deviate at least by 30°. What is the range of the refractive index of the prism?
4. A ray of light falls normally on a refracting face of a prism. Find the angle of prism if the ray just fails to emerge from the prism ($\mu = 3/2$).
5. A ray of light is incident at an angle of 60° on one face of a prism which has an angle of 30°. The ray emerging out of the prism makes an angle of 30° with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges and calculate the refractive index of the material of prism.
6. A ray of light passing through a prism having refractive index $\sqrt{2}$ suffers minimum deviation. It is found that the angle of incidence is double the angle of refraction within the prism. What is the angle of prism?
7. A ray of light undergoes deviation of 30° when incident on an equilateral prism of refractive index $\sqrt{2}$. What is the angle subtended by the ray inside the prism with the base of the prism?
8. Light is incident at an angle i on one planar end of a transparent cylindrical rod of refractive index μ. Find the least value of μ so that the light entering the rod does not emerge from the curved surface of the rod irrespective of the value of i.

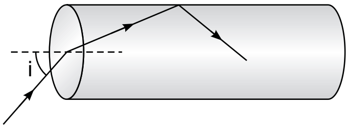


Fig. 31.122

9. The refractive index of the material of a prism of refracting angle 45° is 1.6 for a certain monochromatic ray. What will be the minimum angle of incidence of this ray on the prism so that no TIR takes place as the ray comes out of the prism.

31.9 Deviation

1. **In reflection** In reflection, the deviation is given by

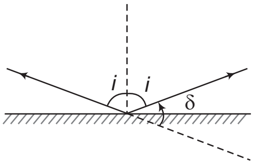


Fig. 31.123

$$\boxed{\delta = 180° - 2i}$$

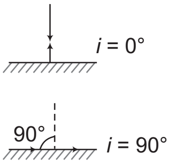


Fig. 31.124

Therefore, δ *versus* i graph is a straight line. The range of i is from 0° to 90°.

At $i = 0°, \delta = 180°$

At $i = 90°, \delta = 0°$

δ *versus* i graph is as shown below:

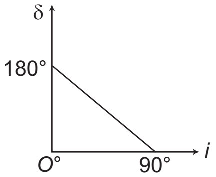


Fig. 31.125

2. Two plane mirrors at 90°, deviate all rays by 180° from their original path.

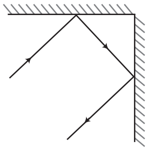


Fig. 31.126

3. **In refraction** Deviation in refraction is given by

$$\boxed{\delta = i - r} \qquad \text{(where, } i > r\text{)}$$

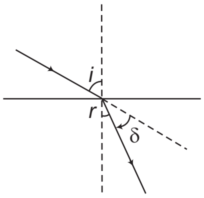


Fig. 31.127

Note *(i) To plot δ versus i graph, first we will have to convert r into i with the help of Snell's law, otherwise there are three variables in the equation,*

$$\delta = i - r$$

(ii) In general, deviation in reflection is more than the deviation in refraction.

4. Deviation by a sphere after two refractions is

$$\delta = 2(i - r)$$

This can be proved as under :

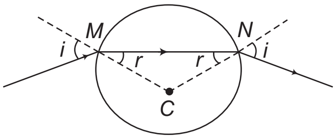


Fig. 31.128

$$MC = NC$$

Therefore, both angles inside the sphere are same (= r). Hence, angles outside the sphere will also be same (= i). Deviation at M as well as N is clockwise.

$$\therefore \quad \delta = \delta_M + \delta_N = (i - r) + (i - r) \quad \text{or} \quad \delta = 2(i - r)$$

5. **Deviation by a prism**

$$\boxed{\delta = (i_1 + i_2) - A}$$

If A and i_1 are small, then

$$\boxed{\delta \approx (\mu - 1)\,A}$$

Example 31.39 ***Theory***

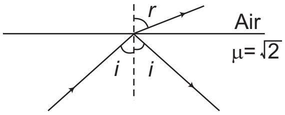


Fig. 31.129

In the figure, i is increased from 0° to 90°. But ray of light is travelling from denser to rarer medium. Therefore, TIR will take place when $i > \theta_C$, *where*

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

From 0° – 45°, refraction and reflection both will take place. After 45°, only reflection will take place.

Question *Plot δ versus i graph between incident ray and refracted ray, for* $i \le 45°$ *and with reflected ray for* $i \ge 45°$.

Solution **For $i \le 45°$**

$$\delta = \delta_{\text{Refraction}} = r - i \qquad (\text{as } r > i)$$

Applying Snell's law, we have

$$\mu = \frac{\sin r}{\sin i} \quad \text{or} \quad \sqrt{2} = \frac{\sin r}{\sin i}$$

$$\therefore \quad r = \sin^{-1}(\sqrt{2}\sin i)$$

Substituting this value of r in the equation, we have

$$\delta = \sin^{-1}(\sqrt{2}\sin i) - i \qquad \ldots\text{(i)}$$

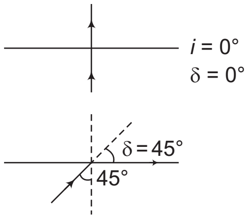


Fig. 31.130

Now, variables are only two δ and i but this is not a known equation. So, we can find some of the coordinates from where graph must pass.

At $i = 0°$, $\delta = 0°$

At $i = 45° = \theta_C$, $\delta = 45°$

For $i \geq 45°$

$$\delta = \delta_{\text{Reflection}}$$

$$\therefore \qquad \delta = 180° - 2i$$

Now, δ *versus* i graph is a straight line.

At $i = 45°$, $\delta = 90°$

At $i = 90°$, $\delta = 0°$

Now, δ *versus* i graph is as shown in Fig. 31.131:

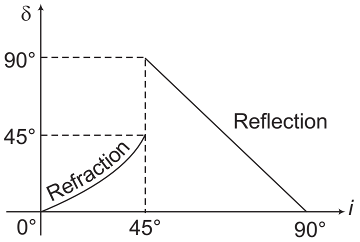


Fig. 31.131

Note *At* $i = 45° = \theta_C$

$\delta_{\text{Refraction}} = 45°$

but $\delta_{\text{Reflection}} = 90°$

This is because deviation in reflection is more than deviation in refraction.

31.10 Optical Instruments

Optical instruments are used to assist the eye in viewing an object. Our eye lens has a power to adjust its focal length to see the nearer objects. This process of adjusting focal length is called accommodation. However, if the object is brought too close to the eye, the focal length cannot be adjusted to form the image on the retina. Thus, there is a minimum distance for the clear vision of an object. This distance is called **least distance of distinct vision (+D).** For normal eye this distance is generally taken to be 25 cm.

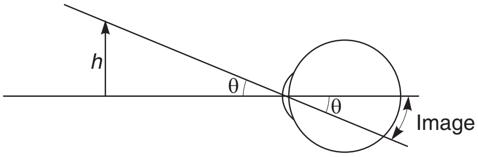


Fig. 31.132

Visual angle The size of an object as sensed by us is related to the size of the image formed on the retina.

The size of the image on the retina is roughly proportional to the angle subtended by the object on the eye. This angle is known as the visual angle. Optical instruments are used to increase this angle artificially in order to improve the clarity.

Magnifying power (M) Magnifying power is the factor by which the image on the retina can be enlarged by using the microscope or telescope. For a microscope and for a telescope the definition of M is slightly different.

For a microscope, $$M = \frac{\text{Visual angle formed by final image}}{\text{Visual angle formed by the object when kept at distance } D}$$

For a telescope, $M = \dfrac{\text{Visual angle formed by final image}}{\text{Visual angle subtended by the object directly when seen from naked eye}}$

Note that M is different from linear magnification $m\left(=\pm\dfrac{v}{u}\right)$ which is the ratio of height of image to that of object while M is the ratio of apparent increase in size of image seen by the eye. Unit of M is X, thus we write an angular magnification of 10 as $10X$.

Simple microscope To view an object with naked eyes, the object must be placed between D and infinity. The maximum angle is subtended when it is placed at D.

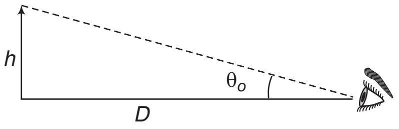


Fig. 31.133

Say this angle is θ_o. Then, $\theta_o = \dfrac{h}{D}$

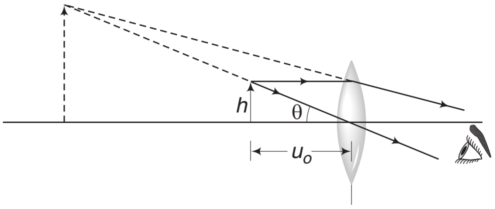


Fig. 31.134

This angle can be further increased if a converging lens of short focal length is placed just in front of the eye. The lens used for this purpose is called simple microscope or a magnifier.

The object is placed at a distance u_o from the lens (between pole and focus of lens). The virtual magnified image is formed as shown. This image subtends a visual angle say θ on the eye. Then,

$$\boxed{\theta = \frac{h}{u_o}}$$

From the definition of magnifying power for a microscope,

$$M = \frac{\theta}{\theta_o} = \frac{h/u_o}{h/D}$$

$$\therefore \qquad \boxed{M = \frac{D}{u_o}}$$

For relaxed eye The final image should be at infinity. Thus, $u_o = f$

$$\therefore \qquad \boxed{M_\infty = \frac{D}{f}}$$

This is also called magnifying power for normal adjustment.

We can see that $M_\infty > 1$ if $f < D$.

Magnifying power when final image is at D In the above case, we saw that M is equal $\frac{D}{f}$. The magnification can be made large by choosing the focal length f small.

The magnifying power can be increased in an another way by moving the object still closer to the lens. Suppose, the final virtual image is formed at a distance D. Then, from the equation $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we have

$$\frac{1}{-D}+\frac{1}{u_o}=\frac{1}{f}$$

or

$$\frac{1}{u_o}=\frac{1}{D}+\frac{1}{f}$$

Substituting this value in the equation, $M=\frac{D}{u_o}$, we have

$$M_D=1+\frac{D}{f}$$

Note *(i) That $M_D > M_\infty$, i.e. when final image is formed at 25 cm, angular magnification is increased but eye is most strained. On the other hand when final image is at infinity, angular magnification is slightly less but eye is relaxed. So, the choice is yours whether you want to see bigger size with strained eye or smaller size with relaxed eye.*

(ii) That M can be increased by decreasing f, but due to several other aberrations the image becomes too defective at large magnification with a simple microscope. Roughly speaking a magnification upto 4 is trouble free.

Compound Microscope Figure shows a simplified version of a compound microscope. It consists of two converging lenses arranged coaxially. The one facing the object is called objective and the one close to eye is called eyepiece. The objective has a smaller aperture and smaller focal length than those of the eyepiece.

The separation between the objective and the eyepiece (called the length of the microscope L) can be varied by appropriate screws fixed on the panel of microscope.

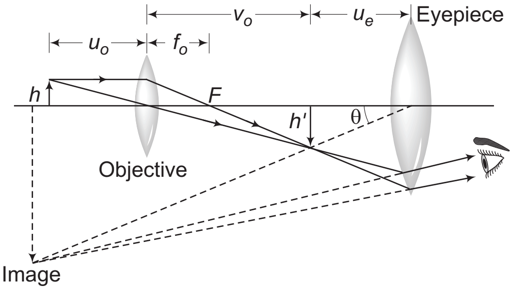


Fig. 31.135

The object is placed beyond first focus of objective, so that an inverted and real image (intermediate image) is formed by the objective. This intermediate image acts as an object for the eyepiece and lies between first focus and pole of eye piece. The final magnified virtual image is formed by the eyepiece. Let θ be the angle subtended by the final image on the eye, then,

$$\theta = \frac{h'}{u_e}$$

Here, h' is the height of the first image and u_e is its distance from the eyepiece.

Further $$\theta_o = \frac{h}{D}$$

∴ Magnifying power of the compound microscope will be

$$M = \frac{\theta}{\theta_o} = \frac{h'}{u_e} \times \frac{D}{h} = \left(\frac{h'}{h}\right)\left(\frac{D}{u_e}\right)$$

Here, $\frac{h'}{h}$ is the linear magnification by the objective. Thus,

$$\frac{h'}{h} = |m_0| = \frac{v_0}{u_0}$$

∴ $$\boxed{M = \frac{v_0}{u_0}\left(\frac{D}{u_e}\right)}$$

Length of the microscope will be

$$\boxed{L = v_0 + u_e}$$

For relaxed eye For relaxed eye final image should be at infinity. Or,

$$\boxed{u_e = f_e \quad \therefore \quad M_\infty = \frac{v_0}{u_0}\frac{D}{f_e}}$$

and $$L_\infty = v_0 + f_e$$

Final image at *D* When the final image (by eyepiece) is formed at D. Then, by the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ we have

$$\frac{1}{-D} + \frac{1}{u_e} = \frac{1}{f_e}$$

∴ $$\frac{1}{u_e} = \frac{1}{D} + \frac{1}{f_e}$$

or $$u_e = \frac{Df_e}{D + f_e}$$

Thus, $$\boxed{M_D = \frac{v_0}{u_0}\left(1 + \frac{D}{f_e}\right)}$$

and $$\boxed{L_D = v_0 + \frac{Df_e}{D + f_e}}$$

Telescopes A microscope is used to view the objects placed closed to it. To look at distant objects such as a star, a planet or a distant tree etc., we use telescopes. There are three types of telescopes in use.

(i) Astronomical telescope,

(ii) Terrestrial telescope and

(iii) Galilean telescope.

(i) **Astronomical telescope** Figure shows the construction and working of an astronomical telescope.

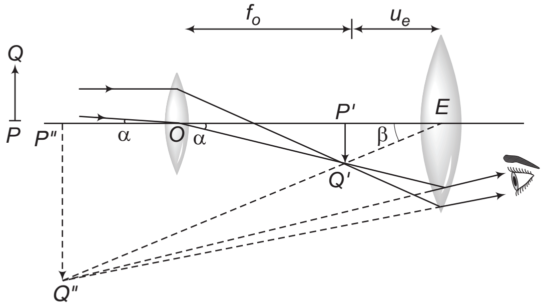


Fig. 31.136

It consists of two converging lenses placed coaxially. The one facing the distant object is called the objective and has a large aperture and large focal length. The other is called the eyepiece, as the eye is placed closed to it. The eyepiece tube can slide within the objective tube, so that the separation between the objective and the eyepiece may be varied.

Magnifying power Although a telescope can also be used to view the objects of few kilometers away but the magnifying power calculated below is for the case when object is at infinity. Rays coming from the object in that case will be almost parallel.

The image formed by objective will be at its second focus. This image called the intermediate image will act as the object for eyepiece. This usually lies between pole and first focus of eyepiece. So that eyepiece forms a virtual and magnified image of it.

$|\alpha|$ = angle subtended by object on objective (or you can say at eye)

$$= \frac{P'Q'}{f_o}$$

$|\beta|$ = angle subtended by final image at eyepiece (or at eye)

$$= \frac{P'Q'}{u_e}$$

From the definition of magnifying power (for telescope),

$$M = \frac{|\beta|}{|\alpha|} = \frac{f_o}{u_e} \quad \text{or} \quad \boxed{M = \frac{f_o}{u_e}}$$

and length of telescope,

$$L = f_o + u_e$$

For relaxed eye For relaxed eye, intermediate image should lie at first focus of eyepiece or

$$u_e = f_e$$

$$\therefore \qquad \boxed{M_\infty = \frac{f_o}{f_e} \quad \text{and} \quad L_\infty = f_o + f_e}$$

Final image at *D* When the final image is at *D*, then using the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ for eyepiece we have,

$$\frac{1}{-D} + \frac{1}{u_e} = \frac{1}{f_e}$$

$$\therefore \qquad \frac{1}{u_e} = \frac{1}{D} + \frac{1}{f_e}$$

$$\text{or} \qquad u_e = \frac{Df_e}{D + f_e}$$

$$\text{Therefore,} \qquad M_D = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right)$$

$$\text{and} \qquad L_D = f_o + \frac{Df_e}{D + f_e}$$

(ii) Terrestrial telescope In an astronomical telescope, the final image is inverted with respect to the object. To remove this difficulty, a convex lens of focal length *f* is included between the objective and the eyepiece in such a way that the focal plane of the objective is a distance 2*f* away from this lens.

The role of the intermediate lens *L* is only to invert the image. The magnification produced by it is – 1. The formula of *M* does not change at all. They remain as it is, as were derived for astronomical telescope. The length of telescope will however increase by 4*f*. Here, you should note that we are talking only about magnitude of *M*. Thus,

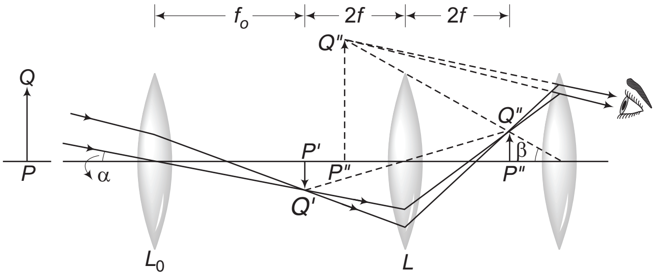


Fig. 31.137

$$\boxed{\begin{aligned} M_\infty &= \frac{f_o}{f_e} \quad \text{and} \quad M_D = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right) \\ L_\infty &= f_o + 4f + f_e \quad \text{and} \quad L_D = f_o + 4f + \frac{Df_e}{D + f_e} \end{aligned}}$$

(iii) Galilean telescope Figure shows a simple model of Galilean telescope. A convergent lens is used as the objective and a divergent lens as the eyepiece. The objective lens forms a real and inverted image $P'Q'$ but the divergent lens comes in between. This intermediate image acts as virtual object for eyepiece. Final image $P''Q''$ is erect and magnified as shown in figure. The intermediate image is formed at second focus of objective.

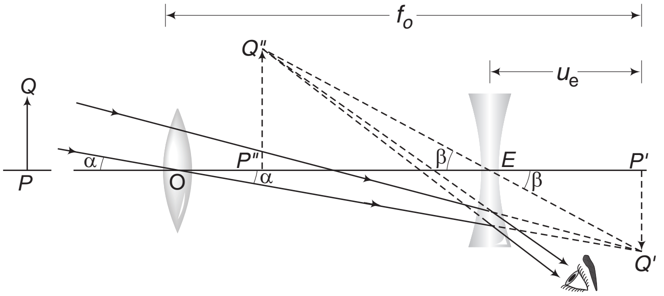


Fig. 31.138

Magnifying power From the figure, we can see that

$$|\alpha| = \frac{P'Q'}{f_O} \quad \text{and} \quad |\beta| = \frac{P'Q'}{u_e}$$

From the definition of magnifying power for telescope,

$$\boxed{M = \frac{|\beta|}{|\alpha|} = \frac{f_O}{u_e}}$$

and length of the telescope,

$$\boxed{L = f_O - u_e}$$

For relaxed eye For relaxed eye intermediate image should lie at first focus of eyepiece. Or,

$$u_e = f_e$$

Hence,

$$\boxed{M_\infty = \frac{f_o}{f_e} \quad \text{and} \quad L_\infty = f_o - f_e}$$

Final image at *D* For the final image to be at a distance D from the eyepiece, we have from the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$$\frac{1}{-D} - \frac{1}{u_e} = \frac{1}{-f_e} \quad \Rightarrow \quad \frac{1}{u_e} = \frac{1}{f_e} - \frac{1}{D}$$

or

$$u_e = \frac{Df_e}{D - f_e}$$

Thus,

$$\boxed{M_D = \frac{f_o}{f_e}\left(1 - \frac{f_e}{D}\right)}$$

and

$$\boxed{L_D = f_o - \frac{f_e D}{D - f_e}}$$

Note *(i) In all above formulae of M, we are considering only the magnitude of M.*

(ii) For telescopes, formulae have been derived when the object is at infinity. For the object at some finite distance different formulae will have to be derived.

(iii) Given below are formulae derived above of M and L in tabular form.

Table 31.5

Name of optical instruments	M	L	M_∞	M_D	L_∞	L_D
Simple microscope	$\frac{D}{u_o}$	—	$\frac{D}{f}$	$1+\frac{D}{f}$	—	—
Compound microscope	$\frac{v_o}{u_o}\frac{D}{u_e}$	v_o+u_e	$\frac{v_o}{u_o}\frac{D}{f_e}$	$\frac{v_o}{u_o}\left(1+\frac{D}{f_e}\right)$	v_o+f_e	$v_o+\frac{Df_e}{D+f_e}$
Astronomical telescope	$\frac{f_o}{u_e}$	f_o+u_e	$\frac{f_o}{f_e}$	$\frac{f_o}{f_e}\left(1+\frac{f_e}{D}\right)$	f_o+f_e	$f_o+\frac{Df_e}{D+f_e}$
Terrestrial telescope	— do —	f_o+4f+u_e	— do —	— do —	f_o+4f+f_e	$f_o+4f+\frac{Df_e}{D+f_e}$
Galilean telescope	$\frac{f_o}{u_e}$	f_o-u_e	$\frac{f_o}{f_e}$	$\frac{f_o}{f_e}\left(1-\frac{f_e}{D}\right)$	f_o-f_e	$f_o-\frac{f_eD}{D-f_e}$

Resolving Power of a Microscope and a Telescope

Microscope The resolving power of a microscope is defined as the reciprocal of the distance between two objects which can be just resolved when seen through the microscope. It depends on the wavelength λ of the light, the refractive index μ of the medium between the object and the objective and the angle θ subtended by a radius of the objective on one of the objects.

$$\boxed{R=\frac{1}{\Delta d}=\frac{2\mu\sin\theta}{\lambda}}$$

To increase R, objective and object are immersed in oil.

Telescope The resolving power of a telescope is defined as the reciprocal of the angular separation between two distant objects which are just resolved by a telescope.

It is given by

$$\boxed{R=\frac{1}{\Delta\theta}=\frac{a}{1.22\lambda}}$$

Here, a is the diameter of the objective. That is why, telescopes with larger objective aperture are used.

Example 31.40 *An astronomical telescope has an angular magnification of magnitude 5 for distant objects. The separation between the objective and eyepiece is 36 cm and the final image is formed at infinity. Determine the focal length of objective and eyepiece.*

Solution For final image at infinity,

$$M_\infty = \frac{f_o}{f_e} \quad \text{and} \quad L_\infty = f_o + f_e$$

$$\therefore \qquad 5 = \frac{f_o}{f_e} \qquad \ldots\text{(i)}$$

$$\text{and} \qquad 36 = f_o + f_e \qquad \ldots\text{(ii)}$$

Solving these two equations, we have

$$f_o = 30\,\text{cm} \quad \text{and} \quad f_e = 6\,\text{cm}$$ **Ans.**

Example 31.41 *A telescope has an objective of focal length 50 cm and an eyepiece of focal length 5 cm. The least distance of distinct vision is 25 cm. The telescope is focused for distinct vision on a scale 2 m away from the objective. Calculate (a) magnification produced and (b) separation between objective and eyepiece.*

Solution Given, $f_o = 50\,\text{cm}$ and $f_e = 5\,\text{cm}$

Note *Here, object is placed at finite distance from the objective. Hence, formulae derived for angular magnification M cannot be applied directly as they have been derived for the object to be at infinity. Here, it will be difficult to find angular magnification. So, only linear magnification can be obtained.*

For objective $$\frac{1}{v_o} - \frac{1}{-200} = \frac{1}{50}$$

$$\therefore \qquad v_o = \frac{200}{3}\,\text{cm}$$

$$m_o = \frac{v_o}{u_o} = \frac{(200/3)}{-200} = -\frac{1}{3}$$

For eyepiece $$\frac{1}{-25} - \frac{1}{u_e} = \frac{1}{5}$$

$$\therefore \qquad u_e = -\frac{25}{6}\,\text{cm}$$

and $$m_e = \frac{v_e}{u_e} = \frac{-25}{-(25/6)} = 6$$

(a) Magnification, $$m = m_o \times m_e = -2$$ **Ans.**

(b) Separation between objective and eyepiece,

$$L = v_o + |u_e| = \frac{200}{3} + \frac{25}{6} = \frac{425}{6}$$

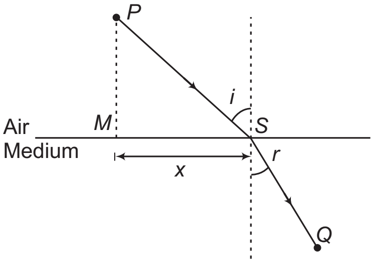
$$= 70.83\text{ cm}$$ **Ans.**

Final Touch Points

Fermat's principle :

1. A ray of light follows that path in reaching from one point to another point along which it takes the shortest time. Let us take an example.

   In figure, P is a point in air and Q a point in a medium of refractive index μ.



In air, speed of light is c and in the medium speed of light is $v = c/\mu$. Suppose that, the ray of light follows a path PSQ where S is at a distance x from M. Then, time taken by the ray of light in reaching from P to Q is

$$t = \frac{PS}{c} + \frac{SQ}{v} = \frac{PS}{c} + \frac{SQ}{(c/\mu)} \quad \text{or} \quad t = \frac{PS}{c} + \frac{\mu SQ}{c}$$

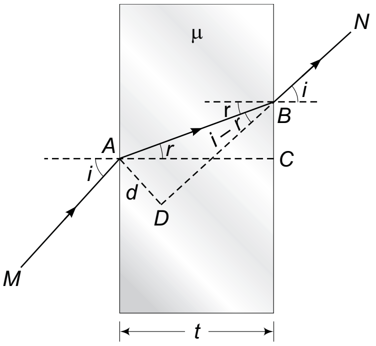
To make the time minimum one has to differentiate it with respect to x and find the point S when t is a minimum. From differentiation (we are skipping here) we find that t is minimum when,

$$\frac{\sin i}{\sin r} = \mu$$

This is nothing but the Snell's law.

2. **Lateral shift** We have already discussed that ray MA is parallel to ray BN. But the emergent ray is displaced laterally by a distance d, which depends on μ, t and i and its value is given by the relation,

$$\boxed{d = t\left(1 - \frac{\cos i}{\sqrt{\mu^2 - \sin^2 i}}\right)\sin i}$$



Proof $AB = \frac{AC}{\cos r} = \frac{t}{\cos r}$ (as $AC = t$)

Now, $d = AB \sin(i - r)$

$$= \frac{t}{\cos r}[\sin i \cos r - \cos i \sin r]$$

or $$d = t[\sin i - \cos i \tan r] \qquad \ldots\text{(i)}$$

Further, $$\mu = \frac{\sin i}{\sin r} \quad \text{or} \quad \sin r = \frac{\sin i}{\mu}$$

∴ $$\tan r = \frac{\sin i}{\sqrt{\mu^2 - \sin^2 i}}$$

Substituting in Eq. (i), we get $$d = \left[1 - \frac{\cos i}{\sqrt{\mu^2 - \sin^2 i}}\right] t \sin i$$ **Hence Proved.**

EXERCISE Show that for small angles of incidence, $d = ti\left(\frac{\mu - 1}{\mu}\right)$.

3. In case of spherical mirrors if object distance x_1 and image distance x_2 are measured from focus instead of pole, $u = (f + x_1)$ and $v = (f + x_2)$ the mirror formula,

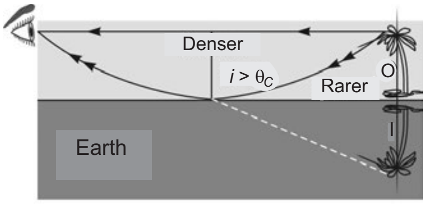
$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f} \text{ reduces to, } \frac{1}{f + x_2} + \frac{1}{f + x_1} = \frac{1}{f}$$

which on simplification gives, $$x_1 x_2 = f^2$$

This formula is called **Newton's formula.**

This formula applies to a lens also, but in that case x_1 is the object distance from first focus and x_2 the image distance from second focus.

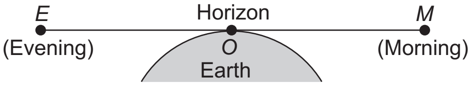
4. Eye is most sensitive to yellow-green light ($\lambda = 5550$ Å).
5. Frequency of visible light is of the order of 10^{15} Hz.
6. Colour of light is determined by its frequency and not the wavelength. During refraction of light frequency and colour of light do not change.
7. **Twinkling of stars** Due to fluctuations in refractive index of atmosphere the refraction of light (reaching to our eye from the star) becomes irregular and the light sometimes reaches the eye and sometimes it does not. This gives rise to twinkling of stars.
8. **Oval shape of sun in the morning and evening** In the morning or evening, the sun is at the horizon. The refractive index decreases with height. Light reaching earth's atmosphere from different parts of vertical diameter of the sun enters at different heights in earth's atmosphere and so travels in media of different refractive indices at the same instant and hence, bends unequally. Due to this unequal bending of light from vertical diameter, the image of the sun gets distorted and it appears oval and larger. However, at noon when the sun is overhead, then due to normal incidence there will be no bending and the sun will appear circular.
9. The sparkling of diamond is due to total internal reflection inside it.
10. **Mirage** Mirage in deserts is caused by total internal reflection.



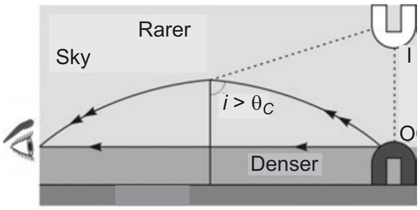
(A) Mirage

Due to heating of the earth, the refractive index of air near the surface of earth becomes lesser than above it. Light from distant objects reaches the surface of earth with $i > \theta_C$. So that total internal reflection will take place and we see the image of an object along with the object as seen in figure, creating an illusion of water near the object.

11. **Duration of sun's visibility** In the absence of atmosphere, the sun will be visible for its positions from M to E as shown in figure. However, in presence of atmosphere, due to **total internal reflection,** the sun will become visible even when it is below the horizon.



12. **Looming** It is also due to **total internal reflection**. This phenomenon is observed in cold deserts and is opposite to that of mirage.



Looming

13. **Scattering of light** If the molecules of a medium after absorbing incoming radiations (light) emit them in all possible directions, the process is called scattering. In scattering if the wavelength of radiation remains unchanged the scattering is called elastic otherwise inelastic.

Rayleigh has shown, theoretically that in case of elastic scattering of light by molecules, the intensity of scattered light depends on both nature of molecules and wavelength of light. According to him,

Intensity of scattered light $\propto \frac{1}{\lambda^4}$

Raman effect was based on inelastic scattering. For this **C.V. Raman** was awarded the Nobel Prize in 1930.

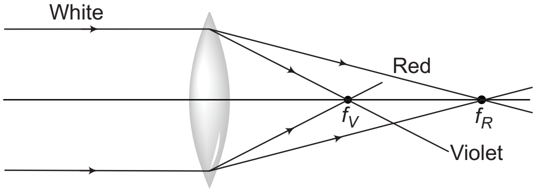
Scattering helps us in understanding the following:

Why sky is Blue When white light from the sun enters the earth's atmosphere, scattering takes place. As scattering is proportional to $\frac{1}{\lambda^4}$, blue is scattered most. When we look at the sky we receive scattered light which is rich in blue and hence, the sky appears blue.

Why sun appears red during sunset and sunrise In the morning and evening when sun is at the horizon, due to oblique incidence, light reaches earth after traversing maximum path in the atmosphere and so suffers maximum scattering. Now, as scattering $\propto \frac{1}{\lambda^4}$, shorter wavelengths are scattered most leaving the longer one. As red light has longest wavelength in the visible region, it is scattered least. This is why sun appears red in the morning and evening. The same reason is why red light is used for danger signals.

14. **Defects of images** Actual image formed by an optical system is usually imperfect. The defects of images are called **aberrations.** The defect may be due to light or optical system. If the defect is due to light, it is called **chromatic aberration,** and if due to optical system, **monochromatic aberration.**

(a) **Chromatic aberration** The image of an object formed by a lens is usually coloured and blurred. This defect of image is called chromatic aberration. This defect arises due to the fact that focal length of a lens is different for different colours. For a lens,



$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

As μ is maximum for violet while minimum for red, violet is focused nearest to the lens while red farthest from it.

The difference between f_R and f_V is a measure of longitudinal chromatic aberration. Thus,

$$\text{LCA} = f_R - f_V$$

Condition of achromatism To get achromatism, we use a pair of two lenses in contact. For two thin lenses in contact we have,

$$\therefore \qquad \boxed{\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0}$$

This is the **condition of achromatism.** From the condition of achromatism, following conclusions can be drawn:

(i) As ω_1 and ω_2 are positive quantities, f_1 and f_2 should have opposite signs, i.e. if one lens is convex, the other must be concave.

(ii) If $\omega_1 = \omega_2$, means both the lenses are of same material. Then,

$$\frac{1}{f_1} + \frac{1}{f_2} = 0 \quad \text{or} \quad \frac{1}{F} = 0 \quad \text{or} \quad F = \infty$$

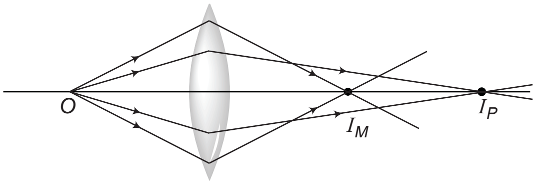
Thus, the combination behaves as a plane glass plate. So, we can conclude that both the lenses should be of different materials or $\omega_1 \neq \omega_2$

(iii) Dispersive power of crown glass (ω_C) is less than that of flint glass (ω_F).

(iv) If we want the combination to behave as a convergent lens, then convex lens should have lesser focal length or its dispersive power should be more. Thus, convex lens should be made of flint glass and concave lens of crown. Thus, combination is converging if convex is made of flint glass and concave of crown. Similarly, for the combination to behave as diverging lens, convex is made of crown glass and concave of flint glass.

(b) **Monochromatic aberration** This is the defect in image due to optical system. Monochromatic aberration is of many types such as, spherical, coma, distortion, curvature and astigmatism. Here, we shall limit ourselves to spherical aberration only.

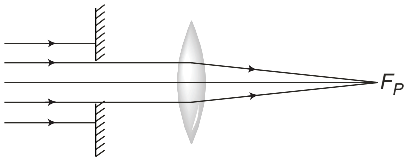
Spherical aberration Spherical aberration arises due to spherical nature of lens (or mirror).



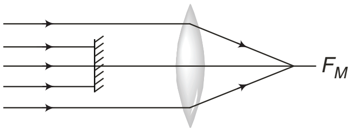
The paraxial rays (close to optic axis) get focused at I_P and marginal rays (away from the optic axis) are focused at I_M. Thus, image of a point object O is not a point.

The inability of the lens to form a point image of an axial point object is called spherical aberration. Spherical aberration can never be eliminated but can be minimised by the following methods:

(i) **By using stops** By using stops either paraxial or marginal rays are cut-off.



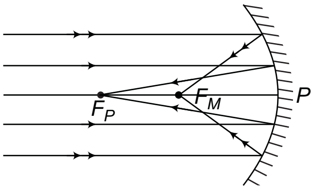
(A)



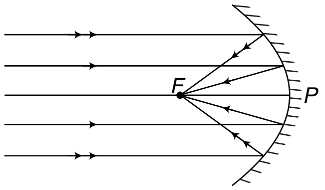
(B)

(ii) **Using two thin lenses separated by a distance** Two thin lenses separated by a distance $d = f_2 - f_1$ has the minimum spherical aberration.

(iii) **Using parabolic mirrors** If spherical mirror is replaced by parabolic mirror, spherical aberration is minimised.



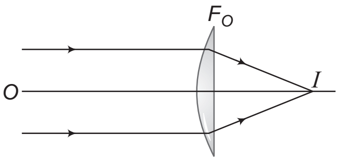
(A) Spherical mirror



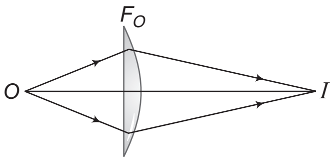
(B) Parabolic mirror

(iv) **Using lens of large focal length** It has been found that spherical aberration varies inversely as the cube of the focal length. So, if f is large, spherical aberration will be reduced.

(v) **Using plano-convex lens** In case of plano-convex lens, spherical aberration is minimised, if its curved surface faces the incident or emergent ray whichever is more parallel.



(A) Telescope



(B) Microscope

This is why in telescope, the curved surface faces the object while in microscope curved surface is towards the image.

(vi) **Using crossed lens** For a single lens with object at infinity, spherical aberration is found to be minimum when R_1 and R_2 have the following ratio,

$$\frac{R_1}{R_2} = \frac{2\mu^2 - \mu - 4}{\mu\,(2\mu + 1)}$$

A lens which satisfies this condition is called a crossed lens.

Defects of vision Regarding eye, the following points are worthnoting:

(a) The human eye is most sensitive to yellow-green light ($\lambda = 5550$ Å).

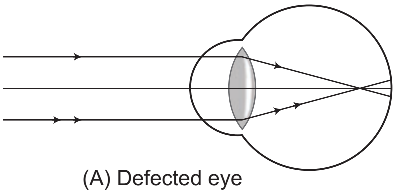
(b) The persistance of vision is $\frac{1}{10}$ sec, i.e. if time interval between two consecutive light pulses is less than 0.1 sec, eye cannot distinguish them separately.

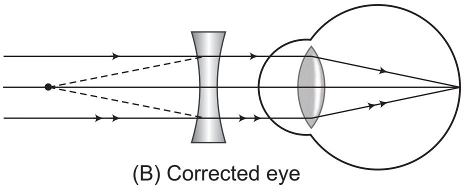
(c) By the eyelens, real, inverted and diminished image is formed on retina.

(d) While testing your eye through reading chart if doctor finds it to 6/12, it implies that you can read a letter from 6 m which the normal eye can read from 12 m. Thus, 6/6 means normal eye sight.

The common defects of vision are as follows:

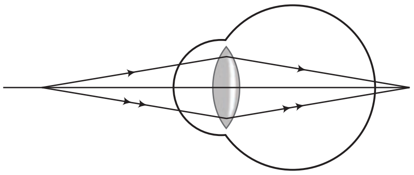
(i) Myopia or short-sightedness Distant objects are not clearly visible in this defect. The image of distant object is formed before the retina.

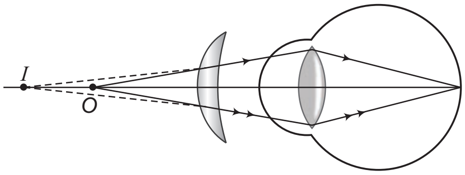

(A) Defected eye


(B) Corrected eye

The defect can be remedied by using a concave lens.

(ii) Hyperopia or far-sightedness The near objects are not clearly visible. Image of near object is formed behind the retina.


(A) Defected eye



(B) Corrected eye

This defect is remedied by using a convex lens.

(iii) Presbyopia In it both near and far objects are not clearly visible. This is remedied either by using two separate lenses or by using single spectacle having bifocal lenses.

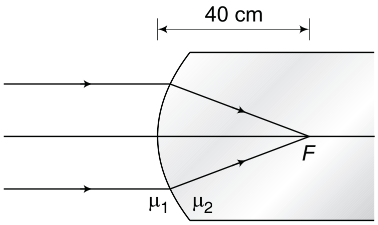
(iv) Astigmatism In this defect, eye cannot see objects in two orthogonal (perpendicular) directions clearly simultaneously. This defect is remedied by using cylindrical lens.

Solved Examples

Example 1 *A spherical convex surface separates object and image space of refractive index 1.0 and* $\frac{4}{3}$. *If radius of curvature of the surface is 10 cm, find its power.*

Solution Let us see where does the parallel rays converge (or diverge) on the principal axis. Let us call it the focus and the corresponding length the focal length f. Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with proper values and signs, we have

$$\frac{4/3}{f} - \frac{1.0}{\infty} = \frac{4/3 - 1.0}{+10} \quad \text{or} \quad f = 40 \text{ cm} = 0.4 \text{ m}$$



Since, the rays are converging, its power should be positive. Hence,

$$P \text{ (in dioptre)} = \frac{+1}{f \text{ (metre)}} = \frac{1}{0.4}$$

or $$P = 2.5 \text{ dioptre}$$ **Ans.**

Example 2 *A ray of light is incident at an angle of* $60°$ *on the face of a prism having refracting angle* $30°$. *The ray emerging out of the prism makes an angle* $30°$ *with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges.*

Solution Given, $i_1 = 60°$, $A = 30°$ and $\delta = 30°$.

From the relation, $$\delta = (i_1 + i_2) - A$$

we have, $$i_2 = 0$$

This means that the emergent ray is perpendicular to the face through which it emerges. **Ans.**

Example 3 *The angle of minimum deviation for a glass prism with* $\mu = \sqrt{3}$ *equals the refracting angle of the prism. What is the angle of the prism?*

Solution Given, $A = \delta_m$

Using, $$\mu = \frac{\sin\left(\frac{A + \delta_m}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

We have, $$\sqrt{3} = \frac{\sin\left(\frac{A + A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

or $$\sqrt{3} = \frac{\sin A}{\sin\left(\frac{A}{2}\right)} = \frac{2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}}{\sin\left(\frac{A}{2}\right)}$$

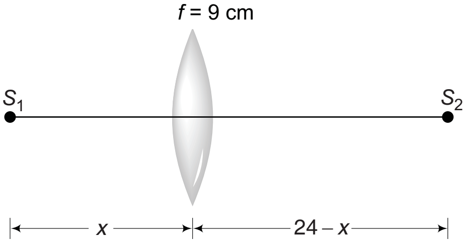
$\therefore$ $$\cos \frac{A}{2} = \frac{\sqrt{3}}{2}$$

$\therefore$ $$\frac{A}{2} = 30°$$

or $$A = 60°$$ **Ans.**

Example 4 *The distance between two point sources of light is 24 cm. Find out where would you place a converging lens of focal length 9 cm, so that the images of both the sources are formed at the same point.*

Solution **For S_1 :** $\frac{1}{v_1} - \frac{1}{-x} = \frac{1}{9}$



$\therefore$ $$\frac{1}{v_1} = \frac{1}{9} - \frac{1}{x} \quad \text{...(i)}$$

For S_2 : $$\frac{1}{v_2} - \frac{1}{-(24 - x)} = \frac{1}{9}$$

$\therefore$ $$\frac{1}{v_2} = \frac{1}{9} - \frac{1}{24 - x} \quad \text{...(ii)}$$

Since, sign convention for S_1 and S_2 are just opposite. Hence,

$$v_1 = - v_2$$

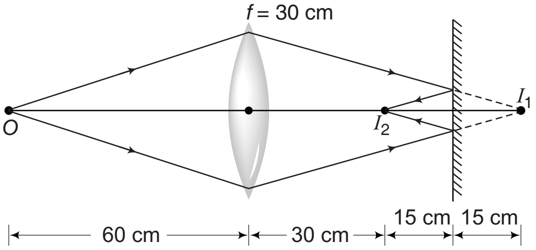
or $$\frac{1}{v_1} = -\frac{1}{v_2}$$

$\therefore$ $$\frac{1}{9} - \frac{1}{x} = \frac{1}{24 - x} - \frac{1}{9}$$

Solving this equation we get, $x = 6$ cm. Therefore, the lens should be kept at a distance of 6 cm from either of the object. **Ans.**

Example 5 *A source of light is located at double focal length from a convergent lens. The focal length of the lens is $f = 30$ cm. At what distance from the lens should a flat mirror be placed, so that the rays reflected from the mirror are parallel after passing through the lens for the second time?*

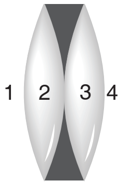
Solution Object is at a distance of $2f = 60$ cm from the lens. Image I_1 formed by lens, should be at a distance 60 cm from the lens. Now I_2, the image formed by plane mirror should lie at focus or at a distance of 30 cm from the lens. Hence, the mirror should be placed at distance 45 cm from the lens as shown in figure.



Example 6 *Two equi-convex lenses of focal lengths 30 cm and 70 cm, made of material of refractive index = 1.5, are held in contact coaxially by a rubber band round their edges. A liquid of refractive index 1.3 is introduced in the space between the lenses filling it completely. Find the position of the image of a luminous point object placed on the axis of the combination lens at a distance of 90 cm from it.*

Solution $|R_1| = |R_2| = f_1 = 30$ cm (As $\mu = 1.5$)

Similarly, $|R_3| = |R_4| = f_2 = 70$ cm



The focal length of the liquid lens (in air).

$$\frac{1}{f_3} = (\mu - 1)\left(\frac{1}{R_2} - \frac{1}{R_3}\right)$$

$$= (1.3 - 1)\left(\frac{1}{-30} - \frac{1}{+70}\right)$$

$$= -\frac{1}{70}$$

Further, equivalent focal length of the combination,

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

$$= \frac{1}{30} + \frac{1}{70} - \frac{1}{70} = \frac{1}{30}$$

Using the lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{F}$, we have

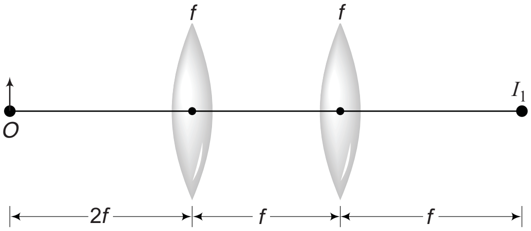
$$\frac{1}{v} - \frac{1}{-90} = \frac{1}{30}$$

$\therefore$ $v = +45$ cm

Thus, image will be formed at a distance of 45 cm from the combination.

Example 7 *Two thin converging lenses are placed on a common axis, so that the centre of one of them coincides with the focus of the other. An object is placed at a distance twice the focal length from the left hand lens. Where will its image be? What is the lateral magnification? The focal of each lens is f.*

Solution



The image formed by first lens will be at a distance $2f$ with lateral magnification $m_1 = -1$. For the second lens this image will behave as a virtual object. Using the lens formula, $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ we have,

$$\frac{1}{v} - \frac{1}{f} = \frac{1}{f} \quad \Rightarrow \quad v = \frac{f}{2}$$

$$m_2 = \frac{v_2}{u_2} = \frac{(f/2)}{f} = \frac{1}{2}$$

Therefore, final image is formed at a distance $\frac{f}{2}$ from the second lens with total lateral magnification,

$$m = m_1 \times m_2 = (-1) \times \left(\frac{1}{2}\right) = -\frac{1}{2}$$ **Ans.**

Example 8 *The refracting angle of a glass prism is* 30°. *A ray is incident onto one of the faces perpendicular to it. Find the angle* δ *between the incident ray and the ray that leaves the prism. The refractive index of glass is* $\mu = 1.5$.

Solution Given, $A = 30°, \mu = 1.5$ and $i_1 = 0°$

Since, $i_1 = 0°$, therefore, r_1 is also equal to 0°.

Further, since, $r_1 + r_2 = A$

$\therefore$ $$r_2 = A = 30°$$

Using, $$\mu = \frac{\sin i_2}{\sin r_2}$$

We have, $$1.5 = \frac{\sin i_2}{\sin 30°}$$

or $$\sin i_2 = 1.5 \sin 30° = 1.5 \times \frac{1}{2} = 0.75$$

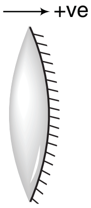
$\therefore$ $$i_2 = \sin^{-1}(0.75) = 48.6°$$

Now, the deviation, $\delta = (i_1 + i_2) - A$

$$= (0 + 48.6) - 30$$

or $$\delta = 18.6°$$ **Ans.**

Example 9 *A biconvex thin lens is prepared from glass of refractive index 3/2. The two bounding surfaces have equal radii of 25 cm each. One of the surfaces is silvered from outside to make it reflecting. Where should an object be placed before this lens so that the image coincides with the object.*



Solution Equivalent focal length of this system which behaves like a mirror is given by

$$\frac{1}{f}=\frac{2(\mu_2/\mu_1)}{R_2}-\frac{2(\mu_2/\mu_1-1)}{R_1}$$

Here, $R_1=+25\text{ cm},\ R_2=-25\text{ cm},\ \mu_1=1$

and $\mu_2=3/2$

Image coincides with object, hence, $u=v=-x$ (say)

Substituting in mirror formula, we have

$$\frac{1}{-x}-\frac{1}{x}=\frac{2(3/2)}{-25}-\frac{2(3/2-1)}{25}$$

or $$\frac{2}{x}=\frac{3}{25}+\frac{1}{25}=\frac{4}{25}$$

$\therefore$ $x=12.5$ cm **Ans.**

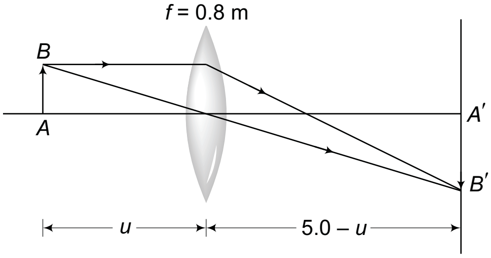
Hence, the object should be placed at a distance 12.5 cm in front of the silvered lens.

Example 10 *An object is 5.0 m to the left of a flat screen. A converging lens for which the focal length is $f=0.8$ m is placed between object and screen.*

(a) Show that two lens positions exist that form images on the screen and determine how far these positions are from the object?

(b) How do the two images differ from each other?

Solution (a) Using the lens formula,



$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

We have, $$\frac{1}{5.0-u}-\frac{1}{-u}=\frac{1}{0.8}\quad\text{or}\quad\frac{1}{5-u}+\frac{1}{u}=1.25$$

$\therefore$ $u+5-u=1.25\,u\,(5-u)$ or $1.25\,u^2-6.25\,u+5=0$

$\therefore$ $$u=\frac{6.25\pm\sqrt{39.0625-25}}{2.5}$$

or $u=4$ m and 1 m **Ans.**

Both the values are real, which means there exist two positions of lens that form images of object on the screen.

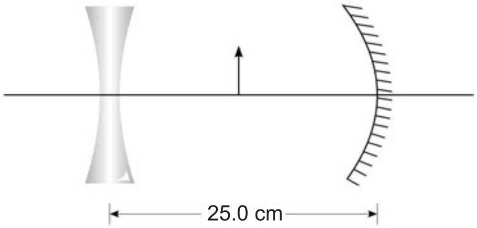
(b) $m = \frac{v}{u}$

$$\therefore \qquad m_1 = \frac{(5.0 - 4.0)}{(-4.0)} = -0.25$$

$$\text{and} \qquad m_2 = \frac{(5.0 - 1.0)}{(-1.0)} = -4.00$$

Hence, both the images are real and inverted, the first has magnification – 0.25 and the second – 4.00. **Ans.**

Example 11 *An object is midway between the lens and the mirror as shown. The mirror's radius of curvature is* 20.0 *cm and the lens has a focal length of* – 16.7 *cm. Considering only the rays that leaves the object and travels first toward the mirror, locate the final image formed by this system. Is this image real or virtual? Is it upright or inverted? What is the overall magnification?*



Solution **Image formed by mirror** Using mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R} \qquad \left(\text{as } f = \frac{R}{2}\right)$$

We have,

$$\frac{1}{v_1} + \frac{1}{-12.5} = \frac{2}{-20}$$

$$\therefore \qquad v_1 = -50 \text{ cm}$$

$$m_1 = -\frac{v}{u} = -\frac{(-50)}{(-12.5)} = -4$$

i.e. image formed by the mirror is at a distance of 50 cm from the mirror to the left of it. It is inverted and four times larger.

Image formed by lens Image formed by mirror acts as an object for lens. It is at a distance of 25.0 cm to the left of lens. Using the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{v_2} - \frac{1}{25} = \frac{1}{-16.7}$$

$$\therefore \qquad v_2 = -50.3 \text{ cm}$$

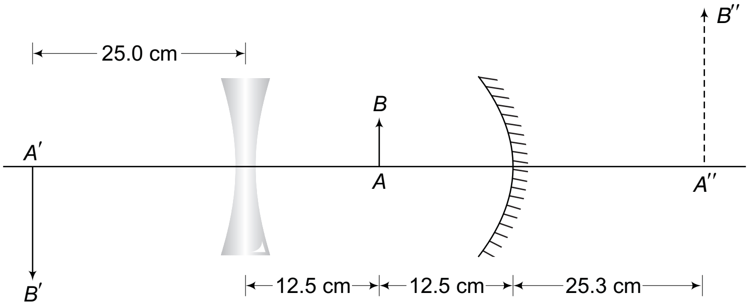
and

$$m_2 = \frac{v}{u} = \frac{-50.3}{25} = -2.012$$

overall magnification is

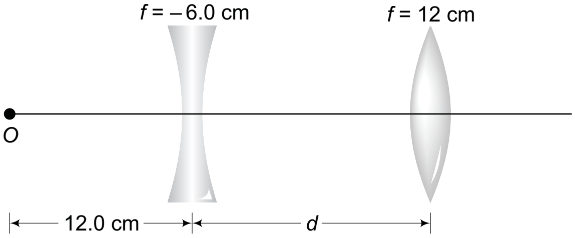
$$m = m_1 \times m_2 = 8.048$$

Thus, the final image is at a distance 25.3 cm to the right of the mirror, virtual, upright enlarged and 8.048 times. Positions of the two images are shown in figure.



Example 12 *An object is placed 12 cm to the left of a diverging lens of focal length – 6.0 cm. A converging lens with a focal length of 12.0 cm is placed at a distance d to the right of the diverging lens. Find the distance d that corresponds to a final image at infinity.*

Solution



Applying lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ twice we have,

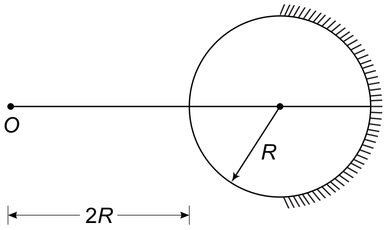
$$\frac{1}{v_1} - \frac{1}{-12} = \frac{1}{-6} \qquad \text{...(i)}$$

$$\frac{1}{\infty} - \frac{1}{v_1 - d} = \frac{1}{12} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have

$$v_1 = -4 \text{ cm} \quad \text{and} \quad d = 8 \text{ cm}$$ **Ans.**

Example 13 *A solid glass sphere with radius R and an index of refraction 1.5 is silvered over one hemisphere. A small object is located on the axis of the sphere at a distance 2R to the left of the vertex of the unsilvered hemisphere. Find the position of final image after all refractions and reflections have taken place.*



Solution The ray of light first gets refracted then reflected and then again refracted. For first refraction and then reflection the ray of light travels from left to right while for the last refraction it travels from right to left. Hence, the sign convention will change accordingly.

First refraction Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with proper sign conventions, we have

$$\frac{1.5}{v_1} - \frac{1.0}{-2R} = \frac{1.5 - 1.0}{+R}$$

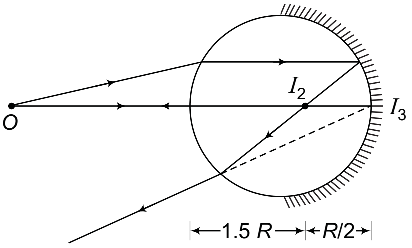
$$\therefore \qquad v_1 = \infty$$

Second reflection Using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$ with proper sign conventions, we have

$$\frac{1}{v_2} + \frac{1}{\infty} = -\frac{2}{R}$$

$$\therefore \qquad v_2 = -\frac{R}{2}$$

Third refraction Again using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with reversed sign convention, we have

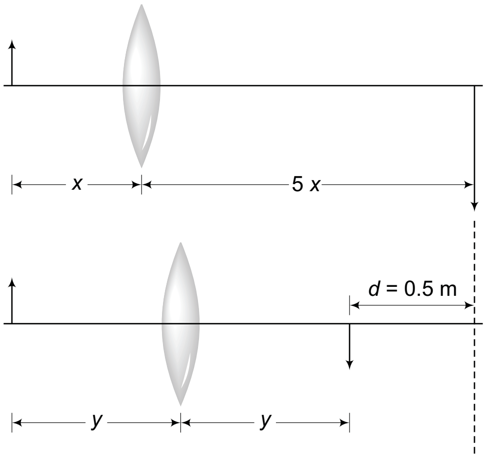


$$\frac{1.0}{v_3} - \frac{1.5}{-1.5R} = \frac{1.0 - 1.5}{-R} \quad \text{or} \quad v_3 = -2R$$

i.e. final image is formed on the vertex of the silvered face.

Example 14 *A converging lens forms a five fold magnified image of an object. The screen is moved towards the object by a distance $d = 0.5$ m, and the lens is shifted so that the image has the same size as the object. Find the power of lens and the initial distance between the object and the screen.*

Solution In the first case image is five times magnified. Hence, $|v| = 5|u|$. In the second case, image and object are of equal size. Hence, $|v| = |u|$. From the two figures,



$$6x = 2y + d \quad \text{or} \quad 6x - 2y = 0.5 \qquad \ldots\text{(i)}$$

Using the lens formula for both the cases,

$$\frac{1}{5x} - \frac{1}{-x} = \frac{1}{f} \quad \text{or} \quad \frac{6}{5x} = \frac{1}{f} \qquad \ldots\text{(ii)}$$

$$\frac{1}{y} - \frac{1}{-y} = \frac{1}{f} \quad \text{or} \quad \frac{2}{y} = \frac{1}{f} \qquad \ldots\text{(iii)}$$

Solving these three equations, we get

$$x = 0.1875 \text{ m} \quad \text{and} \quad f = 0.15625 \text{ m}$$

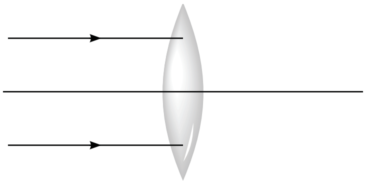
Therefore, initial distance between the object and the screen $= 6x = 1.125$ m **Ans.**

Power of the lens, $$P = \frac{1}{f}$$

$$= \frac{1}{0.15625} = 6.4 \text{ D}$$ **Ans.**

Example 15 *Surfaces of a thin equi-convex glass lens have radius of curvature R. Paraxial rays are incident on it. If the final image is formed after n internal reflections, calculate distance of this image from pole of the lens. Refractive index of glass is μ.*

Solution The rays will first get refracted, then n-times reflected and finally again refracted. So, using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ for first refraction, we have



$$\frac{\mu}{v_i} - \frac{1}{\infty} = \frac{\mu - 1}{R} \Rightarrow v_i = \left(\frac{\mu}{\mu - 1}\right) R$$

For first reflection, let us use $\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$

$$\therefore \quad \frac{1}{v_1} + \left(\frac{\mu - 1}{\mu R}\right) = \frac{-2}{R} \quad \text{or} \quad \frac{1}{v_1} = -\left(\frac{3\mu - 1}{\mu R}\right)$$

For second reflection, $$\frac{1}{v_2} + \frac{3\mu - 1}{\mu R} = \frac{-2}{R} \quad \text{or} \quad \frac{1}{v_2} = -\left(\frac{5\mu - 1}{\mu R}\right)$$

Similarly, after n^{th} reflection, $$\frac{1}{v_n} = -\left[\frac{(2n + 1)\mu - 1}{\mu R}\right]$$

Finally, using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, we have

$$\frac{1}{v_f} - \left\{\frac{(2n + 1)\mu - 1}{R}\right\} = \frac{1 - \mu}{-R}$$

or $$v_f = \frac{R}{2(\mu n + \mu - 1)}$$ **Ans.**

Exercises

LEVEL 1

Note *In different books refractive index has been represented by the symbol n and μ. So, in our book we have used both symbols at different places.*

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** There is a glass slab between Ram and Anoop. Then, Ram appears nearer to Anoop as compared to the actual distance between them.

   Reason : Ray of light starting from Ram will undergo two times refraction before reaching Anoop.

2. **Assertion :** Minimum distance between object and its real image by a convex lens is $4f$.

   Reason : If object distance from a convex lens is $2f$, then its image distance is also $2f$.

3. **Assertion :** In case of single refraction by a plane surface image and object are on same side.

   Reason : If object is real, image will be virtual and *vice-versa*.

4. **Assertion :** Ray of light passing through optical centre of a lens goes undeviated.

   Reason : Ray falls normal at optical centre and in normal incidence, there is no deviation of light.

5. **Assertion :** In displacement method of finding focal length of a convex lens its magnification in one position of lens is +2, then magnification in another position of lens should be $-\frac{1}{2}$.

   Reason : This method can't be applied for a concave lens.

6. **Assertion :** If object is placed at infinity, then a virtual image will be formed at first focus of a concave lens.

   Reason : First focal length of a concave lens is positive.

7. **Assertion :** Minimum deviation by an equilateral prism of refractive index $\sqrt{2}$ is 30°.

   Reason : It is from the relation, $\mu = \dfrac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$

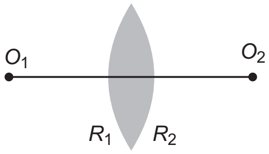
8. **Assertion :** A convex lens and a concave lens are kept in contact. They will behave as a diverging lens if focal length of convex lens is more.

   Reason : Power of a concave lens is always less than the power of a convex lens, as power of concave lens is negative whereas power of convex lens is positive.

9. Assertion : Image of an object is of same size by a convex lens . If a glass slab is placed between object and lens, image will become magnified.

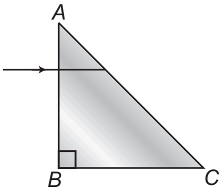
Reason : By inserting the slab, image may be real or virtual.

10. Assertion : In the figure shown $|R_1|>|R_2|$. Two point objects O_1 and O_2 are kept at same distance from the lens. Image distance of O_1 from the lens will be more compared to the image distance of O_2.



Reason : If medium on two sides of the lens is different, we cannot apply lens formulae directly.

11. Assertion : White light is incident on face *AB* of an isosceles right angled prism as shown. Colours, for which refractive index of material of prism is more than 1.414, will be able to emerge from the face *AC*.



Reason : Total internal reflection cannot take place for the light travelling from a rarer medium to a denser medium.

12. Assertion : Image formed by concave lens is not always virtual.

Reason : Image formed by a lens is real if the image is formed in the direction of ray of light with respect to the lens.

13. Assertion : Although the surfaces of goggle lens are curved, it does not have any power.

Reason : In case of goggles, both the curved surfaces have equal radii of curvature and have centre of curvature on the same side.

Objective Questions

1. An endoscope is employed by a physician to view the internal parts of body organ. It is based on the principle of

(a) refraction
(b) reflection
(c) total internal reflection
(d) dispersion

2. Refractive index μ is given as $\mu = A + \frac{B}{\lambda^2}$, where A and B are constants and λ is wavelength, then dimensions of B are same as that of

(a) wavelength
(b) volume
(c) pressure
(d) area

3. A plane glass slab is placed over various coloured letters. The letter which appears to be raised the least is

(a) violet (b) yellow (c) red (d) green

4. Critical angle of light passing from glass to air is least for

(a) red	(b) green
(c) yellow	(d) violet

5. The power in dioptre of an equi-convex lens with radii of curvature of 10 cm and refractive index 1.6 is

(a) +12	(b) +18
(c) +1.2	(d) +1.8

6. The refractive index of water is 4/3. The speed of light in water is

(a) 1.50×10^8 m/s	(b) 1.78×10^8 m/s
(c) 2.25×10^8 m/s	(d) 2.67×10^8 m/s

7. White light is incident from under water on the water-air interface. If the angle of incidence is slowly increased from zero, the emergent beam coming out into the air will turn from

(a) white to violet	(b) white to red
(c) white to black	(d) None of these

8. When light enters from air to water, then its

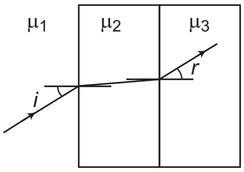
(a) frequency increases and speed decreases

(b) frequency is same, but the wavelength is smaller in water than in air

(c) frequency is same but the wavelength in water is greater than in air

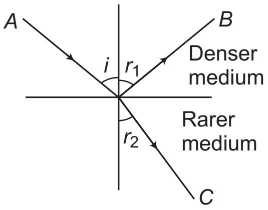
(d) frequency decreases and wavelength is smaller in water than in air

9. In the figure shown $\dfrac{\sin i}{\sin r}$ is equal to



(a) $\dfrac{\mu_2^2}{\mu_3 \mu_1}$ (b) $\dfrac{\mu_3}{\mu_1}$ (c) $\dfrac{\mu_3 \mu_1}{\mu_2^2}$ (d) $\dfrac{\mu_1}{\mu_3}$

10. In figure, the reflected ray B makes an angle 90° with the ray C. If i, r_1 and r_2 are the angles of incidence, reflection and refraction, respectively. Then, the critical angle of the medium is



(a) $\sin^{-1}(\tan i)$	(b) $\sin^{-1}(\cot i)$
(c) r_1	(d) r_2

11. A prism of apex angle $A = 60°$ has the refractive index $\mu = \sqrt{2}$. The angle of incidence for minimum deviation is

(a) 30°	(b) 45°
(c) 60°	(d) None of these

12. A thin equi-convex lens is made of glass of refractive index 1.5 and its focal length is 0.2 m. If it acts as a concave lens of 0.5 m focal length when dipped in a liquid, the refractive index of the liquid is

(a) $\frac{17}{8}$ (b) $\frac{15}{8}$ (c) $\frac{13}{8}$ (d) $\frac{9}{8}$

13. A ray of light, travelling in a medium of refractive index μ, is incident at an angle i on a composite transparent plate consisting of three plates of refractive indices μ_1, μ_2 and μ_3. The ray emerges from the composite plate into a medium of refractive index μ_4, at angle x. Then,

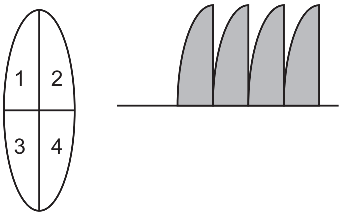
(a) $\sin x = \sin i$

(b) $\sin x = \frac{\mu}{\mu_4} \sin i$

(c) $\sin x = \frac{\mu_4}{\mu} \sin i$

(d) $\sin x = \frac{\mu_1 \mu_3 \mu}{\mu_2 \mu_2 \mu_4} \sin i$

14. The given equi-convex lens is broken into four parts and rearranged as shown. If the initial focal length is f, then after rearrangement the equivalent focal length is



(a) f (b) $f/2$ (c) $f/4$ (d) $4f$

15. A thin convergent glass lens ($\mu_g = 1.5$) has a power of + 5.0 D. When this lens is immersed in a liquid of refractive index μ_l, it acts as a divergent lens of focal length 100 cm. The value of μ_l is

(a) 4/3 (b) 5/3

(c) 5/4 (d) 6/5

16. Two convex lenses of focal length 10 cm and 20 cm respectively placed coaxially and are separated by some distance d. The whole system behaves like a concave lens . One of the possible value of d is

(a) 15 cm (b) 20 cm

(c) 25 cm (d) 40 cm

17. A prism can have a maximum refracting angle of (θ_C = critical angle for the material of prism)

(a) 60° (b) θ_C

(c) $2\theta_C$ (d) slightly less than 180°

18. A ray of light is incident at small angle I on the surface of prism of small angle A and emerges normally from the opposite surface. If the refractive index of the material of the prism is μ, the angle of incidence is nearly equal to

(a) $\frac{A}{\mu}$ (b) $\frac{A}{2\mu}$

(c) μA (d) $\mu A/2$

19. The refractive angle of a prism is A, and the refractive index of the material of the prism is $\cot(A/2)$. The angle of minimum deviation is

(a) $180° - 3A$ (b) $180° + 2A$

(c) $90° - A$ (d) $180° - 2A$

20. A prism of refractive index $\sqrt{2}$ has refractive angle 60°. In order that a ray suffers minimum deviation it should be incident at an angle of

(a) 45° (b) 90°
(c) 30° (d) None

21. The focal length of a combination of two lenses is doubled if the separation between them is doubled. If the separation is increased to 4 times, the magnitude of focal length is

(a) doubled (b) quadrupled
(c) halved (d) same

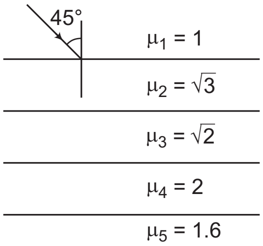
22. A convexo-concave convergent lens is made of glass of refractive index 1.5 and focal length 24 cm. Radius of curvature for one surface is double than that of the other. Then, radii of curvature for the two surfaces are (in cm)

(a) 6, 12 (b) 12, 24
(c) 3, 6 (d) 18, 36

23. An optical system consists of a thin convex lens of focal length 30 cm and a plane mirror placed 15 cm behind the lens. An object is placed 15 cm in front of the lens. The distance of the final image from the object is

(a) 60 cm (b) 30 cm
(c) 75 cm (d) 45 cm

24. In the figure shown, the angle made by the light ray with the normal in the medium of refractive index $\sqrt{2}$ is



(a) 30° (b) 60°
(c) 90° (d) None of these

25. For refraction through a small angled prism, the angle of minimum deviation

(a) increases with increase in refractive index of a prism
(b) will be 2δ for a ray of refractive index 2.4 if it is δ for a ray of refractive index 1.2
(c) is directly proportional to the angle of the prism
(d) will decrease with increase in refractive index of the prism

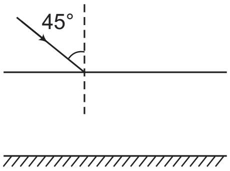
26. A ray of light passes from vacuum into a medium of refractive index n. If the angle of incidence is twice the angle of refraction, then the angle of incidence is

(a) $\cos^{-1}(n/2)$ (b) $\sin^{-1}(n/2)$
(c) $2\cos^{-1}(n/2)$ (d) $2\sin^{-1}(n/2)$

27. A thin convex lens of focal length 30 cm is placed in front of a plane mirror. An object is placed at a distance x from the lens (not in between lens and mirror) so that its final image coincides with itself . Then, the value of x is

(a) 15 cm (b) 30 cm
(c) 60 cm (d) Insufficient data

28. One side of a glass slab is silvered as shown in the figure. A ray of light is incident on the other side at angle of incidence 45°. Refractive index of glass is given as $\sqrt{2}$. The deflection suffered by the ray when it comes out of the slab is

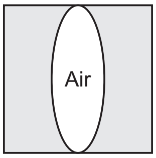


(a) 90° (b) 180° (c) 120° (d) 45°

29. A prism has refractive index $\sqrt{\frac{3}{2}}$ and refractive angle 90°. Find the minimum deviation produced by prism

(a) 60° (b) 45° (c) 30° (d) 15°

30. In figure, an air lens of radius of curvature of each surface equal to 10 cm is cut into a cylinder of glass of refractive index 1.5. The focal length and the nature of lens are



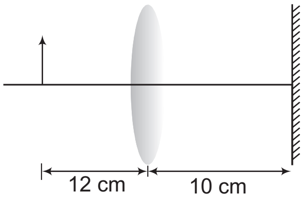
(a) 15 cm diverging (b) 15 cm converging

(c) 10 cm diverging (d) 10 cm converging

31. A point object is placed at a distance of 12 cm from a convex lens of focal length 10 cm. On the other side of the lens, a convex mirror is placed at a distance of 10 cm from the lens such that the image formed by the combination coincides with the object itself. The focal length of the convex mirror is

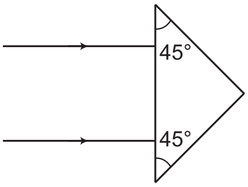
(a) 20 cm (b) 25 cm (c) 15 cm (d) 30 cm

32. An object, a convex lens of focal length 20 cm and a plane mirror are arranged as shown in the figure. How far behind the mirror is the second image formed?



(a) 30 cm (b) 20 cm (c) 40 cm (d) 50 cm

33. Two parallel light rays pass through an isosceles prism of refractive index $\sqrt{3/2}$ as shown in figure. The angle between the two emergent rays is



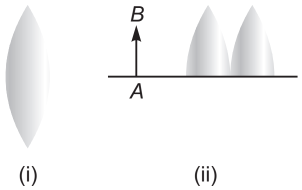
(a) 15° (b) 30°

(c) 45° (d) 60°

34. A prism having refractive index $\sqrt{2}$ and refractive angle 30° has one of the refractive surfaces polished. A beam of light incident on the other surface will trace its path if the angle of incidence is

(a) 0° (b) 30° (c) 45° (d) 60°

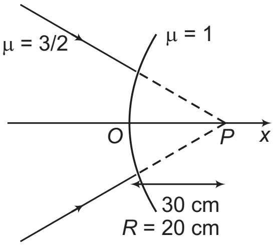
35. In Fig. (i), a lens of focal length 10 cm is shown. It is cut into two parts and placed as shown in Fig. (ii). An object *AB* of height 1 cm is placed at a distance of 7.5 cm. The height of the image will be



(i) (ii)

(a) 2 cm (b) 1 cm (c) 1.5 cm (d) 3 cm

36. The image for the converging beam after refraction through the curved surface is formed at



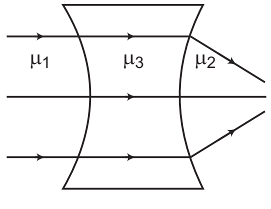
(a) $x = 40$ cm (b) $x = 40/3$ cm

(c) $x = -40/3$ cm (d) $x = 20$ cm

37. A concavo-convex lens is made of glass of refractive index 1.5. The radii of curvature of its two surfaces are 30 cm and 50 cm. Its focal length when placed in a liquid of refractive index 1.4 is

(a) 200 cm (b) 500 cm (c) 800 cm (d) 1050 cm

38. From the figure shown, establish a relation between μ_1, μ_2 and μ_3



(a) $\mu_1 < \mu_2 < \mu_3$ (b) $\mu_3 < \mu_2$; $\mu_3 = \mu_1$

(c) $\mu_3 > \mu_2$; $\mu_3 = \mu_1$ (d) None of these

39. When light of wavelength λ is incident on an equilateral prism, kept on its minimum deviation position, it is found that the angle of deviation equals the angle of the prism itself. The refractive index of the material of the prism for the wavelength λ is

(a) $\sqrt{3}$ (b) $\sqrt{3/2}$

(c) 2 (d) $\sqrt{2}$

Subjective Questions

1. The laws of reflection or refraction are the same for sound as for light. The index of refraction of a medium (for sound) is defined as the ratio of the speed of sound in air 343 m/s to the speed of sound in the medium.

(a) What is the index of refraction (for sound) of water ($v = 1498$ m/s)?

(b) What is the critical angle θ, for total reflection of sound from water?

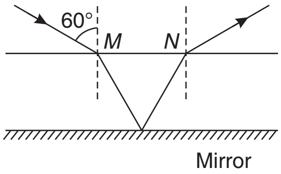
2. Light from a sodium lamp ($\lambda_0 = 589$ nm) passes through a tank of glycerin (refractive index $= 1.47$) 20 m long in a time t_1. If it takes a time t_2 to transverse the same tank when filled with carbon disulfide (index $= 1.63$), determine the difference $t_2 - t_1$.

3. A light beam of wavelength 600 nm in air passes through film 1 ($n_1 = 1.2$) of thickness 1.0 μm, then through film 2 (air) of thickness 1.5 μm, and finally through film 3 ($n_3 = 1.8$) of thickness 1.0 μm

(a) Which film does the light cross in the least time, and what is that least time?

(b) What are the total number of wavelengths (at any instant) across all three films together?

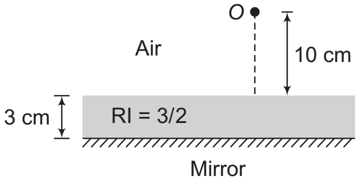
4. A plate with plane parallel faces having refractive index 1.8 rests on a plane mirror. A light ray is incident on the upper face of the plate at 60°. How far from the entry point will the ray emerge after reflection by the mirror. The plate is 6 cm thick?



5. An object is at a distance of $d = 2.5$ cm from the surface of a glass sphere with a radius $R = 10$ cm. Find the position of the final image produced by the sphere. The refractive index of glass is $\mu = 1.5$.

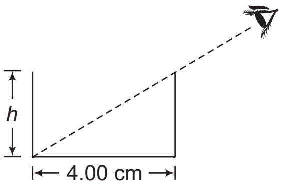
6. An air bubble is seen inside a solid sphere of glass ($n = 1.5$) of 4.0 cm diameter at a distance of 1.0 cm from the surface of the sphere (on seeing along the diameter). Determine the real position of the bubble inside the sphere.

7. Find the position of final image of an object O as shown in figure.

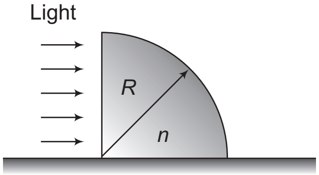


8. One face of a rectangular glass plate 6 cm thick is silvered. An object held 8 cm in front of the unsilvered face forms an image 10 cm behind the silvered face. Find the refractive index of glass. Consider all the three steps.

9. A shallow glass dish is 4.00 cm wide at the bottom as shown in figure. When an observer's eye is positioned as shown, the observer sees the edge of the bottom of the empty dish. When this dish is filled with water, the observer sees the centre of the bottom of the dish. Find the height of the dish $\mu_w = 4/3$.

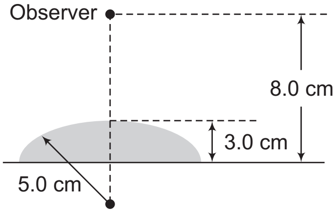


10. A glass prism in the shape of a quarter cylinder lies on a horizontal table. A uniform, horizontal light beam falls on its vertical plane surface as shown in the figure. If the radius of the cylinder is $R = 5$ cm and the refractive index of the glass is $n = 1.5$, where on the table beyond the cylinder, will a path of light be found?



11. A glass sphere with 10 cm radius has a 5 cm radius spherical hole at its centre. A narrow beam of parallel light is directed into the sphere. Where, if anywhere, will the sphere produce an image? The index of refraction of the glass is 1.50.

12. A glass sphere has a radius of 5.0 cm and a refractive index of 1.6. A paperweight is constructed by slicing through the sphere on a plate that is 2.0 cm from the centre of the sphere and perpendicular to a radius of the sphere that passes through the centre of the circle formed by the intersection of the plane and the sphere. The paperweight is placed on a table and viewed from directly above an observer who is 8.0 cm from the table top, as shown in figure. When viewed through the paperweight, how far away does the table top appear to the observer?



13. A fish is rising up vertically inside a pond with velocity 4 cm/s, and notices a bird, which is diving downward and its velocity appears to be 16 cm/ s (to the fish). What is the real velocity of the diving bird, if refractive index of water is 4/ 3?

14. A lens with a focal length of 16 cm produces a sharp image of an object in two positions, which are 60 cm apart. Find the distance from the object to the screen.

15. Two glasses with refractive indices of 1.5 and 1.7 are used to make two identical double convex lenses.

(a) Find the ratio between their focal lengths.

(b) How will each of these lenses act on a ray parallel to its optical axis if the lenses are submerged into a transparent liquid with a refractive index of 1.6?

16. A converging beam of rays is incident on a diverging lens. Having passed through the lens the rays intersect at a point 15 cm from the lens. If the lens is removed, the point where the rays meet, move 5 cm closer to the mounting that holds the lens. Find the focal length of the lens.

17. A parallel beam of rays is incident on a convergent lens with a focal length of 40 cm. Where a divergent lens with a focal length of 15 cm be placed for the beam of rays to remain parallel after passing through the two lenses.

18. An optical system consists of two convergent lenses with focal lengths $f_1 = 20$ cm and $f_2 = 10$ cm. The distance between the lenses is $d = 30$ cm. An object is placed at a distance of 30 cm from the first lens. At what distance from the second lens will the image be obtained?

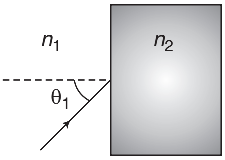
19. Determine the position of the image produced by an optical system consisting of a concave mirror with a focal length of 10 cm and a convergent lens with a focal length of 20 cm. The distance from the mirror to the lens is 30 cm and from the lens to the object is 40 cm. Consider only two steps. Plot the image.

20. A parallel beam of light is incident on a system consisting of three thin lenses with a common optical axis. The focal lengths of the lenses are equal to $f_1 = +10$ cm and $f_2 = -20$ cm, and $f_3 = +9$ cm respectively. The distance between the first and the second lens is 15 cm and between the second and the third is 5 cm. Find the position of the point at which the beam converges when it leaves the system of lenses.

21. A point source of light S is placed at the bottom of a vessel containing a liquid of refractive index $5/3$. A person is viewing the source from above the surface. There is an opaque disc of radius 1 cm floating on the surface. The centre of the disc lies vertically above the source S. The liquid from the vessel is gradually drained out through a tap. What is the maximum height of the liquid for which the source cannot at all be seen from above?

22. A ray of light travelling in glass ($\mu_g = 3/2$) is incident on a horizontal glass-air surface at the critical angle θ_C. If a thin layer of water ($\mu_w = 4/3$) is now poured on the glass-air surface. At what angle will the ray of light emerges into water at glass-water surface?

23. A ray of light is incident on the left vertical face of glass cube of refractive index n_2, as shown in figure. The plane of incidence is the plane of the page, and the cube is surrounded by liquid (refractive index $= n_1$). What is the largest angle of incidence θ_1 for which total internal reflection occurs at the top surface?



24. Light is incident from glass $\left(\mu_g = \frac{3}{2}\right)$ to water $\left(\mu_w = \frac{4}{3}\right)$. Find the range of the angle of deviation for refracted light.

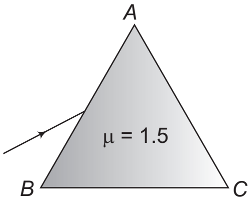
25. The angle of minimum deviation for a glass prism with $\mu = \sqrt{3}$ equals the refracting angle of the prism. What is the angle of the prism?

26. A ray incident on the face of a prism is refracted and escapes through an adjacent face. What is the maximum permissible angle of the prism, if it is made of glass with a refractive index of $\mu = 1.5$?

27. In an equilateral prism of $\mu = 1.5$, the condition for minimum deviation is fulfilled. If face AC is polished

(a) Find the net deviation.

(b) If the system is placed in water what will be the net deviation for same angle of incidence? Refractive index of water $= \frac{4}{3}$.



28. In a certain spectrum produced by a glass prism of dispersive power 0.0305, it is found that the refractive index for the red ray is 1.645 and that for the violet ray is 1.665. What is the refractive index for the yellow ray?

29. An achromatic lens-doublet is formed by placing in contact a convex lens of focal length 20 cm and a concave lens of focal length 30 cm. The dispersive power of the material of the convex lens is 0.18.

(a) Determine the dispersive power of the material of the concave lens.

(b) Calculate the focal length of the lens-doublet.

30. An achromatic convergent lens of focal length 150 cm is made by combining flint and crown glass lenses. Calculate the focal lengths of both the lenses and point out which one is divergent if the ratio of the dispersive power of flint and crown glasses is 3 : 2.

31. The index of refraction of heavy flint glass is 1.68 at 434 nm and 1.65 at 671 nm. Calculate the difference in the angle of deviation of blue (434 nm) and red (671 nm) light incident at 65° on one side of a heavy flint glass prism with apex angle 60°.

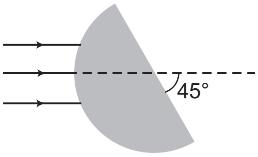
LEVEL 2

Single Correct Option

1. A bird is flying over a swimming pool at a height of 2 m from the water surface. If the bottom is perfectly plane reflecting surface and depth of swimming pool is 1 m, then the distance of final image of bird from the bird itself is $\mu_w = 4/3$

(a) $\frac{11}{3}$ m (b) $\frac{23}{3}$ m (c) $\frac{11}{4}$ m (d) $\frac{11}{2}$ m

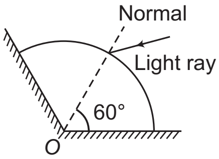
2. A parallel narrow beam of light is incident on the surface of a transparent hemisphere of radius R and refractive index $\mu = 1.5$ as shown. The position of the image formed by refraction at the spherical surface only is



(a) $\frac{R}{2}$ (b) $3R$

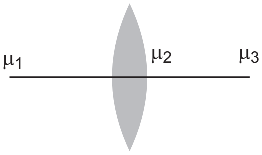
(c) $\frac{R}{3}$ (d) $2\,R$

3. Consider the situation as shown in figure. The point O is the centre . The light ray forms an angle of 60° with the normal. The normal makes an angle 60° with the horizontal and each mirror makes an angle 60° with the normal. The value of refractive index of that spherical portion so that light ray retraces its path is



(a) $\sqrt{2}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{3}{2}$ (d) $\sqrt{3}$

4. The figure shows an equi-convex lens. What should be the condition of the refractive indices so that the lens becomes diverging?



(a) $2\mu_3 > \mu_1 - \mu_2$ (b) $2\mu_2 < \mu_1 + \mu_3$

(c) $2\mu_2 > 2\mu_1 - \mu_3$ (d) None of these

5. An object is kept at a distance of 16 cm from a thin lens and the image formed is real. If the object is kept at a distance of 6 cm from the same lens, the image formed is virtual. If the sizes of the image formed are equal, the focal length of the lens will be

(a) 19 cm (b) 17 cm

(c) 21 cm (d) 11 cm

6. The apparent depth of water in cylindrical water tank of diameter $2R$ cm is reducing at the rate of x cm/min when water is being drained out at a constant rate. The amount of water drained in cc/min is (n_1 = refractive index of air, n_2 = refractive index of water)

(a) $\frac{x\pi R^2 n_1}{n_2}$ (b) $\frac{x\pi R^2 n_2}{n_1}$ (c) $\frac{2\pi R n_1}{n_2}$ (d) $\pi R^2 x$

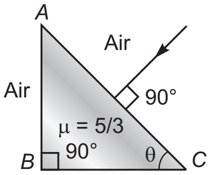
7. A ray of light makes normal incidence on the diagonal face of a right angled prism as shown in figure. If $\theta = 37°$, then the angle of deviation after second step (from AB) is ($\sin 37° = 3/5$)

(a) 53°

(b) 74°

(c) 106°

(d) 90°



8. A bird in air looks at a fish directly below it inside in a transparent liquid in a tank. If the distance of the fish as estimated by the bird is h_1 and that of the bird as estimated by the fish is h_2, then the refractive index of the liquid is

(a) $\dfrac{h_2}{h_1}$ (b) $\dfrac{h_1}{h_2}$

(c) $\dfrac{h_1+h_2}{h_1-h_2}$ (d) $\dfrac{h_1-h_2}{h_1+h_2}$

9. Diameter of the flat surface of a circular plano-convex lens is 6 cm and thickness at the centre is 3 mm. The radius of curvature of the curved part is

(a) 15 cm (b) 20 cm

(c) 30 cm (d) 10 cm

10. When the object is at distances u_1 and u_2 from the optical centre of a convex lens, a real and a virtual image of the same magnification are obtained . The focal length of the lens is

(a) $\dfrac{u_1-u_2}{2}$ (b) u_1+u_2 (c) $\sqrt{u_1u_2}$ (d) $\dfrac{u_1+u_2}{2}$

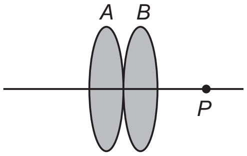
11. Two convex lenses placed in contact form the image of a distant object at P. If the lens B is moved to the right, the image will

(a) move to the left

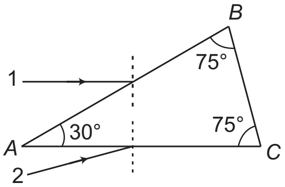
(b) move to the right

(c) remain at P

(d) move either to the left or right, depending upon focal lengths of the lenses



12. Two light rays 1 and 2 are incident on two faces AB and AC on an isosceles prism as shown in the figure. The rays emerge from the side BC. Then,



(a) minimum deviation of ray 1 > minimum deviation of ray 2

(b) minimum deviation of ray 1 < minimum deviation of ray 2

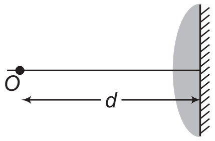
(c) minimum deviation of ray 1 = minimum deviation of ray 2

(d) Cannot be determined

13. Refractive index of a prism is $\sqrt{\dfrac{7}{3}}$ and the angle of prism is 60°. The limiting angle of incidence of a ray that will be transmitted through the prism is approximately

(a) 30° (b) 45° (c) 15° (d) None of these

14. A plano-convex thin lens of focal length 10 cm is silvered at its plane surface. The distance d at which an object must be kept in order to get its image on itself is

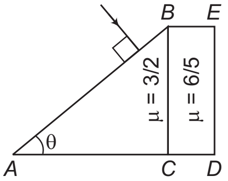


(a) 5 cm (b) 20 cm
(c) 10 cm (d) 2.5 cm

15. There is a small black dot at the centre C of a solid glass sphere of refractive index μ. When seen from outside, the dot will appear to be located

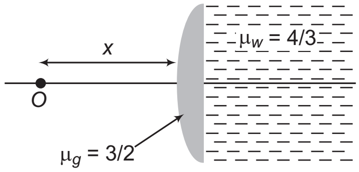
(a) Away from the C for all values of μ
(b) At C for all values of μ.
(c) At C for $\mu = 1.5$, but away from C for μ not equal to 1.5
(d) At C for $2 < \mu < 1.5$

16. In the figure ABC is the cross-section of a right angled prism and $BCDE$ is the cross-section of a glass slab. The value of θ so that light incident normally on the face AB does not cross the face BC is (Given $\sin^{-1} 3/5 = 37°$)



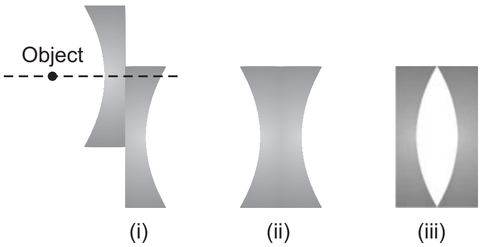
(a) $\theta < 37°$ (b) $\theta < 53°$ (c) $\theta \geq 37°$ (d) $\theta \geq 53°$

17. An object O is kept in air in front of a thin plano-convex lens of radius of curvature 10 cm. Its refractive index is 3/2 and the medium towards right of the plane surface is water of refractive index 4/3. What should be distance x of the object so that the rays become parallel finally?



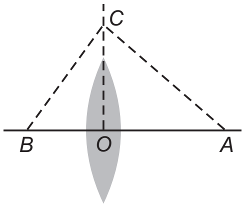
(a) 5 cm (b) 10 cm (c) 20 cm (d) 15 cm

18. If a symmetrical bi-concave thin lens is cut into two identical halves, and they are placed in different ways as shown, then



(a) three images will be formed in case (i)
(b) two images will be formed in case (ii)
(c) the ratio of focal lengths in (ii) and (iii) is 1
(d) the ratio of focal lengths in (ii) and (iii) is 2

19. If an object is placed at $A\,(OA > f)$; where f is the focal length of the lens, the image is formed at B. A perpendicular is erected at O and C is chosen on it such that the angle $\angle BCA$ is a right angle. Then, the value of f will be



(a) $\dfrac{AB}{OC^2}$ (b) $\dfrac{(AC)\,(BC)}{OC}$

(c) $\dfrac{OC^2}{AB}$ (d) $\dfrac{(OC)\,(AB)}{AC + BC}$

20. An object is seen through a glass slab of thickness 36 cm and refractive index 3/2. The observer, and the slab are dipped in water ($\mu = 4/3$). The shift produced in the position of the object is

(a) 12 cm (b) 4 cm

(c) 6 cm (d) 8 cm

21. How much water should be filled in a container of height 21 cm, so that it appears half filled to the observer when viewed from the top of the container ($\mu = 4/3$).

(a) 8 cm (b) 10.5 cm

(c) 12 cm (d) 14 cm

22. Optic axis of a thin equi-convex lens is the x-axis. The co-ordinates of a point object and its image are (– 40 cm, 1 cm) and (50 cm, – 2 cm), respectively. Lens is located at

(a) $x = 20$ cm (b) $x = -30$ cm

(c) $x = -10$ cm (d) origin

23. A thin plano-convex lens acts like a concave mirror of radius of curvature 20 cm when its plane surface is silvered. The radius of curvature of the curved surface if index of refraction of its material is 1.5 will be

(a) 40 cm (b) 30 cm

(c) 10 cm (d) 20 cm

24. A thin lens, made of glass of refractive index 3/2, produces a real and magnified image of an object in air. If the whole system, maintaining the same distance between the object and the lens, is immersed in water (RI = 4/3), then the image formed will be

(a) real, magnified (b) real, diminished

(c) virtual, magnified (d) virtual, diminished

25. The maximum value of refractive index of a prism which permits the transmission of light through it when the refracting angle of the prism is 90°, is given by

(a) 1.500 (b) 1.414

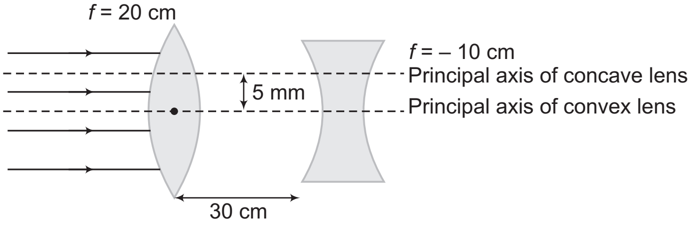
(c) 2.000 (d) 1.732

26. A glass slab of thickness 4 cm contains the same number of waves as 5 cm of water, when both are traversed by the same monochromatic light. If the refractive index of water is 4/3, then refractive index of glass is

(a) 5/3 (b) 5/4

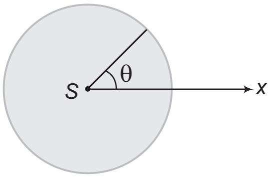
(c) 16/15 (d) 1.5

27. If the optic axis of convex and concave lenses are separated by a distance 5 mm as shown in the figure. Find the coordinate of the final image formed by the combination if parallel beam of light is incident on lens. Origin is at the optical centre of convex lens



(a) (25 cm, 0.5 cm) (b) (25 cm, 0.25 cm)
(c) (25 cm, – 0.5 cm) (d) (25 cm, – 2.5 cm)

28. A light source S is placed at the centre of a glass sphere of radius R and refractive index μ. The maximum angle θ with the x-axis (as shown in the figure) an incident light ray can make without suffering total internal reflection is

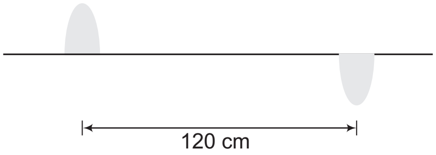


(a) $\cos^{-1}\left(\frac{1}{\mu}\right)$ (b) $\sin^{-1}\left(\frac{1}{\mu}\right)$
(c) $\tan^{-1}\left(\frac{1}{\mu}\right)$ (d) there will never be total internal reflection

29. A sphere $\left(\mu = \frac{4}{3}\right)$ of radius 1 m has a small cavity of diameter 1 cm at its centre. An observer who is looking at it from right, sees the magnification of diameter of the cavity as

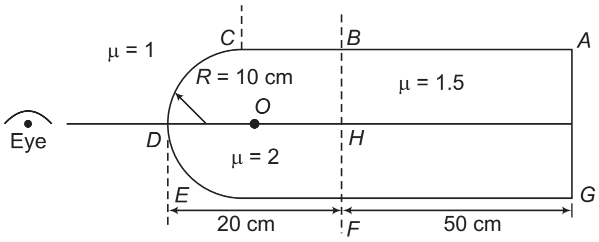
(a) $\frac{4}{3}$ (b) $\frac{3}{4}$
(c) 1 (d) 0.5

30. An equi-convex lens of $\mu = 1.5$ and $R = 20$ cm is cut into two equal parts along its axis. Two parts are then separated by a distance of 120 cm (as shown in figure). An object of height 3 mm is placed at a distance of 30 cm to the left of first half lens. The final image will form at



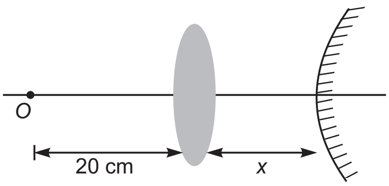
(a) 120 cm to the right of first half lens, 3 mm in size and inverted
(b) 150 cm to the right of first half lens, 3 mm in size and erect
(c) 120 cm to the right of first half lens, 4 mm in size and inverted
(d) 150 cm to the right of first half lens, 4 mm in size and erect

31. As shown in the figure, region *BCDEF* and *ABFG* are of refractive index 2.0 and 1.5 respectively. A particle *O* is kept at the mid of *DH*. Image of the object as seen by the eye is at a distance



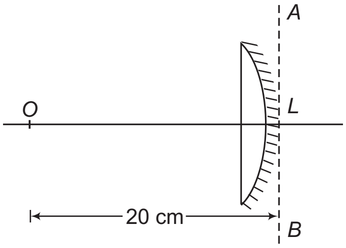
(a) 10 cm from point *D*
(b) 22.5 cm from point *D*
(c) 30 cm from point *D*
(d) 20 cm from point *D*

32. A point object *O* is placed at a distance of 20 cm from a convex lens of focal length 10 cm as shown in the figure. At what distance *x* from the lens should a convex mirror of focal length 60 cm, be placed so that final image coincide with the object?



(a) 10 cm
(b) 40 cm
(c) 20 cm
(d) Final image can never coincide with the object in the given conditions

33. A point object is placed at a distance of 20 cm from a thin plano-convex lens of focal length 15 cm ($\mu = 1.5$). The curved surface is silvered. The image will form at



(a) 60 cm left of *AB*
(b) 30 cm left of *AB*
(c) $\frac{20}{7}$ cm left of *AB*
(d) 60 cm right of *AB*

Note *Neglect thickness of lens.*

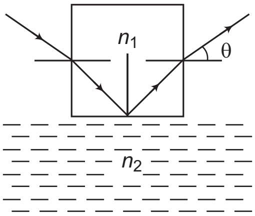
34. A flat glass slab of thickness 6 cm and index 1.5 is placed in front of a plane mirror. An observer is standing behind the glass slab and looking at the mirror. The actual distance of the observer from the mirror is 50 cm. The distance of his image from himself, as seen by the observer is

(a) 94 cm
(b) 96 cm
(c) 98 cm
(d) 100 cm

35. Distance of an object from the first focus of an equi-convex lens is 10 cm and the distance of its real image from second focus is 40 cm. The focal length of the lens is

(a) 25 cm
(b) 10 cm
(c) 20 cm
(d) 40 cm

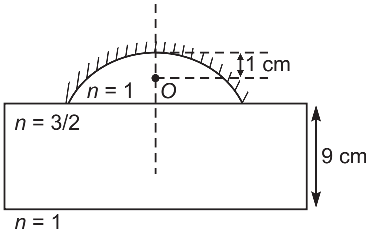
36. A cubical block of glass of refractive index n_1 is in contact with the surface of water of refractive index n_2. A beam of light is incident on vertical face of the block. After refraction a total internal reflection at the base and refraction at the opposite face take place. The ray emerges at angle θ as shown. The value of θ is given by



(a) $\sin\theta < \sqrt{n_1^2 - n_2^2}$
(b) $\cos\theta < \sqrt{n_1^2 - n_2^2}$
(c) $\sin\theta < \dfrac{1}{\sqrt{n_1^2 - n_2^2}}$
(d) $\cos\theta < \dfrac{1}{\sqrt{n_1^2 - n_2^2}}$

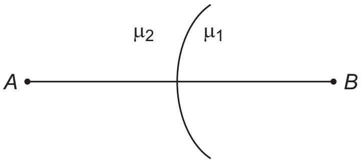
37. A concave mirror of focal length 2 cm is placed on a glass slab as shown in the figure. The image of point object O formed due to reflection at mirror and then refraction by the slab

(a) is virtual and at 2 cm from pole of the concave mirror
(b) is virtual and on the pole of mirror
(c) is real and on the object itself
(d) None of the above



38. Two refracting media are separated by a spherical interface as shown in the figure. AB is the principal axis, μ_1 and μ_2 are the refractive indices of medium of incidence and medium of refraction respectively. Then,

(a) if $\mu_2 > \mu_1$, then there cannot be a real image of real object
(b) if $\mu_2 > \mu_1$, then there cannot be a real image of virtual object
(c) if $\mu_1 > \mu_2$, then there cannot be a real image of virtual object
(d) if $\mu_1 > \mu_2$, then there cannot be a virtual image of virtual object



39. A concavo-convex lens has refractive index 1.5 and the radii of curvature of its surfaces are 10 cm and 20 cm. The concave surface is upwards and is filled with oil of refractive index 1.6. The focal length of the combination will be

(a) 18.18 cm
(b) 15 cm
(c) 22 cm
(d) 28.57 cm

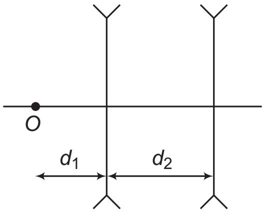
40. A convex spherical refracting surface separates two media glass and air ($\mu_g = 1.5$). If the image is to be real, at what minimum distance u should the object be placed in air if R is the radius of curvature

(a) $u > 3R$
(b) $u > 2R$
(c) $u < 4R$
(d) $u < R$

41. An object is moving towards a converging lens on its axis. The image is also found to be moving towards the lens. Then, the object distance u must satisfy

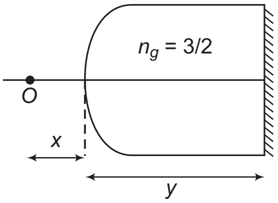
(a) $2f < u < 4f$
(b) $f < u < 2f$
(c) $u > 4f$
(d) $u < f$

42. Two diverging lenses are kept as shown in figure. The final image formed will be



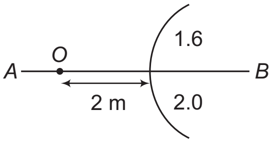
(a) virtual for any value of d_1 and d_2
(b) real for any value of d_1 and d_2
(c) virtual or real depends on d_1 and d_2 only
(d) virtual or real depends on d_1 and d_2 and also on the focal lengths of the lens

43. In the figure shown, a point object O is placed in air on the principal axis. The radius of curvature of the spherical surface is 60 cm. I is the final image formed after all reflections and refractions.



(a) If $x = 120$ cm, then I is formed on O for any value of y
(b) If $x = 240$ cm, then I is formed on O only if $y = 360$ cm
(c) If $x = 240$ cm, then I is formed on O for any value of y
(d) None of the above

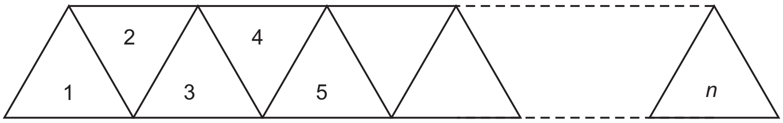
44. In the figure, a point object O is placed in air. A spherical boundary separates two media of radius of curvature 1.0 m. AB is principal axis. The separation between the images formed due to refraction at spherical surface is



(a) 12 m
(b) 20 m
(c) 14 m
(d) 10 m

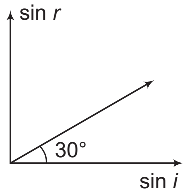
More than One Correct Option

1. n number of identical equilateral prisms are kept in contact as shown in figure. If deviation through a single prism is δ. Then, (n, m are integers)



(a) if $n = 2m$, deviation through n prisms is zero
(b) if $n = 2m + 1$, deviation through system of n prisms is δ
(c) if $n = 2m$, deviation through system of n prisms is δ
(d) if $n = 2m + 1$, deviation through system of n prisms is zero

2. A ray of monochromatic light is incident on the plane surface of separation between two media x and y with angle of incidence i in the medium x and angle of refraction r in the medium y. The graph shows the relation between sin i and sin r.



(a) The speed of light in the medium y is $\sqrt{3}$ times than in medium x

(b) The speed of light in the medium y is $\frac{1}{\sqrt{3}}$ times than in medium x

(c) The total internal reflection can take place when the incidence is in x

(d) The total internal reflection can take place when the incidence is in y

3. Which of the following statement(s) is/are true?

(a) In vacuum the speed of red colour is more than that of violet colour

(b) An object in front of a mirror is moved towards the pole of a spherical mirror from infinity, it is found that image also moves towards the pole. The mirror must be convex

(c) There exist two angles of incidence in a prism for which angles of deviation are same except minimum deviation

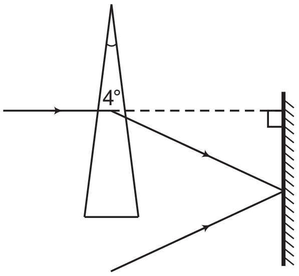
(d) A ray travels from a rarer medium to denser medium. There exist three angles of incidence for which the deviation is same

4. A lens of focal length f is placed in between an object and screen at a distance D. The lens forms two real images of object on the screen for two of its different positions, a distance x apart. The two real images have magnifications m_1 and m_2 respectively ($m_1 > m_2$). Choose the correct statement(s).

(a) $m_1 m_2 = -1$ (b) $m_1 m_2 = 1$

(c) $f = \frac{D^2 - x^2}{4D}$ (d) $D \geq 4f$

5. A small angled prism of apex angle $A = 4°$ and refractive index $\mu = 1.5$ is placed in front of a vertical plane mirror as shown in figure. If the mirror is rotated through an angle θ, then the light ray becomes horizontal either after the mirror or after second time passing from the prism in opposite direction. The value of θ is



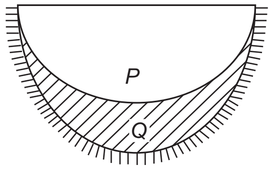
(a) 1° (b) 2°

(c) 4° (d) Not possible

Comprehension Based Questions

Passage : (Q. No. 1 to 3)

A plano-convex lens P and a concavo-convex lens Q are in contact as shown in figure. The refractive index of the material of the lens P and Q is 1.8 and 1.2 respectively. The radius of curvature of the concave surface of the lens Q is double the radius of curvature of the convex surface. The convex surface of Q is silvered.



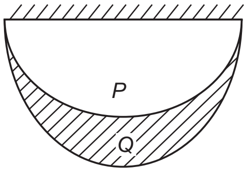
1. An object is placed on the principal axis at a distance 10 cm from the plane surface. The image is formed at a distance 40 cm from the plane surface on the same side. The focal length of the system is

(a) – 8 cm (b) 8 cm (c) $-\frac{40}{3}$ cm (d) $\frac{40}{3}$ cm

2. The radius of curvature of common surface is

(a) 48 cm (b) 24 cm
(c) 12 cm (d) 8 cm

3. If the plane surface of P is silvered as shown in figure, the system acts as



(a) convex mirror of focal length 24 cm
(b) concave mirror of focal length 8 cm
(c) concave mirror of focal length 24 cm
(d) convex mirror of focal length 8 cm

Match the Columns

1. Match the following two columns for a convex lens corresponding to object position shown in Column I.

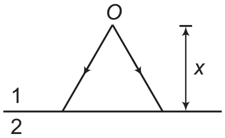
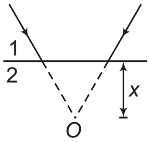
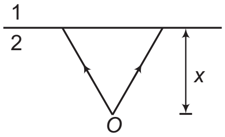
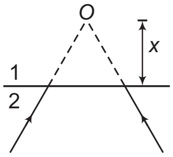
Column I	Column II
(a) Between O and F_1	(p) Real
(b) Between F_1 and $2F_1$	(q) Virtual
(c) Between O and F_2	(r) Erect
(d) Between F_2 and $2F_2$	(s) Inverted

Note *$O \rightarrow$ optical centre, $F_1 \rightarrow$ first focus and $F_2 \rightarrow$ second focus.*

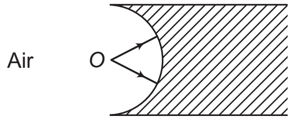
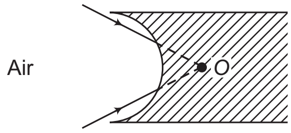
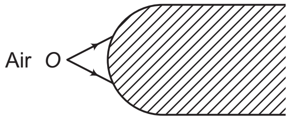
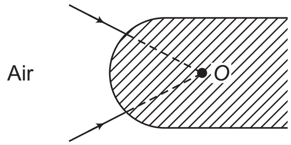
2. Match the following two columns for a concave lens corresponding to object position shown in Column II.

Column I	Column II
(a) Between O and F_1	(p) Real
(b) Between F_1 and $2F_1$	(q) Virtual
(c) Between O and F_2	(r) Erect
(d) Between F_2 and $2F_2$	(s) Inverted

3. Match the following two columns corresponding to single refraction from plane surface. In all cases shown in Column I, $\mu_1 > \mu_2$.

Column I	Column II
(a) 	(p) Image distance greater than *x* from plane surface
(b) 	(q) Image distance less than *x* from plane surface
(c) 	(r) Real image
(d) 	(s) Virtual image

4. Match the following two columns.

Column I	Column II
(a) 	(p) Real image
(b) 	(q) Virtual image
(c) 	(r) May be real or virtual image
(d) 	(s) Image is at infinity

5. A convex lens L_1 and a concave lens L_2 have refractive index 1.5. Match the following two columns.

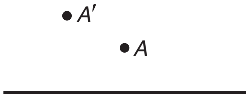
Column I	Column II
(a) L_1 is immersed in a liquid of refractive index 1.4	(p) Lens will behave as convex lens
(b) L_1 is immersed in a liquid of refractive index 1.6	(q) Lens will behave as concave lens
(c) L_2 is immersed in a liquid of refractive index 1.4	(r) Magnitude of power of lens will increase
(d) L_2 is immersed in a liquid of refractive index 1.6	(s) Magnitude of power of lens will decrease

6. Consider a linear extended object that could be real or virtual with its length at right angles to the optic axis of a lens. With regard to image formation by lenses.

Column I	Column II
(a) Image of the same size as the object	(p) Concave lens in case of real object
(b) Virtual image of a size greater than the object	(q) Convex lens in case of real object
(c) Real image of a size smaller than the object	(r) Concave lens in case of virtual object
(d) Real and magnified image	(s) Convex lens in case of virtual object

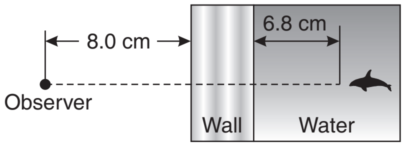
Subjective Questions

1. Figure shows the optical axis of a lens, the point source of light A and its virtual image A'. Trace the rays to find the position of the lens and of its principal focus. What type of lens is it?



2. Solve the problem similar to the previous one if A and A' are interchanged.

3. In figure, a fish watcher watches a fish through a 3.0 cm thick glass wall of a fish tank. The watcher is in level with the fish; the index of refraction of the glass is $8/5$ and that of the water is $4/3$.



(a) To the fish, how far away does the watcher appear to be?

(b) To the watcher, how far away does the fish appear to be?

4. A concave spherical mirror with a radius of curvature of 0.2 m is filled with water. What is the focal length of this system? Refractive index of water is $4/3$.

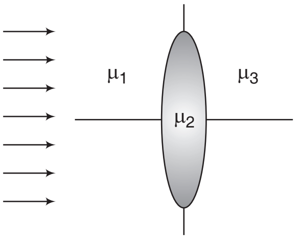
5. A convexo-convex lens has a focal length of $f_1 = 10$ cm. One of the lens surfaces having a radius of curvature of $R = 10$ cm is coated with silver. Construct the image of the object produced by the given optical system and determine the position of the image if the object is at a distance of $a = 15$ cm from the lens. Refractive index of lens $= 1.5$.

6. A lens with a focal length of $f = 30$ cm produces on a screen a sharp image of an object that is at a distance of $a = 40$ cm from the lens. A plane parallel plate with thickness of $d = 9$ cm is placed between the lens and the object perpendicular to the optical axis of the lens. Through what distance should the screen be shifted for the image of the object to remain distinct? The refractive index of the glass of the plate is $\mu = 1.8$.

7. One side of radius of curvature $R_2 = 120$ cm of a convexo-convex lens of material of refractive index $\mu = 1.5$ and focal length $f_1 = 40$ cm is slivered. It is placed on a horizontal surface with silvered surface in contact with it. Another convex lens of focal length $f_2 = 20$ cm is fixed coaxially $d = 10$ cm above the first lens. A luminous point object O on the axis gives rise to an image coincident with it. Find its height above the upper lens.

8. A small object is placed on the principal axis of concave spherical mirror of radius 20 cm at a distance of 30 cm. By how much will the position of the image alter only after mirror, when a parallel-sided slab of glass of thickness 6 cm and refractive index 1.5 is introduced between the centre of curvature and the object? The parallel sides are perpendicular to the principal axis.

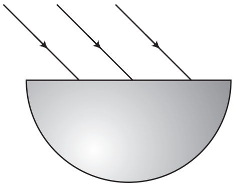
9. A thin glass lens of refractive index $\mu_2 = 1.5$ behaves as an interface between two media of refractive indices $\mu_1 = 1.4$ and $\mu_3 = 1.6$ respectively. Determine the focal length of the lens for the shown arrangement of radius of curvature of both the surfaces 20 cm.



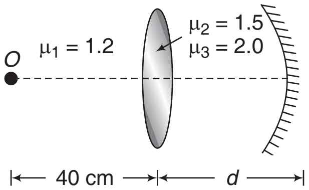
10. A glass hemisphere of radius 10 cm and $\mu = 1.5$ is silvered over its curved surface. There is an air bubble in the glass 5 cms from the plane surface along the axis. Find the position of the images of this bubble seen by observer looking along the axis into the flat surface of the atmosphere.

11. A equilateral prism of flint glass ($\mu_g = 3/2$) is placed inside water ($\mu_w = 4/3$).
    (a) At what angle should a ray of light fall on the face of the prism so that inside the prism the ray is perpendicular to the bisector of the angle of the prism.
    (b) Through what angle will the ray turn after passing through both faces of the prism?

12. Rays of light fall on the plane surface of a half cylinder at an angle 45° in the plane perpendicular to the axis (see figure). Refractive index of glass is $\sqrt{2}$. Discuss the condition that the rays do not suffer total internal reflection.



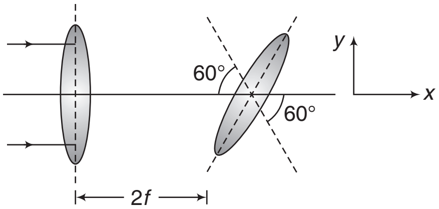
13. The figure shows an arrangement of an equi-convex lens and a concave mirror. A point object O is placed on the principal axis at a distance 40 cm from the lens such that the final image is also formed at the position of the object. If the radius of curvature of the concave mirror is 80 cm, find the distance d. Also draw the ray diagram. The focal length of the lens in air is 20 cm.



14. A convex lens is held 45 cm above the bottom of an empty tank. The image of a point at the bottom of a tank is formed 36 cm above the lens. Now, a liquid is poured into the tank to a depth of 40 cm. It is found that the distance of the image of the same point on the bottom of the tank is 48 cm above the lens. Find the refractive index of the liquid.

15. A parallel beam of light falls normally on the first face of a prism of a small angle. At the second face it is partly transmitted and partly reflected. The reflected beam striking at the first face again emerges from it in a direction making an angle of 6°30′ with the reversed direction of the incident beam. The refracted beam is found to have undergone a deviation of 1°15 from the original direction. Calculate the refractive index of the glass and the angle of the prism.

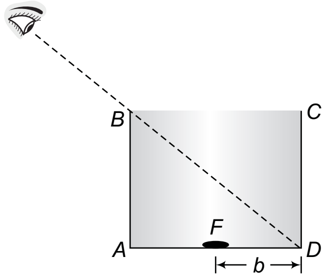
16. Two converging lenses of the same focal length f are separated by a distance $2f$. The axis of the second lens is inclined at angle $\theta = 60°$ with respect to the axis of the first lens. A parallel paraxial beam of light is incident from left side of the lens. Find the coordinates of the final image with respect to the origin of the first lens.



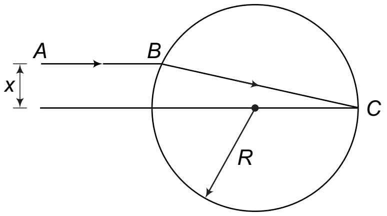
17. A cubical vessel with non-transparent walls is so located that the eye of an observer does not see its bottom but sees all of the wall CD.

To what height should water be poured into the vessel for the observer to see an object F arranged at a distance of $b = 10$ cm from corner D? The face of the vessel is $a = 40$ cm.

Refractive index of water is $\frac{4}{3}$.

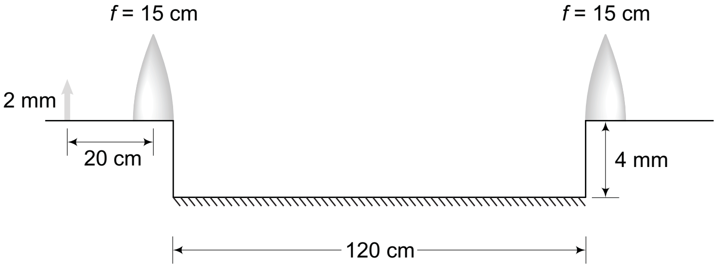


18. A spherical ball of transparent material has index of refraction μ. A narrow beam of light AB is aimed as shown. What must the index of refraction be in order that the light is focused at the point C on the opposite end of the diameter from where the light entered? Given that $x << R$.



19. A ray incident on a droplet of water at an angle of incidence i undergoes two reflections (not total) and emerges. If the deviation suffered by the ray within the drop is minimum and the refractive index of the droplet be μ, then show that $\cos i = \sqrt{\dfrac{\mu^2 - 1}{8}}$.

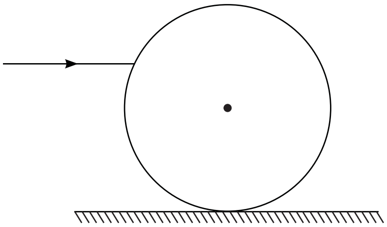
20. A convex lens of focal length 15 cm is split into two halves and the two halves are placed at a separation of 120 cm. Between the two halves of convex lens a plane mirror is placed horizontally and at a distance of 4 mm below the principal axis of the lens halves. An object of length 2 mm is placed at a distance of 20 cm from one half lens as shown in figure.

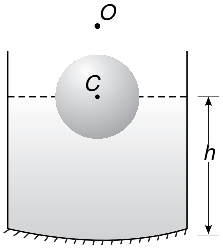


(a) Find the position and size of the final image.

(b) Trace the path of rays forming the image.

21. A cylindrical glass rod of radius 0.1 m and refractive index $\sqrt{3}$ lies on a horizontal plane mirror. A horizontal ray of light moving perpendicular to the axis of the rod is incident on it. At what height from the mirror should the ray be incident so that it leaves the rod at a height of 0.1 m above the plane mirror? At what centre to centre distance a second similar rod, parallel to the first, be placed on the mirror, such that the emergent ray from the second rod is in line with the incident ray on the first rod?





22. A transparent solid sphere of radius 2 cm and density ρ floats in a transparent liquid of density 2ρ kept in a beaker. The bottom of the beaker is spherical in shape with radius of curvature 8 cm and is silvered to make it concave mirror as shown in the figure. When an object is placed at a distance of 10 cm directly above the centre of the sphere C, its final image coincides with it. Find h (as shown in the figure), the height of the liquid surface in the beaker from the apex of the bottom. Consider the paraxial rays only. The refractive index of the sphere is 3/2 and that of the liquid is 4/3.

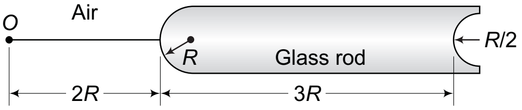
23. A hollow sphere of glass of refractive index μ has a small mark on its interior surface which is observed from a point outside the sphere on the side opposite the centre. The inner cavity is concentric with external surface and the thickness of the glass is everywhere equal to the radius of the inner surface. Prove that the mark will appear nearer than it really is, by a distance $\dfrac{(\mu-1)R}{(3\mu-1)}$, where R is the radius of the inner surface.

24. A ray of light is refracted through a sphere whose material has a refractive index μ in such a way that it passes through the extremities of two radii which make an angle β with each other. Prove that if α is the deviation of the ray caused by its passage through the sphere, then

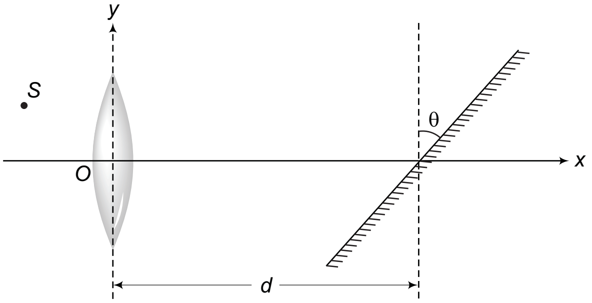
$$\cos\left(\frac{\beta-\alpha}{2}\right)=\mu\cos\left(\frac{\beta}{2}\right)$$

25. A man of height 2.0 m is standing on a level road where because of temperature variation the refractive index of air is varying as $\mu=\sqrt{1+ay}$, where y is height from road. If $a=2.0\times10^{-6}\,\text{m}^{-1}$. Then, find distant point that he can see on the road.

26. A glass rod has ends as shown in figure. The refractive index of glass is μ. The object O is at a distance $2R$ from the surface of larger radius of curvature. The distance between apexes of ends is $3R$. Find the distance of image formed of the point object from right hand vertex. What is the condition to be satisfied if the image is to be real?



27. A thin converging lens of focal length $f=1.5$ m is placed along y-axis such that its optical centre coincides with the origin. A small light source S is placed at (–2.0 m, 0.1 m). Where should a plane mirror inclined at an angle θ, $\tan\theta=0.3$ be placed such that y-coordinate of final image is 0.3 m, i.e. find d. Also find x-coordinate of final image.



Answers

Introductory Exercise 31.1

1. 2

2. The frequency does not change, while the wavelength and speed change by the factor μ_1/μ_2.

3. 1.67

Introductory Exercise 31.2

1. $\sqrt{3}$

2. $\frac{3}{2}$

Introductory Exercise 31.3

1. (a) $\frac{50}{3}$ cm (b) 25 cm

Introductory Exercise 31.4

1. $\frac{125}{3}$ cm

Introductory Exercise 31.5

2. 8.57 cm

3. (a) 45.0 cm (b) –90.0 cm (c) –6.0 cm

4. Inside the bowl at – 9.0 cm

5. 4.0 cm

Introductory Exercise 31.6

1. 39 cm

2. 18.75 cm from the lens, 5.3 cm

4. From the surface to infinity on object side

5. (a) 36 cm (b) 36 cm

6. 40 cm

8. 12 cm

9. 1.5

10. 2 cm

11. 16 cm

12. – 40.0 cm

13. 10.0 cm.

Introductory Exercise 31.7

1. 1.5

2. $\mu = 1.3, \theta_C = \sin^{-1}(0.77)$

3. (a) $\theta = \sin^{-1}\left(\frac{13}{16}\right)$ (b) Yes

Introductory Exercise 31.8

1. 46°

2. C

3. $\mu \le \sqrt{2}$

4. $\sin^{-1}(2/3) = 42^\circ$

5. $\mu = \sqrt{3}$

6. 90°

7. 0°

8. $\sqrt{2}$

9. 9°

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (a,b) **3.** (b) **4.** (c) **5.** (b) **6.** (d) **7.** (a) **8.** (c) **9.** (b) **10.** (d)
11. (d) **12.** (b) **13.** (a)

Objective Questions

1. (c) **2.** (d) **3.** (c) **4.** (d) **5.** (a) **6.** (c) **7.** (c) **8.** (b) **9.** (b) **10.** (a)
11. (b) **12.** (b) **13.** (b) **14.** (b) **15.** (b) **16.** (d) **17.** (c) **18.** (c) **19.** (d) **20.** (a)
21. (a) **22.** (a) **23.** (d) **24.** (a) **25.** (a) **26.** (c) **27.** (b) **28.** (a) **29.** (c) **30.** (a)
31. (b) **32.** (c) **33.** (b) **34.** (c) **35.** (a) **36.** (a) **37.** (d) **38.** (b) **39.** (a)

Subjective Questions

1. (a) 0.229 (b) 13.2° **2.** 1.07×10^{-8} s **3.** (a) First film, $t_{\min} = 4 \times 10^{-15}$ s (b) 7.5

4. 6.6 cm **5.** Final image is formed at 65 cm from first face on the same side of the object.

6. At a distance of 1.2 cm from the surface **7.** 11 cm behind mirror **8.** 1.5

9. 2.4 cm **10.** 1.71 cm **11.** 15 cm from the left edge **12.** 7.42 cm **13.** 9 cm/s

14. 100 cm **15.** (a) 1.4 (b) The first lens will be a diverging and the second a converging one

16. $f = -30$ cm **17.** 25 cm **18.** 7.5 cm **19.** 5 cm from mirror towards the lens

20. Ray will become parallel to the optic axis. **21.** $\frac{4}{3}$ cm **22.** $\sin^{-1}(3/4)$

23. $\sin^{-1}\sqrt{\left(\frac{n_2}{n_1}\right)^2 - 1}$ **24.** 0 to $\cos^{-1}(8/9)$ **25.** 60° **26.** $A_{\max} = 84°$

27. (a) 157.2° (b) 157.2° **28.** $\mu_y = 1.656$ **29.** (a) 0.27 (b) 60 cm

30. 50 cm, –75 cm, divergent lens is of flint glass **31.** 2.8°

LEVEL 2

Single Correct Option

1. (d) **2.** (b) **3.** (d) **4.** (b) **5.** (d) **6.** (b) **7.** (b) **8.** (a) **9.** (a) **10.** (d)
11. (b) **12.** (c) **13.** (a) **14.** (c) **15.** (b) **16.** (a) **17.** (c) **18.** (c) **19.** (c) **20.** (b)
21. (d) **22.** (c) **23.** (c) **24.** (c) **25.** (b) **26.** (a) **27.** (b) **28.** (d) **29.** (a) **30.** (b)
31. (a) **32.** (c) **33.** (c) **34.** (b) **35.** (c) **36.** (a) **37.** (d) **38.** (d) **39.** (d) **40.** (b)
41. (d) **42.** (a) **43.** (a) **44.** (a)

More than One Correct Options

1. (a,b) **2.** (b,d) **3.** (b,c) **4.** (b,c,d) **5.** (a,b)

Comprehension Based Questions

1. (a) **2.** (a) **3.** (c)

Match the Columns

1. (a) → q,r (b) → p,s (c) → p,r (d) → p,r
2. (a) → p,r (b) → q,s (c) → q,r (d) → q,r
3. (a) → q,s (b) → q,r (c) → p,s (d) → p,r
4. (a) → q (b) → r (c) → r (d) → p
5. (a) → p,s (b) → q,s (c) → q,s (d) → p,s
6. (a) → q,r (b) → q,r (c) → q,r (d) → q,r

Subjective Questions

1. Lens is convex

2. Lens is concave

3. (a) 20.0 cm (b) 14.975 cm

4. $F = -7.5$ cm (concave mirror)

5. At a distance of 2.14 cm from the system

6. 60 cm away from the lens

7. 10 cm

8. $\Delta v = 0.55$ cm,

9. $f = \infty$

10. First image at a distance of 3.33 cm from flat surface and the second at infinity

11. (a) 34.2° (b) 8.4°

12. Maximum distance of the incident rays from the centre should be $\sqrt{\frac{2}{3}}R$, where R is the radius of hemisphere

13. 30 cm

14. 1.37

15. $A = 2^\circ$, $\mu = 1.62$

16. (0, 0)

17. 26.65 cm

18. $\mu = 2$

20. (a) Final image is at a distance of 20 cm behind the second half lens and at a distance of 2/3 mm above the principal axis. The size of image is 2 mm and is inverted as compared to the given object

21. 0.186 m, 0.315 m

22. $h = 15$ cm

25. $x_{\max} = 2000$ m = 2 km

26. Distance $= \dfrac{R(9-4\mu)}{(10\mu-9)(\mu-2)}$, for final image to be real μ should lie between 2 and 2.25

27. $d = 5.0$ m, x co-ordinate of final image = 4.0 m

32

Interference and Diffraction of Light

Chapter Contents

32.1 Principle of Superposition

32.2 Resultant amplitude and intensity due to coherent sources

32.3 Interference

32.4 Young's double slit experiment

32.5 Introduction to Diffraction

32.6 Diffraction from narrow slits

32.1 Principle of Superposition

Suppose there are two sources of waves S_1 and S_2.

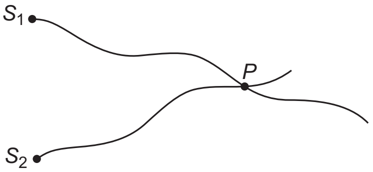


Fig. 32.1

Now, the two waves from S_1 and S_2 meet at some point (say P). Then, according to principle of superposition net displacement at P (from its mean position) at any time is given by

$$\boxed{y = y_1 + y_2}$$

Here, y_1 and y_2 are the displacements of P due to two waves individually.

For example, suppose at 9 AM, displacement of P above its mean position should be 6 mm accordingly to wave-1 and at the same time its displacement should be 2 mm below its mean position accordingly to wave-2, then at 9 AM net displacement of P will be 4 mm above its mean position.

Now, based upon the principle of superposition we have two phenomena in physics, interference and beats. Stationary waves (or standing waves) and Young's double slit experiment (or YDSE) are two examples of interference.

Based on principle of superposition means two or more than two waves meet at one point or several points and at every point net displacement is $y = y_1 + y_2$ or $y = y_1 + y_2 + y_3$ etc.

32.2 Resultant Amplitude and Intensity due to Coherent Sources

In article 32.1, we have seen that the two waves from two sources S_1 and S_2 were meeting at point P. Suppose they meet at P in a phase difference $\Delta\phi$ (or ϕ). If this phase difference remains constant with time, then sources are called coherent, otherwise incoherent.

For sources to be coherent, the frequencies (f, ω or T) of the two sources must be same. This can be understood by the following example.

Suppose the phase difference is 0°. It means they are in same phase. Both reaches their extremes ($+A$ or $-A$) simultaneously. They cross their mean positions (in the same direction) simultaneously. Now, if we want their phase difference to remain constant or we want that the above situation is maintained all the time, then obviously their time periods (or frequencies) must be same.

Resultant Amplitude

1. Consider the superposition of two sinusoidal waves of same frequency (means sources are coherent) at some point. Let us assume that the two waves are travelling in the same direction with same velocity. The equation of the two waves reaching at a point can be written as

$$y_1 = A_1 \sin(kx - \omega t)$$

and

$$y_2 = A_2 \sin(kx - \omega t + \phi)$$

The resultant displacement of the point where the waves meet, is

$$\begin{aligned} y &= y_1 + y_2 \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \phi) \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t)\cos\phi + A_2 \cos(kx - \omega t)\sin\phi \\ &= (A_1 + A_2\cos\phi)\sin(kx - \omega t) + A_2 \sin\phi \cos(kx - \omega t) \\ &= A\cos\theta \sin(kx - \omega t) + A\sin\theta\cos(kx - \omega t) \end{aligned}$$

or $$y = A\sin(kx - \omega t + \theta)$$

Here, $$A_1 + A_2\cos\phi = A\cos\theta$$

and $$A_2\sin\phi = A\sin\theta$$

or $$A^2 = (A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2$$

or $$\boxed{A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}} \qquad \ldots\text{(i)}$$

and $$\tan\theta = \frac{A\sin\theta}{A\cos\theta} = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

2. The above result can be obtained by graphical method also. Assume a vector $\mathbf{A}_1$ of length A_1 to represent the amplitude of first wave.

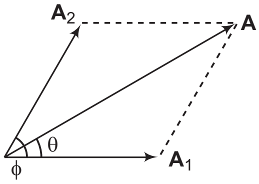


Fig. 32.2

Another vector $\mathbf{A}_2$ of length A_2, making an angle ϕ with $\mathbf{A}_1$ represent the amplitude of second wave. The resultant of $\mathbf{A}_1$ and $\mathbf{A}_2$ represents the amplitude of resulting function y. The angle θ represents the phase difference between the resulting function and the first wave.

Resultant Intensity

In the previous chapter, we have read that intensity of a wave is given by

$$I = \frac{1}{2}\rho\omega^2A^2v \quad \text{or} \quad I \propto A^2$$

So, if ρ, ω and v are same for the both interfering waves then Eq. (i) can also be written as

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi} \qquad \ldots\text{(ii)}$$

Here, proportionality constant ($I \propto A^2$) cancels out on right hand side and left hand side.

Note *(i) Eqs. (i) and (ii) are two equations for finding resultant amplitude and resultant intensity at some point due to two coherent sources.*

(ii) In the above equations ϕ is the constant phase difference at that point as the sources are coherent. Value of this constant phase difference will be different at different points.

(iii) The special case of above two equations is, when the individual amplitudes (or intensities) are equal.

or $A_1 = A_2 = A_0$ *(say)*

$\therefore$ $I_1 = I_2 = I_0$ *(say)*

In this case, Eqs. (i) and (ii) become

$$\boxed{\begin{aligned} A &= 2A_0 \cos\frac{\phi}{2} \quad \text{...(iii)} \\ \text{and} \quad I &= 4I_0 \cos^2\frac{\phi}{2} \quad \text{...(iv)} \end{aligned}}$$

(iv) From Eqs. (i) to (iv), we can see that, for given values of A_1, A_2, I_1 and I_2 the resultant amplitude and the resultant intensity are the functions of only ϕ.

(v) If three or more than three waves (due to coherent sources) meet at some point then there is no direct formula for finding resultant amplitude or resultant intensity. In this case, first of all we will find resultant amplitude by vector method (either by using polygon law of vector addition or component method) and then by the relation $I \propto A^2$, we can also determine the resultant intensity.

For example, if resultant amplitude comes out to be $\sqrt{2}$ times then resultant intensity will become two times.

32.3 Interference

For interference phenomena to take place, sources must be coherent. So, phase difference at some point should remain constant. Value of this constant phase difference will be different at different points. And since the sources are coherent, therefore following four equations can be applied for finding resultant amplitude and intensity (in case of two sources)

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \quad \text{...(i)}$$

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi \quad \text{...(ii)}$$

$$A = 2A_0\cos\frac{\phi}{2} \qquad (\text{if } A_1 = A_2 = A_0) \quad \text{...(iii)}$$

$$I = 4I_0\cos^2\frac{\phi}{2} \qquad (\text{if } I_1 = I_2 = I_0) \quad \text{...(iv)}$$

For given values of A_1, A_2, I_1 and I_2 the resultant amplitude and resultant intensity are the functions of only ϕ.

Now, suppose S_1 and S_2 are two coherent sources, then we can see that the two waves are meeting at several points (P_1, P_2, P_3 … etc). At different points path difference Δx will be different and therefore phase difference $\Delta\phi$ or ϕ will also be different. Because the phase difference depends on the path difference ($\Delta\phi$ or $\phi = \frac{2\pi}{\lambda}\cdot\Delta x$).

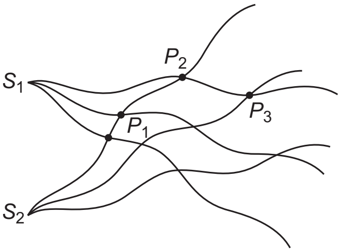


Fig. 32.3

And since phase difference at different points is different, therefore from the above four equations we can see that resultants amplitude and intensity will also be different. But whatever is the intensity at some point, it will remain constant at that point because the sources are coherent and the phase difference is constant at that point.

Constructive Interference

These are the points where resultant amplitude or intensity is maximum or

$A_{\max} = A_1 + A_2$ [from Eq. (i)]

or $A_{\max} = \pm 2A_0$ [from Eq. (iii)]

and $I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$ [from Eq. (ii)]

or $I_{\max} = 4I_0$ [from Eq. (iv)]

at those points where, $\cos\phi = +1$ [from Eqs. (i) or (ii)]

or $\phi = 0, 2\pi, 4\pi, \ldots, 2n\pi$ (where $n = 0, 1, 2$)

$\therefore$ $\Delta x = 0, \lambda, 2\lambda, \ldots, n\lambda$ $\left[\text{as } \Delta x = \phi\left(\dfrac{\lambda}{2\pi}\right)\right]$

Destructive Interference

These are the points where resultant amplitude or intensity is minimum or

$A_{\min} = A_1 \sim A_2$ [from Eq. (i)]

or $A_{\min} = 0$ [from Eq. (iii)]

and $I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$ [from Eq. (ii)]

or $I_{\min} = 0$ [from Eq. (iv)]

at those points where, $\cos\phi = -1$ [from Eqs. (i) or (ii)]

or $\phi = \pi, 3\pi \ldots (2n-1)\pi$ (where $n = 1, 2 \ldots$)

$\therefore$ $\Delta x = \dfrac{\lambda}{2}, \dfrac{3\lambda}{2} \ldots (2n-1)\dfrac{\lambda}{2}$ $\left[\text{as } \Delta x = \phi\left(\dfrac{\lambda}{2\pi}\right)\right]$

✓ Extra Points to Remember

- In amplitude, it hardly matters whether it is $+2A_0$ or $-2A_0$. This is the reason we have taken, $A_{\max} = \pm 2A_0$
- In interference, two or more than two waves from coherent sources meet at several points. At different points Δx, $\Delta\phi$ or ϕ, resultant amplitude and therefore resultant intensity will be different (varying from $I_{\max}$ to $I_{\min}$). But, whatever is the resultant intensity at some point, it remains constant at that point.
- In interference,

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{I_1/I_2} + 1}{\sqrt{I_1/I_2} - 1}\right)^2 = \left(\frac{A_1/A_2 + 1}{A_1/A_2 - 1}\right)^2 = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2$$

- **Coherent sources** In order to produce a stable interference pattern the individual waves must maintain a **constant phase** relationship with one another, i.e. the two interfering sources must emit waves having a constant phase difference between them.

  If the phase difference between two sources does not remain constant, then the places of maxima and minima shift. In case of mechanical waves it is possible to keep a constant phase relationship between two different sources. But in case of light two different light sources can't be coherent This is because of the way light is emitted. In ordinary light sources, atoms gain excess energy by thermal agitation or by impact with accelerated electrons. An atom that is 'excited' in such a way begins to radiate energy and continues until it has lost all the energy it can, typically in a time of the order of 10^{-8} s.

The many atoms in a source ordinarily radiate in an unsynchronized and random phase relationship, and the light that is emitted from two such sources has no definite phase relationship. Hence, to obtain a stable interference in light a single source is split into two coherent sources.

Following are shown some of the methods by which we can split a single light source into two.

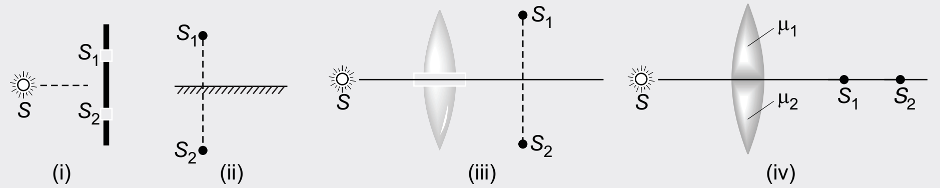


Fig. 32.4

Example 32.1 *In interference, two individual amplitudes are A_0 each and the intensity is I_0 each. Find resultant amplitude and intensity at a point, where:*

(a) phase difference between two waves is 60°

(b) path difference between two waves is $\dfrac{\lambda}{3}$.

Solution (a) Substituting $\phi = 60°$ in the equations,

$$A = 2A_0 \cos\frac{\phi}{2} \quad \text{and} \quad I = 4I_0 \cos^2\frac{\phi}{2}$$

We get, $$A = \sqrt{3}A_0 \text{ and } I = 3I_0$$ **Ans.**

(b) Given, $$\Delta x = \frac{\lambda}{3}$$

$\therefore$ $$\phi \text{ or } \Delta\phi = \left(\frac{2\pi}{\lambda}\right) \cdot \Delta x$$

$$= \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{3}\right) = \frac{2\pi}{3} \text{ or } 120°$$

Now, substituting $\phi = 120°$ in the above two equations, we get

$$A = A_0 \text{ and } I = I_0$$ **Ans.**

Example 32.2 *Three waves from three coherent sources meet at some point. Resultant amplitude of each is A_0. Intensity corresponding to A_0 is I_0. Phase difference between first wave and the second wave is* 60°. *Path difference between first wave and the third wave is $\dfrac{\lambda}{3}$. The first wave lags behind in phase angle from second and the third wave. Find resultant intensity at this point.*

Solution Here, the sources are three. So, we don't have any direct formula for finding the resultant intensity. First we will find the resultant amplitude by vector method and then, by the relation $I \propto A^2$, we can also find the resulting intensity.

Further, a path difference of $\dfrac{\lambda}{3}$ is equivalent to a phase difference of 120° ($\Delta\phi$ or $\phi = \dfrac{2\pi}{\lambda} \cdot \Delta x$).

Hence, the phase difference first and second is 60° and between first and third is 120°. So, vector diagram for amplitude is as shown below.

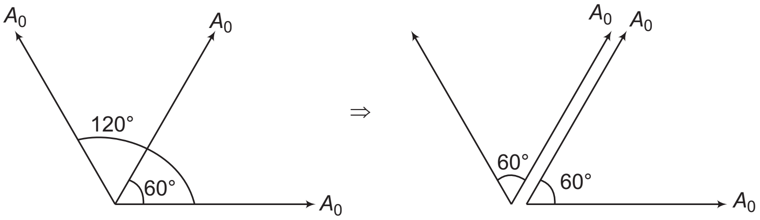


Fig. 32.5

Now, resultant of first and third acting at 120° is also A_0 (as $A = 2A_0 \cos\frac{\phi}{2}$ and $\phi = 120°$) and since the first and third are equal, so this resultant A_0 passes through the bisector line of these two or in the direction of second amplitude vector. Therefore, the resultant amplitude is

$$A = A_0 + A_0 = 2A_0$$

and the resultant intensity is

$$I = 4I_0 \qquad (\text{as } I \propto A^2) \qquad \textbf{Ans.}$$

Example 32.3 *Two waves of equal frequencies have their amplitudes in the ratio of 3 : 5. They are superimposed on each other. Calculate the ratio of maximum and minimum intensities of the resultant wave.*

Solution Given, $\frac{A_1}{A_2} = \frac{3}{5}$

$$\therefore \qquad \sqrt{\frac{I_1}{I_2}} = \frac{3}{5} \qquad (\text{as } I \propto A^2)$$

Maximum intensity is obtained, where

$$\cos\phi = 1$$

and $$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

Minimum intensity is found, where

$$\cos\phi = -1$$

and $$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Hence, $$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}} + 1}{\sqrt{\frac{I_1}{I_2}} - 1}\right)^2$$

$$= \left(\frac{3/5 + 1}{3/5 - 1}\right)^2 = \frac{64}{4} = \frac{16}{1} \qquad \textbf{Ans.}$$

Example 32.4 *In interference,* $\frac{I_{\max}}{I_{\min}} = \alpha$*, find*

(a $\frac{A_{\max}}{A_{\min}}$ (b) $\frac{A_1}{A_2}$ (c) $\frac{I_1}{I_2}$

Solution (a) $\frac{A_{\max}}{A_{\min}} = \sqrt{\frac{I_{\max}}{I_{\min}}} = \sqrt{\alpha}$ **Ans.**

(b) $\frac{A_{\max}}{A_{\min}} = \sqrt{\alpha} = \frac{A_1 + A_2}{A_1 - A_2} = \frac{A_1 / A_2 + 1}{A_1 / A_2 - 1}$

Solving this equation, we get $\frac{A_1}{A_2} = \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}$ **Ans.**

(c) $\frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 = \left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^2$ **Ans.**

INTRODUCTORY EXERCISE 32.1

1. The ratio of intensities of two waves is 9:16. If these two waves interfere, then determine the ratio of the maximum and minimum possible intensities.
2. In interference, two individual amplitudes are 5 units and 3 units. Find
   (a) $\frac{A_{\max}}{A_{\min}}$ (b) $\frac{I_{\max}}{I_{\min}}$
3. Three waves due to three coherent sources meet at one point. Their amplitudes are $\sqrt{2}A_0$, $3A_0$ and $\sqrt{2}A_0$. Intensity corresponding to A_0 is I_0. Phase difference between first and second is 45°. Path difference between first and third is $\frac{\lambda}{4}$. In phase angle, first wave lags behind from the other two waves. Find resultant intensity at this point.

32.4 Young's Double Slit Experiment (YDSE)

(i) YDSE is an experiment (or an example) of interference in light.

(ii) For interference, two coherent sources are required and in light two different light sources are never coherent. Therefore, a single source S is split into two sources as shown below.

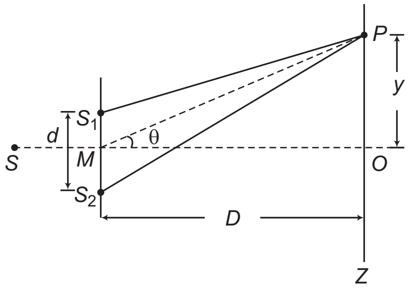


Fig. 32.6

(iii) Z is a two-dimensional screen and OP the centre line of this screen. On different points of this screen, two rays of light (from two coherent sources S_1 and S_2) will interfere. At different points of the screen path difference (and therefore) phase difference will be different. Therefore, resultant intensity will be different. But whatever is the resultant intensity at any point, it remains constant at that point.

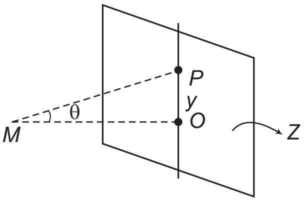


Fig. 32.7

(iv) Mathematically, we will find the resultant intensity at any point on the centre line OP of the screen, as it will have only one variable y (or θ).

(v) The order of λ, d and D is normally

$$\lambda << d << D$$

(vi) Since $d << D$, we can assume that intensities at P due to independent sources S_1 and S_2 are almost equal, or

$$I_1 \approx I_2 = I_0 \quad \text{(say)}$$

Therefore, for resultant intensity we can apply

$$\boxed{I = 4I_0 \cos^2 \frac{\phi}{2}} \qquad \ldots\text{(i)}$$

(vii) In YDSE, our main objective is to find resultant intensity at a general point P on the centre line of the screen. Point P can be generalised in the following four ways.

(a) Directly phase difference ϕ (between two rays interfering at P) can be given. This is the simplest one. As, we can directly apply

$$I = 4I_0 \cos^2 \frac{\phi}{2}$$

for the resultant intensity.

(b) Path difference Δx (between two interfering rays) can be given at P. In this case, first we will convert Δx into phase difference ϕ by the relation,

$$\phi = \frac{2\pi}{\lambda} \Delta x$$

and then, we will apply

$$I = 4I_0 \cos^2 \frac{\phi}{2}$$

(c) Distance OP or y-coordinate of point P can be given.

(d) Angular position θ of point P will be given.

Note *In last two cases, first we will find the path difference Δx in terms of y and θ and then we will find the resultant intensity.*

Path Difference in Terms of *y* and θ

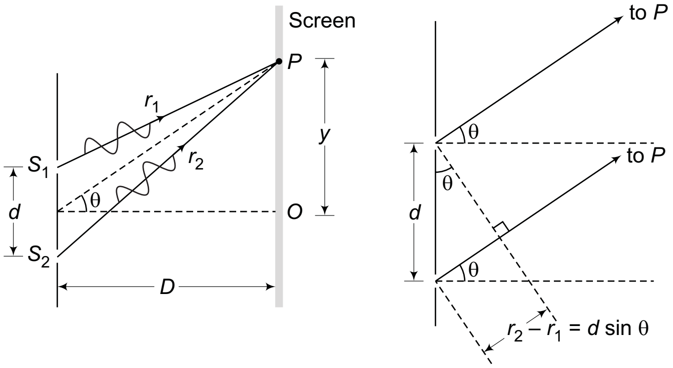


Fig. 32.8 (a) To reach *P*, the light waves from S_1 and S_2 must travel different distances. (b) The path difference between the two rays is *d* sin θ.

Figure shows the light waves from S_1 and S_2 meeting at an arbitrary point *P* on the screen. Since, $D >> d$, the two light rays are approximately parallel, with a path difference,

$$\Delta x = S_2P - S_1P$$

$$\boxed{\Delta x \approx d \sin\theta} \qquad \text{...(ii)}$$

This is basically the expression of Δx in terms of θ.

If θ is small, then $\sin\theta \approx \tan\theta = \dfrac{y}{D}$

Substituting in Eq. (ii), we get

$$\boxed{\Delta x = \frac{yd}{D}} \qquad \text{...(iii)}$$

This is the expression of path difference Δx in terms of *y*.

Maximum and Minimum Intensity

Maximum Intensity

In Eq. (i), we know that maximum intensity is $4I_0$ at points where

$$\Delta x = n\lambda \qquad (n = 0, \pm 1, \pm 2 \ldots)$$

or $$d \sin\theta = n\lambda$$

or $$\frac{yd}{D} = n\lambda$$

or $$\boxed{y = \frac{n\lambda D}{d} = Y_n} \qquad \text{...(iv)}$$

Here, Y_n may be called as *y*-coordinate (with respect to point *O*) of n^{th} order maxima, where

$$n = 0, \pm 1, \pm 2 \ldots$$

For example,

$$Y_0 = 0 \rightarrow y\text{-coordinate of zero order maxima (for } n = 0)$$

$$Y_1 \text{ or } Y_1' = \pm\frac{\lambda D}{d} \rightarrow y\text{-coordinate of first order maxima (for } n = \pm 1)$$

$$Y_2 \text{ or } Y_2' = \pm\frac{2\lambda D}{D} \rightarrow y\text{-coordinate of second order maxima (for } n = \pm 2)$$

and so on.

Minimum Intensity

In Eq. (i), we know that minimum intensity is zero at points where

$$\Delta x = (2n-1)\frac{\lambda}{2} \qquad (n = \pm 1, \pm 2 \ldots)$$

or

$$d \sin\theta = (2n-1)\frac{\lambda}{2}$$

or

$$\frac{yd}{D} = (2n-1)\frac{\lambda}{2}$$

or

$$\boxed{y = \frac{(2n-1)\lambda D}{2d} = y_n} \qquad \ldots(\text{v})$$

Here, y_n may be called as y-coordinate (with respect to point O) of n^{th} order minima, where $n = \pm 1, \pm 2 \ldots$

For example,

$$y_1 \text{ or } y_1' = \pm\frac{\lambda D}{2d} \rightarrow y\text{-coordinate of first order minima (for } n = \pm 1)$$

$$y_2 \text{ or } y_2' = \pm\frac{3\lambda D}{2d} \rightarrow y\text{-coordinate of second order minima (for } n = \pm 2)$$

Fringe Width (ω)

Distance between two adjacent maxima or minima (or bright/dark fringes) is called the fringe width. Thus,

$$\omega = Y_n - Y_{n-1} = \text{ distance between two adjacent maxima}$$

$$= \frac{n\lambda D}{d} - \frac{(n-1)\lambda D}{d}$$

$$= \frac{\lambda D}{d}$$

or

$$\boxed{\omega = \frac{\lambda D}{d}} \qquad \ldots(\text{vi})$$

We can see that distance between two successive minima is also $\frac{\lambda D}{d}$.

Intensity Distribution

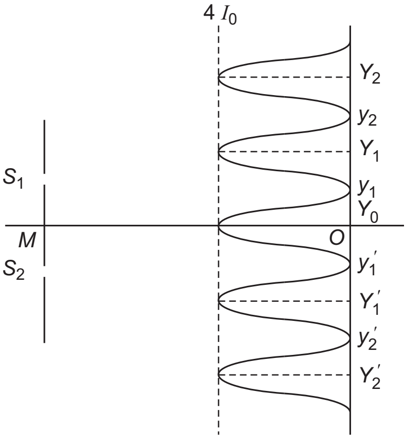


Fig. 32.9

In the shown figure, let us discuss one maxima (say Y_2) on upper side of centre line MO and one minima (say y_2') on lower side.

Y_2 This may be called second order maxima on upper side of line MO. Here, the intensity is $4I_0$. Path difference is

$$\Delta x = S_2P - S_1P = 2\lambda$$

and the phase difference is 4π. Further, y-coordinate of this point is

$$y = \frac{2\lambda D}{d} = 2\omega$$

y_2' This may be called the second order minima on lower side of line MO. Here, the intensity is zero. Path difference is

$$\Delta x = S_1P - S_2P = \frac{3\lambda}{2}$$

and the phase difference is 3π. Further, y-coordinate of this point is

$$y = -\frac{3\lambda D}{2d} = -\frac{3}{2}\omega$$

Intensity Variation within One Fringe

Let us first make the following table corresponding to this point.

Table 32.1

Intensity	Amplitude $A \propto \sqrt{I}$	Phase difference (ϕ) where $A = 2A_0 \cos\frac{\phi}{2}$	Path difference $\Delta x = \frac{\lambda}{2\pi}\phi$	y-coordinate $\Delta x = \frac{yd}{D}$ $\Rightarrow$ $y = \Delta x\left(\frac{D}{d}\right)$
$4I_0$	$2A_0$	2π or 360°	λ	$\lambda D/d = \omega$
$3I_0$	$\sqrt{3}A_0$	$\frac{\pi}{3}$ or 60°	$\lambda/6$	$\lambda D/6d = \omega/6$
$2I_0$	$\sqrt{2}A_0$	$\frac{\pi}{2}$ or 90°	$\lambda/4$	$\lambda D/4d = \omega/4$
I_0	A_0	$\frac{2\pi}{3}$ or 120°	$\lambda/3$	$\lambda D/3d = \omega/3$
0	0	π or 180°	$\lambda/2$	$\lambda D/2d = \omega/2$

Now, according to above table we can make the following figure.

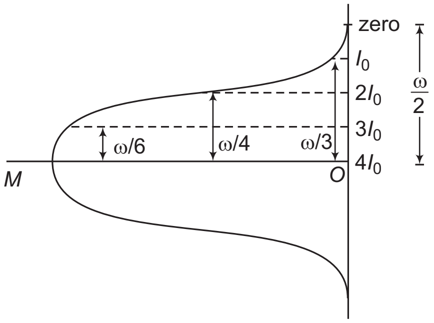


Fig. 32.10

Example 32.5 *In YDSE, the two slits are separated by* 0.1 *mm and they are* 0.5 *m from the screen. The wavelength of light used is* 5000 Å. *Find the distance between* 7*th maxima and* 11*th minima on the upper side of screen.*

Solution Given, $d = 0.1\,\text{mm} = 10^{-4}$ m, $D = 0.5$ m and $\lambda = 5000\,\text{Å} = 5.0 \times 10^{-7}$ m

$$\Delta y = y_{11} - Y_7 = \frac{(2 \times 11 - 1)\,\lambda D}{2d} - \frac{7\lambda D}{d}$$

or

$$\Delta y = \frac{7\lambda D}{2d} = \frac{7 \times 5.0 \times 10^{-7} \times 0.5}{2 \times 10^{-4}}$$

$$= 8.75 \times 10^{-3}\ \text{m}$$

$$= 8.75\ \text{mm}$$ **Ans.**

Example 32.6 *Maximum intensity in YDSE is* I_0. *Find the intensity at a point on the screen where*

(a) the phase difference between the two interfering beams is $\frac{\pi}{3}$.

(b) the path difference between them is $\frac{\lambda}{4}$.

Solution (a) We know that

$$I = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$$

Here, $I_{\max}$ is I_0 (i.e. intensity due to independent sources is $I_0/4$). Therefore, at

$$\phi = \frac{\pi}{3}$$

or

$$\frac{\phi}{2} = \frac{\pi}{6}$$

$$I = I_0 \cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4} I_0$$ **Ans.**

(b) Phase difference corresponding to the given path difference $\Delta x = \frac{\lambda}{4}$ is

$$\phi = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) = \frac{\pi}{2} \quad \text{or} \quad \frac{\phi}{2} = \frac{\pi}{4}$$

$\therefore$ $$I = I_0 \cos^2\left(\frac{\pi}{4}\right) = \frac{I_0}{2}$$ **Ans.**

Dependence of Fringe Width on λ

Fringe width (ω) is given by

$$\omega = \frac{\lambda D}{d} \quad \text{or} \quad \boxed{\omega \propto \lambda}$$

Two conclusions can be drawn from this relation.

(i) If YDSE apparatus is immersed in a liquid of refractive index μ, then wavelength of light and hence, fringe width decreases μ times.

(ii) If white light is used in place of a monochromatic light then coloured fringes are obtained on the screen with red fringes of larger size than that of violet, because $\lambda_{\text{red}} > \lambda_{\text{violet}}$.

But note that centre is still white because path difference there is zero for all colours. Hence, all the wavelengths interfere constructively. At other points light will interfere destructively for those wavelengths for whom path difference is $\lambda/2, 3\lambda/2, \ldots$ etc. and they will interfere constructively for the wavelengths for whom path difference is $\lambda, 2\lambda, \ldots$ etc.

This point can be explained as under,

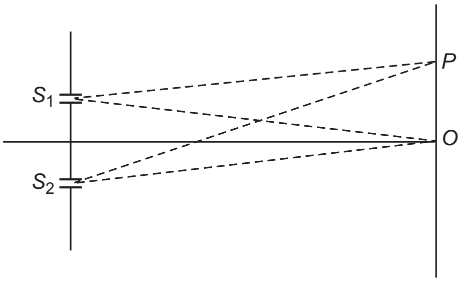


Fig. 32.11

In YDSE let us take a bichromatic light (having two wavelengths). Suppose one wavelength is $\lambda_1 = \lambda$ and the other wavelength is $\lambda_2 = 2\lambda$. At point O,

$S_1O = S_2O \Rightarrow \Delta x = O$ for both wavelengths. Therefore, both wavelengths interfere constructively.

At any other point (say at P) $S_1P \neq S_2P \quad \Rightarrow \quad \Delta x \neq O$ and suppose $\Delta x = \lambda$. Then, this Δx is

$$\Delta x = \lambda_1 \quad \text{and} \quad \Delta x = \frac{\lambda_2}{2}$$

Therefore at P, wavelength λ_1 interferes constructively and λ_2 destructively. Same is the case with white light. At point O, all wavelengths interfere constructively. Therefore, white is produced. At any other point some of the wavelengths interfere constructively, (produce $4I_0$) some destructively (produce zero intensity) and rest intermediately (between zero to $4I_0$ intensity).

Example 32.7 *White light is used to illuminate the two slits in a Young's double slit experiment. The separation between the slits is b and the screen is at a distance $d\,(>>b)$ from the slits. At a point on the screen directly in front of one of the slits, certain wavelengths are missing. Some of these missing wavelengths are* **(JEE 1984)**

(a) $\lambda = b^2/d$ (b) $\lambda = 2b^2/d$ (c) $\lambda = b^2/3d$ (d) $\lambda = 2b^2/3d$

Solution At P (directly in front of S_1) $y = b/2$

$\therefore$ Path difference, $$\Delta x = S_2P - S_1P = \frac{y\cdot(b)}{d} = \frac{\left(\frac{b}{2}\right)(b)}{d} = \frac{b^2}{2d}$$

Those wavelengths will be missing for which

$$\Delta x = \frac{\lambda_1}{2}, \frac{3\lambda_2}{2}, \frac{5\lambda_3}{2} \dots$$

$$\therefore \qquad \lambda_1 = 2\Delta x = \frac{b^2}{d}$$

$$\lambda_2 = \frac{2\Delta x}{3} = \frac{b^2}{3d}$$

$$\lambda_3 = \frac{2\Delta x}{5} = \frac{b^2}{5d}$$

Therefore, the correct options are (a) and (c).

Example 32.8 *Bichromatic light is used in YDSE having wavelengths $\lambda_1 = 400$ nm and $\lambda_2 = 700$ nm. Find minimum order of bright fringe of λ_1 which overlaps with bright fringe of λ_2.*

Solution Let n_1 bright fringe of λ_1 overlaps with n_2 bright fringe of λ_2. Then,

$$\frac{n_1\lambda_1 D}{d} = \frac{n_2\lambda_2 D}{d} \quad \text{or} \quad \frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} = \frac{700}{400} = \frac{7}{4}$$

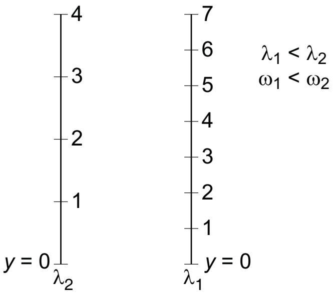


Fig. 32.12

The ratio $\frac{n_1}{n_2} = \frac{7}{4}$ implies that 7th bright fringe of λ_1 will overlap with 4th bright fringe of λ_2.

Similarly, 14th of λ_1 will overlap with 8th of λ_2 and so on.

So, the minimum order of λ_1 which overlaps with λ_2 is 7.

Note *In the above example,*

$$\frac{n_1}{n_2}=\frac{7}{4},\frac{14}{8},\frac{21}{12}\ \textit{etc.}$$

So, to overlap maximas, we can take either of the ratios $\frac{7}{4}$ *or* $\frac{14}{8}$ *or* $\frac{21}{12}$ *etc.*

But in the next example, we will see that if we wish to overlap minimas we cannot take each successive ratio. The reason behind this is that order of fringe n (whether it is order of maxima or minima) should always be an integer.

Example 32.9 *In YDSE, bichromatic light of wavelengths* 400 *nm and* 560 *nm are used. The distance between the slits is* 0.1 *mm and the distance between the plane of the slits and the screen is* 1 *m. The minimum distance between two successive regions of complete darkness is* **(JEE 2004)**

(a) 4 *mm* *(b)* 5.6 *mm* *(c)* 14 *mm* *(d)* 28 *mm*

Solution Let *n*th minima of 400 nm coincides with *m*th minima of 560 nm, then

$$\frac{(2n-1)\lambda_1 D}{2d}=\frac{(2n-1)\lambda_2 D}{2d}$$

or

$$(2n-1)\left(\frac{400}{2}\right)=(2m-1)\left(\frac{560}{2}\right)$$

$$\frac{2n-1}{2m-1}=\frac{7}{5}=\frac{14}{10}=\frac{21}{15}$$

If we take the first ratio, then $n=4$ and $m=3$.

If we take the second ratio, then $n=7.5$ and $m=5.5$.

This is not acceptable. If we take the third ratio, then $n=11$ and $m=8$

i.e. 4th minima of 400 nm coincides with 3rd minima of 560 nm.

Location of this minima is

$$y_1=\frac{(2\times 4-1)(1000)(400\times 10^{-6})}{2\times 0.1}=14\text{ mm}$$

Next 11th minima of 400 nm will coincide with 8th minima of 560 nm.

Location of this minima is

$$y_2=\frac{(2\times 11-1)(1000)(400\times 10^{-6})}{2\times 0.1}=42\text{ mm}$$

∴ Required distance $=y_2-y_1=28\text{ mm}$ **Ans.**

∴ The correct option is (d).

Path Difference Produced by a Slab

Consider two light rays 1 and 2 moving in air parallel to each other. If a slab of refractive index μ and thickness *t* is inserted between the path of one of the rays, then a path difference

$$\boxed{\Delta x=(\mu-1)\,t} \quad \ldots\text{(i)}$$

is produced between them. This can be shown as under,

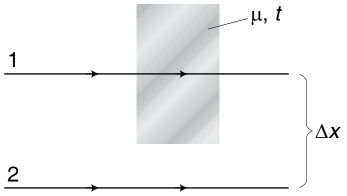


Fig. 32.13

$$\text{Speed of light in air} = c$$

$$\text{Speed of light in medium} = \frac{c}{\mu}$$

Time taken by ray 1 to cross the slab, $t_1 = \frac{t}{c/\mu} = \frac{\mu t}{c}$

and time taken by ray 2 to cross the same thickness t in air will be,

$$t_2 = \frac{t}{c} \quad \text{as} \quad t_1 > t_2$$

Difference in time $\Delta t = t_1 - t_2 = (\mu - 1)\frac{t}{c}$

During this time ray 2 will travel an extra distance, $\Delta x = (\Delta t)\, c = (\mu - 1)\, t$, which is same as Eq. (i).

Exercise 1 A slab of thickness t and refractive index μ_2 is kept in a medium of refractive index μ_1 ($< \mu_2$). Prove that, if two rays parallel to each other passes through such a system, with one ray passing through the slab, then the path difference produced between them due to the slab will be:

$$\boxed{\Delta x = \left(\frac{\mu_2}{\mu_1} - 1\right) t}$$

Exercise 2

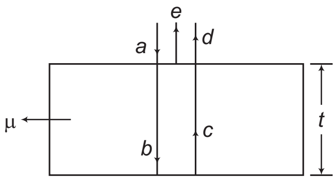


Fig. 32.14

In the figure shown, a is the incident ray, e is reflected from top surface of the slab but d comes after reflecting from bottom surface. Then, prove that path difference between e and d is

$$\boxed{\Delta x = 2\,\mu t}$$

Shifting of Fringes

Suppose a glass slab of thickness t and refractive index μ is inserted onto the path of the ray emanating from source S_1, then the whole fringe pattern shifts upwards by a distance $\frac{(\mu - 1)\, tD}{d}$. This can be shown as under,

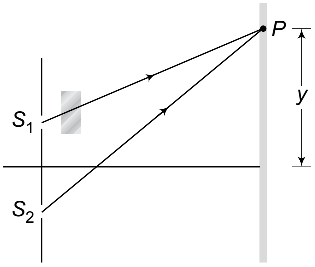


Fig. 32.15

Geometric path difference between S_2P and S_1P is

$$\Delta x_1 = S_2P - S_1P = \frac{yd}{D}$$

Path difference produced by the glass slab,

$$\Delta x_2 = (\mu - 1)\, t$$

Note *Due to the glass slab path of ray 1 gets increased by Δx_2.*

Therefore, net path difference between the two rays is,

$$\Delta x = \Delta x_1 - \Delta x_2 \quad \text{or} \quad \Delta x = \frac{yd}{D} - (\mu - 1)\, t$$

For n^{th} order maxima on upper side, $\Delta x = n\lambda$

or
$$\frac{yd}{D} - (\mu - 1)\, t = n\lambda$$

$\therefore$
$$y = \frac{n\lambda D}{d} + \frac{(\mu - 1)\, tD}{d} = Y_n$$

Earlier without slab it was $\dfrac{n\lambda D}{d}$

$\therefore$
$$\boxed{\text{Shift} = \frac{(\mu - 1) t D}{d}} \quad \ldots\text{(ii)}$$

Following three points are important with regard to Eq. (ii).

(a) Shift is independent of n, (the order of the fringe), i.e.

shift of zero order maximum = shift of 7th order maximum

or shift of 5th order maximum = shift of 9th order minimum and so on

(b) Shift is independent of λ, i.e. if white light is used then,

shift of red colour fringe = shift of violet colour fringe

(c) Number of fringes shifted $= \dfrac{\text{shift}}{\text{fringe width}}$

$$= \frac{(\mu - 1)\, t\, D/d}{\lambda D/d} = \frac{(\mu - 1) t}{\lambda}$$

These numbers are inversely proportional to λ. This is because shift is same for all colours but fringe width of the colour having smaller value of λ is small, so more number of fringes will shift for this colour.

Example 32.10 *In YDSE, find the thickness of a glass slab* ($\mu = 1.5$) *which should be placed in front of the upper slit S_1 so that the central maximum now lies at a point where* 5*th bright fringe was lying earlier (before inserting the slab). Wavelength of light used is* 5000 *Å.*

Solution According to the question,

$$\text{Shift} = 5\,(\text{fringe width})$$

$\therefore$
$$\frac{(\mu - 1)\, tD}{d} = \frac{5\lambda D}{d}$$

$\therefore$
$$t = \frac{5\lambda}{\mu - 1} = \frac{25000}{1.5 - 1}$$

$$= 50000 \text{ Å}$$

Ans.

Example 32.11 *Interference fringes are produced by a double slit arrangement and a piece of plane parallel glass of refractive index* 1.5 *is interposed in one of the interfering beam. If the fringes are displaced through* 30 *fringe widths for light of wavelength* 6×10^{-5} *cm, find the thickness of the plate.*

Solution Path difference due to the glass slab,

$$\Delta x = (\mu - 1)t$$

Thirty fringes are displaced due to the slab. Hence,

$$\Delta x = 30\lambda \quad \Rightarrow \quad (\mu - 1)t = 30\lambda$$

$$\therefore \qquad t = \frac{30\lambda}{\mu - 1} = \frac{30 \times 6 \times 10^{-5}}{1.5 - 1}$$

$$= 3.6 \times 10^{-3} \text{ cm}$$ **Ans.**

INTRODUCTORY EXERCISE 32.2

1. Explain why two flashlights held close together do not produce an interference pattern on a distant screen?
2. Why it is so much easier to perform interference experiments with a laser than with an ordinary light source?
3. In YDSE, $D = 1.2$ m and $d = 0.25$ cm, the slits are illuminated with coherent 600 nm light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75% of the maximum.
4. Slit 1 of a double slit is wider than slit 2, so that the light from slit 1 has an amplitude three times that of the light from slit 2. Show that equation $I = I_{\max} \cos^2 \frac{\phi}{2}$ is replaced by the equation,

$$I = \left(\frac{I_{\max}}{4}\right)\left(1 + 3\cos^2 \frac{\phi}{2}\right)$$

5. In a two-slit interference pattern, the maximum intensity is I_0.
   (a) At a point in the pattern where the phase difference between the waves from the two slits is 60°, what is the intensity?
   (b) What is the path difference for 480 nm light from the two slits at a point where the phase angle is 60°?
6. Two waves of the same frequency and same amplitude 'a' are reaching a point simultaneously. What should be the phase difference between the waves so that the amplitude of the resultant wave be :
   (i) $2a$ (ii) $\sqrt{2}\,a$ (iii) a (iv) zero

32.5 Introduction to Diffraction

When light waves pass through two slits in YDSE, an interference pattern is observed rather than a sharp spot of light. This behaviour indicates that light, once it has passed through the aperture, spreads beyond the narrow path defined by the aperture into regions that would be in shadow if light travelled in straight lines. Other waves, such as sound waves and water waves, also have this property of spreading when passing through apertures or by sharp edges. This phenomenon, known as diffraction, can be described only with a wave model for light.

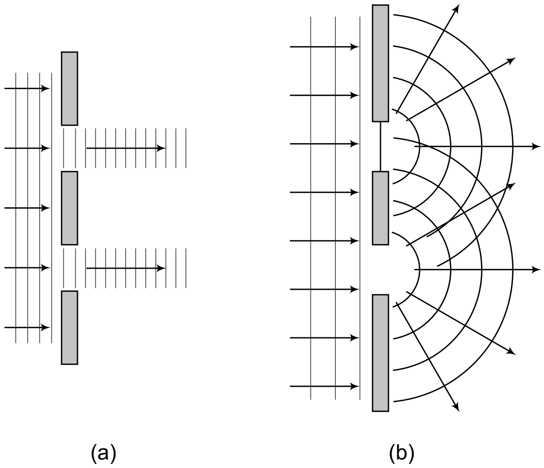


Fig. 32.16 (a) If light waves did not spread out after passing through the slits, no interference would occur (b) The light waves from the two slits overlap as they spread out, filling what we expect to be shadowed regions with light and producing interference fringes.

In general, diffraction occurs when waves pass through small openings, around obstacles, or past sharp edges. As shown in figure, when an opaque object is placed between a point source of light and a screen, no sharp boundary exists on the screen between a shadowed region and an illuminated region. The illuminated region above the shadow of the object contains alternating light and dark fringes. Such a display is called a **diffraction pattern.**

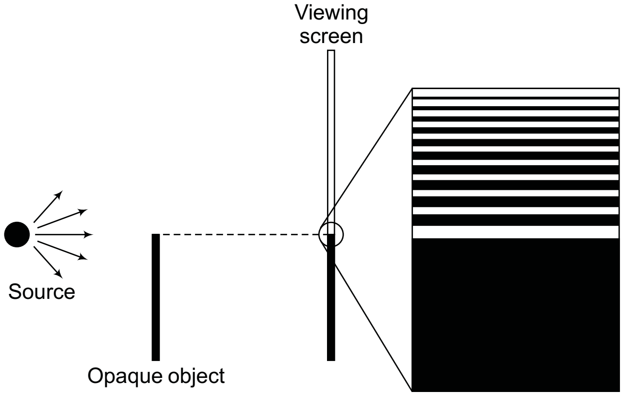


Fig. 32.17 Light from a small source passes by the edge of an opaque object. We might expect no light to appear on the screen below the position of the edge of the object. In reality, light bends around the top edge of the object and enter this region. Because of these effects, a diffraction pattern consisting of bright and dark fringes appears in the region above the edge of the object.

In this chapter, we restrict our attention to **Fraunhofer diffraction**, which occurs, for example, when all the rays passing through a narrow slit are approximately parallel to one another. This can be achieved experimentally either by placing the screen far from the opening used to create the diffraction or by using a converging lens to focus the rays once they pass through the opening, as shown in Fig. 32.18.

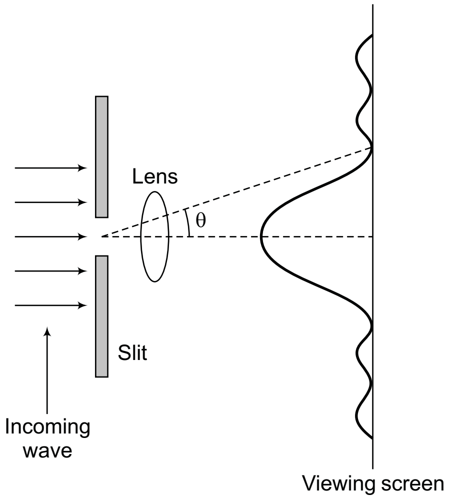


Fig. 32.18 Fraunhofer diffraction pattern of a single slit.

32.6 Diffraction from a Narrow Slit

Until now, we have assumed that slits are point sources of light. In this section, we abandon that assumption and see how the finite width of slits is the basis for understanding Fraunhofer diffraction.

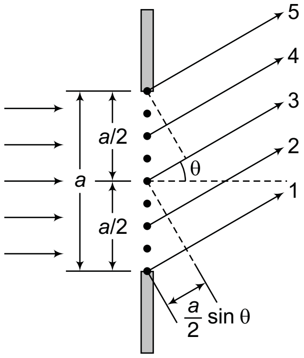


Fig. 32.19 Diffraction of light by a narrow slit of width a. Each portion of the slit acts as a point source of light waves. The path difference between rays 1 and 3 or between rays 2 and 4 is $(a/2)\sin\theta$ (drawing not to scale).

We can deduce some important features of this phenomenon by examining waves coming from various portions of the slit, as shown in Fig. 32.19 According to Huygens's principle, **each portion of the slit acts as a source of light waves.** Hence, light from one portion of the slit can interfere with light from another portion, and the resultant light intensity on a viewing screen depends on the direction θ.

To analyze the diffraction pattern, it is convenient to divide the slit into two halves, as shown in Fig. 32.19. Keeping in mind that all the waves are in phase as they leave the slit, consider rays 1 and 3. As these two rays travel toward a viewing screen far to the right of the Fig. 32.19, ray 1 travels farther than ray 3 by an amount equal to the path difference $(a/2)\sin\theta$, where a is the width of the slit. Similarly, the path difference between rays 2 and 4 is also $(a/2)\sin\theta$. If this path difference is exactly half a wavelength (corresponding to a phase difference of 180°), then the two waves cancel each other and destructive interference results. This is true for any two rays that originate at points separated by half the slit width because the phase difference between two such points is 180°. Therefore, waves from the upper half of the slit interfere destructively with waves from the lower half when

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \quad \text{or when} \quad \sin\theta = \frac{\lambda}{a}$$

If we divide the slit into four equal parts and use similar reasoning, we find that the viewing screen is also dark when

$$\sin\theta = \frac{2\lambda}{a}$$

Likewise, we can divide the slit into six equal parts and show that darkness occurs on the screen when

$$\sin\theta = \frac{3\lambda}{a}$$

Therefore, the general condition for destructive interference is

$$\boxed{\sin\theta = m\frac{\lambda}{a}} \qquad m = \pm 1, \ \pm 2, \pm 3, \ldots \qquad \ldots\text{(i)}$$

This equation gives the values of θ for which the diffraction pattern has zero light intensity—that is, when a dark fringe is formed. However, it tells us nothing about the variation in light intensity along the screen. The general features of the intensity distribution are shown in Fig. 32.20. A broad central bright fringe is observed; this fringe is flanked by much weaker bright fringes alternating with dark fringes. The various dark fringes occur at the values of θ that satisfy Eq. (i). Each bright fringe peak lies approximately halfway between its bordering dark fringe minima. Note that the central bright maximum is twice as wide as the secondary maxima.

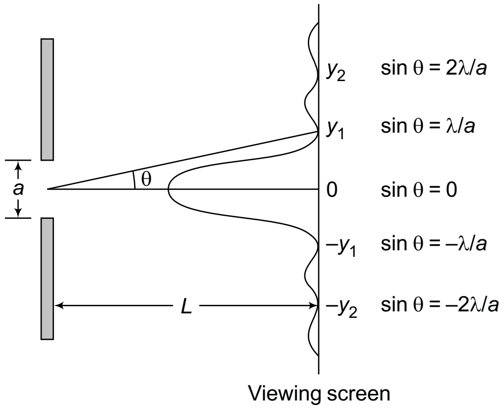


Fig. 32.20 Intensity distribution for a Fraunhofer diffraction pattern from a single slit of width *a*

Intensity of Single Slit Diffraction Patterns

We can use phasor to determine the light intensity distribution for a single slit diffraction pattern. The proof is beyond our syllabus, we are just writing here the intensity at angle θ.

$$I = I_0 \left[\frac{\sin \beta/2}{\beta/2}\right]^2$$

Here, $$\frac{\beta}{2} = \frac{\pi a \sin \theta}{\lambda}$$

or $$\boxed{I = I_0 \left[\frac{\sin (\pi a \sin \theta/\lambda)}{(\pi a \sin \theta)/\lambda}\right]^2} \qquad \text{(at angle } \theta)$$

Here, I_0 is the intensity at $\theta = 0°$ (the central maximum).

From this result, we see that minima occurs when

$$\frac{\pi a \sin \theta}{\lambda} = m\pi$$

or $$\sin \theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \dots$$

This is in agreement with Eq. (i).

Note *That* $\sin \theta = 0$, *corresponds to central maxima while* $\frac{\pi a \sin \theta}{\lambda} = \pi$, *corresponds to first minima.*

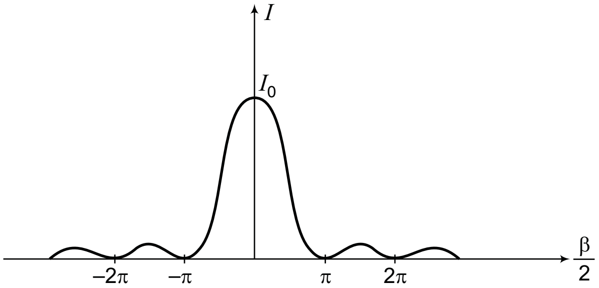


Fig. 32.21

Example 32.12 *A beam of light of wavelength 600 nm from a distant source falls on a single slit 1.0 mm wide and the resulting diffraction pattern is observed on a screen 2 m away. What is the distance between the first dark fringe on either side of the central bright fringe?*

Solution For the diffraction at a single slit, the position of minima is given by

$$d \sin \theta = n\lambda$$

For small value of θ,

$$\sin \theta \approx \theta = \frac{y}{D}$$

$$\therefore \qquad d\frac{y}{D} = \lambda \quad \text{or} \quad y = \frac{D}{d}\lambda$$

Substituting the values, we have

$$y = \frac{2 \times 6 \times 10^{-7}}{1 \times 10^{-3}}$$

$$= 1.2 \times 10^{-3} \text{ m}$$

$$= 1.2 \text{ mm}$$

∴ Distance between first minima on either side of central maxima $= 2y = 2.4$ mm **Ans.**

Example 32.13 *A parallel beam of monochromatic light of wavelength* 450 *nm passes through a slit of width* 0.2 *mm. Find the angular divergence in which most of the light is diffracted.*

Solution Most of the light is diffracted between the two first order minima. These minimas occur at angle given by

$$d \sin \theta = \pm n\lambda$$

$$\therefore \qquad \sin \theta = \pm \frac{\lambda}{d}$$

$$= \pm \frac{450 \times 10^{-9}}{0.2 \times 10^{-3}}$$

$$= \pm 2.25 \times 10^{-3} \text{ rad}$$

∴ The angular divergence $= 4.5 \times 10^{-3}$ rad. **Ans.**

Final Touch Points

1. In YDSE, if one slit is closed then we will not get interference on screen and intensity at every point is almost uniform $(= I_0)$ and this is due to only one slit.
2. In YDSE, if both sources are incoherent, then again we will not get interference and intensity at every point is again uniform $(= I_1 + I_2 \text{ or } 2I_0)$.
3. In YDSE, if width of one slit is slightly increased then I_{max} and I_{min} both will increase. This is because intensity due to the slit of increased width will increase or

$$I_1 = I_0 \text{ but } I_2 = nI_0 \qquad \text{(where } n > 1\text{)}$$

$\therefore$ $$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2 > 0$$

and $$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2 > 4I_0$$

4. **Shape of fringes on complete screen** Until now, we have discussed the fringe pattern on the centre line of the screen. On this centre line maximas (like Y_0, Y_1, Y_2 etc.) or minimas (like y_1, y_2 etc.) are just points. But on the whole screen fringes make a curve and this curve is a locus of points where path difference from two slits is a constant. Or

$$\Delta x = S_1P - S_2P = C \qquad \text{...(i)}$$

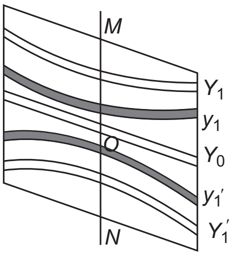
$C = 0$, gives Y_0 fringe. Similarly, $C = \pm\lambda$ gives Y_1 or Y_1' fringe etc.

Now on the centre line, Δx was a function of only one variable coordinate y $\left(\Delta x = \frac{yd}{D}\right)$. But on the whole screen it will become a function of two variable coordinates (say y and z). Therefore, Eq. (i) becomes

$$\Delta x = f(y, z) = C \qquad \text{...(ii)}$$

After proper calculations, we can show that this comes out to be a family of curves of hyperbolas. Thus, on the complete screen, fringes are of the shape of hyperbolas.

For $C = 0$, this hyperbola converts into a straight line. Hence, only Y_0 fringe is a straight line. The fringe pattern is as shown in figure.



MN is the centre line of screen,

on Y_0 fringe, $\Delta x = S_1P - S_2P = 0$

on Y_1 fringe, $\Delta x = S_2P - S_1P = \lambda$

on Y_1' fringe, $\Delta x = S_1P - S_2P = \lambda$ and so on

5. **Types of diffraction** The diffraction phenomenon is divided into two types. **Fresnel diffraction** and **Fraunhofer diffraction.** In the first type either source or screen or both are at finite distance from the diffracting device (obstacle or aperture). In the second type both source and screen are effectively at infinite distance from the diffracting device. Fraunhofer diffraction is a particular limiting case of Fresnel diffraction.
6. **Difference between interference and diffraction** Both interference and diffraction are the results of superposition of waves, so they are often present simultaneously as in Young's double slit experiment. However, interference is the result of superposition of waves from two different wavefronts while diffraction results due to superposition of wavelets from different points of the same wavefronts.

Solved Examples

TYPED PROBLEMS

Based on YDSE

Note *Unless mentioned in the question consider two sources S_1 and S_2 as coherent.*

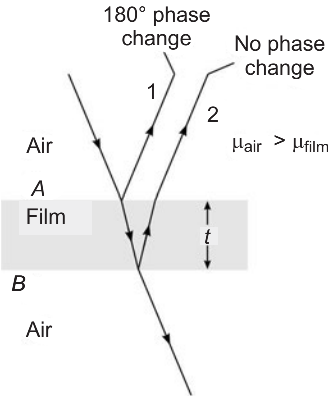
Type 1. *Based on interference by thin films*

Concept

Interference effects are commonly observed in thin films, such as thin layers of oil on water or the thin surface of a soap bubble.

The varied colours observed when white light is incident on such films result from the interference of waves reflected from the two surfaces of the film.

Consider a film of uniform thickness t and index of refraction μ, as shown in figure. Let us assume that the light rays travelling in air are nearly normal to the two surfaces of the film. To determine whether the reflected rays interfere constructively or destructively, we first note the following facts.



(i) The wavelength of light in a medium whose refractive index is μ, is

$$\lambda_\mu = \frac{\lambda}{\mu}$$

where, λ is the wavelength of light in vacuum (or air).

(ii) If a wave is reflected from a denser medium, it undergoes a phase change of 180°. Let us apply these rules to the film shown in figure. The path difference between the two rays 1 and 2 is $2t$ while the phase difference between them is 180°. Hence, condition of **constructive** interference will be

$$2t = (2n - 1)\frac{\lambda_\mu}{2} \qquad (n = 1, 2, 3, \ldots)$$

or

$$2\mu t = \left(n - \frac{1}{2}\right)\lambda \qquad \left(\text{as } \lambda_\mu = \frac{\lambda}{\mu}\right)$$

Similarly, condition of **destructive** interference will be

$$2\mu t = n\lambda \qquad (n = 0, 1, 2, \ldots)$$

Note *Where there is a phase difference of π between two interfering rays, conditions of maximas and minimas are interchanged.*

Example 1 *Calculate the minimum thickness of a soap bubble film* $(\mu = 1.33)$ *that results in constructive interference in the reflected light if the film is illuminated with light whose wavelength in free space is* $\lambda = 600$ *nm.*

Solution For constructive interference in case of soap film,

$$2\mu t = \left(n - \frac{1}{2}\right)\lambda \qquad (n = 1, 2, 3, \ldots)$$

For minimum thickness t, $n = 1$

or

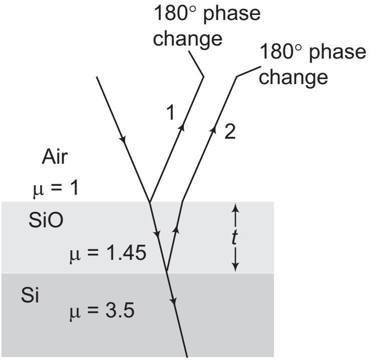
$$2\mu t = \frac{\lambda}{2} \quad \text{or} \quad t = \frac{\lambda}{4\mu} = \frac{600}{4 \times 1.33}$$

$$= 112.78 \text{ nm}$$

Ans.

Example 2 *In solar cells, a silicon solar cell* $(\mu = 3.5)$ *is coated with a thin film of silicon monoxide SiO* $(\mu = 1.45)$ *to minimize reflective losses from the surface. Determine the minimum thickness of SiO that produces the least reflection at a wavelength of* 550 *nm, near the centre of the visible spectrum. Assume approximately normal incidence.*

Solution The reflected light is a minimum when rays 1 and 2 (shown in figure) meet the condition of destructive interference.



Note *Both rays undergo a 180° phase change upon reflection. The net change in phase due to reflection is therefore zero, and the condition for a reflection minimum requires a path difference of* $\lambda_\mu/2$. *Hence,*

$$2t = \frac{\lambda}{2\mu} \quad \text{or} \quad t = \frac{\lambda}{4\mu} = \frac{550}{4\,(1.45)}$$

$$= 94.8 \text{ nm}$$

Ans.

Type 2. *Based on conditions of maxima or minima*

Concept

(i) In the problems of YDSE, our first task is to find the path difference. Let us take a typical case. In the figure shown, path of ray 1 is more than path of ray 2 by a distance,

$$\Delta x_1 = d \sin\alpha$$

and

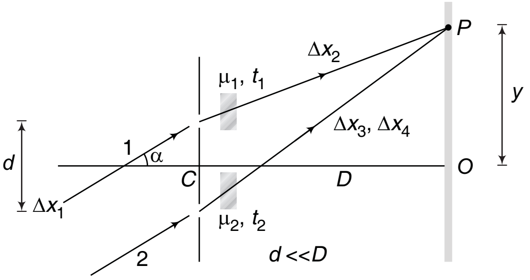
$$\Delta x_2 = (\mu_1 - 1)\, t_1$$

and path of ray 2 is greater than path of ray 1 by a distance,

$$\Delta x_3 = \frac{yd}{D} \quad \text{and} \quad \Delta x_4 = (\mu_2 - 1)\, t_2$$

Therefore, net path difference is

$$\Delta x = (\Delta x_1 + \Delta x_2) \sim (\Delta x_3 + \Delta x_4)$$



(ii) Once the path difference is known, put

$$\Delta x = n\lambda \qquad \text{(for maximum intensity)}$$

and

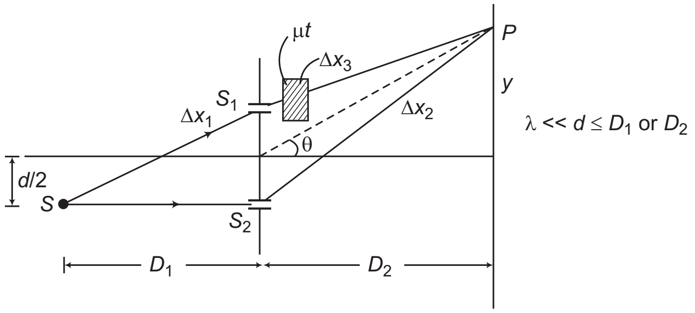
$$\Delta x = (2n - 1)\frac{\lambda}{2} \qquad \text{(for minimum intensity)}$$

Note *If medium is not air or medium is different on two sides of the slits, then for conditions of maxima/minima phase difference between two interfering beams is more important rather than the path difference. For phase difference, we will use*

$$\Delta\phi \quad or \quad \phi = \frac{2\pi}{\lambda}(\Delta x)$$

Here, λ is the wavelength in that particular medium.

Example 3 *Distance between two slits is d. Wavelength of light is λ. There is a source of light S behind S_2 at a distance D_1. A glass slab of thickness t and refractive index μ is kept in front of S_1. Find*



(a) Net path difference at point P at an angular position θ.

(b) Write the equation for finding two angular positions corresponding to second order minima (y_2 and y_2').

(c) If θ in part (b) does not come out to be small, then find two y-coordinates corresponding to them.

Solution (a) $\Delta x_1 = SS_1 - SS_2 = \dfrac{(d/2)(d)}{D_1}$ $\left(\text{Using } \Delta x = \dfrac{yd}{D}\right)$

$$= \frac{d^2}{2D_1}$$

$$\Delta x_2 = d \sin \theta$$

$$\Delta x_3 = (\mu - 1)\, t$$

$$\therefore \quad \Delta x_{\text{net}} = (\Delta x_1 + \Delta x_3) \sim \Delta x_2$$

$$= \left\{\frac{d^2}{2D_1} + (\mu - 1)\, t\right\} \sim d \sin \theta$$

(b) Corresponding to second order minima, net path difference should be $\dfrac{3\lambda}{2}$. So, let θ_1 and θ_2 be the two angular positions, then

$$\left\{\frac{d^2}{2D_1} + (\mu - 1)\, t\right\} - d \sin \theta_1 = \frac{3\lambda}{2}$$

and

$$d \sin \theta_2 - \left\{\frac{d^2}{2D_1} + (\mu - 1)\, t\right\} = \frac{3\lambda}{2}$$

From these two equations, we can find θ_1 and θ_2.

(c) Even if θ is not small, we can write

$$\tan \theta = \frac{y}{D_2}$$

$$\therefore \quad y = D_2 \tan \theta$$

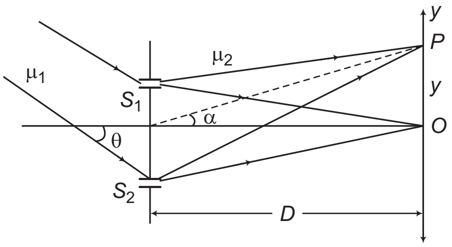
Therefore, two y-coordinates are

$$y_1 = D_2 \tan \theta_1$$

and

$$y_2 = D_2 \tan \theta_2$$

Example 4 *In the figure shown, a parallel beam of light (of wavelength λ_1 in medium μ_1) is incident at an angle θ. Distance $S_1O = S_2O$. Distance between the slits is d.*



Using $d = 1$ mm, $D = 1$ m, $\theta = 30°$, $\lambda_1 = 0.3$ mm, $\mu_1 = 4/3$ and $\mu_2 = 10/9$, find

(a) the y-coordinate of the point where the total phase difference between the interfering waves is zero.

(b) If the intensity due to each light wave at point O is I_0, then find the resultant intensity at O.

(c) Find y-coordinate of the nearest maxima above O.

Solution (a) Wavelength of light in air,

$$\lambda_0 = \mu_1\lambda_1 = 0.4 \text{ mm}$$

Wavelength of light in medium-2,

$$\lambda_2 = \frac{\lambda_0}{\mu_2} = \frac{0.4}{(10/9)} = 0.36 \text{ mm}$$

Let net path difference at some angle α is zero, then

$$\Delta\phi_1 + \Delta\phi_2 = 0$$

or $$\frac{2\pi}{\lambda_1}(\Delta x_1) + \frac{2\pi}{\lambda_2}(\Delta x_2) = 0$$

or $$\frac{\Delta x_1}{\lambda_1} + \frac{\Delta x_2}{\lambda_2} = 0$$

or $$\frac{d\sin\theta}{\lambda_1} + \frac{d\sin\alpha}{\lambda_2} = 0$$

∴ $$\sin\alpha = -\frac{\lambda_2}{\lambda_1}\sin\theta$$

$$= -\frac{0.36}{0.4}\times\frac{1}{2} = -0.45$$

∴ $$\alpha \approx -26.74°$$

$$y = D\tan\alpha$$

$$= (1\text{ m})\tan(-26.74°)$$

$$\approx -\frac{1}{2}m$$ **Ans.**

(b) At O, net phase difference,

$$\Delta\phi \text{ or } \phi = \left(\frac{2\pi}{\lambda_1}\right)\Delta x_1$$

$$= \frac{2\pi}{\lambda_1}(d\sin\theta)$$

$$= \frac{2\pi}{0.3}(1)\sin 30°$$

$$= \frac{10\pi}{3} = 600°$$

Using the equation $I = 4I_0\cos^2\frac{\phi}{2}$, we have

$$I = 4I_0\cos^2(300°)$$

$$= I_0$$ **Ans.**

(c) At O, phase difference $\Delta\phi_1$ is $\frac{10\pi}{3}$ which is greater than 2π but less than 4π. So, to make it 4π, we must have

$$\Delta\phi_1 + \Delta\phi_2 = 4\pi$$

$$\Delta\phi_2 = 4\pi - \frac{10\pi}{3} = \frac{2\pi}{3}$$

$$\Delta\phi_2 = \frac{2\pi}{3} = \frac{2\pi}{\lambda_2}(\Delta x_2) = \frac{2\pi}{\lambda_2}(d\sin\alpha)$$

or $$\sin\alpha = \frac{\lambda_2}{3d} = \frac{0.36}{3 \times 1} = 0.12$$

$\therefore$ $$\alpha \approx 6.89^\circ$$

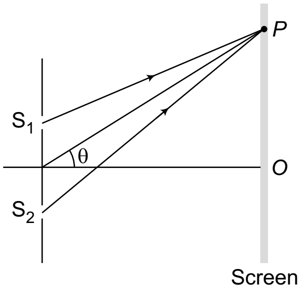
Now, $$y = D\tan\alpha$$
$$= (1\text{ m})\tan(6.89^\circ)$$
$$= 0.12\text{ m}$$ **Ans.**

Type 3. *Based on the concept of $\Delta x = d\cos\theta$ and number of maxima or minima on the screen*

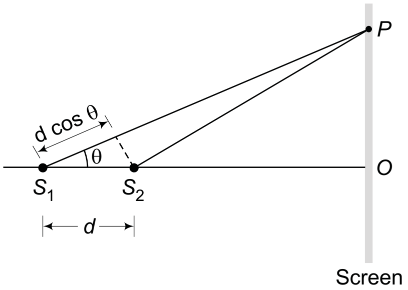
Concept

(i) If the slits are vertical [as in figure (a)], path difference is

$$\Delta x = d\sin\theta$$



(a)



(b)

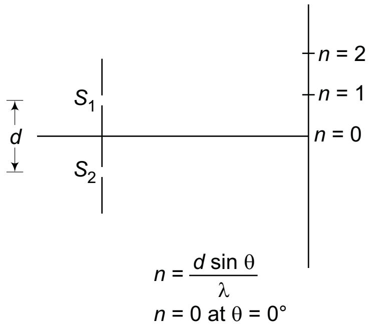
This path difference increases as θ increases. Hence, order of fringe ($d\sin\theta = n\lambda$ or $n = \frac{d\sin\theta}{\lambda}$) increases as we move away from point O on the screen.

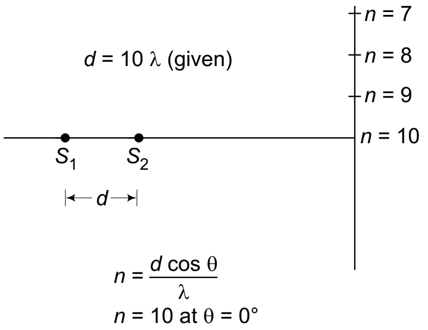
Opposite is the case when the slits are horizontal [as in figure (b)]. Here, path difference is

$$\Delta x = d\cos\theta$$

This path difference decreases as θ increases. Hence, order of fringe $\left(n = \frac{d\cos\theta}{\lambda}\right)$ decreases as we move away from point O.

See the figure below,





(ii) Sometimes maximum number of maximas or minimas are asked in the question which can be obtained on the screen. For this, we use the fact that value of $\sin\theta$ (or $\cos\theta$) can't be greater than 1. For example, in the first case when the slits are vertical,

$$d\sin\theta = n\lambda \quad \text{or} \quad \sin\theta = \frac{n\lambda}{d} \qquad \text{(for maximum intensity)}$$

$$\sin\theta \not> 1$$

$$\therefore \qquad \frac{n\lambda}{d} \not> 1 \quad \text{or} \quad n \not> \frac{d}{\lambda}$$

Suppose in some question $\frac{d}{\lambda}$ is 4.6, then total number of maximas on the screen will be 9. Corresponding to $n = 0, \pm 1, \pm 2, \pm 3$ and ± 4.

(iii) Number of maximas or minimas are normally asked when λ is of the order of d. In this case, fringe size will be large and limited number of maximas and minimas will be obtained on the screen. If $\lambda << d$, then fringe size $\left(\omega = \frac{\lambda D}{d}\right)$ is very small (of the order of μm). So, millions of fringes can be obtained on the screen. So, number of maximas or minimas are normally not asked.

(iv) If $d << D$, then we can apply

$$\Delta x = d\sin\theta \quad \text{or} \quad d\cos\theta$$

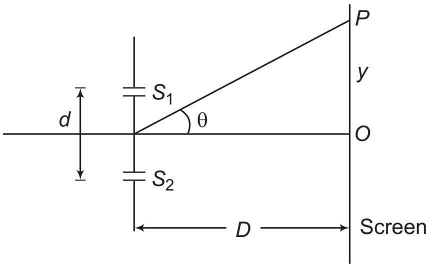
If θ is large, then we cannot use the approximation

$$\sin\theta \approx \tan\theta = \frac{y}{D} \quad \text{or} \quad \Delta x = \frac{yd}{D}$$

But we can use

$$\tan\theta = \frac{y}{D} \quad \text{or} \quad y = D\tan\theta$$

Example 5 *In the YDSE apparatus shown in figure, $d << D$ and $d = 6\lambda$. Find*



(a) total number of maximas and minimas on the screen

(b) two y-coordinates corresponding to third order maxima

Solution (a) In the given set up, $d << D$

Therefore, we can apply

$$\Delta x = d\sin\theta$$

$$(\Delta x)_{\min} = 0 \text{ at } \theta = 0° \text{ and}$$

$$(\Delta x)_{\max} = 6\lambda \text{ at } \theta = 90°$$

Therefore, total number of maximas are **eleven** corresponding to,

$$\Delta x = 0, \pm\lambda, \pm 2\lambda, \pm 3\lambda, \pm 4\lambda \text{ and } \pm 5\lambda$$

Note *$\Delta x = 6\lambda$ will also produce maxima but it is corresponding to $\theta = 90°$ and in the figure, we can see that $\theta = 90°$ point lies outside the screen. So, we will have to ignore this maxima. Similarly, total number of minimas are **twelve** corresponding to :*

$$\Delta x = \pm 0.5\lambda, \pm 1.5\lambda, \pm 2.5\lambda, \pm 3.5\lambda, \pm 4.5\lambda \text{ and } \pm 5.5\lambda$$

(b) Third order minima lies at

$$\Delta x = \pm 3\lambda$$

$$\therefore \quad d\sin\theta = \pm 3\lambda \quad \text{or} \quad 6\lambda\sin\theta = \pm 3\lambda$$

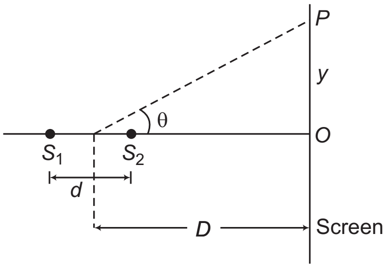
$$\text{or} \quad \sin\theta = \pm\frac{1}{2} \quad \text{or} \quad \theta = \pm 30°$$

$$\text{Now,} \quad \frac{y}{D} = \tan\theta \text{ and } y = D\tan\theta$$

$$\therefore \quad y = \pm D\tan 30° \quad \text{or} \quad y = \pm\frac{D}{\sqrt{3}}$$

Ans.

Example 6 *In the set up shown in figure $d << D$ and $d = 4\lambda$, find*



(a) total number of maximas and minimas on the screen

(b) y-coordinates corresponding to minima nearest to O.

Solution (a) In the given set up, $d << D$, therefore we can use

$$\Delta x = d\cos\theta$$

$$(\Delta x)_{\min} = 0 \text{ at } \theta = 90°$$

$$(\Delta x)_{\max} = d = 4\lambda \text{ at } \theta = 0°$$

Therefore, total number of maximas are **eight** corresponding to

$$\Delta x = \pm 4\lambda, \pm 3\lambda, \pm 2\lambda \text{ and } \pm\lambda$$

Note *$\Delta x = 0$ also produce maxima. But this is corresponding to $\theta = 90°$, which does not lie on screen.*

Similarly, total number of minimas are **eight** corresponding to

$$\Delta x = \pm 3.5\lambda, \pm 2.5\lambda, \pm 1.5\lambda \text{ and } \pm 0.5\lambda$$

(b) Minima nearest to O are corresponding to

$$\Delta x = \pm 3.5\lambda$$

$$\therefore \quad d\cos\theta = \pm 3.5\lambda \quad \text{or} \quad 4\lambda\cos\theta = \pm 3.5\lambda$$

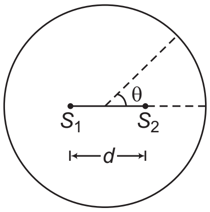
$$\text{or} \quad \theta = \pm\cos^{-1}\left(\frac{3.5\lambda}{4\lambda}\right) \quad \text{or} \quad \theta = \pm\cos^{-1}\left(\frac{7}{8}\right)$$

$$\text{Now,} \quad \frac{y}{D} = \tan\theta \quad \Rightarrow \quad y = D\tan\theta$$

$$\therefore \quad y = \pm D\tan\left\{\cos^{-1}\left(\frac{7}{8}\right)\right\}$$

Ans.

Example 7 *There is a large circle (not a screen this time) around two coherent sources S_1 and S_2 kept at a distance $d = 3.4\lambda$.*



(a) Find total number of maximas on this circle.

(b) Four angular positions corresponding to third order maxima on this circle.

Solution (a) Since, radius of circle << d, therefore we can apply

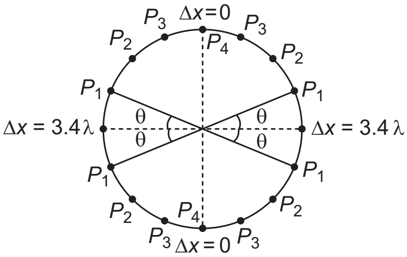
$$\Delta x = d\cos\theta$$

Corresponding to given figure

$$(\Delta x)_{\min} = 0 \text{ at } \theta = 90^\circ$$

and $$(\Delta x)_{\max} = d = 3.4\lambda \text{ at } \theta = 0^\circ$$

Note *This time, $\theta = 90^\circ$ lies on the circle. Therefore, $\Delta x = 0$ corresponding to zero order maxima lie on circle (two in numbers). Hence, the total number of maximas are fourteen as shown below.*



At P_1, $\Delta x = 3\lambda$

At P_2, $\Delta x = 2\lambda$

At P_3, $\Delta x = \lambda$

At P_4, $\Delta x = 0$

(b) Corresponding to third order maxima, (At P_1)

$$\Delta x = 3\lambda$$

or $$d\cos\theta = 3\lambda$$

or $$3.4\cos\theta = 3\lambda$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{3}{3.4}\right)$$ **Ans.**

Four angular positions are as shown in figure.

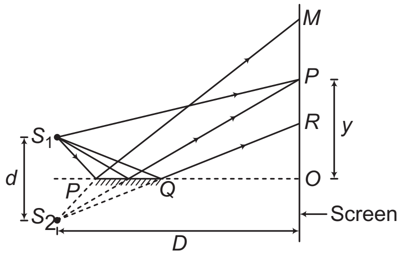
Note *At $\theta = 0^\circ$, $\Delta x = d = 3.4\lambda$. So, this is neither a maxima nor a minima.*

Exercise Find number of minimas on the circle in the above problem.

Type 4. *When one of the interfering rays is reflected from a denser medium*

Concept

In the shown figure, a virtual image S_2 is formed of the real source S_1. Further, $\lambda << d << D$. At any point P on the screen two rays interfere. One is direct from S_1 and other is reflected (or we can assume that it comes from S_2) from a denser medium. So, they will have a phase difference of π or 180° between them.



Whenever there exists a phase difference of π between the two interfering beams of light, conditions of maxima and minima are interchanged, i.e.

$$\Delta x = n\lambda \qquad \text{(for minimum intensity)}$$

and

$$\Delta x = (2n - 1)\,\lambda/2 \qquad \text{(for maximum intensity)}$$

Further, $PQRM$ is the field of view corresponding to S_2 and the plane mirror. Or all reflected rays fall on this region. So, interference will be obtained only between M and R on the screen. Fringe width is still,

$$\omega = \frac{\lambda D}{d}$$

Total number of fringes obtained on the screen will be

$$N = \frac{MR}{\omega}$$

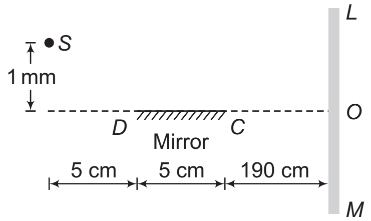
Suppose OP is

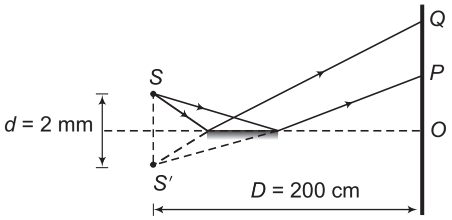
$$y = \frac{n\lambda D}{d} \qquad \text{(where, } n \text{ is an integer)}$$

$$= n\omega$$

and P lies between M and R, then it will become a dark fringe because conditions of maxima and minima have been interchanged.

Example 8 *A source of light of wavelength* 5000 Å *is placed as shown in figure. Considering interference of direct and reflected rays, determine the position of the region where the fringes will be visible and calculate the number of fringes.*





Solution Interference will be obtained between direct rays from S and reflected rays from S' (image of S on mirror).

Since, the reflected rays will lie between region P and Q on the screen. So, interference is obtained in this region only. From geometry will can show that,

$$OP = 1.9 \text{ cm} \quad \text{and} \quad OQ = 3.9 \text{ cm}$$

$$\therefore \qquad PQ = 2 \text{ cm}$$ **Ans.**

$$\text{Fringe width}, \omega = \frac{\lambda D}{d} = \frac{(5000 \times 10^{-8})(200)}{(2 \times 10^{-1})} \text{ cm} = 0.05 \text{ cm}$$

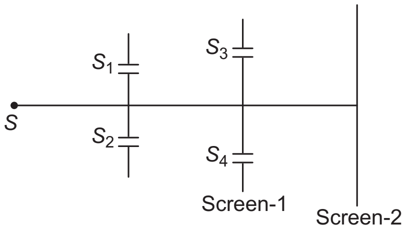
Total number of fringes in the region PQ,

$$N = \frac{PQ}{\omega} = \frac{2}{0.05} = 40$$ **Ans.**

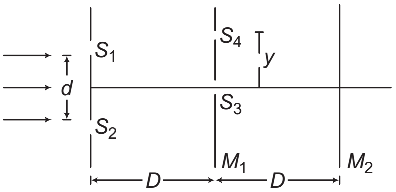
Type 5. *Based on conditions of double interference*

Concept

On screen-1 interference takes place first time and intensity varies between 0 and $4I_0$. There are further two slits on screen-1 at S_3 and S_4. Therefore, second time interference takes place on screen-2 due to two rays of light from S_3 and S_4.



Example 9 *Consider the situation shown in figure. The two slits S_1 and S_2 placed symmetrically around the centre line are illuminated by a monochromatic light of wavelength λ. The separation between the slits is d. The light transmitted by the slits falls on a screen M_1 placed at a distance D from the slits. The slit S_3 is at the centre line and the slit S_4 is at a distance y from S_3. Another screen M_2 is placed at a further distance D away from M_1. Find the ratio of the maximum to minimum intensity observed on M_2 if y is equal to* $(d << D)$



(a) $\dfrac{\lambda D}{2d}$ *(b)* $\dfrac{\lambda D}{d}$ *(c)* $\dfrac{\lambda D}{4d}$

Solution $S_1S_3 = S_2S_3$

$\therefore$ Δx at S_3 is zero. Therefore, intensity at S_3 will be $4I_0$. Let us call it I_1. Thus,

$$I_{S_3} = I_1 = 4I_0$$

Now, I_{S_4} (or I_2) depends on the value of y.

(a) When $y = \dfrac{\lambda D}{2d}$

$$\Delta x = \frac{yd}{D} = \frac{\lambda}{2}$$

$\therefore$ $$I_{S_4} = I_2 = 0$$

or $$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = 1$$ **Ans.**

(b) When $y = \dfrac{\lambda D}{d}$

$$\Delta x = \frac{yd}{D} = \lambda$$

$\therefore$ $$I_{S_4} = I_2 = 4I_0$$

or $$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = \infty$$

(c) When $y = \dfrac{\lambda D}{4d}$

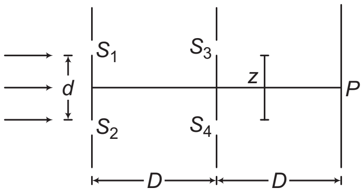
$$\Delta x = \frac{yd}{D} = \frac{\lambda}{4}$$

$\Rightarrow$ $$\Delta\phi = \phi = 90^\circ$$

$\therefore$ $$I_{S_4} = I_2 = 4I_0 \cos^2\frac{\phi}{2} = 2I_0$$

or $$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = 34$$ **Ans.**

Example 10 *Consider the arrangement shown in figure. By some mechanism, the separation between the slits S_3 and S_4 can be changed. The intensity is measured at the point P which is at the common perpendicular bisector of S_1S_2 and S_3S_4. When $z = \dfrac{D\lambda}{2d}$, the intensity measured at P is I. Find the intensity when z is equal to*



(a) $\dfrac{D\lambda}{d}$ (b) $\dfrac{3D\lambda}{2d}$ (c) $\dfrac{2D\lambda}{d}$

Solution $|y_{S_3}| = |y_{S_4}| = \frac{z}{2} = y$ (say)

When $$z = \frac{D\lambda}{2d}, \quad \frac{z}{2} = y = \frac{D\lambda}{4d}$$

$\therefore$ $$\Delta x = \frac{yd}{D} = \frac{\lambda}{4}$$

and we have seen in the above example that, at $\Delta x = \frac{\lambda}{4}$, intensity is $2I_0$.

$\therefore$ $$I_{S_3} = I_{S_4} = 2I_0$$

Now, P is at the perpendicular bisector of S_3S_4. Therefore, intensity at P will be four times of $2I_0$ or $8I_0$.

$\therefore$ $$8I_0 = I \qquad \text{(Given)}$$

Hence, $$I_0 = \frac{I}{8}$$

(a) When $z = \frac{D\lambda}{d}$

$$y = \frac{z}{2} = \frac{D\lambda}{2d}$$

$\therefore$ $$\Delta x = \frac{yd}{D} = \frac{\lambda}{2}$$

or $$I_{S_3} = I_{S_4} = 0$$

Hence, $$I_P = 0$$ **Ans.**

(b) When $$z = \frac{3D\lambda}{2d}$$

$$y = \frac{z}{2} = \frac{3D\lambda}{4d}$$

$$\Delta x = \frac{yd}{D} = \frac{3\lambda}{4}$$

$\therefore$ $$\Delta\phi \quad \text{or} \quad \phi = \frac{2\pi}{\lambda}(\Delta x) = \frac{3\pi}{2}$$

Using $$I = 4I_0 \cos^2 \frac{\phi}{2}$$

We have, $$I_{S_3} = I_{S_4} = 2I_0$$

$\therefore$ $$I_P = 4(2I_0) = 8I_0 = I$$ **Ans.**

(c) When $z = \frac{2D\lambda}{d}$

$$y = \frac{z}{2} = \frac{D\lambda}{d}$$

$\therefore$ $$\Delta x = \frac{yd}{D} = \lambda$$

$\therefore$ $$I_{S_3} = I_{S_4} = 4I_0$$

$$I_P = 4(4I_0) = 16\,I_0 = 2I$$ **Ans.**

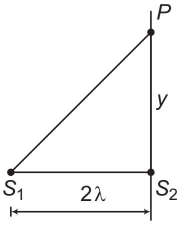
Type 6. *When no approximation can be taken in finding the path difference*

Concept

If D is not very very greater than d, then we cannot apply $\Delta x = d \sin\theta$ or $d\cos\theta$ or $\frac{yd}{D}$. In this case, we will have to find the path difference by using geometry.

Example 11 *Two coherent point sources S_1 and S_2 emit light of wavelength λ. The separation between sources is 2λ. Consider a line passing through S_2 and perpendicular to the line S_1S_2. What is the smallest distance on this line from S_2 where a minimum of intensity occurs?*

Solution At S_2, $\Delta x = 2\lambda$



Therefore, the minima closest to S_2 will be corresponding to the path difference $\Delta x = \frac{3\lambda}{2}$. Suppose this point is P at a distance y from S_2. Then,

$$S_1P - S_2P = \frac{3\lambda}{2}$$

$$\sqrt{y^2 + (S_1S_2)^2} - y = \frac{3\lambda}{2}$$

or

$$\sqrt{y^2 + (2\lambda)^2} = \left(y + \frac{3\lambda}{2}\right)$$

Squaring and then solving this equation, we get

$$y = \frac{7\lambda}{12}$$ **Ans.**

Type 7. *Based on single slit diffraction*

Example 12 *A screen is placed 50 cm from a single slit, which is illuminated with 6000 Å light. If distance between the first and third minima in the diffraction pattern is 3.0 mm, what is the width of the slit?*

Solution Position of first minima on a single slit diffraction pattern is given by

$$d \sin\theta = n\lambda$$

For small value of θ, $\sin\theta \approx \theta = \frac{y}{D}$

$$\therefore \quad \frac{yd}{D} = n\lambda \quad \text{or} \quad y = \frac{n\lambda D}{d}$$

$\therefore$ Distance between third order minima and first order minima will be

$$\Delta y = y_3 - y_1 = \frac{(3-1)(\lambda D)}{d} = \frac{2\lambda D}{d}$$

Substituting the values, we have

$$\Delta y = \frac{(2)(6\times10^{-7})(0.5)}{3\times10^{-3}}$$

$$= 2\times10^{-4}\text{ m}$$

$$= 0.2\text{ mm}$$ **Ans.**

Example 13 *In a single slit diffraction experiment first minima for $\lambda_1 = 660$ nm coincides with first maxima for wavelength λ_2. Calculate the value of λ_2.*

Solution Position of minima in diffraction pattern is given by

$$d \sin\theta = n\lambda$$

For first minima of λ_1, we have

$$d \sin\theta_1 = (1)\,\lambda_1$$

or
$$\sin\theta_1 = \frac{\lambda_1}{d} \qquad \text{...(i)}$$

The first maxima approximately lies between first and second minima. For wavelength λ_2, its position will be

$$d \sin\theta_2 = \frac{3}{2}\lambda_2$$

$\therefore$
$$\sin\theta_2 = \frac{3\lambda_2}{2d} \qquad \text{...(ii)}$$

The two will coincide if

$$\theta_1 = \theta_2$$

or
$$\sin\theta_1 = \sin\theta_2$$

$\therefore$
$$\frac{\lambda_1}{d} = \frac{3\lambda_2}{2d}$$

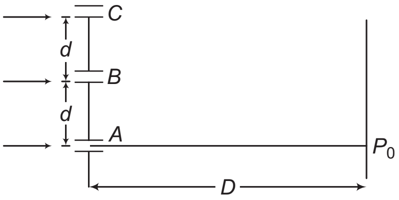
or
$$\lambda_2 = \frac{2}{3}\lambda_1$$

$$= \frac{2}{3} \times 660 \text{ nm}$$

$$= 440 \text{ nm}$$ **Ans.**

Miscellaneous Examples

Example 14 *Figure shows three equidistant slits illuminated by a monochromatic parallel beam of light. Let $BP_0 - AP_0 = \lambda/3$ and $D >> \lambda$*



(a) *Show that $d = \sqrt{2\lambda D/3}$*

(b) *Show that the intensity at P_0 is three times the intensity due to any of the three slits individually.*

Solution (a) Given,
$$BP_0 - AP_0 = \lambda/3$$

$\therefore$
$$\sqrt{D^2 + d^2} - D = \frac{\lambda}{3}$$

or $$\sqrt{D^2 + d^2} = \left(D + \frac{\lambda}{3}\right)$$

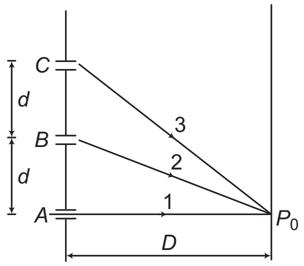
Squaring both sides, we get

$$D^2 + d^2 = D^2 + \frac{\lambda^2}{9} + \frac{2D\lambda}{3}$$

Since $\lambda << D$, we can ignore the term $\frac{\lambda^2}{9}$. By ignoring this term, we get the desired result.

$$d = \sqrt{\frac{2\lambda D}{3}}$$

(b) Given, $$BP_0 - AP_0 = \Delta x_{12} = \frac{\lambda}{3}$$



∴ $$\Delta\phi_{12} \text{ or } \phi_{12} = 120°$$

Now, $$CP_0 - AP_0 = \Delta x_{13} = \sqrt{(2d)^2 + D^2} - D$$

$$= D\left[1 + \left(\frac{2d}{D}\right)^2\right]^{\frac{1}{2}} - D$$

$$\approx D\left[1 + \frac{1}{2} \times \frac{4d^2}{D^2}\right] - D$$

$$= \frac{2d^2}{D}$$

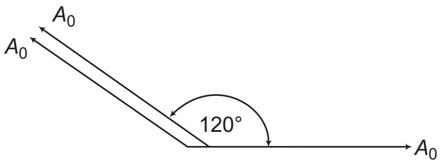
Substituting, $d = \sqrt{\frac{2\lambda D}{3}}$ or $d^2 = \frac{2\lambda D}{3}$

We get, $$\Delta x_{13} = \frac{4}{3}\lambda$$

∴ $$\Delta\phi_{13} \text{ or } \phi_{13} = \frac{4}{3}(360°) = 480°$$

$$= 480° - 360° = 120°$$

Now, we know that in case of coherent sources amplitudes are first added by vector method. So, let individual amplitude is A_0.



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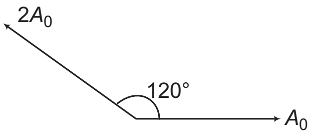


The resultant amplitude will be given by

$$A = \sqrt{A_0^2 + (2A_0)^2 + 2(A_0)(2A_0)\cos 120°}$$

$$= \sqrt{3}\, A_0$$

$$I \propto A^2$$

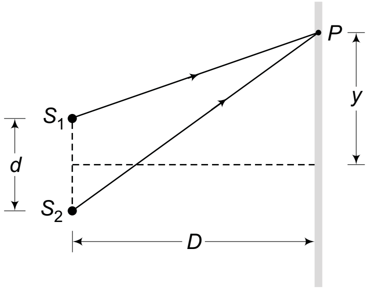
and amplitude has become $\sqrt{3}$ times, therefore resultant intensity will become 3 times.

Example 15 *Young's double slit experiment is carried out using microwaves of wavelength* $\lambda = 3$ *cm. Distance between the slits is* $d = 5$ *cm and the distance between the plane of slits and the screen is* $D = 100$ *cm.*

(a) Find total number of maximas and

(b) their positions on the screen.

Solution (a) The maximum path difference that can be produced = distance between the sources or 5 cm.



Thus, in this case we can have only three maximas, one central maxima and two on its either side for a path difference of λ or 3 cm.

(b) For maximum intensity at P,

$$S_2P - S_1P = \lambda$$

or

$$\sqrt{(y + d/2)^2 + D^2} - \sqrt{(y - d/2)^2 + D^2} = \lambda$$

Substituting $d = 5$ cm, $D = 100$ cm and $\lambda = 3$ cm we get,

$$y = \pm\ 75 \text{ cm}$$

Thus, the three maximas will be at

$$y = 0$$

and $$y = \pm\ 75 \text{ cm}$$ **Ans.**

Example 16 *Two coherent sources are* 0.3 *mm apart. They are* 0.9 *m away from the screen. The second dark fringe is at a distance of* 0.3 *cm from the centre. Find the distance of fourth bright fringe from the centre. Also, find the wavelength of light used.*

Solution Given, $d = 0.3 \times 10^{-3}$ m, $D = 0.9$ m

$$\frac{3\lambda D}{2d} = 0.3 \times 10^{-2} \text{ cm} \qquad \text{(the distance of second dark fringe)}$$

$$\therefore \qquad \frac{\lambda D}{d} = (0.3 \times 10^{-2})\left(\frac{2}{3}\right)$$

$$= 0.2 \times 10^{-2} \text{ m} = 0.2 \text{ cm}$$

(i) Distance of fourth bright fringe from centre $= \dfrac{4\lambda D}{d} = 0.8$ cm **Ans.**

(ii) $$\lambda = \left(\frac{d}{D}\right)(0.2 \times 10^{-2}) \text{ m} = \left(\frac{0.3 \times 10^{-3}}{0.9}\right)(0.2 \times 10^{-2})$$

$$= 6.67 \times 10^{-7} \text{ m}$$ **Ans.**

Example 17 *In a Young's double slit set up, the wavelength of light used is 546 nm. The distance of screen from slits is 1 m. The slit separation is 0.3 mm.*

(a) Compare the intensity at a point P distant 10 mm from the central fringe where the intensity is I_0.

(b) Find the number of bright fringes between P and the central fringe.

Solution Given, $\lambda = 546 \text{ nm} = 5.46 \times 10^{-7}$ m, $D = 1.0$ m and $d = 0.3 \text{ mm} = 0.3 \times 10^{-3}$ m

(a) At a distance $y = 10 \text{ mm} = 10 \times 10^{-3}$ m, from central fringe, the path difference will be

$$\Delta x = \frac{yd}{D} = \frac{(10 \times 10^{-3})(0.3 \times 10^{-3})}{1.0} = 3.0 \times 10^{-6} \text{ m}$$

The corresponding phase difference between the two interfering beams will be

$$\phi = \frac{2\pi}{\lambda} \cdot \Delta x$$

$$= \left(\frac{2\pi}{5.46 \times 10^{-7}}\right)(3.0 \times 10^{-6}) \text{ radian}$$

$$= 1978^\circ$$

$$\therefore \quad \frac{\phi}{2} = 989^\circ$$

$$\therefore \quad I = I_0 \cos^2 \frac{\phi}{2}$$

$$= I_0 \cos^2 (989)$$

$$= 3.0 \times 10^{-4} \, I_0$$

Ans.

(b) Fringe width, $\omega = \frac{\lambda D}{d} = \frac{(5.46 \times 10^{-7})(1.0)}{0.3 \times 10^{-3}}$ m

$$= 1.82 \text{ mm}$$

Since, $\frac{y}{\omega} = \frac{10}{1.82} = 5.49$

Therefore, number of bright fringes between P and central fringe will be 5 (excluding the central fringe). **Ans.**

Example 18 *In a double slit pattern ($\lambda = 6000$ Å), the first order and tenth order maxima fall at 12.50 mm and 14.75 mm from a particular reference point. If λ is changed to 5500 Å, find the position of zero order and tenth order fringes, other arrangements remaining the same.*

Solution Distance between 10 fringes is

$$9\omega = (14.75 - 12.50) \text{ mm} = 2.25 \text{ mm}$$

$\therefore$ Fringe width, $\omega = 0.25$ mm

When the wavelength is changed from 6000 Å to 5500 Å, the new fringe width will become,

$$\omega' = \left(\frac{5500}{6000}\right)\omega = \left(\frac{5500}{6000}\right)(0.25)$$

as fringe width $\propto \lambda$

$$\therefore \quad \omega' = 0.23 \text{ mm}$$

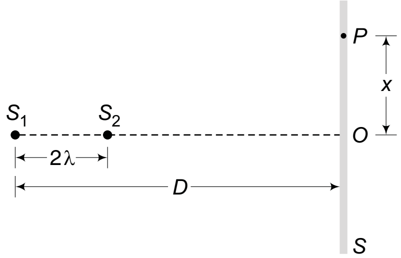
The position of central (or zero order) maxima will remain unchanged. Earlier it was at a position,

$$Y_0 = Y_1 - \omega = (12.50 - 0.25) = 12.25 \text{ mm}$$ **Ans.**

The new position of tenth order maxima will be

$$Y_{10} = Y_0 + 10\,\omega' = (12.25) + (10)\,(0.23) = 14.55 \text{ mm}$$ **Ans.**

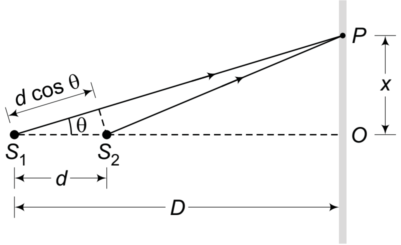
Example 19 *Two coherent narrow slits emitting light of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The light is collected on a screen S which is placed at a distance $D (>> \lambda)$ from the slit S_1 as shown in figure. Find the finite distance x such that the intensity at P is equal to intensity at O.*



Solution Path difference at O,

$$S_1O - S_2O = 2\lambda$$

i.e. maximum intensity is obtained at O. Next maxima will be obtained at point P where,



$$S_1P - S_2P = \lambda$$

or $$d \cos\theta = \lambda$$

or $$2\lambda \cos\theta = \lambda$$

or $$\cos\theta = \frac{1}{2}$$

$\therefore$ $$\theta = 60°$$

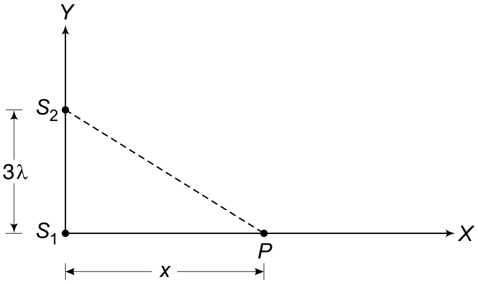
Now in ΔS_1PO, $$\frac{PO}{S_1O} = \tan\theta \quad \text{or} \quad \frac{x}{D} = \tan 60° = \sqrt{3}$$

$\therefore$ $$x = \sqrt{3}D$$ **Ans.**

Note *At point O, path difference is 2λ, i.e. we get second order maxima. At point P, where path difference is λ (i.e $x = \sqrt{3}D$) we get first order maxima. The next, i.e. zero order maxima will be obtained where path difference, i.e. $d \cos\theta = 0$ or $\theta = 90°$. At $\theta = 90°$, $x = \infty$. So, our answer, i.e. finite distance of x should be $x = \sqrt{3}D$, corresponding to first order maxima.*

Example 20 *An interference is observed due to two coherent sources S_1 placed at origin and S_2 placed at $(0,\ 3\lambda,\ 0)$. Here, λ is the wavelength of the sources. A detector D is moved along the positive x-axis. Find x-coordinates on the x-axis (excluding $x = 0$ and $x = \infty$) where maximum intensity is observed.*

Solution At $x = 0$, path difference is 3λ. Hence, third order maxima will be obtained. At $x = \infty$, path difference is zero. Hence, zero order maxima is obtained. In between, first and second order maxima will be obtained.



First order maxima

$$S_2P - S_1P = \lambda \quad \text{or} \quad \sqrt{x^2 + 9\lambda^2} - x = \lambda \quad \text{or} \quad \sqrt{x^2 + 9\lambda^2} = x + \lambda$$

Squaring on both sides, we get

$$x^2 + 9\lambda^2 = x^2 + \lambda^2 + 2x\lambda$$

Solving this, we get $x = 4\lambda$

Second order maxima

$$S_2P - S_1P = 2\lambda \quad \text{or} \quad \sqrt{x^2 + 9\lambda^2} - x = 2\lambda \quad \text{or} \quad \sqrt{x^2 + 9\lambda^2} = (x + 2\lambda)$$

Squaring on both sides, we get

$$x^2 + 9\lambda^2 = x^2 + 4\lambda^2 + 4x\lambda$$

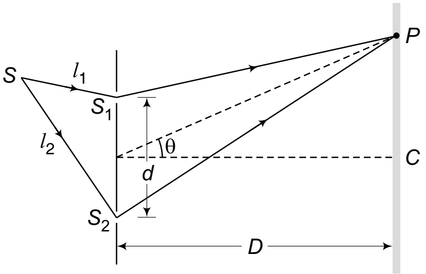
Solving, we get $x = \frac{5}{4}\lambda = 1.25\lambda$

Hence, the desired x-coordinates are

$$x = 1.25\lambda \quad \text{and} \quad x = 4\lambda$$ **Ans.**

Note *(i) As we move along positive x-axis (from origin) order of maxima decreases from $n = 3$ to $n = 0$.*
(ii) Here, we cannot take the path difference $d\cos\theta$ or $d\sin\theta$. Think why?

Example 21 *In a Young's double slit experiment, the light sources is at distance $l_1 = 20\ \mu m$ and $l_2 = 40\ \mu m$ from the main slit. The light of wavelength $\lambda = 500$ nm is incident on slits separated at a distance $10\ \mu m$. A screen is placed at a distance $D = 2$ m away from the slits as shown in figure. Find*



(a) the values of θ relative to the central line where maxima appear on the screen?
(b) how many maxima will appear on the screen?
(c) what should be the minimum thickness of a slab of refractive index 1.5 placed on the path of one of the ray so that minima occurs at C?

Solution (a) The optical path difference between the beams arriving at P,

$$\Delta x = (l_2 - l_1) + d \sin \theta$$

The condition for maximum intensity is

$$\Delta x = n\lambda \qquad [n = 0, \pm 1, \pm 2, \ldots]$$

Thus,

$$\sin \theta = \frac{1}{d}[\Delta x - (l_2 - l_1)] = \frac{1}{d}[n\lambda - (l_2 - l_1)]$$

$$= \frac{1}{10 \times 10^{-6}}[n \times 500 \times 10^{-9} - 20 \times 10^{-6}]$$

$$= 2\left[\frac{n}{40} - 1\right]$$

Hence,

$$\theta = \sin^{-1}\left[2\left(\frac{n}{40} - 1\right)\right] \qquad \textbf{Ans.}$$

(b) $|\sin \theta| \le 1$

$\therefore$

$$-1 \le 2\left[\frac{n}{40} - 1\right] \le 1$$

or

$$-20 \le (n - 40) \le 20$$

or

$$20 \le n \le 60$$

Hence, number of maximas $= 60 - 20 = 40$ **Ans.**

(c) At C, phase difference

$$\phi = \left(\frac{2\pi}{\lambda}\right)(l_2 - l_1) = \left(\frac{2\pi}{500 \times 10^{-9}}\right)(20 \times 10^{-6})$$

$$= 80\,\pi$$

Hence, maximum intensity will appear at C. For minimum intensity at C,

$$(\mu - 1)\, t = \frac{\lambda}{2}$$

or

$$t = \frac{\lambda}{2(\mu - 1)} = \frac{500 \times 10^{-9}}{2 \times 0.5} = 500 \text{ nm} \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

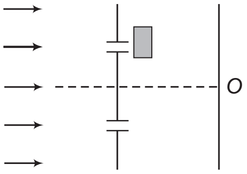
(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Two identical waves due to two coherent sources interfere at a point with a phase difference of $\frac{2\pi}{3}$, then the resultant intensity at this point is equal to the individual intensity of the sources .

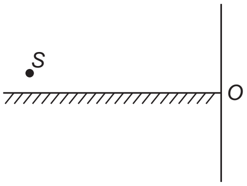
Reason : A phase difference of $\frac{2\pi}{3}$ is equivalent to a path difference of $\frac{\lambda}{3}$.

2. Assertion : In the figure shown, zero order maxima will lie above point *O*.



Reason : Zero order maxima normally means a point where path difference is zero.

3. Assertion : A monochromatic source of light is placed above a plane mirror as shown in figure. Fringes will be obtained at all points above *O* but not below it on the screen.



Reason : All reflected rays will suffer a phase difference of π.

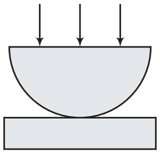
4. Assertion : If width of one slit in Young's double slit experiment is slightly increased, then maximum and minimum both intensities will increase.

Reason : Intensity reaching from that slit on screen will slightly increase.

5. Assertion : If white light is used in place of monochromatic light in YDSE, then central point is white, although at other places coloured fringes will be obtained.

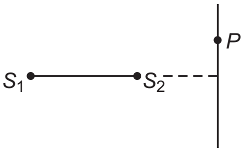
Reason : At centre, path difference is zero for all wavelengths . Hence, all wavelengths will interfere constructively.

6. **Assertion :** A glass hemisphere is placed on a flat plate as shown. The observed interference fringes from this combination shall be circular.



Reason : In all cases fringes are circular .

7. **Assertion :** Two coherent sources S_1 and S_2 are placed in front of a screen as shown in figure. At point P, 10th order maxima is obtained . Then, 11th order maxima will be obtained above P.

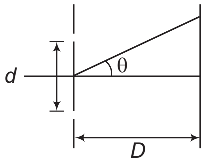


Reason : For 11th order maxima path difference should be more.

8. **Assertion :** Distance between two coherent sources S_1 and S_2 is 4λ. A large circle is drawn around these sources with centre of circle lying at centre of S_1 and S_2. There are total 16 maximas on this circle.

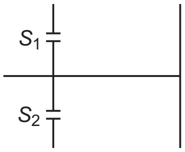
Reason : Total number of minimas on this circle are less, compared to total number of maximas.

9. **Assertion :** In the YDSE apparatus shown in figure $d << D$ and $\frac{d}{\lambda} = 4$, then second order maxima will be obtained at $\theta = 30°$.



Reason : Total seven maximas will be obtained on screen.

10. **Assertion :** White light is used in YDSE. Now, a glass slab is inserted in front of the slit S_1. Then, red fringe will shift less (in upward direction) compared to violet.



Reason : Refractive index for violet colour will be more.

Objective Questions

1. Three coherent waves having amplitudes 12 mm, 6 mm and 4 mm arrive at a given point with successive phase difference of $\pi/2$. Then, the amplitude of the resultant wave is
(a) 7 mm (b) 10 mm (c) 5 mm (d) 4.8 mm

2. Two coherent sources of intensity ratio β^2 interfere. Then, the value of $(I_{\max}-I_{\min})/(I_{\max}+I_{\min})$ is

(a) $\frac{1+\beta}{\sqrt{\beta}}$ (b) $\sqrt{\frac{1+\beta}{\beta}}$

(c) $\frac{1+\beta}{\beta}$ (d) None of these

3. In Young's double slit experiment, distance between two sources is 0.1 mm. The distance of screen from the sources is 20 cm. Wavelength of light used is 5460 Å. Then, angular position of first dark fringe is approximately

(a) 0.08° (b) 0.16°

(c) 0.20° (d) 0.32°

4. Young's double slit experiment is made in a liquid. The tenth bright fringe in liquid lies in screen where 6th dark fringe lies in vacuum. The refractive index of the liquid is approximately

(a) 1.8 (b) 1.54

(c) 1.67 (d) 1.2

5. A plane monochromatic light wave falls normally on a diaphragm with two narrow slits separated by 2.5 mm. The fringe pattern is formed on a screen 100 cm behind the diaphragm. By what distance will these fringes be displaced, when one of the slits is covered by a glass plate ($\mu = 1.5$) of thickness 10 μm?

(a) 2 mm (b) 1 mm

(c) 3 mm (d) 4 mm

6. The distance of *n*th bright fringe to the *n*th dark fringe in Young's experiment is equal to

(a) $\frac{3\lambda D}{2d}$ (b) $\frac{2\lambda D}{d}$

(c) $\frac{\lambda D}{2d}$ (d) $\frac{\lambda D}{d}$

7. When YDSE is conducted with white light, a white fringe is observed at the centre of the screen. When the screen is moved towards the slits by 5 mm, then this white fringe

(a) does not move (b) becomes red

(c) disappears (d) Nothing can be said

8. In Young's double slit experiment, 60 fringes are observed in the central view zone with light of wavelength 4000 Å. The number of fringes that will be observed in the same view zone with the light of wavelength 6000 Å, is

(a) 40 (b) 90

(c) 60 (d) None of these

9. In a two slit experiment with monochromatic light, fringes are obtained on a screen placed at some distance from the slits . If the screen is moved by 5×10^{-2} m, towards the slits, the change in fringe width is 3×10^{-5} m. If separation between the slits is 10^{-3} m, the wavelength of light used is

(a) 6000 Å (b) 5000 Å (c) 3000 Å (d) 4500 Å

10. The ratio of maximum to minimum intensity due to superposition of two waves is $\frac{49}{9}$. Then, the ratio of the intensity of component waves is

(a) $\frac{25}{4}$ (b) $\frac{5}{4}$ (c) $\frac{25}{6}$ (d) $\frac{7}{5}$

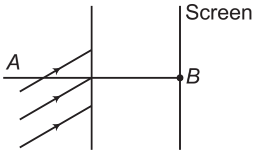
11. With two slits spaced 0.2 mm apart and a screen at a distance of 1 m, the third bright fringe is found to be at 7.5 mm from the central fringe. The wavelength of light used is

(a) 400 nm (b) 500 nm
(c) 550 nm (d) 600 nm

12. A beam of light consisting of two wavelengths 6500 Å and 5200Å is used to obtain interference fringes in YDSE. The distance between slits is 2 mm and the distance of the screen from slits is 120 cm. What is the least distance from central maximum where the bright due to both wavelengths coincide?

(a) 0.156 cm (b) 0.312 cm
(c) 0.078 cm (d) 0.468 cm

13. A beam of light parallel to central line AB is incident on the plane of slits . The number of minima obtained on the large screen is n_1. Now if the beam is tilted by some angle ($\neq 90°$) as shown in figure, then the number of minima obtained is n_2. Then,



(a) $n_1 = n_2$ (b) $n_1 > n_2$
(c) $n_2 > n_1$ (d) n_2 will be zero

Subjective Questions

Note *You can take approximations in the answers.*

1. Two waves of equal frequencies have their amplitude in the ratio of 5 : 3. They are superimposed on each other. Calculate the ratio of the maximum to minimum intensities of the resultant wave.

2. Two coherent sources A and B of radio waves are 5.00 m apart. Each source emits waves with wavelength 6.00 m. Consider points along the line between the two sources. At what distances, if any, from A is the interference (a) constructive (b) destructive?

3. A radio transmitting station operating at a frequency of 120 MHz has two identical antennas that radiate in phase. Antenna B is 9.00 m to the right of antenna A. Consider point P between the antennas and along the line connecting them, a horizontal distance x to the right of antenna A. For what values of x will constructive interference occur at point P ?

4. Coherent light from a sodium-vapour lamp is passed through a filter that blocks everything except for light of a single wavelength. It then falls on two slits separated by 0.460 mm. In the resulting interference pattern on a screen 2.20 m away, adjacent bright fringes are separated by 2.82 mm. What is the wavelength?

5. Find the angular separation between the consecutive bright fringes in a Young's double slit experiment with blue-green light of wavelength 500 nm. The separation between the slits is 2.0×10^{-3} m.

6. A Young's double slit apparatus has slits separated by 0.25 mm and a screen 48 cm away from the slits. The whole apparatus is immersed in water and the slits are illuminated by the red light ($\lambda = 700$ nm in vacuum). Find the fringe width of the pattern formed on the screen. ($\mu_w = 4/3$)

7. In a double slit experiment, the distance between the slits is 5.0 mm and the slits are 1.0 m from the screen. Two interference patterns can be seen on the screen one due to light with wavelength 480 nm, and the other due to light with wavelength 600 nm. What is the separation on the screen between the third order bright fringes of the two interference patterns?

8. Two slits spaced 0.450 mm apart are placed 75.0 cm from a screen. What is the distance between the second and third dark lines of the interference pattern on the screen when the slits are illuminated with coherent light with a wavelength of 500 nm?

9. Coherent light with wavelength 600 nm passes through two very narrow slits and the interference pattern is observed on a screen 3.00 m from the slits. The first order bright fringe is at 4.94 mm from the centre of the central bright fringe. For what wavelength of light will the first order dark fringe be observed at this same point on the screen?

10. Two very narrow slits are spaced 1.80 μm apart and are placed 35.0 cm from a screen. What is the distance between the first and second dark lines of the interference pattern when the slits are illuminated with coherent light of $\lambda = 550$ nm? (Hint : The angle θ is not small).

11. A narrow beam of 100 eV electrons is fired at two parallel slits very close to each other. The distance between the slits is 10 Å. The electron waves after passing through the slits interfere on a screen 3 m away from slits and form interference fringes. Find the width of the fringe.

12. In a Young's double slit set up, the wavelength of light used is 546 nm. The distance of screen from slits is 1 m. The slit separation is 0.3 mm.

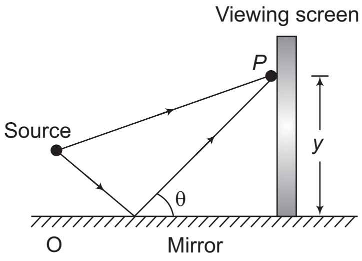
(a) Compare the intensity at a point P distant 10 mm from the central fringe where the intensity is I_0.

(b) Find the number of bright fringes between P and the central fringe.

13. Interference pattern with Young's double slits 1.5 mm apart are formed on a screen at a distance 1.5 m from the plane of slits. In the path of the beam of one of the slits, a transparent film of 10 micron thickness and of refractive index 1.6 is interposed while in the path of the beam from the other slit a transparent film of 15 micron thickness and of refractive index 1.2 is interposed. Find the displacement of the fringe pattern.

14. In a Young's double slit experiment using monochromatic light, the fringe pattern shifts by a certain distance on the screen when a mica sheet of refractive index 1.6 and thickness 1.964 microns is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the slits and screen is doubled. It is found that the distance between successive maxima (or minima) now is the same as observed fringe shift upon the introduction of the mica sheet. Calculate the wavelength of the monochromatic light used in the experiment.

15. Interference effects are produced at point P on a screen as a result of direct rays from a 500 nm source and reflected rays from a mirror, as shown in figure. If the source is 100 m to the left of the screen and 1.00 cm above the mirror, find the distance y (in milimetres) to the first dark band above the mirror.



16. What is the thinnest film of coating with $n = 1.42$ on glass ($n = 1.52$) for which destructive interference of the red component (650 nm) of an incident white light beam in air can take place by reflection?

17. A glass plate $(n = 1.53)$ that is 0.485 μm thick and surrounded by air is illuminated by a beam of white light normal to the plate.

(a) What wavelengths (in air) within the limits of the visible spectrum ($\lambda = 400$ to 700 nm) are intensified in the reflected beam?

(b) What wavelengths within the visible spectrum are intensified in the transmitted light?

18. A thick glass slab $(\mu = 1.5)$ is to be viewed in reflected white light. It is proposed to coat the slab with a thin layer of a material having refractive index 1.3 so that the wavelength 6000 Å is suppressed. Find the minimum thickness of the coating required.

19. An oil film covers the surface of a small pond. The refractive index of the oil is greater than that of water. At one point on the film, the film has the smallest non-zero thickness for which there will be destructive interference in the reflected light when infrared radiation with wavelength 800 nm is incident normal to the film. When this film is viewed at normal incidence at this same point, for what visible wavelengths, if any, will there be constructive interference? (Visible light has wavelengths between 400 nm and 700 nm)

20. A possible means for making an airplane invisible to radar is to coat the plane with an anti reflective polymer. If radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $\mu = 1.5$. How thick is the oil film? Refractive index of the material of airplane wings is greater than the refractive index of polymer.

21. Determine what happens to the double slit interference pattern if one of the slits is covered with a thin, transparent film whose thickness is $\dfrac{\lambda}{2(\mu - 1)}$, where λ is the wavelength of the incident light and μ is the index of refraction of the film.

22. Two slits 4.0×10^{-6} m apart are illuminated by light of wavelength 600 nm. What is the highest order fringe in the interference pattern?

23. Consider an interference experiment using eight equally spaced slits. Determine the smallest phase difference in the waves from adjacent slits such that the resultant wave has zero amplitude.

LEVEL 2

Single Correct Option

1. The intensity of each of the two slits in Young's double slit experiment is I_0. Calculate the minimum separation between the two points on the screen where intensities are $2I_0$ and I_0. Given, the fringe width equal to β.

(a) $\dfrac{\beta}{4}$ (b) $\dfrac{\beta}{3}$ (c) $\dfrac{\beta}{12}$ (d) None of these

2. In Young's double slit experiment, the intensity of light at a point on the screen where path difference is λ is I. If intensity at another point is $I/4$, then possible path differences at this point are

(a) $\lambda/2, \lambda/3$ (b) $\lambda/3, 2\lambda/3$ (c) $\lambda/3, \lambda/4$ (d) $2\lambda/3, \lambda/4$

3. White light is incident normally on a glass plate (in air) of thickness 500 nm and refractive index of 1.5. The wavelength (in nm) in the visible region (400 nm-700 nm) that is strongly reflected by the plate is

(a) 450 (b) 600 (c) 400 (d) 500

4. A double slit of separation 0.1 mm is illuminated by white light. A coloured interference pattern is formed on a screen 100 cm away. If a pin hole is located in this screen at a distance of 2 mm from the central fringe, the wavelengths in the visible spectrum (4000 Å to 7000 Å) which will be absent in the light transmitted through the pin hole is (are)

(a) 4000 Å
(b) 5000 Å
(c) 6000 Å
(d) 7000 Å

5. In a YDSE experiment, $d = 1\text{mm}$, $\lambda = 6000$ Å and $D = 1$ m. The minimum distance between two points on screen having 75% intensity of the maximum intensity will be

(a) 0.50 mm
(b) 0.40 mm
(c) 0.30 mm
(d) 0.20 mm

6. The central fringe of the interference pattern produced by the light of wavelength 6000Å is found to shift to the position of 4th dark fringe after a glass sheet of refractive index 1.5 is introduced. The thickness of glass sheet would be

(a) 4.8 μm
(b) 4.2 μm
(c) 5.4 μm
(d) 3.0 μm

7. Let S_1 and S_2 be the two slits in Young's double slit experiment . If central maxima is observed at P and angle $S_1 P S_2 = \theta$, (θ is small) find the y-coordinates of the 3rd minima assuming the origin at the central maxima . ($\lambda =$ wavelength of monochromatic light used).

(a) $\pm \frac{2\lambda}{\theta}$
(b) $\pm \frac{5\lambda}{2\theta}$
(c) $\pm \frac{5}{2}\lambda\theta$
(d) $\pm 2\lambda\theta$

8. Two monochromatic (wavelength $= a/5$) and coherent sources of electromagnetic waves are placed on the x-axis at the points $(2a, 0)$ and $(-a, 0)$. A detector moves in a circle of radius $R(>>2a)$ whose centre is at the origin. The number of maxima detected during one circular revolution by the detector are

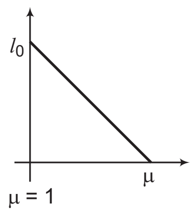
(a) 60
(b) 15
(c) 64
(d) None of these

9. The ratio of the intensity at the centre of a bright fringe to the intensity at a point one quarter of the fringe width from the centre is

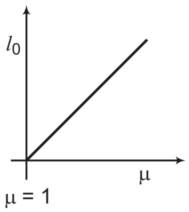
(a) 1
(b) $\frac{1}{2}$
(c) $\frac{3}{4}$
(d) $\frac{1}{4}$

10. In YDSE if a slab whose refractive index can be varied is placed in front of one of the slits . Then, the variation of resultant intensity at mid-point of screen with μ will be best represented by (μ is greater than or equal to 1)

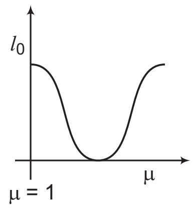
(a)



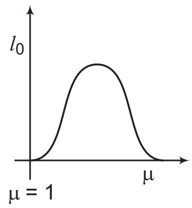
(b)



(c)



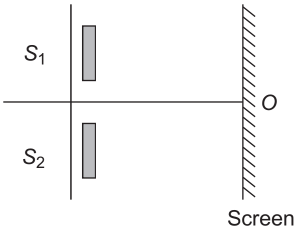
(d)



11. In YDSE, both slits produce equal intensities on the screen. A 100% transparent thin film is placed in front of one of the slits. Now, the intensity on the centre becomes 75% of the previous intensity. The wavelength of light is 6000 Å and refractive index of glass is 1.5. The minimum thickness of the glass slab is

(a) 0.2 μm (b) 0.3 μm
(c) 0.4 μm (d) 0.5 μm

12. YDSE is carried with two thin sheets of thickness 10.4 μm each and refractive index $\mu_1 = 1.52$ and $\mu_2 = 1.40$ covering the slits S_1 and S_2, respectively. If white light of range 400 nm to 780 nm is used, then which wavelength will form maxima exactly at point O, the centre of the screen



(a) 416 nm only (b) 624 nm only
(c) 416 nm and 624 nm only (d) None of these

More than One Correct Options

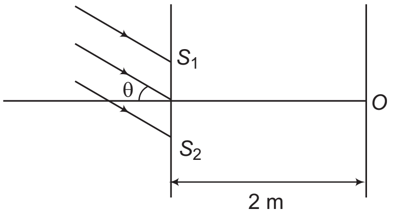
1. A Young's double slit experiment is performed with white light, then

(a) the fringe next to the central will be red
(b) the central fringe will be white
(c) the fringe next to the central will be violet
(d) there will not be a completely dark fringe

2. If one of the slit of a standard Young's double slit experiment is covered by a thin parallel sided glass slab so that it transmits only one-half the light intensity of the other, then

(a) the fringe pattern will get shifted towards the covered slit
(b) the fringe pattern will get shifted away from the covered slit
(c) the bright fringes will be less bright and the dark ones will be more bright
(d) the fringe width will remain unchanged

3. A parallel beam of light ($\lambda = 5000$ Å) is incident at an angle $\theta = 30°$ with the normal to the slit plane in a Young's double slit experiment. The intensity due to each slit is I_0. Point O is equidistant from S_1 and S_2. The distance between slits is 1 mm.

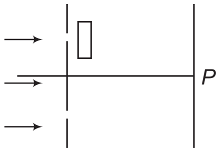


(a) The intensity at O is $4I_0$
(b) The intensity at O is zero
(c) The intensity at a point on the screen 4 mm above O is $4I_0$
(d) The intensity at a point on the screen 4 mm above O is zero

4. In the phenomenon of interference,

(a) sources must be coherent
(b) amplitudes must be same
(c) wavelengths must be same
(d) intensities may be different

5. In YDSE set up shown in figure,



(a) zero order maxima will lie above point P
(b) first order maxima may lie above point P
(c) first order maxima may lie below point P
(d) zero order maxima may lie at point P

6. Bichromatic light of wavelengths $\lambda_1 = 5000$ Å and $\lambda_2 = 7000$ Å are used in YDSE. Then,
(a) 14th order maxima of λ_1 will coincide with 10th order maxima of λ_2
(b) 21st order maxima of λ_2 will coincide with 15th order maxima of λ_1
(c) 11th order minima of λ_1 will coincide with 8th order minima of λ_2
(d) 3rd order minima of λ_1 will coincide with 4th order minima of λ_2

Comprehension Based Questions

Passage : (Q. No. 1 to 4)

A Young's double slit apparatus is immersed in a liquid of refractive index 1.33. *It has slit separation of* 1 *mm and interference pattern is observed on the screen at a distance* 1.33 *m from plane of slits. The wavelength in air is* 6300 *Å.*

1. Calculate the fringe width.
(a) 0.63 mm (b) 1.26 mm
(c) 1.67 mm (d) 2.2 mm

2. Find the distance of seventh bright fringe from third bright fringe lying on the same side of central bright fringe .
(a) 2.52 mm (b) 4.41 mm
(c) 1.89 mm (d) 1.26 mm

3. One of the slits of the apparatus is covered by a thin glass sheet of refractive index 1.53. Find the smallest thickness of the sheet to interchange the position of minima and maxima.
(a) 2.57 mm (b) 1.57 mm
(c) 3.27 mm (d) 4.18 mm

4. One of the slits of the apparatus is covered by a thin glass sheet of refractive index 1.53. Find the fringe width
(a) 0.63 mm (b) 1.26 mm
(c) 1.67 mm (d) 2.2 mm

Match the Columns

1. Two waves from coherent sources meet at a point in a phase difference of ϕ. Both the waves have same intensities. Match the following two columns.

Column I	Column II
(a) If $\phi = 60°$	(p) Resultant intensity will become four times
(b) If $\phi = 90°$	(q) Resultant intensity will become two times
(c) If $\phi = 0°$	(r) Resultant intensity will remain unchanged
(d) If $\phi = 120°$	(s) Resultant intensity will become three times

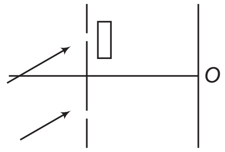
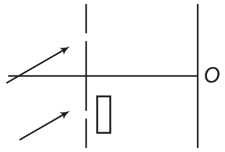
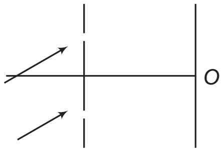
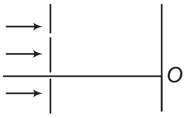
2. Two waves from coherent sources meet at a point in a path difference of Δx. Both the waves have same intensities. Match the following two columns.

Column I	Column II
(a) If $\Delta x = \lambda/3$	(p) Resultant intensity will become three times
(b) If $\Delta x = \lambda/6$	(q) Resultant intensity will remain same
(c) If $\Delta x = \lambda/4$	(r) Resultant intensity will become two times
(d) If $\Delta x = \lambda/2$	(s) Resultant intensity will become zero.

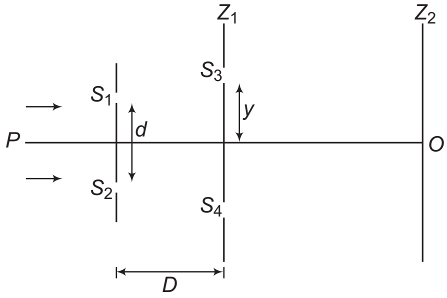
3. In terms of fringe width ω, match the following two columns.

Column I	Column II
(a) Distance between central maxima and third order maxima	(p) $2.5\ \omega$
(b) Distance between central maxima and third order minima	(q) $3.0\ \omega$
(c) Distance between first minima and fourth order maxima	(r) $3.5\ \omega$
(d) Distance between second order maxima and fifth order minima	(s) None of these

4. Match the following two columns.

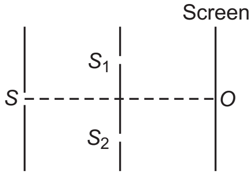
Column I	Column II
(a) 	(p) The zero order maxima will lie above point O
(b) 	(q) The zero order maxima will lie below point O
(c) 	(r) The zero order maxima may lie above or below point O
(d) 	(s) The zero order maxima may lie at point O

5. In the figure shown, Z_1 and Z_2 are two screens. Line PO is the bisector line of S_1S_2 and S_3S_4. When Z_1 is removed, resultant intensity at O due to slits S_1 and S_2 is I. Now, Z_1 is placed. For different values of y given in Column I, match the resultant intensity at O given in Column II.



Column I	Column II
(a) $y = \frac{\lambda D}{2d}$	(p) $3I$
(b) $y = \frac{\lambda D}{6d}$	(q) zero
(c) $y = \frac{\lambda D}{4d}$	(r) I
(d) $y = \frac{\lambda D}{3d}$	(s) None of these

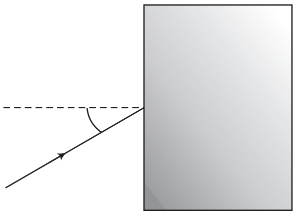
6. Figure shows a set up to perform Young's double slit experiment. A monochromatic source of light is placed at S, S_1 and S_2 and act as coherent sources and interference pattern is obtained on the screen.



Column I	Column II
(a) A thin transparent plate is placed in front of S_1.	(p) Interference fringes disappear
(b) S_1 is closed.	(q) There is a uniform illumination on a large part of the screen
(c) A thin transparent plate is placed in front of S_2.	(r) The zero order fringe will not form at O
(d) S is removed and two different sources emitting light of same wavelength are placed at S_1 and S_2.	(s) Central maxima is formed below O

Subjective Questions

1. A ray of light is incident on the left vertical face of the glass slab. If the incident light has an intensity I and on each reflection the intensity decreases by 90% and on each refraction the intensity decreases by 10%, find the ratio of the intensities of maximum to minimum in reflected pattern.

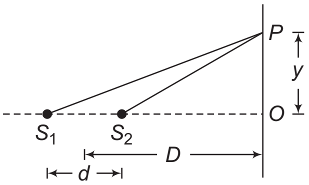


2. A parallel beam of white light falls on a thin film whose refractive index is equal to $4/3$. The angle of incidence $i = 53°$. What must be the minimum film thickness if the reflected light is to be coloured yellow (λ of yellow $= 0.6\,\mu$m) most intensively? ($\tan 53° = 4/3$)

3. A convergent lens with a focal length of $f = 10$ cm is cut into two halves that are then moved apart to a distance of $d = 0.5$ mm (a double lens). Find the fringe width on screen at a distance of 60 cm behind the lens if a point source of monochromatic light ($\lambda = 5000$ Å) is placed in front of the lens at a distance of $a = 15$ cm from it.

4. Two coherent radio point sources that are separated by 2.0 m are radiating in phase with a wavelength of 0.25 m. If a detector moves in a large circle around their mid-point. At how many points will the detector show a maximum signal?

5. In the figure shown, a screen is placed normal to the line joining the two point coherent sources S_1 and S_2. The interference pattern consists of concentric circles.

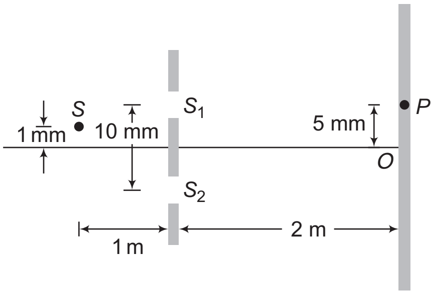


(a) Find the radius of the nth bright ring.

(b) If $d = 0.5$ mm, $\lambda = 5000$ Å and $D = 100$ cm, find the radius of the closest second bright ring.

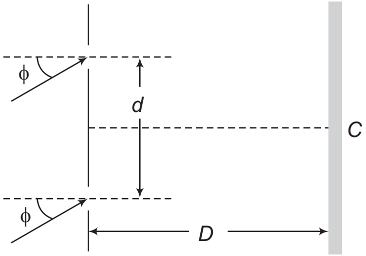
(c) Also, find the value of n for this ring.

6. In the Young's double slit experiment, a point source of $\lambda = 5000$ Å is placed slightly above the central axis as shown in the figure.



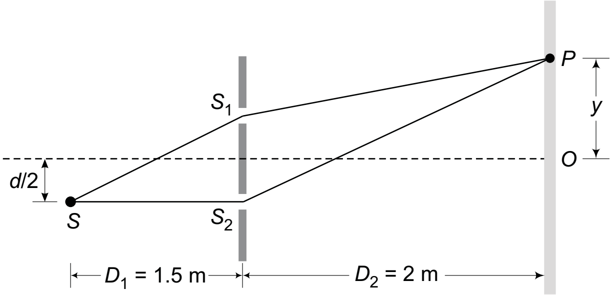
(a) Find the nature and order of the interference at the point P.
(b) Find the nature and order of the interference at O.
(c) Where should we place a film of refractive index $\mu = 1.5$ and what should be its thickness so that maxima of zero order is obtained at O.

7. Light of wavelength $\lambda = 500$ nm falls on two narrow slits placed a distance $d = 50 \times 10^{-4}$ cm apart, at an angle $\phi = 30°$ relative to the slits as shown in figure. On the lower slit a transparent slab of thickness 0.1 mm and refractive index $\frac{3}{2}$ is placed. The interference pattern is observed at a distance $D = 2$ m from the slits. Then, calculate



(a) position of the central maxima.
(b) the order of maxima at point C of screen.
(c) how many fringes will pass C, if we remove the transparent slab from the lower slit?

8. In the YDSE, the monochromatic source of wavelength λ is placed at a distance $\frac{d}{2}$ from the central axis (as shown in the figure), where d is the separation between the two slits S_1 and S_2.

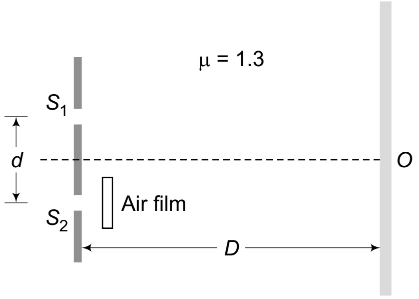


(a) Find the position of the central maxima.
(b) Find the order of interference formed at O.
(c) Now, S is placed on centre dotted line. Find the minimum thickness of the film of refractive index $\mu = 1.5$ to be placed in front of S_2 so that intensity at O becomes $\frac{3}{4}$ th of the maximum intensity.

(Take $\lambda = 6000$ Å ; $d = 6$ mm.)

9. YDSE is carried out in a liquid of refractive index $\mu = 1.3$ and a thin film of air is formed in front of the lower slit as shown in the figure. If a maxima of third order is formed at the origin O, find the thickness of the air film. Find the positions of the fourth maxima.

The wavelength of light in air is $\lambda_0 = 0.78\,\mu\text{m}$ and $D/d = 1000$.



Answers

Introductory Exercise 32.1

1. 49 : 1 **2.** (a) 4 : 1 (b) 16 : 1 **3.** 25 I_0

Introductory Exercise 32.2

1. Because they are incoherent **2.** Because they are highly coherent **3.** 48.0 μm
5. (a) 0.75 I_0 (b) 80 nm **6.** (i) 0° (ii) 90° (iii) 120° (iv) 180°

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (a) **6.** (c) **7.** (d) **8.** (c) **9.** (b) **10.** (d)

Objective Questions

1. (b) **2.** (d) **3.** (b) **4.** (a) **5.** (a) **6.** (c) **7.** (a) **8.** (a) **9.** (a) **10.** (a)
11. (b) **12.** (a) **13.** (a)

Subjective Questions

1. 16 : 1 **2.** (a) 2.50 m (b) 1.0 m, 4.0 m
3. 0.75 m, 2.0 m, 3.25 m, 4.50 m, 5.75 m, 7.0 m, 8.25 m **4.** 590 nm **5.** 0.014°
6. 1.0 mm **7.** 0.072 mm **8.** 0.83 mm **9.** 1200 nm **10.** 12.6 cm **11.** 36.6 cm
12. (a) $I_p = 3.0 \times 10^{-4}\, I_0$ (b) Five **13.** 3 mm **14.** 589 nm **15.** 2.5 mm **16.** 114 nm
17. (a) 424 nm, 594 nm (b) 495 nm **18.** 1154 Å **19.** 533 nm **20.** 0.5 cm
21. Bright and dark fringes interchange positions **22.** 6 **23.** 45°

LEVEL 2

Single Correct Option

1. (c) **2.** (b) **3.** (b) **4.** (a) **5.** (d) **6.** (b) **7.** (b) **8.** (a) **9.** (b) **10.** (c)
11. (a) **12.** (c)

More than One Correct Options

1.(b,c,d) **2.**(a,c,d) **3.**(a,c) **4.**(a,c,d) **5.**(a,b,c) **6.**(a,c)

Comprehension Based Questions

1. (a) **2.** (a) **3.** (b) **4.** (a)

Match the Columns

1. (a) → s (b) → q (c) → p (d) → r
2. (a) → q (b) → p (c) → r (d) → s
3. (a) → q (b) → p (c) → r (d) → p
4. (a) → p (b) → r,s (c) → p (d) → p
5. (a) → q (b) → p (c) → s (d) → r
6. (a) → r (b) → p,q (c) → r,s (d) → p,q

Subjective Questions

1. 361 **2.** 0.14 μm **3.** 0.1 mm
4. 32
5. (a) $D\sqrt{2\left(1-\frac{n\lambda}{d}\right)}$ (b) 6.32 cm (c) 998
6. (a) 70th order maxima (b) 20th order maxima (c) $t = 20\ \mu\text{m}$, in front of S_1
7. (a) At $\theta = 30°$ below C (b) 50 (c) 100
8. (a) 4 mm above O (b) 20 (c) 2000 Å
9. (a) 7.8 μm (b) 4.2 mm, – 0.6 mm

33

Modern Physics-I

Chapter Contents

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33.1 Dual Nature of Electromagnetic Waves

Classical physics treats particles and waves as separate components. The mechanics of particles and the optics of waves are traditionally independent disciplines, each with its own chain of principles based on their results. We regard electrons as particles because they possess charge and mass and behave according to the laws of particle mechanics in such familiar devices as television picture tubes. We shall see, however, that it is just as correct to interpret a moving electron as a wave manifestation as it is to interpret it as a particle manifestation. We regard electromagnetic waves as waves because under suitable circumstances they exhibit diffraction, interference and polarization. Similarly, we shall see that under other circumstances they behave as a stream of particles. Rather we can say that they have the **dual nature.**

The wave nature of light (a part of electromagnetic waves) was first demonstrated by Thomas Young, who observed the interference pattern of two coherent sources. The particle nature of light was first proposed by Albert Einstein in 1905 in his explanation of the photoelectric effect. A particle of light called a **photon** has energy E that is related to the frequency f and wavelength λ of light wave by the Einstein equation,

$$\boxed{E = hf = \frac{hc}{\lambda}} \qquad \text{...(i)}$$

where, c is the speed of light (in vacuum) and h is Planck's constant.

$$h = 6.626 \times 10^{-34} \text{ J-s}$$
$$= 4.136 \times 10^{-15} \text{ eV-s}$$

Since, energies are often given in electron volt $(1 \text{ eV} = 1.6 \times 10^{-19} \text{ J})$ and wavelengths are in Å, it is convenient to the combination hc in eV-Å. We have,

$$hc = 12375 \text{ eV-Å}$$

Hence, Eq. (i) in simpler form can be written as

$$\boxed{E \text{ (in eV)} = \frac{12375}{\lambda \text{ (in Å)}}} \qquad \text{...(ii)}$$

The propagation of light is governed by its wave properties, whereas the exchange of energy between light with matter is governed by its particle properties. The wave particle duality is a general property of nature. For example, electrons (and other so called particles) also propagate as waves and exchange energy as particles.

33.2 Electromagnetic Spectrum

The basic source of electromagnetic wave is an accelerated charge. This produces the changing electric and magnetic fields which constitute an electromagnetic wave. An electromagnetic wave may have its wavelength varying from zero to infinity. Not all of them are known till date. Today we are familiar with electromagnetic waves having wavelengths as small as 30 fm $(1 \text{ fm} = 10^{-15} \text{ m})$ to as large as 30 km. The boundaries separating different regions of spectrum are not sharply defined, with the exception of the visible part of the spectrum. The visible part of the electromagnetic spectrum covers from 4000 Å to 7000 Å. An approximate range of wavelengths is associated with each colour :

violet (4000 Å– 4500 Å), blue (4500 Å–5200 Å), green (5200 Å–5600 Å), yellow (5600 Å–6000 Å), orange (6000 Å– 6250 Å) and red (6250 Å–7000 Å).

Figure shows the spectrum of electromagnetic waves. The classification is based roughly on how the waves are produced and or detected.

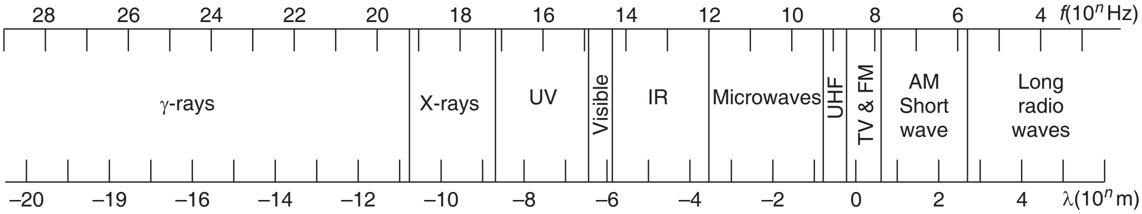


Fig. 33.1. The electromagnetic spectrum

γ-Rays These were identified by P. Villiard in 1900. These are usually produced within the nucleus of an atom and extremely energetic by atomic standards. They cover the range from 0.1 Å down or equivalently from 10^{20} Hz up.

X-Rays X-rays, discovered in 1895 by W. Roentgen extend from 100 Å to 0.1 Å. These are produced by the rapid deceleration of electrons that bombard a heavy metal target. These are also produced by electronic transitions between the energy levels in an atom. X-rays are used to study the atomic structure of crystals or molecules such as DNA. Besides their diagnostic and therapeutic use in medicine they have become an important tool in studying the universe.

Ultraviolet radiation Ultraviolet (UV) rays were first discovered by J.W. Ritter in 1801. The ultraviolet region extends from 4000 Å to 100 Å. It plays a role in the production of vitamin D in our skins. But prolonged doses of UV radiation can induce cancers in humans. Glass absorbs UV radiation and hence, can provide some protection against the sun's rays. If the ozone in our atmosphere did not absorb the UV below 3000 Å, there would be a large number of cell mutations, especially cancerous ones, in humans. For this reason, the depletion of the ozone in our atmosphere by chlorofluorocarbons (CFCs) is now a matter of international concern.

Visible light A lot of discussion has already been done on visible light in previous chapters. As electrons undergo transitions between energy levels in an atom, light is produced at well defined wavelengths. Light covering a continuous range of wavelengths is produced by the random acceleration of electrons in hot bodies. Our sense of vision and the process of photosynthesis in plants have evolved within the range of those wavelengths of sunlight that our atmosphere does not absorb.

Infrared radiation The infrared region (IR) starts at 7000 Å and extends to about 1 mm. It was discovered in 1800 by M. Herchel. It is associated with the vibration and rotation of molecules and is perceived by us as heat. IR is used in the early detection of tumours.

Microwaves Microwaves cover wavelengths from 1 mm to about 15 cm. Microwaves upto about 30 GHz (1 cm) may be generated by the oscillations of electrons in a device called klystron. Microwave ovens are used in kitchens. Modern intercity communications such as phone conversations and TV programs are often carried *via* a cross country network of microwave antennas.

Radio and TV signals Radio waves are generated when charges are accelerating through conducting wires. Their wavelengths lie in the range 10^{14} m to 10 cm. They are generated by *LC* oscillators and are used in radio and television communication systems.

33.3 Momentum and Radiation Pressure

An electromagnetic wave transports linear momentum. We state, without proof that the linear momentum carried by an electromagnetic wave is related to the energy it transports according to

$$\boxed{p = \frac{E}{c}} \qquad \text{...(i)}$$

If the wave is incident in the direction perpendicular to a surface and is completely absorbed, then Eq. (i) tells us the linear momentum imparted to the surface. If surface is perfectly reflecting, the momentum change of the wave is doubled. Consequently, the momentum imparted to the surface is also doubled.

According to Newton's second law, the force exerted by an electromagnetic wave on a surface may be related by the equation

$$F = \frac{\Delta p}{\Delta t}$$

From Eq. (i),

$$\frac{\Delta p}{\Delta t} = \frac{1}{c}\left(\frac{\Delta E}{\Delta t}\right)$$

∴

$$F = \frac{1}{c}\left(\frac{\Delta E}{\Delta t}\right) \qquad \text{...(ii)}$$

Intensity (I) of a wave is the energy transported per unit area per unit time.

or

$$I = \left(\frac{1}{S}\right)\frac{\Delta E}{\Delta t}$$

∴

$$\frac{\Delta E}{\Delta t} = IS$$

Substituting in Eq. (ii),

$$F = \frac{IS}{c}$$

or

$$\frac{F}{S} = \text{pressure} = \frac{I}{c}$$

or

$$p_{\text{rad}} = \frac{I}{c}$$

$\frac{I}{c}$ is also equal to the energy density (energy per unit volume) u.

Hence,

$$\boxed{p_{\text{rad}} = u} \qquad \text{...(iii)}$$

The radiation pressure is thus equal to the energy density ($\text{N/m}^2 = \text{J/m}^3$). At a perfectly reflecting surface, the pressure on the surface is doubled. Thus, we can write

$$p_{\text{rad}} = \frac{I}{c} = u \qquad \text{(wave totally absorbed)}$$

and

$$p_{\text{rad}} = \frac{2I}{c} = 2u \qquad \text{(wave totally reflected)}$$

Example 33.1 *The intensity of direct sunlight before it passes through the earth's atmosphere is* $1.4\ kW/m^2$. *If it is completely absorbed, find the corresponding radiation pressure.*

Solution For completely absorbing surface,

$$p_{\text{rad}} = \frac{I}{c} = \frac{1.4 \times 10^3}{3.0 \times 10^8}$$

$$= 4.7 \times 10^{-6}\ \text{Nm}^{-2}$$ **Ans.**

33.4 de-Broglie Wavelength of Matter Wave

The wave particle nature of electromagnetic waves discussed in article 33.1, led de-Broglie (pronounced de Broy) to suggest that matter might also exhibit this duality and have wave properties. His ideas can be expressed quantitatively by first considering electromagnetic radiation. A photon of frequency f and wavelength λ has energy.

$$E = hf = \frac{hc}{\lambda}$$

By Einstein's energy mass relation, $E = mc^2$ the equivalent mass m of the photon is given by

$$m = \frac{E}{c^2} = \frac{hf}{c^2} = \frac{h}{\lambda c} \qquad \ldots\text{(i)}$$

or

$$\lambda = \frac{h}{mc} \quad \text{or} \quad \frac{h}{p} \qquad \ldots\text{(ii)}$$

Here, p is the momentum of photon. By analogy de-Broglie suggested that a particle of mass m moving with speed v behaves in some ways like waves of wavelength λ given by

$$\boxed{\lambda = \frac{h}{mv} = \frac{h}{p}} \qquad \ldots\text{(iii)}$$

where, p is the momentum of the particle. Momentum is related to the kinetic energy by the equation,

$$p = \sqrt{2Km}$$

and a charge q when accelerated by a potential difference V gains a kinetic energy $K = qV$. Combining all these relations Eq. (iii) can be written as

$$\boxed{\lambda = \frac{h}{mv} = \frac{h}{p} = \frac{h}{\sqrt{2Km}} = \frac{h}{\sqrt{2qVm}} \quad \text{(de-Broglie wavelength)}} \qquad \ldots\text{(iv)}$$

de-Broglie Wavelength for an Electron

If an electron (charge = e) is accelerated by a potential difference of V volts, it acquires a kinetic energy,

$$K = eV$$

Substituting the values of h, m and q in Eq. (iv), we get a simple formula for calculating de-Broglie wavelength of an electron. This is

$$\boxed{\lambda\ (\text{in Å}) = \sqrt{\frac{150}{V\ (\text{in volts})}}} \qquad \ldots(\text{v})$$

Note *If an electron is accelerated by 1 volt, then its kinetic energy becomes 1 eV. Therefore, the above formula can also be written as*

$$\boxed{\lambda\ (\text{in Å}) = \sqrt{\frac{150}{KE(\text{in eV})}}}$$

Example 33.2 *An electron is accelerated by a potential difference of 25 volt. Find the de-Broglie wavelength associated with it.*

Solution For an electron, de-Broglie wavelength is given by

$$\lambda = \sqrt{\frac{150}{V}} = \sqrt{\frac{150}{25}}$$

$$= \sqrt{6} \approx 2.5\ \text{Å}$$ **Ans.**

Example 33.3 *A particle of mass M at rest decays into two particles of masses* m_1 and m_2 *having non-zero velocities. The ratio of the de-Broglie wavelengths of the particles* λ_1/λ_2 *is* **(JEE 1999)**

(a) m_1/m_2 (b) m_2/m_1 (c) 1 (d) $\sqrt{m_2}/\sqrt{m_1}$

Solution From the law of conservation of momentum,

$$p_1 = p_2 \qquad (\text{ in opposite directions})$$

Now, de-Broglie wavelength is given by

$$\lambda = \frac{h}{p}, \text{ where } \quad h = \text{Planck constant}$$

Since magnitude of momentum (p) of both the particles is equal, therefore $\lambda_1 = \lambda_2$

or $$\lambda_1/\lambda_2 = 1$$

Therefore, the correct option is (c).

Example 33.4 *The energy of a photon is equal to the kinetic energy of a proton. The energy of the photon is E. Let* λ_1 *be the de-Broglie wavelength of the proton and* λ_2 *be the wavelength of the photon. The ratio* $\frac{\lambda_1}{\lambda_2}$ *is proportional to*

(a) E^0 (b) $E^{1/2}$ (c) E^{-1} (d) E^{-2} **(JEE 2004)**

Solution $$\frac{\lambda_1}{\lambda_2} = \frac{\dfrac{h}{\sqrt{2mE}}}{\dfrac{hc}{E}} \quad \text{or} \quad \frac{\lambda_1}{\lambda_2} \propto E^{1/2}$$

Therefore, the correct option is (b).

Example 33.5 *An α-particle and a proton are accelerated from rest by a potential difference of 100 V. After this, their de-Broglie wavelengths are λ_α and λ_p respectively. The ratio $\frac{\lambda_p}{\lambda_\alpha}$, to the nearest integer, is* **(JEE 2010)**

Solution $\because \quad \lambda = \frac{h}{p} = \frac{h}{\sqrt{2qVm}}$

or

$$\lambda \propto \frac{1}{\sqrt{qm}}$$

$$\frac{\lambda_p}{\lambda_\alpha} = \sqrt{\frac{q_\alpha}{q_p} \cdot \frac{m_\alpha}{m_p}}$$

$$= \sqrt{\frac{(2)(4)}{(1)(1)}} = 2.828$$

The nearest integer is 3.

$\therefore$ Answer is 3.

Example 33.6 *The potential energy of a particle varies as*

$$U(x) = E_0 \quad \textit{for} \quad 0 \le x \le 1$$
$$= 0 \quad \textit{for} \quad x > 1$$

For $0 \le x \le 1$, de-Broglie wavelength is λ_1 and for $x > 1$ the de-Broglie wavelength is λ_2. Total energy of the particle is $2E_0$. Find $\frac{\lambda_1}{\lambda_2}$. **(JEE 2005)**

Solution For $0 \le x \le 1$, PE $= E_0$

$\therefore$ Kinetic energy K_1 = Total energy – PE

$$= 2E_0 - E_0 = E_0$$

$\therefore$
$$\lambda_1 = \frac{h}{\sqrt{2mE_0}} \qquad \text{...(i)}$$

For $x > 1$, PE $= 0$

$\therefore$ Kinetic energy K_2 = Total energy $= 2E_0$

$\therefore$
$$\lambda_2 = \frac{h}{\sqrt{4mE_0}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\frac{\lambda_1}{\lambda_2} = \sqrt{2}$$

INTRODUCTORY EXERCISE 33.1

1. Find the energy and momentum of a photon of ultraviolet radiation of 280 nm wavelength.
2. A small plate of a metal is placed at a distance of 2 m from a monochromatic light source of wavelength 4.8×10^{-7} m and power 1.0 Watt. The light falls normally on the plate. Find the number of photons striking the metal plate per square metre per second.
3. A proton and a deuteron are accelerated by same potential difference. Find the ratio of their de-Broglie wavelengths.
4. A deuteron and an α-particle have same kinetic energy. Find the ratio of their de-Broglie wavelengths.
5. Two protons are having same kinetic energy. One proton enters a uniform magnetic field at right angles to it. Second proton enters a uniform electric field in the direction of field. After some time their de-Broglie wavelengths are λ_1 and λ_2, then
   (a) $\lambda_1 = \lambda_2$ (b) $\lambda_1 < \lambda_2$ (c) $\lambda_1 > \lambda_2$
   (d) some more information is required
6. Find the de-Broglie wavelengths of
   (a) a 46 g golf ball with a velocity of 30 m/s (b) an electron with a velocity of 10^7 m/s.

33.5 Early Atomic Structures

Every atom consists of a small nucleus of protons and neutrons with a number of electrons some distance away.

In the present article and in the next, our chief concern will be the structure of the atom, since it is this structure that is responsible for nearly all the properties of matter.In nineteenth century many models were present by different scientists, but ultimately the first theory of the atom to meet with any success was put forward in 1913 by Neils Bohr. But before studying Bohr's model of atom let us have a look on other two models of the period one presented by J.J. Thomson in 1898 and the other by Ernest Rutherford in 1911.

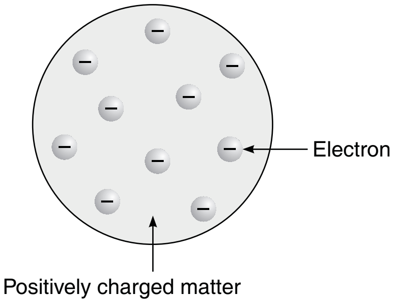


Fig. 33.2 The Thomson model of the atom. The Rutherford scattering experiment showed it to be incorrect.

J.J. Thomson suggested that atoms are just positively charged lumps of matter with electrons embedded in them like raisins in a fruit cake. Thomson's model called the 'plum pudding' model is illustrated in Fig. 33.2.

Thomson had played an important role in discovering the electron, his idea was taken seriously. But, the real atom turned out to be quite different.

Rutherford's Nuclear Atom

The nuclear atom is the basis of the modern theory of atomic structure and was proposed by Rutherford in 1911. He, with his two assistants Geiger and Marsden did an experiment in which they directed a narrow beam of α-particles onto gold foil about $1\ \mu\text{m}$ thick and found that while most of the particles passed straight through, some were scattered appreciably and a very few about 1 in 8000 suffered deflection of more than 90°.

To account for this very surprising result Rutherford suggested that : "All the positive charge and nearly all the mass were concentrated in a very small volume or nucleus at the centre of the atom. The electrons were supposed to move in circular orbits round the nucleus (like planets round the sun). The electrostatic attraction between the two opposite charges being the required centripetal force for such motion.

The large angle scattering of α-particles would then be explained by the strong electrostatic repulsion from the nucleus.

Rutherford's model of the atom, although strongly supported by evidence for the nucleus, is inconsistent with classical physics. An electron moving in a circular orbit round a nucleus is accelerating and according to electromagnetic theory it should therefore, emit radiation continuously and thereby lose energy. If this happened the radius of the orbit would decrease and the electron would spiral into the nucleus in a fraction of second. But atoms do not collapse. In 1913, an effort was made by Neils Bohr to overcome this paradox.

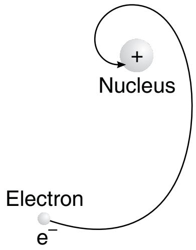


Fig. 33.3 An atomic electron should, classically, spiral rapidly into the nucleus as it radiates energy due to its acceleration

33.6 The Bohr Hydrogen Atom

After Neils Bohr obtained his doctorate in 1911, he worked under Rutherford for a while. In 1913, he presented a model of the hydrogen atom, which has one electron and one proton. He postulated that an electron moves only in certain circular orbits, called stationary orbits. In stationary orbits, electron does not emit radiation, contrary to the predictions of classical electromagnetic theory. According to Bohr, there is a definite energy associated with each stable orbit and an atom radiates energy only when it makes a transition from one of these orbits to another. The energy is radiated in the form of a photon with energy and frequency given by

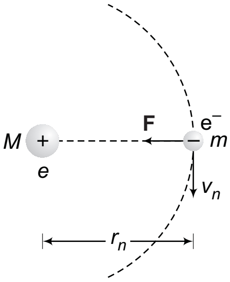


Fig. 33.4

$$\boxed{\Delta E = hf = E_i - E_f} \qquad \ldots(\text{i})$$

Bohr found that the magnitude of the electron's angular momentum is quantised, and this magnitude for the electron must be integral multiple of $\frac{h}{2\pi}$. The magnitude of the angular momentum is $L = mvr$ for a particle with mass m moving with speed v in a circle of radius r. So, according to Bohr's postulate,

$$mvr = \frac{nh}{2\pi} \qquad (n = 1, 2, 3 \ldots)$$

Each value of n corresponds to a permitted value of the orbit radius, which we will denote by r_n and the corresponding speed v_n. The value of n for each orbit is called **principal quantum number** for the orbit. Thus,

$$\boxed{mv_n r_n = \frac{nh}{2\pi}} \qquad \ldots(\text{ii})$$

According to Newton's second law, a radially inward centripetal force of magnitude $F = \frac{mv^2}{r_n}$ is needed to the electron which is being provided by the electrical attraction between the positive proton and the negative electron.

Thus, $$\boxed{\frac{mv_n^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n^2}} \qquad \text{...(iii)}$$

Solving Eqs. (ii) and (iii), we get

$$\boxed{\begin{aligned} r_n &= \frac{\varepsilon_0 n^2 h^2}{\pi m e^2} \quad \begin{pmatrix}n\text{th orbit radius}\\ \text{in Bohr model}\end{pmatrix} && \text{...(iv)}\\ \text{and} \quad v_n &= \frac{e^2}{2\varepsilon_0 n h} \quad \begin{pmatrix}n\text{th orbit speed}\\ \text{in Bohr model}\end{pmatrix} && \text{...(v)}\end{aligned}}$$

The smallest orbit radius corresponds to $n = 1$. We'll denote this minimum radius, called the **Bohr radius** as a_0. Thus,

$$a_0 = \frac{\varepsilon_0 h^2}{\pi m e^2}$$

Substituting values of ε_0, h, π, m and e, we get

$$\boxed{a_0 = 0.529 \times 10^{-10} \text{ m} = 0.529 \text{ Å}} \qquad \text{...(vi)}$$

Eq. (iv), in terms of a_0 can be written as

$$\boxed{r_n = n^2 a_0 \quad \text{or} \quad r_n \propto n^2} \qquad \text{...(vii)}$$

Similarly, substituting values of e, ε_0 and h with $n = 1$ in Eq. (v), we get

$$\boxed{v_1 = 2.19 \times 10^6 \text{ m/s} \approx \frac{c}{137}} \qquad \text{...(viii)}$$

This is the greatest possible speed of the electron in the hydrogen atom. Which is approximately equal to $c/137$, where c is the speed of light in vacuum.

Eq. (v), in terms of v_1 can be written as

$$\boxed{v_n = \frac{v_1}{n} \quad \text{or} \quad v_n \propto \frac{1}{n}} \qquad \text{...(ix)}$$

Energy levels Kinetic and potential energies K_n and U_n in nth orbit are

$$K_n = \frac{1}{2} m v_n^2 = \frac{me^4}{8\varepsilon_0^2 n^2 h^2}$$

and $$U_n = -\frac{1}{4\pi\varepsilon_0}\frac{(e)(e)}{r_n} = -\frac{me^4}{4\varepsilon_0^2 n^2 h^2}$$

The total energy E_n is the sum of the kinetic and potential energies.

$$E_n = K_n + U_n = -\frac{me^4}{8\varepsilon_0{}^2 n^2 h^2}$$

Substituting values of m, e, ε_0 and h with $n = 1$, we get the least energy of the atom in first orbit, which is -13.6 eV. Hence,

$$\boxed{\begin{aligned} E_1 &= -13.6\,\text{eV} \\ E_n &= \frac{E_1}{n^2} = -\frac{13.6}{n^2}\,\text{eV} \end{aligned}} \qquad \begin{aligned} &\ldots\text{(x)} \\ \text{and} \quad &\ldots\text{(xi)} \end{aligned}$$

Substituting $n = 2, 3, 4 \ldots$, etc., we get energies of atom in different orbits.

$$E_2 = -3.40\,\text{eV},\; E_3 = -1.51\,\text{eV}, \ldots E_\infty = 0$$

Ionization energy of the hydrogen atom is the energy required to remove the electron completely. In ground state ($n = 1$), energy of atom is -13.6 eV and energy corresponding to $n = \infty$ is zero. Hence, energy required to remove the electron from ground state is 13.6 eV.

Emission spectrum of hydrogen atom

Under normal conditions the single electron in hydrogen atom stays in ground state ($n = 1$). It is excited to some higher energy state when it acquires some energy from external source. But, it hardly stays there for more than 10^{-8} second.

A photon corresponding to a particular spectrum line is emitted when an atom makes a transition from a state in an excited level to a state in a lower excited level or the ground level.

Let n_i be the initial and n_f the final energy state, then depending on the final energy state following series are observed in the emission spectrum of hydrogen atom.

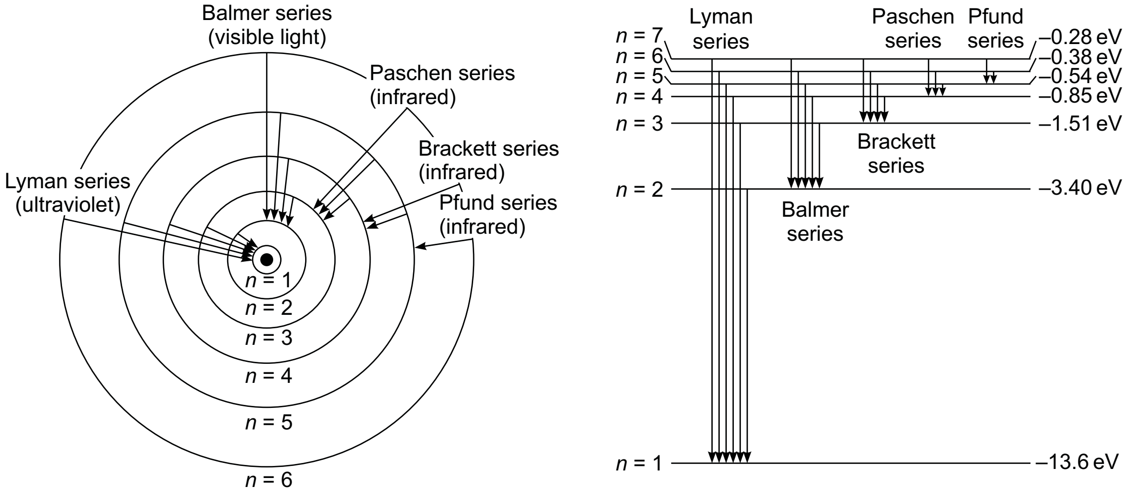


Fig. 33.5

For the Lyman series $n_f = 1$, for Balmer series $n_f = 2$ and so on. The relation of the various spectral series to the energy levels and to electron orbits is shown in figure.

Wavelength of Photon Emitted in De-excitation

According to Bohr when an atom makes a transition from higher energy level to a lower energy level it emits a photon with energy equal to the energy difference between the initial and final levels. If E_i is the initial energy of the atom before such a transition, E_f is its final energy after the transition, and the photon's energy is $hf = \frac{hc}{\lambda}$, then conservation of energy gives

$$\boxed{hf = \frac{hc}{\lambda} = E_i - E_f} \qquad \text{(energy of emitted photon)} \qquad \ldots\text{(xii)}$$

By 1913, the spectrum of hydrogen had been studied intensively. The visible line with longest wavelength, or lowest frequency is in the red and is called H_α, the next line, in the blue-green is called H_β and so on. In 1885, Johann Balmer, a swiss teacher found a formula that gives the wave lengths of these lines. This is now called the Balmer series. The Balmer's formula is

$$\boxed{\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)} \qquad \ldots\text{(xiii)}$$

Here, $n = 3, 4, 5 \ldots$, etc.

$R =$ Rydberg constant $= 1.097 \times 10^7 \text{ m}^{-1}$

and λ is the wavelength of light/photon emitted during transition.

For $n = 3$, we obtain the wavelength of H_α line. Similarly, for $n = 4$, we obtain the wavelength of H_β line. For $n = \infty$, the smallest wavelength ($= 3646$ Å) of this series is obtained. Using the relation $E = \frac{hc}{\lambda}$, we can find the photon energies corresponding to the wavelength of the Balmer series.

Multiplying Eq. (xiii) by hc, we find

$$E = \frac{hc}{\lambda} = hcR\left(\frac{1}{2^2} - \frac{1}{n^2}\right) = \frac{Rhc}{2^2} - \frac{Rhc}{n^2} = E_n - E_2$$

This formula suggests that

$$\boxed{E_n = -\frac{Rhc}{n^2}, \; n = 1, 2, 3 \ldots} \qquad \ldots\text{(xiv)}$$

Comparing this with Eq. (xi) of the same article, we have

$$\boxed{Rhc = 13.60 \text{ eV}} \qquad \ldots\text{(xv)}$$

The wavelengths corresponding to other spectral series (Lyman, Paschen , etc.) can be represented by formula similar to Balmer formula.

Lyman series $\quad \frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right), \; n = 2, 3, 4 \ldots$

Paschen series $\quad \frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right), \; n = 4, 5, 6 \ldots$

Brackett series $\quad \frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right), \; n = 5, 6, 7 \ldots$

Pfund series $$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right), n = 6, 7, 8 \ldots$$

The Lyman series is in the ultraviolet, and the Paschen, Brackett and Pfund series are in the infrared region.

33.7 Hydrogen Like Atoms

The Bohr model of hydrogen can be extended to hydrogen like atoms, i.e. one electron atoms such as singly ionized helium (He^+), doubly ionized lithium (Li^{+2}) and so on. In such atoms, the nuclear charge is $+Ze$, where Z is the atomic number, equal to the number of protons in the nucleus.

The effect in the previous analysis is to replace e^2 everywhere by Ze^2. Thus, the equations for, r_n, v_n and E_n are altered as under

$$r_n = \frac{\varepsilon_0 n^2 h^2}{\pi m Z e^2} = \frac{n^2}{Z} a_0 \quad \text{or} \quad r_n \propto \frac{n^2}{Z} \qquad \ldots\text{(i)}$$

where, $a_0 = 0.529$ Å (radius of first orbit of H)

$$v_n = \frac{Ze^2}{2\varepsilon_0 nh} = \frac{Z}{n} v_1 \quad \text{or} \quad v_n \propto \frac{Z}{n} \qquad \ldots\text{(ii)}$$

where, $v_1 = 2.19 \times 10^6$ m/s (speed of electron in first orbit of H)

$$E_n = -\frac{mZ^2 e^4}{8\varepsilon_0^2 n^2 h^2} = \frac{Z^2}{n^2} E_1 \quad \text{or} \quad E_n \propto \frac{Z^2}{n^2} \qquad \ldots\text{(iii)}$$

where, $E_1 = -13.60$ eV (energy of atom in first orbit of H)

Fig. 33.6 compares the energy levels of H and He^+ which has $Z = 2$. H and He^+ have many spectrum lines that have almost the same wavelengths

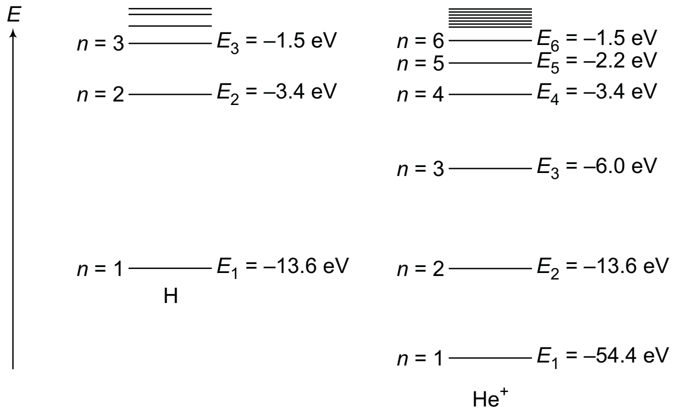


Fig. 33.6 Energy levels of H and He^+. Because of the additional factor Z^2 in the energy expression, the energy of the He^+ ion with a given n is almost exactly four times that of the H-atom with the same n. There are small differences (of the order of 0.05%) because of the different masses.

✓ Extra Points to Remember

- Bohr's theory is applicable for hydrogen and hydrogen like atoms/ions. For such types of atoms/ions number of electron is one. Although, atomic numbers may be different.

  e.g, For ${}_1H^1$, atomic number $Z = 1$, For He^+, atomic number $Z = 2$ and

  For Li^{+2}, atomic number $Z = 3$

  But for all three number of electron is one.

- In nth orbit

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(e)(Ze)}{r^2} \quad \text{and} \quad L_n = mvr = \frac{nh}{2\pi}$$

  After solving these two equations, we will get following results.

  (a) $r \propto \frac{n^2}{Z}$ and $r \propto \frac{1}{m}$

  (b) $v \propto \frac{Z}{n}$ and $v \propto m^0$

  (c) $E \propto \frac{Z^2}{n^2}$ and $E \propto m$

  (d) $r_1^H = 0.529$ Å

  (e) $v_1^H = 2.19 \times 10^6 \text{ m/s} = \frac{c}{137}$

  (f) $E_1^H = -13.6 \text{ eV}$

  (g) $K = |E|$ and $U = 2E$

Note *With the help of above results, we can find any value in any orbit of hydrogen like atoms. In the above expressions, m is the mass of electron*

- Total number of emission lines from some higher energy state n_1 to lower energy state n_2 $(< n_1)$ is given by $\frac{(n_1 - n_2)(n_1 - n_2 + 1)}{2}$.

  For example, total number of lines from $n_1 = n$ to $n_2 = 1$ are $\frac{n(n-1)}{2}$.

- As the principal quantum number n is increased in hydrogen and hydrogen like atoms, some quantities are decreased and some are increased. The table given below shows which quantities are increased and which are decreased.

Table 33.1

Increased	Decreased
Radius	Speed
Potential energy	Kinetic energy
Total energy	Angular speed
Time period	Frequency
Angular momentum	

- Whenever the force obeys inverse square law $\left(F \propto \frac{1}{r^2}\right)$ and potential energy is inversely proportional to $r\left(U \propto \frac{1}{r}\right)$, kinetic energy (K), potential energy (U) and total energy (E) have the following relationships.

$$\boxed{K = \frac{|U|}{2} \quad \text{and} \quad E = -K = \frac{U}{2}}$$

  If force is not proportional to $\frac{1}{r^2}$ or potential energy is not proportional to $\frac{1}{r}$, the above relations do not hold good.

- Total energy of a closed system is always negative and the modulus of this is the binding energy of the system. For instance, suppose a system has a total energy of –100 J. It means that this system will separate if 100 J of energy is supplied to this. Hence, binding energy of this system is 100 J. Thus, total energy of an open system is either zero or greater than zero.
- Kinetic energy of a particle can't be negative, while the potential energy can be zero, positive or negative. It basically depends on the reference point where we have taken it zero. It is customary to take zero potential energy when the electron is at infinite distance from the nucleus. In some problem, suppose we take zero potential energy in first orbit ($U_1 = 0$), then the modulus of actual potential energy in first orbit (when reference point was at infinity) is added in U and E in all energy states, while K remains unchanged. See sample example number 33.11.
- In the transition from n_2 to $n_1 (< n_1)$. The wavelength of emitted photon can be given by the following shortcut formula,

$$\boxed{\lambda \text{ (in Å)} = \frac{12375}{E_{n_2} - E_{n_1}}}$$

where, E_{n_1} and E_{n_2} are in eV. In general, E_n in eV is given by

$$\boxed{E_n = -(13.6)\frac{Z^2}{n^2}}$$

- In hydrogen emission spectrum, Balmer series was first discovered as it lies in visible light.

Example 33.7 *Using the known values for hydrogen atom, calculate*

(a) radius of third orbit for Li^{+2}

(b) speed of electron in fourth orbit for He^+

(c) angular momentum of electron in 3rd orbit of He^+

Solution (a) $Z = 3$ for Li^{+2}. Further we know that $r_n = \frac{n^2}{Z} a_0$

Substituting, $n = 3$, $Z = 3$ and $a_0 = 0.529$ Å

We have r_3 for $\text{Li}^{+2} = \frac{(3)^2}{(3)}(0.529)$ Å $= 1.587$ Å **Ans.**

(b) $Z = 2$ for He^+. Also we know that

$$v_n = \frac{Z}{n} v_1$$

Substituting, $n = 4$, $Z = 2$ and $v_1 = 2.19 \times 10^6$ m/s

We get, v_4 for $\text{He}^+ = \left(\frac{2}{4}\right)(2.19 \times 10^6)$ m/s

$= 1.095 \times 10^6$ m/s **Ans.**

(c) $L_n = n\left(\frac{h}{2\pi}\right)$

For $n = 3$, $L_3 = 3\left(\frac{h}{2\pi}\right)$ **Ans.**

Note *This result is independent of value of Z.*

Example 33.8 *A doubly ionized lithium atom is hydrogen like with atomic number 3. Find the wavelength of the radiation required to excite the electron in Li^{++} from the first to the third Bohr orbit. The ionization energy of the hydrogen atom is 13.6 eV.*

Solution $\because\ E_n = -\dfrac{Z^2}{n^2}(13.6\,\text{eV})$

By putting $Z = 3$, we have
$$E_n = -\frac{122.4}{n^2}\ \text{eV}$$
$$E_1 = -\frac{122.4}{(1)^2} = -122.4\ \text{eV}$$

and
$$E_3 = -\frac{122.4}{(3)^2} = -13.6\ \text{eV}$$

$\therefore$
$$\Delta E = E_3 - E_1 = 108.8\ \text{eV}$$

The corresponding wavelength is
$$\lambda = \frac{12375}{\Delta E\ (\text{in eV})}\ \text{Å} = \frac{12375}{108.8}\ \text{Å}$$
$$= 113.74\ \text{Å}$$ **Ans.**

Example 33.9 *Find variation of angular speed and time period of single electron of hydrogen like atoms with n and Z.*

Solution Angular speed,
$$\omega = \frac{v}{r}$$

Now,
$$v \propto \frac{Z}{n} \quad \text{and} \quad r \propto \frac{n^2}{Z}$$

$\therefore$
$$\omega \propto \frac{(Z/n)}{(n^2/Z)} \quad \text{or} \quad \omega \propto \frac{Z^2}{n^3}$$ **Ans.**

Time period,
$$T = \frac{2\pi}{\omega} \quad \text{or} \quad T \propto \frac{1}{\omega}$$

$\therefore$
$$T \propto \frac{n^3}{Z^2}$$ **Ans.**

Example 33.10 *Find kinetic energy, electrostatic potential energy and total energy of single electron in 2nd excited state of Li^{+2} atom.*

Solution $\because\ E_I^H = -13.6$ eV

Further, $E \propto \dfrac{Z^2}{n^2}$, For Li^{+2}, $Z = 3$

and for 2nd excited state $n = 3$

$\therefore$ $$E = -13.6\left(\frac{3}{3}\right)^2 = -13.6 \text{ eV}$$ **Ans.**

$$K = |E| = 13.6 \text{ eV}$$ **Ans.**

$$U = 2E = -27.2 \text{ eV}$$ **Ans.**

Note *In the above expressions E is the total energy, K is the kinetic energy and U is the potential energy.*

Example 33.11 *Find the kinetic energy, potential energy and total energy in first and second orbit of hydrogen atom if potential energy in first orbit is taken to be zero.*

Solution $E_1 = -13.60\,\text{eV}$, $K_1 = -E_1 = 13.60\,\text{eV}$, $U_1 = 2E_1 = -27.20\,\text{eV}$

$E_2 = \dfrac{E_1}{(2)^2} = -3.40\,\text{eV}$, $K_2 = 3.40\,\text{eV}$ and $U_2 = -6.80\,\text{eV}$

Now, $U_1 = 0$, i.e. potential energy has been increased by 27.20 eV. So, we will increase U and E in all energy states by 27.20 eV, while kinetic energy will remain unchanged. Changed values in tabular form are as under.

Table 33.2

Orbit	*K* (eV)	*U* (eV)	*E* (eV)
First	13.60	0	13.60
Second	3.40	20.40	23.80

Example 33.12 *A small particle of mass m moves in such a way that the potential energy $U = ar^2$, where a is constant and r is the distance of the particle from the origin. Assuming Bohr model of quantization of angular momentum and circular orbits, find the radius of nth allowed orbit.*

Solution The force at a distance r is

$$F = -\frac{dU}{dr} = -2ar$$

Suppose r be the radius of nth orbit. Then, the necessary centripetal force is provided by the above force. Thus,

$$\frac{mv^2}{r} = 2ar \qquad \text{...(i)}$$

Further, the quantization of angular momentum gives

$$mvr = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii) for r, we get

$$r = \left(\frac{n^2h^2}{8\,am\,\pi^2}\right)^{1/4}$$ **Ans.**

Example 33.13 *Calculate (a) the wavelength and (b) the frequency of the* H_β *line of the Balmer series for hydrogen.*

Solution (a) H_β line of Balmer series corresponds to the transition from $n = 4$ to $n = 2$ level.

Using Eq. (xiii), the corresponding wavelength for H_β line is

$$\frac{1}{\lambda} = (1.097 \times 10^7)\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = 0.2056 \times 10^7$$

$$\therefore \qquad \lambda = 4.9 \times 10^{-7} \text{ m} \qquad \textbf{Ans.}$$

(b)

$$f = \frac{c}{\lambda} = \frac{3.0 \times 10^8}{4.9 \times 10^{-7}} = 6.12 \times 10^{14} \text{ Hz} \qquad \textbf{Ans.}$$

Example 33.14 *Find the largest and shortest wavelengths in the Lyman series for hydrogen. In what region of the electromagnetic spectrum does each series lie?*

Solution The transition equation for Lyman series is given by

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right), \quad n = 2, 3, \ldots$$

The largest wavelength is corresponding to $n = 2$

$$\therefore \qquad \frac{1}{\lambda_{\max}} = 1.097 \times 10^7 \left(\frac{1}{1} - \frac{1}{4}\right) = 0.823 \times 10^7$$

$$\therefore \qquad \lambda_{\max} = 1.2154 \times 10^{-7} \text{ m} = 1215 \text{ Å} \qquad \textbf{Ans.}$$

The shortest wavelength corresponds to $n = \infty$

$$\therefore \qquad \frac{1}{\lambda_{\min}} = 1.097 \times 10^7 \left(\frac{1}{1} - \frac{1}{\infty}\right)$$

$$\text{or} \qquad \lambda_{\min} = 0.911 \times 10^{-7} \text{ m} = 911 \text{ Å} \qquad \textbf{Ans.}$$

Both of these wavelengths lie in ultraviolet (UV) region of electromagnetic spectrum.

Example 33.15 *In a hypothetical atom, mass of electron is doubled, value of atomic number is* $Z = 4$. *Find wavelength of photon when this electron jumps from* 3*rd excited state to* 2*nd orbit.*

Solution $E \propto \dfrac{Z^2}{n^2} \propto m$

Mass is doubled, $Z = 4$ and 3rd excited state means $n = 4$, second orbit means $n = 2$. For these values, we have

$$E_4 = -13.6 \times 2\left(\frac{4}{4}\right)^2 = -27.2 \text{ eV} \quad \text{and} \quad E_2 = -13.6 \times 2\left(\frac{4}{2}\right)^2 = -108.8 \text{ eV}$$

$$\lambda (\text{in Å}) = \frac{12375}{\Delta E (\text{in eV})} = \frac{12375}{E_4 - E_2} = \frac{12375}{-27.2 + 108.8}$$

$$\therefore \qquad \lambda = 151.65 \text{ Å} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 33.2

1. Find the ionisation energy of a doubly ionized lithium atom.

2. A hydrogen atom is in a state with energy –1.51 eV. In the Bohr model, what is the angular momentum of the electron in the atom with respect to an axis at the nucleus?

3. As an electron makes a transition from an excited state to the ground state of a hydrogen like atom/ion **(JEE 2015)**
   (a) kinetic energy, potential energy and total energy decrease
   (b) kinetic energy decreases, potential energy increases but total energy remains same
   (c) kinetic energy and total energy decrease but potential energy increases
   (d) its kinetic energy increases but potential energy and total energy decrease

4. The wavelength of the first spectral line in the Balmer series of hydrogen atom is 6561 Å. The wavelength of the second spectral line in the Balmer series of singly ionized helium atom is
   (a) 1215 Å (b) 1640 Å (c) 2430 Å (d) 4687 Å **(JEE 2011)**

5. The largest wavelength in the ultraviolet region of the hydrogen spectrum is 122 nm. The smallest wavelength in the infrared region of the hydrogen spectrum (to the nearest integer) is
   (a) 802 nm (b) 823 nm
   (c) 1882 nm (d) 1648 nm **(JEE 2007)**

6. A photon collides with a stationary hydrogen atom in ground state inelastically. Energy of the colliding photon is 10.2 eV. After a time interval of the order of micro second another photon collides with same hydrogen atom inelastically with an energy of 15 eV. What will be observed by the detector? **(JEE 2005)**
   (a) 2 photons of energy 10.2 eV
   (b) 2 photons of energy 1.4 eV
   (c) One photon of energy 10.2 eV and an electron of energy 1.4 eV
   (d) One photon of energy 10.2 eV and another photon of energy 1.4 eV

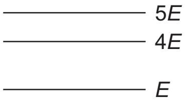
7. A hydrogen atom and a Li^{2+} ion are both in the second excited state. If l_H and l_{Li} are their respective electronic angular momenta, and E_H and E_{Li} their respective energies, then
   (a) $l_H > l_{Li}$ and $|E_H| > |E_{Li}|$ (b) $l_H = l_{Li}$ and $|E_H| < |E_{Li}|$ **(JEE 2002)**
   (c) $l_H = l_{Li}$ and $|E_H| > |E_{Li}|$ (d) $l_H < l_{Li}$ and $|E_H| < |E_{Li}|$

8. The transition from the state $n = 4$ to $n = 3$ in a hydrogen like atom results in ultraviolet radiation. Infrared radiation will be obtained in the transition **(JEE 2001)**
   (a) $2 \to 1$ (b) $3 \to 2$ (c) $4 \to 2$ (d) $5 \to 4$

9. As per Bohr model, the minimum energy (in eV) required to remove an electron from the ground state of doubly ionized Li atom ($Z = 3$) is **(JEE 1997)**
   (a) 1.51 (b) 13.6 (c) 40.8 (d) 122.4

10. Consider the spectral line resulting from the transition $n = 2 \to n = 1$ in the atoms and ions given below. The shortest wavelength is produced by **(JEE 1983)**
    (a) hydrogen atom (b) deuterium atom
    (c) singly ionized helium (d) doubly ionized lithium

11. The energy levels of a certain atom are shown in figure. If a photon of frequency f is emitted when there is an electron transition from $5E$ to E, what frequencies of photons could be produced by other energy level transitions?



12. Find the longest wavelength present in the Balmer series of hydrogen.

33.8 X-Rays

Electromagnetic radiation with wavelengths from 0.1 Å to 100 Å falls into the category of X-rays. The boundaries of this category are not sharp. The shorter wavelength end overlaps gamma rays and the longer wavelength end overlaps ultraviolet rays. Photoelectric effect (will be discussed later) provides convincing evidence that photons of light can transfer energy to electrons. Is the inverse process also possible? That is, can part or all of the kinetic energy of a moving electron be converted into a photon? Yes, it is possible. In 1895 **Wilhelm Roentgen** found that a highly penetrating radiation of unknown nature is produced when fast moving electrons strike a target of high atomic number and high melting point. These radiations were given a name X-rays as their nature was unknown (in mathematics an unknown quantity is normally designated by X). Later, it was discovered that these are high energy photons (or electromagnetic waves).

Production of X-Rays Figure shows a diagram of a X-ray tube, called the coolidge tube. A cathode (a plate connected to negative terminal of a battery), heated by a filament through which an electric current is passed, supplies electrons by thermionic emission. The high potential difference V maintained between the cathode and a metallic target accelerate the electrons toward the later. The face of the target is at an angle relative to the electron beam, and the X-rays that leave the target pass through the side of the tube. The tube is evacuated to permit the electrons to get to the target unimpedded.

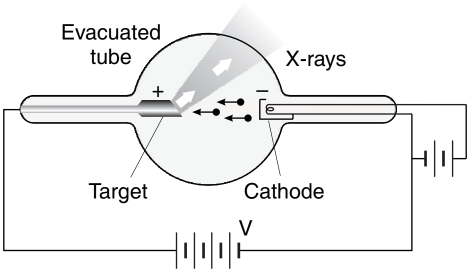


Fig. 33.7 An X-ray tube. The higher the accelerating voltage *V*, the faster the electrons and the shorter the wavelengths of the X-rays

Continuous and characteristic X-rays X-rays so produced by the coolidge tube are of two types, continuous and characteristic. While the former depends only on the accelerating voltage V, the later depends on the target used.

Continuous X-rays Electromagnetic theory predicts that an accelerated electric charge will radiate electromagnetic waves, and a rapidly moving electrons when suddenly brought to rest is certainly accelerated (of course negative). X-rays produced under these circumstances is given the German name **bremsstrahlung** (braking radiation). Energy loss due to bremsstrahlung is more important for electrons than for heavier particles because electrons are more violently accelerated when passing near nuclei in their paths. The continuous X-rays (or bremsstrahlung X-rays) produced at a given accelerating potential V vary in wavelength, but none has a wavelength shorter than a certain value λ_{min}. This minimum wavelength corresponds to the maximum energy of the X-rays which in turn is equal to the maximum kinetic energy qV or eV of the striking electrons. Thus,

$$\frac{hc}{\lambda_{\text{min}}} = eV \quad \text{or} \quad \lambda_{\text{min}} = \frac{hc}{eV}$$

After substituting values of h, c and e we obtain the following simple formula for λ_{min}.

$$\boxed{\lambda_{\text{min}} \text{ (in Å)} = \frac{12375}{V \text{(in volts)}}} \qquad \ldots\text{(i)}$$

Increasing V decreases λ_{min}. This wavelength is also known as the cut off wavelength or the threshold wavelength.

Characteristic X-rays The X-ray spectrum typically consists of a broad continuous band containing a series of sharp lines as shown in Fig. 33.8.

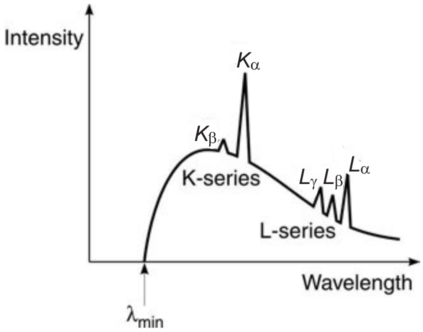


Fig. 33.8 X-ray spectrum

As discussed above the continuous spectrum is the result of collisions between incoming electrons and atoms in the target. The kinetic energy lost by the electrons during the collisions emerges as the energy of the X-ray photons radiated from the target.

The sharp lines superimposed on the continuous spectrum are known as **characteristic X-rays** because they are characteristic of the target material. They were discovered in 1908, but their origin remained unexplained until the details of atomic structure, particularly the shell structure of the atom, were discovered.

Characteristic X-ray emission occurs when a bombarding electron that collides with a target atom has sufficient energy to remove an inner shell electron from the atom. The vacancy created in the shell is filled when an electron from a higher level drops down into it. This transition is accompanied by the emission of a photon whose energy equals the difference in energy between the two levels.

Let us assume that the incoming electron has dislodged an atomic electron from the innermost shell-the K shell. If the vacancy is filled by an electron dropping from the next higher shell the L shell, the photon emitted has an energy corresponding to the K_α characteristic X-ray line. If the vacancy is filled by an electron dropping from the M shell, the K_β line is produced. An L_α line is produced as an electron drops from the M shell to the L-shell, and an L_β line is produced by a transition from the N-shell to the L-shell.

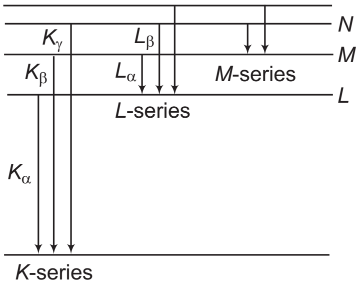


Fig. 33.9

Moseley's Law for Characteristic Spectrum

Although multi-electron atoms cannot be analyzed with the Bohr model, Henery G.J. Moseley in 1914 made an effort towards this. Moseley measured the frequencies of characteristic X-rays from a large number of elements and plotted the square root of the frequency $\sqrt{f}$ against the atomic number

Z of the element. He discovered that the plot is very close to a straight line. He plotted the square root of the frequency of the K_α line *versus* the atomic number Z.

As figure shows, Moseley's plot did not pass through the origin. Let us see why. It can be understood from Gauss's law. Consider an atom of atomic number Z in which one of the two electrons in the K-shell has been ejected. Imagine that we draw a Gaussian sphere just inside the most probable radius of the L-electrons. The effective charge inside the Gaussian surface is the positive nuclear charge and one negative charge due to the single K-electron. If we ignore the interactions between L-electrons, a single L electron behaves as if it experiences an electric field due to a charge $(Z-1)$ enclosed by the Gaussian surface.

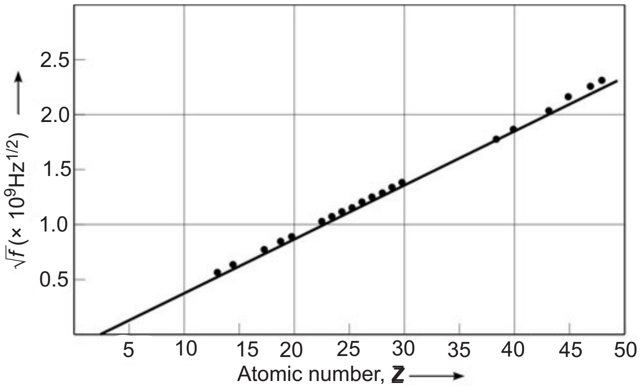


Fig. 33.10 A plot of the square root of the frequency of the K_α lines *versus* atomic number using Moseley's data

Thus, Moseley's law of the frequency of K_α line is

$$\sqrt{f_{K_\alpha}} = a\,(Z-1) \qquad \ldots\text{(ii)}$$

where, a is a constant that can be related to Bohr theory. The above law in general can be stated as under

$$\sqrt{f} = a\,(Z-b) \quad \text{or} \quad \boxed{\sqrt{f} \propto (Z-b)} \qquad \ldots\text{(iii)}$$

For K_α line,
$$\Delta E = hf = Rhc\,(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right)$$

or
$$\sqrt{f} = \sqrt{\frac{3Rc}{4}}\,(Z-1)$$

or
$$a = \sqrt{\frac{3Rc}{4}} \quad \text{and} \quad b = 1$$

After substituting values of R and c, we get $a = 4.98 \times 10^7\ (\text{Hz})^{1/2}$

Eq. (iii) can also be written as
$$f = a^2\,(Z-b)^2 \qquad \ldots\text{(iv)}$$

For K_α line,
$$a^2 = \frac{3Rc}{4} = (2.48 \times 10^{15}\ \text{Hz}) \quad \text{and} \quad b = 1$$

Hence,
$$f_{K_\alpha} = (2.48 \times 10^{15}\ \text{Hz})\,(Z-1)^2$$

Extra Points to Remember

- In continuous X-ray spectrum, all wavelengths greater than $\lambda_{\min}$ are obtained. Characteristic X-ray spectrum is discrete. Certain fixed wavelengths like λ_{K_α}, λ_{K_β} etc. are obtained and these wavelengths are different for different target elements. The mixed spectrum of continuous and characteristic X-rays is as shown in figure 33.8.

- In general, if we compare between different series, then

$$E_K > E_L > E_M \quad \text{or} \quad f_K > f_L > f_M \quad \text{or} \quad \lambda_K < \lambda_L < \lambda_M$$

  And if we compare between α, β and γ, then

$$E_\alpha < E_\beta < E_\gamma \quad \text{or} \quad f_\alpha < f_\beta < f_\gamma \quad \text{or} \quad \lambda_\alpha > \lambda_\beta > \lambda_\gamma$$

- In Moseley's law,

$$\sqrt{f} \propto (Z - b)$$

$$b = 1 \quad \text{for} \quad K_\alpha \text{ line}$$

  and $$b = 7.4 \quad \text{for} \quad L_\alpha \text{ line}$$

  Thus, $\sqrt{f} = 0$ at $Z = b = 1$ for K_α line

  and $\sqrt{f} = 0$ at $Z = b = 7.4$ for K_β line

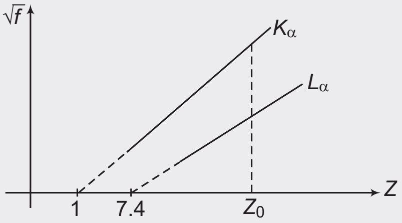


Fig. 33.11

  For lower atomic numbers lines are shown dotted. This is because X-rays are obtained only at high atomic numbers. Further, we can see that for a given atomic number (say Z_0).

$$\sqrt{f_K} > \sqrt{f_L}$$

- **Screening effect** The energy levels, in general, depend on the principal quantum number (n) and orbital quantum number (l). Let us take sodium ($Z = 11$) as an example. According to Gauss's law, for any spherically symmetric charge distribution the electric field magnitude at a distance r from the centre is $\frac{1}{4\pi\varepsilon_0}\frac{q_{\text{encl}}}{r^2}$, where q_{encl} is the total charge enclosed within a sphere with radius r. Mentally, remove the outer (valence) electron from a sodium atom. What you have left is a spherically symmetric collection of 10 electrons (filling the K and L shells) and 11 protons. So,

$$q_{\text{encl}} = -10e + 11e = +e$$

  If the eleventh is completely outside this collection of charges, it is attracted by an effective charge of $+e$, not $+11e$.

  This effect is called **screening**, the 10 electrons screen 10 of the 11 protons leaving an effective net charge of $+e$. In general, an electron that spends all its time completely outside a positive charge $Z_{\text{eff}}e$ has energy levels given by the hydrogen expression with e^2 replaced by $Z_{\text{eff}}\, e^2$. i.e.

$$E_n = -\frac{Z_{\text{eff}}^2}{n^2}(13.6 \text{ eV}) \quad \text{(energy levels with screening)}$$

  If the eleventh electron in the sodium atom is completely outside the remaining charge distribution, then $Z_{\text{eff}} = 1$.

  We can estimate the frequency of K_α X-ray photons using the concept of screening. A K_α X-ray photon is emitted when an electron in the L-shell ($n = 2$) drops down to fill a hole in the K-shell ($n = 1$). As the electron drops down, it is attracted by the Z protons in the nucleus screened by one remaining electron in the K-shell. Thus,

$$Z_{\text{eff}} = (Z-1), \quad n_i = 2 \quad \text{and} \quad n_f = 1$$

The energy before transition is

$$E_i = -\frac{(Z-1)^2}{2^2}(13.6\text{ eV}) = -(Z-1)^2\,(3.4\text{ eV})$$

and energy after transition is

$$E_f = -\frac{(Z-1)^2}{1^2}(13..6\text{ eV}) = -(Z-1)^2\,(13.6\text{ eV})$$

The energy of the K_α X-ray photon is

$$E_{K_\alpha} = E_i - E_f = (Z-1)^2\,(10.2\text{ eV})$$

The frequency of K_α X-ray photon is therefore,

$$f_{K_\alpha} = \frac{E_{K_\alpha}}{h} = \frac{(Z-1)^2\,(10.2\text{ eV})}{(4.136\times 10^{-15}\text{ eV -s})}$$

$$= (2.47\times 10^{15}\text{ Hz})(Z-1)^2$$

This relation agrees almost exactly with Moseley's experimental law.

- The target (or anode) used in the Coolidge tube should be of high melting point. This is because less than 0.5% of the kinetic energy of the electrons is converted into X-rays. The rest of the kinetic energy converts into internal energy of the target which simultaneously has to be kept cool by circulating oil or water.
- Atomic number of the target material should be high. This is because X-rays are high energy photons and as we have seen above energy of the X-rays increases as Z increases.
- X-rays are basically electromagnetic waves. So, they possess all the properties of electromagnetic waves.

Example 33.16 *Find the cut off wavelength for the continuous X-rays coming from an X-ray tube operating at 40 kV.*

Solution Cut off wavelength $\lambda_{\min}$ is given by

$$\lambda_{\min}\ (\text{in Å}) = \frac{12375}{V(\text{in volts})} = \frac{12375}{40\times 10^3}$$

$$= 0.31\ \text{Å}$$ **Ans.**

Example 33.17 *Use Moseley's law with $b = 1$ to find the frequency of the K_α X-rays of La $(Z = 57)$ if the frequency of the K_α X-rays of Cu $(Z = 29)$ is known to be 1.88×10^{18} Hz.*

Solution Using the equation, $\sqrt{f} = a\,(Z-b)$ $\quad(b = 1)$

$$\frac{f_{\text{La}}}{f_{\text{Cu}}} = \left(\frac{Z_{\text{La}}-1}{Z_{\text{Cu}}-1}\right)^2 \quad \text{or} \quad f_{\text{La}} = f_{\text{Cu}}\left(\frac{Z_{\text{La}}-1}{Z_{\text{Cu}}-1}\right)^2$$

$$= 1.88\times 10^{18}\left(\frac{57-1}{29-1}\right)^2$$

$$= 7.52\times 10^{18}\text{ Hz}$$ **Ans.**

Example 33.18 *Electrons with de-Broglie wavelength λ fall on the target in an X-ray tube. The cut off wavelength of the emitted X-rays is* **(JEE 2007)**

(a) $\lambda_0 = \dfrac{2mc\lambda^2}{h}$

(b) $\lambda_0 = \dfrac{2h}{mc}$

(c) $\lambda_0 = \dfrac{2m^2c^2\lambda^3}{h^2}$

(d) $\lambda_0 = \lambda$

Solution Momentum of bombarding electrons,

$$p = \frac{h}{\lambda}$$

$\therefore$ Kinetic energy of bombarding electrons,

$$K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

This is also maximum energy of X-ray photons.

Therefore, $$\frac{hc}{\lambda_0} = \frac{h^2}{2m\lambda^2}$$

or $$\lambda_0 = \frac{2m\lambda^2 c}{h}$$

$\therefore$ Correct option is (a).

Example 33.19 *Electrons with energy* 80 *keV are incident on the tungsten target of an X-ray tube. K-shell electrons of tungsten have* 72.5 *keV energy. X-rays emitted by the tube contain only* **(JEE 2000)**

(a) a continuous X-ray spectrum (Bremsstrahlung) with a minimum wavelength of ≈ 0.155 *Å*

(b) a continuous X-ray spectrum (Bremsstrahlung) with all wavelengths

(c) the characteristic X-ray spectrum of tungsten

(d) a continuous X-ray spectrum (Bremsstrahlung) with a minimum wavelength of ≈ 0.155 *Å and the characteristic X-ray spectrum of tungsten*

Solution Minimum wavelength of continuous X-ray spectrum is given by λ_{min} (in Å)

$$= \frac{12375}{E\,(\text{in eV})}$$

Here, $E =$ energy of incident electrons (in eV)

$=$ energy corresponding to minimum wavelength λ_{min} of X-ray

$$E = 80\,\text{keV} = 80 \times 10^3 \text{ eV}$$

$\therefore$ $$\lambda_{\text{min}}\ (\text{in Å}) = \frac{12375}{80 \times 10^3} \approx 0.155$$

Also the energy of the incident electrons (80 keV) is more than the ionization energy of the *K*-shell electrons (i.e. 72.5 keV). Therefore, characteristic X-ray spectrum will also be obtained because energy of incident electron is high enough to knock out the electron from *K* or *L*-shells.

$\therefore$ The correct option is (d).

INTRODUCTORY EXERCISE 33.3

1. If λ_{Cu} is the wavelength of K_α, X-ray line of copper (atomic number 29) and λ_{Mo} is the wavelength of the K_α X-ray line of molybdenum (atomic number 42), then the ratio $\lambda_{Cu}/\lambda_{Mo}$ is close to **(JEE 2014)**

   (a) 1.99 (b) 2.14

   (c) 0.50 (d) 0.48

2. Which one of the following statements is wrong in the context of X-rays generated from an X-ray tube? **(JEE 2008)**

   (a) Wavelength of characteristic X-rays decreases when the atomic number of the target increases

   (b) Cut off wavelength of the continuous X-rays depends on the atomic number of the target

   (c) Intensity of the characteristic X-rays depends on the electrical power given to the X-ray tube

   (d) Cut off wavelength of the continuous X-rays depends on the energy of the electrons in the X-ray tube

3. K_α wavelength emitted by an atom of atomic number $Z = 11$ is λ. Find the atomic number for an atom that emits K_α radiation with wavelength 4λ **(JEE 2005)**

   (a) $Z = 6$ (b) $Z = 4$

   (c) $Z = 11$ (d) $Z = 44$

4. The intensity of X-rays from a coolidge tube is plotted against wavelength λ as shown in the figure. The minimum wavelength found is λ_c and the wavelength of the K_α line is λ_k. As the accelerating voltage is increased **(JEE 2001)**

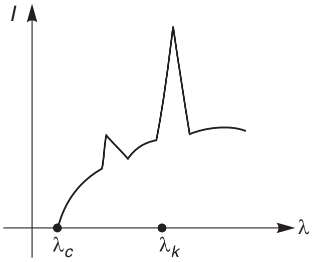


Fig. 33.12

   (a) $\lambda_k - \lambda_c$ increases (b) $\lambda_k - \lambda_c$ decreases

   (c) λ_k increases (d) λ_k decreases

5. X-rays are produced in an X-ray tube operating at a given accelerating voltage. The wavelength of the continuous X-rays has values from **(JEE 1998)**

   (a) 0 to ∞

   (b) λ_{min} to ∞, where $\lambda_{min} > 0$

   (c) 0 to λ_{max}, where $\lambda_{max} < \infty$

   (d) λ_{min} to λ_{max}, where $0 < \lambda_{min} < \lambda_{max} < \infty$

6. Characteristic X-rays of frequency 4.2×10^{18} Hz are produced when transitions from *L*-shell to *K*-shell take place in a certain target material. Use Mosley's law to determine the atomic number of the target material. Given Rydberg constant $R = 1.1 \times 10^7 \text{ m}^{-1}$. **(JEE 2003)**

33.9 Emission of Electrons

At room temperature the free electrons move randomly within the conductor, but they don't leave the surface of the conductor due to attraction of positive charges. Some external energy is required to emit electrons from a metal surface. Minimum energy is required to emit the electrons which are just on the surface of the conductor. This minimum energy is called the **work-function** (denoted by W) of the conductor. Work-function is the property of the metallic surface.

The energy required to liberate an electron from metal surface may arise from various sources such as heat, light, electric field etc. Depending on the nature of source of energy, the following methods are possible :

(i) **Thermionic emission** The energy to the free electrons can be given by heating the metal. The electrons so emitted are known as **thermions**.

(ii) **Field emission** When a conductor is put under strong electric field, the free electrons on it experience an electric force in the opposite direction of field. Beyond a certain limit, electrons start coming out of the metal surface. Emission of electrons from a metal surface by this method is called the field emission.

(iii) **Secondary emission** Emission of electrons from a metal surface by the bombardment of high speed electrons or other particles is known as secondary emission.

(iv) **Photoelectric emission** Emission of free electrons from a metal surface by falling light (or any other electromagnetic wave which has an energy greater than the work-function of the metal) is called photoelectric emission. The electrons so emitted are called **photoelectrons**.

33.10 Photoelectric Effect

When light of an appropriate frequency (or correspondingly of an appropriate wavelength) is incident on a metallic surface, electrons are liberated from the surface. This observation is known as **photoelectric effect**. Photoelectric effect was first observed in 1887 by Hertz. For photoemission to take place, energy of incident light photons should be greater than or equal to the work-function of the metal.

or $$\boxed{E \geq W} \qquad \text{...(i)}$$

∴ $$hf \geq W$$

or $$f \geq \frac{W}{h}$$

Here, $\frac{W}{h}$ is the minimum frequency required for the emission of electrons. This is known as threshold frequency f_0.

Thus, $$\boxed{f_0 = \frac{W}{h} \quad \text{(threshold frequency)}} \qquad \text{...(ii)}$$

Further, Eq. (i) can be written as

$$\frac{hc}{\lambda} \geq W \quad \text{or} \quad \lambda \leq \frac{hc}{W}$$

Here, $\frac{hc}{W}$ is the largest wavelength beyond which photoemission does not take place. This is called the threshold wavelength λ_0.

Thus, $$\boxed{\lambda_0 = \frac{hc}{W} \quad \text{(threshold wavelength)}} \qquad \ldots\text{(iii)}$$

Hence, for the photoemission to take place either of the following conditions must be satisfied.

$$\boxed{E \ge W \quad \text{or} \quad f \ge f_0 \quad \text{or} \quad \lambda \le \lambda_0} \qquad \ldots\text{(iv)}$$

Stopping Potential and Maximum Kinetic Energy of Photoelectrons

When the frequency f of the incident light is greater than the threshold frequency, some electrons are emitted from the metal with substantial initial speeds. Suppose E is the energy of light incident on a metal surface and W $(< E)$ the work-function of metal. As minimum energy is required to extract electrons from the surface, they will have the maximum kinetic energy which is $E - W$.

Thus, $$K_{\max} = E - W \qquad \ldots\text{(v)}$$

This value $K_{\max}$ can experimentally be found by keeping the metal plate P (from which electrons are emitting) at higher potential relative to an another plate Q placed in front of P. Some electrons after emitting from plate P, reach the plate Q despite the fact that Q is at lower potential and it is repelling the electrons from reaching in itself. This is because the electrons emitted from plate P possess some kinetic energy and due to this energy they reach the plate Q and current i flows in the circuit in the direction shown in figure.

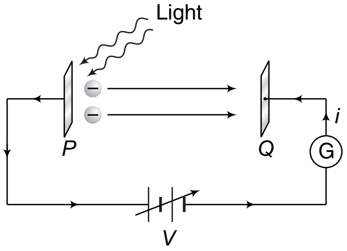


Fig. 33.13

As the potential V is increased, the force of repulsion to the electrons gets increased and less number of electrons reach the plate Q and current in the circuit gets decreased. At a certain value V_0 electrons having maximum kinetic energy ($K_{\max}$) also get stopped and current in the circuit becomes zero. This is called the **stopping potential**.

As an electron moves from P to Q, the potential decreases by V_0 and negative work $-eV_0$ is done on the (negatively charged) electron, the most energetic electron leaves plate P with kinetic energy $K_{\max} = \frac{1}{2} mv_{\max}^2$ and has zero kinetic energy at Q. Using the work energy theorem, we have

$$W_{\text{ext}} = -eV_0 = \Delta K = 0 - K_{\max}$$

or $$\boxed{K_{\max} = \frac{1}{2} mv_{\max}^2 = eV_0} \qquad \ldots\text{(vi)}$$

Photoelectric Current

Figure shows an apparatus used to study the variation of photocurrent i with the intensity and frequency of light falling on metal plate P. Photoelectrons are emitted from plate P which are being attracted by the positive plate Q and a photoelectric current i flows in the circuit, which can be measured by the galvanometer G.

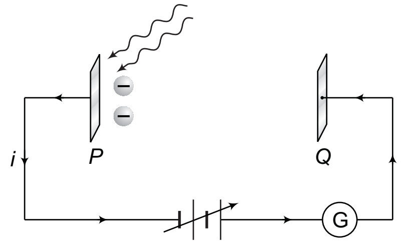
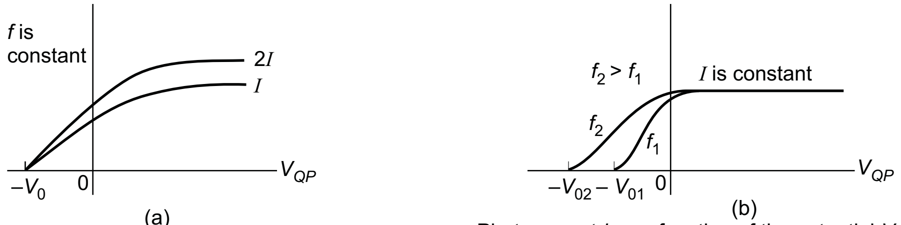


Fig. 33.14

Figure 33.15 (a) shows graphs of photocurrent as a function of potential difference V_{QP} for light of constant frequency and two different intensities. When V_{QP} is sufficiently large and positive the current becomes constant, showing that all the emitted electrons are being collected by the anode plate Q. The stopping potential difference $-V_0$ needed to reduce the current to zero is shown. If the intensity of light is increased, (or we can say the number of photons incident per unit area per unit time is increased) while its frequency is kept the same, the current becomes constant at a higher value, showing that more electrons are being emitted per unit time. But the stopping potential is found to be the same.



(a)

Photocurrent i as a function of the potential V_{QP} of the anode with respect to the cathode for a constant light frequency f, the stopping potential V_0 is independent of the light intensity I.

(b)

Photocurrent i as a function of the potential V_{QP} of the anode with respect to a cathode for two different light frequencies f_1 and f_2 with the same intensity. The stopping potential V_0 (and therefore the maximum kinetic energy of the photoelectrons) increases linearly with frequency.

Fig. 33.15

Figure 33.15 (b) shows current as a function of potential difference for two different frequencies with the same intensity in each case. We see that when the frequency of the incident monochromatic light is increased, the stopping potential V_0 gets increased. Of course, V_0 turn out to be a linear function of the frequency f.

Graph between K_{max} and f

Let us plot a graph between maximum kinetic energy K_{max} of photoelectrons and frequency f of incident light. The equation between K_{max} and f is

$$K_{max} = hf - W$$

Comparing it with $y = mx + c$, the graph between K_{max} and f is a straight line with positive slope and negative intercept.

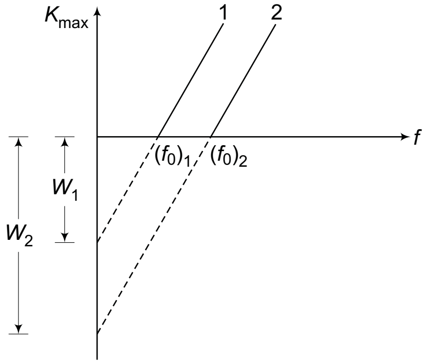


Fig. 33.16

From the graph, we can note the following points

(i) $K_{\text{max}} = 0$ at $f = f_0$

(ii) Slope of the straight line is h, a universal constant, i.e. if graph is plotted for two different metals 1 and 2, slope of both the lines is same.

(iii) The negative intercept of the line is W, the work-function which is characteristic of a metal, i.e. intercepts for two different metals will be different. Further,

$$W_2 > W_1$$

$$\therefore \quad (f_0)_2 > (f_0)_1 \qquad [\text{as } W = hf_0]$$

Here, $\quad f_0 =$ threshold frequency

Graph between V_0 and f

Let us now plot a graph between the stopping potential V_0 and the incident frequency f. The equation between them is

$$eV_0 = hf - W$$

or

$$V_0 = \left(\frac{h}{e}\right) f - \left(\frac{W}{e}\right)$$

Again comparing with $y = mx + c$, the graph between V_0 and f is a straight line with positive slope $\frac{h}{e}$ (a universal constant) and negative intercept $\frac{W}{e}$ (which depends on the metal). The corresponding graph is shown in figure.

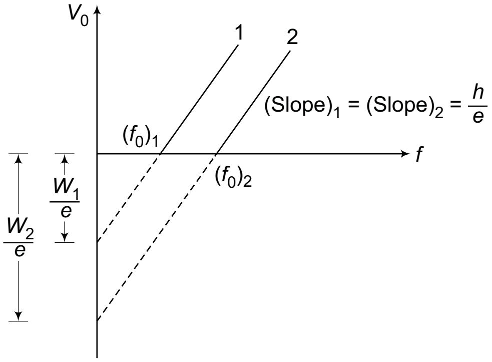


Fig. 33.17

Extra Points to Remember

- The major features of the photoelectric effect could not be explained by the wave theory of light which were later explained by Einstein's photon theory.
  - (i) Wave theory suggests that the kinetic energy of the photoelectrons should increase with the increase in intensity of light. However equation, $K_{max} = eV_0$ suggests that it is independent of the intensity of light.
  - (ii) According to wave theory, the photoelectric effect should occur for any frequency of the light, provided that the light is intense enough. However, the above equation suggests that photoemission is possible only when frequency of incident light is either greater than or equal to the threshold frequency f_0.
  - (iii) If the energy to the photoelectrons is obtained by soaking up from the incident wave, it is not likely that the effective target area for an electron in the metal is much more than a few atomic diameters. (see example 33.20) between the impinging of the light on the surface and the ejection of the photoelectrons. During this interval the electron should be "soaking up" energy from the beam until it had accumulated enough energy to escape. However, no detectable time lag has ever been measured.
- **Einstein's photon theory** Einstein succeeded in explaining the photoelectric effect by making a remarkable assumption, that the energy in a light beam travels through space in concentrated bundles, called photons. The energy E of a single photon is given by

$$E = hf$$

Applying the photon concept to the photoelectric effect, Einstein wrote

$$hf = W + K_{max} \qquad \text{(already discussed)}$$

Consider how Einstein's photon hypothesis meets the three objections raised against the wave theory interpretation of the photoelectric effect.

As for objection 1 (the lack of dependence of K_{max} on the intensity of illumination), doubling the light intensity merely doubles the number of photons and thus doubles the photoelectric current, it does not change the energy of the individual photons

Objection 2 (the existence of a cutoff frequency) follows from equation $hf = W + K_{max}$. If K_{max} equals zero, We have $hf_0 = W$ which asserts that the photon has just enough energy to eject the photoelectrons and none extra to appear as kinetic energy. The quantity W is called the work-function of the substance. If f is reduced below f_0, the individual photons, no matter how many of them there are (that is, no matter how intense the illumination), will not have enough energy to eject photo electrons.

Objection 3 (the absence of a time lag) follows from the photon theory because the required energy is supplied in a concentrated bundle. It is not spread uniformly over a large area, as in the wave theory. Although, the photon hypothesis certainly fits the facts of photoelectricity, it seems to be in direct conflict with the wave theory of light. Out modern view of the nature of light is that it has a dual character, behaving like a wave under some circumstances and like a particle or photon under others.

Example 33.20 *A metal plate is placed 5 m from a monochromatic light source whose power output is 10^{-3} W. Consider that a given ejected photoelectron may collect its energy from a circular area of the plate as large as ten atomic diameters (10^{-9} m) in radius. The energy required to remove an electron through the metal surface is about 5.0 eV. Assuming light to be a wave, how long would it take for such a 'target' to soak up this much energy from such a light source.*

Solution The target area is $S_1 = \pi(10^{-9})^2 = \pi \times 10^{-18}$ m^2. The area of a 5 m sphere centered on the light source is, $S_2 = 4\pi(5)^2 = 100\,\pi$ m^2. Thus, if the light source radiates uniformly in all directions the rate P at which energy falls on the target is given by

$$P = (10^{-3}\ \text{watt})\left(\frac{S_1}{S_2}\right) = (10^{-3})\left(\frac{\pi \times 10^{-18}}{100 \times \pi}\right) = 10^{-23}\ \text{J/s}$$

Assuming that all power is absorbed, the required time is

$$t = \left(\frac{5\,\text{eV}}{10^{-23}\ \text{J/s}}\right)\left(\frac{1.6 \times 10^{-19}\ \text{J}}{1\ \text{eV}}\right) \approx 20\,\text{h}$$ **Ans.**

Example 33.21 *The photoelectric work-function of potassium is* 2.3 *eV. If light having a wavelength of* 2800 Å *falls on potassium, find*

(a) the kinetic energy in electron volts of the most energetic electrons ejected.
(b) the stopping potential in volts

Solution Given, $W = 2.3\,\text{eV}$, $\lambda = 2800\ \text{Å}$

$$\therefore \qquad E\ (\text{in eV}) = \frac{12375}{\lambda\ (\text{in Å})} = \frac{12375}{2800} = 4.4\ \text{eV}$$

(a) $K_{\max} = E - W$

$$= (4.4 - 2.3)\,\text{eV} = 2.1\ \text{eV}$$ **Ans.**

(b) $K_{\max} = eV_0$

$$\therefore \qquad 2.1\,\text{eV} = eV_0 \quad \text{or} \quad V_0 = 2.1\,\text{volt}$$ **Ans.**

Example 33.22 *When a beam of* 10.6 *eV photons of intensity* 2.0 W/m^2 *falls on a platinum surface of area* 1.0×10^{-4} m^2 *and work-function* 5.6 *eV*, 0.53% *of the incident photons eject photoelectrons. Find the number of photoelectrons emitted per second and their minimum and maximum energies (in eV). Take* $1\,eV = 1.6 \times 10^{-19}$ *J.*

Solution Number of photoelectrons emitted per second

$$= \frac{(\text{Intensity})\,(\text{Area})}{(\text{Energy of each photon})} \times \frac{0.53}{100}$$

$$= \frac{(2.0)\,(1.0 \times 10^{-4})}{(10.6 \times 1.6 \times 10^{-19})} \times \frac{0.53}{100}$$

$$= 6.25 \times 10^{11}$$ **Ans.**

Minimum kinetic energy of photoelectrons,

$$K_{\min} = 0$$

and maximum kinetic energy is, $K_{\max} = E - W = (10.6 - 5.6)\,\text{eV}$

$$= 5.0\ \text{eV}$$ **Ans.**

Example 33.23 *Maximum kinetic energy of photoelectrons from a metal surface is* K_0 *when wavelength of incident light is* λ. *If wavelength is decreased to* $\lambda/2$, *the maximum kinetic energy of photoelectrons becomes*

(a) $= 2K_0$ *(b)* $> 2K_0$ *(c)* $< 2K_0$

Solution Using the equation,

$$K_{\max} = E - W, \text{ we have}$$

$$K_0 = \frac{hc}{\lambda} - W \qquad \dots \text{(i)}$$

with wavelength $\frac{\lambda}{2}$, suppose the maximum kinetic energy is K_0', then

$$K_0' = \frac{hc}{\lambda/2} - W = 2\frac{hc}{\lambda} - W$$

$$= 2\left(\frac{hc}{\lambda} - W\right) + W$$

but $\frac{hc}{\lambda} - W$ is K_0. Therefore,

$$K_0' = 2K_0 + W$$

or $$K_0' > 2K_0$$

∴ The correct option is (b). **Ans.**

Example 33.24 *Intensity and frequency of incident light both are doubled. Then, what is the effect on stopping potential and saturation current.*

Solution By increasing the frequency of incident light energy of incident light will increase. So, maximum kinetic energy of photoelectrons will also increase. Hence, stopping potential will increase.

Further, by doubling the frequency of incident light energy of each photon will be doubled. So, intensity itself becomes two times without increasing number of photons incident per unit area per unit time. Therefore, saturation current will remain unchanged.

Example 33.25 *When a monochromatic point source of light is at a distance of* 0.2 *m from a photoelectric cell, the cut off voltage and the saturation current are respectively* 0.6 *V and* 18.0 *mA. If the same source is placed* 0.6 *m away from the photoelectric cell, then* **(JEE 1992)**

(a) the stopping potential will be 0.2 *V*

(b) the stopping potential will be 0.6 *V*

(c) the saturation current will be 6.0 *mA*

(d) the saturation current will be 2.0 *mA*

Solution (b) Stopping potential depends on two factors – one the energy of incident light and the other the work-function of the metal. By increasing the distance of source from the cell, neither of the two change. Therefore, stopping potential remains the same.

(d) Saturation current is directly proportional to the intensity of light incident on cell and for a point source, intensity $I \propto 1/r^2$

When distance is increased from 0.2 m to 0.6 m (three times), the intensity and hence the saturation current will decrease 9 times, i.e. the saturation current will be reduced to 2.0 mA.

∴ The correct options are (b) and (d).

Example 33.26 *The threshold wavelength for photoelectric emission from a material is 5200 Å. Photoelectrons will be emitted when this material is illuminated with monochromatic radiation from a* **(1982, 3M)**

(a) 50 W infrared lamp *(b) 1 W infrared lamp*

(c) 50 W ultraviolet lamp *(d) 1 W ultraviolet lamp*

Solution For photoemission to take place, wavelength of incident light should be less than the threshold wavelength. Wavelength of ultraviolet light < 5200 Å while that of infrared radiation > 5200 Å.

$\therefore$ The correct options are (c) and (d).

INTRODUCTORY EXERCISE 33.4

1. Light of wavelength 2000 Å is incident on a metal surface of work-function 3.0 eV. Find the minimum and maximum kinetic energy of the photoelectrons.

2. Is it correct to say that $K_{\max}$ is proportional to f? If not, what would a correct statement of the relationship between $K_{\max}$ and f?

3. When a metal is illuminated with light of frequency f, the maximum kinetic energy of the photoelectrons is 1.2 eV. When the frequency is increased by 50%, the maximum kinetic energy increases to 4.2 eV. What is the threshold frequency for this metal?

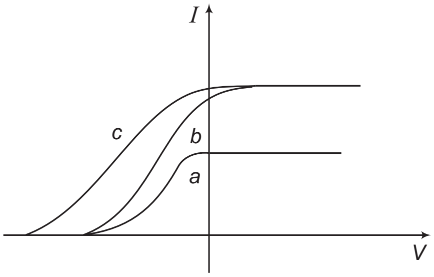
4. A metal surface is illuminated by light of two different wavelengths 248 nm and 310 nm. The maximum speeds of the photoelectrons corresponding to these wavelengths are u_1 and u_2, respectively. If the ratio $u_1 : u_2 = 2 : 1$ and $hc = 1240$ eV nm, the work-function of the metal is nearly **(JEE 2014)**

(a) 3.7 eV (b) 3.2 eV (c) 2.8 eV (d) 2.5 eV

5. The figure shows the variation of photocurrent with anode potential for a photosensitive surface for three different radiations. Let I_a, I_b and I_c be the intensities and f_a, f_b and f_c be the frequencies for the curves a, b and c, respectively **(JEE 2004)**

(a) $f_a = f_b$ and $I_a \neq I_b$ (b) $f_a = f_c$ and $I_a = I_c$

(c) $f_a = f_b$ and $I_a = I_b$ (d) $f_b = f_c$ and $I_b = I_c$



6. The work-function of a substance is 4.0 eV. The longest wavelength of light that can cause photoelectron emission from this substance is approximately **(JEE 1998)**

(a) 540 nm (b) 400 nm (c) 310 nm (d) 220 nm

7. The maximum kinetic energy of photoelectrons emitted from a surface when photons of energy 6 eV fall on it is 4 eV. The stopping potential in volt is **(JEE 1997)**

(a) 2 (b) 4 (c) 6 (d) 10

8. Photoelectric effect supports quantum nature of light because **(JEE 1987)**

(a) there is a minimum frequency of light below which no photoelectrons are emitted

(b) the maximum kinetic energy of photoelectrons depends only on the frequency of light and not on its intensity

(c) even when the metal surface is faintly illuminated, the photoelectrons leave the surface immediately

(d) electric charge of the photoelectrons is quantised

Final Touch Points

1. In hydrogen and hydrogen like atoms :

   Circumference of n^{th} orbit $= n$ (wavelength of single electron in that orbit)

   Proof

   According to Bohr's assumption,

   $$L_n = n\left(\frac{h}{2\pi}\right)$$

   or $$mvr = n\left(\frac{h}{2\pi}\right)$$

   $\therefore$ $$(2\pi r) = n\left(\frac{h}{mv}\right)$$

   or circumference $= n$(de-Broglie wavelength of electron)

2. $Rhc = 13.6\,\text{eV}$

   Therefore, Rhc has the dimensions of energy or

   $$[Rhc] = [\text{ML}^2\text{T}^{-2}]$$

3. 1 Rhc is also called 1 Rydberg. It is not Rydberg constant R.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on de-Broglie wavelength*

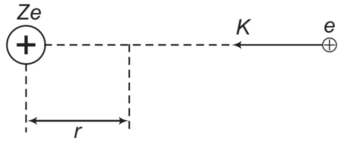
Example 1 *A proton is fired from very far away towards a nucleus with charge $Q = 120\ e$, where e is the electronic charge. It makes a closest approach of 10 fm to the nucleus. The de-Broglie wavelength (in units of fm) of the proton at its start is [Take the proton mass, $m_p = (5/3)\times 10^{-27}$ kg, $h/e = 4.2\times 10^{-15}$ J-s/C,*

$\dfrac{1}{4\pi\varepsilon_0} = 9\times 10^9$ m/F, 1 fm = 10^{-15} m]

(JEE 2013)

Solution r = closest distance = 10 fm

From energy conservation, we have



$$K_i + U_i = K_f + U_f$$

or
$$K + 0 = 0 + \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r}$$

or
$$K = \frac{1}{4\pi\varepsilon_0}\cdot\frac{(120\ e)\ (e)}{r} \qquad \text{...(i)}$$

de-Broglie wavelength,

$$\lambda = \frac{h}{\sqrt{2Km}} \qquad \text{...(ii)}$$

Substituting the given values in above two equations, we get

$$\lambda = 7\times 10^{-15}\ \text{m} = 7\ \text{fm}$$

Example 2 *Find de-Broglie wavelength of single electron in 2 nd orbit of hydrogen atom by two methods.*

Solution **Method 1** Kinetic energy of single electron in 2nd orbit is 3.4 eV using the equation,

$$\lambda = \sqrt{\frac{150}{\text{KE (in eV)}}} = \sqrt{\frac{150}{3.4}} = 6.64\ \text{Å} \qquad \textbf{Ans.}$$

Method 2 Circumference of nth orbit = $n\lambda$

∴
$$2\pi r = 2\lambda$$

or
$$\lambda = \pi r$$

Now,
$$r \propto n^2$$

$$r_1 = 0.529\ \text{Å}$$

∴
$$\lambda = \pi(0.529)(2)^2\ \text{Å}$$

$$= 6.64\ \text{Å} \qquad \textbf{Ans.}$$

Type 2. *Based on Bohr's atomic models*

Example 3 *The electric potential between a proton and an electron is given by* $V = V_0 \ln \frac{r}{r_0}$*, where* r_0 *is a constant. Assuming Bohr model to be applicable, write variation of* r_n *with n, being the principal quantum number.* **(JEE 2003)**

(a) $r_n \propto n$ (b) $r_n \propto \frac{1}{n}$ (c) $r_n \propto n^2$ (d) $r_n \propto \frac{1}{n^2}$

Solution $\because \quad U = eV = eV_0 \ln\left(\frac{r}{r_0}\right)$

$$|F| = \left|-\frac{dU}{dr}\right| = \frac{eV_0}{r}$$

This force will provide the necessary centripetal force. Hence, $\frac{mv^2}{r} = \frac{eV_0}{r}$

or $$v = \sqrt{\frac{eV_0}{m}} \qquad \text{...(i)}$$

Moreover, $$mvr = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we have

$$mr = \left(\frac{nh}{2\pi}\right)\sqrt{\frac{m}{eV_0}} \quad \text{or} \quad r_n \propto n$$

Example 4 *Imagine an atom made up of proton and a hypothetical particle of double the mass of the electron but having the same charge as the electron. Apply the Bohr atom model and consider all possible transitions of this hypothetical particle to the first excited level. The longest wavelength photon that will be emitted has wavelength λ (given in terms of the Rydberg constant R for the hydrogen atom) equal to* **(JEE 2000)**

(a) $9/5R$ (b) $36/5R$ (c) $18/5R$ (d) $4/R$

Solution In hydrogen atom, $E_n = -\frac{Rhc}{n^2}$

Also, $$E_n \propto m$$

where, m is the mass of the electron.

Here, the electron has been replaced by a particle whose mass is double of an electron. Therefore, for this hypothetical atom energy in nth orbit will be given by

$$E_n = -\frac{2Rhc}{n^2}$$

The longest wavelength $\lambda_{\max}$ (or minimum energy) photon will correspond to the transition of particle from $n = 3$ to $n = 2$.

$$\therefore \quad \frac{hc}{\lambda_{\max}} = E_3 - E_2 = 2Rhc\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$$

This gives, $\lambda_{\max} = 18/5R$

$\therefore$ The correct option is (c).

Example 5 *The recoil speed of a hydrogen atom after it emits a photon is going from $n = 5$ state to $n = 1$ state is .... m/s.* **(JEE 1997)**

Solution From conservation of linear momentum,

|Momentum of recoil hydrogen atom| = |Momentum of emitted photon|

or $$mv = \frac{\Delta E}{c}$$

Here, $$\Delta E = E_5 - E_1 = -13.6\left[\frac{1}{5^2} - \frac{1}{1^2}\right]\text{eV}$$

$$= (13.6)\,(24/25)\text{ eV} = 13.056\text{ eV}$$

$$= 13.056 \times 1.6 \times 10^{-19}\text{ J} = 2.09 \times 10^{-18}\text{ J}$$

and $$m = \text{mass of hydrogen atom} = 1.67 \times 10^{-27}\text{ kg}$$

$\therefore$ $$v = \frac{\Delta E}{mc} = \frac{2.09 \times 10^{-18}}{(1.67 \times 10^{-27})\,(3 \times 10^8)}$$

$$v \approx 4.17\text{ m/s}$$

Example 6 *A hydrogen like atom (described by the Bohr model) is observed to emit six wavelengths, originating from all possible transitions between a group of levels. These levels have energies between – 0.85 eV and –0.544 eV (including both these values).* **(JEE 2002)**

(a) Find the atomic number of the atom.

(b) Calculate the smallest wavelength emitted in these transitions.

(Take, $hc = 1240$ eV-nm, ground state energy of hydrogen atom $= -13.6$ eV)

Solution (a) Total 6 lines are emitted. Therefore,

$$\frac{n\,(n-1)}{2} = 6 \quad \text{or} \quad n = 4$$

So, transition is taking place between mth energy state and $(m+3)$ th energy state.

$$E_m = -0.85\text{ eV} \quad \text{or} \quad -13.6\left(\frac{Z^2}{m^2}\right) = -0.85$$

or $$\frac{Z}{m} = 0.25 \qquad \text{...(i)}$$

Similarly, $E_{m+3} = -0.544$ eV

or $$-13.6\,\frac{Z^2}{(m+3)^2} = -0.544$$

or $$\frac{Z}{(m+3)} = 0.2 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii) for Z and m, we get

$$m = 12 \text{ and } Z = 3$$ **Ans.**

(b) Smallest wavelength corresponds to maximum difference of energies which is obviously $E_{m+3} - E_m$

$\therefore$ $$\Delta E_{\max} = -0.544 - (-0.85) = 0.306\text{ eV}$$

$\therefore$ $$\lambda_{\min} = \frac{hc}{\Delta E_{\max}} = \frac{1240}{0.306} = 4052.3\text{ nm}$$ **Ans.**

Example 7 *A hydrogen like atom of atomic number Z is in an excited state of quantum number 2n. It can emit a maximum energy photon of 204 eV. If it makes a transition to quantum state n, a photon of energy 40.8 eV is emitted. Find n, Z and the ground state energy (in eV) of this atom. Also calculate the minimum energy (in eV) that can be emitted by this atom during de-excitation. Ground state energy of hydrogen atom is –13.6 eV.* **(JEE 2000)**

Solution Let ground state energy (in eV) be E_1.

Then, from the given condition

$$E_{2n} - E_1 = 204 \text{ eV}$$

or
$$\frac{E_1}{4n^2} - E_1 = 204 \text{ eV}$$

or
$$E_1\left(\frac{1}{4n^2} - 1\right) = 204 \text{ eV} \qquad \text{...(i)}$$

and
$$E_{2n} - E_n = 40.8 \text{ eV}$$

or
$$\frac{E_1}{4n^2} - \frac{E_1}{n^2} = 40.8 \text{ eV}$$

or
$$E_1\left(\frac{-3}{4n^2}\right) = 40.8 \text{ eV} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1 - \dfrac{1}{4n^2}}{\dfrac{3}{4n^2}} = 5 \quad \text{or} \quad 1 = \frac{1}{4n^2} + \frac{15}{4n^2}$$

or
$$\frac{4}{n^2} = 1 \quad \text{or} \quad n = 2$$

From Eq. (ii),

$$E_1 = -\frac{4}{3}n^2 (40.8) \text{ eV}$$

$$= -\frac{4}{3}(2)^2 (40.8) \text{ eV}$$

or
$$E_1 = -217.6 \text{ eV}$$

$$E_1 = -(13.6)Z^2$$

$\therefore$
$$Z^2 = \frac{E_1}{-13.6} = \frac{-217.6}{-13.6} = 16$$

$\therefore$
$$Z = 4$$

$$E_{\min} = E_{2n} - E_{2n-1} = \frac{E_1}{4n^2} - \frac{E_1}{(2n-1)^2}$$

$$= \frac{E_1}{16} - \frac{E_1}{9} = -\frac{7}{144}E_1$$

$$= -\left(\frac{7}{144}\right)(-217.6) \text{ eV}$$

$\therefore$
$$E_{\min} = 10.58 \text{ eV}$$

Example 8 *A hydrogen like atom (atomic number Z) is in a higher excited state of quantum number n. The excited atom can make a transition to the first excited state by successively emitting two photons of energy* 10.2 *eV and* 17.0 *eV, respectively. Alternatively, the atom from the same excited state can make a transition to the second excited state by successively emitting two photons of energies* 4.25 *eV and* 5.95 *eV, respectively.* **(JEE 1994)**

Determine the values of n and Z.

(Ionization energy of H-atom = 13.6 *eV)*

Solution From the given conditions,

$$E_n - E_2 = (10.2 + 17)\text{ eV} = 27.2\text{ eV} \qquad \text{...(i)}$$

and

$$E_n - E_3 = (4.25 + 5.95)\text{ eV} = 10.2\text{ eV} \qquad \text{...(ii)}$$

Eq. (i) – Eq. (ii) gives

$$E_3 - E_2 = 17.0\text{ eV} \quad \text{or} \quad Z^2\,(13.6)\left(\frac{1}{4} - \frac{1}{9}\right) = 17.0$$

$$\Rightarrow \qquad Z^2\,(13.6)\,(5/36) = 17.0$$

$$\Rightarrow \qquad Z^2 = 9 \quad \text{or} \quad Z = 3$$

From Eq. (i),

$$Z^2\,(13.6)\left(\frac{1}{4} - \frac{1}{n^2}\right) = 27.2 \quad \text{or} \quad (3)^2\,(13.6)\left(\frac{1}{4} - \frac{1}{n^2}\right) = 27.2$$

or

$$\frac{1}{4} - \frac{1}{n^2} = 0.222 \quad \text{or} \quad 1/n^2 = 0.0278$$

or

$$n^2 = 36 \Rightarrow n = 6$$

Type 3. *Based on X-rays*

Example 9 *Determine the energy of the characteristic X-ray* (K_β) *emitted from a tungsten* $(Z = 74)$ *target when an electron drops from the M-shell* $(n = 3)$ *to a vacancy in the K-shell* $(n = 1)$.

Solution Energy associated with the electron in the K-shell is approximately

$$E_K = -\,(74 - 1)^2\,(13.6\text{ eV}) = -\,72474\text{ eV}$$

An electron in the M-shell is subjected to an effective nuclear charge that depends on the number of electrons in the $n = 1$ and $n = 2$ states because these electrons shield the M electrons from the nucleus. Because there are eight electrons in the $n = 2$ state and one remaining in the $n = 1$ state, roughly nine electrons shield M electrons from the nucleus.

So,

$$Z_{\text{eff}} = z - 9$$

Hence, the energy associated with an electron in the M-shell is

$$E_M = \frac{-13.6\,Z_{\text{eff}}^2}{3^2}\text{ eV} = \frac{-13.6\,(Z - 9)^2}{3^2}\text{ eV}$$

$$= -\frac{(13.6)\,(74 - 9)^2}{9}\text{ eV} = -\,6384\text{ eV}$$

Therefore, emitted X-ray has an energy equal to

$$E_M - E_K = \{-6384 - (-72474)\}\text{ eV} = 66090\text{ eV} \qquad \textbf{Ans.}$$

Example 10 *The potential difference applied to an X-ray tube is 5 kV and the current through it is 3.2 mA. Then, the number of electrons striking the target per second is* **(JEE 2002)**

(a) 2×10^{16} *(b)* 5×10^{6}

(c) 1×10^{17} *(d)* 4×10^{15}

Solution $\because \quad i=\dfrac{q}{t}=\dfrac{ne}{t}$

$\therefore \quad n=\dfrac{i\,t}{e}$

Substituting $i=3.2\times10^{-3}$ A,

$e=1.6\times10^{-19}$ C and $t=1$ s

We get, $n=2\times10^{16}$

$\therefore$ The correct answer is (a).

Example 11 *X-rays are incident on a target metal atom having 30 neutrons. The ratio of atomic radius of the target atom and* ${}^4_2\text{He}$ *is* $(14)^{1/3}$. **(JEE 2005)**

(a) Find the mass number of target atom.

(b) Find the frequency of K_α *line emitted by this metal.*

Hint : *Radius of a nucleus (r) has the following relation with mass number (A).*

$$r\propto A^{1/3}$$

$(R=1.1\times10^{7}\,m^{-1}, c=3\times10^{8}\ m/s)$

Solution (a) From the relation $r\propto A^{1/3}$,

We have, $$\frac{r_2}{r_1}=\left(\frac{A_2}{A_1}\right)^{1/3}$$

or $$\left(\frac{A_2}{4}\right)^{1/3}=(14)^{1/3}$$

$\therefore$ $$A_2=56$$

(b) $Z_2=A_2-$ number of neutrons

$$=56-30=26$$

$\therefore$ $$f=Rc\,(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right)=\frac{3Rc}{4}\,(Z-1)^2$$

Substituting the given values of R, c and Z, we get

$$f=1.55\times10^{18}\text{ Hz}$$

Example 12 *Stopping potential of* 24, 100, 110 *and* 115 *kV are measured for photoelectrons emitted from a certain element when it is radiated with monochromatic X-ray. If this element is used as a target in an X-ray tube, what will be the wavelength of* K_α*-line?*

Solution Stopping potentials are 24, 100, 110 and 115 kV, i.e. if the electrons are emitted from conduction band, maximum kinetic energy of photoelectrons would be 115×10^{3} eV. If they are emitted from next inner shell, maximum kinetic energy of photoelectrons would be 110×10^{3} eV and so on.

For photoelectrons of L- shell it would be 100×10^3 eV and for K-shell it is 24×10^3 eV. Therefore, difference between energy of L-shell and K-shell is

$$\Delta E = E_L - E_K$$
$$= (100-24)\times10^3 \text{ eV}$$
$$= 76\times10^3 \text{ eV}$$

$\therefore$ Wavelength of K_α-line (transition of electron from L-shell to K-shell) is,

$$\lambda_{K_\alpha} \text{ (in Å)} = \frac{12375}{\Delta E \text{ (in eV)}} = \frac{12375}{76\times10^3}$$
$$= 0.163 \text{ Å}$$ **Ans.**

Example 13 *In Moseley's equation $\sqrt{f} = a(Z-b)$, a and b are constants. Find their values with the help of the following data.*

Element	Z	Wavelength of K_α X-rays
Mo	42	0.71 Å
Co	27	1.785 Å

Solution $\sqrt{f} = a\,(Z-b)$

or
$$\sqrt{\frac{c}{\lambda_1}} = a\,(Z_1-b) \qquad \ldots\text{(i)}$$

and
$$\sqrt{\frac{c}{\lambda_2}} = a\,(Z_2-b) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we have

$$\sqrt{c}\left[\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right] = a\,(Z_1-Z_2) \qquad \ldots\text{(iii)}$$

Solving above three equations with $c = 3.0\times10^8$ m/s, $\lambda_1 = 0.71\times10^{-10}$ m

$\lambda_2 = 1.785\times10^{-10}$ m, $Z_1 = 42$ and $Z_2 = 27$, we get

$a = 5\times10^7\ (\text{Hz})^{1/2}$ and $b = 1.37$ **Ans.**

Type 4. *Based on photoelectric effect*

Example 14 *A monochromatic light source of frequency f illuminates a metallic surface and ejects photoelectrons. The photoelectrons having maximum energy are just able to ionize the hydrogen atoms in ground state. When the whole experiment is repeated with an incident radiation of frequency $\frac{5}{6}f$, the photoelectrons so emitted are able to excite the hydrogen atom beam which then emits a radiation of wavelength 1215 Å.*

(a) What is the frequency of radiation?

(b) Find the work-function of the metal.

Solution (a) Using Einstein's equation of photoelectric effect,

$$K_{\max} = hf - W$$

Here, $$K_{\max} = 13.6 \text{ eV}$$

$\therefore$ $$hf - W = 13.6 \text{ eV} \qquad \ldots\text{(i)}$$

Further, $$h\left(\frac{5}{6}f\right) - W = \frac{12375}{1215} = 10.2 \text{ eV} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we have

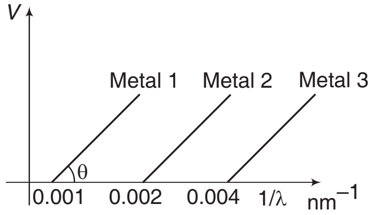
$$\frac{hf}{6} = 3.4 \text{ eV} \quad \text{or} \quad f = \frac{(6)\,(3.4)\,(1.6 \times 10^{-19})}{(6.63 \times 10^{-34})}$$

$$= 4.92 \times 10^{15} \text{ Hz}$$ **Ans.**

(b) $$W = hf - 13.6 \qquad \text{[from Eq. (i)]}$$

$$= 6\,(3.4) - 13.6 = 6.8 \text{ eV}$$ **Ans.**

Example 15 *The graph between* $1/\lambda$ *and stopping potential* (V) *of three metals having work-functions* ϕ_1, ϕ_2 *and* ϕ_3 *in an experiment of photoelectric effect is plotted as shown in the figure. Which of the following statement(s) is/are correct? (Here,* λ *is the wavelength of the incident ray).* **(JEE 2006)**



(a) *Ratio of work-functions* $\phi_1 : \phi_2 : \phi_3 = 1 : 2 : 4$

(b) *Ratio of work-functions* $\phi_1 : \phi_2 : \phi_3 = 4 : 2 : 1$

(c) $\tan\theta$ *is directly proportional to* hc/e*, where h is Planck constant and c is the speed of light*

(d) *The violet colour light can eject photoelectrons from metals 2 and 3*

Solution From the relation,

$$eV = \frac{hc}{\lambda} - \phi \quad \text{or} \quad V = \left(\frac{hc}{e}\right)\left(\frac{1}{\lambda}\right) - \frac{\phi}{e}$$

This is equation of straight line. Slope is $\tan\theta = \frac{hc}{e}$.

Further $V = 0$ at $\phi = \frac{hc}{\lambda}$

$\therefore$ $$\phi_1 : \phi_2 : \phi_3 = \frac{hc}{\lambda_{01}} : \frac{hc}{\lambda_{02}} : \frac{hc}{\lambda_{03}} = \frac{1}{\lambda_{01}} : \frac{1}{\lambda_{02}} : \frac{1}{\lambda_{03}} = 1 : 2 : 4$$

$$\frac{1}{\lambda_{01}} = 0.001 \text{ nm}^{-1} \quad \text{or} \quad \lambda_{01} = 10000 \text{ Å}$$

$$\frac{1}{\lambda_{02}} = 0.002 \text{ nm}^{-1} \quad \text{or} \quad \lambda_{02} = 5000 \text{ Å}$$

$$\frac{1}{\lambda_{03}} = 0.004 \text{ nm}^{-1} \quad \text{or} \quad \lambda_{03} = 2500 \text{ Å}$$

Violet colour has wavelength 4000 Å.

So, violet colour can eject photoelectrons from metal 1 and metal 2.

$\therefore$ The correct options are (a) and (c).

Example 16 *A beam of light has three wavelengths* 4144 Å, 4972 Å *and* 6216 Å *with a total intensity of* $3.6 \times 10^{-3}\ \text{Wm}^{-2}$ *equally distributed amongst the three wavelengths. The beam falls normally on an area* 1.0 cm^2 *of a clean metallic surface of work-function* 2.3 *eV. Assume that there is no loss of light by reflection and that each energetically capable photon ejects one electron. Calculate the number of photoelectrons liberated in two seconds.* **(JEE 1989)**

Solution Energy of photon having wavelength 4144 Å,

$$E_1 = \frac{12375}{4144}\ \text{eV}$$

$$= 2.99\ \text{eV}$$

Similarly,

$$E_2 = \frac{12375}{4972}\ \text{eV}$$

$$= 2.49\ \text{eV and}$$

$$E_3 = \frac{12375}{6216}\ \text{eV}$$

$$= 1.99\ \text{eV}$$

Since, only E_1 and E_2 are greater than the work-function $W = 2.3$ eV, only first two wavelengths are capable for ejecting photoelectrons. Given intensity is equally distributed in all wavelengths. Therefore, intensity corresponding to each wavelength is

$$\frac{3.6 \times 10^{-3}}{3} = 1.2 \times 10^{-3}\ \text{W/m}^2$$

Or energy incident per second in the given area ($A = 1.0\ \text{cm}^2 = 10^{-4}\ \text{m}^2$) is

$$P = 1.2 \times 10^{-3} \times 10^{-4}$$

$$= 1.2 \times 10^{-7}\ \text{J/s}$$

Let n_1 be the number of photons incident per unit time in the given area corresponding to first wavelength. Then,

$$n_1 = \frac{P}{E_1}$$

$$= \frac{1.2 \times 10^{-7}}{2.99 \times 1.6 \times 10^{-19}}$$

$$= 2.5 \times 10^{11}$$

Similarly,

$$n_2 = \frac{P}{E_2}$$

$$= \frac{1.2 \times 10^{-7}}{2.49 \times 1.6 \times 10^{-19}}$$

$$= 3.0 \times 10^{11}$$

Since, each energetically capable photon ejects one electron, total number of photoelectrons liberated in 2 s.

$$= 2(n_1 + n_2)$$

$$= 2\,(2.5 + 3.0) \times 10^{11}$$

$$= 1.1 \times 10^{12}$$

Ans.

Miscellaneous Examples

Example 17 *Two metallic plates A and B each of area $5 \times 10^{-4}\ m^2$, are placed parallel to each other at a separation of 1 cm. Plate B carries a positive charge of 33.7×10^{-12} C. A monochromatic beam of light, with photons of energy 5 eV each, starts falling on plate A at $t = 0$ so that 10^{16} photons fall on it per square metre per second. Assume that one photoelectron is emitted for every 10^6 incident photons. Also assume that all the emitted photoelectrons are collected by plate B and the work-function of plate A remains constant at the value 2 eV.*

Determine **(JEE 2002)**

(a) the number of photoelectrons emitted upto $t = 10$ s,

(b) the magnitude of the electric field between the plates A and B at $t = 10$ s and

(c) the kinetic energy of the most energetic photoelectrons emitted at $t = 10$s when it reaches plate B.

Neglect the time taken by the photoelectron to reach plate B.

(Take, $\varepsilon_0 = 8.85 \times 10^{-12} C^2/N\text{-}m^2$).

Solution Area of plates, $= 5 \times 10^{-4}\ \text{m}^2$

Distance between the plates, $d = 1\ \text{cm} = 10^{-2}\ \text{m}$

(a) Number of photoelectrons emitted upto $t = 10$ s are

$$n = \frac{(\text{number of photons falling on unit area in unit time}) \times (\text{area} \times \text{time})}{10^6}$$

$$= \frac{1}{10^6}\,[(10)^{16} \times (5 \times 10^{-4}) \times (10)]$$

$$= 5.0 \times 10^7$$ **Ans.**

(b) At time $t = 10$ s,

Charge on plate *A*, $q_A = +\,ne = (5.0 \times 10^7)\,(1.6 \times 10^{-19})$

$$= 8.0 \times 10^{-12}\ \text{C}$$

and charge on plate *B*,

$$q_B = (33.7 \times 10^{-12} - 8.0 \times 10^{-12})$$

$$= 25.7 \times 10^{-12}\ \text{C}$$

$\therefore$ Electric field between the plates, $E = \dfrac{(q_B - q_A)}{2A\varepsilon_0}$

or

$$E = \frac{(25.7 - 8.0) \times 10^{-12}}{2 \times (5 \times 10^{-4})\,(8.85 \times 10^{-12})}$$

$$= 2 \times 10^3\ \text{N/C}$$

(c) Energy of most energetic photoelectrons at plate *A*,

$$= E - W = (5 - 2)\ \text{eV} = 3\ \text{eV}$$

Increase in energy of photoelectrons

$$= (eEd) \text{ joule} = (Ed) \text{ eV}$$
$$= (2 \times 10^3)(10^{-2}) \text{ eV} = 20 \text{ eV}$$

Energy of photoelectrons at plate B

$$= (20 + 3) \text{ eV} = 23 \text{ eV}$$ **Ans.**

Example 18 *Photoelectrons are emitted when 400 nm radiation is incident on a surface of work-function 1.9 eV. These photoelectrons pass through a region containing α-particles. A maximum energy electron combines with an α-particle to form a He$^+$ ion, emitting a single photon in this process. He$^+$ ions thus formed are in their fourth excited state. Find the energies in eV of the photons lying in the 2 to 4 eV range, that are likely to be emitted during and after the combination. [Take, $h = 4.14 \times 10^{-15}$ eV-s]* **(JEE 1999)**

Solution Given work-function $W = 1.9$ eV

Wavelength of incident light, $\lambda = 400$ nm

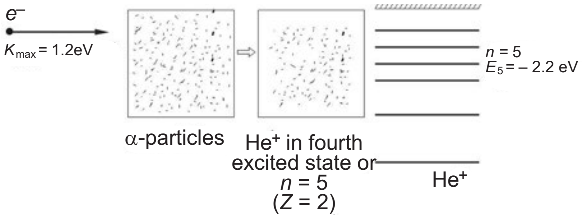
$\therefore$ Energy of incident light, $E = \dfrac{hc}{\lambda} = 3.1$ eV

(Substituting the values of h, c and λ)

Therefore, maximum kinetic energy of photoelectrons

$$K_{\max} = E - W = (3.1 - 1.9) = 1.2 \text{ eV}$$

Now, the situation is as shown in figure.



Energy of electron in 4th excited state of He$^+$ ($n = 5$) will be

$$E_5 = -13.6 \frac{Z^2}{n^2} \text{ eV}$$

$$\Rightarrow \qquad E_5 = -(13.6)\frac{(2)^2}{(5)^2} = -2.2 \text{ eV}$$

Therefore, energy released during the combination

$$= 1.2 - (-2.2) = 3.4 \text{ eV}$$

Similarly, energies in other energy states of He$^+$ will be

$$E_4 = -13.6 \frac{(2)^2}{(4)^2} = -3.4 \text{ eV}$$

$$E_3 = -13.6 \frac{(2)^2}{(3)^2} = -6.04 \text{ eV}$$

$$E_2 = -13.6 \frac{(2)^2}{(2)^2} = -13.6 \text{ eV}$$

The possible transitions are

$$\Delta E_{5\to 4} = E_5 - E_4 = 1.2 \text{ eV} < 2 \text{ eV}$$
$$\Delta E_{5\to 3} = E_5 - E_3 = 3.84 \text{ eV}$$
$$\Delta E_{5\to 2} = E_5 - E_2 = 11.4 \text{ eV} > 4 \text{ eV}$$
$$\Delta E_{4\to 3} = E_4 - E_3 = 2.64 \text{ eV}$$
$$\Delta E_{4\to 2} = E_4 - E_2 = 10.2 \text{ eV} > 4 \text{ eV}$$

Hence, the energy of emitted photons in the range of 2 eV and 4 eV are 3.4 eV during combination and 3.84 eV and 2.64 after combination.

Example 19 *Light from a discharge tube containing hydrogen atoms falls on the surface of a piece of sodium. The kinetic energy of the fastest photoelectrons emitted from sodium is* 0.73 *eV. The work-function for sodium is* 1.82 *eV.* **(JEE 1992)**

Find

(a) the energy of the photons causing the photoelectrons emission.

(b) the quantum numbers of the two levels involved in the emission of these photons.

(c) the change in the angular momentum of the electron in the hydrogen atom, in the above transition, and

(d) the recoil speed of the emitting atom assuming it to be at rest before the transition. (Ionization potential of hydrogen is 13.6 *eV.)*

Solution (a) From Einstein's equation of photoelectric effect,

Energy of photons causing the photoelectric emission

= Maximum kinetic energy of emitted photons + work-function

or $$E = K_{\max} + W = (0.73 + 1.82) \text{ eV}$$

or $$E = 2.55 \text{ eV}$$ **Ans.**

(b) In case of a hydrogen atom,

$$E_1 = -13.6 \text{ eV}, E_2 = -3.4 \text{ eV}, E_3 = -1.5 \text{ eV}$$
$$E_4 = -0.85 \text{ eV}$$

Since, $$E_4 - E_2 = 2.55 \text{ eV}$$

Therefore, quantum numbers of the two levels involved in the emission of these photons are 4 and 2$(4 \to 2)$.

(c) Change in angular momentum in transition from 4 to 2 will be

$$\Delta L = L_2 - L_4 = 2\left(\frac{h}{2\pi}\right) - 4\left(\frac{h}{2\pi}\right) \quad \text{or} \quad \Delta L = -\frac{h}{\pi}$$

(d) From conservation of linear momentum

| Momentum of hydrogen atom | = | Momentum of emitted photon |

or $$mv = \frac{E}{c}$$ (m = mass of hydrogen atom)

or $$v = \frac{E}{mc} = \frac{(2.55 \times 1.6 \times 10^{-19} \text{ J})}{(1.67 \times 10^{-27} \text{kg})\,(3.0 \times 10^8 \text{ m/s})}$$

$$v = 0.814 \text{ m/s}$$ **Ans.**

Example 20 *If an X-ray tube operates at the voltage of* $10\ kV$*, find the ratio of the de-Broglie wavelength of the incident electrons to the shortest wavelength of X-rays produced. The specific charge of electron is* $1.8 \times 10^{11}\ C/kg$.

Solution de-Broglie wavelength when a charge q is accelerated by a potential difference of V volts is

$$\lambda_b = \frac{h}{\sqrt{2qVm}} \quad \text{...(i)}$$

For cut off wavelength of X-rays, we have $qV = \dfrac{hc}{\lambda_m}$

or $$\lambda_m = \frac{hc}{qV} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $$\frac{\lambda_b}{\lambda_m} = \frac{\sqrt{\frac{qV}{2m}}}{c}$$

For electron $\dfrac{q}{m} = 1.8 \times 10^{11}$ C/kg (given). Substituting the values the desired ratio is

$$\frac{\lambda_b}{\lambda_m} = \frac{\sqrt{\frac{1.8 \times 10^{11} \times 10 \times 10^3}{2}}}{3 \times 10^8} = 0.1$$ **Ans.**

Example 21 *The wavelength of the first line of Lyman series for hydrogen is identical to that of the second line of Balmer series for some hydrogen like ion x. Calculate energies of the first four levels of x.*

Solution Wavelength of the first line of Lyman series for hydrogen atom will be given by the equation

$$\frac{1}{\lambda_1} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3R}{4} \quad \text{...(i)}$$

The wavelength of second Balmer line for hydrogen like ion x is

$$\frac{1}{\lambda_2} = RZ^2\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = \frac{3RZ^2}{16} \quad \text{...(ii)}$$

Given that $\lambda_1 = \lambda_2$ or $\dfrac{1}{\lambda_1} = \dfrac{1}{\lambda_2}$

i.e. $$\frac{3R}{4} = \frac{3RZ^2}{16}$$

$\therefore$ $Z = 2$

i.e. x ion is He^+. The energies of first four levels of x are

$$E_1 = -(13.6)\,Z^2 = -54.4 \text{ eV}$$

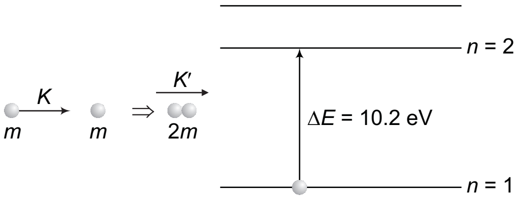
$$E_2 = \frac{E_1}{(2)^2} = -13.6 \text{ eV}$$

$$E_3 = \frac{E_1}{(3)^2} = -6.04 \text{ eV}$$

and $$E_4 = \frac{E_1}{(4)^2} = -3.4 \text{ eV}$$ **Ans.**

Example 22 *A moving hydrogen atom makes a head on collision with a stationary hydrogen atom. Before collision both atoms are in ground state and after collision they move together. What is the minimum value of the kinetic energy of the moving hydrogen atom, such that one of the atoms reaches one of the excited state?*

Solution Let K be the kinetic energy of the moving hydrogen atom and K', the kinetic energy of combined mass after collision.



From conservation of linear momentum,

$$p = p' \quad \text{or} \quad \sqrt{2Km} = \sqrt{2K'(2m)}$$

or $$K = 2K' \qquad \text{...(i)}$$

From conservation of energy, $$K = K' + \Delta E \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $$\Delta E = \frac{K}{2}$$

Now, minimum value of ΔE for hydrogen atom is 10.2 eV.

or $$\Delta E \geq 10.2 \text{ eV}$$

$\therefore$ $$\frac{K}{2} \geq 10.2$$

$\therefore$ $$K \geq 20.4 \text{ eV}$$

Therefore, the minimum kinetic energy of moving hydrogen is 20.4 eV. **Ans.**

Example 23 *An imaginary particle has a charge equal to that of an electron and mass* 100 *times the mass of the electron. It moves in a circular orbit around a nucleus of charge* + 4e*. Take the mass of the nucleus to be infinite. Assuming that the Bohr model is applicable to this system.*

(a) Derive an expression for the radius of nth Bohr orbit.

(b) Find the wavelength of the radiation emitted when the particle jumps from fourth orbit to the second orbit.

Solution (a) We have $$\frac{m_p v^2}{r_n} = \frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{r_n^2} \qquad \text{...(i)}$$

The quantization of angular momentum gives

$$m_p v r_n = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$r = \frac{n^2 h^2 \varepsilon_0}{Z\pi m_p e^2}$$

Substituting $$m_p = 100 \text{ m}$$

where, m = mass of electron and $Z = 4$

We get, $$r_n = \frac{n^2 h^2 \varepsilon_0}{400\,\pi\, me^2}$$ **Ans.**

(b) As we know, $$E_1^H = -13.60 \text{ eV}$$

and $$E_n \propto \left(\frac{Z^2}{n^2}\right) m$$

For the given particle, $$E_4 = \frac{(-13.60)\,(4)^2}{(4)^2} \times 100$$

$$= -1360 \text{ eV}$$

and $$E_2 = \frac{(-13.60)\,(4)^2}{(2)^2} \times 100$$

$$= -5440 \text{ eV}$$

$$\Delta E = E_4 - E_2$$

$$= 4080 \text{ eV}$$

$\therefore$ $$\lambda \text{ (in Å)} = \frac{12375}{\Delta E \text{ (in eV)}}$$

$$= \frac{12375}{4080}$$

$$= 3.0 \text{ Å}$$ **Ans.**

Example 24 *The energy levels of a hypothetical one electron atom are given by*

$$E_n = -\frac{18.0}{n^2} \, eV$$

where $n = 1, 2, 3, \ldots$

(a) Compute the four lowest energy levels and construct the energy level diagram.

(b) What is the first excitation potential

(c) What wavelengths (Å) can be emitted when these atoms in the ground state are bombarded by electrons that have been accelerated through a potential difference of 16.2 *V?*

(d) If these atoms are in the ground state, can they absorb radiation having a wavelength of 2000 *Å?*

(e) What is the photoelectric threshold wavelength of this atom?

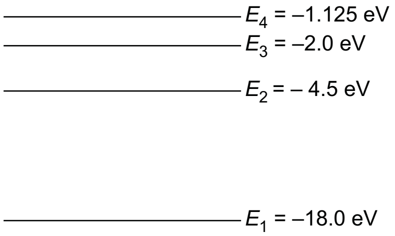
Solution (a) $E_1 = \dfrac{-18.0}{(1)^2} = -18.0 \text{ eV}$

$$E_2 = \frac{-18.0}{(2)^2} = -4.5 \text{ eV}$$

$$E_3 = \frac{-18.0}{(3)^2} = -2.0 \text{ eV}$$

and $$E_4 = \frac{-18.0}{(4)^2} = -1.125 \text{ eV}$$

The energy level diagram is shown in figure.



(b) $E_2 - E_1 = 13.5$ eV

$\therefore$ First excitation potential is 13.5 V.

(c) Energy of the electron accelerated by a potential difference of 16.2 V is 16.2 eV. With this energy the electron can excite the atom from $n = 1$ to $n = 3$ as

$$E_4 - E_1 = -1.125 - (-18.0) = 16.875 \text{ eV} > 16.2 \text{ eV}$$

and

$$E_3 - E_1 = -2.0 - (-18.0) = 16.0 \text{ eV} < 16.2 \text{ eV}$$

Now,

$$\lambda_{32} = \frac{12375}{E_3 - E_2} = \frac{12375}{-2.0 - (-4.5)}$$

$$= 4950 \text{ Å}$$ **Ans.**

$$\lambda_{31} = \frac{12375}{E_3 - E_1} = \frac{12375}{16} = 773 \text{ Å}$$ **Ans.**

and

$$\lambda_{21} = \frac{12375}{E_2 - E_1} = \frac{12375}{-4.5 - (-18.0)}$$

$$= 917 \text{ Å}$$ **Ans.**

(d) No, the energy corresponding to $\lambda = 2000$ Å is

$$E = \frac{12375}{2000} = 6.1875 \text{ eV}$$ **Ans.**

The minimum excitation energy is 13.5 eV ($n = 1$ to $n = 2$).

(e) Threshold wavelength for photoemission to take place from such an atom is

$$\lambda_{\min} = \frac{12375}{18}$$

$$= 687.5 \text{ Å}$$ **Ans.**

Example 25 *In a photocell the plates P and Q have a separation of 5 cm, which are connected through a galvanometer without any cell. Bichromatic light of wavelengths 4000 Å and 6000 Å are incident on plate Q whose work-function is 2.39 eV. If a uniform magnetic field B exists parallel to the plates, find the minimum value of B for which the galvanometer shows zero deflection.*

Solution Energy of photons corresponding to light of wavelength $\lambda_1 = 4000$ Å is

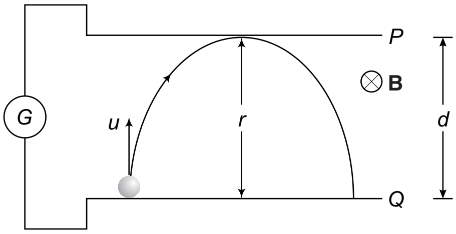
$$E_1 = \frac{12375}{4000} = 3.1 \text{ eV}$$

and that corresponding to $\lambda_2 = 6000$ Å is

$$E_2 = \frac{12375}{6000} = 2.06 \text{ eV}$$

As, $E_2 < W$ and $E_1 > W$ (W = work-function)

Photoelectric emission is possible with λ_1 only.



Photoelectrons experience magnetic force and move along a circular path. The galvanometer will indicate zero deflection if the photoelectrons just complete semicircular path before reaching the plate P.

Thus, $d = r = 5$ cm

$\therefore$ $r = 5 \text{ cm} = 0.05 \text{ m}$

Further, $$r = \frac{mv}{Bq} = \frac{\sqrt{2Km}}{Bq}$$

$\therefore$ $$B_{\min} = \frac{\sqrt{2Km}}{rq}$$

Here, $K = E_1 - W = (3.1 - 2.39)$

$= 0.71$ eV

Substituting the values, we have

$$B_{\min} = \frac{\sqrt{2 \times 0.71 \times 1.6 \times 10^{-19} \times 9.109 \times 10^{-31}}}{(0.05)\,(1.6 \times 10^{-19})}$$

$$= 5.68 \times 10^{-5} \text{ T}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. **Assertion :** X-rays cannot be deflected by electric or magnetic fields.

Reason : These are electromagnetic waves.

2. **Assertion :** If wavelength of light is doubled, energy and momentum of photons are reduced to half.

Reason : By increasing the wavelength, speed of photons will decrease.

3. **Assertion :** We can increase the saturation current in photoelectric experiment without increasing the intensity of light.

Reason : Intensity can be increased by increasing the frequency of incident photons.

4. **Assertion :** Photoelectric effect proves the particle nature of light.

Reason : Photoemission starts as soon as light is incident on the metal surface, provided frequency of incident light is greater than or equal to the threshold frequency.

5. **Assertion :** During de-excitation from $n = 6$ to $n = 3$, total six emission lines may be obtained.

Reason : From $n = n$ to $n = 1$, total $\frac{n\,(n-1)}{2}$ emission lines are obtained.

6. **Assertion :** If frequency of incident light is doubled, the stopping potential will also become two times.

Reason : Stopping potential is given by

$$V_0 = \frac{h}{e}(\nu - \nu_0)$$

7. **Assertion :** X-rays cannot be obtained in the emission spectrum of hydrogen atom.

Reason : Maximum energy of photons emitted from hydrogen spectrum is 13.6 eV.

8. **Assertion :** If applied potential difference in coolidge tube is increased, then difference between K_α wavelength and cut off wavelength will increase.

Reason : Cut off wavelength is inversely proportional to the applied potential difference in coolidge tube.

9. **Assertion :** In $n = 2$, energy of electron in hydrogen like atoms is more compared to $n = 1$.

Reason : Electrostatic potential energy in $n = 2$ is more.

10. **Assertion :** In continuous X-ray spectrum, all wavelengths can be obtained.

Reason : Accelerated (or retarded) charged particles radiate energy. This is the cause of production of continuous X-rays.

Objective Questions

1. According to Einstein's photoelectric equation, the plot of the maximum kinetic energy of the emitted photoelectrons from a metal *versus* frequency of the incident radiation gives a straight line whose slope

(a) depends on the nature of metal used

(b) depends on the intensity of radiation

(c) depends on both intensity of radiation and the nature of metal used

(d) is the same for all metals and independent of the intensity of radiation

2. The velocity of the electron in the first Bohr orbit as compared to that of light is about

(a) 1/300 (b) 1/500

(c) 1/137 (d) 1/187

3. ${}_{86}A^{222} \to {}_{84}B^{210}$. In this reaction, how many α and β particles are emitted?

(a) $6\alpha, 3\beta$ (b) $3\alpha, 4\beta$

(c) $4\alpha, 3\beta$ (d) $3\alpha, 6\beta$

4. An X-ray tube is operated at 20 kV. The cut off wavelength is

(a) 0.89 Å (b) 0.75 Å

(c) 0.62 Å (d) None of these

5. An X-ray tube is operated at 18 kV. The maximum velocity of electron striking the target is

(a) 8×10^7 m/s (b) 6×10^7 m/s

(c) 5×10^7 m/s (d) None of these

6. What is the ratio of de-Broglie wavelength of electron in the second and third Bohr orbits in the hydrogen atoms?

(a) 2/3 (b) 3/2

(c) 4/3 (d) 3/4

7. The energy of a hydrogen like atom (or ion) in its ground state is – 122.4 eV. It may be

(a) hydrogen atom (b) He^+

(c) Li^{2+} (d) Be^{3+}

8. The operating potential in an X-ray tube is increased by 2%. The percentage change in the cut off wavelength is

(a) 1% increase (b) 2% increase

(c) 2% decrease (d) 1% decrease

9. The energy of an atom or ion in the first excited state is –13.6 eV. It may be

(a) He^+ (b) Li^{++}

(c) hydrogen (d) deuterium

10. In order that the short wavelength limit of the continuous X-ray spectrum be 1 Å, the potential difference through which an electron must be accelerated is

(a) 124 kV (b) 1.24 kV

(c) 12.4 kV (d) 1240 kV

11. The momentum of an X-ray photon with $\lambda = 0.5$ Å is

(a) 13.26×10^{-26} kg-m/s

(b) 1.326×10^{-26} kg-m/s

(c) 13.26×10^{-24} kg-m/s

(d) 13.26×10^{-22} kg-m/s

12. The work-function of a substance is 1.6 eV. The longest wavelength of light that can produce photoemission from the substance is

(a) 7750 Å (b) 3875 Å
(c) 5800 Å (d) 2900 Å

13. Find the binding energy of an electron in the ground state of a hydrogen like atom in whose spectrum the third Balmer line is equal to 108.5 nm.

(a) 54.4 eV (b) 13.6 eV
(c) 112.4 eV (d) None of these

14. Let the potential energy of hydrogen atom in the ground state be zero. Then, its energy in the first excited state will be

(a) 10.2 eV (b) 13.6 eV
(c) 23.8 eV (d) 27.2 eV

15. Light of wavelength 330 nm falling on a piece of metal ejects electrons with sufficient energy with required voltage V_0 to prevent them from reaching a collector. In the same set up, light of wavelength 220 nm ejects electrons which require twice the voltage V_0 to stop them in reaching a collector. The numerical value of voltage V_0 is

(a) $\frac{16}{15}$ V (b) $\frac{15}{16}V$
(c) $\frac{15}{8}V$ (d) $\frac{8}{15}V$

16. Maximum kinetic energy of a photoelectron is E when the wavelength of incident light is λ. If energy becomes four times when wavelength is reduced to one-third, then work-function of the metal is

(a) $\frac{3\,hc}{\lambda}$ (b) $\frac{hc}{3\lambda}$
(c) $\frac{hc}{\lambda}$ (d) $\frac{hc}{2\lambda}$

17. If the frequency of K_α X-ray emitted from the element with atomic number 31 is f, then the frequency of K_α X-ray emitted from the element with atomic number 51 would be

(a) $\frac{5f}{3}$ (b) $\frac{51f}{31}$
(c) $\frac{9f}{25}$ (d) $\frac{25f}{9}$

18. According to Moseley's law, the ratio of the slope of graph between $\sqrt{f}$ and Z for K_β and K_α is

(a) $\sqrt{\frac{32}{27}}$ (b) $\sqrt{\frac{27}{32}}$
(c) $\sqrt{\frac{5}{36}}$ (d) $\sqrt{\frac{36}{5}}$

19. If the electron in hydrogen orbit jumps from third orbit to second orbit, the wavelength of the emitted radiation is given by

(a) $\lambda = \frac{R}{6}$ (b) $\lambda = \frac{5}{R}$
(c) $\lambda = \frac{36}{5R}$ (d) $\lambda = \frac{5R}{36}$

20. A potential of 10000 V is applied across an X-ray tube. Find the ratio of de-Broglie wavelength associated with incident electrons to the minimum wavelength associated with X-rays. (Given, $e/m = 1.8 \times 10^{11}$ C/kg for electrons)

(a) 10 (b) 20
(c) 1/10 (d) 1/20

21. When a metallic surface is illuminated with monochromatic light of wavelength λ, the stopping potential is $5\,V_0$. When the same surface is illuminated with the light of wavelength 3λ, the stopping potential is V_0. Then, the work-function of the metallic surface is

(a) $hc/6\lambda$ (b) $hc/5\lambda$
(c) $hc/4\lambda$ (d) $2hc/4\lambda$

22. The threshold frequency for a certain photosensitive metal is ν_0. When it is illuminated by light of frequency $\nu = 2\nu_0$, the stopping potential for photoelectric current is V_0. What will be the stopping potential when the same metal is illuminated by light of frequency $\nu = 3\nu_0$?

(a) $1.5\,V_0$ (b) $2\,V_0$
(c) $2.5\,V_0$ (d) $3\,V_0$

23. The frequency of the first line in Lyman series in the hydrogen spectrum is ν. What is the frequency of the corresponding line in the spectrum of doubly ionized Lithium?

(a) ν (b) $3\,\nu$
(c) $9\,\nu$ (d) $2\,\nu$

24. Which energy state of doubly ionized lithium (Li^{++}) has the same energy as that of the ground state of hydrogen?

(a) $n = 1$ (b) $n = 2$
(c) $n = 3$ (d) $n = 4$

25. Two identical photo-cathodes receive light of frequencies ν_1 and ν_2. If the velocities of the photoelectrons (of mass m) coming out are v_1 and v_2 respectively, then

(a) $v_1 - v_2 = \left[\frac{2h}{m}(\nu_1 - \nu_2)\right]^{1/2}$ (b) $v_1^2 - v_2^2 = \frac{2h}{m}(\nu_1 - \nu_2)$

(c) $v_1 + v_2 = \left[\frac{2h}{m}(\nu_1 - \nu_2)\right]^{1/2}$ (d) $v_1^2 + v_2^2 = \frac{2h}{m}(\nu_1 - \nu_2)$

26. The longest wavelength of the Lyman series for hydrogen atom is the same as the wavelength of a certain line in the spectrum of He^+ when the electron makes a transition from $n \to 2$. The value of n is

(a) 3 (b) 4 (c) 5 (d) 6

27. The wavelength of the K_α- line for the uranium is ($Z = 92$) ($R = 1.0973 \times 10^7\ \text{m}^{-1}$)

(a) 1.5 Å (b) 0.5 Å (c) 0.15 Å (d) 2.0 Å

28. The frequencies of K_α, K_β and L_α X-rays of a material are γ_1, γ_2 and γ_3 respectively. Which of the following relation holds good?

(a) $\gamma_2 = \sqrt{\gamma_1 \gamma_3}$ (b) $\gamma_2 = \gamma_1 + \gamma_3$
(c) $\gamma_2 = \frac{\gamma_1 + \gamma_3}{2}$ (d) $\gamma_3 = \sqrt{\gamma_1 \gamma_2}$

29. A proton and an α-particle are accelerated through same potential difference. Then, the ratio of de-Broglie wavelength of proton and α-particle is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $2\sqrt{2}$ (d) None of these

30. If E_1, E_2 and E_3 represent respectively the kinetic energies of an electron , an α-particle and a proton each having same de-Broglie wavelength, then

(a) $E_1 > E_3 > E_2$ (b) $E_2 > E_3 > E_1$
(c) $E_1 > E_2 > E_3$ (d) $E_1 = E_2 = E_3$

31. If the potential energy of a hydrogen atom in the ground state is assumed to be zero, then total energy of $n = \infty$ is equal to

(a) 13.6 eV (b) 27.2 eV
(c) zero (d) None of these

32. A 1000 W transmitter works at a frequency of 880 kHz. The number of photons emitted per second is

(a) 1.7×10^{28} (b) 1.7×10^{30}
(c) 1.7×10^{23} (d) 1.7×10^{25}

33. Electromagnetic radiation of wavelength 3000 Å is incident on an isolated platinum surface of work-function 6.30 eV. Due to the radiation, the

(a) sphere becomes positively charged
(b) sphere becomes negatively charged
(c) sphere remains neutral
(d) maximum kinetic energy of the ejected photoelectrons would be 2.03 eV

34. The energy of a hydrogen atom in its ground state is – 13.6 eV. The energy of the level corresponding to the quantum number $n = 5$ is

(a) – 0.54 eV (b) – 5.40 eV
(c) – 0.85 eV (d) – 2.72 eV

35. Ultraviolet radiation of 6.2 eV falls on an aluminium surface (work-function = 4.2 eV). The kinetic energy in joule of the fastest electrons emitted is

(a) 3.2×10^{-21} (b) 3.2×10^{-19}
(c) 3.2×10^{-17} (d) 3.2×10^{-15}

36. What should be the velocity of an electron so that its momentum becomes equal to that of a photon of wavelength 5200 Å?

(a) 700 m/s (b) 1000 m/s
(c) 1400 m/s (d) 2800 m/s

37. Photoelectric work-function of a metal is 1 eV. Light of wavelength $\lambda = 3000$ Å falls on it. The photoelectrons come out with maximum velocity

(a) 10 m/s (b) 10^3 m/s
(c) 10^4 m/s (d) 10^6 m/s

Subjective Questions

Note *You can take approximations in the answers.*

$h = 6.62 \times 10^{-34}$ J-s, $c = 3.0 \times 10^8$ m/s, $m_e = 9.1 \times 10^{-31}$ kg and 1 eV $= 1.6 \times 10^{-19}$ J

1. For a given element the wavelength of the K_α-line is 0.71 nm and of the K_β-line it is 0.63 nm. Use this information to find wavelength of the L_α-line.

2. The energy of the $n = 2$ state in a given element is $E_2 = -2870$ eV. Given that the wavelengths of the K_α and K_β-lines are 0.71 nm and 0.63 nm respectively, determine the energies E_1 and E_3.

3. 1.5 mW of 400 nm light is directed at a photoelectric cell. If 0.1% of the incident photons produce photoelectrons, find the current in the cell.

4. A photon has momentum of magnitude 8.24×10^{-28} kg-m/s.

(a) What is the energy of this photon? Give your answer in joules and in electron volts.

(b) What is the wavelength of this photon? In what region of the electromagnetic spectrum does it lie?

5. A 75 W light source emits light of wavelength 600 nm.

(a) Calculate the frequency of the emitted light.

(b) How many photons per second does the source emit?

6. An excited nucleus emits a gamma-ray photon with energy of 2.45 MeV.

(a) What is the photon frequency? (b) What is the photon wavelength?

7. (a) A proton is moving at a speed much less than the speed of light. It has kinetic energy K_1 and momentum p_1. If the momentum of the proton is doubled, so $p_2 = 2p_1$, how is its new kinetic energy K_2 related to K_1?

(b) A photon with energy E_1 has momentum p_1. If another photon has momentum p_2 that is twice p_1, how is the energy E_2 of the second photon related to E_1?

8. A parallel beam of monochromatic light of wavelength 500 nm is incident normally on a perfectly absorbing surface. The power through any cross-section of the beam is 10 W. Find

(a) the number of photons absorbed per second by the surface and

(b) the force exerted by the light beam on the surface.

9. A beam of white light is incident normally on a plane surface absorbing 70% of the light and reflecting the rest. If the incident beam carries 10 W of power, find the force exerted by it on the surface.

10. A parallel beam of monochromatic light of wavelength 663 nm is incident on a totally reflecting plane mirror. The angle of incidence is 60° and the number of photons striking the mirror per second is 1.0×10^{19}. Calculate the force exerted by the light beam on the mirror.

11. **Wavelength of Bullet**. Calculate the de-Broglie wavelength of a 5.00 g bullet that is moving at 340 m/s. Will it exhibit wave like properties?

12. (a) An electron moves with a speed of 4.70×10^6 m/s. What is its de-Broglie wavelength?

(b) A proton moves with the same speed. Determine its de-Broglie wavelength.

13. An electron has a de-Broglie wavelength of 2.80×10^{-10} m. Determine

(a) the magnitude of its momentum,

(b) its kinetic energy (in joule and in electron volt).

14. Find de-Broglie wavelength corresponding to the root-mean square velocity of hydrogen molecules at room temperature (20°C).

15. An electron, in a hydrogen like atom, is in excited state. It has a total energy of –3.4 eV, find the de-Broglie wavelength of the electron.

16. In the Bohr model of the hydrogen atom, what is the de-Broglie wavelength for the electron when it is in

(a) the $n = 1$ level?

(b) the $n = 4$ level? In each case, compare the de-Broglie wavelength to the circumference $2\pi r_n$ of the orbit.

17. The binding energy of an electron in the ground state of He atom is equal to $E_0 = 24.6$ eV. Find the energy required to remove both electrons from the atom.

18. Hydrogen atom in its ground state is excited by means of monochromatic radiation of wavelength 1023 Å. How many different lines are possible in the resulting spectrum? Calculate the longest wavelength among them. You may assume the ionization energy of hydrogen atom as 13.6 eV.

19. A doubly ionized lithium atom is hydrogen like with atomic number 3. Find the wavelength of the radiation required to excite the electron in Li^{++} from the first to the third Bohr orbit (ionization energy of the hydrogen atom equals 13.6 eV).

20. Find the quantum number n corresponding to nth excited state of He^+ ion if on transition to the ground state the ion emits two photons in succession with wavelengths 108.5 nm and 30.4 nm. The ionization energy of the hydrogen atom is 13.6 eV.

21. A hydrogen like atom (described by the Bohr model) is observed to emit ten wavelengths, originating from all possible transitions between a group of levels. These levels have energies between –0.85 eV and –0.544 eV (including both these values).

(a) Find the atomic number of the atom.

(b) Calculate the smallest wavelength emitted in these transitions.

(Take ground state energy of hydrogen atom = – 13.6 eV)

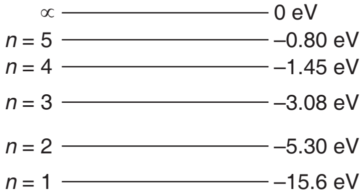
22. The energy levels of a hypothetical one electron atom are shown in the figure.

(a) Find the ionization potential of this atom.

(b) Find the short wavelength limit of the series terminating at $n = 2$.

(c) Find the excitation potential for the state $n = 3$.

(d) Find wave number of the photon emitted for the transition $n = 3$ to $n = 1$.



23. (a) An atom initially in an energy level with $E = -6.52$ eV absorbs a photon that has wavelength 860 nm. What is the internal energy of the atom after it absorbs the photon? (b) An atom initially in an energy level with $E = -2.68$ eV emits a photon that has wavelength 420 nm. What is the internal energy of the atom after it emits the photon?

24. A silver ball is suspended by a string in a vacuum chamber and ultraviolet light of wavelength 2000 Å is directed at it. What electrical potential will the ball acquire as a result? Work function of silver is 4.3 eV.

25. A small particle of mass m moves in such a way that the potential energy $U = \frac{1}{2} m^2 \omega^2 r^2$, where ω is a constant and r is the distance of the particle from the origin. Assuming Bohr model of quantization of angular momentum and circular orbits, show that radius of the nth allowed orbit is proportional to $\sqrt{n}$.

26. Wavelength of K_α-line of an element is λ_0. Find wavelength of K_β-line for the same element.

27. X-rays are produced in an X-ray tube by electrons accelerated through an electric potential difference of 50.0 kV. An electron makes three collisions in the target coming to rest and loses half its remaining kinetic energy in each of the first two collisions. Determine the wavelength of the resulting photons. (Neglecting the recoil of the heavy target atoms).

28. From what material is the anode of an X-ray tube made, if the K_α-line wavelength of the characteristic spectrum is 0.76 Å?

29. A voltage applied to an X-ray tube being increased $\eta = 1.5$ times, the short wave limit of an X-ray continuous spectrum shifts by $\Delta\lambda = 26$ pm. Find the initial voltage applied to the tube.

30. The K_α X-rays of aluminium ($Z = 13$) and zinc ($Z = 30$) have wavelengths 887 pm and 146 pm, respectively. Use Moseley's equation $\sqrt{v} = a(Z - b)$ to find the wavelength of the K_a X-ray of iron ($Z = 26$).

31. Characteristic X-rays of frequency 4.2×10^{18} Hz are produced when transitions from L shell take place in a certain target material. Use Moseley's law and determine the atomic number of the target material. Given, Rydberg constant is $R = 1.1 \times 10^7\ \text{m}^{-1}$.

32. The electric current in an X-ray tube operating at 40 kV is 10 mA. Assume that on an average 1% of the total kinetic energy of the electrons hitting the target are converted into X-rays.

(a) What is the total power emitted as X-rays and

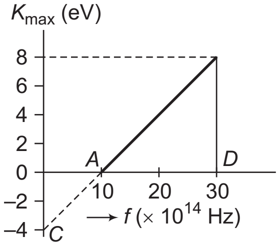
(b) How much heat is produced in the target every second?

33. The stopping potential for the photoelectrons emitted from a metal surface of work-function 1.7 eV is 10.4 V. Find the wavelength of the radiation used. Also, identify the energy levels in hydrogen atom, which will emit this wavelength.

34. What will be the maximum kinetic energy of the photoelectrons ejected from magnesium (for which the work-function $W = 3.7$ eV) when irradiated by ultraviolet light of frequency $1.5 \times 10^{15}\ \text{s}^{-1}$.

35. A metallic surface is irradiated with monochromatic light of variable wavelength. Above a wavelength of 5000 Å, no photoelectrons are emitted from the surface. With an unknown wavelength, stopping potential of 3 V is necessary to eliminate the photocurrent. Find the unknown wavelength.

36. A graph regarding photoelectric effect is shown between the maximum kinetic energy of electrons and the frequency of the incident light . On the basis of data as shown in the graph, calculate



(a) threshold frequency, (b) work-function, (c) planck constant

37. A metallic surface is illuminated alternatively with light of wavelengths 3000 Å and 6000 Å. It is observed that the maximum speeds of the photoelectrons under these illuminations are in the ratio 3 : 1. Calculate the work-function of the metal and the maximum speed of the photoelectrons in two cases.

38. Light of wavelength 180 nm ejects photoelectrons from a plate of metal whose work-function is 2 eV. If a uniform magnetic field of 5×10^{-5} T be applied parallel to the plate, what would be the radius of the path followed by electrons ejected normally from the plate with maximum energy.

39. Light described at a place by the equation $E = (100\ \text{V/m})\ [\sin(5 \times 10^{15}\ \text{s}^{-1})t + \sin(8 \times 10^{15}\text{s}^{-1})t]$ falls on a metal surface having work-function 2.0 eV. Calculate the maximum kinetic energy of the photoelectrons.

40. The electric field associated with a light wave is given by $E = E_0 \sin[(1.57 \times 10^7\ \text{m}^{-1})(x - ct)]$. Find the stopping potential when this light is used in an experiment on photoelectric effect with a metal having work-function 1.9 eV.

LEVEL 2

Single Correct Option

1. If we assume only gravitational attraction between proton and electron in hydrogen atom and the Bohr quantization rule to be followed, then the expression for the ground state energy of the atom will be (the mass of proton is M and that of electron is m.)

(a) $\frac{G^2M^2m^2}{h^2}$

(b) $-\frac{2\pi^2G^2M^2m^3}{h^2}$

(c) $-\frac{2\pi^2GM^2m^3}{h^2}$

(d) None of these

2. An electron in a hydrogen atom makes a transition from first excited state to ground state. The magnetic moment due to circulating electron

(a) increases two times

(b) decreases two times

(c) increases four times

(d) remains same

3. The excitation energy of a hydrogen like ion to its first excited state is 40.8 eV. The energy needed to remove the electron from the ion in the ground state is

(a) 54.4 eV

(b) 62.6 eV

(c) 72.6 eV

(d) 58.6 eV

4. An electron in a hydrogen atom makes a transition from first excited state to ground state. The equivalent current due to circulating electron

(a) increases 4 times

(b) decreases 4 times

(c) increases 8 times

(d) decreases 8 times

5. In a sample of hydrogen like atoms all of which are in ground state, a photon beam containing photons of various energies is passed. In absorption spectrum, five dark lines are observed. The number of bright lines in the emission spectrum will be
(assume that all transitions take place)

(a) 21

(b) 10

(c) 15

(d) None of these

6. Let A_n be the area enclosed by the nth orbit in a hydrogen atom. The graph of $\ln(A_n/A_1)$ against $\ln(n)$

(a) will not pass through origin

(b) will be a straight line with slope 4

(c) will be a rectangular hyperbola

(d) will be a parabola

7. In the hydrogen atom, an electron makes a transition from $n = 2$ to $n = 1$. The magnetic field produced by the circulating electron at the nucleus

(a) decreases 16 times

(b) increases 4 times

(c) decreases 4 times

(d) increases 32 times

8. A stationary hydrogen atom emits photon corresponding to the first line of Lyman series. If R is the Rydberg constant and M is the mass of the atom, then the velocity acquired by the atom is

(a) $\frac{3Rh}{4M}$

(b) $\frac{4M}{3Rh}$

(c) $\frac{Rh}{4M}$

(d) $\frac{4M}{Rh}$

9. Light wave described by the equation 200 V/m $\sin(1.5\times10^{15}\ \text{s}^{-1})\,t\cos(0.5\times10^{15}\ \text{s}^{-1})\,t$ falls on metal surface having work-function 2.0 eV. Then, the maximum kinetic energy of photoelectrons is

(a) 3.27 eV (b) 2.2 eV
(c) 2.85 eV (d) None of these

10. A hydrogen like atom is excited using a radiation. Consequently, six spectral lines are observed in the spectrum. The wavelength of emission radiation is found to be equal or smaller than the radiation used for excitation. This concludes that the gas was initially at

(a) ground state (b) first excited state
(c) second excited state (d) third excited state

11. The time period of the electron in the ground state of hydrogen atom is two times the time period of the electron in the first excited state of a certain hydrogen like atom (Atomic number Z). The value of Z is

(a) 2 (b) 3
(c) 4 (d) None of these

12. The wavelengths of K_α X-rays from lead isotopes $\text{Pb}^{204}, \text{Pb}^{206}$ and Pb^{208} are λ_1, λ_2 and λ_3 respectively. Choose the correct alternative.

(a) $\lambda_1 < \lambda_2 < \lambda_3$ (b) $\lambda_1 > \lambda_2 > \lambda_3$
(c) $\lambda_1 = \lambda_2 = \lambda_3$ (d) None of these

13. In case of hydrogen atom, whenever a photon is emitted in the Balmer series,

(a) there is a probability of emitting another photon in the Lyman series
(b) there is a probability of emitting another photon of wavelength 1213 Å
(c) the wavelength of radiation emitted in Lyman series is always shorter than the wavelength emitted in the Balmer series
(d) All of the above

14. An electron of kinetic energy K collides elastically with a stationary hydrogen atom in the ground state. Then,

(a) $K > 13.6$ eV (b) $K > 10.2$ eV
(c) $K < 10.2$ eV (d) data insufficient

15. In a stationary hydrogen atom, an electron jumps from $n = 3$ to $n = 1$. The recoil speed of the hydrogen atom is about

(a) 4 m/s (b) 4 cm/s
(c) 4 mm/s (d) 4×10^{-4} m/s

16. An X-ray tube is operating at 150 kV and 10 mA. If only 1% of the electric power supplied is converted into X-rays, the rate at which the target is heated in calorie per second is

(a) 3.55 (b) 35.5
(c) 355 (d) 3550

17. An electron revolves round a nucleus of atomic number Z. If 32.4 eV of energy is required to excite an electron from the $n = 3$ state to $n = 4$ state, then the value of Z is

(a) 5 (b) 6
(c) 4 (d) 7

18. If the de-Broglie wavelength of a proton is 10^{-13} m, the electric potential through which it must have been accelerated is

(a) 4.07×10^{4} V (b) 8.15×10^{4} V (c) 8.15×10^{3} V (d) 4.07×10^{5} V

19. If E_n and L_n denote the total energy and the angular momentum of an electron in the nth orbit of Bohr atom, then

(a) $E_n \propto L_n$
(b) $E_n \propto \frac{1}{L_n}$
(c) $E_n \propto L_n^2$
(d) $E_n \propto \frac{1}{L_n^2}$

20. An orbital electron in the ground state of hydrogen has the magnetic moment μ_1. This orbital electron is excited to 3rd excited state by some energy transfer to the hydrogen atom. The new magnetic moment of the electron is μ_2, then

(a) $\mu_1 = 4\mu_2$
(b) $2\mu_1 = \mu_2$
(c) $16\mu_1 = \mu_2$
(d) $4\mu_1 = \mu_2$

21. A moving hydrogen atom makes a head-on collision with a stationary hydrogen atom. Before collision, both atoms are in ground state and after collision they move together. The minimum value of the kinetic energy of the moving hydrogen atom, such that one of the atoms reaches one of the excitation state is

(a) 20.4 eV
(b) 10.2 eV
(c) 54.4 eV
(d) 13.6 eV

22. In an excited state of hydrogen like atom an electron has total energy of – 3.4 eV. If the kinetic energy of the electron is E and its de-Broglie wavelength is λ, then

(a) $\lambda = 6.6$ Å
(b) $E = 3.4$ eV
(c) Both are correct
(d) Both are wrong

More than One Correct Options

1. If the potential difference of coolidge tube producing X-ray is increased, then choose the correct option (s).

(a) the interval between $\lambda_{K\alpha}$ and $\lambda_{K\beta}$ increases
(b) the interval between $\lambda_{K\alpha}$ and λ_0 increases
(c) the interval between $\lambda_{K\beta}$ and λ_0 increases
(d) λ_0 does not change

Here, λ_0 is cut off wavelength and λ_{K_α} and λ_{K_β} are wavelengths of K_α and K_β characteristic X-rays.

2. In Bohr model of the hydrogen atom, let R, v and E represent the radius of the orbit, speed of the electron and the total energy of the electron respectively. Which of the following quantities are directly proportional to the quantum number n?

(a) vR
(b) RE
(c) $\frac{v}{E}$
(d) $\frac{R}{E}$

3. The magnitude of angular momentum, orbital radius and time period of revolution of an electron in a hydrogen atom corresponding to the quantum number n are L, r and T respectively. Which of the following statement(s) is/are correct?

(a) $\frac{rL}{T}$ is independent of n
(b) $\frac{L}{T} \propto \frac{1}{n^2}$
(c) $\frac{T}{r} \propto n$
(d) $Lr \propto \frac{1}{n^3}$

4. In which of the following cases the heavier of the two particles has a smaller de-Broglie wavelength? The two particles

(a) move with the same speed

(b) move with the same linear momentum

(c) move with the same kinetic energy

(d) have the same change of potential energy in a conservative field

5. Hydrogen atom absorbs radiations of wavelength λ_0 and consequently emit radiations of 6 different wavelengths, of which two wavelengths are longer than λ_0. Choose the correct alternative(s).

(a) The final excited state of the atoms is $n = 4$

(b) The initial state of the atoms is $n = 2$

(c) The initial state of the atoms is $n = 3$

(d) There are three transitions belonging to Lyman series

6. In coolidge tube, if f and λ represent the frequency and wavelength of K_α-line for a metal of atomic number Z, then identify the statement which represents a straight line

(a) $\sqrt{f}$ *versus* Z

(b) $\frac{1}{\sqrt{\lambda}}$ *versus* Z

(c) f *versus* Z

(d) λ *versus* Z

Comprehension Based Questions

Passage I (Q. No. 1 to 3)

When a surface is irradiated with light of wavelength 4950 Å, *a photocurrent appears which vanishes if a retarding potential greater than* 0.6 *volt is applied across the phototube. When a second source of light is used, it is found that the critical retarding potential is changed to* 1.1 *volt.*

1. The work-function of the emitting surface is

(a) 2.2 eV

(b) 1.5 eV

(c) 1.9 eV

(d) 1.1 eV

2. The wavelength of the second source is

(a) 6150 Å

(b) 5150 Å

(c) 4125 Å

(d) 4500 Å

3. If the photoelectrons (after emission from the source) are subjected to a magnetic field of 10 tesla, the two retarding potentials would

(a) uniformly increase

(b) uniformly decrease

(c) remain the same

(d) None of these

Passage II (Q. No. 4 to 6)

In an experimental set up to study the photoelectric effect a point source of light of power 3.2×10^{-3} *W was taken. The source can emit monoenergetic photons of energy* 5 *eV and is located at a distance of* 0.8 *m from the centre of a stationary metallic sphere of work-function* 3.0 *eV. The radius of the sphere is* $r = 8 \times 10^{-3}$ *m. The efficiency of photoelectric emission is one for every* 10^6 *incident photons.*

Based on the information given above answer the questions given below.

(Assume that the sphere is isolated and photoelectrons are instantly swept away after the emission).

4. de-Broglie wavelength of the fastest moving photoelectron is

(a) 6.63 Å (b) 8.69 Å
(c) 2 Å (d) 5.26 Å

5. It was observed that after some time emission of photoelectrons from the sphere stopped. Charge on the sphere when the photon emission stops is

(a) $16\pi\varepsilon_0 r$ coulomb (b) $8\pi\varepsilon_0 r$ coulomb
(c) $15\pi\varepsilon_0 r$ coulomb (d) $20\pi\varepsilon_0 r$ coulomb

6. Time after which photoelectric emission stops is

(a) 100 s (b) 121 s
(c) 111 s (d) 141 s

Match the Columns

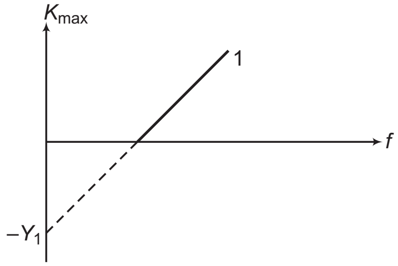
1. Match the following two columns for hydrogen spectrum.

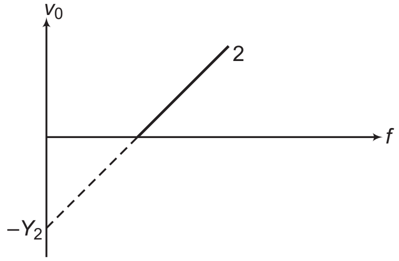
Column I	Column II
(a) Lyman series	(p) infrared region
(b) Balmer series	(q) visible region
(c) Paschen series	(r) ultraviolet region
(d) Brackett series	(s) X-rays

2. Ionization energy from first excited state of hydrogen atom is E. Match the following two columns for He^+ atom.

Column I	Column II
(a) Ionization energy from ground state	(p) $4E$
(b) Electrostatic potential energy in first excited state.	(q) $-16E$
(c) Kinetic energy of electron in ground state.	(r) $-8E$
(d) Ionization energy from first excited state.	(s) $16E$

3.





Maximum kinetic energy *versus* frequency of incident light and stopping potential *versus* frequency of incident light graphs are shown in figure. Match the following two columns.

Column I	Column II
(a) Slope of line-1	(p) h/e
(b) Slope of line-2	(q) h
(c) Y_1	(r) W
(d) Y_2	(s) W/e

Here, h = Planck constant, $e = 1.6 \times 10^{-19}$ C and W = work-function.

4. For hydrogen and hydrogen type atoms, match the following two columns.

Column I	Column II
(a) Time period	(p) Proportional to n/Z
(b) Angular momentum	(q) Proportional to n^2/Z
(c) Speed	(r) Proportional to n^3/Z^2
(d) Radius	(s) None of these

5. In hydrogen atom wavelength of second line of Balmer series is λ. Match the following two columns corresponding to the wavelength.

Column I	Column II
(a) First line of Balmer series	(p) (27/20) λ
(b) Third line of Balmer series	(q) (λ/4)
(c) First line of Lyman series	(r) (25/12) λ
(d) Second line of Lyman series	(s) None of these

6. Match the following (Give most appropriate one matching)

Column I	Column II
(a) Characteristic X-ray	(p) Inverse process of photoelectric effect
(b) X-ray production	(q) Potential difference
(c) Cut off wavelength	(r) Moseley's law
(d) Continuous X-ray	(s) None of these

7. In a photoelectric effect experiment. If *f* is the frequency of radiations incident on the metal surface and *I* is the intensity of the incident radiations, then match the following columns.

Column I	Column II
(a) If *f* is increased keeping *I* and work-function constant	(p) Stopping potential increases
(b) If distance between cathode and anode is increased	(q) Saturation current increases
(c) If *I* is increased keeping *f* and work-function constant	(r) Maximum kinetic energy of photoelectron increases
(d) Work-function is decreased keeping *f* and *I* constant	(s) Stopping potential remains same

Subjective Questions

1. The wavelength for $n = 3$ to $n = 2$ transition of the hydrogen atom is 656.3 nm. What are the wavelengths for this same transition in (a) positronium, which consists of an electron and a positron (b) singly ionized helium (**Note:** A positron is a positively charged electron).

2. (a) Find the frequencies of revolution of electrons in $n = 1$ and $n = 2$ Bohr orbits.

(b) What is the frequency of the photon emitted when an electron in an $n = 2$ orbit drops to an $n = 1$ hydrogen orbit?

(c) An electron typically spends about 10^{-8}s in an excited state before it drops to a lower state by emitting a photon. How many revolutions does an electron in an $n = 2$ Bohr hydrogen orbit make in 1.00×10^{-8} s?

3. A **muon** is an unstable elementary particle whose mass is 207 m_e and whose charge is either ^{+}e or ^{-}e. A negative muon (μ^{-}) can be captured by a nucleus to form a muonic atom.

(a) A proton captures a μ^{-}. Find the radius of the first Bohr orbit of this atom.

(b) Find the ionization energy of the atom.

4. (a) A gas of hydrogen atoms in their ground state is bombarded by electrons with kinetic energy 12.5 eV. What emitted wavelengths would you expect to see?

(b) What if the electrons were replaced by photons of same energy?

5. A source emits monochromatic light of frequency 5.5×10^{14} Hz at a rate of 0.1 W. Of the photons given out, 0.15% fall on the cathode of a photocell which gives a current of 6 μA in an external circuit.

(a) Find the energy of a photon.

(b) Find the number of photons leaving the source per second.

(c) Find the percentage of the photons falling on the cathode which produce photoelectrons.

6. The hydrogen atom in its ground state is excited by means of monochromatic radiation. Its resulting spectrum has six different lines. These radiations are incident on a metal plate. It is observed that only two of them are responsible for photoelectric effect. If the ratio of maximum kinetic energy of photoelectrons in the two cases is 5 then find the work-function of the metal.

7. Electrons in hydrogen like atoms $(Z = 3)$ make transitions from the fifth to the fourth orbit and from the fourth to the third orbit. The resulting radiations are incident normally on a metal plate and eject photoelectrons. The stopping potential for the photoelectrons ejected by the shorter wavelength is 3.95 volts. Calculate the work-function of the metal and the stopping potential for the photoelectrons ejected by the longer wavelength.

8. Find an expression for the magnetic dipole moment and magnetic field induction at the centre of Bohr's hypothetical hydrogen atom in the nth orbit of the electron in terms of universal constant.

9. An electron and a proton are separated by a large distance and the electron approaches the proton with a kinetic energy of 2 eV. If the electron is captured by the proton to form a hydrogen atom in the ground state, what wavelength photon would be given off?

10. Hydrogen gas in the atomic state is excited to an energy level such that the electrostatic potential energy of H-atom becomes –1.7 eV. Now, a photoelectric plate having work-function $W = 2.3$ eV is exposed to the emission spectra of this gas. Assuming all the transitions to be possible, find the minimum de-Broglie wavelength of the ejected photoelectrons.

11. A gas of hydrogen like atoms can absorb radiation of 68 eV. Consequently, the atom emits radiations of only three different wavelengths. All the wavelengths are equal or smaller than that of the absorbed photon.

(a) Determine the initial state of the gas atoms.

(b) Identify the gas atoms.

(c) Find the minimum wavelength of the emitted radiations.

(d) Find the ionization energy and the respective wavelength for the gas atoms.

12. A photon with energy of 4.9 eV ejects photoelectrons from tungsten. When the ejected electron enters a constant magnetic field of strength $B = 2.5$ mT at an angle of 60° with the field direction, the maximum pitch of the helix described by the electron is found to be 2.7 mm. Find the work-function of the metal in electron-volt. Given that specific charge of electron is 1.76×10^{11} C/kg.

13. For a certain hypothetical one-electron atom, the wavelength (in Å) for the spectral lines for transitions originating at $n = p$ and terminating at $n = 1$ are given by

$$\lambda = \frac{1500\,p^2}{p^2 - 1}, \quad \text{where} \quad p = 2, 3, 4$$

(a) Find the wavelength of the least energetic and the most energetic photons in this series.

(b) Construct an energy level diagram for this element showing the energies of the lowest three levels.

(c) What is the ionization potential of this element?

14. A photocell is operating in saturation mode with a photocurrent 4.8 mA when a monochromatic radiation of wavelength 3000 Å and power of 1 mW is incident. When another monochromatic radiation of wavelength 1650 Å and power 5 mW is incident, it is observed that maximum velocity of photoelectron increases to two times. Assuming efficiency of photoelectron generation per incident photon to be same for both the cases, calculate

(a) the threshold wavelength for the cell

(b) the saturation current in second case

(c) the efficiency of photoelectron generation per incident photon

15. Wavelengths belonging to Balmer series for hydrogen atom lying in the range of 450 nm to 750 nm were used to eject photoelectrons from a metal surface whose work-function is 2.0 eV. Find (in eV) the maximum kinetic energy of the emitted photoelectrons.

16. Assume that the de-Broglie wave associated with an electron can form a standing wave between the atoms arranged in a one-dimensional array with nodes at each of the atomic sites. It is found that one such standing wave is formed if the distance d between the atoms of the array is 2 Å. A similar standing wave is again formed if d is increased to 2.5 Å but not for any intermediate value of d. Find the energy of the electron in eV and the least value of d for which the standing wave of the type described above can form.

17. The negative muon has a charge equal to that of an electron but a mass that is 207 times as great. Consider hydrogen like atom consisting of a proton and a muon.

(a) What is the reduced mass of the atom?

(b) What is the ground-level energy (in eV)?

(c) What is the wavelength of the radiation emitted in the transition from the $n = 2$ level to the $n = 1$ level?

18. Assume a hypothetical hydrogen atom in which the potential energy between electron and proton at separation r is given by $U = [k \ln r - (k/2)]$, where k is a constant. For such a hypothetical hydrogen atom, calculate the radius of nth Bohr orbit and energy levels.

19. An electron is orbiting in a circular orbit of radius r under the influence of a constant magnetic field of strength B. Assuming that Bohr postulate regarding the quantisation of angular momentum holds good for this electron, find

(a) the allowed values of the radius r of the orbit.

(b) the kinetic energy of the electron in orbit

(c) the potential energy of interaction between the magnetic moment of the orbital current due to the electron moving in its orbit and the magnetic field B.

(d) the total energy of the allowed energy levels.

(e) the total magnetic flux due to the magnetic field B passing through the nth orbit.

(Assume that the charge on the electron is $-e$ and the mass of the electron is m).

20. A mixture of hydrogen atoms (in their ground state) and hydrogen like ions (in their first excited state) are being excited by electrons which have been accelerated by same potential difference V volts. After excitation when they come directly into ground state, the wavelengths of emitted light are found in the ratio 5 : 1. Then, find

(a) the minimum value of V for which both the atoms get excited after collision with electrons.

(b) atomic number of other ion.

(c) the energy of emitted light.

21. When a surface is irradiated with light of $\lambda = 4950$ Å a photocurrent appears which vanishes if a retarding potential 0.6 V is applied. When a different source of light is used, it is found that critical retarding potential is changed to 1.1 volt. Find the work-function of emitting surface and wavelength of second source. If photoelectrons after emission from surface are subjected to a magnetic field of 10 tesla, what changes will be observed in the above two retarding potentials?

22. In an experiment on photoelectric effect light of wavelength 400 nm is incident on a metal plate at the rate of 5 W. The potential of the collector plate is made sufficiently positive with respect to emitter so that the current reaches the saturation value. Assuming that on the average one out of every 10^6 photons is able to eject a photoelectron, find the photocurrent in the circuit.

23. A light beam of wavelength 400 nm is incident on a metal of work-function 2.2 eV. A particular electron absorbs a photon and makes 2 collisions before coming out of the metal

(a) Assuming that 10% of existing energy is lost to the metal in each collision find the final kinetic energy of this electron as it comes out of the metal.

(b) Under the same assumptions find the maximum number of collisions, the electron should suffer before it becomes unable to come out of the metal.

Answers

Introductory Exercise 33.1

1. 4.6 eV, $2.45 \times 10^{-27} \dfrac{\text{kg}\cdot\text{m}}{\text{s}}$ **2.** 4.82×10^{16} per $\text{m}^2\cdot\text{s}$ **3.** $\sqrt{2}$

4. $\sqrt{2}$ **5.** (c) **6.** (a) 4.81×10^{-34}m (b) 7.12×10^{-11}m

Introductory Exercise 33.2

1. 122.4 eV **2.** 3.16×10^{-34} kg-m^2/s **3.** (d) **4.** (a) **5.** (b)

6. (c) **7.** (b) **8.** (d) **9.** (d) **10.** (d) **11.** $\dfrac{3f}{4}, \dfrac{f}{4}$

12. 651 nm

Introductory Exercise 33.3

1. (b) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** 42

Introductory Exercise 33.4

1. Zero, 3.19 eV **2.** $K_{\text{max}} \propto (f - f_0)$ **3.** 1.16×10^{15} Hz **4.** (a)

5. (a) **6.** (c) **7.** (b) **8.** (a)

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (c) **3.** (a or b) **4.** (a) **5.** (a or b) **6.** (d) **7.** (a) **8.** (b) **9.** (b) **10.** (d)

Objective Questions

1. (d) **2.** (c) **3.** (b) **4.** (c) **5.** (a) **6.** (a) **7.** (c) **8.** (c) **9.** (a) **10.** (c)
11. (c) **12.** (a) **13.** (a) **14.** (c) **15.** (c) **16.** (b) **17.** (d) **18.** (a) **19.** (c) **20.** (c)
21. (a) **22.** (b) **23.** (c) **24.** (c) **25.** (b) **26.** (b) **27.** (c) **28.** (b) **29.** (c) **30.** (a)
31. (b) **32.** (b) **33.** (c) **34.** (a) **35.** (b) **36.** (c) **37.** (d)

Subjective Questions

1. 5.59 nm **2.** $E_1 = -4613$ eV, $E_3 = -2650$ eV **3.** 0.48 μA

4. (a) 2.47×10^{-19} J = 1.54 eV (b) 804 nm, infrared

5. (a) 5.0×10^{14} Hz (b) 2.3×10^{20} photons/s **6.** (a) 5.92×10^{20} Hz (b) 5.06×10^{-13} m

7. (a) $K_2 = 4K_1$ (b) $E_2 = 2E_1$ **8.** (a) 2.52×10^{19} (b) 3.33×10^{-8} N **9.** 4.3×10^{-8} N

10. 10^{-8} N **11.** 3.90×10^{-34} m, No **12.** (a) 1.55×10^{-10} m (b) 8.44×10^{-14} m

13. (a) $2.37 \times 10^{-24} \dfrac{\text{kg}\cdot\text{m}}{\text{s}}$ (b) 3.07×10^{-18} J = 19.2 eV **14.** 1.04 Å **15.** 6.663 Å

16. (a) 3.32×10^{-10} m (b) 1.33×10^{-9} m **17.** 79 eV **18.** 3, 6513 Å **19.** 113.74 Å

20. $n = 5$ **21.** (a) $Z = 4$ (b) $\lambda_{\text{min}} = 40441$ Å

22. (a) 15.6 volt (b) 2335 Å (c) 12.52 V (d) $1.01 \times 10^7\ \text{m}^{-1}$ **23.** (a) –5.08 eV (b) –5.63 eV

24. 1.9 V **26.** $\frac{27}{32}\lambda_0$ **27.** 49.5 pm, 99.0 pm **28.** $Z \approx 41$ **29.** 15865 V

30. 198 pm **31.** $Z = 42$ **32.** (a) 4 W (b) 396 J/s **33.** 1022 Å, $n = 3$ to $n = 1$

34. 2.51 eV **35.** 2260 Å **36.** (a) 10^{15} Hz (b) 4 eV (c) 6.4×10^{-34} J-s

37. 1.81 eV, 9.0×10^5 m/s, 3.0×10^5 m/s **38.** 0.148 m **39.** 3.27 eV **40.** 1.2 V

LEVEL 2

Single Correct Option

1. (b) **2.** (b) **3.** (a) **4.** (c) **5.** (c) **6.** (b) **7.** (d) **8.** (a) **9.** (d) **10.** (c)
11. (c) **12.** (c) **13.** (d) **14.** (c) **15.** (a) **16.** (c) **17.** (d) **18.** (b) **19.** (d) **20.** (d)
21. (a) **22.** (c)

More than One Correct Options

1.(b,c) **2.**(a,c) **3.**(a,b,c) **4.**(a,c) **5.**(a,b,d) **6.**(a,b)

Comprehension Based Questions

1. (c) **2.** (c) **3.** (c) **4.** (b) **5.** (b) **6.** (c)

Match the Columns

1. (a) → r (b) → q (c) → p (d) → p
2. (a) → s (b) → r (c) → s (d) → p
3. (a) → q (b) → p (c) → r (d) → s
4. (a) → r (b) → s (c) → s (d) → q
5. (a) → p (b) → s (c) → q (d) → s
6. (a) → r (b) → p (c) → q (d) → q
7. (a) → p,r (b) → s (c) → q,s (d) → p,r

Subjective Questions

1. (a) 1.31 μm (b) 164 nm

2. (a) 6.58×10^{15} Hz, 0.823×10^{15} Hz (b) 2.46×10^{15} Hz (c) 8.23×10^6 revolutions

3. (a) 2.55×10^{-13} m (b) 2.81 keV **4.** (a) 102 nm, 122 nm, 651 nm (b) No lines

5. (a) 2.27 eV (b) 2.75×10^{17} (c) 9% **6.** $W = 11.925$ eV **7.** 2 eV, 0.754 V

8. $\frac{neh}{4\pi m}, \frac{\mu_0 \pi m^2 e^7}{8\varepsilon_0 h^5 n^5}$ **9.** 793.3 Å **10.** 3.8 Å

11. (a) $n_i = 2$ (b) $Z = 6$ (c) 28.43 Å (d) 489.6 eV, 25.3 Å **12.** 4.5 eV

13. (a) 2000 Å, 1500 Å (b) $E_1 = -8.25$ eV, $E_2 = -2.05$ eV and $E_3 = -0.95$ eV (c) 8.25 V

14. (a) 4125 Å (b) 34 μA (c) 5.1% **15.** 0.55 eV **16.** 150 eV, 0.5 Å

17. (a) 1.69×10^{-28} kg (b) –2.53 keV (c) 0.653 nm **18.** $r_n = \frac{nh}{2\pi\sqrt{mk}}$, $E_n = k\ln\left\{\frac{nh}{2\pi\sqrt{mk}}\right\}$

19. (a) $r_n = \sqrt{\frac{nh}{2\pi Be}}$ (b) $K = \frac{nhBe}{4\pi m}$ (c) $U = \frac{nheB}{4\pi m}$ (d) $E = \frac{nheB}{2\pi m}$ (e) $\frac{nh}{2e}$

20. (a) 10.2 volt (b) $Z = 2$ (c) 10.2 eV and 51 eV **21.** 1.9 eV, 4125 Å, No change is observed

22. 1.6μA **23.** (a) 0.31 eV (b) 4

34

Modern Physics-II

Chapter Contents

34.1 Nuclear Stability and Radioactivity
34.2 The radioactive decay law
34.3 Successive disintegration
34.4 Equivalence of mass and energy
34.5 Binding energy and nuclear stability
34.6 Nuclear fission (Divide and conquer)
34.7 Nuclear fusion

34.1 Nuclear Stability and Radioactivity

Among about 1500 known nuclides, less than 260 are stable. The others are unstable that decay to form other nuclides by emitting α and β-particles and γ-electromagnetic waves. This process is called radioactivity. It was discovered in 1896 by Henry Becquerel.

Whilst the chemical properties of an atom are governed entirely by the number of protons in the nucleus (i.e. the proton number Z), the stability of an atom appears to depend on both the number of protons and the number of neutrons. For light nuclei, the greatest stability is achieved when the numbers of protons and neutrons are approximately equal ($N \approx Z$).

For heavier nuclei, instability caused by electrostatic repulsion between the protons is minimized when there are more neutrons than protons.

Figure shows a plot of N *versus* Z for the stable nuclei. For mass numbers upto about $A = 40$, we see that $N \approx Z$. $_{40}$Ca is the heaviest stable nucleus for which $N = Z$. For larger values of Z, the (short range) nuclear force is unable to hold the nucleus together against the (long range) electrical repulsion of the protons unless the number of neutrons exceeds the number of protons. At Bi ($Z = 83$, $A = 209$), the neutron excess is $N - Z = 43$. There are no stable nuclides with $Z > 83$.

The nuclide $^{209}_{83}$Bi is the heaviest stable nucleus.

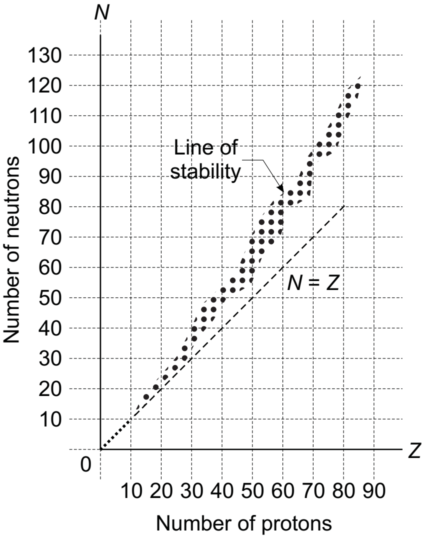


Fig. 34.1 The stable nuclides plotted on a graph of neutron number, *N*, *versus* proton number, *Z*. Note that for heavier nuclides, *N* is larger relative to *Z*. The stable nuclides group along a curve called the line of stability.

Atoms are radioactive if their nuclei are unstable and spontaneously (and randomly) emit various particles, the α, β and or γ radiations. When naturally occurring nuclei are unstable, we call the phenomena **natural radioactivity.** Other nuclei can be transformed into radioactive nuclei by various means, typically involving irradiation by neutrons, this is called **artificial radioactivity.**

A radioactive nucleus is called a **parent nucleus,** the nucleus resulting from its decay by particle emission is called **daughter nucleus.** Daughter nuclei also might be granddaughter nuclei, and so on. There are no son or grandson nuclei. For unstable nuclides and radioactivity, the following points can be made.

(i) Disintegrations tend to produce new nuclides near the stability line and continue until a stable nuclide is formed.

(ii) Radioactivity is a nuclear property, i.e. α, β and γ emission take place from the nucleus.

(iii) Nuclear processes involve huge amount of energy so the particle emission rate is independent of temperature and pressure. The rate depends solely on the concentration of the number of atoms of the radioactive substance.

(iv) A radioactive substance is either an α-emitter or a β-emitter, γ-rays emit with both.

Alpha Decay

An alpha particle is a helium nucleus. Thus, a nucleus emitting an alpha particle loses two protons and two neutrons. Therefore, the atomic number Z decreases by 2, the mass number A decreases by 4 and the neutron number N decreases by 2. The decay can be written as

$$\boxed{{}^{A}_{Z}X = {}^{A-4}_{Z-2}Y + {}^{4}_{2}\text{He}}$$

where, X is the parent nucleus and Y the daughter nucleus. As examples U^{238} and Ra^{226} are both alpha emitters and decay according to

$$ {}^{238}_{92}\text{U} \longrightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He}$$

$$ {}^{226}_{88}\text{Ra} \longrightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$$

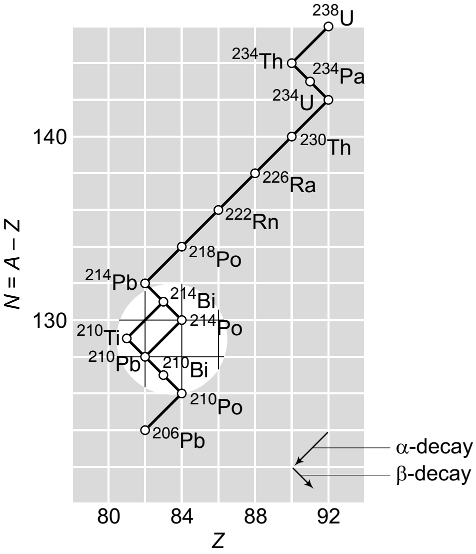


Fig. 34.2 The uranium decay series ($A = 4n + 2$). The decay of $^{214}_{83}$Bi may proceed either by alpha emission and then beta emission or in the reverse order.

As a general rule, in any decay sum of mass numbers A and atomic numbers Z must be the same on both sides.

Note that a nuclide below the stability line in Fig. 34.2 disintegrates in such a way that its proton number decreases and its neutron to proton ratio increases. In heavy nuclides, this can occur by alpha emission.

If the original nucleus has a mass number A that is 4 times an integer, the daughter nucleus and all those in the chain will also have mass numbers equal to 4 times an integer. (Because in α-decay A decreases by 4 and in β-decay it remains the same). Similarly, if the mass number of the original nucleus is $4n+1$, where n is an integer, all the nuclei in the decay chain will have mass numbers given by $4n+1$ with n decreasing by 1 in each α-decay. We can see therefore, that there are four possible α-decay chains, depending on whether A equals $4n, 4n+1, 4n+2$ or $4n+3$, where n is an integer.

Series $4n+1$ is now not found. Because its longest lived member (other than the stable end product Bi^{209}) is Np^{237} which has a half-life of only 2×10^{6} years. Because this is much less than the age of the earth, this series has disappeared. Figure shows the uranium $(4n+2)$ series. The series branches at $\text{Bi}^{214,}$ which decays either by α-decay to Ti^{210} or β-decay to Po^{214}. The branches meet at the lead isotope Pb^{210}. Table 34.1 lists the four radioactive series.

Table 34.1 Four Radioactive Series

Mass numbers	Series	Parent	Half-life, Years	Stable product
$4n$	Thorium	$^{232}_{90}$Th	1.39×10^{10}	$^{208}_{82}$Pb
$4n + 1$	Neptunium	$^{237}_{93}$Np	2.25×10^{6}	$^{209}_{83}$Bi
$4n + 2$	Uranium	$^{238}_{92}$U	4.47×10^{9}	$^{206}_{82}$Pb
$4n + 3$	Actinium	$^{235}_{92}$U	7.07×10^{8}	$^{207}_{82}$Pb

Beta Decay

Beta decay can involve the emission of either electrons or positrons. A positron is a form of antimatter which has a charge equal to $+e$ and mass equal to that of an electron. The electrons or positrons emitted in β-decay do not exist inside the nucleus. They are only created at the time of emission, just as photons are created when an atom makes a transition from a higher to a lower energy state.

In β^- decay a neutron in the nucleus is transformed into a proton, an electron and an antineutrino.

$$\boxed{n \longrightarrow p + e^- + \bar{\nu}}$$

To conserve energy and momentum in the process, the emission of an antineutrino ($\bar{\nu}$) (alongwith proton and electron) was first suggested by W. Pauli in 1930, but it was first observed experimentally in 1957.

Thus, a parent nucleus with atomic number Z and mass number A decays by β^- emission into a daughter with atomic number $Z+1$ and the same mass number A.

$$\boxed{{}^{A}_{Z}X \xrightarrow{\beta^-} {}^{A}_{Z+1}Y}$$

β^- **decay** occurs in nuclei that have too many neutrons. An example of β^- decay is the decay of carbon-14 into nitrogen,

$$\boxed{{}^{14}_{6}\text{C} \longrightarrow {}^{14}_{7}\text{N} + e^- + \bar{\nu}}$$

In β^+ decay, a proton changes into a neutron with the emission of a positron (and a neutrino)

$$\boxed{p \longrightarrow n + e^+ + \nu}$$

Positron (e^+) emission from a nucleus decreases the atomic number Z by 1 while keeping the same mass number A.

$$\boxed{{}^{A}_{Z}X \xrightarrow{\beta^+} {}^{A}_{Z-1}Y}$$

B^+ decay occurs in nuclei that have too few neutrons. A typical β^+ decay is

$${}^{13}_{7}\text{N} \longrightarrow {}^{13}_{6}\text{C} + e^+ + \nu$$

Electron capture Electron capture is competitive with positron emission since both processes lead to the same nuclear transformation. This occurs when a parent nucleus captures one of its own orbital electrons and emits a neutrino.

$$\boxed{{}^{A}_{Z}X + e^{-1} \longrightarrow {}^{A}_{Z-1}Y + \nu}$$

In most cases, it is a K-shell electron that is captured, and for this reason the process is referred to as ***K*-capture.** One example is the capture of an electron by ${}_4\text{Be}^7$

$${}^{7}_{4}\text{Be} + e^- \longrightarrow {}^{7}_{3}\text{Li} + \nu$$

Gamma Decay

Very often a nucleus that undergoes radioactive decay (α or β-decay) is left in an excited energy state (analogous to the excited states of the orbiting electrons, except that the energy levels associated with the nucleus have much larger energy differences than those involved with the atomic electrons). The typical half-life of an excited nuclear state is 10^{-10} s. The excited nucleus (X^*) then undergoes to a lower energy state by emitting a high energy photon, called the γ-ray photon. The following sequence of events represents a typical situation in which γ-decay occurs.

$$ {}^{12}_{5}\mathrm{B} \longrightarrow {}^{12}_{6}\mathrm{C}^* + e^- + \bar{\nu} $$

$$ {}^{12}_{6}\mathrm{C}^* \longrightarrow {}^{12}_{6}\mathrm{C} + \gamma $$

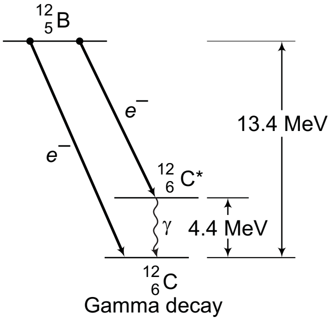


Fig. 34.3

Figure shows decay of B^{12} nucleus, which undergoes β-decay to either of two levels of C^{12}. It can either decay directly to the ground state of C^{12} by emitting a 13.4 MeV electron or undergo β-decay to an excited state of ${}^{12}_{6}\mathrm{C}^*$ followed by γ-decay to the ground state. The later process results in the emission of a 9.0 MeV electron and a 4.4 MeV photon. The various pathways by which a radioactive nucleus can undergo decay are summarized in Table 34.2.

Note *In both α and β-decay, the Z value of a nucleus changes and the nucleus of one element becomes the nucleus of a different element. In γ-decay, the element does not change, the nucleus merely goes from an excited state to a less excited state.*

Table 34.2 Various Decay Pathways

Alpha decay	${}^{A}_{Z}X \longrightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\mathrm{He}$
Beta decay (β^-)	${}^{A}_{Z}X \longrightarrow {}_{Z+1}^{\;\;A}Y + e^- + \bar{\nu}$
Beta decay (β^+)	${}^{A}_{Z}X \longrightarrow {}_{Z-1}^{\;\;A}Y + e^+ + \nu$
Electron capture	${}^{A}_{Z}X + e^- \longrightarrow {}_{Z-1}^{\;\;A}Y + \nu$
Gamma decay	${}^{A}_{Z}X^* \longrightarrow {}^{A}_{Z}X + \gamma$

✓ Extra Points to Remember

- After emission of one alpha particle and two beta particles isotopes are produced. This is because after the emission of one alpha particle, atomic number decreases by 2. Further, after the emission of two beta particles atomic number increases by 2. So, finally atomic number remains unchanged.
- From beta emission mass number does not change. Therefore, isobars will be produced.

Example 34.1 *Mass number of a nucleus X is A and atomic number is Z. Find mass number and atomic number of the new nucleus (say Y) after the emission of m-alpha particles and n-beta particles.*

Solution By the emission of one alpha particle, atomic number decreases by 2 and by the emission of one beta particle atomic number increases by 1. Therefore, the atomic number of nucleus Y is $Z = Z - 2m + n$

Further, by the emission of one alpha particle mass number decreases by 4 and by the emission of beta particle, mass number does not change. Therefore, the mass number of Y is

$$A' = A - 4m$$

34.2 Radioactive Decay Law

Radioactive decay is a random process. Each decay is an independent event and one cannot tell when a particular nucleus will decay. When a particular nucleus decays, it is transformed into another nuclide, which may or may not be radioactive. When there is a very large number of nuclei in a sample, the rate of decay is proportional to the number of nuclei, N, that are present

$$\left(-\frac{dN}{dt}\right) \propto N$$

or

$$\left(-\frac{dN}{dt}\right) = \lambda N$$

where, λ is called the **decay constant.** This equation may be expressed in the form $\frac{dN}{N} = -\lambda dt$ and integrated,

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt$$

or

$$\ln\left(\frac{N}{N_0}\right) = -\lambda t$$

where, N_0 is the initial number of parent nuclei at $t = 0$. The number that survives at time t is therefore,

$$\boxed{N = N_0 e^{-\lambda t}} \qquad \text{...(i)}$$

This function is plotted in Fig. 34.4.

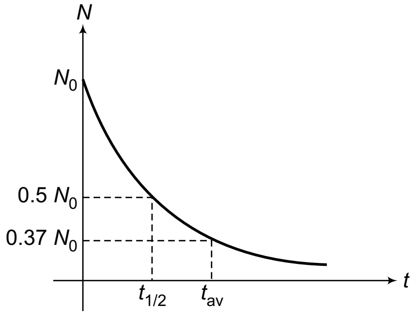


Fig. 34.4

Half-life The time required for the number of parent nuclei to fall to 50% is called half-life $t_{1/2}$ and may be related to λ as follows.

$$0.5N_0 = N_0 e^{-\lambda t_{1/2}}$$

We have $$\lambda t_{1/2} = \ln(2) = 0.693$$

$\therefore$ $$\boxed{t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{0.693}{\lambda}} \qquad \text{...(ii)}$$

Mean life The average or mean life t_{av} is the reciprocal of the decay constant.

$$\boxed{t_{av} = \frac{1}{\lambda}} \qquad \text{...(iii)}$$

The mean life is analogous to the time constant in the exponential decrease in the charge on a capacitor in an *RC* circuit. After a time equal to the mean life time, the number of radioactive nuclei decreases to $\frac{1}{e}$ times or approximately 37% of their original values.

Activity of a Radioactive Substance

The decay rate R of a radioactive substance is the number of decays per second. And as we have seen above

$$-\frac{dN}{dt} \propto N \qquad \text{or} \qquad -\frac{dN}{dt} = \lambda N$$

Thus, $$R = -\frac{dN}{dt} \qquad \text{or} \qquad R \propto N$$

or $$R = \lambda N \qquad \text{or} \qquad R = \lambda N_0 e^{-\lambda t}$$

or $$\boxed{R = R_0 e^{-\lambda t}} \qquad \text{...(iv)}$$

where, $R_0 = \lambda N_0$ is the activity of the radioactive substance at time $t = 0$. The activity *versus* time graph is shown in Fig. 34.5.

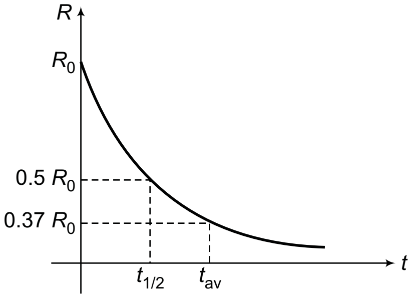


Fig. 34.5

Thus, the number of nuclei and hence the activity of the radioactive substance also decreases exponentially with time.

Units of activity The SI unit for the decay rate is the Becquerel (Bq), but the curie (Ci) and rutherford (rd) are often used in practice.

$$\boxed{1\,\text{Bq} = 1\,\text{decays/s}, \quad 1\,\text{Ci} = 3.7 \times 10^{10}\,\text{Bq} \quad \text{and} \quad 1\,\text{rd} = 10^6\,\text{Bq}}$$

Extra Points to Remember

- After n half-lives,

  (a) number of nuclei left $= N_0\left(\frac{1}{2}\right)^n$

  (b) fraction of nuclei left $= \left(\frac{1}{2}\right)^n$ and

  (c) percentage of nuclei left $= 100\left(\frac{1}{2}\right)^2$

- Number of nuclei decayed after time t,

$$= N_0 - N$$
$$= N_0 - N_0 e^{-\lambda t} = N_0\,(1 - e^{-\lambda t})$$

  The corresponding graph is as shown in Fig. 34.6.

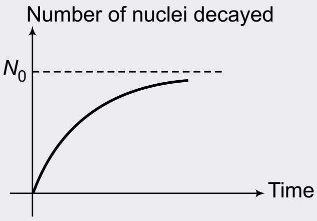


Fig. 34.6

- Probability of a nucleus for survival upto time t,

$$P\,(\text{survival}) = \frac{N}{N_0} = \frac{N_0 e^{-\lambda t}}{N_0} = e^{-\lambda t}$$

  The corresponding graph is shown in Fig. 34.7.

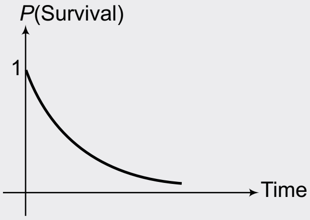


Fig. 34.7

- Probability of a nucleus to disintegrate in time t is,

$$P\,(\text{disintegration}) = 1 - P\,(\text{survival}) = 1 - e^{-\lambda t}$$

  The corresponding graph is as shown in Fig. 34.8.

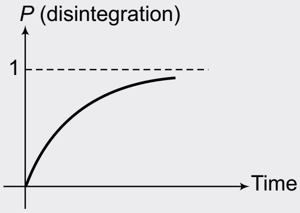


Fig. 34.8

- Half-life and mean life are related to each other by the relation,

$$\boxed{t_{1/2} = 0.693\,t_{av} \quad \text{or} \quad t_{av} = 1.44\,t_{1/2}}$$

- As we discussed above number of nuclei decayed in time t are $N_0\,(1 - e^{-\lambda t})$. This expression involves power of e.

  So, to avoid it we can use

$$\boxed{\Delta N = \lambda N \Delta t}$$

  where, ΔN are the number of nuclei decayed in time Δt at the instant when total number of nuclei are N. But, this can be applied only when $\Delta t << t_{1/2}$.

  Proof $$-\frac{dN}{dt} = \lambda N \quad \Rightarrow \quad -dN = \lambda N dt$$

  or $$\Delta N = \lambda N \Delta t$$

- In same interval of time, equal percentage (or fraction) of nuclei are decayed (or left undecayed).

Example 34.2 *At time $t = 0$, number of nuclei of a radioactive substance are 100. At $t = 1$ s these numbers become 90. Find the number of nuclei at $t = 2$ s.*

Solution In 1 second, 90% of the nuclei have remained undecayed, so in another 1 second 90% of 90, i.e. 81 nuclei will remain undecayed.

Example 34.3 *At time $t = 0$, activity of a radioactive substance is 1600 Bq, at $t = 8$ s activity remains 100 Bq. Find the activity at $t = 2$ s.*

Solution $R = R_0\left(\frac{1}{2}\right)^n$

Here, n is the number of half-lives.

Given, $$R = \frac{R_0}{16}$$

$\therefore$ $$\frac{R_0}{16} = R_0\left(\frac{1}{2}\right)^n \quad \text{or} \quad n = 4$$

Four half-lives are equivalent to 8 s. Hence, 2 s is equal to one half-life. So, in one half-life activity will remain half of 1600 Bq, i.e. 800 Bq.

Example 34.4 *From a radioactive substance n_1 nuclei decay per second at an instant when total number of nuclei are n_2. Find half-life of the radioactive substance.*

Solution Using the equation,

$$-\frac{dN}{dt} = \lambda N$$

We have, $$n_1 = \lambda n_2$$

$\therefore$ $$\lambda = \frac{n_1}{n_2}$$

Now, half-life is given by

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{\ln 2}{(n_1/n_2)}$$

$$= \left(\frac{n_2}{n_1}\right)\ln 2 \qquad \textbf{Ans.}$$

Example 34.5 *Half-life of a radioactive substance is T. At time t_1 activity of a radioactive substance is R_1 and at time t_2 it is R_2. Find the number of nuclei decayed in this interval of time.*

Solution Half-life is given by

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

$\therefore$ $$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{T}$$

Activity $R = \lambda N$

$\therefore$ $$N = \frac{R}{\lambda} = \frac{RT}{\ln 2}$$

When activity is R_1, numbers of nuclei are $N_1 = \dfrac{R_1 T}{\ln 2}$

Similarly, $$N_2 = \frac{R_2 T}{\ln 2}$$

$\therefore$ Numbers decayed

$= N_1 - N_2 = \dfrac{(R_1 - R_2)T}{\ln 2}$ **Ans.**

Example 34.6 *Two radioactive materials X_1 and X_2 have decay constants 10λ and λ, respectively. If initially they have the same number of nuclei, then the ratio of the number of nuclei of X_1 to that of X_2 will be $1/e$ after a time* **(JEE 2000)**

(a) $1/10\lambda$ *(b)* $1/11\lambda$ *(c)* $11/10\lambda$ *(d)* $1/9\lambda$

Solution $$\frac{N_{x_1}(t)}{N_{x_2}(t)} = \frac{1}{e}$$

or $$\frac{N_0 e^{-10\lambda t}}{N_0 e^{-\lambda t}} = \frac{1}{e}$$ (Initially, both have same number of nuclei say N_0)

or $$e^{-9\lambda t} = e^{-1} \quad \text{or} \quad e = e^{9\lambda t}$$

or $$9\lambda t = 1 \quad \text{or} \quad t = \frac{1}{9\lambda}$$

$\therefore$ The correct option is (d).

Example 34.7 *The half-life period of a radioactive element x is same as the mean life time of another radioactive element y. Initially, both of them have the same number of atoms. Then,* **(JEE 1999)**

(a) x and y have the same decay rate initially

(b) x and y decay at the same rate always

(c) y will decay at a faster rate than x

(d) x will decay at a faster rate than y

Solution $(t_{1/2})_x = (t_{\text{mean}})_y$

or $$\frac{0.693}{\lambda_x} = \frac{1}{\lambda_y}$$

$\therefore$ $$\lambda_x = 0.693\,\lambda_y$$
$$\lambda_x < \lambda_y$$

or Rate of decay $= \lambda N$

Initially, number of atoms (N) of both are equal but since $\lambda_y > \lambda_x$, therefore, y will decay at a faster rate than x.

$\therefore$ The correct option is (c).

Example 34.8 *A radioactive sample consists of two distinct species having equal number of atoms initially. The mean life of one species is τ and that of the other is 5τ. The decay products in both cases are stable. A plot is made of the total number of radioactive nuclei as a function of time. Which of the following figure best represents the form of this plot?* **(JEE 2001)**

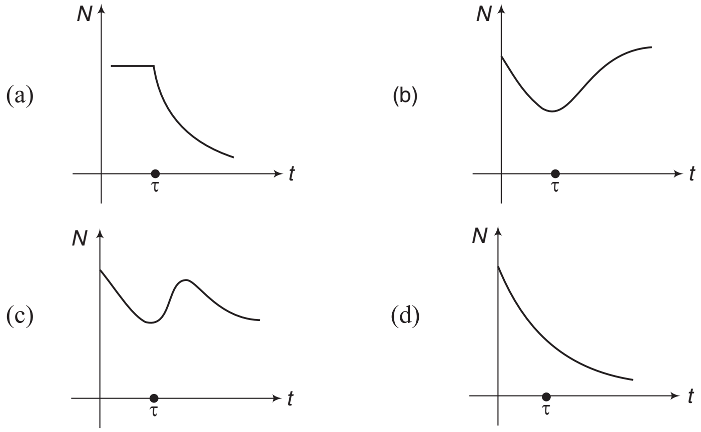


Fig. 34.9

Solution The total number of atoms can neither remain constant (as in option a) nor can ever increase (as in options b and c). They will continuously decrease with time. Therefore, (d) is the appropriate option.

INTRODUCTORY EXERCISE 34.1

1. The decay constant of a radioactive sample is λ. The half-life and mean life of the sample are respectively given by **(JEE 1989)**
(a) $1/\lambda$ and $(\ln 2)/\lambda$
(b) $(\ln 2)/\lambda$ and $1/\lambda$
(c) $\lambda (\ln 2)$ and $1/\lambda$
(d) $\lambda/(\ln 2)$ and $1/\lambda$

2. Consider α-particles, β-particles and γ-rays each having an energy of 0.5 MeV. In increasing order of penetrating powers, the radiations are **(JEE 1994)**
(a) α, β, γ
(b) α, γ, β
(c) β, γ, α
(d) γ, β, α

3. Which of the following is a correct statement? **(JEE 1999)**
(a) Beta rays are same as cathode rays
(b) Gamma rays are high energy neutrons
(c) Alpha particles are singly ionized helium atoms
(d) Protons and neutrons have exactly the same mass

4. The electron emitted in beta radiation originates from **(JEE 2001)**
(a) inner orbits of atom
(b) free electrons existing in nuclei
(c) decay of a neutron in a nucleus
(d) photon escaping from the nucleus

5. During a negative beta decay, **(JEE 1987)**

(a) an atomic electron is ejected

(b) an electron which is already present within the nucleus is ejected

(c) a neutron in the nucleus decays emitting an electron

(d) a part of the binding energy of the nucleus is converted into an electron

6. A freshly prepared radioactive source of half-life 2 h emits radiation of intensity which is 64 times the permissible safe level. The minimum time after which it would be possible to work safely with this source is **(JEE 1988)**

(a) 6 h (b) 12 h (c) 24 h (d) 128 h

7. A radioactive sample S_1 having an activity of 5 μCi has twice the number of nuclei as another sample S_2 which has an activity of 10 μCi. The half-lives of S_1 and S_2 can be **(JEE 2008)**

(a) 20 yr and 5 yr, respectively (b) 20 yr and 10 yr, respectively

(c) 10 yr each (d) 5 yr each

8. Half-life of a radioactive substance A is 4 days. The probability that a nucleus will decay in two half-lives is **(JEE 2006)**

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) 1

9. After 280 days, the activity of a radioactive sample is 6000 dps. The activity reduces to 3000 dps after another 140 days. The initial activity of the sample in dps is **(JEE 2004)**

(a) 6000 (b) 9000 (c) 3000 (d) 24000

10. The half-life of ^{215}At is 100 μs. The time taken for the activity of a sample of ^{215}At to decay to $\frac{1}{16}$ th of its initial value is **(JEE 2002)**

(a) 400 μs (b) 63 μs

(c) 40 μs (d) 300 μs

11. The half-life of the radioactive radon is 3.8 days. The time, at the end of which 1/20 th of the radon sample will remain undecayed, is (given $\log_{10} e = 0.4343$) **(JEE 1981)**

(a) 3.8 days (b) 16.5 days

(c) 33 days (d) 76 days

12. Activity of a radioactive substance decreases from 8000 Bq to 1000 Bq in 9 days. What is the half-life and average life of the radioactive substance?

13. A radioactive substance has a half-life of 64.8 h. A sample containing this isotope has an initial activity ($t = 0$) of 40 μCi. Calculate the number of nuclei that decay in the time interval between $t_1 = 10.0$ h and $t_2 = 12.0$ h.

14. A freshly prepared sample of a certain radioactive isotope has an activity of 10 mCi. After 4.0 h its activity is 8.00 mCi.

(a) Find the decay constant and half-life

(b) How many atoms of the isotope were contained in the freshly prepared sample?

(c) What is the sample's activity 30.0 h after it is prepared?

15. A radioactive substance contains 10^{15} atoms and has an activity of 6.0×10^{11} Bq. What is its half-life?

16. Two radioactive elements X and Y have half-life periods of 50 minutes and 100 minutes, respectively. Initially, both of them contain equal number of atoms. Find the ratio of atoms left N_X/N_Y after 200 minutes.

34.3 Successive Disintegration

Suppose a parent radioactive nucleus A (decay constant $=\lambda_a$) has number of atoms N_0 at time $t=0$. After disintegration it converts into a nucleus B (decay constant $=\lambda_b$) which is further radioactive. Initially ($t=0$), number of atoms of B are zero. We are interested in finding N_b, the number of atoms of B at time t.

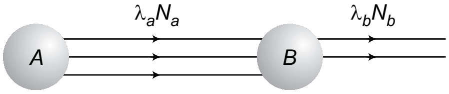


Fig. 34.10

	A	B
At $t=0$	N_0	0
At $t=t$	$N_a=N_0e^{-\lambda_a t}$	$N_b=?$

At time t, net rate of formation of B = rate of disintegration of A – rate of disintegration of B

$$\therefore \qquad \frac{dN_b}{dt}=\lambda_a N_a-\lambda_b N_b$$

$$\text{or} \qquad \frac{dN_b}{dt}=\lambda_a N_0 e^{-\lambda_a t}-\lambda_b N_b \qquad (\text{as } N_a=N_0e^{-\lambda_a t})$$

$$\text{or} \qquad dN_b+\lambda_b N_b\, dt=\lambda_a N_0 e^{-\lambda_a t}$$

Multiplying this equation by $e^{\lambda_b t}$, we have

$$e^{\lambda_b t}dN_b+e^{\lambda_b t}\lambda_b N_b\, dt=\lambda_a N_0 e^{(\lambda_b-\lambda_a)t}$$

$$\therefore \qquad d\{N_b e^{\lambda_b t}\}=\lambda_a N_0 e^{(\lambda_b-\lambda_a)t}dt$$

Integrating both sides, we get

$$N_b e^{\lambda_b t}=\left(\frac{\lambda_a}{\lambda_b-\lambda_a}\right)N_0 e^{(\lambda_b-\lambda_a)t}+C \qquad \ldots\text{(i)}$$

where, C is the constant of integration, which can be found as under.

At time, $t=0$, $N_b=0$

$$\therefore \qquad C=-\left(\frac{\lambda_a}{\lambda_b-\lambda_a}\right)N_0$$

Substituting this value in Eq. (i), we have

$$\boxed{N_b=\frac{N_0\lambda_a}{\lambda_b-\lambda_a}(e^{-\lambda_a t}-e^{-\lambda_b t})} \qquad \ldots\text{(ii)}$$

Now, the following conclusions may be drawn from the above discussion.

1. From Eq. (ii) we can see that $N_b=0$ at time $t=0$ (it was given) and at $t=\infty$ (because B is also radioactive)

2. N_a will continuously decrease with time while N_b will first increase (until $\lambda_a N_a > \lambda_b N_b$), reaches to a maximum value (when $\lambda_a N_a = \lambda_b N_b$) and then decreases (when $\lambda_b N_b > \lambda_a N_a$). The two graphs for N_a and N_b with time are shown below.

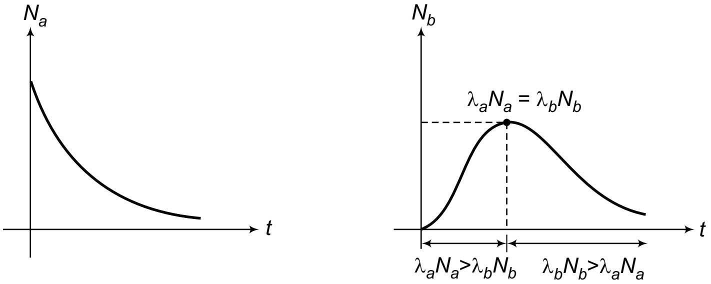


Fig. 34.11

Example 34.9 *A radio nuclide X is produced at constant rate* α. *At time* $t = 0$, *number of nuclei of X are zero. Find*

(a) the maximum number of nuclei of X.

(b) the number of nuclei at time t.

Decay constant of X is λ.

Solution (a) Let N be the number of nuclei of X at time t.

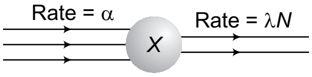


Fig. 34.12

Rate of formation of $X = \alpha$ (given)

Rate of disintegration $= \lambda N$

Number of nuclei of X will increase until both the rates will become equal. Therefore,

$$\alpha = \lambda N_{\max}$$

$$\therefore \qquad N_{\max} = \frac{\alpha}{\lambda} \qquad \textbf{Ans.}$$

(b) Net rate of formation of X at time t is

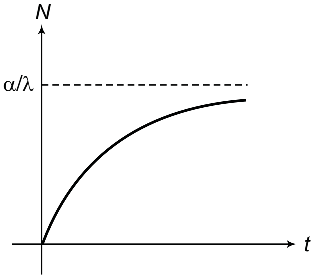


Fig. 34.13

$$\frac{dN}{dt} = \alpha - \lambda N$$

$$\therefore \qquad \frac{dN}{\alpha - \lambda N} = dt$$

Integrating with proper limits, we have

$$\int_0^N \frac{dN}{\alpha - \lambda N} = \int_0^t dt$$

or $$N = \frac{\alpha}{\lambda}(1 - e^{-\lambda t}) \qquad \textbf{Ans.}$$

This expression shows that number of nuclei of X are increasing exponentially from 0 to $\frac{\alpha}{\lambda}$.

Example 34.10 *In the above problem if each decay produces E_0 energy, then find*

(a) power produced at time t

(b) total energy produced upto time t

Solution (a) $\because \quad N = \frac{\alpha}{\lambda}(1 - e^{-\lambda t})$

At time t, number of decays per second $= \lambda N = \alpha\,(1 - e^{-\lambda t})$.

Each decay produces E_0 energy. Therefore, energy produced per second or power.

= (number of decays per second) (energy produced in each decay)

$$= (\lambda N)\,E_0$$

$$= \alpha E_0\,(1 - e^{-\lambda t})$$

or $$P = \alpha E_0\,(1 - e^{-\lambda t}) \qquad \textbf{Ans.}$$

(b) Power is a function of time. Therefore, total energy produced upto time t can be obtained by integrating this power or

$$E_{\text{Total}} = \int_0^t P\,dt$$

Alternate Method

Energy is produced only in decay. Upto time t total αt nuclei are produced and N nuclei are left. So, total number of nuclei decayed.

$$N_d = \alpha t - N = \alpha t - \frac{\alpha}{\lambda}(1 - e^{-\lambda t})$$

$$= \alpha\left[t - \frac{1}{\lambda}(1 - e^{-\lambda t})\right]$$

Each decay produces E_0 energy. Therefore, total energy produced upto time t,

$$E_{\text{Total}} = N_d E_0$$

$$= \alpha E_0\left[t - \frac{1}{\lambda}(1 - e^{-\lambda t})\right] \qquad \textbf{Ans.}$$

34.4 Equivalence of Mass and Energy

In 1905, while developing his special theory of relativity, Einstein made the suggestion that energy and mass are equivalent. He predicted that if the energy of a body changes by an amount E, its mass changes by an amount m given by the equation,

$$\boxed{E = mc^2}$$

where, c is the speed of light. Everyday examples of energy gain are much too small to produce detectable changes of mass. But in nuclear physics this plays an important role. Mass appears as energy and the two can be regarded as equivalent. In nuclear physics, mass is measured in **unified atomic mass units (u)**, $1\,\text{u}$ being one-twelfth of the mass of carbon-12 atom and equals 1.66×10^{-27} kg. It can readily be shown using $E = mc^2$ that, 1 u mass has energy 931.5 MeV.

Thus, $$\boxed{1\,\text{u} \equiv 931.5\ \text{MeV/c}^2 \ \text{ or } \ c^2 = 931.5\ \text{MeV/u}}$$

A unit of energy may therefore be considered to be a unit of mass. For example, the electron has a rest mass of about 0.5 MeV.

If the principle of conservation of energy is to hold for nuclear reactions it is clear that mass and energy must be regarded as equivalent. The implication of $E = mc^2$ is that any reaction producing an appreciable mass decrease is a possible source of energy.

Example 34.11 *Find the increase in mass of water when* 1.0 *kg of water absorbs* 4.2×10^3 *J of energy to produce a temperature rise of* 1 *K.*

Solution

$$m = \frac{E}{c^2} = \frac{4.2 \times 10^3}{(3.0 \times 10^8)^2}\ \text{kg}$$

$$= 4.7 \times 10^{-14}\ \text{kg}$$ **Ans.**

34.5 Binding Energy and Nuclear Stability

The existence of a stable nucleus means that the nucleons (protons and neutrons) are in a bound state. Since, the protons in a nucleus experience strong electrical repulsion, there must exist a stronger attractive force that holds the nucleus together. The **nuclear force** is a short range interaction that extends only to about 2 fm. (In contrast, the electromagnetic interaction is a long-range interaction). An important feature of the nuclear force is that it is essentially the same for all nucleons, independent of charge.

The **binding energy** (E_b) of a nucleus is the energy required to completely separate the nucleons. The origin of the binding energy may be understood with the help of mass-energy relation, $\Delta E = \Delta mc^2$, where Δm is the difference between the total mass of the separated nucleons and the mass of the stable nucleus. The mass of the stable nucleus is less than the sum of the mass of its nucleons. The binding energy of a nuclide ${}_Z X^A$ is thus,

$$\boxed{E_b = [Zm_P + (A - Z)\, m_N - m_X]\, c^2} \qquad \ldots\text{(i)}$$

where, m_P = mass of proton, m_N = mass of neutron and m_X = mass of nucleus

Note *(i) $\Delta m = [Zm_P + (A - Z)\, m_N - m_X]$ is called the mass defect. This much mass is lost during the formation of a nucleus. Energy $\Delta E = (\Delta m)\, c^2$ is liberated during the making of the nucleus. This is the energy due to which nucleons are bound together. So, to break the nucleus in its constituent nucleons this much energy has to be given to the nucleus.*

(ii) ***Stability*** *: Although nuclides with Z values upto Z = 92 (uranium) occur naturally, not all of these are stable. The nuclide $^{209}_{83}\text{Bi}$ is the heaviest stable nucleus. Even though uranium is not stable, however, its long lived isotope ^{238}U, has a half-life of some 4 billion year.*

Binding energy per nucleon If the binding energy of a nucleus is divided by its mass number, the binding energy per nucleon is obtained. A plot of binding energy per nucleon E_b / A as a function of mass number A for various stable nuclei is shown in figure.

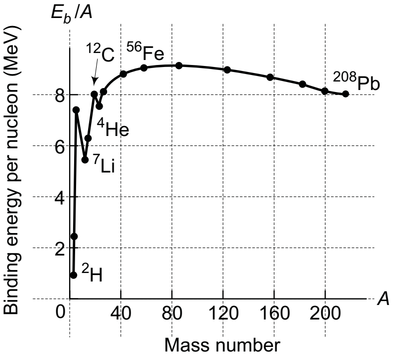


Fig. 34.14 The binding energy per nucleon, E_b / A, as a function of the mass number A

Note *That it is the binding energy per nucleon which is more important for stability of a nucleus rather than the total binding energy.*

Following conclusions can be drawn from the above graph.

1. The greater the binding energy per nucleon the more stable is the nucleus. The curve reaches a maximum of about 8.75 MeV in the vicinity of $^{56}_{26}\text{Fe}$ and then gradually falls to 7.6 MeV for $^{238}_{92}\text{U}$.
2. In a nuclear reaction energy is released if total binding energy is increasing. Let us take an example.

   Suppose a nucleus X, which has total binding energy of 100 MeV converts into some another nucleus Y which has total binding energy 120 MeV. Then, in this process 20 MeV energy will be released. This is because 100 MeV energy has already been released during the formation of X while in case of Y it is 120 MeV. So, the remaining 20 MeV will be released now.

   $$\boxed{\text{Energy is released if } \Sigma E_b \text{ is increasing.}}$$

3. ΣE_b in a nuclear process is increased if binding energy per nucleon of the daughter products gets increased. Let us take an example. Consider a nucleus X $(A_X = 100)$ breaks into lighter nuclei Y $(A_Y = 60)$ and $Z (A_Z = 40)$.

   $$X \to Y + Z$$

   Binding energy per nucleon of these three are say, 7 MeV, 7.5 MeV and 8.0 MeV. Then, total binding energy of X is $100 \times 7 = 700$ MeV and that of $Y + Z$ is $(60 \times 7.5) + (40 \times 8.0) = 770$ MeV. So, in this process 70 MeV energy will be released.

4. Binding energy per nucleon is increased if two or more lighter nuclei combine to form a heavier nucleus. This process is called **nuclear fusion.**

Fig. 34.15

In **nuclear fission** a heavy nucleus splits into two or more lighter nuclei of almost equal mass.

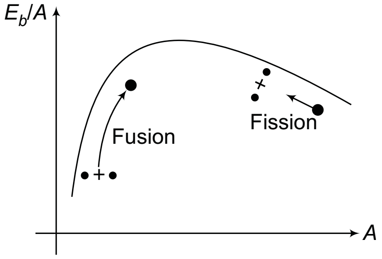


Fig. 34.16

In both the processes E_b/A is increasing. Thus, energy will be released.

34.6 Nuclear Fission (Divide and Conquer)

As we saw in the above article nuclear fission occurs when a heavy nucleus such as ${}^{235}\text{U}$, splits into two lighter nuclei. In nuclear fission, the combined mass of the daughter nuclei is less than the mass of the parent nucleus. The difference is called the mass defect. Fission is initiated when a heavy nucleus captures a thermal neutron (slow neutrons). Multiplying the mass defect by c^2 gives the numerical value of the released energy. Energy is released because the binding energy per nucleon of the daughter nuclei is about 1 MeV greater than that of the parent nucleus.

The fission of ${}^{235}\text{U}$ by thermal neutrons can be represented by the equation,

$$\boxed{{}_0^1n + {}_{92}^{235}\text{U} \longrightarrow {}_{92}^{236}\text{U}^* \longrightarrow X + Y + \text{neutrons}}$$

where, ${}^{236}\text{U}^*$ is an intermediate excited state that lasts only for 10^{-12} s before breaking into nuclei X and Y, which are called fission fragments. In any fission equation there are many combinations of X and Y that satisfy the requirements of conservation of energy and charge with uranium, for example, there are about 90 daughter nuclei that can be formed. Fission also results in the production of several neutrons, typically two or three. On the average, about 2.5 neutrons are released per event.

A typical fission reaction for uranium is

$$\boxed{{}_0^1n + {}_{92}^{235}\text{U} \longrightarrow {}_{56}^{141}\text{Ba} + {}_{36}^{92}\text{Kr} + 3{}_0^1n}$$

About 200 MeV is released in the fission of a heavy nucleus. The fission energy appears mostly as kinetic energy of the fission fragments (e.g. barium and krypton nuclei) which fly apart at great speed. The kinetic energy of the fission neutrons also makes a slight contribution. In addition one or both of the large fragments are highly radioactive and small amount of energy takes the form of beta and gamma radiation.

Chain Reaction Shortly after nuclear fission was discovered, it was realized that, the fission neutrons can cause further fission of ^{235}U and a chain reaction can be maintained.

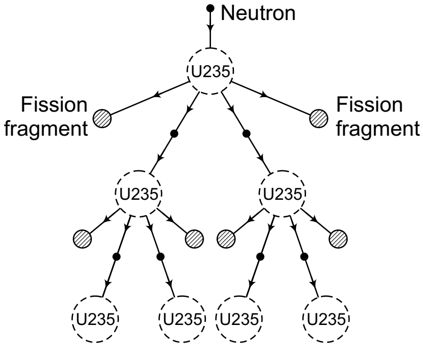


Fig. 34.17 A chain reaction

In practice only a proportion of the fission neutrons is available for new fissions since, some are lost by escaping from the surface of the uranium before colliding with another nucleus. The ratio of neutrons escaping to those causing fission decreases as the size of the piece of uranium-235 increases and there is a **critical size** (about the size of a cricket ball) which must be attained before a chain reaction can start.

In the **'atomic bomb'** an increasing uncontrolled chain reaction occurs in a very short time when two pieces of uranium-235 are rapidly brought together to form a mass greater than the critical size.

Nuclear Reactors In a nuclear reactor the chain reaction is steady and controlled so that on average only one neutron from each fission produces another fission. The reaction rate is adjusted by inserting neutron absorbing rods of boron steel into the uranium 235.

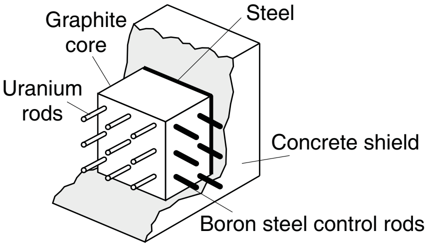


Fig. 34.18 Nuclear reactor

Graphite core is used as a **moderator** to slow down the neutrons. Natural uranium contains over 99% of ^{238}U and less than 1% of ^{235}U. The former captures the medium speed fission neutrons without fissioning. It fissions with very fast neutrons. On the other hand ^{235}U (and plutonium-239) fissions with slow neutrons and the job of moderator is to slow down the fission neutrons very quickly so that most escape capture by ^{238}U and then cause the fission of ^{235}U.

A bombarding particle gives up most energy when it has an elastic collision with a particle of similar mass. For neutrons, hydrogen atoms would be most effective but they absorb the neutrons. But deuterium (in heavy water) and carbon (as graphite) are both suitable as moderator.

To control the power level **control rods** are used. These rods are made of materials such as cadmium, that are very efficient in absorbing neutrons. The first nuclear reactor was built by Enrico Fermi and his team at the University of Chicago in 1942.

34.7 Nuclear Fusion

Binding energy for light nuclei ($A < 20$) is much smaller than the binding energy for heavier nuclei. This suggests a process that is the reverse of fission. When two light nuclei combine to form a heavier nucleus, the process is called nuclear fusion. The union of light nuclei into heavier nuclei also lead to a transfer of mass and a consequent liberation of energy. Such a reaction has been achieved in **'hydrogen bomb'** and it is believed to be the principal source of the sun's energy.

A reaction with heavy hydrogen or deuterium which yields 3.3 MeV per fusion is

$$\boxed{{}_1^2\text{H} + {}_1^2\text{H} \rightarrow {}_2^3\text{He} + {}_0^1 n}$$

By comparison with the 200 MeV per fission of ^{235}U this seems small, but per unit mass of material it is not. Fusion of two deuterium nuclei, i.e. deuterons, will only occur if they overcome their mutual electrostatic repulsion. This may happen, if they collide at very high speed when, for example, they are raised to a very high temperature ($10^8 - 10^9$ K). So much high temperature is obtained by using an atomic (fission) bomb to trigger off fusion. If a controlled fusion reaction can be achieved, an almost unlimited supply of energy will become available from deuterium in the water of the oceans.

✓ Extra Points to Remember

- **Q-value of a nuclear reaction (optional)** Consider a nuclear reaction in which a target nucleus *X* is bombarded by a particle *'a'* resulting in a daughter nucleus *Y* and a particle *b*.

$$\boxed{a + X \rightarrow Y + b}$$

Sometimes this reaction is written as $X(a, b)Y$

The reaction energy *Q* associated with a nuclear reaction is defined as the total energy released as a result of the reaction. Thus,

$$\boxed{Q = (M_a + M_X - M_Y - M_b)c^2}$$

A reaction for which *Q* is positive is called **exothermic.** A reaction for which *Q* is negative is called **endothermic.**

In an exothermic reaction, the total mass of incoming particles is greater than that of the outgoing particles and the *Q*-value is positive. If the total mass of the incoming particles is less than that of the outgoing particles, energy is required for reaction to take place and the reaction is said to be endothermic. Thus, an endothermic reaction does not occur unless the bombarding particle has a kinetic energy greater than $|Q|$. The minimum energy necessary for such a reaction to occur is called **threshold energy K_{th}**. The threshold energy is somewhat greater than $|Q|$ because the outgoing particles must have some kinetic energy to conserve momentum.

Thus, $$\boxed{K_{th} > |Q| \quad \text{(in endothermic reaction)}}$$

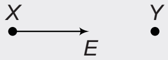


Fig. 34.19

Consider a bombarding particle *X* of mass m_1 and a target *Y* of mass m_2 (at rest). The threshold energy of *X* for endothermic reaction (negative value of Q) to take place is

$$\boxed{K_{th} = |Q|\left(\frac{m_1}{m_2} + 1\right)}$$

Example 34.12 *In the fusion reaction* ${}_{1}^{2}\text{H} + {}_{1}^{2}\text{H} \longrightarrow {}_{2}^{3}\text{He} + {}_{0}^{1}n$*, the masses of deuteron, helium and neutron expressed in amu are* 2.015, 3.017 *and* 1.009 *respectively. If* 1 *kg of deuterium undergoes complete fusion, find the amount of total energy released.* 1 *amu* $\equiv 931.5\ MeV/c^2$.

Solution $\Delta m = 2\,(2.015) - (3.017 + 1.009) = 0.004$ amu

$\therefore$ Energy released $= (0.004 \times 931.5)$ MeV $= 3.726$ MeV

Energy released per deuteron $= \dfrac{3.726}{2} = 1.863$ MeV

Number of deuterons in 1 kg $= \dfrac{6.02 \times 10^{26}}{2} = 3.01 \times 10^{26}$

$\therefore$ Energy released per kg of deuterium fusion $= (3.01 \times 10^{26} \times 1.863) = 5.6 \times 10^{26}$ MeV

$\approx 9.0 \times 10^{13}$ J **Ans.**

Example 34.13 *A nucleus with mass number* 220 *initially at rest emits an* α*-particle. If the Q-value of the reaction is* 5.5 *MeV, calculate the kinetic energy of the* α*-particle.* **(JEE 2003)**

(a) 4.4 *MeV* *(b)* 5.4 *MeV* *(c)* 5.6 *MeV* *(d)* 6.5 *MeV*

Solution Given that $K_1 + K_2 = 5.5$ MeV ...(i)

From conservation of linear momentum,

$$p_1 = p_2 \quad \text{or} \quad \sqrt{2K_1\,(216m)} = \sqrt{2K_2\,(4m)} \quad \text{as} \quad p = \sqrt{2Km}$$

$$\therefore \qquad K_2 = 54\,K_1 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get K_2 = KE of α-particle = 5.4 MeV

$\therefore$ The correct option is (b).

Example 34.14 *Binding energy per nucleon versus mass number curve for nuclei is shown in figure. W, X, Y and Z are four nuclei indicated on the curve. The process that would release energy is* **(JEE 1999)**

(a) $Y \to 2Z$

(b) $W \to X + Z$

(c) $W \to 2Y$

(d) $X \to Y + Z$

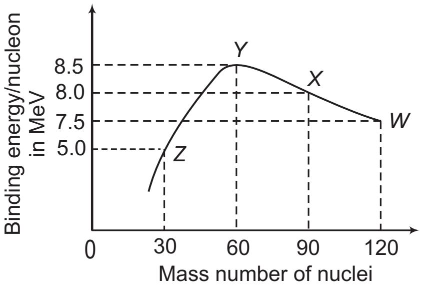


Fig. 34.20

Solution Energy is released in a process when total binding energy of the nucleus (= binding energy per nucleon × number of nucleons) is increased or we can say, when total binding energy of products is more than the reactants. By calculation we can see that only in option (c), this happens. Given, $W \to 2Y$

Binding energy of reactants $= 120 \times 7.5 = 900$ MeV

and binding energy of products $= 2\,(60 \times 8.5) = 1020$ MeV $>$ 900 MeV

$\therefore$ The correct option is (b).

Example 34.15 *A star initially has* 10^{40} *deuterons. It produces energy via the processes* ${}_1H^2 + {}_1H^2 \rightarrow {}_1H^3 + p$ *and* ${}_1H^2 + {}_1H^3 \rightarrow {}_2He^4 + n$. *If the average power radiated by the star is* 10^{16} *W, the deuteron supply of the star is exhausted in a time of the order of* **(JEE 1993)**

(a) 10^6 *s* *(b)* 10^8 *s* *(c)* 10^{12} *s* *(d)* 10^{16} *s*

The masses of the nuclei are as follows

$$M(H^2) = 2.014 \text{ amu}; M(n) = 1.008 \text{ amu},$$
$$M(p) = 1.007 \text{ amu}; M(He^4) = 4.001 \text{ amu}$$

Solution The given reactions are

$${}_1H^2 + {}_1H^2 \longrightarrow {}_1H^3 + p$$
$${}_1H^2 + {}_1H^3 \longrightarrow {}_2He^4 + n$$

$\Rightarrow$
$$3\,{}_1H^2 \longrightarrow {}_2He^4 + n + p$$

Mass defect,
$$\Delta m = (3 \times 2.014 - 4.001 - 1.007 - 1.008) \text{ amu}$$
$$= 0.026 \text{ amu}$$
$$\text{Energy released} = 0.026 \times 931 \text{ MeV}$$
$$= 0.026 \times 931 \times 1.6 \times 10^{-13} \text{ J}$$
$$= 3.87 \times 10^{-12} \text{ J}$$

This is the energy produced by the consumption of three deuteron atoms.

$\therefore$ Total energy released by 10^{40} deuterons

$$= \frac{10^{40}}{3} \times 3.87 \times 10^{-12} \text{ J} = 1.29 \times 10^{28} \text{ J}$$

The average power radiated is $P = 10^{16}$ W or 10^{16} J/s.

Therefore, total time to exhaust all deuterons of the star will be

$$t = \frac{1.29 \times 10^{28}}{10^{16}} = 1.29 \times 10^{12} \text{ s} \approx 10^{12} \text{ s}$$

$\therefore$ The correct option is (c).

Example 34.16 *Assume that the nuclear binding energy per nucleon* (B/A) *versus mass number (A) is as shown in the figure. Use this plot to choose the correct choice(s) given below.* **(JEE 2008)**

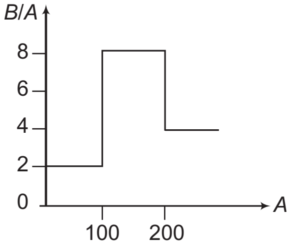


Fig. 34.21

(a) Fusion of two nuclei with mass numbers lying in the range of $1 < A < 50$ *will release energy.*

(b) Fusion of two nuclei with mass numbers lying in the range of $51 < A < 100$ *will release energy.*

(c) Fission of a nucleus lying in the mass range of $100 < A < 200$ *will release energy when broken into two equal fragments.*

(d) Fission of a nucleus lying in the mass range of $200 < A < 260$ *will release energy when broken into two equal fragments.*

Solution In fusion, two or more lighter nuclei combine to make a comparatively heavier nucleus. In fission, a heavy nucleus breaks into two or more comparatively lighter nuclei. Further, energy will be released in a nuclear process if total binding energy increases.

$\therefore$ The correct options are (b) and (d).

INTRODUCTORY EXERCISE 34.2

1. If a star can convert all the He nuclei completely into oxygen nuclei. The energy released per oxygen nuclei is (Mass of the helium nucleus is 4.0026 amu and mass of oxygen nucleus is 15.9994 amu) **(JEE 2005)**

(a) 7.6 MeV (b) 56.12 MeV
(c) 10.24 MeV (d) 23.4 MeV

2. Fast neutrons can easily be slowed down by **(JEE 1994)**

(a) the use of lead shielding
(b) passing them through heavy water
(c) elastic collisions with heavy nuclei
(d) applying a strong electric field

3. During a nuclear fusion reaction, **(JEE 1987)**

(a) a heavy nucleus breaks into two fragments by itself
(b) a light nucleus bombarded by thermal neutrons breaks up
(c) a heavy nucleus bombarded by thermal neutrons breaks up
(d) two light nuclei combine to give a heavier nucleus and possibly other products

4. The equation $4\,{}_1^1\text{H} \longrightarrow {}_2^4\text{He}^{2+} + 2e^- + 26\,\text{MeV}$ represents **(JEE 1983)**

(a) β-decay (b) γ-decay
(c) fusion (d) fission

5. (a) How much mass is lost per day by a nuclear reactor operated at a 10^9 watt power level?
(b) If each fission releases 200 MeV, how many fissions occur per second to yield this power level?

6. Find energy released in the alpha decay,

$${}_{92}^{238}\text{U} \longrightarrow {}_{90}^{234}\text{Th} + {}_2^4\text{He}$$

Given,

$$M({}_{92}^{238}\text{U}) = 238.050784\ \text{u}$$

$$M({}_{90}^{234}\text{Th}) = 234.043593\ \text{u}$$

$$M({}_2^4\text{He}) = 4.002602\ \text{u}$$

7. Complete the nuclear reactions.

(a) ${}_3^6\text{Li} + ? \longrightarrow {}_4^7\text{Be} + {}_0^1 n$ (b) ${}_{17}^{35}\text{Cl} + ? \longrightarrow {}_{16}^{32}\text{S} + {}_2^4\text{He}$
(c) ${}_4^9\text{Be} + {}_2^4\text{He} \longrightarrow 3({}_2^4\text{He}) + ?$ (d) ${}_{35}^{79}\text{Br} + {}_1^2\text{H} \longrightarrow ? + 2({}_0^1 n)$

8. Consider the reaction ${}_1^2\text{H} + {}_1^2\text{H} = {}_2^4\text{He} + Q$. Mass of the deuterium atom = 2.0141u. Mass of helium atom = 4.0024 u. This is a nuclear .... reaction in which the energy Q released is .... MeV. **(JEE 1996)**

9. The binding energies per nucleon for deuteron (${}_1\text{H}^2$) and helium (${}_2\text{He}^4$) are 1.1 MeV and 7.0 MeV respectively. The energy released when two deuterons fuse to form a helium nucleus (${}_2\text{He}^4$) is .... **(JEE 1988)**

Final Touch Points

1. **Classification of nuclei** The nuclei have been divided in isotopes, isobars and isotones on the basis of number of protons (atomic number) or the total number of nucleons (mass number).

   Isotopes The elements having the same number of protons but different number of neutrons are called isotopes. In other words, isotopes have same value of atomic number (Z) but different values of mass number (A). Almost every element has isotopes. Because of the same atomic number isotopes of an element have the same place in the periodic table. The isotopes of some elements are given below.

Element	Its isotopes	Number of protons	Number of neutrons
Hydrogen	${}_1H^1$	1	0
	${}_1H^2$	1	1
	${}_1H^3$	1	2
Oxygen	${}_8O^{16}$	8	8
	${}_8O^{17}$	8	9
	${}_8O^{18}$	8	10
Chlorine	${}_{17}Cl^{35}$	17	18
	${}_{17}Cl^{37}$	17	20
Uranium	${}_{92}U^{235}$	92	143
	${}_{92}U^{238}$	92	146

   In nature, the isotopes of chlorine (${}_{17}Cl^{35}$ and ${}_{17}Cl^{37}$) are found in the ratio 75.4% and 24.6%. When chlorine is prepared in laboratory, its atomic mass is found to be

$$M = (35 \times 0.754) + (37 \times 0.246) = 35.5$$

Note *Since, the isotopes have the same atomic number, they have the same chemical properties. Their physical properties are different as they have different mass numbers. Two isotopes, thus cannot be separated by chemical method, but they can be separated from the physical methods.*

   Isobars The elements having the same mass number (A) but different atomic number (Z) are called isobars. They have different places in periodic table. Their chemical (as well as physical) properties are different.

   ${}_1H^3$ and ${}_2He^3$, ${}_8O^{17}$ and ${}_9F^{17}$ are examples of isobars.

   Isotones Elements having the equal number of neutrons $(A - Z)$ are called isotones.

   ${}_3Li^7$ and ${}_4Be^8$, ${}_1H^3$ and ${}_2He^4$ are examples of isotones.

2. **Nuclear forces** In nucleus the positively charged protons and the uncharged neutrons are held together in an extremely small space $(\approx 10^{-15}\text{ m})$ in spite of the strong electrostatic repulsion between the protons. Obviously, there are some strong attractive forces operating within the nucleus between the nucleons. The nuclear forces are non-electric and non gravitational forces. These forces are extremely short-range forces. They become operative only when the distance between two nucleons is a small multiple of 10^{-15} m. They do not exist when the distance is appreciably larger than 10^{-15} m and become repulsive when the distance is appreciably smaller than 10^{-15} m. Nuclear forces

between protons and protons between neutrons and neutrons and between protons and neutrons are all essentially the same in magnitude. Thus, we can say that nuclear forces are charge independent.

Yukawa's meson theory of nuclear forces A Japanese scientist Yukawa in 1935 suggested that the nuclear forces are 'exchange forces. Which are produced by the exchange of new particles called π-mesons between nucleons. These particles were later on actually discovered in cosmic radiation. There are three types of π-mesons, π^+, π^- and π^0. There is a continuous exchange of π-mesons between protons and neutrons due to which they continue to be converted into one another. When a π^+ - meson jumps from a proton to a neutron, the proton is converted into a neutron and the neutron is converted into a proton.

$$p - \pi^+ \longrightarrow n$$

and

$$n + \pi^+ \longrightarrow p$$

Conversely, when a π^- - meson jumps from a neutron to a proton, then neutron is converted into a proton and the proton is converted into a neutron. Thus,

$$n - \pi^- \longrightarrow p$$

and

$$p + \pi^- \longrightarrow n$$

The exchange of π^+ and π^- - mesons between protons and neutrons is responsible for the origin of nuclear forces between them. Similarly, nuclear forces between two protons and between two neutrons are generated by a continuous exchange of π^0-mesons between them. Thus, the basis of nuclear forces is the exchange of mesons and hence these are called 'exchange forces'.

3. **Size and shape of the nucleus** The Rutherford scattering experiment established that mass of an atom is concentrated within a small positively charged region at the centre which is called the nucleus of the atom. The nuclear radius is given by

$$R = R_0 A^{1/3}$$

Here, A is the mass number of the particular nucleus and $R_0 = 1.3$ fm (fermi) $= 1.3 \times 10^{-15}$ m. This means that the nucleus radius is of the order of 10^{-15} m.

Here, $R_0 = 1.3$ fm is the distance of closest approach to the nucleus and is also known as nuclear unit radius.

4. **Nuclear density** Let us consider the nucleus of an atom having the mass number A.

Mass of nucleus $\approx A \times 1.67 \times 10^{-27}$ kg

Volume of the nucleus $= \frac{4}{3} \pi R^3$

$$= \frac{4}{3} \pi (R_0 A^{1/3})^3 = \frac{4}{3} \pi R_0^3 A$$

$\therefore$ Density of the nucleus, $\rho = \frac{\text{mass}}{\text{volume}}$

or

$$\rho = \frac{A \times 1.67 \times 10^{-27}}{\frac{4}{3} \times \pi \times (1.3 \times 10^{-15})^3 \times A}$$

$$= 1.8 \times 10^{17} \text{ kg/m}^3$$

Thus, density of a nucleus is independent of the mass number A and of the order of 10^{17} kg/m^3.

5. **Magic numbers** We know that the electrons in an atom are grouped in 'shells' and 'sub-shells'. Atoms with 2, 10, 18, 36, 54 and 86 electrons have all of their shells completely filled. Such atoms are unusually stable and chemically inert. A similar situation exists with nuclei also. Nuclei having 2, 8, 20, 28, 50, 82 and 126 nucleons of the same kind (either protons or neutrons) are more stable than nuclei of neighbouring mass numbers. These numbers are called as 'magic numbers'.

6. **Fundamental particles** The particles which are not constituted by any other particles are called fundamental particles. A brief discussion of important fundamental particles is as follows.
   (i) **Electron** It was discovered in 1897 by Thomson. Its charge is $-e$ and mass is 9.1×10^{-31} kg. Its symbol is e^- (or ${}_{-1}\beta^0$).
   (ii) **Proton** It was discovered in 1919 by Rutherford in artificial nuclear disintegration. It has a positive charge $+e$ and its mass is 1836 times (1.673×10^{-27} kg) the mass of electron. In free state, the proton is a stable particle. Its symbol is p^+. It is also written as ${}_1\text{H}^1$.
   (iii) **Neutron** It was discovered in 1932 by Chadwick. Electrically, it is a neutral particle. Its mass is 1839 times (1.675×10^{-27} kg) the mass of electron. In free state the neutron is unstable (mean life ≈ 17 minutes) but it constitutes a stable nucleus with the proton. Its symbol is n or ${}_0n^1$.
   (iv) **Positron** It was discovered by Anderson in 1932. It is the antiparticle of electron, i.e. its charge is $+e$ and its mass is equal to that of the mass of electron. Its symbol is e^+ (or ${}_{+1}\beta^0$).
   (v) **Antiproton** It is the antiparticle of proton. It was discovered in 1955. Its charge is $-e$ and its mass is equal to that of the mass of proton. Its symbol is p^-.
   (vi) **Antineutron** It was discovered in 1956. It has no charge and its mass is equal to the mass of neutron. The only difference between neutron and antineutron is that if they spin in the same direction, their magnetic momenta will be in opposite directions. The symbol for antineutron is $\bar{n}$.
   (vii) **Neutrino and antineutrino** The existence of these particles was predicted in 1930 by Pauli while explaining the emission of β-particles from radioactive nuclei, but these particles were actually observed experimentally in 1956. Their rest mass and charge are both zero but they have energy and momentum. These are mutually antiparticles of each other. They have the symbol ν and $\bar{\nu}$.
   (viii) **Pi-mesons** The existence of pi-mesons was predicted by Yukawa in 1935, but they were actually discovered in 1947 in cosmic rays. Nuclear forces are explained by the exchange of pi-mesons between the nucleons. pi-mesons are of three types, positive π-mesons (π^+), negative pi-mesons (π^-) and neutral π-mesons (π^0). Charge on $\pi^\pm$ is $\pm e$. Whereas mass of $\pi^\pm$ is 274 times the mass of electron. π^0 has mass nearly 264 times the electronic mass.
   (ix) **Mu-Mesons** These were discovered in 1936 by Anderson and Neddermeyer. These are found in abundance in the cosmic rays at the ground level. There are two types of mu-mesons. Positive mu-meson (μ^+) and negative mu-meson (μ^-). There is no neutral mu-meson. Both the mu-mesons have the same rest mass 207 times the rest mass of the electron.
   (x) **Photon** These are bundles of electromagnetic energy and travel with the speed of light. Energy and momentum of a photon of frequency ν are $h\nu$ and $\frac{h\nu}{c}$, respectively.

Antiparticles For every fundamental particle there exists an identical fundamental particle just opposite in some property. For example electron and positron are identical in all respects, except that charges on them are opposite.

The following table shows various particles and their antiparticles. Some particles are their own antiparticles. For example π^0 and γ.

Name of particle	Symbol	Antiparticle	Mass in comparison to mass of electron	Average life (in seconds) for the unstable particles
Electron	e^-	e^{+1}	1	stable
Proton	p^+	p^-	1836	stable
Neutron	n	$\bar{n}$	1839	1010
Neutrino	ν	$\bar{\nu}$	0	stable

Name of particle	Symbol	Antiparticle	Mass in comparison to mass of electron	Average life (in seconds) for the unstable particles
Pi-Mesons	π^+	π^-	274	2.6×10^{-8}
	π^0	π^0	264	0.9×10^{-16}
Mu-Mesons	μ^-	μ^+	207	2.2×10^{-6}
Photon	γ	γ	0	stable

7. If an unstable nucleus decays by two different processes and decay constants in two processes are λ_1 and λ_2, then effective value of λ is

$$\boxed{\lambda = \lambda_1 + \lambda_2}$$

Now, the above equation can also be written as

$$\frac{\ln 2}{T} = \frac{\ln 2}{T_1} + \frac{\ln 2}{T_2} \qquad (T = \text{half-life})$$

or

$$\frac{1}{T} = \frac{1}{T_1} + \frac{1}{T_2}$$

$\Rightarrow$

$$T = \frac{T_1 T_2}{T_1 + T_2}$$

Proof Suppose at some instant, the unstable nucleus has N number of nuclei, then

net rate of decay = decay in process 1 + decay in process 2

or

$$-\frac{dN}{dt} = \left(-\frac{dN}{dt}\right)_1 + \left(-\frac{dN}{dt}\right)_2$$

$\therefore$

$$\lambda N = \lambda_1 N + \lambda_2 N$$

or

$$\lambda = \lambda_1 + \lambda_2$$

Hence Proved.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on radioactivity*

Example 1 *At a given instant there are 25% undecayed radioactive nuclei in a sample. After 10 s the number of undecayed nuclei reduces to 12.5%. Calculate*

(a) mean life of the nuclei, (JEE 1996)

(b) the time in which the number of undecayed nuclei will further reduce to 6.25% of the reduced number.

Solution (a) In 10 s, number of nuclei has been reduced to half (25% to 12.5%).

Therefore, its half-life is

$$t_{1/2} = 10 \text{ s}$$

Relation between half-life and mean life is

$$t_{\text{mean}} = \frac{t_{1/2}}{\ln 2} = \frac{10}{0.693} \text{ s}$$

$$t_{\text{mean}} = 14.43 \text{ s}$$ **Ans.**

(b) From initial 100% to reduction till 6.25%, it takes four half-lives.

$$100\% \xrightarrow{t_{1/2}} 50\% \xrightarrow{t_{1/2}} 25\% \xrightarrow{t_{1/2}} 12.5\% \xrightarrow{t_{1/2}} 6.25\%$$

$$\therefore \quad t = 4\, t_{1/2} = 4\,(10) \text{ s} = 40 \text{ s}$$

$$t = 40 \text{ s}$$ **Ans.**

Example 2 *A radioactive element decays by β-emission. A detector records n beta particles in 2 s and in next 2 s it records 0.75 n beta particles. Find mean life correct to nearest whole number. Given ln |2| = 0.6931, ln |3| = 1.0986.* (JEE 2003)

Solution Let n_0 be the number of radioactive nuclei at time $t = 0$. Number of nuclei decayed in time t are given by $n_0\,(1 - e^{-\lambda t})$, which is also equal to the number of beta particles emitted during the same interval of time. For the given condition,

$$n = n_0\,(1 - e^{-2\lambda}) \quad \text{...(i)}$$

$$(n + 0.75n) = n_0\,(1 - e^{-4\lambda}) \quad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we get

$$1.75 = \frac{1 - e^{-4\lambda}}{1 - e^{-2\lambda}}$$

or

$$1.75 - 1.75\, e^{-2\lambda} = 1 - e^{-4\lambda}$$

$$\therefore \quad 1.75\, e^{-2\lambda} - e^{-4\lambda} = \frac{3}{4} \quad \text{...(iii)}$$

Let us take $e^{-2\lambda} = x$

Then, the above equation is

$$x^2 - 1.75\,x + 0.75 = 0$$

or $$x = \frac{1.75 \pm \sqrt{(1.75)^2 - (4)\,(0.75)}}{2}$$

or $$x = 1 \text{ and } \frac{3}{4}$$

∴ From Eq. (iii) either

$$e^{-2\lambda} = 1$$

or $$e^{-2\lambda} = \frac{3}{4}$$

but $e^{-2\lambda} = 1$ is not acceptable because which means $\lambda = 0$.

Hence, $$e^{-2\lambda} = \frac{3}{4}$$

or $$-2\lambda \ln (e) = \ln (3) - \ln (4) = \ln (3) - 2 \ln (2)$$

∴ $$\lambda = \ln (2) - \frac{1}{2} \ln (3)$$

Substituting the given values,

$$\lambda = 0.6931 - \frac{1}{2} \times (1.0986) = 0.14395 \text{ s}^{-1}$$

∴ Mean life, $t_{\text{mean}} = \frac{1}{\lambda} = 6.947$ s

∴ The correct answer is 7. **Ans.**

Example 3 *A small quantity of solution containing* Na^{24} *radio nuclide* ($half$*-life* $= 15$ *h*) *of activity* 1.0 *microcurie is injected into the blood of a person. A sample of the blood of volume* $1\ cm^3$ *taken after 5h shows an activity of* 296 *disintegrations per minute. Determine the total volume of the blood in the body of the person. Assume that the radioactive solution mixes uniformly in the blood of the person.* **(JEE 1994)**

(1 *curie* $= 3.7 \times 10^{10}$ *disintegrations per second)*

Solution ∵ λ = Disintegration constant

$$\frac{0.693}{t_{1/2}} = \frac{0.693}{15}\ \text{h}^{-1} = 0.0462\ \text{h}^{-1}$$

Let R_0 = initial activity = 1 microcurie = 3.7×10^4 disintegrations per second

r = Activity in 1 cm^3 of blood at $t = 5$ h

$= \frac{296}{60}$ disintegration per second

$= 4.93$ disintegration per second, and

R = Activity of whole blood at time $t = 5$ h

Total volume of blood should be

$$V = \frac{R}{r} = \frac{R_0 e^{-\lambda t}}{r}$$

Substituting the values, we have

$$V = \left(\frac{3.7 \times 10^4}{4.93}\right) e^{-(0.0462)(5)} \text{ cm}^3$$

$$V = 5.95 \times 10^3 \text{ cm}^3 \text{ or } V = 5.95 \text{ L}$$ **Ans.**

Example 4 *A radioactive nucleus X decays to a nucleus Y with a decay constant $\lambda_X = 0.1\ s^{-1}$, Y further decays to a stable nucleus Z with a decay constant $\lambda_Y = 1/30\, s^{-1}$. Initially, there are only X nuclei and their number is $N_0 = 10^{20}$. Set up the rate equations for the populations of X, Y and Z. The population of Y nucleus as a function of time is given by $N_Y(t) = \{N_0 \lambda_X/(\lambda_X - \lambda_Y)\} [exp(-\lambda_Y t) - exp(-\lambda_X t)]$. Find the time at which N_Y is maximum and determine the populations X and Z at that instant.* **(JEE 2001)**

Solution (a) Let at time $t = t$, number of nuclei of Y and Z are N_Y and N_Z. Then,

Rate equations of the populations of X, Y and Z are

$$\left(\frac{dN_X}{dt}\right) = -\lambda_X N_X \qquad \text{...(i)}$$

$$\left(\frac{dN_Y}{dt}\right) = \lambda_X N_X - \lambda_Y N_Y \qquad \text{...(ii)}$$

and $$\left(\frac{dN_Z}{dt}\right) = \lambda_Y N_Y \qquad \text{...(iii)}$$

(b) Given, $N_Y(t) = \dfrac{N_0 \lambda_X}{\lambda_X - \lambda_Y} [e^{-\lambda_Y t} - e^{-\lambda_X t}]$

For N_Y to be maximum

$$\frac{dN_Y(t)}{dt} = 0$$

i.e $$\lambda_X N_X = \lambda_Y N_Y \qquad \text{...(iv)} \qquad \text{[from Eq. (ii)]}$$

or $$\lambda_X (N_0 e^{-\lambda_X t}) = \lambda_Y \frac{N_0 \lambda_X}{\lambda_X - \lambda_Y} [e^{-\lambda_Y t} - e^{-\lambda_X t}]$$

or $$\frac{\lambda_X - \lambda_Y}{\lambda_Y} = \frac{e^{-\lambda_Y t}}{e^{-\lambda_X t}} - 1$$

$$\frac{\lambda_X}{\lambda_Y} = e^{(\lambda_X - \lambda_Y)t}$$

or $$(\lambda_X - \lambda_Y)\, t \ln(e) = \ln\left(\frac{\lambda_X}{\lambda_Y}\right)$$

or $$t = \frac{1}{\lambda_X - \lambda_Y} \ln\left(\frac{\lambda_X}{\lambda_Y}\right)$$

Substituting the values of λ_X and λ_Y, we have

$$t = \frac{1}{(0.1 - 1/30)} \ln\left(\frac{0.1}{1/30}\right) = 15 \ln(3)$$

or $$t = 16.48 \text{ s}$$ **Ans.**

(c) The population of X at this moment,

$$N_X = N_0\, e^{-\lambda_X t} = (10^{20})\, e^{-(0.1)(16.48)}$$

$$N_X = 1.92 \times 10^{19}$$

$$N_Y = \frac{N_X \lambda_X}{\lambda_Y} \qquad \text{[From Eq. (iv)]}$$

$$= (1.92 \times 10^{19}) \frac{(0.1)}{(1/30)}$$

$$= 5.76 \times 10^{19}$$

$$N_Z = N_0 - N_X - N_Y$$

$$= 10^{20} - 1.92 \times 10^{19} - 5.76 \times 10^{19}$$

or $$N_Z = 2.32 \times 10^{19}$$

Type 2. *Based on nuclear physics*

Example 5 *In a nuclear reactor* ^{235}U *undergoes fission liberating* 200 *MeV of energy. The reactor has a* 10% *efficiency and produces* 1000 *MW power. If the reactor is to function for* 10 *yr, find the total mass of uranium required.* **(JEE 2001)**

Solution The reactor produces 1000 MW power or 10^9 J/s. The reactor is to function for 10 yr. Therefore, total energy which the reactor will supply in 10 yr is

$$E = \text{(power) (time)}$$

$$= (10^9\,\text{J/s})\,(10 \times 365 \times 24 \times 3600\,\text{s})$$

$$= 3.1536 \times 10^{17}\ \text{J}$$

But since the efficiency of the reactor is only 10%, therefore actual energy needed is 10 times of it or 3.1536×10^{18} J. One uranium atom liberates 200 MeV of energy or $200 \times 1.6 \times 10^{-13}$ J or 3.2×10^{-11} J of energy. So, number of uranium atoms needed are

$$\frac{3.1536 \times 10^{18}}{3.2 \times 10^{-11}} = 0.9855 \times 10^{29}$$

or number of kg-moles of uranium needed are

$$n = \frac{0.9855 \times 10^{29}}{6.02 \times 10^{26}} = 163.7$$

Hence, total mass of uranium required is

$$m = (n)M = (163.7)\,(235)\ \text{kg}$$

or $$m \approx 38470\ \text{kg}$$

or $$m = 3.847 \times 10^4\ \text{kg}$$

Example 6 *The element curium* $^{248}_{96}Cm$ *has a mean life of* 10^{13} *s. Its primary decay modes are spontaneous fission and* α*-decay, the former with a probability of 8% and the later with a probability of 92%, each fission releases* 200 *MeV of energy. The masses involved in decay are as follows* **(JEE 1997)**

$^{248}_{96}Cm = 248.072220\ u$, $^{244}_{94}Pu = 244.064100\ u$ *and* $^{4}_{2}He = 4.002603\ u$. *Calculate the power output from a sample of* 10^{20} *Cm atoms.* $(1\ u = 931\ MeV/c^2)$

Solution The reaction involved in α-decay is

$$^{248}_{96}\text{Cm} \rightarrow\ ^{244}_{94}\text{Pu} +\ ^{4}_{2}\text{He}$$

Mass defect, $\Delta m = \text{mass of }^{248}_{96}\text{Cm} - \text{ mass of }^{244}_{94}\text{Pu} - \text{mass of }^{4}_{2}\text{He}$

$$= (248.072220 - 244.064100 - 4.002603)\ \text{u}$$

$$= 0.005517\ \text{u}$$

Therefore, energy released in α-decay will be

$$E_\alpha = (0.005517 \times 931)\ \text{MeV} = 5.136\ \text{MeV}$$

Similarly, $E_{\text{fission}} = 200\ \text{MeV}$ (given)

Mean life is given as $t_{\text{mean}} = 10^{13}\ \text{s} = 1/\lambda$

$\therefore$ Disintegration constant $\lambda = 10^{-13}\ \text{s}^{-1}$

Rate of decay at the moment when number of nuclei are 10^{20}

$$= \lambda N = (10^{-13})\,(10^{20})$$

$$= 10^{7}\ \text{disintegration per second}$$

Of these, 8% are in fission and 92% are in α-decay.

Therefore, energy released per second

$$= (0.08 \times 10^{7} \times 200 + 0.92 \times 10^{7} \times 5.136)\ \text{MeV}$$

$$= 2.074 \times 10^{8}\ \text{MeV}$$

$\therefore$ Power output (in watt) $= \text{energy released per second (J/s)}$

$$= (2.074 \times 10^{8})\,(1.6 \times 10^{-13})$$

$$= 3.32 \times 10^{-5}\ \text{Js}^{-1}$$

$\therefore$ Power output $= 3.32 \times 10^{-5}$ W

Example 7 *A nucleus X, initially at rest, undergoes alpha-decay according to the equation.* **(JEE 1991)**

$$^{A}_{92}X \rightarrow\ ^{228}_{Z}Y + \alpha$$

(a) Find the values of A and Z in the above process.

(b) The alpha particle produced in the above process is found to move in a circular track of radius 0.11m in a uniform magnetic field of 3 T. Find the energy (in MeV) released during the process and the binding energy of the parent nucleus X.

Given that $m\ (Y) = 228.03\ u$, $m\,(^{1}_{0}n) = 1.009\ u$

$$m\,(^{4}_{2}He) = 4.003\ u,\ m(^{1}_{1}H) = 1.008\ u$$

Solution (a) $A - 4 = 228$

$$\therefore \quad A = 232$$

$$92 - 2 = Z$$

or $$Z = 90$$

(b) From the relation,

$$r = \frac{\sqrt{2Km}}{Bq} \quad \Rightarrow \quad K_\alpha = \frac{r^2B^2q^2}{2m}$$

$$= \frac{(0.11)^2(3)^2\,(2 \times 1.6 \times 10^{-19})^2}{2 \times 4.003 \times 1.67 \times 10^{-27} \times 1.6 \times 10^{-13}}\ \text{MeV}$$

$$= 5.21\ \text{MeV}$$

From the conservation of momentum,

$$p_Y = p_\alpha \quad \text{or} \quad \sqrt{2K_Y m_Y} = \sqrt{2K_\alpha m_\alpha}$$

$$\therefore \qquad K_Y = \left(\frac{m_\alpha}{m_Y}\right) K_\alpha = \frac{4.003}{228.03} \times 5.21$$

$$= 0.09 \text{ MeV}$$

$\therefore$ Total energy released $= K_\alpha + K_Y = 5.3$ MeV

Total binding energy of daughter products

$$= [92 \times (\text{mass of proton}) + (232 - 92)\ (\text{mass of neutron}) - (m_Y) - (m_\alpha)] \times 931.48 \text{ MeV}$$

$$= [(92 \times 1.008) + (140)\ (1.009) - 228.03 - 4.003]\ 931.48 \text{ MeV}$$

$$= 1828.5 \text{ MeV}$$

$\therefore$ Binding energy of parent nucleus

= binding energy of daughter products – energy released

$= (1828.5 - 5.3)$ MeV $= 1823.2$ MeV

Example 8 *It is proposed to use the nuclear fusion reaction,*

$$ {}_1^2H + {}_1^2H \rightarrow {}_2^4He$$

in a nuclear reactor 200 *MW rating. If the energy from the above reaction is used with a* 25 *per cent efficiency in the reactor, how many grams of deuterium fuel will be needed per day? (The masses of* ${}_1^2H$ *and* ${}_2^4He$ *are* 2.0141 *atomic mass units and* 4.0026 *atomic mass units respectively.)* **(JEE 1990)**

Solution Mass defect in the given nuclear reaction,

$$\Delta m = 2\ (\text{mass of deuterium}) - (\text{mass of helium})$$

$$= 2\ (2.0141) - (4.0026) = 0.0256$$

Therefore, energy released

$$\Delta E = (\Delta m)\ (931.48) \text{ MeV} = 23.85 \text{ MeV}$$

$$= 23.85 \times 1.6 \times 10^{-13} \text{ J} = 3.82 \times 10^{-12} \text{ J}$$

Efficiency is only 25%, therefore,

$$25\% \text{ of } \Delta E = \left(\frac{25}{100}\right) (3.82 \times 10^{-12}) \text{ J}$$

$$= 9.55 \times 10^{-13} \text{ J}$$

i.e. by the fusion of two deuterium nuclei, 9.55×10^{-13} J energy is available to the nuclear reactor.

Total energy required in one day to run the reactor with a given power of 200 MW,

$$E_{\text{Total}} = 200 \times 10^6 \times 24 \times 3600 = 1.728 \times 10^{13} \text{J}$$

$\therefore$ Total number of deuterium nuclei required for this purpose,

$$n = \frac{E_{\text{Total}}}{\Delta E/2} = \frac{2 \times 1.728 \times 10^{13}}{9.55 \times 10^{-13}}$$

$$= 0.362 \times 10^{26}$$

$\therefore$ Mass of deuterium required = (Number of g-moles of deuterium required) × 2 g

$$= \left(\frac{0.362 \times 10^{26}}{6.02 \times 10^{23}}\right) \times 2 = 120.26 \text{ g}$$

Miscellaneous Examples

Example 9 *Find the minimum kinetic energy of an* α*-particle to cause the reaction* $^{14}N(\alpha, p)^{17}O$*. The masses of* ^{14}N*,* ^{4}He*,* ^{1}H *and* ^{17}O *are respectively* 14.00307 *u,* 4.00260 *u,* 1.00783 *u and* 16.99913 *u.*

Solution Since, the masses are given in atomic mass units, it is easiest to proceed by finding the mass difference between reactants and products in the same units and then multiplying by 931.5 MeV/u. Thus, we have

$$Q = (14.00307\text{ u} + 4.00260\text{ u} - 1.00783\text{ u} - 16.99913\text{ u})\left(931.5\,\frac{\text{MeV}}{\text{u}}\right)$$

$$= -\,1.20\text{ MeV}$$

Q-value is negative. It means reaction is endothermic.

So, the minimum kinetic energy of α-particle to initiate this reaction would be

$$K_{\min} = |Q|\left(\frac{m_\alpha}{m_N} + 1\right) = (1.20)\left(\frac{4.00260}{14.00307} + 1\right)$$

$$= 1.54\text{ MeV}$$ **Ans.**

Example 10 *Neon*-23 *decays in the following way,*

$$^{23}_{10}Ne \longrightarrow {}^{23}_{11}Na + {}^{0}_{-1}e + \overline{\nu}$$

Find the minimum and maximum kinetic energy that the beta particle $({}^{0}_{-1}e)$ *can have. The atomic masses of* ^{23}Ne *and* ^{23}Na *are* 22.9945 *u and* 22.9898 *u, respectively.*

Solution Here, atomic masses are given (not the nuclear masses), but still we can use them for calculating the mass defect because mass of electrons get cancelled both sides. Thus,

Mass defect $\Delta m = (22.9945 - 22.9898) = 0.0047\text{ u}$

$$\therefore \quad Q = (0.0047\text{ u})(931.5\text{ MeV/u})$$

$$= 4.4\text{ MeV}$$

Hence, the energy of beta particles can range from 0 to 4.4 MeV. **Ans.**

Example 11 *The mean lives of an unstable nucleus in two different decay processes are* 1620 *yr and* 405 *yr, respectively. Find out the time during which three-fourth of a sample will decay.*

Solution Let at some instant of time t, number of nuclei are N. Then,

$$\left(\frac{-dN}{dt}\right)_{\text{net}} = \left(\frac{-dN}{dt}\right)_1 + \left(\frac{-dN}{dt}\right)_2$$

If the effective decay constant is λ, then

$$\lambda N = \lambda_1 N + \lambda_2 N$$

or

$$\lambda = \lambda_1 + \lambda_2 = \frac{1}{1620} + \frac{1}{405} = \frac{1}{324}\text{ year}^{-1}$$

Now, $$\frac{N_0}{4} = N_0\, e^{-\lambda t}$$

∴ $$-\lambda t = \ln\left(\frac{1}{4}\right) = -1.386$$

or $$\left(\frac{1}{324}\right) t = 1.386$$

∴ $$t = 449 \text{ yr}$$ **Ans.**

Example 12 *In the chemical analysis of a rock the mass ratio of two radioactive isotopes is found to be* $100:1$. *The mean lives of the two isotopes are* 4×10^9 *years and* 2×10^9 *years, respectively. If it is assumed that at the time of formation the atoms of both the isotopes were in equal proportional, calculate the age of the rock. Ratio of the atomic weights of the two isotopes is* $1.02:1$.

Solution At the time of observation ($t = t$),

$$\frac{m_1}{m_2} = \frac{100}{1} \qquad \text{(given)}$$

Further it is given that $$\frac{A_1}{A_2} = \frac{1.02}{1}$$

Number of atoms, $$N = \frac{m}{A}$$

∴ $$\frac{N_1}{N_2} = \frac{m_1}{m_2} \times \frac{A_2}{A_1} = \frac{100}{1.02} \qquad \text{...(i)}$$

Let N_0 be the number of atoms of both the isotopes at the time of formation, then

$$\frac{N_1}{N_2} = \frac{N_0\, e^{-\lambda_1 t}}{N_0\, e^{-\lambda_2 t}} = e^{(\lambda_2 - \lambda_1)\, t} \qquad \text{...(ii)}$$

Eq. (i) and Eq. (ii), we have

$$e^{(\lambda_2 - \lambda_1)\, t} = \frac{100}{1.02}$$

or $$(\lambda_2 - \lambda_1) t = \ln 100 - \ln 1.02$$

∴ $$t = \frac{\ln 100 - \ln 1.02}{\left(\dfrac{1}{2 \times 10^9} - \dfrac{1}{4 \times 10^9}\right)}$$

Substituting the values, we have

$$t = 1.834 \times 10^{10} \text{ yr}$$ **Ans.**

Example 13 *A proton is bombarded on a stationary lithium nucleus. As a result of the collision, two* α*-particles are produced. If the direction of motion of the* α*-particles with the initial direction of motion makes an angle* $\cos^{-1}(1/4)$*, find the kinetic energy of the striking proton. Given, binding energies per nucleon of* Li^7 *and* He^4 *are* 5.60 *and* 7.06 *MeV, respectively.*
(Assume mass of proton ≈ mass of neutron).

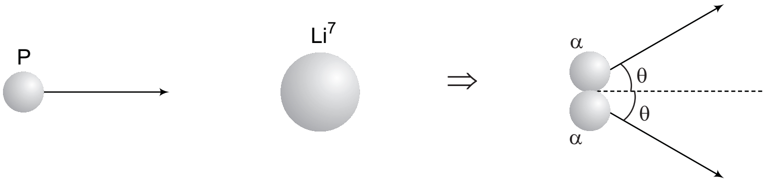
Solution Q-value of the reaction is

$$Q = (2 \times 4 \times 7.06 - 7 \times 5.6)\ \text{MeV} = 17.28\ \text{MeV}$$

Applying conservation of energy for collision,

$$K_p + Q = 2K_\alpha \qquad \ldots\text{(i)}$$

(Here, K_p and K_α are the kinetic energies of proton and α-particle respectively)



From conservation of linear momentum,

$$\sqrt{2m_pK_p} = 2\sqrt{2m_\alpha K_\alpha}\cos\theta \qquad \ldots\text{(ii)}$$

$$\therefore \quad K_p = 16K_\alpha\cos^2\theta = (16K_\alpha)\left(\frac{1}{4}\right)^2 \qquad (\text{as } m_\alpha = 4m_p)$$

$$\therefore \quad K_\alpha = K_p \qquad \ldots\text{(iii)}$$

Solving Eqs. (i) and (iii) with $Q = 17.28$ MeV

We get $K_p = 17.28$ MeV **Ans.**

Example 14 *A* ^{7}Li *target is bombarded with a proton beam current of* 10^{-4} *A for* 1 *hour to produce* ^{7}Be *of activity* 1.8×10^8 *disintegrations per second. Assuming that one* ^{7}Be *radioactive nucleus is produced by bombarding* 1000 *protons, determine its half-life.*

Solution At time t, let say there are N atoms of ^{7}Be (radioactive). Then, net rate of formation of ^{7}Be nuclei at this instant is

$$\frac{dN}{dt} = \frac{10^{-4}}{1.6 \times 10^{-19} \times 1000} - \lambda N$$

or

$$\frac{dN}{dt} = 6.25 \times 10^{11} - \lambda N$$

or

$$\int_0^{N_0} \frac{dN}{6.25 \times 10^{11} - \lambda N} = \int_0^{3600} dt$$

where, N_0 are the number of nuclei at $t = 1$ h or 3600 s.

$$\therefore \quad -\frac{1}{\lambda}\ln\left(\frac{6.25 \times 10^{11} - \lambda N_0}{6.25 \times 10^{11}}\right) = 3600$$

λN_0 = activity of ^{7}Be at $t = 1$ h $= 1.8 \times 10^8$ disintegrations/s

$$\therefore \quad -\frac{1}{\lambda}\ln\left(\frac{6.25 \times 10^{11} - 1.8 \times 10^8}{6.25 \times 10^{11}}\right) = 3600$$

$$\therefore \quad \lambda = 8.0 \times 10^{-8}\ \text{sec}^{-1}$$

Therefore, half-life

$$t_{1/2} = \frac{0.693}{8.0 \times 10^{-8}} = 8.66 \times 10^6\ \text{s}$$

$$= 100.26\ \text{days}$$ **Ans.**

Example 15 *A ^{118}Cd radio nuclide goes through the transformation chain.*

$$^{118}Cd \xrightarrow[30\,min]{} {}^{118}\,\text{ln} \xrightarrow[45\,min]{} {}^{118}Sn \ \ (stable)$$

The half-lives are written below the respective arrows. At time $t = 0$ only Cd was present. Find the fraction of nuclei transformed into stable over 60 minutes.

Solution At time $t = t$, $N_1 = N_0\, e^{-\lambda_1 t}$ and $N_2 = \dfrac{N_0 \lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$ (see Article 34.3)

$$\therefore \qquad N_3 = N_0 - N_1 - N_2$$

$$= N_0 \left[1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}) \right]$$

$$\therefore \qquad \frac{N_3}{N_0} = 1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\lambda_1 = \frac{0.693}{30} = 0.0231 \text{ min}^{-1}$$

$$\lambda_2 = \frac{0.693}{45} = 0.0154 \text{ min}^{-1}$$

and $$t = 60 \text{ min}$$

$$\therefore \qquad \frac{N_3}{N_0} = 1 - e^{-0.0231 \times 60} - \frac{0.0231}{0.0154 - 0.0231}(e^{-0.0231 \times 60} - e^{-0.0154 \times 60})$$

$$= 1 - 0.25 + 3\,(0.25 - 0.4)$$

$$= 0.31$$ **Ans.**

Example 16 *Natural uranium is a mixture of three isotopes $^{234}_{92}U$, $^{235}_{92}U$ and $^{238}_{92}U$ with mass percentage 0.01%, 0.71% and 99.28% respectively. The half-life of three isotopes are 2.5×10^5 yr, 7.1×10^8 yr and 4.5×10^9 yr respectively. Determine the share of radioactivity of each isotope into the total activity of the natural uranium.*

Solution Let R_1, R_2 and R_3 be the activities of U^{234}, U^{235} and U^{238} respectively.

Total activity, $$R = R_1 + R_2 + R_3$$

Share of U^{234}, $$\frac{R_1}{R} = \frac{\lambda_1 N_1}{\lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3}$$

Let m be the total mass of natural uranium.

Then, $$m_1 = \frac{0.01}{100} m, \quad m_2 = \frac{0.71}{100} m \quad \text{and} \quad m_3 = \frac{99.28}{100} m$$

Now, $$N_1 = \frac{m_1}{M_1}, \quad N_2 = \frac{m_2}{M_2} \quad \text{and} \quad N_3 = \frac{m_3}{M_3}$$

where M_1, M_2 and M_3 are atomic weights.

$$\therefore \qquad \frac{R_1}{R} = \frac{\left(\dfrac{m_1}{M_1}\right)\dfrac{1}{T_1}}{\dfrac{m_1}{M_1}\dfrac{1}{T_1} + \dfrac{m_2}{M_2}\cdot\dfrac{1}{T_2} + \dfrac{m_3}{M_3}\cdot\dfrac{1}{T_3}}$$

$$= \frac{\dfrac{(0.01/100)}{234} \times \dfrac{1}{2.5 \times 10^5 \text{ years}}}{\left(\dfrac{0.01/100}{234}\right)\left(\dfrac{1}{2.5 \times 10^5}\right) + \left(\dfrac{0.71/100}{235}\right)\left(\dfrac{1}{7.1 \times 10^8}\right) + \left(\dfrac{99.28/100}{238}\right)\left(\dfrac{1}{4.5 \times 10^9}\right)}$$

$$= 0.648 \approx 64.8\,\%$$

Similarly, share of $U^{235} = 0.016\,\%$

and of $U^{238} = 35.184\,\%$ **Ans.**

Example 17 *Uranium ores on the earth at the present time typically have a composition consisting of* 99.3% *of the isotope* ${}_{92}U^{238}$ *and* 0.7% *of the isotope* ${}_{92}U^{235}$. *The half-lives of these isotopes are* 4.47×10^9 *yr and* 7.04×10^8 *yr, respectively. If these isotopes were equally abundant when the earth was formed, estimate the age of the earth.*

Solution Let N_0 be number of atoms of each isotope at the time of formation of the earth $(t = 0)$ and N_1 and N_2 the number of atoms at present $(t = t)$. Then,

$$N_1 = N_0 e^{-\lambda_1 t} \qquad \text{...(i)}$$

and

$$N_2 = N_0 e^{-\lambda_2 t} \qquad \text{...(ii)}$$

$\therefore$

$$\frac{N_1}{N_2} = e^{(\lambda_2 - \lambda_1)t} \qquad \text{...(iii)}$$

Further it is given that

$$\frac{N_1}{N_2} = \frac{99.3}{0.7} \qquad \text{...(iv)}$$

Equating Eqs. (iii) and (iv) and taking log on both sides, we have

$$(\lambda_2 - \lambda_1)\, t = \ln\left(\frac{99.3}{0.7}\right)$$

$\therefore$

$$t = \left(\frac{1}{\lambda_2 - \lambda_1}\right) \ln\left(\frac{99.3}{0.7}\right)$$

Substituting the values, we have

$$t = \frac{1}{\dfrac{0.693}{7.04 \times 10^8} - \dfrac{0.693}{4.47 \times 10^9}} \ln\left(\frac{99.3}{0.7}\right)$$

or

$$t = 5.97 \times 10^9 \text{ yr}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Rate of radioactivity cannot be increased or decreased by increasing or decreasing pressure or temperature.

Reason : Rate depends on the number of nuclei present in the radioactive sample.

2. **Assertion :** Only those nuclei which are heavier than lead are radioactive.

Reason : Nuclei of elements heavier than lead are unstable.

3. **Assertion :** After emission of one α-particle and two β-particles, atomic number remains unchanged.

Reason : Mass number changes by four.

4. **Assertion :** γ-rays are produced by the transition of a nucleus from some higher energy state to some lower energy state.

Reason : Electromagnetic waves are always produced by the transition process.

5. **Assertion :** During β-decay a proton converts into a neutron and an electron. No other particle is emitted.

Reason : During β-decay linear momentum of system should remain constant.

6. **Assertion :** If we compare the stability of two nuclei, then that nucleus is more stable whose total binding energy is more.

Reason : More the mass defect during formation of a nucleus more will be the binding energy.

7. **Assertion :** In a nuclear process energy is released if total binding energy of daughter nuclei is more than the total binding energy of parent nuclei.

Reason : If energy is released then total mass of daughter nuclei is less than the total mass of parent nuclei.

8. **Assertion :** Binding energy per nucleon is of the order of MeV.

Reason : $1\text{ MeV} = 1.6 \times 10^{-13}$ J.

9. **Assertion :** 1 amu is equal to 931.48 MeV.

Reason : 1 amu is equal to $\frac{1}{12}$th the mass of C^{12} atom.

10. **Assertion :** Between α, β and γ radiations, penetrating power of γ-rays is maximum.

Reason : Ionising power of γ-rays is least.

11. **Assertion :** The nuclear energy can be obtained by the nuclear fission of heavier nuclei as well as by fusion of lighter nuclei.

Reason : As the mass number increases, the binding energy per nucleon, first increases and then decreases.

Objective Questions

1. For uranium nucleus how does its mass vary with volume? **(JEE 2003)**

(a) $m \propto V$
(b) $m \propto 1/V$
(c) $m \propto \sqrt{V}$
(d) $m \propto V^2$

2. Order of magnitude of density of uranium nucleus is ($m_p = 1.67 \times 10^{-27}$ kg) **(JEE 1999)**

(a) 10^{20} kg/m^3
(b) 10^{17} kg/m^3
(c) 10^{14} kg/m^3
(d) 10^{11} kg/m^3

3. During a beta decay,

(a) an atomic electron is ejected
(b) an electron present inside the nucleus is ejected
(c) a neutron in the nucleus decays emitting an electron
(d) a part of the binding energy is converted into electron

4. In the nucleus of helium if F_1 is the net force between two protons, F_2 is the net force between two neutrons and F_3 is the net force between a proton and a neutron. Then,

(a) $F_1 = F_2 = F_3$
(b) $F_1 > F_2 > F_3$
(c) $F_2 > F_3 > F_1$
(d) $F_2 = F_3 > F_1$

5. What are the respective number of α and β-particles emitted in the following radioactive decay?

$$^{200}_{90}X \rightarrow {}^{168}_{80}Y$$

(a) 6 and 8
(b) 6 and 6
(c) 8 and 8
(d) 8 and 6

6. If an atom of $^{235}_{92}$U, after absorbing a slow neutron, undergoes fission to form an atom of $^{138}_{54}$Xe and an atom of $^{94}_{38}$Sr, the other particles produced are

(a) one proton and two neutrons
(b) three neutrons
(c) two neutrons
(d) one proton and one neutron

7. Nucleus A is converted into C through the following reactions,

$$A \rightarrow B + \alpha$$
$$B \rightarrow C + 2\beta$$

then,

(a) A and B are isotopes
(b) A and C are isobars
(c) A and B are isobars
(d) A and C are isotopes

8. The binding energy of α-particle is

(if $m_p = 1.00785$ u, $m_n = 1.00866$ u and $m_\alpha = 4.00274$ u)

(a) 56.42 MeV
(b) 2.821 MeV
(c) 28.21 MeV
(d) 32.4 MeV

9. $\frac{7}{8}$th of the active nuclei present in a radioactive sample has decayed in 8 s. The half-life of the sample is

(a) 2 s
(b) 1 s
(c) 7 s
(d) $\frac{8}{3}$ s

10. A radioactive element disintegrates for a time interval equal to its mean life. The fraction that has disintegrated is

(a) $\frac{1}{e}$ (b) $1-\frac{1}{e}$

(c) $\frac{0.693}{e}$ (d) $0.693\left(1-\frac{1}{e}\right)$

11. Starting with a sample of pure ^{66}Cu, $\frac{3}{4}$ of it decays into Zn in 15 minutes. The corresponding half-life is

(a) 5 minutes (b) 7.5 minutes

(c) 10 minutes (d) 3.5 minutes

12. A sample of radioactive substance loses half of its activity in 4 days. The time in which its activity is reduced to 5% is

(a) 12 days (b) 8.3 days

(c) 17.3 days (d) None of these

13. On bombardment of U^{235} by slow neutrons, 200 MeV energy is released. If the power output of atomic reactor is 1.6 MW, then the rate of fission will be

(a) 5×10^{16} per second (b) 10×10^{16} per second

(c) 15×10^{16} per second (d) 20×10^{16} per second

14. Atomic masses of two heavy atoms are A_1 and A_2. Ratio of their respective nuclear densities will be approximately

(a) $\frac{A_1}{A_2}$ (b) $\left(\frac{A_1}{A_2}\right)^{1/3}$ (c) $\left(\frac{A_2}{A_1}\right)^{1/3}$ (d) 1

15. A radioactive element is disintegrating having half-life 6.93 s. The fractional change in number of nuclei of the radioactive element during 10 s is

(a) 0.37 (b) 0.63

(c) 0.25 (d) 0.50

16. The activity of a radioactive sample goes down to about 6% in a time of 2 hour. The half-life of the sample in minute is about

(a) 30 (b) 15 (c) 60 (d) 120

17. What is the probability of a radioactive nucleus to survive one mean life?

(a) $\frac{1}{e}$ (b) $\frac{1}{e+1}$ (c) $1-\frac{1}{e}$ (d) $\frac{1}{e}-1$

Subjective Questions

Note *You can take approximations in the answers.*

1. The disintegration rate of a certain radioactive sample at any instant is 4750 disintegrations per minute. Five minutes later the rate becomes 2700 per minute. Calculate

(a) decay constant and (b) half-life of the sample

2. A radioactive sample contains 1.00×10^{15} atoms and has an activity of 6.00×10^{11} Bq. What is its half-life?

3. Obtain the amount of ^{60}Co necessary to provide a radioactive source of 8.0 Ci strength. The half-life of ^{60}Co is 5.3 years?

4. The half-life of ${}^{238}_{92}\text{U}$ against alpha decay is 4.5×10^9 year. How much disintegration per second occurs in 1 g of ${}^{238}_{92}\text{U}$?

5. What is the probability that a radioactive atom having a mean life of 10 days decays during the fifth day?

6. In an ore containing uranium, the ratio of ${}^{238}\text{U}$ to ${}^{206}\text{Pb}$ nuclei is 3. Calculate the age of the ore, assuming that all the lead present in the ore is the final stable product of ${}^{238}\text{U}$. Take the half-life of ${}^{238}\text{U}$ to be 4.5×10^9 years..

7. The half-lives of radioisotopes P^{32} and P^{33} are 14 days and 25 days respectively. These radioisotopes are mixed in the ratio of 4 : 1 of their atoms. If the initial activity of the mixed sample is 3.0 mCi, find the activity of the mixed isotopes after 60 years.

8. Complete the following reactions.

(a) ${}^{226}_{88}\text{Ra} \to \alpha +$ (b) ${}^{19}_{8}\text{O} \to {}^{19}_{9}\text{F} +$ (c) ${}^{25}_{13}\text{AI} \to {}^{25}_{12}\text{Mg} +$

9. Consider two decay reactions.

(a) ${}^{236}_{92}\text{U} \to {}^{206}_{82}\text{Pb} + 10 \text{ protons} + 20 \text{ neutrons}$ (b) ${}^{238}_{92}\text{U} \to {}^{206}_{82}\text{Pb} + 8\,{}^{4}_{2}\text{He} + 6 \text{ electrons}$

Are both the reactions possible?

10. Obtain the binding energy of a nitrogen nucleus from the following data :

$$m_H = 1.00783 \text{ u}, \; m_N = 1.00867 \text{ u}, \; m({}^{14}_{7}\text{N}) = 14.00307 \text{ u}$$

Give your answer in units of MeV. [Remember $1 \text{ u} = 931.5 \text{ MeV}/c^2$]

11. 8 protons and 8 neutrons are separately at rest. How much energy will be released if we form ${}^{16}_{8}\text{O}$ nucleus?

Given :

$$\text{Mass of } {}^{16}_{8}\text{O atom} = 15.994915 \text{ u}$$
$$\text{Mass of neutron} = 1.008665 \text{ u}$$
$$\text{Mass of hydrogen atom} = 1.007825 \text{ u}$$

12. Assuming the splitting of U^{235} nucleus liberates 200 MeV energy, find

(a) the energy liberated in the fission of 1 kg of U^{235} and

(b) the mass of the coal with calorific value of 30 kJ/g which is equivalent to 1 kg of U^{235}.

13. ${}^{212}_{83}\text{Bi}$ decays as per following equation.

$${}^{212}_{83}\text{Bi} \to {}^{208}_{82}\text{Ti} + {}^{4}_{2}\text{He}$$

The kinetic energy of α-particle emitted is 6.802 MeV. Calculate the kinetic energy of Ti recoil atoms.

14. In a neutron induced fission of ${}_{92}\text{U}^{235}$ nucleus, usable energy of 185 MeV is released. If ${}_{92}\text{U}^{235}$ reactor is continuously operating it at a power level of 100 MW power, how long will it take for 1 kg of uranium to be consumed in this reactor?

15. Calculate the Q-values of the following fusion reactions :

(a) ${}^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \to {}^{3}_{1}\text{H} + {}^{1}_{1}\text{H}$ (b) ${}^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \to {}^{3}_{2}\text{He} + n$ (c) ${}^{2}_{1}\text{H} + {}^{3}_{1}\text{H} \to {}^{4}_{2}\text{He} + n$

Atomic masses are $m({}^{2}_{1}\text{H}) = 2.014102 \text{ u}, \; m({}^{3}_{1}\text{H}) = 3.016049 \text{ u},$

$$m({}^{3}_{2}\text{He}) = 3.016029 \text{ u}, \; m({}^{4}_{2}\text{He}) = 4.002603 \text{ u},$$
$$m({}^{1}_{1}\text{H}) = 1.007825 \text{ u}$$

16. Calculate the Q-value of the fusion reaction,

$$^4\text{He} + {^4\text{He}} \rightarrow {^8\text{Be}}$$

Is such a fusion energetically favourable? Atomic mass of ^{8}Be is 8.0053 u and that of ^{4}He is 4.0026 u.

17. When fission occurs, several neutrons are released and the fission fragments are beta radioactive, why?

LEVEL 2

Single Correct Option

1. The count rate observed from a radioactive source at t second was N_0 and at $4t$ second it was $\frac{N_0}{16}$. The count rate observed at $\left(\frac{11}{2}\right) t$ second will be

(a) $\frac{N_0}{128}$ (b) $\frac{N_0}{64}$

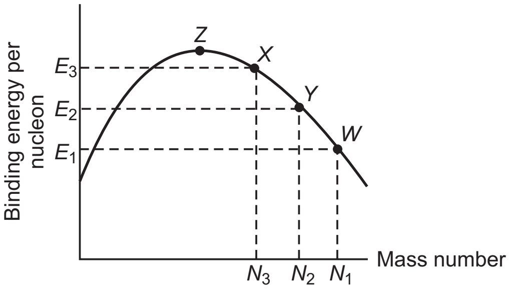
(c) $\frac{N_0}{32}$ (d) None of these

2. The half-lives of a radioactive sample are 30 years and 60 years for two decay processes. If the sample decays by both the processes simultaneously. The time after which, only one-fourth of the sample will remain is

(a) 10 years (b) 20 years

(c) 40 years (d) 60 years

3. Consider the nuclear fission reaction $W \rightarrow X + Y$. What is the Q-value (energy released) of the reaction?



(a) $E_1N_1 - (E_2N_2 + E_3N_3)$ (b) $(E_2N_2 + E_3N_3 - E_1N_1)$

(c) $E_2N_2 + E_1N_1 - E_3N_3$ (d) $E_1N_1 + E_3N_3 - E_2N_2$

4. Consider the following nuclear reaction,

$$X^{200} \rightarrow A^{110} + B^{90} + \text{Energy}$$

If the binding energy per nucleon for X, A and B are 7.4 MeV, 8.2 MeV and 8.2 MeV respectively, the energy released will be

(a) 90 MeV (b) 110 MeV

(c) 200 MeV (d) 160 MeV

5. The energy released by the fission of a single uranium nucleus is 200 MeV. The number of fissions of uranium nucleus per second required to produce 16 MW of power is
(Assume efficiency of the reactor is 50%)

(a) 2×10^6 (b) 2.5×10^6
(c) 5×10^6 (d) None of these

6. A radioactive isotope is being produced at a constant rate A. The isotope has a half-life T. Initially, there are no nuclei, after a time $t >> T$, the number of nuclei becomes constant. The value of this constant is

(a) AT (b) $\frac{A}{T} \ln 2$
(c) $AT \ln 2$ (d) $\frac{AT}{\ln 2}$

7. A bone containing 200 g carbon-14 has a β-decay rate of 375 decay/min. Calculate the time that has elapsed since the death of the living one. Given the rate of decay for the living organism is equal to 15 decay per min per gram of carbon and half-life of carbon-14 is 5730 years.

(a) 27190 years (b) 1190 years
(c) 17190 years (d) None of these

8. Two identical samples (same material and same amount) P and Q of a radioactive substance having mean life T are observed to have activities A_P and A_Q respectively at the time of observation. If P is older than Q, then the difference in their age is

(a) $T \ln \left(\frac{A_P}{A_Q}\right)$ (b) $T \ln \left(\frac{A_Q}{A_P}\right)$
(c) $T \left(\frac{A_P}{A_Q}\right)$ (d) $T \left(\frac{A_Q}{A_P}\right)$

9. A star initially has 10^{40} deuterons. It produces energy *via* the processes ${}_1^2\text{H} + {}_1^2\text{H} \to {}_1^3\text{H} + p$ and ${}_1^2\text{H} + {}_1^3\text{H} \to {}_2^4\text{He} + n$. Where the masses of the nuclei are $m({}^2\text{H}) = 2.014$ amu, $m(p) = 1.007$ amu, $m(n) = 1.008$ amu and $m({}^4\text{He}) = 4.001$ amu. If the average power radiated by the star is 10^{16} W, the deuteron supply of the star is exhausted in a time of the order of

(a) 10^6 s (b) 10^8 s (c) 10^{12} s (d) 10^{16} s

10. Two radioactive samples of different elements (half-lives t_1 and t_2 respectively) have same number of nuclei at $t = 0$. The time after which their activities are same is

(a) $\frac{t_1 t_2}{0.693 (t_2 - t_1)} \ln \frac{t_2}{t_1}$ (b) $\frac{t_1 t_2}{0.693} \ln \frac{t_2}{t_1}$
(c) $\frac{t_1 t_2}{0.693 (t_1 + t_2)} \ln \frac{t_2}{t_1}$ (d) None of these

11. A nucleus X initially at rest, undergoes alpha decay according to the equation

$${}_Z^{232}X \to {}_{90}^{A}Y + \alpha$$

What fraction of the total energy released in the decay will be the kinetic energy of the alpha particle?

(a) $\frac{90}{92}$ (b) $\frac{228}{232}$ (c) $\sqrt{\frac{228}{232}}$ (d) $\frac{1}{2}$

12. A stationary nucleus of mass 24 amu emits a gamma photon. The energy of the emitted photon is 7 MeV. The recoil energy of the nucleus is

(a) 2.2 keV
(b) 1.1 keV
(c) 3.1 keV
(d) 22 keV

13. A radioactive material of half-life T was kept in a nuclear reactor at two different instants. The quantity kept second time was twice of that kept first time. If now their present activities are A_1 and A_2 respectively, then their age difference equals

(a) $\frac{T}{\ln 2} \ln \frac{2A_1}{A_2}$
(b) $T \ln \frac{A_1}{A_2}$
(c) $\frac{T}{\ln 2} \ln \frac{A_2}{2A_1}$
(d) $T \ln \frac{A_2}{2A_1}$

More than One Correct Options

1. At $t = 0$, number of radioactive nuclei of a radioactive substance are x and its radioactivity is y. Half-life of radioactive substance is T. Then,

(a) $\frac{x}{y}$ is constant throughout
(b) $\frac{x}{y} > T$
(c) value of xy remains half after one half-life
(d) value of xy remains one fourth after one half-life

2. Choose the correct options.

(a) Isotopes have same number of atomic number
(b) Isobars have same atomic weight
(c) Isotones have same number of neutrons
(d) In neutral isotope atoms number of electrons are same

3. Choose the correct options.

(a) By gamma radiations atomic number is not changed
(b) By gamma radiations mass number is not changed
(c) By the emission of one α and two β particles isotopes are produced
(d) By the emission of one α and four β particles isobars are produced

4. Two radioactive substances have half-lives T and $2T$. Initially, they have equal number of nuclei. After time $t = 4T$, the ratio of their number of nuclei is x and the ratio of their activity is y. Then,

(a) $x = 1/8$
(b) $x = 1/4$
(c) $y = 1/2$
(d) $y = 1/4$

5. Regarding the nuclear forces, choose the correct options.

(a) They are short range forces
(b) They are charge independent forces
(c) They are not electromagnetic forces
(d) They are exchange forces

6. Regarding a nucleus choose the correct options.

(a) Density of a nucleus is directly proportional to mass number A
(b) Density of all the nuclei is almost constant of the order of 10^{17} kg/m^3
(c) Nucleus radius is of the order of 10^{-15} m
(d) Nucleus radius $\propto A$

Comprehension Based Questions

Passage : (Q. No. 1 to 3)

The atomic masses of the hydrogen isotopes are

$$\text{Hydrogen } m_1\text{H}^1 = 1.007825 \text{ amu}$$
$$\text{Deuterium } m_1\text{H}^2 = 2.014102 \text{ amu}$$
$$\text{Tritium } m_1\text{H}^3 = 3.016049 \text{ amu}$$

1. The energy released in the reaction,

$${}_1\text{H}^2 + {}_1\text{H}^2 \rightarrow {}_1\text{H}^3 + {}_1\text{H}^1 \text{ is nearly}$$

(a) 1 MeV
(b) 2 MeV
(c) 4 MeV
(d) 8 MeV

2. The number of fusion reactions required to generate 1 kWh is nearly

(a) 10^8
(b) 10^{18}
(c) 10^{28}
(d) 10^{38}

3. The mass of deuterium, ${}_1\text{H}^2$ that would be needed to generate 1 kWh

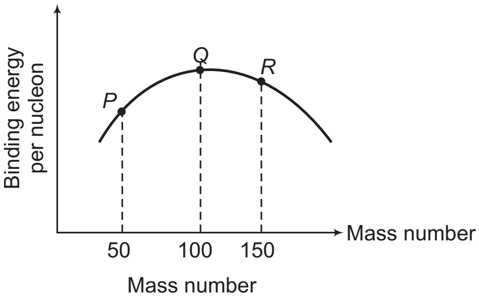
(a) 3.7 kg
(b) 3.7 g
(c) 3.7×10^{-5} kg
(d) 3.7×10^{-8} kg

Match the Columns

1. At $t = 0$, x nuclei of a radioactive substance emit y nuclei per second. Match the following two columns.

Column I	Column II
(a) Decay constant λ	(p) $(\ln 2)(x/y)$
(b) Half-life	(q) x/y
(c) Activity after time $t = \frac{1}{\lambda}$	(r) y/e
(d) Number of nuclei after time $t = \frac{1}{\lambda}$	(s) None of these

2. Corresponding to the graph shown in figure, match the following two columns.



Column I	Column II
(a) $P + P = Q$	(p) energy is released
(b) $P + P + P = R$	(q) energy is absorbed
(c) $P + R = 2Q$	(r) No energy transfer will take place
(d) $P + Q = R$	(s) data insufficient

3. In the following chain,

$$A \rightarrow B \rightarrow C$$

A and B are radioactive, while C is stable. Initially, we have only A and B nuclei. There is no nucleus of C. As the time passes, match the two columns.

Column I	Column II
(a) Nuclei of $(A + B)$	(p) will increase continuously
(b) Nuclei of B	(q) will decrease continuously
(c) Nuclei of $(C + B)$	(r) will first increase then decrease
(d) Nuclei of $(A + C)$	(s) data insufficient

4. Match the following two columns.

Column I	Column II
(a) After emission of one α and one β particles	(p) atomic number will decrease by 3.
(b) After emission of two α and one β particle	(q) atomic number will decrease by 2
(c) After emission of one α and two β particles	(r) mass number will decrease by 8
(d) After emission of two α and two β-particles.	(s) mass number will decrease by 4

5. Match the following two columns.

Column I	Column II
(a) The energy of air molecules at room temperature	(p) 0.02 eV
(b) Binding energy of heavy nuclei per nucleon	(q) 2 eV
(c) X-ray photon energy	(r) 10 keV
(d) Photon energy of visible light	(s) 7 MeV

Subjective Questions

1. A F^{32} radio nuclide with half-life $T = 14.3$ days is produced in a reactor at a constant rate $q = 2 \times 10^9$ nuclei per second. How soon after the beginning of production of that radio nuclide will its activity be equal to $R = 10^9$ disintegration per second?

2. Consider a radioactive disintegration according to the equation $A \rightarrow B \rightarrow C$. Decay constant of A and B is same and equal to λ. Number of nuclei of A, B and C are N_0, 0, 0 respectively at $t = 0$. Find

(a) number of nuclei of B as function of time t.

(b) time t at which the activity of B is maximum and the value of maximum activity of B.

3. Nuclei of a radioactive element A are being produced at a constant rate α. The element has a decay constant λ. At time $t = 0$, there are N_0 nuclei of the element.

(a) Calculate the number N of nuclei of A at time t.

(b) If $\alpha = 2N_0\lambda$, calculate the number of nuclei of A after one half-life of A, and also the limiting value of N as $t \rightarrow \infty$.

4. A solution contains a mixture of two isotopes A (half-life = 10 days) and B (half-life = 5 days). Total activity of the mixture is 10^{10} disintegration per second at time $t = 0$. The activity reduces to 20% in 20 days. Find (a) the initial activities of A and B, (b) the ratio of initial number of their nuclei.

5. A radio nuclide with disintegration constant λ is produced in a reactor at a constant rate α nuclei per second. During each decay energy E_0 is released. 20% of this energy is utilized in increasing the temperature of water. Find the increase in temperature of m mass of water in time t. Specific heat of water is s. Assume that there is no loss of energy through water surface.

6. A stable nuclei C is formed from two radioactive nuclei A and B with decay constant of λ_1 and λ_2 respectively. Initially, the number of nuclei of A is N_0 and that of B is zero. Nuclei B are produced at a constant rate of P. Find the number of the nuclei of C after time t.

7. Polonium ($^{210}_{84}Po$) emits ^4_2He particles and is converted into lead ($^{206}_{82}Pb$). This reaction is used for producing electric power in a space mission. Po^{210} has half-life of 138.6 days. Assuming an efficiency of 10% for the thermoelectric machine, how much ^{210}Po is required to produce 1.2×10^7 J of electric energy per day at the end of 693 days. Also find the initial activity of the material.

Given : Masses of nuclei

$^{210}Po = 209.98264$ amu, $^{206}Pb = 205.97440$ amu, $^4_2He = 4.00260$ amu,

1 amu = 931 MeV/c^2 and Avogadro's number = 6×10^{23}/mol

8. A radio nuclide consists of two isotopes. One of the isotopes decays by α-emission and other by β-emission with half-lives $T_1 = 405\,s$ and $T_2 = 1620\,s$, respectively. At $t = 0$, probabilities of getting α and β-particles from the radio nuclide are equal. Calculate their respective probabilities at $t = 1620\,s$. If at $t = 0$, total number of nuclei in the radio nuclide are N_0. Calculate the time t when total number of nuclei remained undecayed becomes equal to $N_0/2$.

$$\log_{10} 2 = 0.3010, \ \log_{10} 5.94 = 0.7742 \text{ and } x^4 + 4x - 2.5 = 0, \ x = 0.594$$

9. Find the amount of heat generated by 1 mg of Po^{210} preparation during the mean life period of these nuclei if the emitted alpha particles are known to possess kinetic energy 5.3 MeV and practically all daughter nuclei are formed directly in the ground state.

10. In an agricultural experiment, a solution containing 1 mole of a radioactive material ($T_{1/2} = 14.3$ days) was injected into the roots of a plant. The plant was allowed 70 h to settle down and then activity was measured in its fruit. If the activity measured was 1 μCi, what percentage of activity is transmitted from the root to the fruit in steady state?

Answers

Introductory Exercise 34.1

1. (b) **2.** (a) **3.** (a) **4.** (c) **5.** (c)
6.(b) **7.** (a) **8.** (b) **9.** (d) **10.** (a)
11. (b) **12.** 3 days, 4.32 days **13.** 9.47×10^9 nuclei
14. (a) 1.55×10^{-5} /s, 12.4 h (b) 2.39×10^{13} atoms (c) 1.87 mCi **15.** 1.16×10^3 s
16. $\frac{1}{4}$

Introductory Exercise 34.2

1. (c) **2.** (b) **3.** (d) **4.** (c)
5. (a) 9.6×10^{-4} kg (b) 3.125×10^{19} **6.** 4.27 MeV
7. (a) ${}_1^2H$ (b) ${}_1^1H$ (c) ${}_0^1n$ (d) ${}_{36}^{79}Kr$ **8.** Fusion, 24 **9.** 23.6 MeV

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (d) **3.** (b) **4.** (c) **5.** (d) **6.** (d) **7.** (a or b) **8.** (b) **9.** (d) **10.** (b)
11. (a or b)

Objective Questions

1. (a) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (b) **12.** (c) **13.** (a) **14.** (d) **15.** (b) **16.** (a) **17.** (a)

Subjective Questions

1. (a) $0.113\ \text{min}^{-1}$ (b) 6.132 min **2.** 19.25 min **3.** 7.11×10^{-3} g
4. 1.23×10^4 dps **5.** 0.39 **6.** 1.88×10^9 yr
7. 0.205 mCi **8.** (a) ${}_{86}Rn^{222}$ (b) $\bar{e} + \bar{\nu}$ (c) $e^+ + \nu$
9. Reaction (a) is possible (b) is not possible **10.** 104.72 MeV **11.** 127.6 MeV
12. (a) 8.19×10^{13} J (b) 2.7×10^6 kg **13.** 0.1308 MeV **14.** 8.78 day
15. (a) 4.05 MeV (b) 3.25 MeV (c) 17.57 MeV **16.** – 93.1 keV, No
17. N/P ratio required for stability decreases with decreasing *A*, hence there is an excess of neutrons when fission occurs. Some of the excess neutrons are released directly, and others change to protons by beta decay in the fission products.

LEVEL 2

Single Correct Option

1.(b) **2.**(c) **3.**(b) **4.**(d) **5.**(d) **6.**(d) **7.**(c) **8.**(b) **9.**(c) **10.**(a)
11.(b) **12.**(b) **13.**(c)

More than One Correct Options

1.(a,b,d) **2.**(a,b,c,d) **3.**(a,b,c) **4.**(b,c) **5.**(a,b,c,d) **6.**(b,c)

Comprehension Based Questions

1.(c) **2.**(b) **3.**(d)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → s

2. (a) → p (b) → p (c) → p (d) → s

3. (a) → q (b) → s (c) → p (d) → s

4. (a) → s (b) → p,r (c) → s (d) → q, r

5. (a) → p (b) → s (c) → r (d) → q

Subjective Questions

1. 14.3 h **2.** (a) $N_B = \lambda N_0 (te^{-\lambda t})$ (b) $t = \dfrac{1}{\lambda}$, $R_{\max} = \dfrac{\lambda N_0}{e}$

3. (a) $\dfrac{1}{\lambda}[\alpha - (\alpha - \lambda N_0) e^{-\lambda t}]$ (b) $\dfrac{3}{2} N_0$, $2N_0$

4. (a) 0.73×10^{10} dps, 0.27×10^{10} dps (b) 5.4

5. $\dfrac{0.2\, E_0 \left[\alpha t - \dfrac{\alpha}{\lambda}(1 - e^{-\lambda t}) \right]}{ms}$

6. $N_c = N_0 (1 - e^{-\lambda_1 t}) + P \left(t + \dfrac{e^{-\lambda_2 t} - 1}{\lambda_2} \right)$

7. 10 g, 4.57×10^{21} disintegrations/day

8. $\dfrac{1}{9}$, $\dfrac{8}{9}$, 1215 s

9. 1.55×10^{6} J

10. $1.26 \times 10^{-11}\%$

35

Semiconductors

Chapter Contents

35.1 Introduction
35.2 Energy Bands in Solids
35.3 Intrinsic and Extrinsic Semiconductors
35.4 *p-n* Junction Diode
35.5 Junction Diode as a Rectifier
35.6 Applications of *p-n* Junction Diodes
35.7 Junction Transistors
35.8 Transistor as an Amplifier
35.9 Digital Electronics and Logic Gates

35.1 Introduction

Solids can be classified in three types as per their electrical conductivity. (i) conductors, (ii) insulators and (iii) semiconductors. In a conductor, large number of free electrons are present. They are always in *zig-zag* motion inside the conductor. In an insulator, all the electrons are tightly bound to the nucleus. If an electric field is applied inside a conductor, the free electrons experience force due to the field and acquire a drift speed. This results in an electric current. The conductivity of a conductor such as copper decreases as the temperature is increased. This is because as the temperature is increased, the random collisions of the free electrons with the particles in the conductor become more frequent. This results in a decrease in the drift speed and hence the conductivity decreases.

In insulators, almost zero current is obtained unless a very high electric field is applied. Semiconductors conduct electricity when an electric field is applied, but the conductivity is very small as compared to the usual metallic conductors. Silicon, germanium, carbon etc., are few examples of semiconductors.

Conductivity of silicon is about 10^{11} times smaller than that of copper and is about 10^{13} times larger than that of fused quartz. Conductivity of a semiconductor increases as the temperature is increased.

Extra Points to Remember

- Before the discovery of transistors (in 1948) mostly vacuum tubes (also called valves) were used in all electrical circuits.
- The order of electrical conductivity (σ) and resistivity $\left(\rho = \frac{1}{\sigma}\right)$ of metals, semiconductors and insulators are given below in tabular form.

Table 35.1

S.No	Types of solid	$\rho\ (\Omega\text{-m})$	$\sigma\ (\Omega^{-1}\text{-m}^{-1})$
1.	Metals	$10^{-2} - 10^{-8}$	$10^{2} - 10^{8}$
2.	Semiconductors	$10^{-5} - 10^{6}$	$10^{5} - 10^{-6}$
3.	Insulators	$10^{11} - 10^{19}$	$10^{-11} - 10^{-19}$

35.2 Energy Bands In Solids

To understand the energy bands in solids, let us consider the electronic configuration of sodium atom which has 11 electrons. The configuration is $(1s)^2$, $(2s)^2$, $(2p)^6$ and $(3s)^1$. The levels $1s$, $2s$ and $2p$ are completely filled. The level $3s$ is half filled and the levels above $3s$ are empty. Consider a group of N sodium atoms all in ground state separated from each other by large distances such as in sodium vapour. There are total $11N$ electrons. Each atom has two energy states in $1s$ energy level. So, there are $2N$ identical energy states lebelled $1s$ and all them are filled from $2N$ electrons. Similarly, energy level $2p$ has $6N$ identical energy states which are also completely filled. In $3s$ energy levels N of the $2N$ states are filled by the electrons and the remaining N states are empty.

These ideas are shown in the table given below.

Table 35.2

Energy level	Total available energy states	Total occupied states
1*s*	2*N*	2*N*
2*s*	2*N*	2*N*
2*p*	6*N*	6*N*
3*s*	2*N*	*N*
3*p*	6*N*	0

In the above discussion, we have assumed that *N* sodium atoms are widely spread and hence the electrons of one atom do not interact with others. As a result energy states of different states (e.g. 1*s*) are identical. When atoms are drawn closer to one another, electron of one atom starts interacting with the electrons of the neighbouring atoms of the same energy states. For example 1*s* electrons of one atom interact with 1*s* electrons of the other. Due to interaction of electrons, the energy states are not identical, but a sort of energy band is formed. These bands are shown in figure.

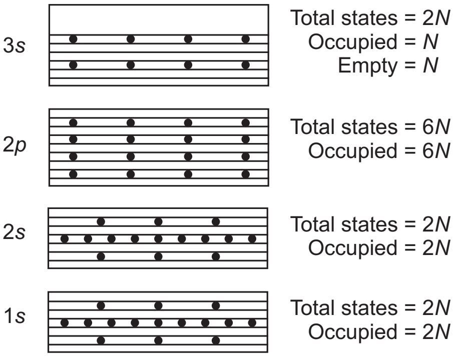


Fig. 35.1

The difference between the highest energy in a band and the lowest energy in the next higher band is called the **band gap** between the two energy bands.

Thus, we can conclude that energy levels of an electron in a solid consists of bands of allowed states. There are regions of energy, called gaps, where no states are possible. In each allowed band, the energy levels are very closely spaced. Electrons occupy states which minimize the total energy. Depending on the number of electrons and on the arrangement of the bands, a band may be fully occupied or partially occupied.

Now, electrical conductivity of conductors, insulators and semiconductors can be explained by these energy bands.

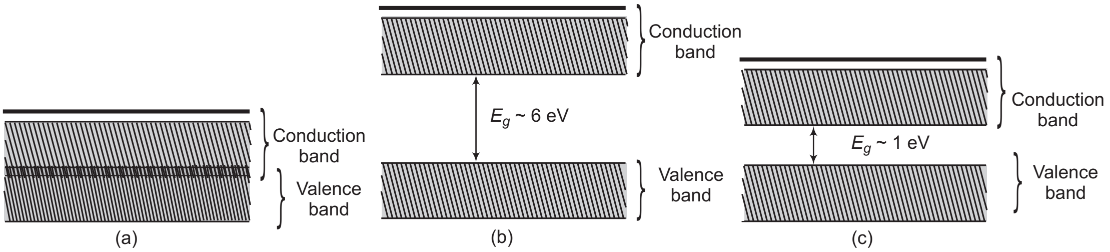


Fig 35.2 Energy band diagram for a (a) metal, (b) insulator and (c) semiconductor.
Note that one can have a metal either when the conduction band is partially filled or when the conduction and valence bands overlap in energy.

Conductors The energy band structure of a conductor is shown in figure (a). The last occupied band of energy level (called conduction band) is only partially filled. In conductors, this band overlaps with completely filled valence band.

Insulators The energy band structure of an insulator is shown in figure (b). The conduction band is separated from the valence band by a wide energy gap (e.g. 6 eV for diamond). But at any non-zero temperature, some electrons can be excited to the conduction band.

Semiconductors The energy band structure of a semiconductor is shown in figure (c). It is similar to that of an insulator but with a comparatively small energy gap. At absolute zero temperature, the conduction band of semiconductors is totally empty, and all the energy states in the valence band are filled. The absence of electrons in the conduction band at absolute zero does not allow current to flow under the influence of an electric field. Therefore, they are insulators at low temperatures. However at room temperatures some valence electrons acquire thermal energy greater than the energy gap E_g and move to the conduction band where they are free to move under the influence of even a small electric field. Thus, a semiconductor originally an insulator at low temperatures becomes slightly conducting at room temperature. Unlike conductors the resistance of semiconductors decreases with increasing temperature. We are generally concerned with only the highest valence band and the lowest conduction band. So, when we say valence band, it means the highest valence band. Similarly, when we say conduction band, it means the lowest conduction band.

Example 35.1 *What is the energy band gap of : (i) silicon and (ii) germanium?*

Solution The energy band gap of silicon is 1.1 eV and of germanium is about 0.7 eV.

Example 35.2 *In a good conductor, what is the energy gap between the conduction band and the valence band.*

Solution In a good conductor, conduction band overlaps with the valence band. Therefore, the energy gap between them is zero.

35.3 Intrinsic and Extrinsic Semiconductors

As discussed above, in semiconductors the conduction band and the valence band are separated by a relatively small energy gap. For silicon, this gap is 1.1 eV and for germanium it is 0.7 eV.

Silicon has an atomic number 14 and electronic configuration $1s^2, 1s^2, 2p^6, 3s^2, 3p^2$.

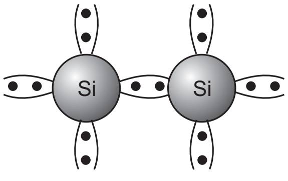


Fig. 35.3

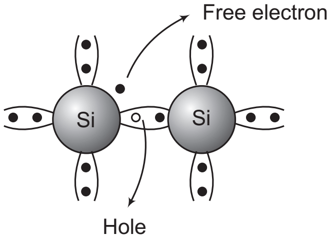


Fig. 35.4

The chemistry of silicon tells us that it has a valency 4. Each silicon atom makes covalent bonds with the four neighbouring silicon atoms. On the basis of bonds the atoms make with their neighbouring atoms, semiconductors are divided in two groups.

Intrinsic Semiconductors A pure (free from impurity) semiconductor which has a valency 4 is called an intrinsic semiconductor. Pure germanium, silicon or carbon in their natural state are intrinsic semiconductors. As discussed above, each atom makes four covalent bonds with their neighbouring atoms. At temperature close to zero, all valence electrons are tightly bound and so no free electrons are available to conduct electricity through the crystal. At room temperature, however a few of the covalent bonds are broken due to thermal agitation and thus some of the valence electrons become free. Thus, we can say that a valence electrons is shifted to conduction band leaving a **hole** (vacancy of electron) in valence band. In intrinsic semiconductors,

$$\text{Number of holes} = \text{Number of free electrons} \quad \text{or} \quad \boxed{n_h = n_e}$$

Extrinsic Semiconductors The conductivity of an intrinsic semiconductor is very poor (unless the temperature is very high). At ordinary temperature, only one covalent bond breaks in 10^9 atoms of Ge. Conductivity of an intrinsic (pure) semiconductor is significantly increased, if some pentavalent or trivalent impurity is mixed with it. Such impure semiconductors are called extrinsic or doped semiconductors. Extrinsic semiconductors are again of two types (i) *p*-type and (ii) *n*-type.

(i) ***p*-type semiconductors** When a trivalent (e.g. boron, aluminium, gallium or indium) is added to a germanium or silicon crystal it replaces one of the germanium or silicon atom. Its three valence electrons form covalent bonds with neighbouring three Ge (or Si) atoms while the fourth valence electron of Ge (or Si) is not able to form the bond. Thus, there remains a hole (an empty space) on one side of the impurity atom.

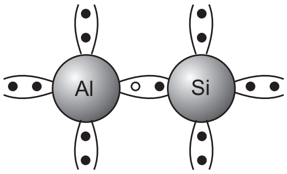


Fig. 35.5

The trivalent impurity atoms are called **acceptor atoms** because they create holes which accept electrons. Following points are worthnoting regarding *p*-type semiconductors.

(a) Holes are the majority charge carriers and electrons are minority charge carriers in case of *p*-type semiconductors or number of holes are much greater than the number of electrons.

$$\boxed{n_h >> n_e}$$

(b) *p*-type semiconductor is electrically neutral.

(c) *p*-type semiconductor can be shown as

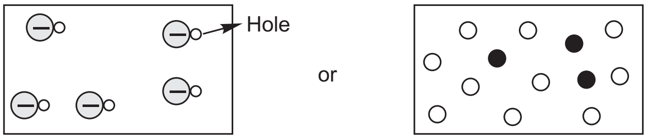


Fig. 35.6

(ii) ***n*-type semiconductors** When a pentavalent impurity atom (antimony, phosphorus or arsenic is added to a Ge (or Si) crystal it replaces a Ge (or Si) atom. Four of the five valence electrons of the impurity atom form covalent bonds with four neighbouring Ge (or Si) atoms and the fifth valence electron becomes free to move inside the crystal lattice. Thus, by doping pentavalent impurity number of free electrons increases.

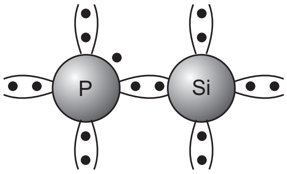


Fig. 35.7

The impurity (pentavalent) atoms are called **donor atoms** because they donate conduction electrons inside the crystal. Following points are worthnoting regarding *n*-type semiconductors,

(a) Electrons are the majority charge carriers and holes are minority or number of electrons are much greater than the number of holes

$$\boxed{n_e >> n_h}$$

(b) n-type semiconductor is also electrically neutral.

(c) n-type semiconductor can be shown as

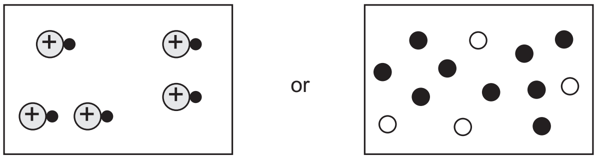


Fig. 35.8

Electrical Conduction through Semiconductors

When a battery is connected across a semiconductor (whether intrinsic or extrinsic) a potential difference is developed across its ends. Due to the potential difference an electric field is produced inside the semiconductor. A current (although very small) starts flowing through the semiconductor. This current may be due to the motion of (i) free electrons and (ii) holes. Electrons move in opposite direction of electric field while holes move in the same direction.

The motion of holes towards right (in the figure) take place because electrons from right hand side come to fill this hole, creating a new hole in their own position. Thus, we can say that holes are moving from left to right. Thus, current in a semiconductor can be written as,

$$\boxed{i = i_e + i_h}$$

But it should be noted that mobility of holes is less than the mobility of electrons.

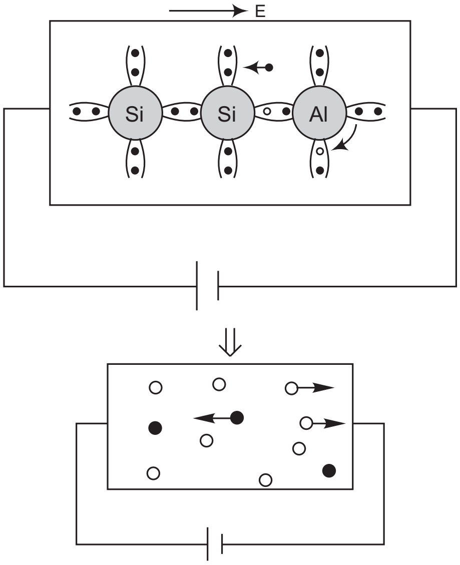


Fig. 35.9

Example 35.3 *C, Si and Ge have same lattice structure. Why is C insulator while Si and Ge intrinsic semiconductors?*

Solution *The energy gap between conduction band and valence band is least for Ge, followed by Si and highest for C. Hence, number of free electrons are negligible for C. This is why carbon is insulator.*

Example 35.4 *In an n-type silicon, which of the following statements is true?*

(a) Electrons are majority carriers and trivalent atoms are the dopants.

(b) Electrons are minority carriers and pentavalent atoms are the dopants.

(c) Holes are minority carriers and pentavalent atoms are the dopants.

(d) Holes are majority carriers and trivalent atoms are the dopants.

Solution (c) Holes are minority charge carriers and pentavalent atoms are the dopants in an n-type silicon.

Example 35.5 *Which of the statements given in above example is true for p-type semiconductors?*

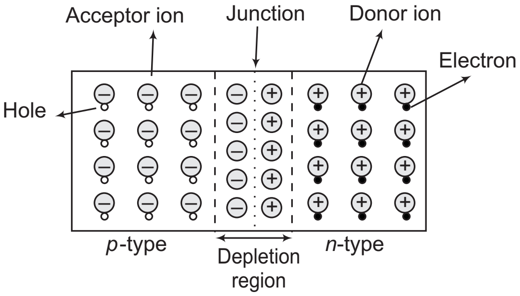
Solution (d) Holes are majority carriers and trivalent atoms are the dopants in an p–type semiconductors.

INTRODUCTORY EXERCISE 35.1

1. Carbon, silicon and germanium have four valence electrons each. These are characterised by valence and conduction bands separated by energy band-gap respectively equal to $(E_g)_{\text{C}}$, $(E_g)_{\text{Si}}$ and $(E_g)_{\text{Ge}}$. Which of the following statements is true?

   (a) $(E_g)_{\text{Si}} < (E_g)_{\text{Ge}} < (E_g)_{\text{C}}$
   (b) $(E_g)_{\text{C}} < (E_g)_{\text{Ge}} > (E_g)_{\text{Si}}$
   (c) $(E_g)_{\text{C}} > (E_g)_{\text{Si}} > (E_g)_{\text{Ge}}$
   (d) $(E_g)_{\text{C}} = (E_g)_{\text{Si}} = (E_g)_{\text{Ge}}$

35.4 *p-n* Junction Diode

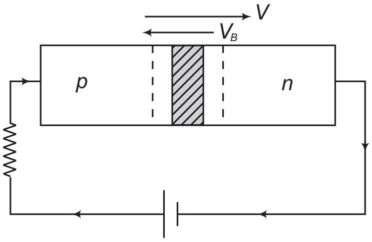


(a) Formation of *p-n* junction

A *p*-type or *n*-type silicon crystal can be made by adding appropriate impurity as discussed above. These crystals are cut into thin slices called the wafer. Semiconductor devices are usually made of these wafers.

If on a wafer of *n*-type silicon, an aluminium film is placed and heated to a high temperature, aluminium diffuses into silicon.

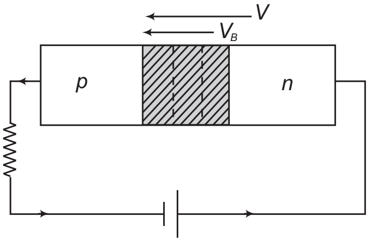
In this way, a *p*-type semiconductor is formed on an *n*-type semiconductor. Such a formation of *p*-region on *n*-region is called the *p-n*-junction. Another way to make a *p-n* junction is by diffusion of phosphorus into a *p*-type semiconductor. Such *p-n* junctions are used in a host of semiconductor devices of practical applications. The simplest of the semiconductor devices is a *p-n* junction diode.



(b) Forward biased *p-n* junction

Biasing of a diode In a *p-n* junction diode, holes are majority carriers on *p*-side and electrons on *n*-side. Holes, thus diffuse to *n*-side and electrons to *p*-side.

This diffusion causes an excess positive charge in the *n* region and an excess negative charge in the *p* region near the junction. This double layer of charge creates an electric field which exerts a force on the electrons and holes, against their diffusion. In the equilibrium position, there is a barrier, for charge motion with the *n*-side at a higher potential than the *p*-side.



(c) Reverse biased *p-n* junction

Fig. 35.10

The junction region has a very low density of either *p* or *n*-type carriers, because of inter diffusion. It is called **depletion region.** There is a barrier V_B associated with it, as described above. This is called **potential barrier.**

Now suppose a DC voltage source is connected across the *p-n* junction. The polarity of this voltage can lead to an electric field across the *p-n* junction that is opposite to the already present electric field. The potential drop across the junction decreases and the diffusion of electrons and holes is thereby increased, resulting in a current in the circuit. This is called **forward biasing.**

The depletion layer effectively becomes smaller. In the opposite case, called **reverse biasing** the barrier increases, the depletion region becomes larger, current of electrons and holes is greatly reduced.

Thus, the *p-n* junction allows a much larger current flow in forward biasing than in reverse biasing. This is crudely, the basis of the action of a *p-n* junction as a rectifier. The symbol of *p-n* junction diode is ( )

Diffusion Current and Drift Current

Because of concentration difference, holes try to diffuse from the *p*-side to the *n*-side at the *p-n* junction. This diffusion give rise to a current from *p*-side to *n*-side called **diffusion current.** Because of thermal collisions, electron-hole pair are created at every part of a diode.

However, if an electron-hole pair is created in the depletion region, the electron is pushed by the electric field towards the *n*-side and the hole towards the *p*-side. This gives rise to a current from *n*-side to *p*-side called the **drift current.**

Thus, $I_{df} \longrightarrow$ from *p*-side to *n*-side

$I_{dr} \longrightarrow$ from *n*-side to *p*-side

When diode is unbiased $I_{df} = I_{dr}$ or $I_{\text{net}} = 0$.

When diode is forward biased $I_{df} > I_{dr}$ or I_{net} is from *p*-side to *n*-side.

When diode is reverse biased $I_{dr} > I_{df}$ or I_{net} is from *n*-side to *p*-side.

Characteristic Curve of a *p-n* Junction Diode

(a) Circuit for obtaining the characteristics of a forward biased diode and (b) Circuit for obtaining the characteristics of a reverse bias diode.

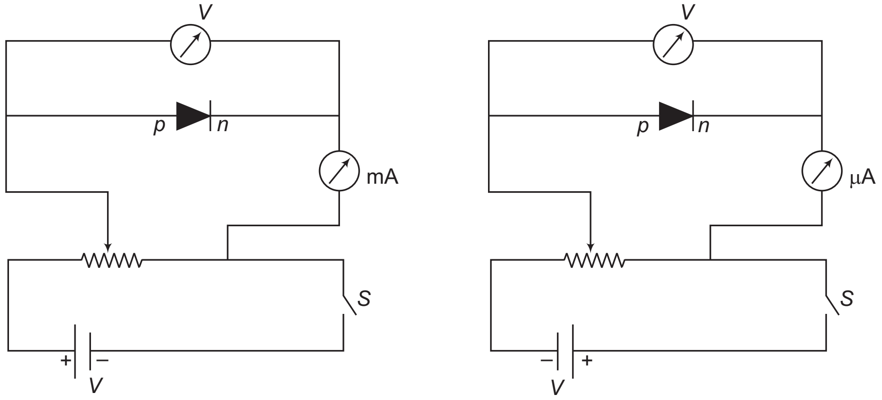


Fig. 35.11

When the diode is forward biased i.e. *p*-side is kept at higher potential, the current in the diode changes with the voltage applied across the diode. The current increases very slowly till the voltage across the diode crosses a certain value.

After this voltage, the diode current increases rapidly, even for very small increase in the diode voltage. This voltage is called the **threshold voltage or cut-off voltage.** The value of the cut-off voltage is about 0.2 V for a germanium diode and 0.7 V for a silicon diode.

When the diode is reverse biased, a very small current (about a few micro amperes) produces in the circuit which remains nearly constant till a characteristic voltage called the breakdown voltage, is reached. Then the reverse current suddenly increases to a large value. This phenomenon is called **avalanche breakdown**. The reverse voltage beyond which current suddenly increases is called the **breakdown voltage**.

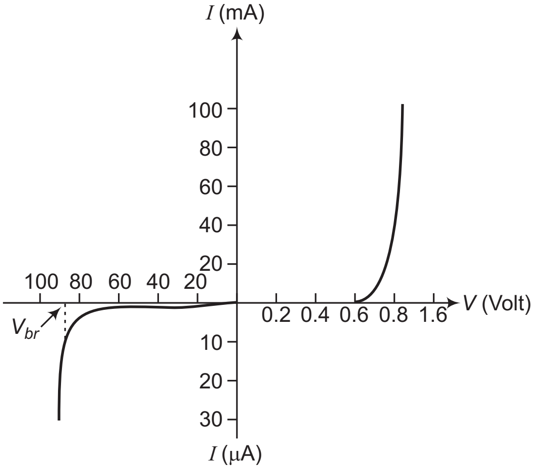


Fig. 35.12

Example 35.6 *Can we take one slab of p-type semiconductor and physically join it to another n-type semiconductor to get p-n junction?*

Solution No. Any slab will have some roughness. Hence continuous contact at the atomic level will not be possible. For the charge carriers, the junction will behave as a discontinuity.

Example 35.7 *Find current passing through* $2\,\Omega$ *and* $4\,\Omega$ *resistance in the circuit shown in figure.*

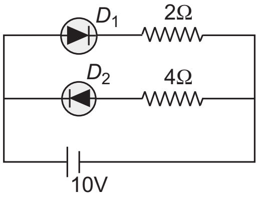


Fig. 35.13

Solution In the given circuit diode D_1 is forward biased and D_2 reverse biased. Hence, D_1 will conduct but D_2 not. Therefore, current through $4\,\Omega$ resistance will be zero while through $2\,\Omega$ resistance will be,

$$\frac{10}{2} = 5\text{ A.}$$

INTRODUCTORY EXERCISE 35.2

1. In an unbiased *p-n* junction, holes diffuse from the *p*-region to *n*-region because
   - (a) free electrons in the *n*-region attract them
   - (b) they move across the junction by the potential difference
   - (c) hole concentration in *p*-region is more as compared to *n*-region
   - (d) All of the above
2. When a forward bias is applied to a *p* - *n* junction. It
   - (a) raises the potential barrier
   - (b) reduces the majority carrier current to zero
   - (c) lowers the potential barrier
   - (d) All of the above

35.5 Junction Diode as a Rectifier

A rectifier is a device which converts an alternating current (or voltage) into a direct (or unidirectional) current (or voltage). A *p-n* junction diode can work as an excellent rectifier. It offers a low resistance for the current to flow when it is forward biased, but a very high resistance when reverse biased. Thus, it allows current through it only in one direction and acts as a rectifier. The junction diode can be used either as an half-wave rectifier or as a full-wave rectifier.

(i) ***p-n* junction diode as half-wave rectifier** A simple rectifier circuit called the half-wave rectifier, using only one diode is shown in figure.

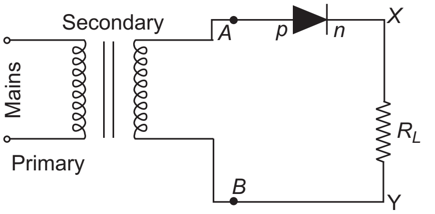


Fig. 35.14

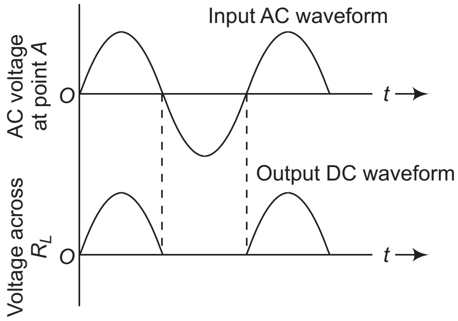
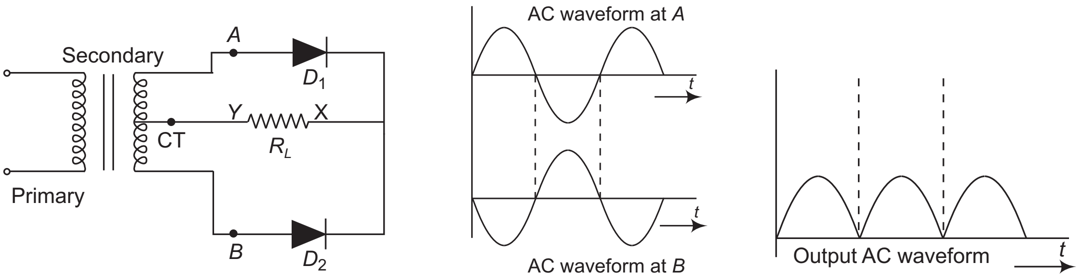


Fig. 35.15

When the voltage at A is positive, the diode is forward biased and it conducts and when the voltage at A is negative, the diode is reverse biased and does not conduct. Since, the diode conducts only in the positive half cycles, the voltage between X and Y or across R_L will be DC but in pulses. When this is given to a circuit called filter (normally a capacitor), it will smoothen the pulses and will produce a rather steady DC voltage.

(ii) ***p-n* junction diode as full-wave rectifier** Figure shows a circuit which is used in full-wave rectification. Two diodes are used for this purpose.

The secondary coil of the transformer is wound in two parts and the junction is called a Centre-Tap (CT). During one-half cycle D_1 is forward biased and D_2 is reverse biased. Therefore, D_1 conducts but D_2 does not, current flows from X to Y through load resistance R_L. During another half cycle D_2 is forward biased and D_1 reverse biased. Therefore, D_2 conducts and D_1 does not. In this half cycle also current through R_L flows from X to Y. Thus, current through R_L in both the half cycles is in one direction, i.e. from X to Y.



(a) Ful-wave rectifier

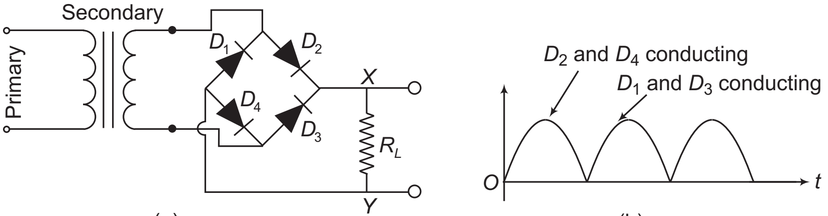
(b) AC voltage waveforms at points *A* and *B*

(c) Output DC waveforms of a full-wave rectifier.

Fig. 35.16

Bridge rectifier Another full-wave rectifier called the bridge rectifier which uses four diodes is shown in figure.

For one-half cycle diodes D_1 and D_3 are forward biased and D_2 and D_4 are reverse biased. So, D_1 and D_3 conduct but D_2 and D_4 don't. Current through R_L flows from X to Y. In another half cycle D_2 and D_4 are forward biased and D_1 and D_3 are reverse biased. So, in this half cycle D_2 and D_4 conduct but D_1 and D_3 do not. Current again flows from X to Y through R_L. Thus, we see that current through R_L always flows in one direction from X to Y.



(a) (b)

Fig. 35.17 (a) Bridge rectifier and (b) output waveforms for a bridge rectifier

Note *Even after rectification ripples are present in the output which can be removed upto great extent by a filter circuit. A filter circuit consists of a capacitor.*

Example 35.8 *In half-wave rectification, what is the output frequency, if the input frequency is 50 Hz? What is the output frequency of a full-wave rectifier for the same input frequency?*

Solution A half-wave rectifier conducts once during a cycle. Therefore frequency of AC output is also the frequency of AC input i.e. 50 Hz. A full-wave rectifier rectifies both the half cycles of the AC output i.e. it conducts twice during a cycle.

So, Frequency of AC output $= 2\times$ frequency of AC input

$$= 2\times 50 = 100 \text{ Hz}$$ **Ans.**

Example 35.9 *In the figure, the input is across the terminals A and C and the output is across B and D. Then the output is*

(a) zero

(b) same as the input

(c) full-wave rectified

(d) half-wave rectified

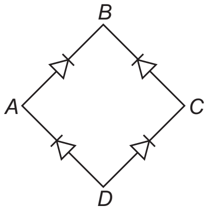


Fig. 35.18

Solution (c) During the half cycle when $V_M > V_N$, D_1 and D_3 are forward biased. Hence, the path of current is *MABPQDCNM*.

In the second half cycle, when $V_N > V_M$, D_2 and D_4 are forward biased while D_1 and D_3 are reverse biased. Hence, the path of current is *NCBPQDAMN*.

Therefore, in both half cycles current flows from P to Q from load resistance R_L. Or, it is a full-wave rectifier.

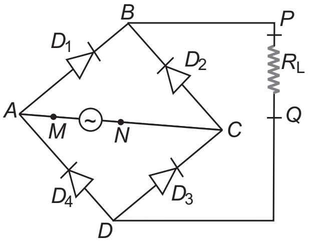


Fig. 35.19

35.6 Applications of *p-n* Junction Diodes

Zener Diode

A diode meant to operate under reverse bias in the breakdown region is called an **avalanche diode** or a **zener diode**. Such diode is used as a voltage regulator. The symbol of zener diode is shown in figure.

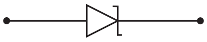


Fig. 35.20

Once the breakdown occurs, the potential difference across the diode does not increase even if, there is large change in the current. Figure shows a zener diode in reverse biasing.

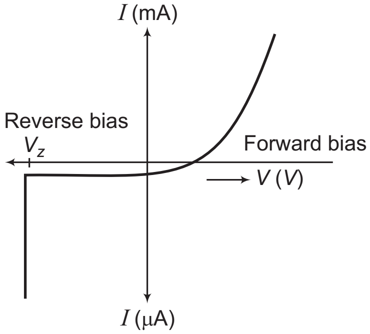


Fig. 35.21

An input voltage V_i is connected to the zener diode through a series resistance R such that the zener diode is reverse biased.

If the input voltage increases, the current through R and zener diode also increases. This increases the voltage drop across R without any change in the voltage across the zener diode. Similarly, if the input voltage decreases the current through R and zener diode also decreases. The voltage drop across R decreases without any change in the voltage across the zener diode.

Thus any increase/decrease in the input voltage results in increase/decrease of the voltage drop across R without any change in voltage across the zener diode (and hence across load resistance R_L). Thus, the zener diode acts as a voltage regulator.

We have to select the zener diode according to the required output voltage and accordingly the series resistance R.

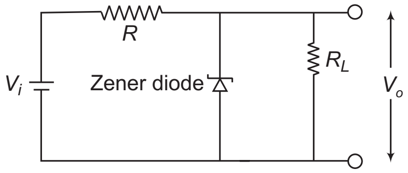


Fig. 35.22

Optoelectronic Devices

Semiconductor diodes in which carriers are generated by photons (photo excitation) are called optoelectronic devices. Examples of optoelectronic devices are, photodiodes, Light Emitting Diodes (LED) and photovoltaic devices, etc.

(a) Photodiodes Photodiodes are used as photodetector to detect optical signals. They are operated in reverse biased connections.

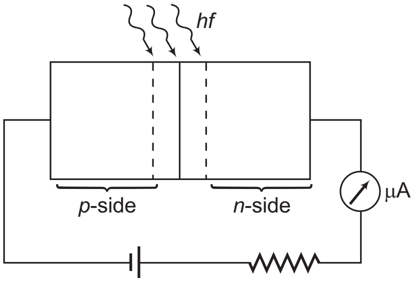


Fig. 35.23

When light of energy greater than the energy gap falls on the depletion region of the diode, electron-hole pairs are generated. Due to the electric field of junction, electrons and holes are separated before they recombine. Electrons reach *n*-side and holes reach *p*-side giving rise to an emf. When an external load is connected, current flows. The magnitude of the photocurrent depends on the intensity of incident light.

(b) Light Emitting Diode (LED) It is heavily doped *p*-*n* junction diode which under forward bias emits spontaneous radiation. LEDs that can emit red, yellow, orange, green and blue light are commercially available. These LEDs find extensive use in remote controls, burglar alarm systems, optical communications, etc.

Extensive research is being done for developing white LEDs which can replace incandescent lamps.

LED have the following advantages over conventional incandescent power lamps.

(i) Long life

(ii) Low operational voltage and less power

(iii) No warm up time is required. So fast on-off switching capability.

(c) Solar Cell It works on the same principle as the photodiode. It is basically a *p*-*n* junction which generates emf when solar radiation falls on the *p*-*n* junction. The difference between a photodiode and a solar cell is that no external bias is applied and the junction area is kept much larger for solar radiation to be incident because we require more power.

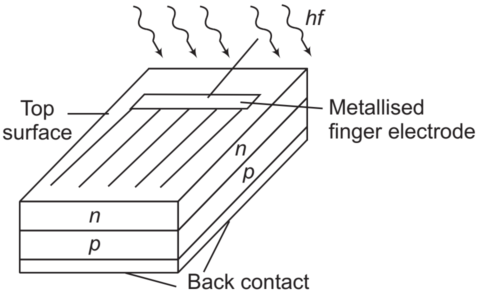


Fig. 35.24 Typical *p*-*n* junction Solar cell

The generation of emf by a solar cell (when light falls on it) is due to the following three processes.

(i) Generation Generation of electron-hole pairs due to light ($hf > E_g$) falling on it.

(ii) Separation Separation of electrons and holes due to electric field of the depletion region.

(iii) Collection Electrons are swept to n-side and holes to p-side. Thus, p-side becomes positive and n-side becomes negative giving rise to photovoltage.

Solar cells are used to power electronic devices in satellites and space vehicles and also as power supply to some calculators.

Example 35.10 *In a zener regulated power supply a zener diode with* $V_Z = 6.0\ V$ *is used for regulation. The load current is to be* 4.0 *mA and the unregulated input is* 10.0 *V. What should be the value of series resistor R?*

Solution Zener current I_Z should be sufficiently larger than load current I_L.

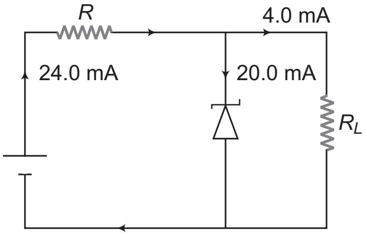


Fig. 35.25

Given, $I_L = 4.0\,\text{mA}$

So, let us take I_Z to be five times I_L or $I_Z = 20\,\text{mA}$

$$\text{Total current } I = I_Z + I_L = 24.0\,\text{mA}$$

$$\text{Input voltage } V_{\text{in}} = 10\,\text{V}$$

$$\text{Zener diode voltage } V_Z = 6\,\text{V}$$

$\therefore$ Voltage drop across resistance, $V_R = V_{\text{in}} - V_Z$

or $V_R = (10 - 6)\,\text{V} = 4\ \text{V}$

Now, $$R = \frac{V_R}{I_R} = \frac{4}{24 \times 10^{-3}} = 167\,\Omega$$

The nearest value of carbon resistor is $150\,\Omega$. So, a series resistor of $150\,\Omega$ is appropriate.

Example 35.11 *The current in the forward bias is known to be more* (in *mA*) *than the current in the reverse bias* (in μ*A*). *What is the reason then to operate the photodiodes in reverse bias ?*

Solution Let us take an example of p-type semiconductor.

Without illumination

number of holes $(n_h) >>$ number of electrons (n_e) ...(i)

This is because holes are the majority charge carriers in p-type semiconductor.

On illumination, let Δn_e and Δn_h are the excess electrons and holes generated.

$$\Delta n_e = \Delta n_h \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$\frac{\Delta n_e}{n_e} >> \frac{\Delta n_h}{n_h}$$

From here, we can say that the fractional change due to illumination on the minority carrier dominated reverse bias current is more easily measurable than the fractional change in the forward bias current.

35.7 Junction Transistors

A junction transistor is formed by sandwiching a thin wafer of one type of semiconductor between two layers of another type. The *n-p-n* transistor has a *p*-type wafer between two *n*-type layers. Similarly, the *p-n-p* transistor has a *n*-type wafer between two *p*-type layers.

p-n-p Transistor

Figure shows a *p-n-p* transistor, in which a thin layer of *n*-type semiconductor is sandwiched between two *p*-type semiconductors. The middle layer (called the base) is very thin (of the order of 1 μm) as compared to the widths of the two layers at the sides. Base is very lightly doped. One of the side layer (called emitter) is heavily doped and the other side layer (called collector) is moderately doped. Figure (c) shows the symbol of *p-n-p* transistor.

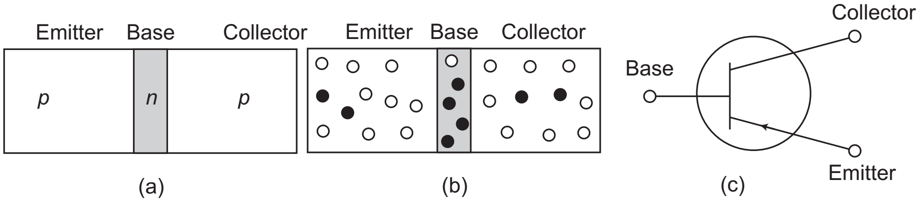


Fig. 35.26

n-p-n Transistor

In *n-p-n* transistor, *p*-type semiconductor is sandwiched between two *n*-type semiconductors. Symbol of *n-p-n* transistor is shown in figure (f).

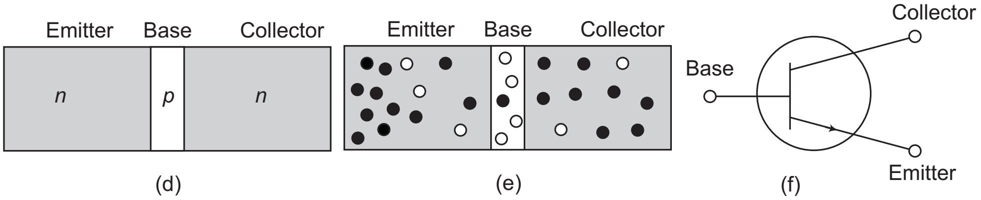


Fig. 35.27

More points about a transistor A transistor is basically a three-terminal device. Terminals come out from the emitter, base and the collector for external connections. In normal operation of a transistor, the emitter-base junction is always forward biased and collector-base junction is reverse biased.

The arrow on the emitter-base line shows the direction of current between emitter and base. In an *n-p-n* transistor for example, there are a large number of conduction electrons in the emitter and a large number of holes in the base. If the junction is forward biased the electrons will diffuse from emitter to the base and holes will diffuse from the base to the emitter. The direction of electric current at this junction is therefore from the base to the emitter. A transistor can be operated in three different modes.

(i) Common emitter (or grounded-emitter)

(ii) Common collector (or grounded-collector) and

(iii) Common base (or grounded-base)

In common emitter mode, emitter is kept at zero potential. Similarly in common collector mode collector is at zero potential and so on.

Working of a *p-n-p* Transistor

Let us consider the working of a *p-n-p* transistor in common base mode. In emitter (*p*-type) holes are in majority. Since, emitter-base is forward biased, holes move toward base. Few of them combine with electrons in the base and rest go to the collector. Since, base-collector is reverse biased, holes coming from base move toward the terminal of collector. They combine with equal number of electrons entering from collector terminal.

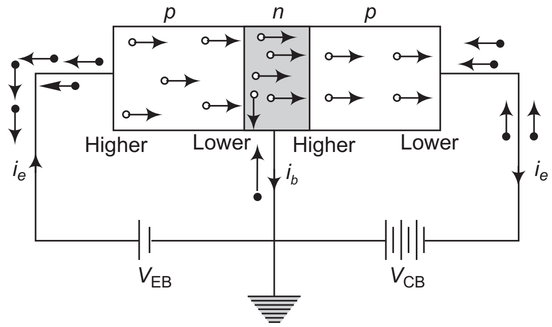


Fig. 35.28

Let us take an example with some numerical values.

Suppose 5 holes enter from emitter to base. This deficiency of 5 holes in emitter is compensated when 5 electrons emit from emitter and give rise to i_e. One out of five holes which reach the base combine with one electron entering from base (the equivalent current is i_b). Rest four holes enter the collector and move towards its terminal. On the other hand, 5 electrons which leave the emitter (as i_e) come to the base, emitter and collector junctions. One electron of it goes to base and rest four to collector. These four electrons give rise to i_c (the collector current) and combine with the four holes coming from the base, and thus circuit is complete. From the figure, we can see that,

$$\boxed{i_e = i_c + i_b}$$

Note *that i_b is only about 2% of i_e, or roughly around 2% of holes coming from emitter to base combine with the electrons. Rest 98% move to collector.*

Working of *n-p-n* Transistor

A common base circuit of an *n-p-n* transistor is shown in figure. Majority charge carriers in the emitter (*n*-type) are electrons. Since, emitter-base circuit is forward biased. The electrons rush from emitter to base. Few of them leave the base terminal (comprising i_b) and rest move to collector. These electrons finally leave the collector terminal (give rise to i_c). Electrons coming from base and from collector meet at junction O and they jointly move to emitter, which gives rise to i_e.

Thus, here also we can see that $\boxed{i_e = i_b + i_c}$

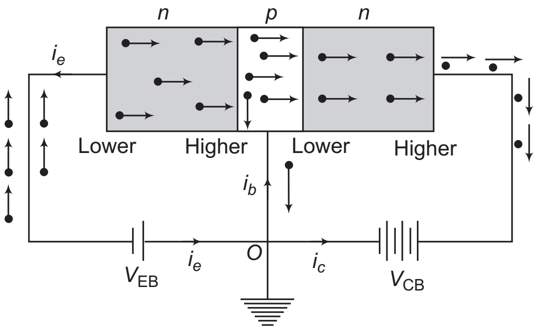


Fig. 35.29

Note *that although the working principle of p-n-p and n-p-n transistors are similar but the current carriers in p-n-p transistor are mainly holes whereas in n-p-n transistors the current carriers are mainly electrons. Mobility of electrons are however more than the mobility of holes, therefore n-p-n transistors are used in high frequency and computer circuits where the carriers are required to respond very quickly to signals.*

α and β-parameters: α and β-parameters of a transistor are defined as,

$$\boxed{\alpha = i_c / i_e} \quad \text{and} \quad \boxed{\beta = i_c / i_b}$$

As i_b is about 1 to 5% of i_e, α is about 0.95 to 0.99 and β is about 20 to 100. By simple mathematics we can prove that,

$$\boxed{\beta = \frac{\alpha}{1-\alpha}}$$

35.8 Transistor As An Amplifier

A transistor can be used for amplifying a weak signal.

When a transistor is to be operated as amplifier, three different basic circuit connections are possible. These are

(i) common base, (ii) common emitter and (iii) common collector circuits.

Whichever circuit configuration, the emitter-base junction is always forward biased, while the collector-base junction is always reverse biased.

(a) **Common base amplifier using a *p-n-p* transistor** In common base amplifier, the input signal is applied across the emitter and the base, while the amplified output signal is taken across the collector and the base. This circuit provides a very low input resistance, a very high output resistance and a current gain of just less than 1. Still it provides a good voltage and power amplification. There is no phase difference between input and output signals.

The common base amplifier circuit using a *p-n-p* transistor is shown in figure. The emitter base input circuit is forward biased by a low voltage battery V_{EB}. The collector base output circuit is reversed biased by means of a high voltage battery V_{CC}. Since, the input circuit is forward biased, resistance of input circuit is small. Similarly, output circuit is reverse biased, hence resistance of output circuit is high.

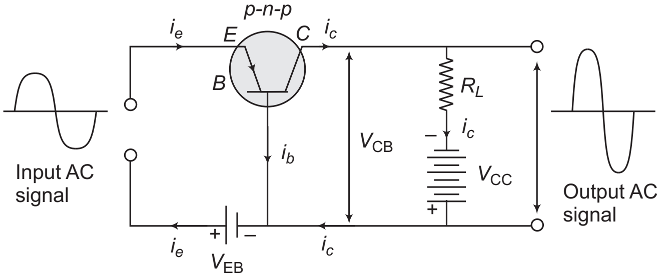


Fig. 35.30

The weak input AC voltage signal is superimposed on V_{EB} and the amplified output signal is obtained across collector-base circuit. In the figure we can see that,

$$V_{\text{CB}} = V_{\text{CC}} - i_c R_L$$

The input AC voltage signal changes net value of V_{EB}. Due to fluctuations in V_{EB}, the emitter current i_e also fluctuates which in turn fluctuates i_c. In accordance with the above equation there are fluctuations in V_{CB}, when the input signal is applied and an amplified output is obtained.

Current gain, Voltage gain and Power gain

(i) **Current gain** Also called AC current gain (α_{ac}), is defined as the ratio of the change in the collector current to the change in the emitter current at constant collector-base voltage.

Thus, α_{ac} or simply $\boxed{\alpha = \dfrac{\Delta i_c}{\Delta i_e}}$ ($V_{\text{CB}} = \text{constant}$)

As stated earlier also, α is slightly less than 1.

(ii) **Voltage gain** It is defined as the ratio of change in the output voltage to the change in the input voltage. It is denoted by A_V. Thus,

$$A_V = \frac{\Delta i_c \times R_{\text{out}}}{\Delta i_e \times R_{\text{in}}}$$

but $\dfrac{\Delta i_c}{\Delta i_e} = \alpha$, the current gain.

$\therefore$ $$\boxed{A_V = \frac{a\,R_{\text{out}}}{R_{\text{in}}}}$$

Since, $R_{\text{out}} >> R_{\text{in}}$, A_V is quite high, although α is slightly less than 1.

(iii) **Power gain** It is defined as the change in the output power to the change in the input power.

Since $P = Vi$

Therefore, power gain = current gain × voltage gain

or $$\boxed{\text{Power gain} = \alpha^2 \cdot \frac{R_{\text{out}}}{R_{\text{in}}}}$$

Extra Points to Remember

- The output voltage signal is in phase with the input voltage signal.
- The common base amplifier is used to amplify high (radio)-frequency signals and to match a very low source impedance (~20 Ω) to a high load impedance (~100 k Ω).

(b) **Common emitter amplifier using a *p-n-p* transistor** Figure shows a *p-n-p* transistor as an amplifier in common emitter mode. The emitter is common to both input and output circuits. The input (base-emitter) circuit is forward biased by a low voltage battery V_{BE}. The output (collector-emitter) circuit is reverse biased by means of a high voltage battery V_{CC}.

Since, the base-emitter circuit is forward biased, input resistance is low. Similarly, collector-emitter circuit is reverse biased, therefore output resistance is high. The weak input AC signal is superimposed on V_{BE} and the amplified output signal is obtained across the collector-emitter circuit.

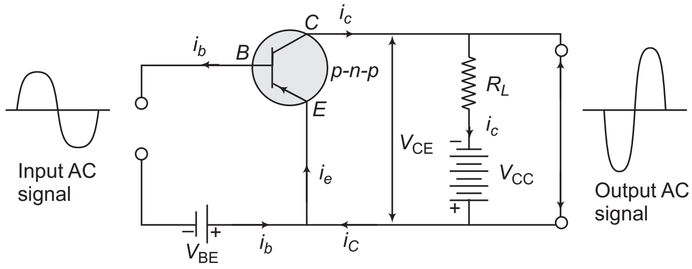


Fig. 35.31

In the figure we can see that, $V_{CE} = V_{CC} - i_c R_L$

When the input AC voltage signal is applied across the base-emitter circuit, it fluctuates V_{BE} and hence the emitter current i_e. This in turn changes the collector current i_c consequently V_{CE} varies in accordance with the above equation. This variation in V_{CE} appears as an amplified output.

Current Gain, Voltage Gain and Power Gain

(i) **Current gain** Also called ac current gain (β_{ac}), is defined as the ratio of the collector current to the base current at constant collector to emitter voltage.

$$\boxed{\beta_{\mathrm{ac}} \text{ or simply } \beta = \left(\frac{\Delta i_c}{\Delta i_b}\right)} \qquad (V_{\mathrm{CE}} = \text{constant})$$

(ii) **Voltage gain** It is defined as the ratio of the change in the output voltage to the change in the input voltage. It is denoted by A_V. Thus,

$$A_V = \frac{\Delta i_c \times R_{\mathrm{out}}}{\Delta i_b \times R_{\mathrm{in}}} \quad \text{or} \quad \boxed{A_V = \beta\left(\frac{R_{\mathrm{out}}}{R_{\mathrm{in}}}\right)}$$

(iii) **Power gain** It is defined as the ratio of change in output power to the change in the input power. Since,

$$P = Vi$$

Therefore, power gain = current gain × voltage gain or $\boxed{\text{Power gain} = \beta^2\left(\frac{R_{\mathrm{out}}}{R_{\mathrm{in}}}\right)}$

Extra Points to Remember

- The value of current gain β is from 15 to 50 which is much greater than α.
- The voltage gain in common-emitter amplifier is larger compared to that in common base amplifier.
- The power gain in common-emitter amplifier is extremely large compared to that in common base amplifier.
- The output voltage signal is 180° out of phase with the input voltage signal in the common-emitter amplifier.

Transconductance (g_m) There is one more term called transconductance (g_m) in common-emitter mode. It is defined as the ratio of the change in the collector current to the change in the base to emitter voltage at constant collector to emitter voltage. Thus,

$$\boxed{g_m = \left(\frac{\Delta i_c}{\Delta V_{\text{BE}}}\right)} \qquad (V_{\text{CE}} = \text{constant})$$

The unit of g_m is Ω^{-1} or siemen (S). By simple calculation we can prove that,

$$\boxed{g_m = \frac{\beta}{R_{\text{in}}}}$$

Advantages of a transistor over a triode valve A transistor is similar to a triode valve in the sense that both have three elements. While the elements of a triode are, cathode, plate and grid. The three elements of a transistor are emitter, collector and base. Emitter of a transistor can be compared with the cathode of the triode, the collector with the plate and the base with the grid.

Transistor has following advantages over a triode valve

(i) A transistor is small and cheap as compared to a triode valve. They can bear mechanical shocks.

(ii) A transistor has much longer life as compared to a triode valve.

(iii) Loss of power in a transistor is less as it operates at a much lower voltage.

(iv) In a transistor no heating current is required. So, unlike a triode valve, a transistor starts functioning immediately as soon as the switch is opened. In case of valves, they come in operation after some time of opening the switch (till cathode gets heated).

Drawbacks of a transistor over a triode valve Transistor have following drawbacks as compared to valves.

(i) Since, the transistors are made of semiconductors they are temperature sensitive. We cannot work on transistors at high temperatures.

(ii) In transistors noise level is high. Keeping all the factors into consideration, transistors have replaced the valve from most of the modern electronic devices.

Example 35.12 *The current gain of a transistor in a common base arrangement in* 0.98. *Find the change in collector current corresponding to a change of* 5.0 *mA in emitter current. What would be the change in base current?*

Solution Given, $\alpha = 0.98$ and $\Delta i_e = 5.0$ mA

From the definition of, $\alpha = \dfrac{\Delta i_c}{\Delta i_e}$

Change in collector current, $\Delta i_c = (\alpha)(\Delta i_e) = (0.98)(5.0)\text{ mA} = 4.9\text{ mA}$

Further, change in base current, $\Delta i_b = \Delta i_e - \Delta i_c = 0.1\, mA$ **Ans.**

Example 35.13 *A transistor is connected in common emitter configuration. The collector supply is 8 V and the voltage drop across a resistor of* $800\,\Omega$ *in the collector circuit is 0.5 V. If the current gain factor* (α) *is 0.96, find the base current.*

Solution $\beta = \dfrac{\alpha}{1-\alpha} = \dfrac{0.96}{1-0.96} = 24$

The collector current is,

$$i_c = \frac{\text{voltage drop across collector resistor}}{\text{resistance}} = \frac{0.5}{800}\,\text{A} = 0.625\times 10^{-3}\,\text{A}$$

From the definition of $\beta = \dfrac{i_c}{i_b}$

the base current $i_b = \dfrac{i_c}{\beta} = \dfrac{0.625\times 10^{-3}}{24}\,\text{A}$

$= 26\,\mu\text{A}$ **Ans.**

Example 35.14 *In a common emitter amplifier, the load resistance of the output circuit is 500 times the resistance of the input circuit. If* $\alpha = 0.98$, *then find the voltage gain and power gain.*

Solution Given $\alpha = 0.98$ and $\dfrac{R_{\text{out}}}{R_{\text{in}}} = 500 \Rightarrow \beta = \dfrac{\alpha}{1-\alpha} = \dfrac{0.98}{1-0.98} = 49$

(i) Voltage gain $= (\beta)\dfrac{R_{\text{out}}}{R_{\text{in}}} = (49)(500) = 24500$

(ii) Power gain $= (\beta^2)\dfrac{R_{\text{out}}}{R_{\text{in}}} = (49)^2\,(500) = 1200500$

INTRODUCTORY EXERCISE 35.3

1. For transistor action, which of the following statements are correct?
   (a) Base, emitter and collector regions should have similar size and doping concentrations
   (b) The base region must be very thin and lightly doped
   (c) The emitter junction is forward biased and collector junction is reverse biased
   (d) Both the emitter junction as well as the collector junction are forward biased
2. For a transistor amplifier, the voltage gain
   (a) remains constant for all frequencies
   (b) is high at high and low frequencies and constant in the middle frequency range
   (c) is low at high and low frequencies and constant at mid frequencies
   (d) None of the above
3. For a CE-transistor amplifier, the audio signal voltage across the collector resistance of $2\,\text{k}\Omega$ is 2 V. Suppose the current amplification factor of the transistor is 100. Find the input signal voltage and base current, if the base resistance is $1\,\text{k}\Omega$.

35.9 Digital Electronics and Logic Gates

(i) **Binary system** There are a number of questions which have only two answers Yes or No. A statement can be either True or False. A switch can be either ON or OFF. These values may be represented by two symbols 0 and 1. In a number system, in which we have only two digits is called a binary system. (decimal system for example has ten digits).

In binary system usually we write 1 for positive response (e.g. when a switch is ON) and 0 for negative (when switch is OFF).

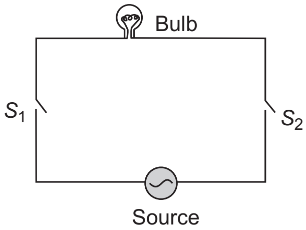


Fig. 35.32

(ii) **Truth table** To understand the concept of truth table let us take an example. A bulb is connected to an AC source *via* two switches S_1 and S_2.

In binary system, we will write 0, if the switch (or bulb) is off and write 1 if it is on. Further let us write

A for state of switch S_1

B for state of switch S_2

and C for state of the bulb.

Now, let us make a table (called truth table) which is self explanatory.

Table 35.3

Switch S_1	Switch S_2	Bulb	A	B	C
Off	On	Off	0	1	0
On	Off	Off	1	0	0
Off	Off	Off	0	0	0
On	On	On	1	1	1

Exercise *Make a truth table corresponding to the circuit shown in figure.*

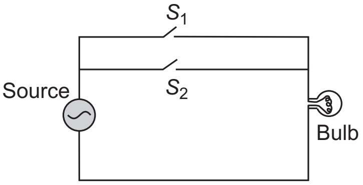


Fig. 35.33

Table 35.4

Switch S_1	Switch S_2	Bulb	A	B	C
On	Off	On	1	0	1
Off	On	On	0	1	1
Off	Off	Off	0	0	0
On	On	On	1	1	1

(iii) **Logical function** A variable (e.g. state of a switch or state of a bulb) which can assume only two values (0 and 1) is called a logical variable. A function of logical variables is called a logical function. **AND, OR** and **NOT** represent three basic operations on logical variables.

'AND' function Suppose C is a function of A and B, then it will be said an 'AND' function when C has value 1 when both A and B have value 1. Truth table corresponding to Table 35.3 is an example of 'AND' function. The function is written as,

$$C = A \text{ and } B$$

AND function is also denoted as $C = A \cdot B$

'OR' function C, a function of A and B will be said an 'OR' function when C has value 1 when either of A or B has value 1. Truth table corresponding to Table 35.4 is an example of 'OR' function. The function is written as,

$$C = A \text{ OR } B$$

OR function is also denoted as,

$$C = A + B$$

'NOT' function 'NOT' function is a function of a single variable.

A bulb is short circuited by a switch. If the switch is open, the current goes through the bulb and it is on. If the switch is closed the current goes through the switch and the bulb is off. The truth table corresponding to the above situation (or NOT function) is as under.

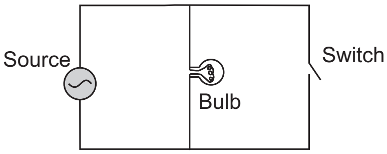


Fig. 35.34

Table 35.5

Switch	Bulb	A	B
Open	On	0	1
Closed	Off	1	0

'NOT' function is denoted as,

$$B = \text{NOT } A \text{ or } B = \overline{A}$$

Example 35.15 *Write the truth table for the logical function $D = (A \text{ OR } B) \text{ AND } B$.*

Solution A OR B is a logical function, say it is equal to X, *i.e.,*

$$X = A \text{ OR } B$$

Now, $$D = X \text{ AND } B$$

The corresponding truth table is as under.

Table 35.6

A	B	$X = A$ OR B	$D = (A$ OR $B)$ AND B
1	0	1	0
0	1	1	1
0	0	0	0
1	1	1	1

Note *that the given function can also be written as,*

$$D = (A + B) \cdot B$$

(iv) **Logic gates** Logic gates are important building blocks in digital electronics. These are circuits with one or more inputs and one output. The basic gates are OR, AND, NOT, NAND, NOR and XOR. As we know, in digital electronics only two voltage levels are present. Conventionally, these are 5V and 0V, referred to as 1 and 0 respectively or *vice-versa.* They are also referred as high and low.Figure given are the symbols of six basic gates.

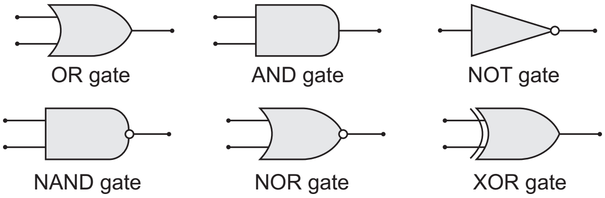


Fig. 35.35

OR gate The truth table of 'OR' gate is given below.

Table 35.7

A	*B*	*X*
0	0	0
0	1	1
1	0	1
1	1	1

The output X will be 1 (*i.e.,* 5V) when the A input is 1, OR when the B input is 1, OR when both are 1. This is written as,

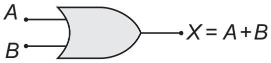


Fig. 35.36

$$X = A + B$$

Figure shows construction of an OR gate using two diodes

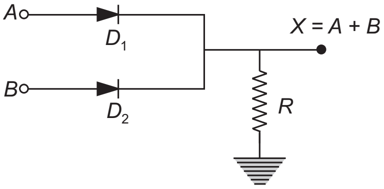


Fig. 35.37

When either of point A or point B (or both) has potential +5V, diodes D_1 or D_2 (or both) are forward biased and the potential at X is the same as the common potential at A and B which is 5V.

AND gate The truth table of 'AND' gate is given below.

Table 35.8

A	*B*	*Y*
0	0	0
0	1	0
1	0	0
1	1	1

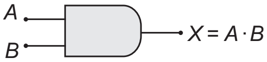


Fig. 35.38

The output X will be 1 (i.e. 5V) when both the inputs A and B is 1. This is written as,

$$X = A \cdot B$$

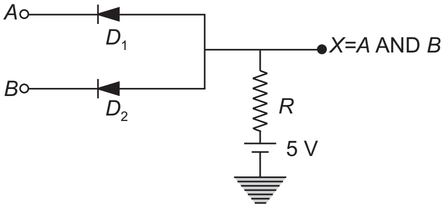


Fig. 35.39

Figure shows construction for an AND gate using two diodes D_1 and D_2.

When potentials at A and B both are zero, then both the diodes are forward biased and offer no resistance. The potential at X in this position is equal to the potential at A or B i.e. 0. Thus $X = 0$, when both A and B are zero. Now suppose potential at A is zero but at B is 5V, then D_1 is forward biased. In this situation potential at X is also zero.

Thus, $X = 0$ when $A = 0$. Similarly, we can see that $X = 0$ when $B = 0$. Lastly when potentials at both A and B are 5V, so that both the diodes are unbiased and there will be no current through R and the potential at X will be equal to 5V. Thus, $X = 1$ when A and B both are 1.

NOT gate This has one input and one output. The output is the inverse of the input. When the input A is 1, the output X will be 0 and *vice-versa.* The truth table for 'NOT' gate is given below.

Table 35.9

A	X
0	1
1	0

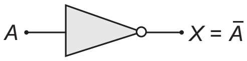


Fig. 35.40

Note *A NOT gate cannot be constructed with diodes. Transistor is used for realisation of a NOT gate, but at this stage students do not require it. A NOT gate is written as* $X = \bar{A}$.

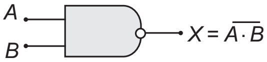


Fig. 35.41

NAND gate The function, $X =$ NOT (A and B) of two logical variables A and B is called NAND function. It is written as $X = A$ NAND B. It is also written as,

$$X = \overline{A \cdot B}$$

or

$$X = \overline{AB}$$

The truth table of a 'NAND' gate is given below.

Table 35.10

A	B	$A \cdot B$	$X = \overline{A \cdot B}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

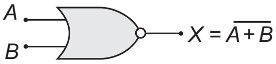


Fig. 35.42

NOR gate The function $X =$ NOT (A OR B) is called a NOR function and is written as $X = A$ NOR B. It is also written as, $X = \overline{A + B}$. The truth table for a NOR gate is given below.

Table 35.11

A	B	$A+B$	$X=\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

XOR gate It is also called the exclusive OR function. It is a function of two logical variables A and B which evaluates to 1 if one of two variables is 0 and the other is 1. The function is zero, if both the variables are 0 or 1.

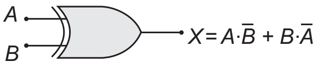


Fig. 35.43

$$A \text{ XOR } B = A \cdot \overline{B} + \overline{A} \cdot B$$

The truth table for XOR is given below.

Table 35.12

A	B	$\overline{A}$	$\overline{B}$	$A \cdot \overline{B}$	$\overline{A} \cdot B$	$A = A \cdot \overline{B} + \overline{A} \cdot B$
0	0	1	1	0	0	0
0	1	1	0	0	1	1
1	0	0	1	1	0	1
1	1	0	0	0	0	0

Example 35.16 *Construct the truth table for the function X of A and B represented by figure shown here.*

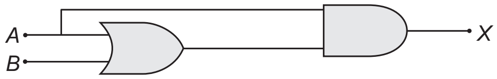


Fig. 35.44

Solution The output X in terms of the input A and B can be written as, $X = A \cdot (A + B)$

Let us make the truth table corresponding to this function.

Table 35.13

A	B	$A+B$	$X = A \cdot (A+B)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

Example 35.17 *Make the output waveform (Y) of the OR gate for the following inputs A and B.*

Table 35.14

Time	A	B
For $t < t_1$	0	0
From t_1 to t_2	1	0
From t_2 to t_3	1	1
From t_3 to t_4	0	1
From t_4 to t_5	0	0
From t_5 to t_6	1	0
For $t > t_6$	0	1

Solution Output value Y corresponding to OR gate is given in the following table.

Table 35.15

Time	A	B	$Y = A + B$
For $t < t_1$	0	0	0
From t_1 to t_2	1	0	1
From t_2 to t_3	1	1	1
From t_3 to t_4	0	1	1
From t_4 to t_5	0	0	0
From t_5 to t_6	1	0	1
For $t > t_6$	0	1	1

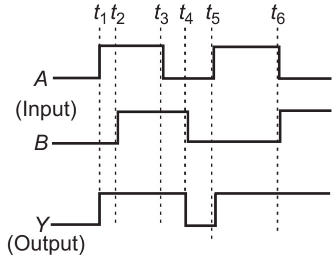


Fig. 35.45

Therefore, the waveform Y will be as shown in the figure.

Example 35.18 *Take A and B inputs similar to that in above example. Sketch the output waveform obtained from AND gate.*

Solution Output value, Y corresponding to AND gate is given in the following table.

Table 35.16

Time	A	B	$Y = A \cdot B$
For $t < t_1$	0	0	0
From t_1 to t_2	1	0	0
From t_2 to t_3	1	1	1
From t_3 to t_4	0	1	0
From t_4 to t_5	0	0	0
From t_5 to t_6	1	0	0
For $t > t_6$	0	1	0

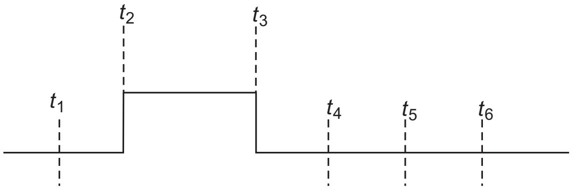


Fig. 35.46

Based on the above table, the output waveform Y for AND gate can be drawn as in figure 35.46.

INTRODUCTORY EXERCISE 35.4

1. Make the output waveform Y of the NAND gate for the following inputs A and B.

Table 35.17

Time	A	B
For $t < t_1$	1	1
From t_1 to t_2	0	0
From t_2 to t_3	0	1
From t_3 to t_4	1	0
From t_4 to t_5	1	1
From t_5 to t_6	0	0
For $t > t_6$	0	1

2. You are given two circuits. Identify the logic operation carried out by the two circuits.

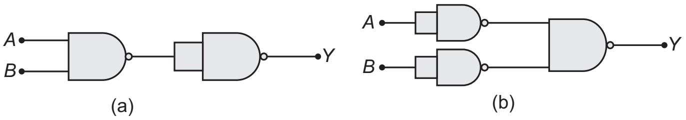


Fig. 35.47

Final Touch Points

1. **Integrated Circuits** The short form of integrated circuit is IC. It is revolutionised the electronics technology. The entire electronic circuit (consisting of many passive components like *R* and *C* and active devices like diode and transistor) is fabricated on a small single block (called chip) of a semiconductor. Such circuits are more reliable and less shock proof compared to conventional circuits used before. The chip dimensions are as small as 1 mm ×1 mm or it could be even smaller than this.

   Depending on the nature of input signals, ICs are of two types.

   (i) Linear or analogue IC
   (ii) Digital IC

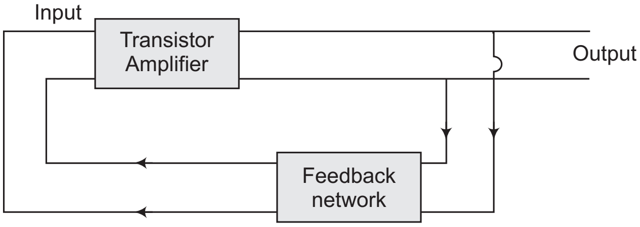
   Linear ICs process analogue signals which change over a range of values between a maximum and a minimum. The digital ICs process signals that have only two values. They contain circuits such as logic gates.

   IC is the heart of all computer systems. It is used in almost all electronic devices like, cell phones, televisions, cars etc.

   It was first invented in 1958 by **Jack Kilky** and he was awarded Nobel prize for this in the year 2000. Growth of semiconductor industry is very fast. From current trends it is expected that by 2020 computers will operate at 40 GHz and would be much smaller, more efficient and less expensive than present day computers.

2. **Feedback amplifier and transistor oscillator** In an amplifier, a sinusoidal input is given which gets amplified as an output. Hence, an external input is necessary to sustain AC signal in the output.

   In an oscillator, we get AC output without any external input signal. A portion of the output power is returned back (feedback) to the input (in phase) with the starting power. In other words, the output in an oscillator is self sustained.



Principle of a transistor amplifier with positive feedback working as an oscillator

3. In transistors, the base region is narrow and lightly doped, otherwise the electrons or holes coming from the input side (say emitter in CE-configuration) will not be able to reach the collector.

Solved Examples

Example 1 *Sn, C, Si and Ge are all group XIV elements. Yet, Sn is a conductor, C is an insulator while Si and Ge are semiconductors. Why ?*

Solution It all depends on energy gap between valence band and conduction band. The energy gap for Sn is 0 eV, for C is 5.4 eV, for Si is 1.1 eV and for Ge is 0.7 eV.

Example 2 *Three photodiodes D_1, D_2 and D_3 are made of semiconductors having band gaps of 2.5 eV, 2 eV and 3 eV, respectively. Which one will be able to detect light of wavelength* $6000\ \overset{o}{\text{A}}$ *?*

Solution Energy of incident light

$$E\text{ (in eV)} = \frac{12375}{\lambda\text{ (in }\overset{o}{\text{A}})}$$

$$E = \frac{12375}{6000}\text{ eV}$$

or

$$E = 2.06\text{ eV}$$

For the incident radiation to be detected by the photodiode energy of incident radiation should be greater than the band gap. This is true only for D_2. Therefore only D_2 will detect this radiation.

Example 3 *What is the range of energy gap (E_g) in insulators, semiconductors and conductors?*

Solution For insulators $E_g > 3$ eV, for semiconductors, $E_g = 0.2$ eV to 3eV while for conductors (or metals) $E_g = 0$.

Example 4 *n-type extrinsic semiconductor is negatively charged, while p-type extrinsic semiconductor is positively charged. Is this statement true or false?*

Solution False. Intrinsic as well as extrinsic semiconductors are electrically neutral.

Example 5 *What is resistance of an intrinsic semiconductor at 0K ?*

Solution At 0K number of holes (or number of free electrons) in an intrinsic semiconductor become zero. Therefore, resistance of an intrinsic semiconductor becomes infinite at 0 K.

Example 6 *Consider an amplifier circuit using a transistor. The output power is several times greater than the input power. Where does the extra power come from?*

Solution The extra power required for amplified output is obtained from the DC source.

Example 7 *A piece of copper and the other of germanium are cooled from the room temperature to 80 K. What will happen to their resistance?*

Solution Copper is conductor and germanium is semiconductor. With decrease in temperature resistance of a conductor decreases and that of semiconductor increases. Therefore resistance of copper will decrease and that of semiconductor will increase.

Example 8 *A transistor has three impurity regions, emitter, base and collector. Arrange them in order of increasing doping levels.*

Solution The order of increasing doping levels is

base > collector > emitter.

Example 9 *Name two gates which can be used repeatedly to produce all the basic or complicated gates.*

Solution NAND and NOR gates can be used repeatedly to produce all the basic or complicated gates. This is why these gates are called digital building blocks.

Example 10 *A change of 8.0 mA in the emitter current brings a change of 7.9 mA in the collector current. How much change in the base current is required to have the same change 7.9 mA in the collector current ? Find the values of* α *and* β.

Solution We know that,

$$i_e = i_b + i_c$$

$\therefore$ $$\Delta i_e = \Delta i_b + \Delta i_c$$

or $$\Delta i_b = \Delta i_e - \Delta i_c$$

Substituting the given values of the question,

We have $$\Delta i_b = (8.0 - 7.9)\text{ mA} = 0.1\text{ mA}$$

Hence, a change of 0.1 mA in the base current is required to have a change of 7.9 mA in the collector current.

Further, $$\alpha = \frac{i_c}{i_e} \text{ or } \frac{\Delta i_c}{\Delta i_e} = \frac{7.9}{8.0}$$

$$= 0.99$$ **Ans.**

$$\beta = \frac{i_c}{i_b} \text{ or } \frac{\Delta i_c}{\Delta i_b} = \frac{7.9}{0.1}$$

$$= 79$$ **Ans.**

Example 11 *A transistor is used in common-emitter mode in an amplifier circuit. When a signal of 20 mV is added to the base-emitter voltage, the base current changes by* 20 μA *and the collector current changes by 2 mA. The load resistance is* 5 kΩ. *Calculate (a) the factor* β *(b) the input resistance* R_{in} *(c) the transconductance and (d) the voltage gain.*

Solution (a) **Factor** β

$$\beta = \frac{\Delta i_c}{\Delta i_b}$$

Substituting the given values, we have

$$\beta = \frac{2 \times 10^{-3}}{20 \times 10^{-6}} = 100$$ **Ans.**

(b) **Input Resistance R_{in}**

$$R_{\text{in}} = \frac{\Delta V_{\text{BE}}}{\Delta i_b} = \frac{20 \times 10^{-3}}{20 \times 10^{-6}}$$

$$= 10^3\ \Omega$$

$$= 1\ \text{k}\Omega$$ **Ans.**

(c) **Transconductance g_m**

$$g_m = \frac{\Delta i_c}{\Delta V_{\text{BE}}} = \frac{2 \times 10^{-3}}{20 \times 10^{-3}}$$

$$= 0.1 \text{ mho}$$ **Ans.**

(d) **Voltage Gain A_V**

$$A_V = \beta \left(\frac{R_{\text{out}}}{R_{\text{in}}} \right)$$

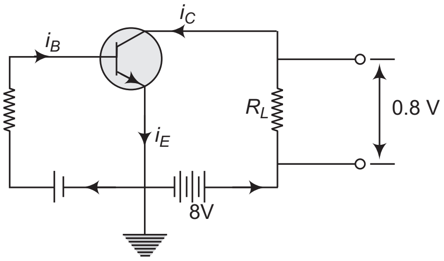
Substituting the values we have,

$$A_V = (100) \left(\frac{5 \times 10^3}{1 \times 10^3} \right)$$

$$= 500$$ **Ans.**

Example 12 *An n-p-n transistor is connected in common-emitter configuration in which collector supply is* 8V *and the voltage drop across the load resistance of* 800 Ω *connected in the collector circuit is* 0.8 V. *If current amplification factor is* 25, *determine collector-emitter voltage and base current. If the internal resistance of the transistor is* 200 Ω, *calculate the voltage gain and the power gain.*

Solution The corresponding circuit is shown in figure.



Voltage across $R_L = i_c R_L = 0.8$ V (given)

$$\therefore \qquad i_c = \frac{0.8}{R_L} = \frac{0.8}{800} \text{ A} = 1 \text{ mA}$$

Further it is given that,

$$\beta = 25 = \frac{i_c}{i_b}$$

$$\therefore \qquad i_b = \frac{i_c}{25} = 40\ \mu\text{A}$$ **Ans.**

Collector-Emitter Voltage (V_{CE})

Applying Kirchhoff's second law in emitter-collector circuit, we have

$$V_{CE} = (8 - 0.8) \text{ V} = 7.2 \text{ V}$$ **Ans.**

Voltage gain (A_V)

Voltage gain, $$A_V = \beta\left(\frac{R_{\text{out}}}{R_{\text{in}}}\right)$$

or $$A_V = 25\left(\frac{800}{200}\right) = 100$$ **Ans.**

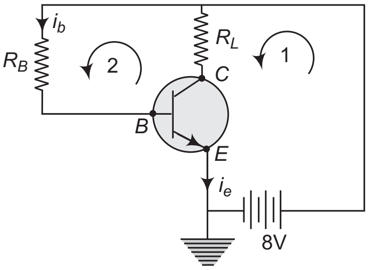
Power gain

$$\text{Power gain} = \beta^2\left(\frac{R_{\text{out}}}{R_{\text{in}}}\right) = (25)^2\left(\frac{800}{200}\right) = 2500$$ **Ans.**

Note *Kirchhoff's laws can be applied in a transistor circuit in the similar manner as is done in normal circuits.*

Example 13 *An n-p-n transistor in a common-emitter mode is used as a simple voltage amplifier with a collector current of 4 mA. The positive terminal of a 8 V battery is connected to the collector through a load resistance R_L and to the base through a resistance R_B. The collector-emitter voltage $V_{CE} = 4$ V, the base-emitter voltage $V_{BE} = 0.6$ V and the current amplification factor $\beta = 100$. Calculate the values of R_L and R_B.*

Solution Given, $i_c = 4$ mA



Applying Kirchhoff's second law in loop 1, we have

$$V_{CE} = 8 - i_c R_L$$

∴ $$R_L = \frac{8 - V_{CE}}{i_c} = \frac{8-4}{4\times10^{-3}}$$

$$= 1000\,\Omega$$

$$= 1\text{ k}\Omega$$ **Ans.**

Further, $$\beta = \frac{i_c}{i_b}$$

∴ $$i_b = \frac{i_c}{\beta} = \frac{4\times10^{-3}}{100}\text{ A} = 40\ \mu\text{A}$$

Now, $$V_{BE} = 8 - i_b R_B$$

∴ $$R_B = \frac{8 - V_{BE}}{i_b}$$

$$= \frac{8-0.6}{40\times10^{-6}}$$

$$= 1.85\times10^5\ \Omega$$ **Ans.**

Example 14 *Let* $X = A \cdot \overline{BC}$. *Evaluate X for*

(a) $A=1, B=0, C=1$, *(b)* $A=B=C=1$ *and* *(c)* $A=B=C=0$.

Solution (a) When, $A=1, B=0$ and $C=1$

$$BC = 0$$

$\therefore$ $$\overline{BC} = 1$$

or $$A \cdot \overline{BC} = 1$$ **Ans.**

(b) When, $$A = B = C = 1$$

Then, $$BC = 1$$

or $$\overline{BC} = 0$$

$\therefore$ $$A \cdot \overline{BC} = 0$$ **Ans.**

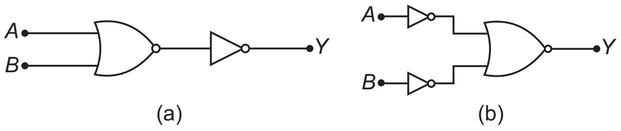
(c) When, $$A = B = C = 0$$

Then, $$BC = 0$$

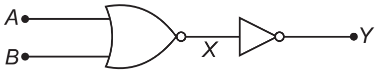
$\therefore$ $$\overline{BC} = 1$$

or $$A \cdot \overline{BC} = 0$$ **Ans.**

Example 15 *Show that given circuit (a) acts as OR gate while the given circuit (b) acts as AND gate.*



Solution (a) The first gate is NOR gate then NOT gate



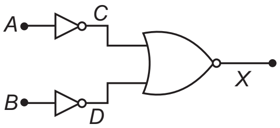
Thus, $X = \overline{A+B}$ and $Y = \overline{X}$

The truth table can be made as under.7

Table 35.18

A	B	A + B	$X = \overline{A+B}$	$Y = \overline{X}$
1	0	1	0	1
0	1	1	0	1
1	1	1	0	1
0	0	0	1	0

The last column of Y is similar to third column of $A + B$ which is the truth table corresponding to OR gate.



(b) First two gates are NOT gates and the last gate is NOR gate.

Thus, $$C = \overline{A} \text{ and } D = \overline{B}$$

$$X = \overline{C + D}$$

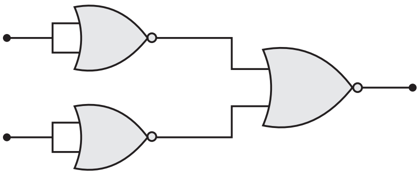
The truth table corresponding to this can be made as under.

Table 35.19

A	B	$A \cdot B$	$C = \overline{A}$	$D = \overline{B}$	$C + D$	$X = \overline{C + D}$
1	0	0	0	1	1	0
0	1	0	1	0	1	0
1	1	1	0	0	0	1
0	0	0	1	1	1	0

The last column of X is similar to third column of $A \cdot B$, which is the truth table corresponding to AND gate.

Example 16 *Write the truth table for the circuit given in figure consisting of NOR gates. Identify the logic operations (OR, AND, NOT) performed by the the circuits.*



Solution The truth table corresponding to given circuit of logic gates is

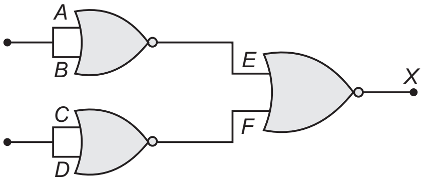


Table 35.20

A	B	$A + B$	$E = \overline{A + B}$	C	D	$C + D$	$F = \overline{C + D}$	$E + F$	$X = \overline{E + F}$
1	1	1	0	1	1	1	0	0	1
0	0	0	1	0	0	0	1	1	0

Corresponding to input columns of A, B, C and D we can see that output column of X is of AND gate,

$$X = A + B + C + D$$

Exercises

Single Correct Option

1. The conductivity of a semiconductor increases with increase in temperature because

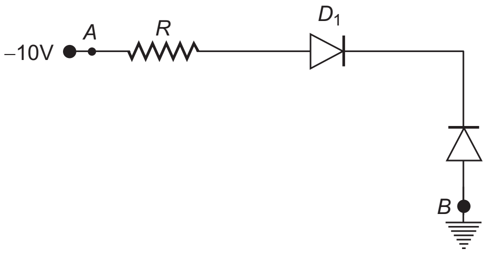
(a) number density of free current carriers increases

(b) relaxation time increases

(c) both number density of carriers and relaxation time decreases but effect of decrease in relaxation time is much less than increase in number density.

(d) number density of current carriers increases, relaxation time decreases but effect of decrease in relaxation time is much less than increase in number density

2. In figure, assuming the diodes to be ideal,



(a) D_1 is forward biased and D_2 is reverse biased and hence current flows from A to B.

(b) D_2 is forward biased and D_1 is reverse biased and hence no current flows from B to A and *vice-versa*.

(c) D_1 and D_2 are both forward biased and hence current flows from A to B.

(d) D_1 and D_2 are both reverse biased and hence no current flows from A to B and *vice-versa*.

3. Hole is

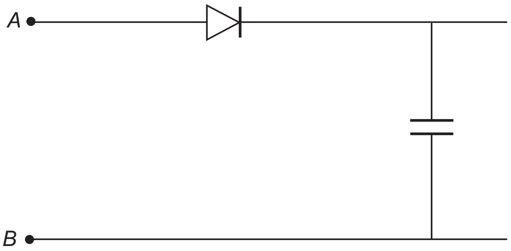
(a) an anti-particle of electron

(b) a vacancy created when an electron leaves a covalent bond

(c) absence of free electrons

(d) an artificially created particle

4. A 220 V AC supply is connected between points A and B. What will be the potential difference V across the capacitor?



(a) 220 V

(b) 110 V

(c) 0V

(d) 220 $\sqrt{2}$ V

More than One Correct Options

5. When an electric field is applied across a semiconductor,

(a) electrons move from lower energy level to higher energy level in the conduction band.

(b) electrons move from higher energy level to lower energy level in the conduction band.

(c) holes in the valence band move from higher energy level to lower energy level.

(d) holes in the valence band move from lower energy level to higher energy level.

6. Consider an *n*-*p*-*n* transistor with its base-emitter junction forward biased and collector base junction reverse biased. Which of the following statements are true?

(a) Electrons crossover from emitter to collector.

(b) Holes move from base to collector.

(c) Electrons move from emitter to base.

(d) Electrons from emitter move out of base without going to the collector.

7. In an *n*-*p*-*n* transistor circuit, the collector current is 10 mA. If 95 per cent of the electrons emitted reach the collector, which of the following statements are true?

(a) The emitter current will be 8 mA.
(b) The emitter current will be 10.53 mA.

(c) The base current will be 0.53 mA.
(d) The base current will be 2 mA.

8. In the depletion region of a diode,

(a) there are no mobile charges

(b) equal number of holes and electrons exist, making the region neutral

(c) recombination of holes and electrons has taken place

(d) immobile charged ions exist.

9. What happens during regulation action of a Zener diode?

(a) The current and voltage across the Zener remains fixed.

(b) The current through the series resistance (R) changes.

(c) The Zener resistance is constant.

(d) The resistance offered by the Zener changes.

10. The breakdown in a reverse biased *p*-*n* junction diode is more likely to occur due to

(a) large velocity of the minority charge carriers if the doping concentration is small

(b) large velocity of the minority charge carriers if the doping concentration is large

(c) strong electric field in a depletion region if the doping concentration is small

(d) strong electric field in the depletion region if the doping concentration is large.

Subjective Questions

11. Can the potential barrier across a *p*-*n* junction be measured by simply connecting a voltmeter across the junction?

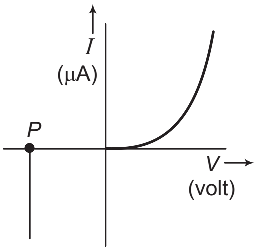
12. Two car garages have a common gate which needs to open automatically when a car enters either of the garages or cars enter both. Devise a circuit that resembles this situation using diodes for this situation.

13. Two amplifiers are connected one after the other in series (cascaded). The first amplifier has a voltage gain of 10 and the second has a voltage gain of 20. If the input signal is 0.01 V, calculate the output AC signal.

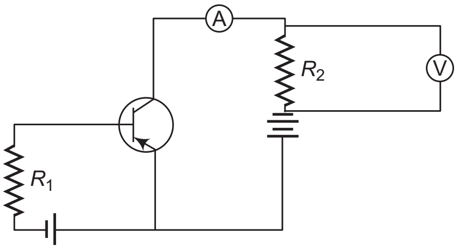
14. A *p* -*n* photodiode is fabricated from a semiconductor with band-gap of 2.8 eV. Can it detect a wavelength of 6000 nm?

15. (i) Name the type of a diode whose characteristics are shown in figure.

(ii) What does the point P in figure represent?

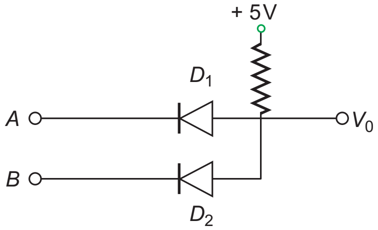


16. If the resistance R_1 is increased, how will the readings of the ammeter and voltmeter change?

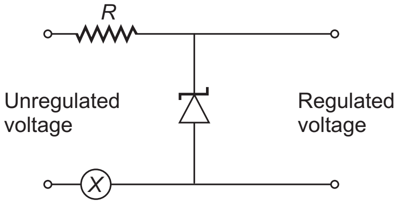


17. How would you set up a circuit to obtain NOT gate using a transistor?

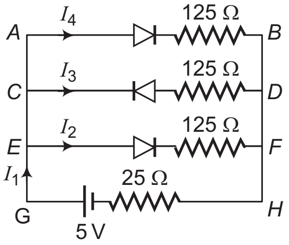
18. Write the truth table for the circuit shown in figure. Name the gate that the circuit resembles.



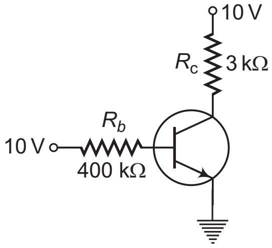
19. A Zener of power rating 1 W is to be used as a voltage regulator. If Zener has a breakdown of 5V and it has to regulate voltage which fluctuated between 3V and 7V, what should be the value of R for safe operation.



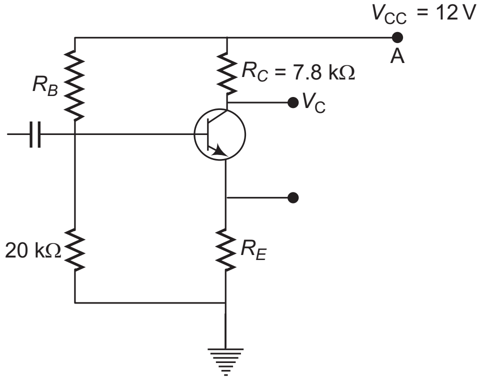
20. If each diode in figure has a forward bias resistance of 25Ω and infinite resistance in reverse bias, what will be the values of the current I_1, I_2, I_3 and I_4?



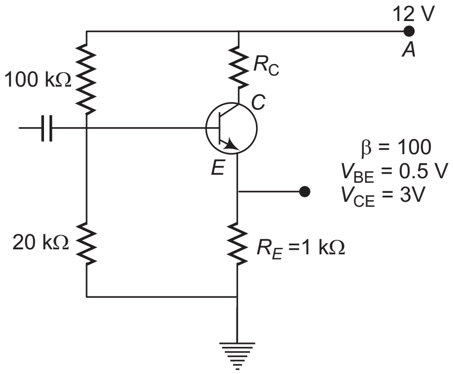
21. In the circuit shown in figure when the input voltage of the base resistance is 10 V. Find the values of I_b, I_c and β.



22. For the transistor circuit shown in figure, evaluate V_E, R_B and R_E. Given $I_C = 1$ mA, $V_{\mathrm{CE}} = 3$ V, $V_{\mathrm{BE}} = 0.5$ V, $V_{\mathrm{CC}} = 12$ V and $\beta = 100$.



23. In the circuit shown in figure, find the value of R_C.



Answers

Introductory Exercise 35.1

1. (c)

Introductory Exercise 35.2

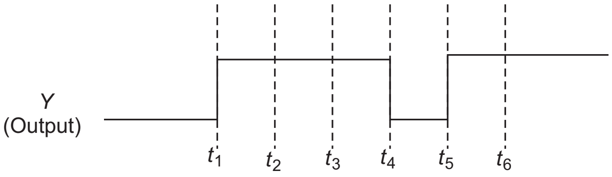
1. (c) **2.** (c)

Introductory Exercise 35.3

1. (b,c) **2.** (c) **3.** $V_i = 0.01\text{ V}$, $i_b = 10\ \mu\text{A}$

Introductory Exercise 35.4

1.



2. (a) AND (b) OR

Exercises

1. (d) **2.** (b) **3.** (b) **4.** (d) **5.** (a,c) **6.** (a,c)

7. (b,c) **8.** (a,b,c) **9.** (b,d) **10.** (a,d) **11.** No **12.** OR gate

13. 2 V **14.** No

15. (i) Zener junction diode and solar cell (ii) Zener breakdown voltage

16. Both readings will decrease

18. AND gate

19. $10\ \Omega$

20. $I_1 = 0.05\text{ A}$, $I_2 = 0.025\text{ A}$, $I_3 = 0$, $I_4 = 0.025\text{ A}$

21. $I_B = 25\ \mu\text{A}$, $I_C = 3.33\text{ mA}$, $\beta = 133$

22. $V_E = 1.2\text{ V}$, $R_B = 108\text{ k}\Omega$, $R_E = 1.2\text{ k}\Omega$

23. $0.56\text{ k}\Omega$

36

Communication System

Chapter Contents

36.1 Introduction
36.2 Different Terms Used in Communication System
36.3 Bandwidth of Signals
36.4 Bandwidth of Transmission Medium
36.5 Propagation of Electromagnetic Waves or Communication Channels
36.6 Modulation
36.7 Amplitude Modulation
36.8 Production of Amplitude Modulated Wave
36.9 Detection of Amplitude Modulated Wave

36.1 Introduction

Communication refers to the transfer of information or message from one point to another point. In modern communication systems, the information is first converted into electrical signals and then sent electronically. This has the advantage of speed, reliability and possibility of communicating over long distances. We are using these every day such as telephones, TV and radio transmission, satellite communication etc. Historically, long distance communication started with the advent of telegraphy in early nineteenth century. The milestone in trans-atlantic radio transmission in 1901 is credited to **Marconi.** However, the concept of radio transmission was first demonstrated by Indian physicist **JC Bose**. Satellite communication started in 1962 with the launching of **Telstar** satellite. The first geostationary satellite **Early Bird** was launched in 1965. Around 1970, optical fibre communication entered in USA, Europe and Japan. The basic units of any communication systems are shown in Fig. 36.1

The transmitter is located at one place. The receiver is located at some other place. Transmission channel connects the transmitter and the receiver. A channel may be in the form of wires or cables or it may be wireless. Transmitter converts message signals produced by the source of information into a form suitable for transmission through the channel.

In any communication system, a non-electrical signal (like voice signal) is first converted into an electrical signal by a device called **transducer.** Most of the speech or information signal cannot be directly transmitted to long distances. For this an intermediate step of **modulation** is necessary in which the information signal is loaded or superimposed on a high frequency wave which acts as a carrier wave.

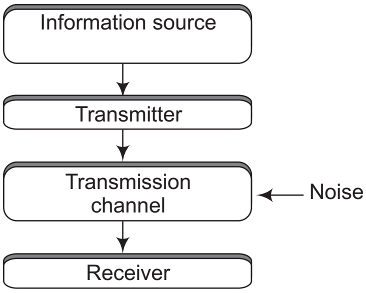


Fig. 36.1 Block diagram of communication system

✓ Extra Points to Remember

- There are basically two communication modes : point to point and broadcast.
- **Point to point** In this mode, communication takes place between a single receiver and transmitter. For example : telephonic call between two persons is a point to point communication.
- **Broadcast** In this mode, there are a large number of receivers corresponding to a single transmitter. Radio and television are examples of this type of communication.

36.2 Different Terms Used in Communication System

Following basic terminology is used in any communication system. Now let us discuss them in detail.

Electrical Transducer As discussed earlier also a transducer converts a non-electrical signal (like a voice signal) into an electrical signal.

Signal Any information in electrical form suitable for transmission is called a signal. Signals can be either analog or digital. Analog signals are continuous variations of voltage or current. Sine functions of time are fundamental analog signal. Digital signals are those which can take only discrete values. Binary system is extensively used in digital electronics. In binary system 0 corresponds to low level and 1 corresponds to high level of voltage or current.

Noise Unwanted signals which are mixed with the main signals are referred as noise.

Transmitter A transmitter makes the incoming message signal suitable for transmission through a channel.

Receiver The signal sent by transmitter through channels is received by the receiver.

Attenuation When the signal propagates from transmitter to receiver it loses some strength and it becomes weaker. This is known as attenuation.

Amplification The signal received by receiver is weaker than the signal sent by transmitter (due to attenuation). The amplitude of this signal is increased by an amplifier. The energy needed for additional signal is obtained from a DC power source.

Range This is the largest distance from the transmitter up to which signal can be received with sufficient strength.

Bandwidth This is the width of the range of frequencies that an electronic signal uses on a given transmission medium. It is expressed in terms of the difference between the highest frequency signal component and the lowest frequency signal component.

Modulation The low frequency message signals cannot be transmitted to long distances by their own. They are superimposed on a high frequency wave (also called a carrier wave). This process is called modulation.

Demodulation This is reverse process of modulation. At the receiver end information is retrieved from the carrier wave. This process is known as demodulation.

Repeater Repeaters are used to extend the range of a communication system. It is a combination of a receiver and a transmitter. Receiver (or a repeater) first receives the original signals, then amplifies it and retransmits it to other places (sometimes with a different carrier frequency).

36.3 Bandwidth of Signals

Message signals (such as voice, picture or computer data) have different range of frequencies. The type of communication system depends on the bandwidth (discussed in the above article). Some frequency range and their corresponding bandwidth are given below.

(i) **For telephonic communication** A bandwidth of 2800 Hz is required. As, the signals range from 300 Hz to 3100 Hz and their difference is 2800 Hz.

(ii) **For music channels** A bandwidth of approximately 20 kHz is required. Because, the audible range of frequencies extends from 20 Hz to 20 kHz and their difference is approximately 20 kHz.

(iii) **For TV signals** A TV signal consists both audio and video. A bandwidth of approximately 6 MHz is required for its transmission.

36.4 Bandwidth of Transmission Medium

Like bandwidths of message signals different types of transmission media offer different bandwidths. Commonly used transmission media are optical fibres, free space and wire. The International Telecommunication Union (ITU) administers the present system of frequency allocations.

(i) Coaxial cables offers a bandwidth of approximately 750 MHz.

(ii) Optical fibres offers a frequency range of 1 THz to 1000 THz.

(iii) Communication through free space (using radio waves) offers a bandwidth varying from few hundreds of kHz to a few GHz. These frequencies are further subdivided for various services as given in following table.

Table 36.1

S.No.	Service	Frequency Bands
1.	AM radio broadcast	540 – 1600 kHz
2.	FM radio broadcast	88 – 108 MHz
3.	Television	54 – 890 MHz
4.	Cellular Phones	840 – 935 MHz
5.	Satellite communication	3.7 – 6.425 GHz

36.5 Propagation of Electromagnetic Waves or Communication Channels

Physical medium through which signals propagate between transmitting and receiving station is called the communication channel. There are basically two types of communications.

(i) Space communication

(ii) Line communication

As per syllabus, we are here discussing only space communication.

Space Communication

Consider two friends playing with a ball in a closed room. One friend throws the ball (transmitter) and the other receives the ball (receiver). There are three ways in which the ball can be sent to the receiver.

(a) By rolling it along the ground

(b) Throwing directly and

(c) Throwing towards roof and then reflected towards the receiver. Similarly, there are three ways of transmitting an information from one place to the other using physical space around the earth.

Line Communication

(a) Along the ground (ground waves).

(b) Directly in a straight line through intervening topographic space (space wave, or tropospheric wave or surface wave) and

(c) Upwards in sky followed by reflection from the ionosphere (sky wave).

These three modes are discussed below.

(i) **Ground Wave or Surface Wave Propagation** Information can be transmitted through this mode when the transmitting and receiving antenna are close to the surface of the earth.

The radio waves which progress along the surface of the earth are called ground waves or surface waves. These waves are vertically polarised in order to prevent short-circuiting of the electric component. The electrical field due to the wave induce charges in the earth's surface as shown in figure. As the wave travels, the induced charges in the earth also travel along it. This constitutes a current in the earth's surface. As the ground wave passes over the surface of the earth, it is weakened as a result of energy absorbed by the earth. Due to these losses, the ground waves are not suited for very long range communication. Further these losses are higher for high frequency. Hence, ground wave propagation can be sustained only at low frequencies (500 kHz to 1500 kHz).

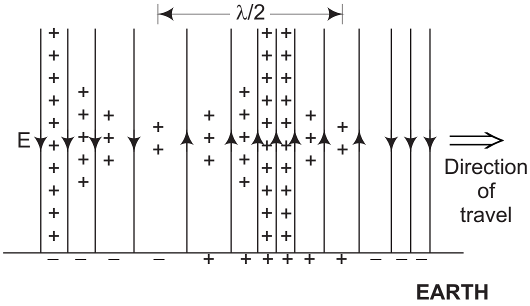


Fig. 36.2. Vertically polarised wave travelling over the surface of the earth. The solid lines represent the electric field (E) of the electromagnetic wave.

Space Wave Propagation or Tropospheric Wave Propagation Television signal (80 MHz to 200 MHz) waves neither follow the curvature of the earth nor get reflected by ionosphere. Surface wave or sky wave cannot be employed in television communication. Television signals can be reflected from geostationary satellite or tall receiver antennas.

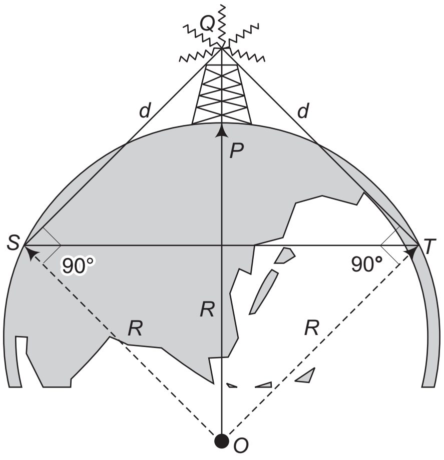


Fig. 36.3

Height of Transmitting Antenna The transmitted waves, travelling in a straight line, directly reach the receiver end and are then picked up by the receiving antenna as shown in figure. Due to finite curvature of the earth, such waves cannot be seen beyond the tangent points S and T.

Suppose h is the height of antenna PQ. Let R be the radius of earth.

Further, let $QT = QS = d$, $PQ = h$, $OQ = R + h$

From the right angled triangle OQT,

$$OQ^2 = OT^2 + QT^2$$

$$\therefore \quad (R+h)^2 = R^2 + d^2$$

$$\therefore \quad d^2 = h^2 + 2Rh$$

Since, $R >> h,\ h^2 + 2Rh \approx 2Rh$

$$\therefore \quad \boxed{d \approx \sqrt{2Rh}}$$

This distance is of the order of 40 km. Area covered for TV transmission

$$\boxed{A = \pi d^2 = 2\pi Rh}$$

If height of receiving antenna is also given in the question, then the maximum line of sight distance d_M is given by

$$\boxed{d_M = \sqrt{2Rh_T} + \sqrt{2Rh_R}}$$

where, h_T = height of transmitting antenna

and h_R = height of receiving antenna

Further $\boxed{\text{population covered} = \text{population density} \times \text{area covered.}}$

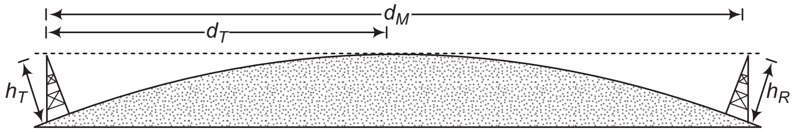


Fig. 36.4 Line of sight communication by space waves

Example 36.1 *A TV tower has a height of 60 m. What is the maximum distance and area up to which TV transmission can be received? (Take radius of earth as 6.4×10^6 m.)*

Solution (i) Distance $d = \sqrt{2Rh} = \sqrt{2 \times 6.4 \times 10^6 \times 60}$ m

$= 27.7$ km **Ans.**

(ii) Area covered $= \pi d^2 = 2\pi Rh = (2 \times 3.14 \times 6.4 \times 10^6 \times 60)\,\text{m}^2$

$= 2411\ \text{km}^2$ **Ans.**

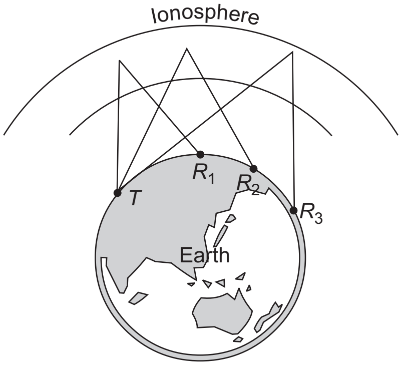


Fig. 36.5

Sky Wave Propagation or Ionospheric Propagation If one wishes to send signals at far away stations, then either repeater transmitting stations are necessary or height of the antenna is to be increased. However much before the advent of satellites, radio broadcast covered long distances by the reflection of signals from the ionosphere. This mode of transmission is called ionospheric propagation or sky wave propagation.

$T \rightarrow$ Transmitter, $R \rightarrow$ Receiver

The ionosphere extends from a height of 80 km to 300 km. The refractive index of ionosphere is less than its free space value. That is, it behaves as a rare medium. As, we go deep into the ionosphere, the refractive index keeps on decreasing. The bending of beam (away from the normal) will continue till it reaches critical angle after which it will be reflected back. The different points on earth receive signals reflected from different depths of the ionosphere. There is a critical frequency f_c (5 to 100 MHz) beyond which the waves cross the ionosphere and do not return back to earth.

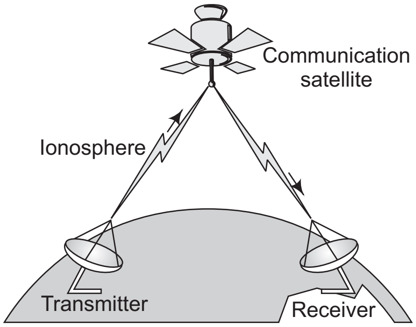


Fig. 36.6 Principle of satellite communication

Satellite Communication Long distance communication beyond 10 to 20 MHz was not possible before 1960 because all the three modes of communication discussed above failed (ground waves due to conduction losses, space wave due to limited line of sight and sky wave due to the penetration of the ionosphere by the high frequencies beyond f_c). Satellite communication made this possible.

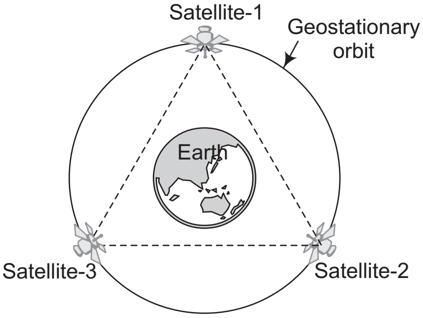


Fig. 36.7

The basic principle of satellite communication is shown in figure. A communication satellite is a spacecraft placed in an orbit around the earth. The frequencies used in satellite communication lie in UHF/microwave regions. These waves can cross the ionosphere and reach the satellite.

For steady, reliable transmission and reception it is preferred that satellite should be geostationary. A geostationary satellite is one that appears to be stationary relative to the earth. It has a circular orbit lying in the equatorial plane of the earth at an approximate height of 36,000 km. Its time period is 24 hours.

If we use three geostationary satellites placed at the vertices of an equilateral triangle as shown in figure. The entire earth can be covered by the communication network.

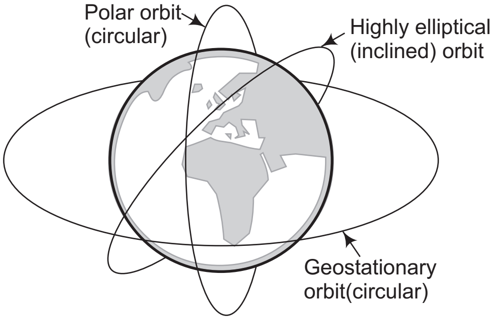

Polar orbit (circular)
Highly elliptical (inclined) orbit
Geostationary orbit(circular)

Fig. 36.8 A schematic diagram of various satellite orbits used in satellite communication

In addition to geostationary equatorial orbits, there are two more orbits which are being used for communication. These are

(a) **Polar circular orbit** This orbit passes over or very close to the poles. It is approximately at a height of 1000 km from earth.

(b) **Highly elliptical (inclined) orbits**

Remote Sensing

Remote sensing is an application of satellite communication. It is the art of obtaining information about an object or area acquired by a sensor that is not in direct contact with the target of investigation. Any photography is a kind of remote sensing. If we want to cover large areas for which information is required, we have to take photographs from larger distances. This is called aerial photography. Town and country planning can also be done by remote sensing.

A satellite equipped with appropriate sensors is used for remote sensing. Taking photograph of any object relies on the reflected wave from the object. We use visible light in normal photography. In principle, waves of any wavelength in the electromagnetic spectrum can be used for this purpose by using suitable sensors.

Some applications of remote sensing include meteorology (development of weather systems and weather forecasting), climatology (monitoring climate changes), and oceanography etc.

36.6 Modulation

In this section, we will discuss in detail about modulation. What is it ? What is the need of modulation or how the modulation is done etc.

No signal in general is a single frequency signal but it spreads over a range of frequencies called the signal bandwidth. Suppose we wish to transmit an electronic signal in the Audio Frequency (20 Hz-20 kHz) range over a long distance. Can we do it ? No it cannot because of the following problems.

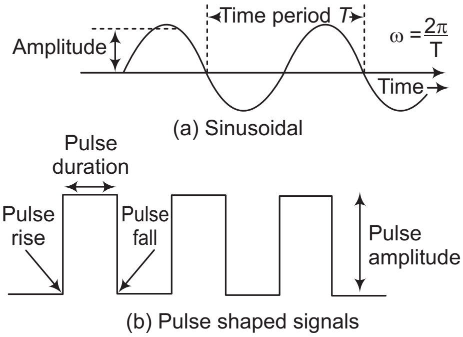
(i) **Size of antenna** For transmitting a signal we need an antenna. This antenna should have a size comparable to the wavelength of the signal. For an electromagnetic wave of frequency 20 kHz, the wavelength is 15 km. Obviously such a long antenna is not possible and hence direct transmission of such signal is not practical.

(ii) **Effective power radiated by antenna** Power radiated by an antenna $\propto \frac{1}{(\lambda)^2}$.

Therefore, power radiated by large wavelength would be small. For good transmission, we require high power and hence need of high frequency transmission is required.

(iii) **Mixing up of signal from different transmitters** Another problem in transmitting baseband signals directly is of intermixing of different signals. Suppose many people are talking at the same time or many transmitters are transmitting baseband information signals simultaneously. All these signals will get mixed and there is no simple way to distinguish between them. A possible solution to all above problems is using communication at high frequencies and allotting a band of frequencies to each message signal for its transmission.

Thus in the process of modulation the original low frequency information signal is attached with the high frequency carrier wave. The carrier wave may be continuous (sinusoidal) or in the form of pulses as shown in figure.



(a) Sinusoidal

(b) Pulse shaped signals

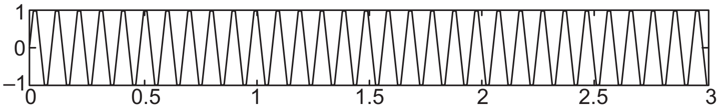
Fig. 36.9

Modulation Types

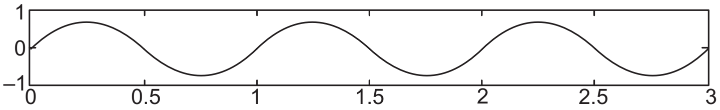
Different types of modulation depend upon the specific characteristic of the carrier wave which is being varied in accordance with the message signal.

We know that a sinusoidal carrier wave can be expressed as

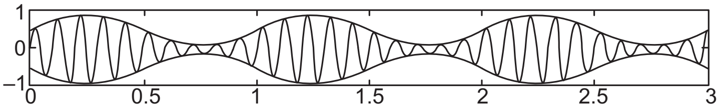
$$E = E_0 \sin(\omega t + \phi)$$



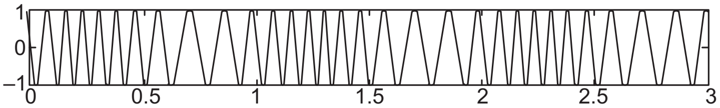
(a) A sinusoidal carrier wave



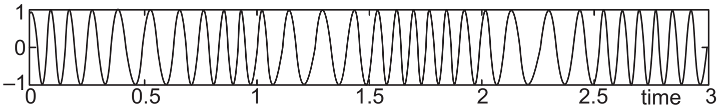
(b) A modulating signal



(c) Amplitude modulation



(d) Frequency modulation, and



(e) Phase modulation

Fig. 36.10 Modulation of a carrier wave

The three distinct characteristics are Amplitude (E_0), angular frequency (ω) and phase angle (ϕ). Either of these three characteristics can be varied in accordance with the signal. The three types of modulation are, amplitude modulation, frequency modulation and phase modulation.

Similarly, the characteristics of a pulse are, Pulse amplitude, pulse duration or pulse width and pulse position (time of rise or fall of the pulse amplitude).

Hence, different types of pulse modulation are, Pulse Amplitude Modulation (PAM), Pulse Duration Modulation (PDM) or Pulse Width Modulation (PWM) and Pulse Position Modulation (PPM).

In this chapter, we shall confine to amplitude modulation (of continuous wave or sinusoidal wave) only.

36.7 Amplitude Modulation

In this type of modulation, the amplitude of the carrier signal varies in accordance with the information signal. The high frequency carrier wave (Fig.a) is superimposed on low frequency information signal (Fig. b). As a result, in the amplitude modulated carrier wave, amplitude no longer remains constant, but its envelope has similar sinusoidal variation as that of the low frequency or modulating signal. The carrier wave frequency ranges from 0.5 to 2.0 MHz. AM signals are noisy because electrical noise signals significantly affect this.

Let $C = A_c \sin \omega_c t$ represents a carrier wave

and $S = A_s \sin \omega_s t$ represents the signal wave

Then after making calculations we see that the modulated signal wave equation can be written as

$$m = A_c \sin \omega_c t + \frac{\mu A_c}{2} \cos (\omega_c - \omega_s) t - \frac{\mu A_c}{2} \cos (\omega_c + \omega_s) t \qquad \text{...(i)}$$

where, $\mu = \frac{A_s}{A_c}$ is called the **modulation index.** In practice, μ is kept ≤ 1 to avoid distortion.

In Eq. (i), $\omega_c - \omega_s$ and $\omega_c + \omega_s$ are respectively called the lower side and upper side frequencies.

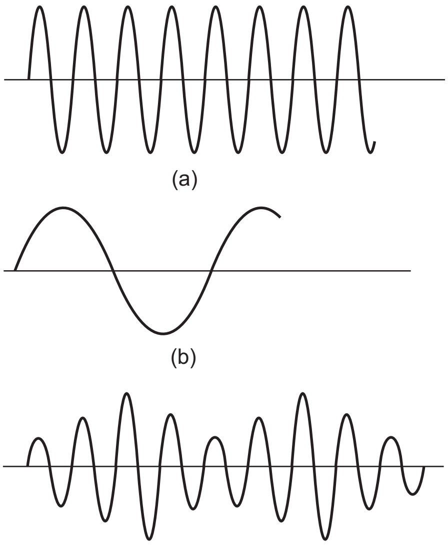


Fig. 36.11

The modulated signal therefore consists of the carrier wave of frequency ω_c plus two sinusoidal waves each with a frequency slightly different from ω_c, known as side bands.

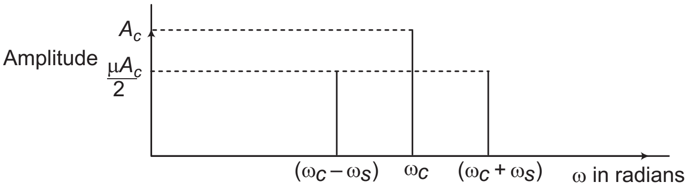


Fig. 36.12 A plot of amplitude *versus* ω for an amplitude modulated signal

Example 36.2 *A message signal of frequency 10 kHz and peak voltage of 10 V is used to modulate a carrier wave of frequency 1 MHz and peak voltage of 20 V. Determine*

(a) modulation index, (b) the side bands produced.

Solution (a) **Modulation Index** $\mu = \frac{A_s}{A_c} = \frac{10}{20} = 0.5$ **Ans.**

(b) **Side Bands** 1 MHz = 1000 kHz

The side bands are, $(\omega_c - \omega_s)$ and $(\omega_c + \omega_s)$

or we can write $(f_c - f_s)$ and $(f_c + f_s)$

$\therefore$ $(f_c - f_s) = (1000 - 10) = 990$ kHz and

$(f_c + f_s) = (1000 + 10) = 1010$ kHz **Ans.**

36.8 Production of Amplitude Modulated Wave

Amplitude modulation can be produced by a variety of methods. A simple method is shown in the block diagram of Fig.36.13.

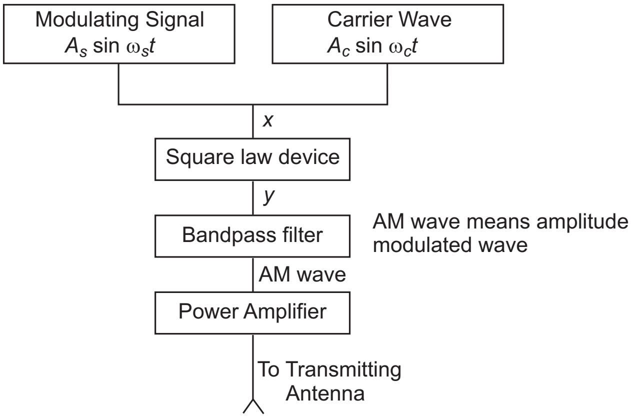


Fig. 36.13

In the y function shown in block diagram there is a DC term $\frac{C}{2}(A_c^2 + A_s^2)$ and sinusoids of frequencies $\omega_s, 2\omega_s, 2\omega_c, \omega_c - \omega_s$ and $\omega_c + \omega_s$. As shown in block diagram, this signal y is passed through a band pass filter which rejects DC and the sinusoids of frequencies $\omega_s, 2\omega_s$ and $2\omega_c$. After bandpass filter, the frequencies remaining are $\omega_c, \omega_c - \omega_s$ and $\omega_c + \omega_s$. The output (AM wave) of the band pass filter therefore is of the same form as Eq. (i) of previous article.

It is further to be noticed that this modulated signal cannot be transmitted as such. This signal is passed through a power amplifier and then the signal is fed to transmitter antenna.

36.9 Detection of Amplitude Modulated Wave

Signal received from the receiving antenna is first passed through an amplifier because the signal becomes weak in travelling from transmitting antenna to receiving antenna. For further processing, the signal is passed through intermediate frequency (IF) stage preceding the detection.

At this stage, the carrier frequency is usually changed to a lower frequency. The output signal from detector may not be strong enough. So, it is further passed through an amplifier for final use. The block diagram of all steps is shown in Fig. 36.14.

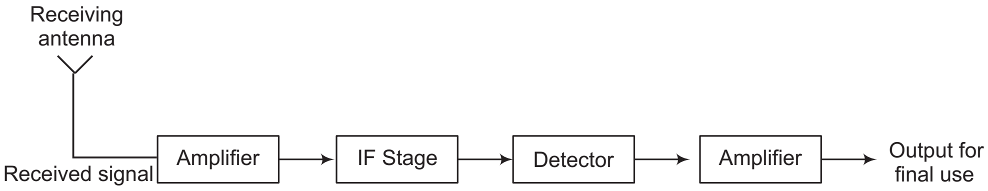
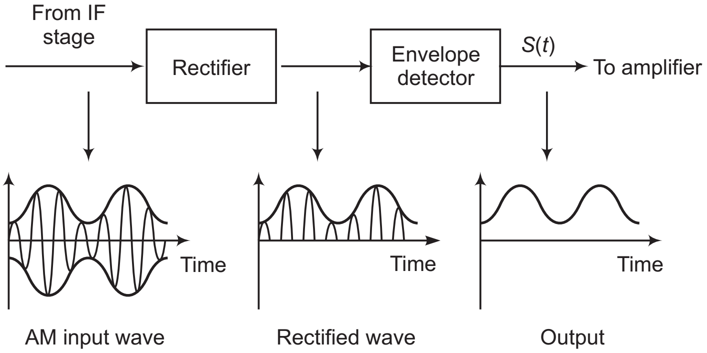


Fig. 36.14 Block diagram of a receiver

Inside the Detector

Detection is the process of recovering the signal from the carrier wave. In the previous two articles, we have seen that the modulated carrier wave contains the frequencies ω_c, $\omega_c + \omega_s$ and $\omega_c - \omega_s$. In order to obtain the original message signal $S\ (= A_s \sin \omega_s t)$ of angular frequency ω_s a simple method is shown in the form of a block diagram as shown below.



AM input wave | Rectified wave | Output

Fig. 36.15 Block diagram of a detector for AM signal

Note *that the quantity on y-axis can be current or voltage.*

Extra Points to Remember

- **The Internet** Everyone is well aware with internet. It has billions of users worldwide. It was started in 1960's and opened for public use in 1990's. Its applications include, E-mail, file transfer, website, E-commerce and chatting etc.
- **Facsimile (FAX)** It first scans the image of contents of a document. Then those are converted into electronic signals. These electronic signals are sent to another FAX machine using telephone lines. At the destination, signals are reconverted into a replica of the original document.

Solved Examples

Example 1 *Name the device fitted in the satellite which receives signals from Earth station and transmits them in different directions after amplification.*

Solution Transponder.

Example 2 *An electromagnetic wave of frequency* 28 *MHz passes through the lower atmosphere of Earth and gets incident on the ionosphere. Shall the ionosphere reflects these waves?*

Solution Yes. The ionosphere reflects back electromagnetic waves of frequency less than 30 MHz.

Example 3 *Which waves constitute amplitude-modulated band?*

Solution Electromagnetic waves of frequency less than 30 MHz constitute amplitude-modulated band.

Example 4 *Give the frequency ranges of the following (i) High frequency band (HF) (ii) Very high frequency band (VHF) (iii) Ultra high frequency band (UHF) (iv) Super high frequency band (SHF).*

Solution (i) 3 MHz to 30 MHz (ii) 30 MHz to 300 MHz (iii) 300 MHz to 3000 MHz (iv) 3000 MHz to 30,000 MHz.

Example 5 *State the two functions performed by a modem.*

Solution (i) Modulation (ii) Demodulation.

Example 6 *Why is the transmission of signals using ground waves restricted up to a frequency of* 1500 *kHz?*

Solution This is because at frequencies higher than 1500 kHz, there is an increase in the absorption of signal by the ground.

Example 7 *How does the effective power radiated by an antenna vary with wavelength?*

Solution Power radiated by an antenna $\propto \left(\frac{1}{\lambda}\right)^2$.

Example 8 *Why is it necessary to use satellites for long distance TV transmission?*

Solution Television signals are not properly reflected by the ionosphere. So, reflection is affected by satellites.

Example 9 *Why long distance radio broadcasts use shortwave bands?*

Solution This is because ionosphere reflects waves in these bands.

Example 10 *What is a channel bandwidth?*

Solution Channel bandwidth is the range of frequencies that a system can transmit with efficient fidelity.

Example 11 *Give any one difference between FAX and e-mail systems of communication.*

Solution Electronic reproduction of a document at a distant place is known as FAX. In e-mail system, message can be created, processed and stored. Such facilities are not there in Fax system.

Example 12 *Why ground wave propagation is not suitable for high frequency?*

Solution At high frequency, the absorption of the signal by the ground is appreciable. So, ground wave propagation is not suitable for high frequency.

Example 13 *What is the purpose of modulating a signal in transmission?*

Solution A low frequency signal cannot be transmitted to long distances because of many practical difficulties. On the other hand, effective transmission is possible at high frequencies. So, modulation is always done in communication systems.

Example 14 *What is a transducer?*

Solution A device which converts energy in one form to another is called a transducer.

Example 15 *Why do we need a higher bandwidth for transmission of music compared to that for commercial telephone communication?*

Solution As compared to speech signals in telephone communication, the music signals are more complex and correspond to higher frequency range.

Example 16 *From which layer of the atmosphere, radio waves are reflected back?*

Solution The electromagnetic waves of radio frequencies are reflected by ionosphere.

Example 17 *Why sky waves are not used in the transmission of television signals?*

Solution The television signals have frequencies in 100-200 MHz range. As the ionosphere cannot reflect radio waves of frequency greater than 40 MHz back to the earth, the sky waves cannot be used in the transmission of TV signals.

Example 18 *Why are short waves used in long distance broadcasts?*

Solution The short waves (wavelength less than 200 m or frequencies greater than 1,00 kHz) are absorbed by the earth due to their high frequency but are effectively reflected by Flayer in ionosphere. After reflection from the ionosphere, the short waves reach the surface of earth back only at a large distance from the transmitter. For this reason, short waves are used in long distance transmission.

Example 19 *Define the term critical frequency in relation to sky wave propagation of electromagnetic waves.*

Solution The highest value of the frequency of radio waves, which on being radiated towards the ionosphere at some angle are reflected back to the earth is called critical frequency.

Example 20 *What mode of communication is employed for transmission of TV signals?*

Solution Space wave communication.

Exercises

Single Correct Option

1. Three waves A, B and C of frequencies 1600 kHz, 5 MHz and 60 MHz, respectively are to be transmitted from one place to another. Which of the following is the most appropriate mode of communication?

(a) A is transmitted *via* space wave while B and C transmitted *via* sky wave.

(b) A is transmitted *via* ground wave, B *via* sky wave and C *via* space wave.

(c) B and C are transmitted *via* ground wave while A is transmitted via sky wave.

(d) B is transmitted *via* ground wave while A and C are transmitted *via* space wave.

2. A 100 m long antenna is mounted on a 500 m tall building. The complex can become a transmission tower for waves with λ

(a) ~ 400 m
(b) ~ 25 m
(c) ~ 150 m
(d) ~ 2400 m

3. A speech signal of 3 kHz is used to modulate a carrier signal of frequency 1 MHz, using amplitude modulation. The frequencies of the side bands will be

(a) 1.003 MHz and 2.997 MHz
(b) 3001 kHz and 2997 kHz
(c) 1003 kHz and 1000 kHz
(d) 1 MHz and 0.997 MHz

4. A message signal of frequency ω_m is superposed on a carrier wave of frequency ω_c to get an amplitude modulated wave (AM). The frequency of the AM wave will be

(a) ω_m
(b) ω_c
(c) $\frac{\omega_c + \omega_m}{2}$
(d) $\frac{\omega_c - \omega_m}{2}$

5. A basic communication system consists of

(A) transmitter (B) information source (C) user of information (D) channel (E) receiver

Choose the correct sequence in which these are arranged in a basic communication system.

(a) ABCDE
(b) BADEC
(c) BDACE
(d) BEADC

6. Which of the following frequencies will be suitable for beyond the horizon communication using sky waves?

(a) 10 kHz
(b) 10 MHz
(c) 1 GHz
(d) 1000 GHz

7. Frequencies in the UHF range normally propagate by means of

(a) ground waves
(b) sky waves
(c) surface waves
(d) space waves

8. Digital signals

(i) do not provide a continuous set of values
(ii) represent values as discrete steps
(iii) can utilize binary system and
(iv) can utilize decimal as well as binary systems

Which of the above statements are true?

(a) (i) and (ii) only
(b) (ii) and (iii) only
(c) (i), (ii) and (iii) but not (iv)
(d) All of (i), (ii), (iii) and (iv)

More than One Correct Options

9. A TV transmission tower has a height of 240 m. Signals broadcast from this tower will be received by LOS communication at a distance of (assume the radius of earth to be 6.4×10^6m)

(a) 100 km
(b) 24 km
(c) 55 km
(d) 50 km

10. An audio signal of 15 kHz frequency cannot be transmitted over long distances without modulation because

(a) the size of the required antenna would be at least 5 km which is not convenient
(b) the audio signal cannot be transmitted through sky waves
(c) the size of the required antenna would be at least 20 km, which is not convenient
(d) effective power transmitted would be very low, if the size of the antenna is less than 5 km

11. Audio sine waves of 3 kHz frequency are used to amplitude modulate a carrier signal of 1.5 MHz. Which of the following statements are true?

(a) The sideband frequencies are 1506 kHz and 1494 kHz
(b) The bandwidth required for amplitude modulation is 6 kHz
(c) The bandwidth required for amplitude modulation is 3 MHz
(d) The sideband frequencies are 1503 kHz and 1497 kHz

12. In amplitude modulation, the modulation index μ, is kept less than or equal to 1 because

(a) $\mu > 1$ will result in interference between carrier frequency and message frequency, resulting into distortion.
(b) $\mu > 1$ will result in overlapping of both sidebands resulting into loss of information.
(c) $\mu > 1$ will result in change in phase between carrier signal and message signal.
(d) $\mu > 1$ indicates amplitude of message signal greater than amplitude of carrier signal resulting into distortion.

Subjective Questions

13. Compute the *LC* product of a tuned amplifier circuit required to generate a carrier wave of 1 MHz for amplitude modulation.

14. A carrier wave of peak voltage 12 V is used to transmit a message signal. What should be the peak voltage of the modulating signal in order to have a modulation index of 75%?

15. Which of the following would produce analog signals and which would produce digital signals?

(i) A vibrating tuning fork
(ii) Musical sound due to a vibrating sitar string
(iii) Light pulse
(iv) Output of NAND gate

16. Two waves *A* and *B* of frequencies 2 MHz and 3 MHz, respectively are beamed in the same direction for communication *via* sky wave. Which one of these is likely to travel longer distance in the ionosphere before suffering total internal reflection?

17. The maximum amplitude of an AM wave is found to be 15 V while its minimum amplitude is found to be 3V. What is the modulation index?

18. Why is an AM signal likely to be more noisy than a FM signal upon transmission through a channel?

19. Is it necessary for a transmitting antenna to be at the same height as that of the receiving antenna for line of sight communication? A TV transmitting antenna is 81 m tall. How much service area can it cover, if the receiving antenna is at the ground level?

20. A TV transmission tower antenna is at a height of 20 m. How much service area can it cover if the receiving antenna is (i) at ground level, (ii) at a height of 25 m? Calculate the percentage increase in area covered in case (ii) relative to case (i).

21. If the whole earth is to be connected by LOS communication using space waves (no restriction of antenna size or tower height), what is the minimum number of antennas required? Calculate the tower height of these antennas in terms of earth's radius?

22. The maximum frequency for reflection of sky waves from a certain layer of the ionosphere is found to be $f_{\max} = 9(N_{\max})^{1/2}$, where $N_{\max}$ is the maximum electron density at that layer of the ionosphere. On a certain day it is observed that signals of frequencies higher than 5 MHz are not received by reflection from the F_1 layer of the ionosphere while signals of frequencies higher than 8 MHz are not received by reflection from the F_2 layer of the ionosphere. Estimate the maximum electron densities of the F_1 and F_2 layers on that day.

23. A 50 MHz sky wave takes 4.04 ms to reach a receiver *via* re-transmission from a satellite 600 km above earth's surface. Assuming re-transmission time by satellite negligible, find the distance between source and receiver. If communication between the two was to be done by Line of Sight (LOS) method, what should be the size of transmitting antenna?

Answers

1. (b) **2.** (a) **3.** (a) **4.** (b) **5.** (b) **6.** (b) **7.** (d)
8. (c) **9.** (b,c,d) **10.** (a,b,d) **11.** (b,d)
12. (b,d) **13.** $2.53 \times 10^{-14}\ \text{s}^2$ **14.** 9 V
15. (i) analog (ii) analog (iii) digital (iv) digital
16. 3 MHz **17.** 2/3
19. No, 3258 km^2 **20.** (i) 804 km^2 (ii) 3608 km^2 (iii) 349 %
21. Six, h = radius of earth **22.** $3.086 \times 10^{11}\ \text{m}^{-3}$, $7.9 \times 10^{11}\ \text{m}^{-3}$
23. 170 km, 565 m

29. Electromagnetic Waves

INTRODUCTORY EXERCISE 29.1

1.
$$\begin{aligned} i_d &= \varepsilon_0 \frac{d\phi_E}{dt} \\ &= \varepsilon_0 \frac{d}{dt}\left(E\frac{A}{2}\right) \\ &= \frac{\varepsilon_0 A}{2}\frac{d}{dt}(E) \\ &= \frac{\varepsilon_0 A}{2}\frac{d}{dt}\left(\frac{\sigma}{\varepsilon_0}\right) \\ &= \frac{\varepsilon_0 A}{2}\frac{d}{dt}\left(\frac{q}{A\varepsilon_0}\right) \\ &= \frac{1}{2}\frac{dq}{dt} \\ &= \frac{i}{2} \qquad \left(\because \frac{dq}{dt} = i\right) \end{aligned}$$

INTRODUCTORY EXERCISE 29.2

1. $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ = speed of light in vacuum

$\therefore$ Unit of $\dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ should be the unit of c or speed or m/s.

2. There is a change in potential difference between the plates of capacitor.

Therefore, change in electric field between its plates. Now, change in electric field produces a magnetic field.

3. $\dfrac{E_0}{B_0} = c$

$$\begin{aligned} \therefore \quad B_0 &= \frac{E_0}{c} = \frac{810}{3 \times 10^8} \\ &= 2.7 \times 10^{-6}\ \text{T} \\ &= 2.7\ \mu\text{T} \end{aligned}$$

4.
$$\begin{aligned} u_E = u_B &= \frac{1}{2}\varepsilon_0 E^2 \\ &= \frac{1}{2}\varepsilon_0\left(\frac{E_0}{\sqrt{2}}\right)^2 \\ &= \frac{1}{4}\varepsilon_0 E_0^2 \\ &= \frac{1}{4}(8.86 \times 10^{-12})(50)^2 \\ &= 5.54 \times 10^{-9}\ \text{J/m}^3 \end{aligned}$$

Total energy $= (u_E + u_B)$ (Given volume)

$$\begin{aligned} &= 2 \times 5.54 \times 10^{-9} \times (10 \times 10^{-4})(0.50) \\ &= 5.55 \times 10^{-12}\ \text{J} \end{aligned}$$

Exercises

Single Correct Option

1. 11 eV energy radiation lies in UV range.

4. In case of perfectly non-reflecting surface,

$$\Delta p = \frac{E}{c}$$

where, $E = 20 \times 30 \times 30 \times 60$

$= 1.08 \times 10^6$ J

$$\therefore \quad \Delta p = \frac{1.08 \times 10^6}{3 \times 10^8}$$

$= 36 \times 10^{-4}$ kg-m/s

More than One Correct Options

5. **E**, **B** and velocity of electromagnetic waves are mutually perpendicular.

6. Every accelerated charged particle produces electromagnetic waves.

8. $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8}{10^9} = 0.3$ m

This wavelength lies in radio waves region.

Subjective Questions

11. $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8}{30 \times 10^6} = 10$ m

12. $\lambda = \dfrac{c}{f}$

$$\lambda_1 = \frac{3 \times 10^8}{7.5 \times 10^6} = 40\ \text{m}$$

$$\lambda_2 = \frac{3 \times 10^8}{12 \times 10^6} = 25\ \text{m}$$

13. $c = \frac{E_0}{B_0}$

$$E_0 = cB_0$$
$$= (3 \times 10^8)(510 \times 10^{-9})$$
$$= 153 \text{ V/m}$$

14. (a) $C = \frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0(\pi R^2)}{d}$

$$= \frac{(8.86 \times 10^{-12})(3.14)(0.12)^2}{5 \times 10^{-2}}$$
$$= 8 \times 10^{-12}\text{F} = 8 \text{ pF}$$
$$V = \frac{q}{C}$$
$$\therefore \quad \frac{dV}{dt} = \frac{1}{C}\left(\frac{dq}{dt}\right) = \frac{i}{C}$$
$$= \frac{0.15}{8 \times 10^{-12}}$$
$$= 1.875 \times 10^{10} \text{ V/s}$$

(b) $i_d = i_c = 0.15$ A

15. (a) $c = \frac{E_0}{B_0}$

$$\therefore \quad B_0 = \frac{E_0}{c}$$
$$\omega = 2\pi f$$
$$c = v = \frac{\omega}{k}$$
$$\therefore \quad k = \frac{\omega}{c}$$
$$\lambda = \frac{2\pi}{k}$$

16. $X_C = \frac{1}{\omega C}$

With decrease in frequency, X_C will increase, so current will decrease.

Now,

$$i_d = i_c$$

i_c has decreased, so i_d will also decrease.

17.

$$I_E = \frac{I}{2} = \frac{1}{2}\varepsilon_0 E^2 c$$
$$= \frac{1}{2}\varepsilon_0\left(\frac{\varepsilon_0}{\sqrt{2}}\right)^2 c$$
$$\therefore \quad E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}}$$
$$= \sqrt{\frac{2 \times 2.5 \times 10^{14}}{8.86 \times 10^{-12} \times 3 \times 10^8}}$$
$$= 4.3 \times 10^8 \text{ N/C}$$

Now, $c = \frac{E_0}{B_0}$

or $B_0 = \frac{E_0}{c}$

18. (a) $\lambda = \frac{c}{f}$

(b) $c = \frac{E_0}{B_0}$

or $B_0 = \frac{E_0}{c}$

(c) $u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{E_0}{\sqrt{2}}\right)^2 = \frac{1}{4}\varepsilon_0 E_0^2$

Similarly, $u_B = \frac{B_0^2}{4\mu_0}$

After substituting the values we get,

$$u_E = u_B$$

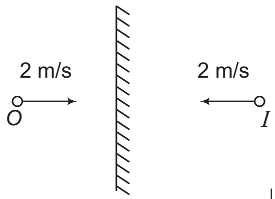
19. $i_d = i_c = \frac{dq}{dt}$

$$= -2\pi q_0 f \sin(2\pi f t)$$

30. Reflection of Light

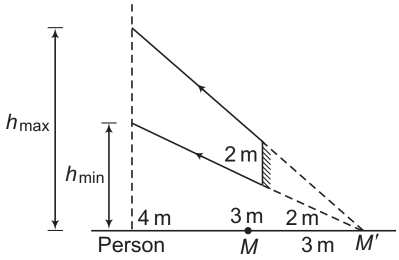
INTRODUCTORY EXERCISE 30.1

1.



$$|\mathbf{v}_{OI}| = 4\ \text{m/s}$$

2.



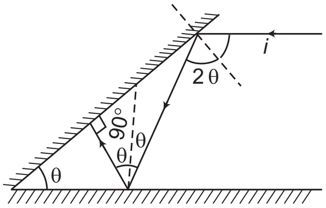
$$\frac{h_{\text{min}}}{10} = \frac{2}{3}$$

$$\therefore \qquad h_{\text{min}} = \frac{20}{3}\ \text{m}$$

$$\frac{h_{\text{max}}}{10} = \frac{4}{3}$$

$$\therefore \qquad h_{\text{max}} = \frac{40}{3}\ \text{m}$$

3.



$i = 2\theta$

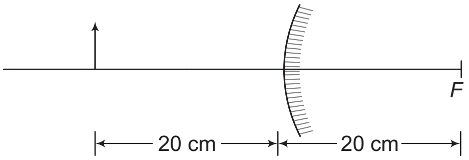
Exercises

LEVEL 1

Assertion and Reason

1. Convex mirror can make real images of virtual objects.

2.

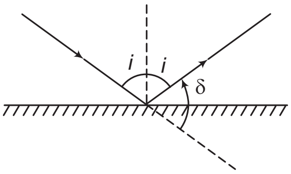


Real object is in front of mirror, not at F. So, image is not at infinity.

4. Field of view of convex mirror is large.

5. $m = -2$, means image is real, inverted an 2-times magnified.

6.



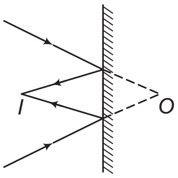
$$\delta = 180° - 2i$$

For normal incidence, $i = 0°$

$$\therefore \qquad \delta = 180°$$

Objective Questions

1. Image is real and incident beam is convergent.

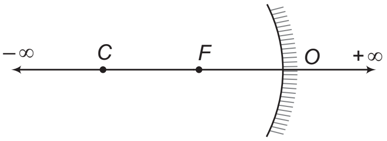


4. $$\frac{1}{v} + \frac{1}{u} = \frac{1}{f} \qquad \text{...(i)}$$

For virtual object u is positive and for concave mirror f is negative.

Substituting these signs in Eq. (i), we can see that v is always negative or image is always real.

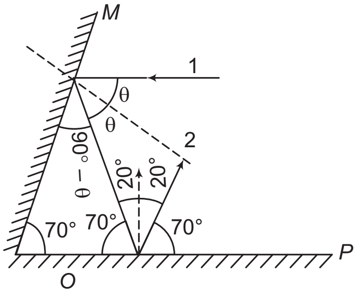
5.



S.No.	*u*	*v*
1.	From O to F or from O to $-f$	From O to $+\infty$
2.	From F to C from $-f$ to $-2f$	From $-\infty$ to C or from $-\infty$ to $-2f$
3.	From C to $-\infty$ or from $-2f$ to $-\infty$	From C to F or from $-2f$ to $-f$

In the above example, we have seen that for virtual object ($u = +$ ve), image is always real ($v = -$ ve) corresponding graph is shown in Fig. (b).

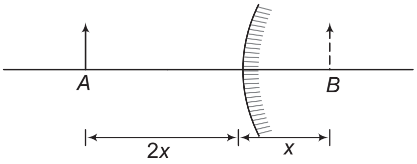
6.



$$90° - \theta + 70° + 70° = 180°$$

$$\therefore \qquad \theta = 50°$$

7.



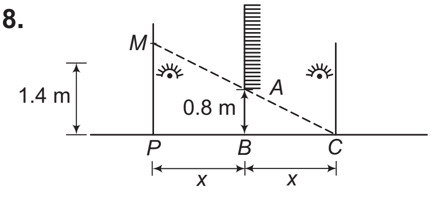
$$f = \frac{R}{2} = 30 \text{ cm}$$

$$\frac{1}{+x} + \frac{1}{-2x} = \frac{1}{+30}$$

$$\therefore \qquad x = 15 \text{ cm}$$

Hence, $AB = 3x = 45$ cm

8.



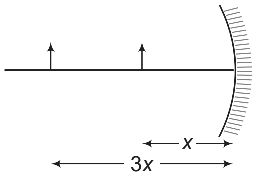
$$\frac{MP}{PC} = \frac{AB}{BC}$$

$$\therefore \qquad \frac{MP}{2x} = \frac{0.8}{x}$$

or $MP = 1.6 \text{ m} > 1.4 \text{ m}$

Hence, the boy cannot see his feet.

9. Magnified image is formed only by concave mirror. But this image may be real or virtual.



For real image,

$$2x = 80 \text{ cm}$$

$$\therefore \qquad x = 40 \text{ cm}$$

$$\frac{1}{-120} + \frac{1}{-40} = \frac{1}{f}$$

Solving, we get $f = -30$ cm

Similarly, we can check for virtual image.

10. Let, $u = -x$

Then, $v = +\frac{x}{n}$

$$\frac{1}{(+x/n)} + \frac{1}{-x} = \frac{1}{+f}$$

Solving, we get $x = (n-1)f$

11. $\frac{1}{v} + \frac{1}{-60} = \frac{1}{-24}$

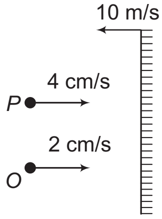
$$\therefore \qquad v = -40 \text{ cm}$$

$$m = -\frac{v}{u} = -\frac{(40)}{(-60)} = \frac{-2}{3}$$

Image speed is m^2-times the object speed and opposite to the direction of object velocity.

12. For real objects image formed by a convex mirror is always virtual, erect and diminished.

13.



$v_{PM} = 14$ cm/s towards right

$v_{IM} = 14$ cm/s towards left

So, actual speed of image = 24 cm/s towards left

$\therefore \quad v_{IO} = 26$ cm/s towards left.

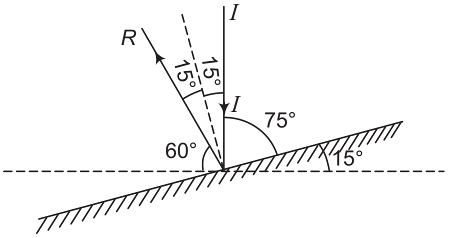
Subjective Questions

1. Reflected rays are neither converging nor diverging. Hence, mirror is a plane.

2. Image distance from plane mirror = object distance.

Lateral magnifications = 1

3.



where, I = incident ray

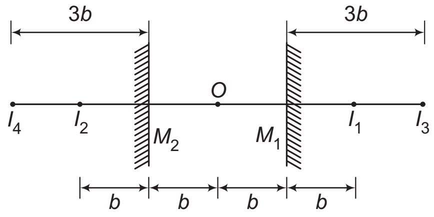
R = reflected ray

Angle of incidence = 15°

Angle between reflected ray and horizontal = 60°

4. Image from one mirror will behave like object for other mirror.

5.



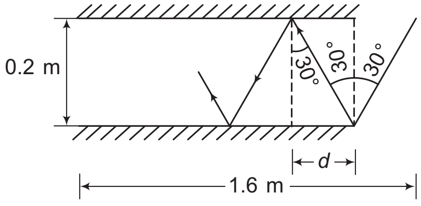
$$OI_1 = OI_2 = 2b$$

I_3 is the image of I_2 from mirror M_1 similarly I_4 is the image of I_1 from mirror M_2.

$$OI_3 = OI_4 = 4b$$

6. Given in the theory.

7.



$$\frac{d}{0.2} = \tan 30° = \frac{1}{\sqrt{3}}$$

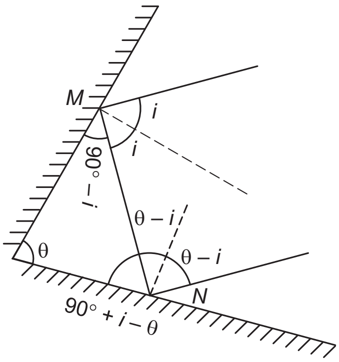
$$\therefore \qquad d = \frac{0.2}{\sqrt{3}}$$

$$N = \frac{1.6}{d} = 8\sqrt{3}$$

$$= 13.85$$

Therefore, actual number of reflections required are 14.

8.



$$\delta_{\text{Total}} = \delta_M + \delta_N$$

$$= (180° - 2i) + [180° - 2\,(\theta - i)] = 360° - 2\theta$$

9. Apply $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

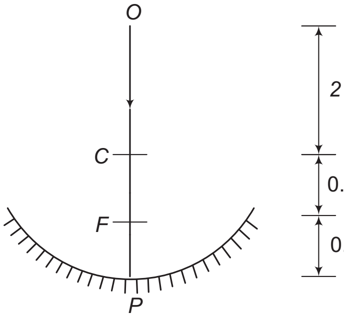
(a) $f = -10$ cm, $u = -25$ cm

Solving, we get $v = -16.7$ cm

Since, v is negative, image is in front of mirror. So, it is real. Similarly, we can solve for other parts.

10. (a) From O to F, image is real.

From F to P, image is virtual.



At *O* $\qquad \frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\therefore \qquad \frac{1}{v} + \frac{1}{-3.0} = \frac{1}{-0.5}$$

$$v = -0.6 \text{ m}$$

(b) When object is at C and P, image coincides with object.

Using

$$s = \frac{1}{2} g t^2$$

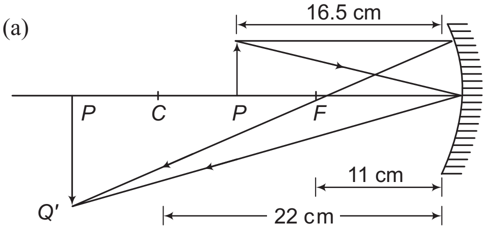
or

$$t = \sqrt{\frac{2s}{g}}$$

At *C* $t=\sqrt{\frac{2\times 2}{9.8}}=0.639$ s

At *P* $t=\sqrt{\frac{2\times 3}{9.8}}=0.782$ s

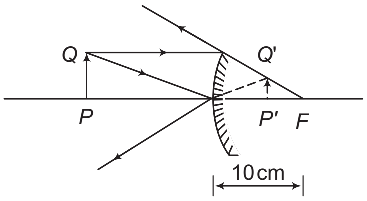
11. (a)



(b) Apply $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

and $m=-\frac{v}{u}$

12. (a)



(b) Apply, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

and $m=-\frac{v}{u}$

13. $f=\frac{R}{2}=-18$ cm

Let $u=-x$ cm

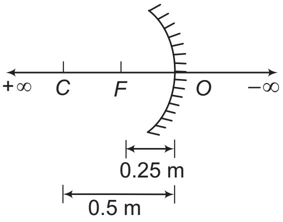
Then, $v=-\frac{x}{9}$ cm for real image of $\frac{1}{9}$th size.

Using, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

We have, $\frac{1}{(-x/9)}+\frac{1}{(-x)}=\frac{1}{-18}$

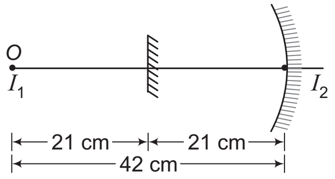
Solving we get, $x=180$ cm

14.



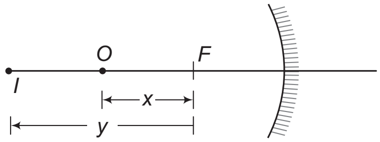
Object	Image
From O to F or from O to + 0.25 m	From O to $-\infty$
From F to C or from + 0.25 m to + 0.50 m	From $-\infty$ to C or from $+\infty$ to + 0.50 m
From C to $+\infty$ or from + 0.50 cm to $+\infty$	From C to F or from + 0.50 m to + 0.25 m

15. O is placed at centre of curvature of concave mirror (= 42 cm). Therefore, image from this mirror I_1 will coincide with object O.



Now, plane mirror will make its image I_2 at the same distance from itself.

16.



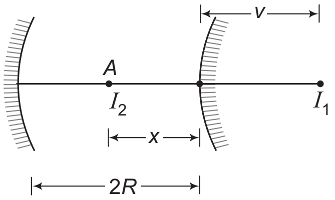
Using $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$\therefore$ $\frac{1}{-(y+f)}+\frac{1}{-(x+f)}=\frac{1}{-f}$

Solving this equation, we get

$$xy=f^2$$

17.



For convex mirror

$$\frac{1}{v}+\frac{1}{-x}=\frac{1}{+R/2}$$

$\therefore$ $\frac{1}{v}=\frac{2}{R}+\frac{1}{x}$ or $v=\frac{Rx}{R+2x}$

Now, applying mirror formula for concave mirror we have

$$\frac{1}{-(2R-x)}+\frac{1}{-(2R+v)}=\frac{1}{-R/2}$$

Solving this equation, we can find value of x.

18. (a) Image has to be taken on a screen. So, it should be real. Hence, mirror should be concave.

(b)

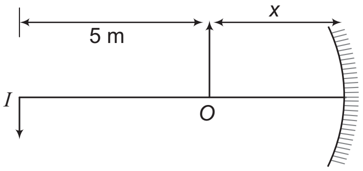


Image is 5 times magnified.

Hence, $|v| = 5|u|$

or $(5 + x) = 5x$

Solving we get,

$$x = 1.25 \text{ cm}$$

Now using,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$$

We have,

$$\frac{1}{-6.25} - \frac{1}{1.25} = \frac{2}{R}$$

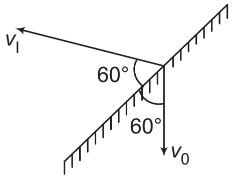
Solving we get,

$$R = -2.08 \text{ m}$$

LEVEL 2

Single Correct Option

1.

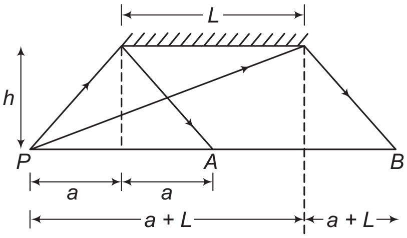


$$v_0 = v_I = \omega A = \left(\sqrt{\frac{k}{M}}\right) A$$

$$v_r = \sqrt{v_0^2 + v_I^2 - 2v_0 \cos 120°}$$

$$= \sqrt{3}\, A \sqrt{\frac{k}{M}}$$

2.



$$AB = PB - PA = 2(a + L) - 2a$$

$$= 2L = \text{ constant}$$

3. $m = -\dfrac{v}{u} = -\dfrac{(-20)}{(-10)} = -2$

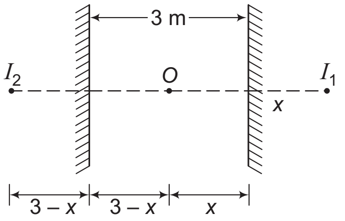
$$|\Delta v| = m^2 |\Delta u|$$

$$= (-2)^2 (0.1 \text{ cm})$$

$$= 0.4 \text{ cm}$$

Object and image travel in opposite directions (along the axis).

4.

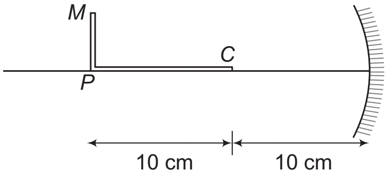


$$I_1 I_2 = (3 - x) + (3) + x$$

$$= 4 \text{ m}$$

Solving we get, $x = 1$ m

5. Image of c will coincide with this.



For P :

$$\frac{1}{v} + \frac{1}{-20} = \frac{1}{-5}$$

$\therefore$ $v = -\dfrac{20}{3}$ cm

$$m = -\frac{v}{u} = -\frac{(-20/3)}{(-20)} = -\frac{1}{3}$$

Length of image of PM :

$$I_1 = 10 \times \frac{1}{3} = \frac{10}{3} \text{ cm}$$

Length of image of PC :

$$I_2 = 10 - \frac{20}{3} = \frac{10}{3} \text{ cm}$$

$\therefore$ $\dfrac{I_1}{I_2} = 1$

6. $\dfrac{1}{v} + \dfrac{1}{-15} = \dfrac{1}{-10}$

$\therefore$ $v = -30$ cm

$$m = -\frac{v}{u} = -\frac{(-30)}{(-15)} = -2$$

$$|\Delta v| = m^2 |\Delta u| = (-2)^2 \text{ (2 mm)}$$
$$= 8 \text{ mm}$$

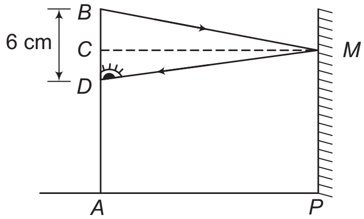
7. Let i is the angle between incident ray and original reflected ray. Then, initial angle between them will be $180° - 2i$. When mirror is rotated by 20°, then reflected ray will rotate by 40°.

$\therefore \quad 180° - 2i \pm 40° = 45°$

Solving we get,

$$i = 47.5° \text{ or } 87.5°$$

8.



$$PM = AC = AD + DC$$
$$= AD + \frac{DB}{2} = 164 + \frac{6}{2} = 167 \text{ cm}$$

9. $a_{\text{LHS}} = a_1 = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{3\,mg}{4\,m} = \dfrac{3}{4}g$

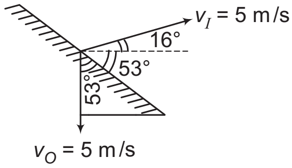
$$a_{\text{RHS}} = a_2$$
$$= \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{2\,mg}{3\,m}$$
$$= \frac{2}{3}g$$

The relative acceleration is therefore :

$$\frac{3}{4}g + \frac{2}{3}g = \frac{17}{12}g$$

10. See the hint of Q.No-7 of subjective questions for Level 1.

11.



$$\mathbf{v}_I = (5\cos 16°)\,\hat{\mathbf{i}} + (5\sin 16°)\,\hat{\mathbf{j}}$$
$$= (4.8\,\hat{\mathbf{i}} + 1.4\,\hat{\mathbf{j}}) \text{ m/s}$$

12. $u_y = \sqrt{2}\sin 45° = 1 \text{ m/s}$

In vertical direction,

$$s_1 = \text{displacement of particle}$$
$$= (1)(0.5) - \frac{1}{2}gt^2 = 0.5 - \frac{1}{2}gt^2$$

$$s_2 = \text{displacement of mirror}$$
$$= -\frac{1}{2}gt^2$$

$\therefore$ Vertical distance of particle from mirror,

$$s = s_1 - s_2 = 0.5 \text{ m}$$

Hence, distance between particle and its image

$$= 25 = 1 \text{ m}$$

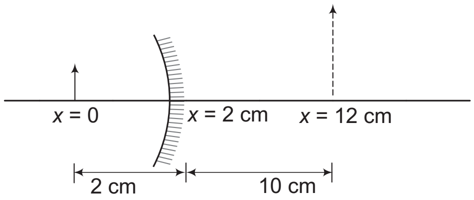
13. $\mathbf{v}_{OM} = \mathbf{v}_O - \mathbf{v}_M = (-\hat{\mathbf{i}} - 3\hat{\mathbf{k}})$

$\therefore \quad \mathbf{v}_{IM} = (-\hat{\mathbf{i}} + 3\hat{\mathbf{k}}) = \mathbf{v}_I - \mathbf{v}_M$

or $\quad \mathbf{v}_I = (-\hat{\mathbf{i}} + 3\hat{\mathbf{k}}) + \mathbf{v}_M$

$$= (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 11\hat{\mathbf{k}})$$

14.



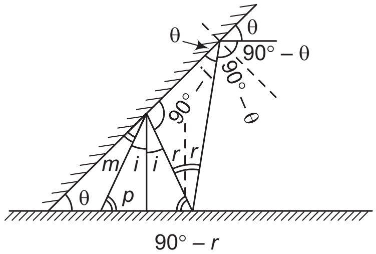
$$m = +5$$

$\therefore$ v should be positive and 5 times $|u|$.

15. $90° - i\theta + 2r = 180°$

$$\therefore \quad r = \left(\frac{90° + i - \theta}{2}\right)$$

$$p = 180° - (90° - r) - 2i$$
$$= 180° - 90° + \left(\frac{90° + i - \theta}{2}\right) - 2i$$



$$= \frac{180° - 4i + 90° + i - \theta}{2}$$
$$= \left(\frac{270° - 3i - \theta}{2}\right)$$

$$m = p - \theta = 180° - (90° - i) - 2i$$

$$\therefore \quad \left(\frac{270° - 3i - \theta}{2}\right) - \theta = 90° - i$$

Substituting $\theta = 20°$, we get

$$i = 30°$$

16. Ray passing through c is only correct.

More than One Correct Options

1. For real image,

Let $u = -x$, then $v = -2x$

$$\therefore \qquad \frac{1}{-2x} + \frac{1}{-x} = \frac{1}{-20}$$

Solving, we get $x = 30$ cm.

For virtual image,

Let $u = -x$, then $v = +2x$

$$\therefore \qquad \frac{1}{2x} + \frac{1}{-x} = \frac{1}{-20}$$

or $\qquad x = 10$ cm

2. Inverted and real image is formed by concave mirror.

Let $u = -x$, then $v = +x/2$

$$\therefore \qquad \frac{1}{+x/2} + \frac{1}{-x} = \frac{1}{+f}$$

$$\therefore \qquad x = 3f$$

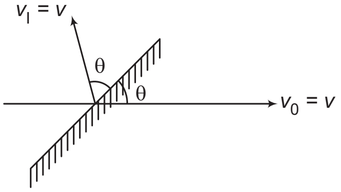
Erect and virtual image is formed by convex mirror.

Let $\quad u = -x$, $\quad$ then $\quad v = +\dfrac{x}{2}$

$$\frac{1}{-x/2} + \frac{1}{-x} = \frac{1}{+f}$$

$$x = +f$$

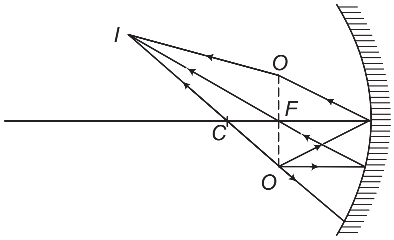
3.



$$v_r = \sqrt{v^2 + v^2 - 2v \cdot v \cos 2\theta}$$

$$= 2v \sin \theta$$

4. Ray diagram is as shown in figure.



5. $\dfrac{1}{v} + \dfrac{1}{-30} = \dfrac{1}{-20}$

Solving we get, $\quad v = -60$ cm

$$m = \frac{v}{u} = -\frac{(-60)}{(-30)} = -2$$

Speed of image (in event -1) is m^2 times and m times in event-2.

6. **In event -1**

For concave mirror,

$$\frac{1}{v} + \frac{1}{-3f} = \frac{1}{-f}$$

$$\therefore \qquad v = -1.5f$$

For convex mirror,

$$\frac{1}{v} + \frac{1}{-3f} = \frac{1}{+f}$$

$$v = \frac{3}{4}f = 0.75f$$

For plane mirror,

$$v = +3f$$

In event-2

For concave mirror,

$$\frac{1}{v} + \frac{1}{-1.5f} = \frac{1}{-f}$$

$$\therefore \qquad v = -3f$$

For convex mirror,

$$\frac{1}{v} + \frac{1}{-1.5f} = \frac{1}{+f}$$

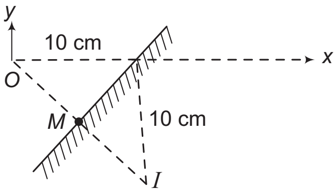
$$\therefore \qquad v = \frac{3f}{5} = 0.6f$$

For plane mirror,

$$v = +1.5f$$

Comprehension Based Questions

1.



$$OM = MI$$

Coordinates of I are (10 cm, -10 cm).

2. Object is placed at centre at curvature of mirror. Hence, image is at the same point, real, inverted and of same size. Hence, coordinates are (20 cm, 0).

3. $\dfrac{1}{v} + \dfrac{1}{-20} = \dfrac{1}{+10}$

$$\therefore \qquad v = +\frac{20}{3} \text{ cm}$$

$$m = -\frac{v}{u} = -\frac{(+20/3)}{(-20)} = +\frac{1}{3}$$

$$\therefore \quad I = 10\left(\frac{1}{3}\right) = \frac{10}{3}\text{ cm}$$

$$x = 10 - \frac{10}{3} = \frac{20}{3}\text{ cm}$$

and $$y = 20 + \frac{20}{3} = \frac{80}{3}\text{ cm}$$

4. Plane mirror forms image at equal distance on opposite sides.

Hence, $x = 0,\ y = 40$ cm

Match the Columns

1. (a) $m = -2$ means image is real, inverted and 2-times magnified. So, mirror should be concave. Same logic can be given for other options also.

2. $\frac{1}{v} + \frac{1}{+u} = \frac{1}{f}$ (as u is positive for virtual objects)

$$\therefore \quad \frac{1}{v} = \frac{1}{f} - \frac{1}{u}$$

For plane mirror, $f = \infty$. So, v is always negative. Hence, image is always real.

For concave mirror, f is negative. So, v is again negative. Therefore, image is always real.

For convex mirror, f is positive. So, v may be positive or negative.

Hence, image may be virtual or real.

3. (a) Image is inverted, real and diminished. Hence, mirror is concave.

Same logic can be applied for other options too.

4. (a) $\frac{1}{v} + \frac{1}{-20} = \frac{1}{-20}$

$$\therefore \quad v = \infty$$

$$m = -\frac{v}{u} = \infty$$

Same formulae can be applied for other options too.

5. (a) See the hint of Q.No-1 of more than one correct options section.

(b) Half size image is formed only in case of real image.

Let $u = -x$, then $v = -\frac{x}{2}$

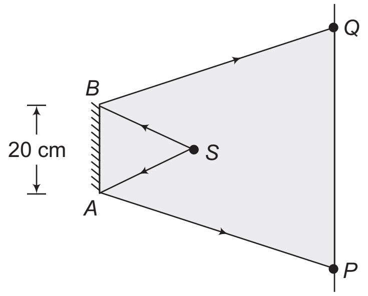
Now, $$\frac{1}{-x/2} + \frac{1}{-x} = \frac{1}{-20}$$

$$\therefore \quad x = 60\text{ cm}$$

In the similar manner, other options can be solved.

Subjective Questions

1.



Insect can see the image of source S in the mirror, so far as it remains in field of view of image overlapping with the road.

Shaded portion is the field of view, which overlaps with the road upto length PQ.

By geometry we can see that, $PQ = 3AB = 60$ cm

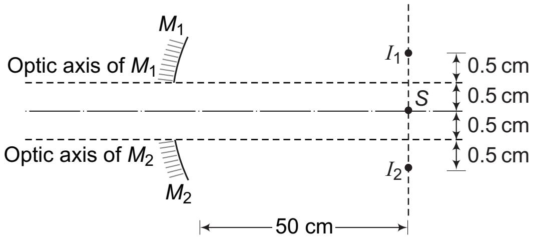
$$\therefore \quad t = \frac{\text{Distance}}{\text{Speed}} = \frac{60}{10} = 6\text{ s}$$ **Ans.**

2. Using mirror formula, $\left(\frac{1}{v} + \frac{1}{u} = \frac{1}{f}\right)$

$$\frac{1}{v} - \frac{1}{50} = \frac{-1}{25}$$

$$\therefore \quad v = -50\text{ cm}$$

$$m = -\frac{v}{u} = -1$$ **Ans.**

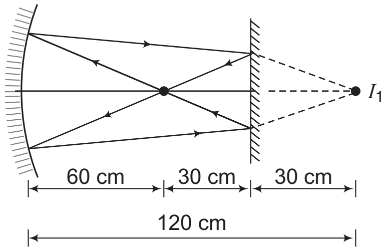


3. Using mirror formula $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ for concave mirror first, we have

$$\frac{1}{v} - \frac{1}{60} = \frac{1}{-40} \qquad \left(\because f = \frac{R}{2}\right)$$

or $$v = -120\text{ cm}$$

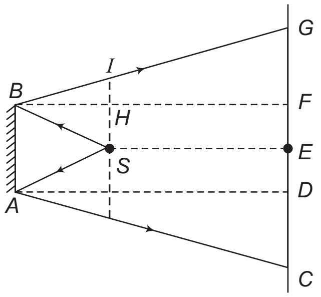
First image I_1 at 120 cm from concave mirror will act as virtual object for plane mirror. Plane mirror will form real image of I_1 at S.



Ray diagram is shown in figure.

Distance between two mirrors is 90 cm. **Ans.**

4.



$$\frac{FG}{BF} = \frac{IH}{BH} = \frac{HS}{BH}$$

$$\therefore \quad FG = (BF)\left(\frac{HS}{BH}\right)$$

$$= (5)\left(\frac{1.0}{0.5}\right)$$

$$= 10 \text{ m}$$

$$FC = 2 + 10 = 12 \text{ m}$$

The boy has dropped himself at point F. So, his velocity is 20 m/s in upward direction.

Let us first find the time to move from F to topmost point and then from topmost point to point C. From $s = ut + \frac{1}{2}at^2$, we have

$$-12 = (20t) + \frac{1}{2}(-10)\,t^2$$

Solving this equation we get, $t_1 = 4.53$ s.

Velocity of boy at point G,

$$v = \sqrt{(20)^2 - 2 \times 10 \times 10}$$

$$= 14.14 \text{ m/s} \qquad (\because\ v^2 = u^2 - 2gh)$$

Time taken to move the boy from G to topmost point and then from topmost point to G will be

$$t_2 = \frac{2v}{g} = 2.83 \text{ s}$$

$\therefore$ The required time is

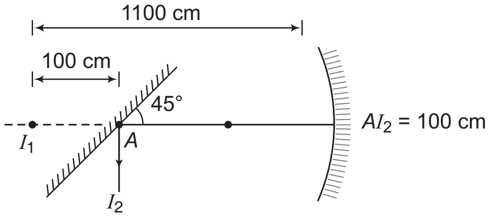
$$t = t_1 - t_2 = 1.7 \text{ s}$$ **Ans.**

5. Applying mirror formula for concave mirror first $\left(\frac{1}{v} + \frac{1}{u} = \frac{1}{f}\right)$ we have,

$$\frac{1}{v} - \frac{1}{110} = \frac{1}{-100}$$

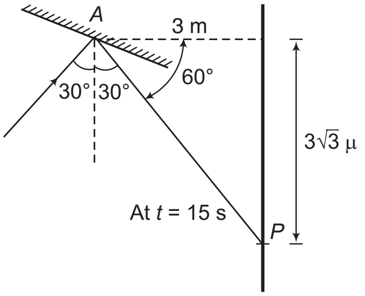
$$v = -1100 \text{ cm}$$

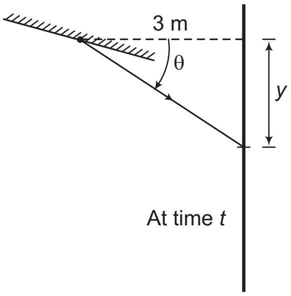
$AI_1 = 100$ cm. Therefore, final image will be real and at distance 100 cm below point A at I_2.



6. In 15 seconds, mirror will rotate 15° in clockwise direction.

Hence, the reflected ray will rotate 30° in clockwise direction.





$$y = 3\tan\theta$$

$$\therefore \quad \frac{dy}{dt} = (3\sec^2\theta).\frac{d\theta}{dt} \qquad \ldots\text{(i)}$$

Here, $\frac{dy}{dt} = v_P,\ \frac{d\theta}{dt} = 2^\circ$ per second

$$= \frac{2\times\pi}{180} = \frac{\pi}{90} \text{ rad per second}$$

At $t = 15$ s and $\theta = 60^\circ$

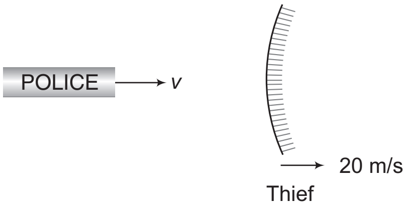
Substituting the values in Eq. (i), we have

$$v_P = \{3\sec^2 60^\circ\}\left\{\frac{\pi}{90}\right\}$$

$$= 3\times 4\times\frac{\pi}{90}$$

$$= \frac{2\pi}{15} \text{ m/s}$$ **Ans.**

7. (a) Differentiating the mirror formula, (with respect to time)



$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We get velocity of image,

$$v_I = (m)^2 v_O \qquad \ldots\text{(i)}$$

Here, v_O = relative velocity of object with respect to mirror

v_I = relative velocity of image

m = linear magnification

Here, $v_O = (v - 20)$ m/s

$v_I = 1$ cm/s $= 0.01$ m/s

$$m = \frac{1}{10}$$

Substituting in Eq. (i) we have,

$$0.01 = \left(\frac{1}{10}\right)^2 (v - 20)$$

$$\therefore \quad v = 21 \text{ m/s}$$

(b) $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

Multiplying with u we get,

$$\frac{u}{v} + 1 = \frac{u}{f}$$

or $$\frac{1}{m} = \frac{u}{f} - 1 = \frac{u - f}{f}$$

$$\therefore \quad m = \frac{f}{u - f}$$

Differentiating we have,

$$\left(\frac{dm}{dt}\right) = -\frac{f}{(u - f)^2}.\frac{du}{dt} \qquad \ldots\text{(ii)}$$

Using mirror formula to find u with magnification

$m = \frac{1}{10}$ we get,

$$\frac{1}{u/10} - \frac{1}{u} = \frac{1}{10}$$

or $u = 90$ m (with sign $u = -90$ m)

Substituting in Eq. (ii) we have,

$$\frac{dm}{dt} = -\frac{(10)}{(-90 - 10)^2}(-1)$$

$$= 10^{-3} \text{ per second}$$ **Ans.**

Note *u is decreasing at a rate of v – 20 or (21 – 20) or 1 m/s*

$$\therefore \quad \frac{du}{dt} = -1 \text{ m/s}$$

8. (a) At $t = t$,

$$u = -(2f + x)$$

$$= -(2f + f\cos\omega t)$$

Using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have,

$$\frac{1}{v} - \frac{1}{2f + f\cos\omega t} = \frac{-1}{f}$$

$$\therefore \quad v = -\left(\frac{2 + \cos\omega t}{1 + \cos\omega t}\right)f$$

i.e. distance of image from mirror at time t is

$$\left(\frac{2 + \cos\omega t}{1 + \cos\omega t}\right)f$$ **Ans.**

(b) Ball coincides with its image at centre of curvature, i.e. at $x = 0$.

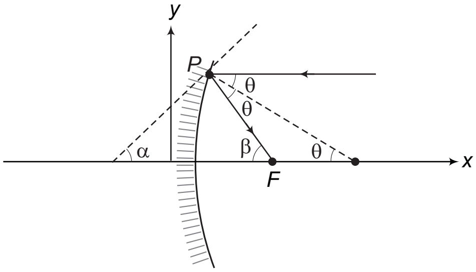
(c) At $t = T/2$

or $\omega t = \pi,\ x = -f$

i.e. $u = -f$ or ball is at focus. So, its image is at ∞.

$$m = \infty$$

9. Let the ray is incident at a point $P = (x_1, y_1)$ on the mirror. Then, slope at P,



$$\tan\alpha = \left(\frac{dy}{dx}\right)_{(x_1 y_1)} = \frac{1}{4by_1} \qquad \text{...(i)}$$

$$\alpha = 90° - \theta$$

and $\beta = 2\theta$

Now, the reflected ray is passing through $P(x_1 y_1)$ and has a slope $-\tan\beta$. Hence, the equation will be

$$\left(\frac{y-y_1}{x-x_1}\right) = -\tan\beta$$

$$= -\tan 2\theta = \frac{-2\tan\theta}{1-\tan^2\theta}$$

$$\therefore \quad y - y_1 = \frac{2\cot\alpha}{1+\cot^2\alpha}(x_1 - x) \qquad \text{...(ii)}$$

Further, $x_1 = 4by_1^2$ ...(iii)

At F, $x = 0$ ...(iv)

From Eq. (i) to Eq. (iv),

we get $x = \dfrac{1}{8b}$

It shows that the coordinates of F are unique $\left(\dfrac{1}{8b}, 0\right)$.

Hence, the reflected ray passing through one focus and the focal length is $\dfrac{1}{8b}$. **Hence proved.**

31. Refraction of Light

INTRODUCTORY EXERCISE 31.1

1. ${}_1\mu_2 \times {}_2\mu_3 \times {}_3\mu_1 = 1$

$$\therefore \quad \frac{4}{3} \times \frac{3}{2} = \frac{1}{{}_3\mu_1} = {}_1\mu_3 \quad \text{or} \quad {}_1\mu_3 = 2$$

3. $\mu = \dfrac{c}{v} = \dfrac{c}{f\lambda} = \dfrac{3 \times 10^8}{6 \times 10^{14} \times 300 \times 10^{-9}} = 1.67$

INTRODUCTORY EXERCISE 31.2

1. ${}_1\mu_2 = \dfrac{\mu_2}{\mu_1} = \dfrac{\sin i_1}{\sin i_2} = \dfrac{\sin 60°}{\sin 30°} = \sqrt{3}$

2. ${}_1\mu_2 = 1.5 = \dfrac{\lambda_1}{\lambda_2}$

INTRODUCTORY EXERCISE 31.3

1. (a) $d_{\text{app}} = 10 + \dfrac{10}{1.5} = \dfrac{50}{3}$ cm

(b) $h_{\text{app}} = 10 + (1.5)(10) = 25$ cm

INTRODUCTORY EXERCISE 31.4

1. Total shift $= \left(1 - \dfrac{1}{\mu_1}\right) t_1 + \left(1 - \dfrac{1}{\mu_2}\right) t_2$

$$= \left(1 - \frac{2}{3}\right)10 + \left(1 - \frac{1}{2}\right)10$$

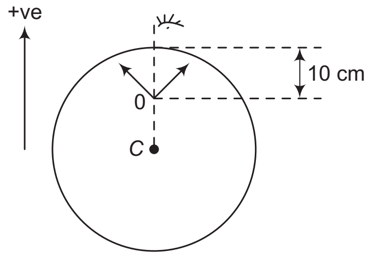
$$= \frac{25}{3} \text{ cm}$$

$\therefore$ Image distance $= 50 - \dfrac{25}{3} = \dfrac{125}{3}$ cm

INTRODUCTORY EXERCISE 31.5

1. All rays starting from centre pass undeviated as they fall normal to the surface.

2.

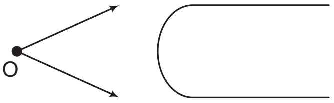


Using $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$, we get

$$\frac{1}{v} - \frac{1.5}{-10} = \frac{1 - 1.5}{-15}$$

Solving, we get $v = -8.57$ cm

3.



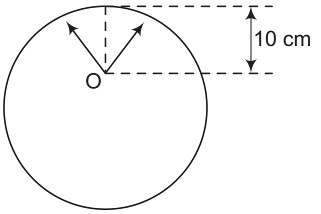
(a) Using $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$, we get

$$\frac{1.5}{v} - \frac{1.0}{-20} = \frac{1.5 - 1.0}{+6}$$

Solving, we get $v = +45$ cm

Similarly other parts can be solved.

4.

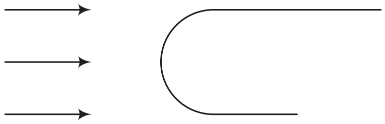


Using $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$, we get

$$\frac{1}{v} - \frac{4/3}{-10} = \frac{1 - 4/3}{-15}$$

Solving, we get $v = -9.0$ cm

5.



Using the equation

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}, \quad \text{we get}$$

$$\frac{1.44}{v} - \frac{1.0}{\infty} = \frac{1.44 - 1.0}{+1.25}$$

$$\therefore \quad v = 4.0 \text{ cm}$$

INTRODUCTORY EXERCISE 31.6

1. $\dfrac{1}{v} - \dfrac{1}{u} = \dfrac{1}{f} = (\mu - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$$\therefore \quad \frac{1}{-20} - \frac{1}{-60} = (1.65 - 1)\left(\frac{1}{-R} - \frac{1}{+R}\right)$$

Solving we get, $R = 39$ cm

2. $\frac{1}{-50} - \frac{1}{u} = \frac{1}{+30}$

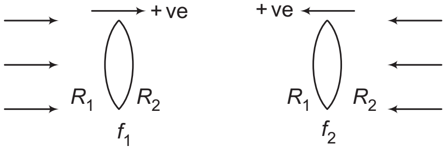
Solving, we get $u = -18.75$ cm

$$m = \frac{v}{u} = \frac{(-50)}{(-18.75)} = 2.67$$

$$I = m(O) = 2.67 \times 2$$

$$= 5.33 \text{ cm}$$

3.

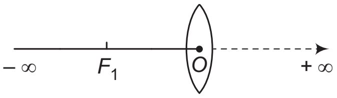


$$\frac{1}{f_1} = (\mu - 1)\left(\frac{1}{+R_1} - \frac{1}{-R_2}\right) \qquad \ldots\text{(i)}$$

$$\frac{1}{f_2} = (\mu - 1)\left(\frac{1}{+R_2} - \frac{1}{-R_1}\right) \qquad \ldots\text{(ii)}$$

Solving these two equations, we can see that $f_1 = f_2$.

4.



When object is moved from O to F_1, its virtual, erect and magnified image should vary from O to $-\infty$.

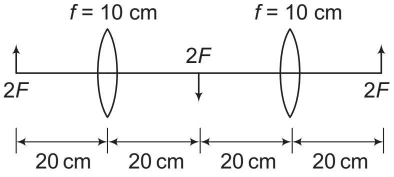
5. (a) $\frac{1}{f} = \left(\frac{1.3}{1.8} - 1\right)\left(\frac{1}{-20} - \frac{1}{+20}\right)$

$$\therefore \qquad f = +36 \text{ cm}$$

(b) Between O and F_1 image is virtual. Hence, for real image.

$$|\mu| < f \text{ or } 36 \text{ cm}$$

6.



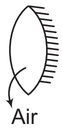
7. $\because \quad \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

Differentiating this equation, we get

$$-v^{-2}\cdot dv + u^{-2}\cdot du = 0 \qquad \text{(as } f = \text{constant)}$$

$$\therefore \qquad dv = \left(-\frac{v^2}{u^2}\right)\cdot du$$

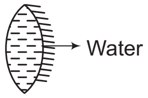
8. It is just like a concave mirror.



$$|f| = 0.2 \text{ m}$$

$$\therefore \qquad |R| = 0.4 \text{ m}$$

Focal length of this equivalent mirror is



$$\frac{1}{F} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1} \qquad \text{(extra points)}$$

$$= \frac{2(4/3)}{-0.4} - \frac{2(4/3 - 1)}{+0.4}$$

or $F = -0.12$ m or -12 cm

9. $|R| = 0.5$ m (from first case)

In the shown figure, object appears at distance

$$d = \mu_e(0.2) + 0.2$$

Now, for image to further coincide with the object,

$$d = |R|$$

Solving we get, $\mu_e = 1.5$

10. $O = \sqrt{I_1 I_2}$ (Displacement method)

$$= \sqrt{6 \times \frac{2}{3}} = 2 \text{ cm}$$

11. Virtual, magnified and erect image is formed by convex lens.

Let $u = -x$

Then, $v = -3x$

Now, $\frac{1}{-3x} - \frac{1}{-x} = \frac{1}{+12}$

$\therefore \qquad x = 8$ cm

Distance between object and image

$$= 3x - x = 2x = 16 \text{ cm}$$

12. Diminished erect image is formed by concave lens.

Let $u = -x$, then $v = -\frac{x}{2}$

Now, $|u| - |v| = 20$ cm

$\therefore \quad \frac{x}{2} = 20$ cm or $x = 40$ cm

$\therefore \quad u = -40$ cm and $v = -20$ cm

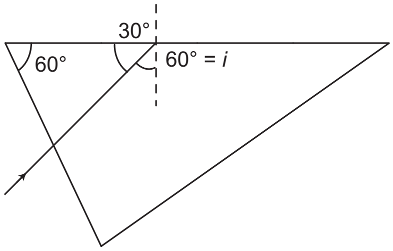
$$\frac{1}{f} = \frac{1}{-20} - \frac{1}{-40}$$

or $f = -40$ cm

13. If an object is placed at focus of lens (= 10 cm), rays become parallel and fall normal on plane mirror. So, rays retrace their path.

INTRODUCTORY EXERCISE 31.7

1.



Critical angle = $i = 60° = \theta_C$

$$\sin\theta_C = \frac{\mu_R}{\mu_D} \Rightarrow \sin 60^\circ = \frac{\mu}{\sqrt{3}}$$

Solving we get, $\mu = 1.5$

2. $\mu = \dfrac{c}{v} = \dfrac{3\times10^8}{2.3\times10^8} = 1.3$

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{1}{1.3}\right) = \sin^{-1}(0.77)$$

3. (a) $\mu_1 \sin i_1 = \mu_2 \sin i_2 = \mu_2 \sin\theta_C$

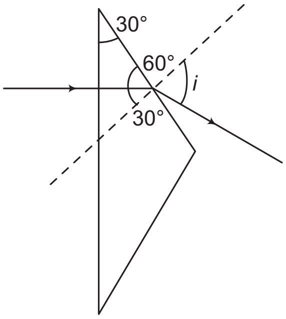
$$(1.6)\sin\theta = (1.80)\left(\frac{1.30}{1.80}\right)$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{13}{16}\right)$$

(b) If θ is decreased, then i_2 will decrease from the value θ_C. Hence, refraction will take place in medium-3.

INTRODUCTORY EXERCISE 31.8

1.



$$\mu = \frac{\sin i}{\sin 30^\circ}$$

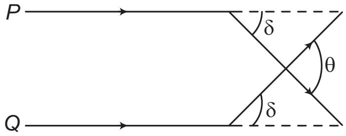
$$\therefore \quad \sin i = \mu \sin 30^\circ$$

$$= (1.6)\left(\frac{1}{2}\right) = 0.8$$

$$\therefore \quad i = 53^\circ$$

P ray deviates from its original path by an angle,

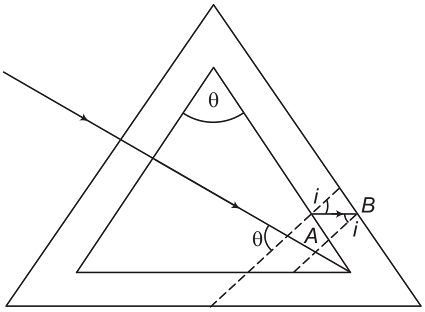
$$\delta = i - 30^\circ = 23^\circ$$



$\therefore$ Angle between two rays, $\theta = 2\delta$

$$= 46^\circ$$

2.



$$i = \theta_C$$

$$\therefore \quad \sin i = \sin\theta_C = \frac{1}{\mu_g} = \frac{2}{3}$$

Applying Snell's law at point A,
We have

$$\mu_w \sin\theta = \mu_g \sin i$$

$$\therefore \quad \frac{4}{3}\sin\theta = \frac{3}{2}\times\frac{2}{3}$$

$$\therefore \quad \sin\theta = \frac{3}{4}$$

3. $\mu = \dfrac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin(A/2)}$, $\delta_m = 30^\circ$

4. $i_1 = 0^\circ \Rightarrow r_1 = 0^\circ$ or $r_2 = A$

Now, $r_2 = \theta_C = A$

$$\therefore \quad \sin A = \sin\theta_C = \frac{1}{\mu} = \frac{2}{3}$$

or $\quad A = \sin^{-1}\left(\dfrac{2}{3}\right)$

5. $\delta = i_1 + i_2 - A$

$$30^\circ = 60^\circ + i_2 - 30^\circ$$

$\therefore \quad i_2 = 0$ or $r_2 = 0$ **Hence proved.**

$$\Rightarrow \qquad r_1 = A = 30°$$

Now, $$\mu = \frac{\sin i_1}{\sin r_1} = \frac{\sin 60°}{\sin 30°} = \sqrt{3}$$

6. $$\sqrt{2} = \frac{\sin i_1}{\sin (i_1/2)}$$
$$= \frac{2\sin (i_1/2)\cos (i_1/2)}{\sin (i_1/2)}$$

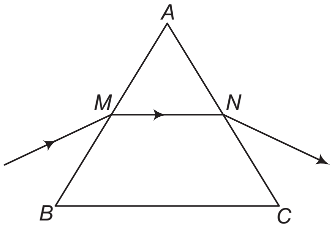
Solving this, we get $i_1 = 90°$ and $r_1 = \frac{i_1}{2} = 45°$

At minimum deviation,

$$r_2 = r_1 = 45°$$
$$\therefore \qquad A = r_1 + r_2 = 90°$$

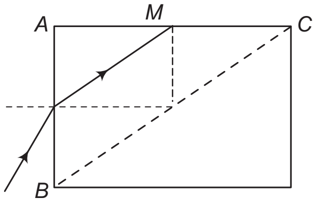
7. From $\mu = \sin\left(\frac{A+\delta_m}{2}\right) / \sin (A/2)$

We can see that given deviation is the minimum deviation.



At minimum deviation, MN is parallel to BC is $\angle B = \angle C$.

8. ABC can be treated as a prism with angle of prism $A = 90°$. Condition of no emergence is



$$A \geq 2\theta_C$$

or $$\sin \theta_C \leq \sin\left(\frac{A}{2}\right)$$

or $$\frac{1}{\mu} \leq \sin 45°$$

or $$\frac{1}{\mu} \leq \frac{1}{\sqrt{2}}$$

$$\therefore \qquad \mu \geq \sqrt{2}$$

9. $$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{1}{1.6}\right) = 38.7°$$

$$r_2 = \theta_C = 38.7°$$
$$\therefore \qquad r_1 = A - r_2 = 45° - 38.7° = 6.3°$$

Now applying, $\mu = \frac{\sin i_1}{\sin r_1}$

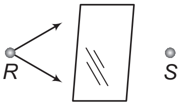
we can find i_1.

Exercises

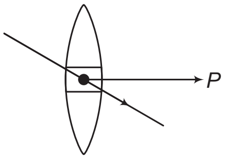
LEVEL 1

Assertion and Reason

1. Shift due to a slab $= \left(1 - \frac{1}{\mu}\right) t$ is in the direction of ray of light. Hence, Ram appears nearer to Anoop by that much distance.



4. P can be assumed a slab of negligible thickness. Deviation is almost negligible.



5. $m_1 m_2 = 1$

6. Image is formed at second focus (not the first focus).

7. $A = 60°$ and $\delta_m = 30°$

Substituting the values, we get

$$\mu = \sqrt{2}$$

8. $$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

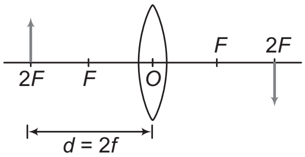
f_1 = Focal length of convex lens (+ ve)

f_2 = Focal length of concave lens (– ve)

If $f_1 > |f_2|$, then F comes out to be negative. So, it becomes a diverging lens.

Negative power of a diverging lens is just for its diverging lens, nothing else.

9. By inserting a slab between the lens and the object, effective distance between object and lens decreases. So, object now comes between F and $2F$ or between F and O.



In the first case, image is real and magnified and in the second case it is virtual and magnified.

10. If medium on both sides of the lens is same, then it doesn't matter, which side the object is kept.

11. $i = 45°$

For $\mu > \sqrt{2}$ or $1.414 \Rightarrow \theta_C < 45°$

$\therefore$ They get TIR on face AC.

12. For virtual object a concave lens can form a real image.

13. $$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{R}\right)$$

$$f = \infty$$

$$\therefore \quad P = \frac{1}{f} = 0$$

Objective Questions

2. μ, A and $\frac{B}{\lambda^2}$ are dimensionless.

Hence, $[B] = [\lambda^2] = [L^2]$

3. $$d_{\text{app.}} = \frac{d}{\mu}$$

μ of red is least. So, $d_{\text{app.}}$ for red is maximum. So, they appear to be raised least.

4. $$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right)$$

μ for violet is maximum, so θ_C for violet is least.

5. $$\frac{1}{f} = P = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= (1.6 - 1)\left(\frac{1}{0.1} - \frac{1}{-0.1}\right)$$

$$= +12\text{ D}$$

6. $$v = \frac{c}{\mu} = \frac{3 \times 10^8}{(4/3)} = 2.25 \times 10^8\text{ m/s}$$

7. After a certain angle all colours get total internally reflected.

8. Frequency does not change during refraction, but wavelength and speed decrease in a denser (here water) medium.

9. $\mu_1 \sin i_1 = \mu_3 \sin i_3$

$$\therefore \quad \mu_1 \sin i = \mu_3 \sin r \quad \text{or} \quad \frac{\sin i}{\sin r} = \frac{\mu_3}{\mu_1}$$

10. $r_1 = i$ and $r_2 = 90° - r_1 = 90° - i$

Now, $\mu_R \sin i_R = \mu_D \sin i_D$

$$\text{or} \quad \frac{\mu_R}{\mu_D} = \sin\theta_C = \frac{\sin i_D}{\sin i_R} = \frac{\sin i}{\sin(90° - i)}$$

$$\text{or} \quad \sin\theta_C = \frac{\sin i}{\cos i} = \tan i$$

$$\Rightarrow \quad \theta_C = \sin^{-1}(\tan i)$$

11. At minimum deviation,

$$r_1 = r_2 = \frac{A}{2} = 30°$$

Now using, $\mu = \frac{\sin i_1}{\sin r_1}$ or $\sqrt{2} = \frac{\sin i_1}{\sin 30°}$

We get $i_1 = 45°$

12. $$\frac{1}{0.2} = (1.5 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(i)}$$

$$\frac{1}{-0.5} = \left(\frac{1.5}{\mu} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(ii)}$$

Here, μ = refractive index of medium or liquid.
Dividing Eq. (i) by Eq. (ii), we get

$$-5 = \frac{1}{(1.5/\mu) - 1} \quad \text{or} \quad \mu = \frac{15}{8}$$

13. From Snell's law,

$$\mu \sin i = \mu_4 \sin x$$

$$\therefore \quad \sin x = \frac{\mu}{\mu_4} \sin i$$

14. Focal length of any one part will be $2f$.

$$\therefore \quad \frac{1}{F} = \frac{1}{2f} + \frac{1}{2f} + \frac{1}{2f} + \frac{1}{2f} \quad \text{or} \quad F = \frac{f}{2}$$

15. In air, focal length is

$$f = \frac{100}{P} = \frac{100}{5} = 20\text{ cm}$$

$$\frac{1}{20} = (1.5 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(i)}$$

$$\frac{1}{-100} = \left(\frac{1.5}{\mu_e} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$-5=\frac{0.5}{(1.5/\mu_e)-1}$$

or $$\mu_e=\frac{1.5}{0.9}=\frac{5}{3}$$

16. $$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}$$

To behave like concave lens, F should be negative.

So, $$\frac{d}{f_1f_2}>\frac{1}{f_1}+\frac{1}{f_2}$$

or $$\frac{d}{f_1f_2}>\frac{f_1+f_2}{f_1f_2}$$

or $$d>(f_1+f_2) \text{ or } 30\,\text{cm}$$

17. Condition of no emergence from opposite face is

$$A>2\theta_C$$

18. $r_2=0$, $r_1=A$

$$\mu=\frac{\sin i_1}{\sin r_1}\approx\frac{i_1}{r_1}=\frac{i_1}{A}\quad\Rightarrow\quad i_1=\mu A$$

19. $$\mu=\frac{\sin\left(\frac{A+\delta_m}{2}\right)}{\sin(A/2)}$$

Given, $$\mu=\cot\left(\frac{A}{2}\right)$$

Solving, we get $\sin=180°-2A$

20. At minimum deviation,

$$r_1=r_2=\frac{A}{2}=30°$$

Now, $$\mu=\frac{\sin i_1}{\sin r_1}\text{ or }\sqrt{2}=\frac{\sin i_1}{\sin 30°}$$

Solving, we get $i_1=45°$

21. $$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}$$

$$=\frac{f_1+f_2}{f_1\,f_2}-\frac{d}{f_1f_2}$$

$$\frac{1}{F}=\frac{(f_1+f_2)-d}{f_1f_2}$$

$$\therefore\quad F=\frac{f_1f_2}{(f_1+f_2)-d}\qquad\ldots\text{(i)}$$

If d is doubled, focal length is doubled or denominator becomes half.

$$\therefore\quad (f_1+f_2)-2d=\frac{1}{2}[(f_1+f_2)-d]$$

or $$(f_1+f_2)=3d$$

Substituting in Eq. (i), we have

Originally,

$$F=\frac{f_1f_2}{(f_1+f_2)-d}=\frac{f_1f_2}{3d-d}=\frac{f_1f_2}{2d}$$

When d is made 4 times.

$$F'=\frac{f_1f_2}{(f_1+f_2)-4d}=\frac{f_1f_2}{3d-4d}=-\frac{f_1f_2}{d}$$

$$=-2F$$

22. $$\frac{1}{24}=(1.5-1)\left(\frac{1}{R}-\frac{1}{2R}\right)$$

$$R=6\text{ cm}$$

$$\therefore\quad 2R=12\text{ cm}$$

23. **Lens formula:**

$$\frac{1}{v_1}-\frac{1}{-15}=\frac{1}{30}$$

$$\Rightarrow\quad v_1=-30\text{ cm}$$

Mirror formula:

$$v_2=-u_2=-(-45)=45\text{ cm}$$

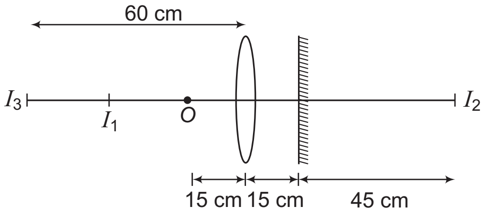
Lens formula:

$$\frac{1}{v_3}-\frac{1}{-60}=\frac{1}{30}$$

$$\therefore\quad v_3=+60\text{ cm}$$

Hence, distance of final image from object

$$=60-15=45\text{ cm}$$



24. $n_1\sin i_1=n_3\sin i_3$

$$\therefore\quad (1)(\sin 45°)=\sqrt{2}\sin i_3$$

$$\Rightarrow\quad \sin i_3=\frac{1}{2}\text{ or } i_3=30°$$

25. $$\mu=\frac{\sin\left(\frac{A+\delta_m}{2}\right)}{\sin(A/2)}$$

For small angled prism, $\sin\frac{A}{2}\approx\frac{A}{2}$

$$\therefore\quad \sin\left(\frac{A+\delta_m}{2}\right)=\frac{\mu A}{2}$$

With increase in μ, $\sin\left(\frac{A+\delta_m}{2}\right)$ will increase.

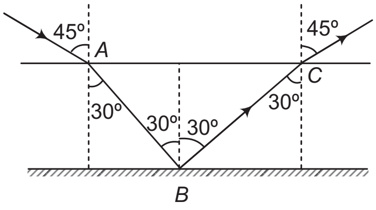
Hence, δ_m will increase.

26. $n = \dfrac{\sin i}{\sin (i/2)} = \dfrac{2 \sin (i/2) \cos (i/2)}{\sin (i/2)}$

Solving we get, $i = 2 \cos^{-1} (n/2)$

27. If object is placed at focus of convex lens, rays become parallel and they are incident normally on plane mirror. So, ray of light will retrace its path.

28. $\mu = \dfrac{\sin i}{\sin r} \Rightarrow \sqrt{2} = \dfrac{\sin 45°}{\sin r}$ or $r = 30°$



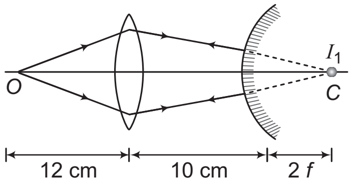
$$\begin{aligned}\delta_{\text{Total}} &= \delta_A - \delta_B + \delta_C \\ &= (45° - 30°) - [180° - 2(30°)] + (45° - 30°) \\ &= -90° \quad \text{or} \quad |\delta_{\text{Total}}| = 90°\end{aligned}$$

29. $\mu = \dfrac{\sin\left(\dfrac{A + \delta_m}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$

30. $\dfrac{1}{f} = \left(\dfrac{1}{1.5} - 1\right)\left(\dfrac{1}{10} - \dfrac{1}{-10}\right)$

Solving, we get $f = -15$ cm

31. Image formed by convex lens I_1 should coincide at C_1, the centre of curvature C of convex mirror.



For convex lens,

$$\frac{1}{+(10 + 2f)} - \frac{1}{-12} = \frac{1}{+10}$$

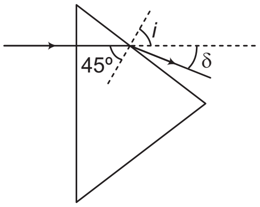
Solving this equation we get, $f = 25$ cm

32. For convex lens,

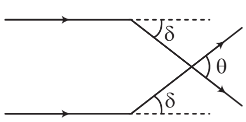
$$\frac{1}{v} - \frac{1}{-12} = \frac{1}{+20} \Rightarrow v = -30 \text{ cm}$$

This image I_1 is therefore, (30 + 10) cm or 40 cm towards left of plane mirror. Therefore, second image I_2 (by the plane mirror) will be formed 40 cm behind the mirror.

33. $\mu = \dfrac{\sin i}{\sin 45°}$ or $\sqrt{\dfrac{3}{2}} = \dfrac{\sin i}{(1/\sqrt{2})}$ or $i = 60°$



$\therefore \qquad \delta = 60° - 45° = 15°$



$$\theta = 2\delta = 30°$$

34. $r_2 = 0°$ and $r_1 = A = 30°$

Applying, $\mu = \dfrac{\sin i_1}{\sin r_1}$

$$\sqrt{2} = \frac{\sin i_1}{\sin 30°}$$

Solving, we get $i_1 = 45°$

35. $\dfrac{1}{F} = \dfrac{1}{f_1} + \dfrac{1}{f_2} = \dfrac{1}{10} + \dfrac{1}{10}$

$\therefore \qquad F = 5$ cm

$$\frac{1}{v} - \frac{1}{-7.5} = \frac{1}{+5}$$

$\therefore \qquad v = +15$ cm

$$m = \frac{v}{u} = \frac{+15}{-7.5} = -2$$

$$\begin{aligned} I &= mO = (-2)(1 \text{ cm}) \\ &= -2 \text{ cm}\end{aligned}$$

36. Applying, $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$

$\therefore \qquad \dfrac{1}{v} - \dfrac{3/2}{+30} = \dfrac{1 - 3/2}{+20}$

or $\qquad \dfrac{1}{v} = \dfrac{1}{20} - \dfrac{3}{40}$

$\therefore \qquad v = 40$ cm

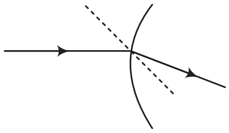
37. $\dfrac{1}{f} = \left(\dfrac{1.5}{1.4} - 1\right)\left(\dfrac{1}{-30} - \dfrac{1}{-50}\right)$

$\therefore \qquad f = -1050$ cm

38. Between 1 and 3 there is no deviation. Hence,

$$\mu_1 = \mu_3$$

Between 3 and 2



Ray to light binds towards normal. Hence,

$$\mu_2 > \mu_3.$$

39. $A = \delta_m = 60° \Rightarrow \mu = \dfrac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin(A/2)}$

Subjective Questions

1. (a) $\mu = \dfrac{343}{1498} = 0.229$

(b) $\theta_C = \sin^{-1}(0.229) = 13.2°$

2. $t_1 = \dfrac{l}{v_1} = \dfrac{l}{c/\mu_1} = \dfrac{\mu_1 l}{e} \Rightarrow t_2 = \dfrac{\mu_2 l}{c}$

$$\therefore \quad t_2 - t_1 = (\mu_2 - \mu_1)\frac{l}{c}$$

$$= \frac{(1.63 - 1.47)(20)}{3 \times 10^8}$$

$$= 1.07 \times 10^{-8} \text{ s}$$

3. (a) $t_1 = \dfrac{l}{v_1} = \dfrac{l}{c/n_1} = \dfrac{n_1 l}{c} = \text{minimum}$

$$= \frac{1.2 \times 10^{-6}}{3 \times 10^8}$$

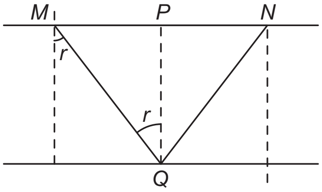
$$= 4 \times 10^{-15} \text{ s}$$

(b) $\lambda = \dfrac{\lambda_0}{n}$

Number of wavelengths across any film,

$$N = \frac{l}{\lambda} = \frac{ln}{\lambda_0}$$

4. $\mu = \dfrac{\sin i}{\sin r}$



$$1.8 = \frac{\sin 60°}{\sin r}$$

Solving, we get $r = 28.8°$

$$MP = PQ \tan r$$

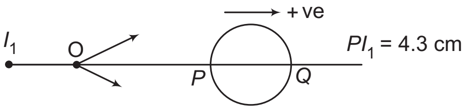
$$\therefore \quad MN = 2(MP) = 2PQ \cdot \tan r$$

$$= 2 \times 6 \times \tan 28.8°$$

$$= 6.6 \text{ cm}$$

$$\mu = \frac{\sin 45°}{\sin r} = \frac{4}{3}$$

5.



Applying $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$ two times

$$\frac{1.5}{v_1} - \frac{1.0}{-2.5} = \frac{1.5 - 1.0}{+10}$$

Solving, we get

$$v_1 \approx -4.3 \text{ cm}$$

Again applying the same equation, we get

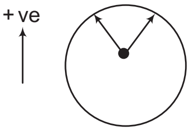
$$\frac{1.0}{v_2} - \frac{1.5}{-24.3} = \frac{1.0 - 1.5}{-10}$$

Solving, we get

$$v_2 \approx -85 \text{ cm}$$

or $QI_2 = 85$ cm (towards left)

6.



Applying $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$ we get,

$$\frac{1.0}{-1.0} - \frac{1.5}{-u} = \frac{1.0 - 1.5}{-2.0}$$

Solving we get,

$$u = 1.2 \text{ cm}$$

7. For plane surface, $h_{\text{app}} = \mu h$

$$= \frac{3}{2} \times 10 = 15 \text{ cm}$$

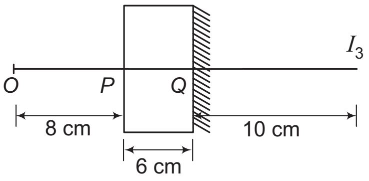
This first image is at a distance, (15 + 3) cm from the plane mirror. So, mirror will make its second image at a distance 18 cm below the mirror or 21 cm below the plane surface.

Now further applying,

$$d_{\text{app}} = \frac{d}{\mu} = \frac{21}{1.5} = 14 \text{ cm}$$

below the plane surface.

8.



Applying h_{app} for first refraction,

$$PI_1 = 8\mu \quad \text{(towards left)}$$

$$QI_1 = PQ + PI_1 = (6 + 8\mu)$$

$$\therefore \quad QI_2 = QI_1 = (6 + 8\mu), \text{ towards right.}$$

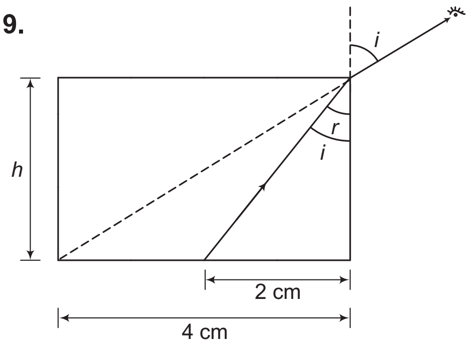
$$PI_2 = PQ + QI_2 = (12 + 8\mu)$$

Applying $d_{\text{app}} = \dfrac{d}{\mu}$ for third and last refraction we have,

$$PI_3 = \left(\frac{12 + 8\mu}{\mu}\right) = 16$$

$$\therefore \quad \mu = 1.5$$

9.



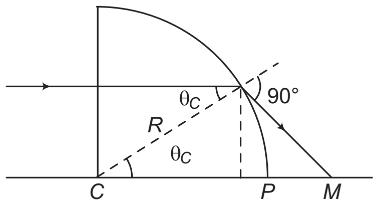
Applying, $\mu = \dfrac{\sin i}{\sin r}$ we get

$$\frac{4}{3} = \frac{4/\sqrt{16 + h^2}}{2/\sqrt{4 + h^2}}$$

Solving this equation we get,

$$h = 2.4\,\text{cm}$$

10.



$$\theta_C = \sin^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{2}{3}\right) = 41.8°$$

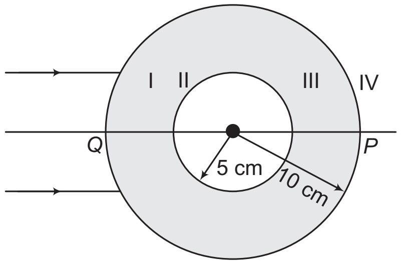
$$CM = R\sec\theta_C = \frac{R}{\cos\theta_C}$$

$$= \frac{5\,\text{cm}}{\cos 41.8°} = 6.7 \text{ cm}$$

$$\therefore \quad PM = CM - CP = (6.7 - 5.0)$$

$$= 1.7 \text{ cm}$$

11.



Apply $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$ four times,

We have $\dfrac{1.5}{v_1} - \dfrac{1.0}{\infty} = \dfrac{1.5 - 1.0}{+10}$

$$\therefore \quad v_1 = +30 \text{ cm}$$

$$\frac{1.0}{v_2} - \frac{1.5}{+25} = \frac{1.0 - 1.5}{-5.0}$$

$$v_2 = -25 \text{ cm}$$

$$\frac{1.5}{v_3} - \frac{1.0}{-35} = \frac{1.5 - 1.0}{-5.0}$$

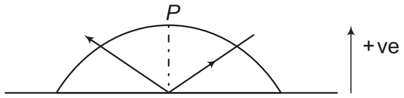
$$v_3 = -11.67 \text{ cm}$$

$$\frac{1.0}{v_4} - \frac{1.5}{-16.67} = \frac{1.0 - 1.5}{-10}$$

$$\Rightarrow \quad v_4 = -25 \text{ cm}$$

∴`Final image is at 25 cm to the left of P.

12. Applying $\dfrac{\mu_2}{v} = \dfrac{\mu_1}{u} = \dfrac{\mu_2 - \mu_1}{R}$ we have,



$$\frac{1.0}{v} - \frac{1.6}{-3.0} = \frac{1.0 - 1.6}{-5.0}$$

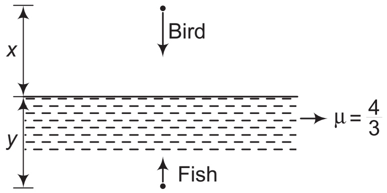
Solving we get, $v = -2.42$ cm

This is distance from P (downwards).

$\therefore$ Distance from observer $= 5 + 2.42$

$= 7.42$ cm

13.



Given, $-\dfrac{dy}{dt} = 4$ cm/s

Distance of bird as observed by fish

$$Z = y + \mu x = y + \frac{4}{3}x$$

$$\therefore \qquad \frac{-dZ}{dt} = \frac{-dy}{dt} + \frac{4}{3}\left(\frac{-dx}{dt}\right) \qquad \ldots\text{(i)}$$

Given, $\dfrac{-dZ}{dt} = 16$ cm/s

Substituting in Eq. (i), we get

$$\frac{-dx}{dt} = 9 \text{ cm/s}$$

14. $f = \dfrac{d^2 - x^2}{4d}$ (displacement method)

$f = 16$ cm, $x = 60$ cm

Substituting the values we get,

$$d = 100 \text{ cm}$$

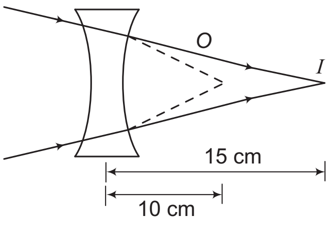
15. (a) $\dfrac{1}{f} = (\mu - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$$\therefore \qquad f \propto \left(\frac{1}{\mu - 1}\right)$$

or $\quad \dfrac{f_1}{f_2} = \dfrac{\mu_2 - 1}{\mu_1 - 1} = \dfrac{1.7 - 1.0}{1.5 - 1.0} = 1.4$

(b) If refraction index of the liquid (or the medium) is greater than the refraction index of lens it changes its nature or converging lens behaves as diverging.

16.

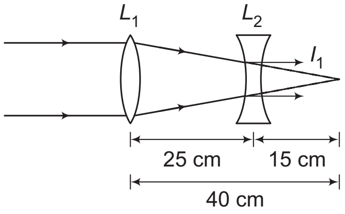


Using $\quad \dfrac{1}{f} = \dfrac{1}{v} - \dfrac{1}{u}$,

or $\quad \dfrac{1}{f} = \dfrac{1}{+15} - \dfrac{1}{+10}$

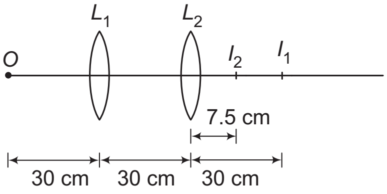
Solving, we get $f = -30$ cm

17.



I_1 is formed at second focus of L_1 and first focus of L_2.

18.



For L_1,

$$\frac{1}{v} - \frac{1}{-30} = \frac{1}{+20}$$

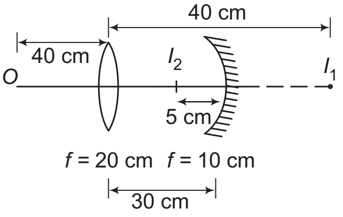
$$\therefore \qquad v = +60 \text{ cm}$$

For L_2,

$$\frac{1}{v} - \frac{1}{+30} = \frac{1}{+10}$$

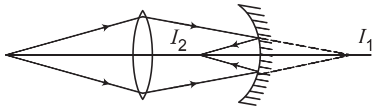
$$\therefore \qquad v = 7.5 \text{ cm}$$

19. Object is placed at distance $2f$ from the lens. Hence, image is also formed at distance $2f$ on other side. For mirror,



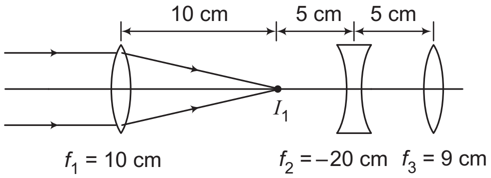
$$\frac{1}{v} + \frac{1}{+10} = \frac{1}{-10}$$

Ray diagram is as shown below.



20. For second lens,

$$\frac{1}{v}-\frac{1}{-5}=\frac{1}{-20} \Rightarrow v=-4\text{ cm}$$



For third lens,

$$\frac{1}{v}-\frac{1}{-(4+9)}=\frac{1}{+9}$$

$$\therefore \qquad v=\infty$$

21. See the result of sample example 31.30

$$R=\frac{h}{\sqrt{\mu^2-1}}$$

$$h=R\sqrt{\mu^2-1}$$

$$=(1\text{ cm})\sqrt{(5/3)^2-1}=\frac{4}{3}\text{ cm}$$

22. $\sin\theta_C=\dfrac{1}{\mu_1}=\dfrac{2}{3}$

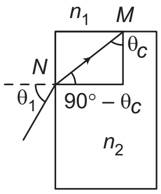
Now, $\quad \mu_1\sin i_1=\mu_2\sin i_2$

or $\quad \mu_1\sin\theta_C=\mu_2\sin\theta$

$$\therefore \qquad \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)=\left(\frac{4}{3}\right)\sin\theta$$

or $\quad \theta=\sin^{-1}\left(\dfrac{3}{4}\right)$

23.



n_1 should be less than n_2 for TIR at M.

$$\sin\theta_C=\frac{n_1}{n_2}$$

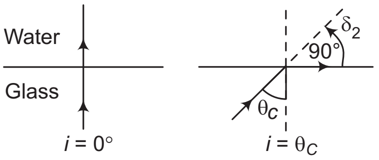
At N, $\quad n_1\sin\theta_1=n_2\sin(90^\circ-\theta_C)=n_2\cos\theta_C$

$$\therefore \qquad \sin\theta_1=\frac{n_2}{n_1}\cos\theta_C$$

$$\sin\theta_1=\frac{n_2}{n_1}\sqrt{1-\sin^2\theta_C}$$

$$=\frac{\sqrt{n_2^2-n_1^2}}{n_1}$$

24. $\theta_C=\sin^{-1}\left(\dfrac{\mu_R}{\mu_D}\right)=\sin^{-1}\left(\dfrac{4/3}{3/2}\right)=\sin^{-1}(8/9)$



$\therefore \quad \delta_1=0^\circ \qquad \therefore \quad \delta_2=90^\circ-\theta_C$

$$\cos\delta_2=\cos(90^\circ-\theta_C)=\sin\theta_C=\frac{8}{9}$$

$$\therefore \qquad \delta_2=\cos^{-1}\left(\frac{8}{9}\right)$$

25. Using $\mu=\dfrac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$,

Given, $\quad \delta_m=A$

$$\therefore \qquad \sqrt{3}=\frac{\sin A}{\sin\dfrac{A}{2}}=\frac{2\sin\dfrac{A}{2}\cdot\cos\dfrac{A}{2}}{\sin\dfrac{A}{2}}$$

Solving this equation, we get $A=60^\circ$.

26. Condition of no emergence is,

$$A>2\theta_C$$

$$\therefore \qquad A_{\max}=2\theta_C=2\sin^{-1}\left(\frac{1}{1.5}\right)$$

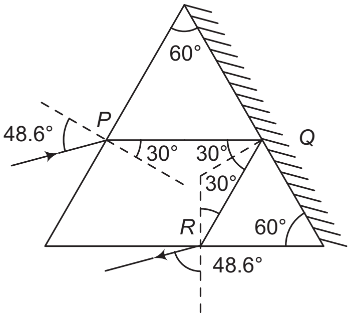
$$=84^\circ$$

27. (a) At minimum deviation,

$$r_1=r_2=\frac{A}{2}=30^\circ$$

Applying $\quad \mu=\dfrac{\sin i_1}{\sin r_1} \quad$ or $\quad 1.5=\dfrac{\sin i_1}{\sin 30^\circ}$

We get $\quad i_1=48.6^\circ$



$$\delta_{\text{Total}}=\delta_P+\delta_Q+\delta_R$$

$$= (48.6°-30°) + (180°-2\times30°) + (48.6°-30°)$$
$$= 157.2°$$

(b) $\frac{4}{3}\sin(48.6°) = 1.5 \sin r_1$

Solving we get, $r_1 = 41.8°$

$$r_2 = 60° - r_1 = 18.2°$$
$$r_3 = 60° - r_2 = 41.8°$$
$$\therefore \quad i_3 = 48.6°$$

Hence,

$$\delta_{\text{Total}} = (48.6° - 41.8°) + (180° - 2\times18.2°) + (48.6° - 41.8°)$$
$$= 157.2°$$

28. Dispersion power,

$$\omega = \frac{\mu_v - \mu_r}{\mu_y - 1}$$

29. (a) $\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$

$$\therefore \quad \omega_2 = \left(-\frac{f_2}{f_1}\right)\omega_1$$
$$= -\frac{(-30)}{(+20)}(0.18) = 0.27$$

(b) $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{20} - \frac{1}{30}$

or $F = +60$ cm

30. $\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$

$$\therefore \quad \left|\frac{\omega_1}{\omega_2}\right| = \left|\frac{f_1}{f_2}\right|$$

Let 1-stands for flint glass and 2-stands for crown glass. Then,

$$\left|\frac{f_1}{f_2}\right| = \frac{3}{2} \text{ or } |f_1| = 1.5\,|f_2|$$

Focal length of flint glass is more. So, its power is less. Combined focal length (and hence combined power) is positive. So, convex lens (converging lens) should be made up of crown glass (having more positive power).

Now, $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$

$$\therefore \quad \frac{1}{150} = \frac{1}{-1.5f} + \frac{1}{f}$$

Solving this equation, we get

$$f_2 = f = 50 \text{ cm and}$$
$$f_1 = -1.5,\ f = -75 \text{ cm}$$

31. For blue light,

$$\mu = \frac{\sin i_1}{\sin r_1}$$

or $1.68 = \frac{\sin 65°}{\sin r_1}$

Solving this equation, we get

$$r_1 = 32.6°$$
$$r_2 = A - r_1 = 27.4°$$

Again applying,

$$\mu = \frac{\sin i_2}{\sin r_2} \quad\text{or}\quad 1.68 = \frac{\sin i_2}{\sin 27.4°}$$

Solving this equation, we get

$$i_2 = 50.6°$$

Now, $\delta_B = i_1 + i_2 - A$

or $\delta_B = 65° + 50.6° - 60° = 55.6°$ ...(i)

For red light,

$$1.65 = \frac{\sin 65°}{\sin r_1}$$

or $r_1 = 33.3°$

$\therefore$ $r_2 = A - r_1 = 26.7°$

Now, $1.65 = \frac{\sin i_2}{\sin 26.7°}$

$\therefore$ $i_2 = 47.8°$

$\therefore$ $\delta_R = i_1 + i_2 - A$

or $\delta_R = 65° + 47.8° - 60° = 52.8°$ ...(ii)

From Eqs. (i) and (ii), we get

$$\delta_B - \delta_R = 2.8°$$

LEVEL 2

Single Correct Option

1. $h_{\text{app}} = \mu h = \frac{4}{3}\times 2 = \frac{8}{3}$ m

Distance from mirror $= 1 + h_{\text{app}} = \frac{11}{3}$ m. So, mirror will make image at same distance (= 11/3 m from itself). Now in third refraction, depth of second image,

$$d = \frac{11}{3} + 1 = \frac{14}{3} \text{ m}$$
$$d_{\text{app}} = \frac{d}{\mu} = \frac{14}{3}\times\frac{3}{4} = \frac{7}{2} \text{ m}$$

The desired distance is therefore,

$$(d_{\text{app}} + h)$$

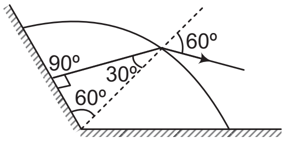
or $\left(\frac{7}{2} + 2\right)$ m or $\frac{11}{2}$ m

2. Applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ we get,

$$\frac{1.5}{v} - \frac{1}{\infty} = \frac{1.5 - 1.0}{+R}$$

$$\therefore \qquad v = +3R$$

3.



$$\mu = \frac{\sin 60^\circ}{\sin 30^\circ} = \sqrt{3}$$

4. Refraction from first surface,

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{\propto} = \frac{\mu_2 - \mu_2}{+R}$$

Refraction from second surface,

$$\frac{\mu_3}{f} - \frac{\mu_2}{v_1} = \frac{\mu_3 - \mu_2}{-R} \qquad (\text{as } v_2 = f)$$

Adding these two equations, we get

$$\frac{\mu_3}{f} = \frac{\mu_2 - \mu_1}{R} - \frac{\mu_3 - \mu_2}{R}$$

Lens becomes diverging if f is negative or

$$\mu_3 - \mu_2 > \mu_2 - \mu_1$$

or $$\mu_3 + \mu_1 > 2\mu_2$$

Same result is obtained if parallel beam of light is incident from RHS.

5. In first case,

$u = -16$ cm, then $v = (+16n)$ cm

$$\therefore \qquad \frac{1}{16n} - \frac{1}{-16} = \frac{1}{f} \qquad \ldots\text{(i)}$$

In second case,

$u = -6$ cm, then $v = -(6n)$ cm

$$\therefore \qquad \frac{1}{-6n} - \frac{1}{-6} = \frac{1}{f} \qquad \ldots\text{(ii)}$$

Solving these two equations, we get

$$f = 11 \text{ cm}$$

6. $d_{\text{app}} = \frac{d}{(n_1/n_2)}$

$$\therefore \qquad d = \frac{n_2}{n_1} d_{\text{app}}$$

$$-\frac{d}{dt}(d) = \frac{n_2}{n_1}\left(-\frac{d}{dt} d_{\text{app}}\right) = \left(\frac{n_2}{n_1}\right) x$$

$$\frac{dv}{dt} = A\left[-\frac{d}{dt}(d)\right] = \frac{x\,\pi R^2\, n_2}{n_1}$$

7. $i_1 = r_1 = 0^\circ$

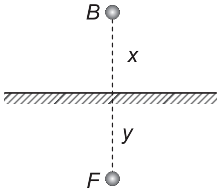
$$r_2 = 90^\circ - \theta = 53^\circ$$

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = 37^\circ$$

Since, $r_2 > \theta_C$, T/R will take place on the face AB

$$\delta = \delta_{AC} + \delta_{AB}$$
$$= 0 + 180^\circ - 2r_2$$
$$= 180^\circ - 2 \times 53^\circ = 74^\circ$$

8.



$$h_1 = x + \frac{y}{\mu} \qquad \ldots\text{(i)}$$

$$h_2 = y + \mu x \qquad \ldots\text{(ii)}$$

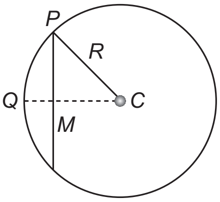
Eq. (i) can be written as

$$\frac{\mu x + y}{\mu} = h_1 \quad \text{or} \quad \frac{h_2}{\mu} = h_1 \quad \text{or} \quad \mu = \frac{h_2}{h_1}$$

9. $MC = QC - QM = (R - 0.3)$ cm

$$PC^2 = MC^2 + PM^2$$

$$R^2 = (R - 0.3)^2 + (3)^2$$



Solving this equation, we get

$$R = 15 \text{ cm}$$

10. See the hint of Q.No-5 of the same section. In that example,

$$f = \frac{16 + 6}{2} = 11 \text{ cm}$$

11. $\frac{1}{F_1} = \frac{1}{f_1} + \frac{1}{f_2}$

$$\frac{1}{F_2} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$F_2 > F_1$$

So, image of distant object will be formed to the right of P.

12. $$\mu = \frac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin(A/2)}$$

μ and A for both rays are same. Hence, value of δ_m is also same for both rays.

13. $$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{3}{7}}\right) = 40.9^\circ$$

$$r_2 = \theta_C = 40.9^\circ$$
$$r_1 = A - r_2 = 19.1^\circ$$
$$\mu = \frac{\sin i_1}{\sin r_1}$$
$$\sqrt{\frac{7}{3}} = \frac{\sin i_1}{\sin r_1}$$

Solving this equation, we get $i_1 = 30^\circ$

14. If object is placed at focus of plano-convex lens, then it will make rays parallel. Now, these rays fall normal on plane mirror. So, they retrace their path.

15. If object is placed at centre of the sphere, then all rays starting from C fall normal on spherical surface and pass understand.

16. $$\sin\theta_C = \frac{\mu_R}{\mu_D} = \frac{6/5}{3/2} = \frac{4}{5}$$

$$r_1 = 0^\circ \Rightarrow r_2 = < B = 90^\circ - \theta$$

Now, $$r_2 > \theta_c$$

or $$\sin r_2 > \sin\theta_C$$

$\therefore$ $$\sin(90^\circ - \theta) > \frac{4}{5}$$

or $$\cos\theta > \frac{4}{5} \qquad \ldots(\text{i})$$

$$\cos 37^\circ = \frac{4}{5}$$

From Eq. (i), we see that

$$\theta < 37^\circ$$

17. From the first refraction, rays should become parallel. Or, image should formed at infinity.

Applying,

$$\frac{\mu_2}{u} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

$$\frac{3/2}{\infty} - \frac{1.0}{-x} = \frac{3/2 - 1.0}{+10}$$

Solving, we get

$$x = 20\,\text{cm}$$

18. (a) Two images are formed in case (i).

(b) One image is formed in case (ii).

(c) $$\frac{1}{f_2} = (\mu - 1)\left(\frac{1}{-R} - \frac{1}{R}\right) = -\frac{2}{R}(\mu - 1) \qquad \ldots(\text{i})$$

$$\frac{1}{f_3} = \frac{1}{F_1} + \frac{1}{F_2}$$

$$= (\mu - 1)\left(\frac{1}{\infty} - \frac{1}{R}\right) + (\mu - 1)\left(\frac{1}{-R} - \frac{1}{\infty}\right) \ldots(\text{ii})$$

From Eqs. (i) and (ii), we can see that $f_2 = f_3$.

19. $$\frac{1}{f} = \frac{1}{OB} - \frac{1}{-OA} = \frac{1}{OB} + \frac{1}{OA}$$

$$f = \frac{(OA)(OB)}{AB + OB}$$

$\therefore$ $$f = \frac{(OA)(OB)}{AB} \qquad \ldots(\text{i})$$

Now, $$AB^2 = AC^2 + BC^2$$

or $$(OA + OB)^2 = AC^2 + BC^2$$

or $$OA^2 + OB^2 + 2(OA)(OB) = AC^2 + BC^2$$

$\therefore$ $$(AC^2 - OC^2) + (BC^2 - OC^2) + 2(OA)(OB) = AC^2 + BC^2$$

Solving, we get

$$(OA)(OB) = OC^2$$

Substituting in Eq. (i), we get

$$f = \frac{OC^2}{AB}$$

20. $$\text{Shift} = \left(1 - \frac{\mu_{\text{medium}}}{\mu_{\text{slab}}}\right)t$$

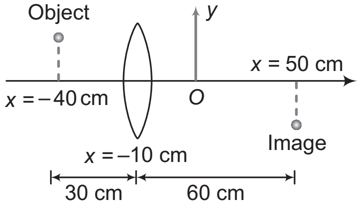
$$= \left[1 - \frac{4/3}{3/2}\right] \times 36 = 4\ \text{cm}$$

21. $$\frac{h}{\mu} = \frac{21}{2} \quad \text{or} \quad \frac{h}{(4/3)} = \frac{21}{2}$$

Solving, we get $h = 14$ cm.

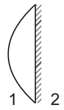
22. $$m = -\frac{2\,\text{cm}}{1\,\text{cm}} = -2$$

$$\Rightarrow \quad |v| = m\,|u|$$



23. Using the equation,

$$\frac{1}{F} = \frac{2\,(\mu_2/\mu_1)}{R_2} - 2\,\frac{(\mu_2/\mu_1 - 1)}{R_1}$$



we get

$$\frac{1}{-10} = \frac{2\,(1.5)}{\infty} - \frac{1\,(1.5-1)}{R_1} \qquad \left(\text{as } F = \frac{R}{2}\right)$$

Solving, we get $R_1 = 10$ cm.

24. In air, object lies between F and $2F$.

In liquid, focal length will become 4 times. So, object will now lie between optical centre and focus.

25. Condition of no emergence,

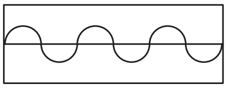
$$A > 2\theta_C$$

$$\therefore \qquad \theta_C < \frac{A}{2} \text{ or } 45°$$

$$\therefore \qquad \sin\theta_C \text{ or } \frac{1}{\mu} < \sin 45° \text{ or } \frac{1}{\sqrt{2}}$$

For no emergence, $\mu > \sqrt{2}$

26. $t = n\lambda = \dfrac{n\,\lambda_0}{\mu}$ $\qquad \left(\text{as } \lambda = \dfrac{\lambda_0}{\mu}\right)$



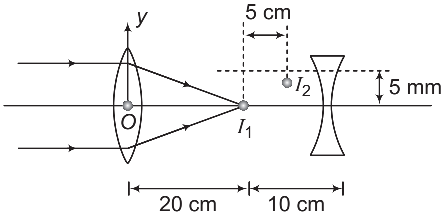
$$\text{or} \qquad t \propto \frac{1}{\mu}$$

$$\therefore \qquad \frac{\mu_2}{\mu_1} = \frac{t_1}{t_2}$$

$$\text{or} \qquad \mu_2 = \frac{t_1}{t_2}\,\mu_1$$

$$= \left(\frac{5}{4}\right)\left(\frac{4}{3}\right) = \frac{5}{3}$$

27. For concave lens,



$$\frac{1}{v} = \frac{1}{-10} = -\frac{1}{10} \quad \Rightarrow \quad v = -5 \text{ cm}$$

$$m = \frac{v}{u} = \frac{(-5\text{ cm})}{(-10\text{ cm})} = \frac{1}{2}$$

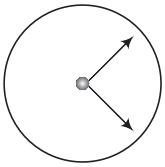
$$X_{I_2} = 20 + 5 = 25 \text{ cm}$$

$$Y_{I_2} = (5\text{ mm})\left(\frac{1}{2}\right) = 2.5 \text{ mm}$$

$$= 0.25 \text{ cm}$$

28. $\angle i = 0°$ for all values of θ, as the rays fall normal to sphere at all points.

29. Cavity of placed at centre. Hence, image of cavity is also formed at centre, as all rays fall normal to surface at all points. (so pass undedicated).



$$\text{Now,} \qquad m = \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{v}{u}\right)$$

$$= \frac{(4/3)}{1}\left(\frac{1\text{ m}}{1\text{ m}}\right)$$

$$= \frac{4}{3}$$

30. $\dfrac{1}{f} = (1.5-1)\left(\dfrac{1}{20} - \dfrac{1}{-20}\right)$

$$\Rightarrow \qquad f = +20 \text{ cm}$$

$$\frac{1}{v_1} - \frac{1}{-30} = \frac{1}{+20}$$

$$\therefore \qquad v_1 = +60 \text{ cm}$$

$$m_1 = \frac{v_1}{u_1} = \frac{+60}{-30} = -2$$

For second lens,

$$\frac{1}{v_2} - \frac{1}{-60} = \frac{1}{+20}$$

$$v_2 = +30 \text{ cm}$$

$$m_2 = \frac{v_2}{u_2} = \frac{(+30)}{(-60)} = -\frac{1}{2}$$

$$m = m_1 m_2 = 1$$

Final image is 30 cm to the right of second lens or 150 cm to the right of first lens.

$$m = 1$$

Hence, image height = object height

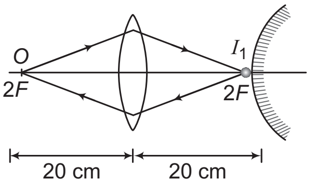
= 3 mm

31. Apply $\dfrac{\mu_2}{v} - \dfrac{\mu_1}{v} = \dfrac{\mu_2 - \mu_1}{R}$

We get $\dfrac{1}{v} - \dfrac{2}{-10} = \dfrac{1-2}{-10}$

Solving we get, $\quad v = -10$ cm

32.



33. $\frac{1}{15} = (1.5 - 1)\left(\frac{1}{R} - \frac{1}{\infty}\right)$

∴ $R = 7.5$ cm

With polished surface,

$$\frac{1}{F} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1}$$

$$= \frac{2(1.5)}{-7.5} - \frac{2(1.5-1)}{\infty}$$

$F = -2.5$ cm

Now, using mirror formula,

$$\frac{1}{v} + \frac{1}{-20} = \frac{1}{-2.5}$$

Solving, we get $v = -\frac{20}{7}$ cm

34. Shift $= \left(1 - \frac{1}{\mu}\right) t = \left(1 - \frac{2}{3}\right)(6) = 2$ cm

For mirror, object distance = 50 − 2 = 48 cm.

So, mirror will make image at a distance 48 cm behind it. In return journey of ray of light, we will have to farther take 2 cm shift in the direction of ray of light.

So, image distance as observed by observer

$= (50 + 48) - 2 = 96$ cm

35. Using $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$, we get

$$\frac{1}{+(f+40)} - \frac{1}{-(f+10)} = \frac{1}{+f}$$

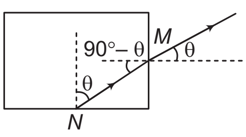
Solving this equation, we get

$f = +20$ cm

36. Applying Snell's law at point M, we get

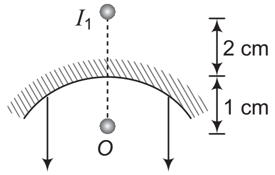
$$n_1 = \frac{\sin\theta}{\sin(90° - \theta_C)} \quad \text{or} \quad \sin\theta = n_1 \cos\theta_C$$

$$= n_1\sqrt{1 - \sin^2\theta_C} = n_1\sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{n_1^2 - n_2^2}$$



If angle of θ is less than this value, then angle of incidence at N will be greater than θ_C. Hence, TIR will take place at N.

37. Reflection from the concave mirror, I_1



$$\frac{1}{v} + \frac{1}{-1} = \frac{1}{-2} \quad \Rightarrow \quad v = +2 \text{ cm}$$

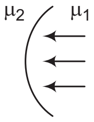
Shift due to slab $= \left(1 - \frac{1}{n}\right) t$

$= (1 - 2/3)(9 \text{ cm})$

$= 3$ cm

Now, slab will make next image at a distance 3 cm from I_1 in the direction of ray of light, i.e. at O itself but it is virtual, as the ray of light has crossed the slab and we are making image behind the slab.

38.

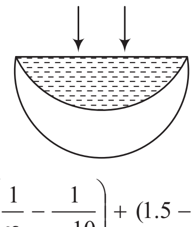


$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{-R}$$

∴ $$\frac{\mu_2}{v} = \frac{\mu_1 - \mu_2}{R} + \frac{\mu_1}{u}$$

If $\mu_1 > \mu_2$ and u is positive (i.e. virtual object), then v is always positive or image is always real.

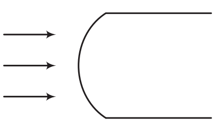
39. $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$



$$= (1.6 - 1)\left(\frac{1}{\infty} - \frac{1}{-10}\right) + (1.5 - 1)\left(\frac{1}{-10} - \frac{1}{-20}\right)$$

Solving we get, $F = 28.57$ cm

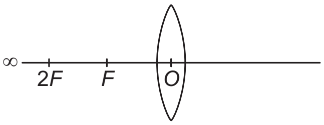
40.



$$\frac{1.5}{v} - \frac{1.0}{-u} = \frac{1.5 - 1.0}{+R}$$

or $$\frac{1.5}{v} = \frac{-1}{u} + \frac{1}{2R}$$

For image to real (for negative value of u) v should be positive. Hence, $\frac{1}{u} < \frac{1}{2R}$ or $u > 2R$.

41. 

When object moves towards F to O virtual erect and magnified image moves from ∞ to O.

42. $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d_2}{f_1/f_2}$

f_1 and f_2 both are negative. Hence, F is also negative. Object is real so combined lens (having negative focal length) will always make, its virtual image.

43. Using $\frac{\mu_2}{u} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ we get

$$\frac{3/2}{\propto} - \frac{1.0}{-x} = \frac{3/2 - 1}{+60}$$

$\therefore$ $$x = 120 \text{ cm}$$

Hence for $x = 120$ cm, rays of light become parallel to principal axis and fall normal to polished surface.

Hence, rays retrace their path.

44. $\frac{1.6}{v_1} - \frac{1.0}{-2} = \frac{1.6 - 1.0}{+1.0}$

$\Rightarrow$ $$v_1 = 16 \text{ m}$$

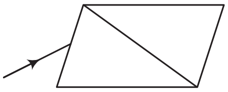
$$\frac{2.0}{v_2} - \frac{1.0}{-2} = \frac{2.0 - 1.0}{+1.0}$$

$\therefore$ $$v_2 = 4 \text{ m}$$

$$I_2 I_2 = v_1 - v_2$$
$$= 12 \text{ m}$$

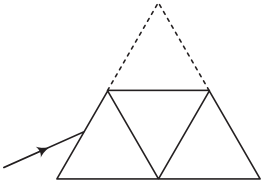
More than One Correct Options

1. For $n = 2m$, it is just like slab.



$\therefore$ Deviation = 0

For $n = 2m + 1$, it is just like an identical prism of larger size.



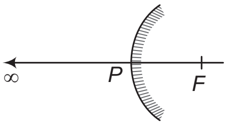
2. $\frac{\sin i_1}{\sin i_2} = \frac{v_1}{v_2}$

$\therefore$ $$\frac{\sin r}{\sin i} = \text{slope} = \frac{1}{\sqrt{3}} = \frac{v_y}{v_x}$$

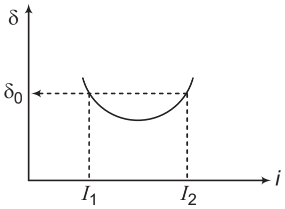
$\therefore$ $$v_y = \frac{v_x}{\sqrt{3}}$$

Speed of light in medium-y is less. So, it is denser. TIR takes place when ray of travels from denser to rarer medium.

3. (a) If vacuum speed of light of all colours is same.

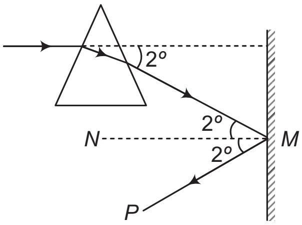
(b) 

If object moves from ∞ to P, then its virtual, erect and diminished image move from F to P.

(c) 

4. Displacement method of finding focal length of convex lens.

5. Deviation, $\delta = (\mu - 1) A = (1.5 - 1) 4° = 2°$

(a) 

To rotate ray MP by 2° (to make it parallel to MN) we will have to rotate the mirror only by 1°.

(b) Without prism, ray of light was falling normal to plane mirror. So, ray of light was retracing its path prism has deviated it 2°. So, if we rotate the mirror by 2°, ray of light further falls normal to plan mirror and retraces its path.

Comprehension Based Questions

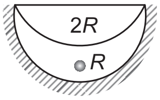
1. Using the mirror formula, $\frac{1}{f} = \frac{1}{v} + \frac{1}{u} = \frac{1}{-40} - \frac{1}{10}$

Solving we get, $f = -8$ cm

2. Focal length of lenses is

$$\frac{1}{F} = (1.8-1)\left(\frac{1}{\infty} - \frac{1}{-2R}\right) + (1.2-1)\left(\frac{1}{-2R} - \frac{1}{-R}\right)$$

$$F = 2R$$



Now combined power of system,

$$P = 2P_L + P_m$$

or $$-\frac{1}{f} = 2P_L + P_m$$

or $$\frac{1}{8} = 2\left(\frac{1}{2R}\right) + \frac{2}{R}$$

or $$R = 24 \text{ cm}$$

∴ Radius of curvature of common surface

$$= 2R = 48 \text{ cm}$$

3. Combined focal length of lens,

$$\frac{1}{f} = (1.2-1)\left(\frac{1}{+24} - \frac{1}{+48}\right) + (1.8-1)\left(\frac{1}{+48} - \frac{1}{\infty}\right)$$

$$= \frac{1}{48}$$

Now, combined power of system

∴ $$-\frac{1}{F} = 2\left(\frac{1}{f}\right) + \infty = 2\left(\frac{1}{48}\right)$$

∴ $$F = -24 \text{ cm}$$

Match the Columns

1. (c) and (d) $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

∴ $$\frac{1}{v} = \frac{1}{u} + \frac{1}{f}$$

Between O and F_2 or between F_2 and $2F_2$, u is positive. So, v is also positive.

∴ $$m = \frac{u}{v} \quad \text{is positive}$$

Therefore, image is real (as v is positive) and erect (as m is also positive).

2. (a) $\frac{1}{v} - \frac{1}{u} = \frac{1}{-f}$

∴ $$\frac{1}{u} = \frac{1}{u} - \frac{1}{f}$$

Between O and F_1, u is positive and less than f. So, v is positive (therefore image is real).

Further from

$$m = \frac{v}{u}$$

We can see that m is allow positive. So, it is erect also.

(b) $\frac{1}{v} - \frac{1}{u} = \frac{1}{-f}$

∴ $$\frac{1}{v} = \frac{1}{u} - \frac{1}{f}$$

Between F_1 and $2F_1$, μ positive and greater than f. So, v negative (therefore image is virtual).

Further from

$$m = \frac{v}{u}$$

We can see that m is negative so, image is inverted.

3. (a) and (b)

$$v = \frac{\mu_2}{\mu_1} u$$

⇒ $$|v| < |u|$$

as $$\mu_1 > \mu_2$$

i.e v and u are of same sign.

Or they are on same side of plane surface. From plane surface, if object is real, image is virtual and *vice-versa.*

(c) and (d)

$$v = \frac{\mu_1}{\mu_2} u$$

⇒ $$|v| > |u|$$

Other explanations are same.

4. (a) $\frac{\mu}{v} - \frac{1}{-u} = \frac{\mu - 1}{-R}$

$$\therefore \qquad \frac{\mu}{v} = \frac{-1}{u} - \frac{\mu - 1}{R}$$

Therefore, v is always negative. Or image is always virtual.

(b) $$\frac{\mu}{v} - \frac{1}{+u} = \frac{\mu - 1}{-R}$$

or $$\frac{\mu}{v} = \frac{1}{u} - \frac{\mu - 1}{R}$$

So, v may be positive or negative. Hence, image may be real or virtual.

Same logic can be applied for two options. For them R is positive. In option (c), u is negative and in option (d) u is positive.

5. $$\frac{1}{f_1} \text{ or } P_1 = (1.5 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{...(i)}$$

$$\frac{1}{f_2} \text{ or } P_2 = \left(\frac{1.5}{\mu_e} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we get

$$P_2 = \left(\frac{3}{\mu_e} - 2\right) P_1 \qquad \text{...(iii)}$$

(a) $\mu_e = 1.4$, then P_2 is positive and less than P_1.

Other options can be checked from Eq. (iii).

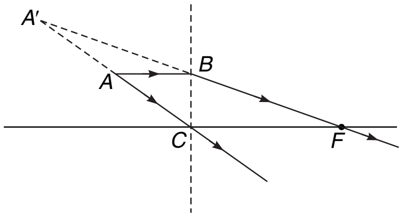
6. Concave lens can make only virtual, erect and diminished images of real objects.

Convex lens can make real, inverted and diminished size or real inverted and magnified or virtual, erect and magnified images of real objects.

Rest type of cases are possible with virtual, objects.

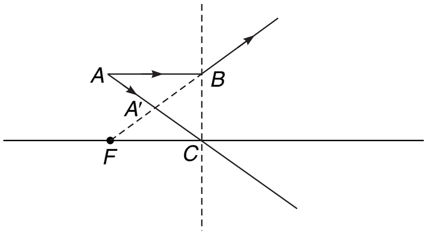
Subjective Questions

1. First draw a ray AA' until it intersects with the principal optical axis and find the centre of the lens C. Since, the virtual image is magnified, the lens is convex.



Draw a ray AB parallel to the principal optical axis. It is refracted by the lens so that it passes through its focus and its continuation passes through the virtual image. The ray $A'B$ intersects the principal optical axis at point F, the focus of the lens.

2. The paths of the rays are shown in figure. Since, the virtual image is diminished, the lens is concave.

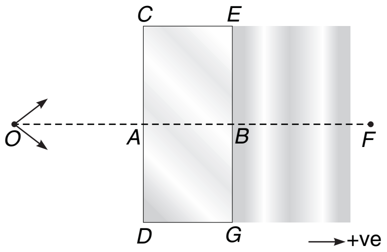


3. (a) $OA = 8.0$ cm

$$\therefore \quad AI_1 = (n_g)(OA)$$

$$= \left(\frac{8}{5}\right)(8.0) = 12.8 \text{ cm}$$

For refraction at EG ($R = \infty$), using



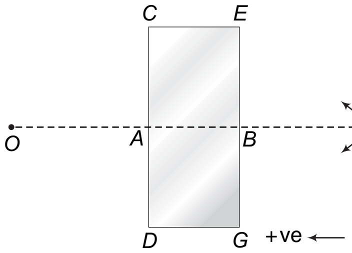
$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$$

$$\therefore \qquad \frac{4/3}{BI_2} - \frac{8/5}{-(12.8 + 3)} = 0$$

$$\therefore \qquad BI_2 = -(15.8)(4/3)(5/8) = -13.2 \text{ cm}$$

$$\therefore \qquad FI_2 = 13.2 + 6.8 = 20.0 \text{ cm}$$ **Ans.**

(b) **For face EF**



$$\frac{8/5}{BI_1} - \frac{4/3}{-6.8} = 0$$

$$\therefore \qquad BI_1 = -(6.8)(8/5)(3/4)$$

$$= -8.16 \text{ cm}$$

For face *CD*

$$\frac{1.0}{AI_2}-\frac{8/5}{-11.16}=0$$

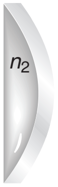
$$\therefore \quad AI_2=-(11.16)(5/8)$$

$$=-6.975 \text{ cm}$$

$$\therefore \quad FI_2=8+6.975$$

$$=14.975 \text{ cm} \qquad \textbf{Ans.}$$

4. The system behaves like a mirror of focal length given by



$$\frac{1}{F}=\frac{2(n_2/n_1)}{R_2}-\frac{2(n_2/n_1-1)}{R_1}$$

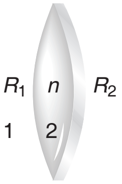
Substituting the values with proper sign.

$$\frac{1}{F}=\frac{2\times 4/3}{-20} \qquad (\because R_1=\infty)$$

$$\text{or} \qquad F=-7.5 \text{ cm} \qquad \textbf{Ans.}$$

i.e. system behaves as a concave mirror of focal length 7.5 cm.

5. $\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$



$$\frac{1}{10}=(1.5-1)\left(\frac{1}{R_1}-\frac{1}{-10}\right)$$

$$\therefore \quad R_1=+10 \text{ cm}$$

Now using,

$$\frac{1}{v}+\frac{1}{u}=\frac{2(n_2/n_1)}{R_2}-\frac{2(n_2/n_1-1)}{R_1}$$

Substituting the values,

$$\frac{1}{v}-\frac{1}{-15}=\frac{2(1.5)}{-10}-\frac{2(1.5-1)}{+10}$$

$$\therefore \quad v=-2.14 \text{ cm} \qquad \textbf{Ans.}$$

6. Using lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$

$$\frac{1}{v_1}-\frac{1}{-40}=\frac{1}{30}$$

$$\therefore \quad v_1=120 \text{ cm}$$

Shift due to the slab,

$$\Delta x=\left(1-\frac{1}{\mu}\right)d=\left(1-\frac{1}{1.8}\right)9=4 \text{ cm}$$

$$\therefore \quad u'=-(40-\Delta x)$$

$$=-36 \text{ cm}$$

$$\therefore \quad \frac{1}{v_2}-\frac{1}{-36}=\frac{1}{30}$$

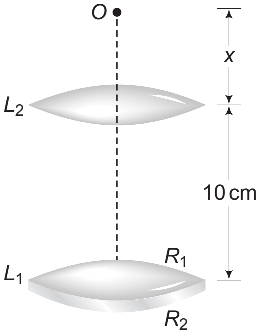
$$\therefore \quad v_2=180 \text{ cm}$$

Therefore, we will have to shift the screen a distance $x=v_2-v_1=60$ cm away from lens. **Ans.**

7. $\frac{1}{40}=(1.5-1)\left(\frac{1}{120}+\frac{1}{R_1}\right)$

Solving we get, $R_2=24$ cm

Applying lens formula, for L_2



$$\frac{1}{v_1}+\frac{1}{x}=\frac{1}{20} \qquad \text{...(i)}$$

Using $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$ for unsilvered side of L_1.

$$\frac{1.5}{-120}-\frac{1.0}{v_1-10}=\frac{1.5-1.0}{24} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $x=10$ cm **Ans.**

8. ***Case* I** $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$$\therefore \quad \frac{1}{v_1}-\frac{1}{30}=\frac{1}{-10}$$

$$\therefore \quad v_1=-15 \text{ cm}$$

***Case* II** Shift $=\left(1-\frac{1}{\mu}\right)t=\left(1-\frac{1}{1.5}\right)6=2$ cm

$$\therefore \quad \frac{1}{v_2}-\frac{1}{28}=\frac{1}{-10}$$

$$\therefore \quad v_2=-15.55 \text{ cm}$$

$$\therefore \quad \Delta v=0.55 \text{ cm} \qquad \textbf{Ans.}$$

9. Using $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$, twice with $u=\infty$,

we have

$$\frac{1.5}{v_1}=\frac{1.5-1.4}{+20} \qquad \text{...(i)}$$

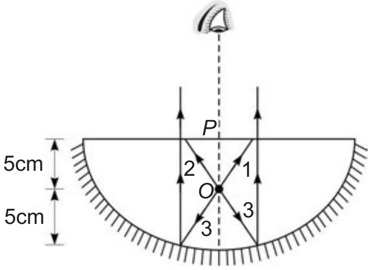
$$\frac{1.6}{v_2}-\frac{1.5}{v_1}=\frac{1.6-1.5}{-20} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $f=v_2=\infty$, i.e. the system behaves like a glass plate.

10. First image will be formed by direct rays 1 and 2, etc.

$$PI_1 = \frac{PO}{\mu} = \frac{5}{1.5} = 3.33 \text{ cm} \qquad \textbf{Ans.}$$

Second image will be formed by reflected rays 3 and 4, etc.

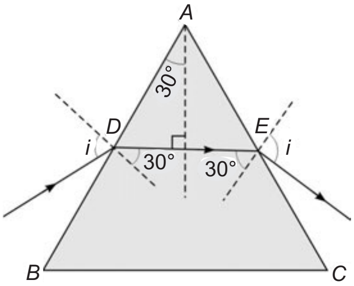


Object is placed at the focus of the mirror. Hence, I_2 is formed at infinity.

11. (a) Applying Snell's law at D,

$$\frac{4}{3} \sin i = \frac{3}{2} \sin 30°$$

$$\therefore \qquad i = 34.2° \qquad \textbf{Ans.}$$

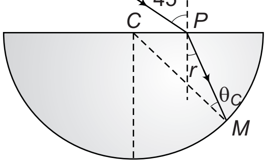


(b) $\delta = \delta_D + \delta_E = 2\delta_D$

$= 2(i - 30°)$

$= 8.4°$ **Ans.**

12. $$\sqrt{2} = \frac{\sin 45°}{\sin r}$$



$r = 30°$

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = 45°$$

Applying sine law in ΔCPM,

$$\frac{CP}{\sin \theta_C} = \frac{CM}{\sin (90° + r)}$$

$$\therefore \qquad \frac{CP}{(1/\sqrt{2})} = \frac{R}{\cos r} \qquad [R = \text{radius}]$$

$$\therefore \qquad CP = \sqrt{\frac{2}{3}}R$$

As we move away from C, angle PMC will increase. Therefore, $CP \ngtr \sqrt{\frac{2}{3}}R$. Same is the case on left side of C.

13. $$\frac{1}{20} = (1.5 - 1)\left(\frac{1}{R} - \frac{1}{-R}\right)$$

$\therefore \quad R = 20$ cm

Applying $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ twice with the condition that rays must fall normally on the concave mirror.

$$\frac{1.5}{v_1} - \frac{1.2}{-40} = \frac{1.5 - 1.2}{+20} \qquad \text{...(i)}$$

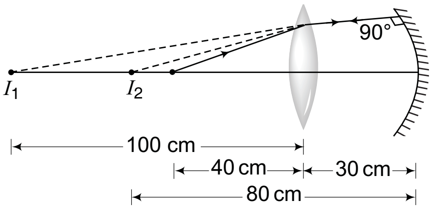
$$\frac{2.0}{d - 80} - \frac{1.5}{v_1} = \frac{2.0 - 1.5}{-20} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$d = 30 \text{ cm} \qquad \textbf{Ans.}$$

and $\qquad v_1 = -100$ cm

The ray diagram is as shown in figure.



14. Using lens formula,

$$\frac{1}{36} - \frac{1}{-45} = \frac{1}{f}$$

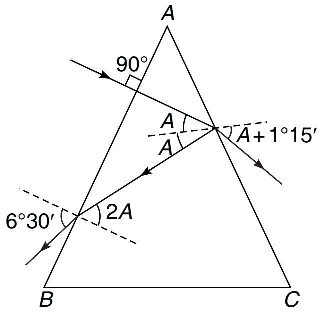
$$\therefore \qquad f = 20 \text{ cm}$$

In the second case, let μ be the refractive index of the liquid, then

$$\frac{1}{48} - \frac{1}{-\left(5 + \frac{40}{\mu}\right)} = \frac{1}{20}$$

Solving this, we get $\mu = 1.37$ **Ans.**

15. As the angles are small we can take,



$$\sin\theta \approx \theta$$

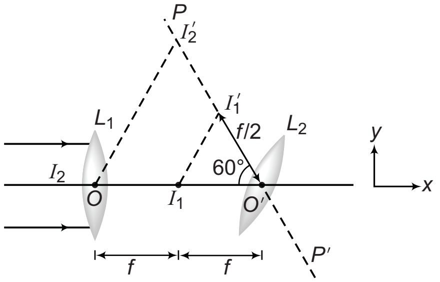
Now, $$\mu = \frac{A + 1°15'}{A} = \frac{6°30'}{2A}$$

Solving this equation, we get

$$A = 2°$$

and $\mu = 1.62$ **Ans.**

16. Using lens formula for L_2,



$$\frac{1}{v} - \frac{1}{-f/2} = \frac{1}{f}$$

or $v = -f$

This f length will be along PP' from point O' (towards P).

$\therefore$ $O'I_2' = f$

On x-axis this distance will be $f \sec 60° = 2f$.

Since, $OO' = 2f$, therefore image will be formed at origin.

Note *A ray passing through O and then O′ goes undeviated.*

Therefore I_1 and I_2 both should be on this line, which is also the x-axis. That's why for final image we have taken projection of $O'I_2'$ on x-axis.

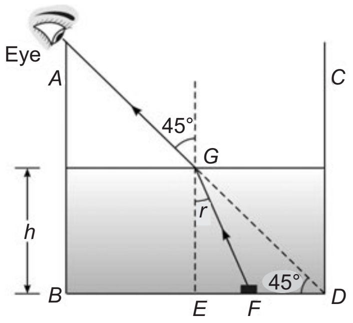
17. Since, the vessel is cubical, $\angle GDE = 45°$

and $GE = ED = h$ (say)

then $EF = ED - FD = (h - 10)$

Further, $$\frac{4}{3} = \frac{\sin 45°}{\sin r}$$

$\therefore$ $r = 32°$



Now, $$\frac{EF}{GE} = \tan r = \tan 32°$$

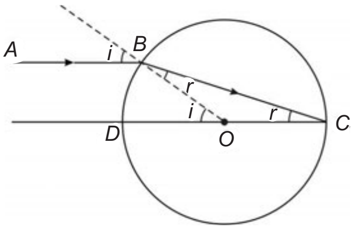
$\therefore$ $$\frac{h-10}{h} = 0.62$$

Solving this, we get

$h = 26.65$ cm **Ans.**

18. $BO = OC$

$\therefore$ $\angle OBC = \angle BCO = r$ (say)



Let angle of incidence be i,

$i = r + r = 2r$ (external angle)

$\therefore$ $$\mu = \frac{\sin i}{\sin r} = \frac{\sin 2r}{\sin r} \approx \frac{2r}{r} = 2$$ **Ans.**

19. $\delta_{\text{Total}} = \delta_{\text{Refraction}} + 2\delta_{\text{Reflection}} + \delta_{\text{Refraction}}$

or $\delta = (i - r) + 2(180° - 2r) + (i - r)$

$= 360° + 2i - 6r$

$$= 360° + 2i - 6\sin^{-1}\left(\frac{\sin i}{\mu}\right)$$

For deviation to be minimum, $\frac{d\delta}{di} = 0$

By putting first derivative of δ (w.r.t. i) equal to zero, we get the desired result.

20. (a) For refraction at first half lens $\left(\frac{1}{v} - \frac{1}{u} = \frac{1}{f}\right)$

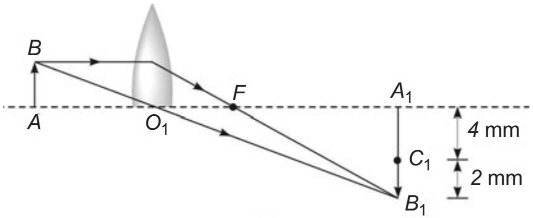
$$\frac{1}{v} - \frac{1}{-20} = \frac{1}{15}$$

$\therefore \quad v = 60$ cm

Magnification, $m = \frac{v}{u} = \frac{60}{-20} = -3$

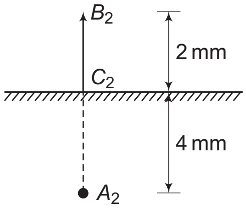
The image formed by first half lens is shown in figure (a).

$AB = 2$ mm, $A_1B_1 = 6$ mm, $AO_1 = 20$ cm, $O_1F = 15$ cm and $O_1A_1 = 60$ cm.



(a)

Point B_1 is 6 mm below the principal axis of the lenses. Plane mirror is 4 mm below it.



(b)

Hence, 4 mm length of A_1B_1 (i.e. A_1C_1) acts as real object for mirror. Mirror forms its virtual image A_2C_2. 2 mm length of A_1B_1 (i.e. C_1B_1) acts as virtual object for mirror. Real image C_2B_2 is formed of this part. Image formed by plane mirror is shown in figure (b).

For the second half lens, $\frac{1}{v} - \frac{1}{-60} = \frac{1}{15}$

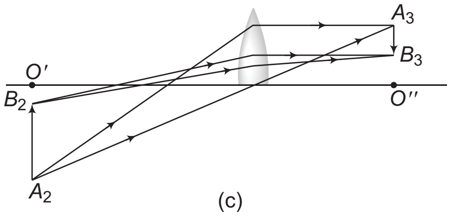
$\therefore \quad v = +20$

$$m = \frac{v}{u} = \frac{20}{-60} = -\frac{1}{3}$$

So, length of final image

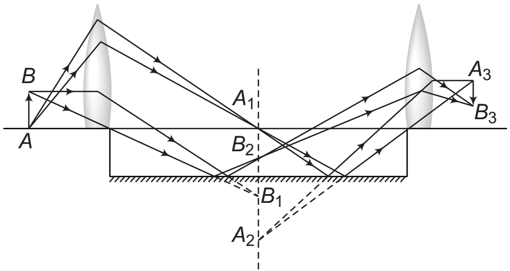
$$A_3B_3 = \frac{1}{3}A_2B_2 = 2 \text{ mm}$$

Point B_2 is 2 mm below the optic axis of second half lens. Hence, its image B_3 is formed 2/3 mm above the principal axis. Similarly, point A_2 is 8 mm below the principal axis. Hence, its image is 8/3 mm above it. Therefore, image is at a distance of 20 cm behind the second half lens and at a distance of 2/3 mm above the principal axis. The size of image is 2 mm and is inverted as compared to the given object. Image formed by second half lens is shown in figure (c).



(c)

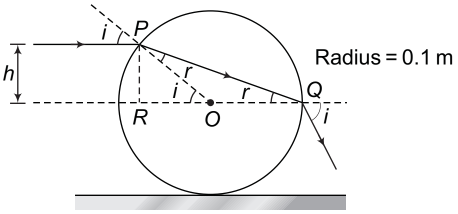
(b) Ray diagram for final image is shown in figure (d).



(d)

21. (i) $PO = OQ$

$\therefore \quad \angle OPQ = \angle OQP = r$ (say)



Also, $\quad i = r + r = 2r$

In ΔPOR, $\quad h = OP \sin i = 0.1 \sin i$

$\quad = 0.1 \sin 2r$

or $\quad h = 0.2 \sin r \cos r \quad$ ...(i)

Also, $\quad \sqrt{3} = \frac{\sin i}{\sin r} = \frac{2\sin r \cos r}{\sin r}$

$$= 2\cos r$$

$$\therefore \quad r = 30°$$

Substituting in Eq. (i), we get

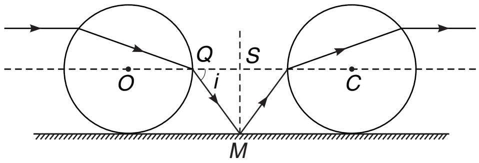
$$h = 0.2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= 0.086 \text{ m}$$

Hence, height from the mirror is

$$0.1 + 0.086 = 0.186 \text{ m}$$

(ii) Use the principle of reversibility.



$$i = 2r = 60°$$

Now, $\frac{QS}{MS} = \cot i = \cot 60° = \frac{1}{\sqrt{3}}$

$$\therefore \quad QS = \frac{MS}{\sqrt{3}} = \frac{0.1}{\sqrt{3}}$$

∴ The desired distance,

$$OC = 2 \times 0.1 + \frac{2 \times 0.1}{\sqrt{3}}$$

$$= 0.315$$ **Ans.**

22. $\frac{V_i}{V} = \frac{\rho_S}{\rho_L} = \frac{\rho}{2\rho} = \frac{1}{2}$

i.e. half the sphere is inside the liquid. For the image to coincide with the object light should fall normally on the sphere.

Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ twice, we have

$$\frac{3/2}{v_1} - \frac{1}{-8} = \frac{3/2 - 1}{+2}$$

$$\therefore \quad v_1 = 12 \text{ cm}$$

Further, $\frac{4/3}{h - 10} - \frac{3/2}{8} = \frac{4/3 - 3/2}{-2}$

Solving this equation, we get

$$h = 15 \text{ cm}$$ **Ans.**

23. We have to see the image of O from the other side.

Applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ twice, we have

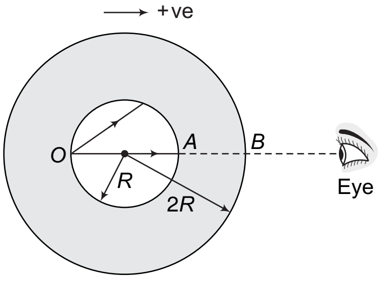
$$\frac{\mu}{AI_1} - \frac{1}{-2R} = \frac{\mu - 1}{-R}$$

$$\therefore \quad AI_1 = \frac{2\mu R}{1 - 2\mu}$$

Further, $\frac{1}{BI_2} - \frac{\mu}{(AI_1 - R)} = \frac{1 - \mu}{-R}$

Solving this equation, we get

$$BI_2 = -\frac{2R(4\mu - 1)}{3\mu - 1}$$

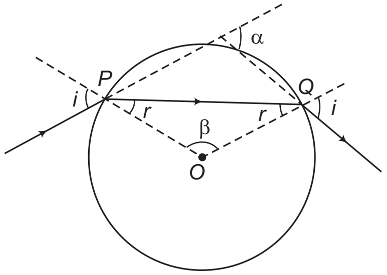


∴ Distance between the final image and object is

$$d = 3R - \frac{2R(4\mu - 1)}{3\mu - 1}$$

$$= \frac{(\mu - 1)R}{(3\mu - 1)}$$ **Hence proved.**

24. $\delta_{\text{Total}} = \delta_P + \delta_Q$



$$\therefore \quad \alpha = (i - r) + (i - r)$$

or $$i - r = \frac{\alpha}{2} \quad \text{...(i)}$$

Further, in ΔOPQ, $r + r + \beta = 180°$

$$\therefore \quad r = 90° - \frac{\beta}{2} \quad \text{...(ii)}$$

From Eq. (i),

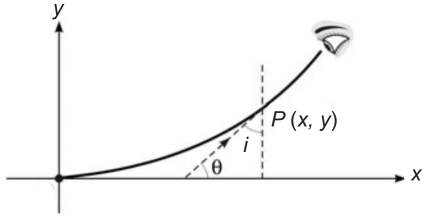
$$i = r + \frac{\alpha}{2} = 90° + \left(\frac{\alpha - \beta}{2}\right) \quad \text{...(iii)}$$

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin\left[90° + \left(\frac{\alpha - \beta}{2}\right)\right]}{\sin\left(90° - \frac{\beta}{2}\right)}$$

$$= \frac{\cos\left(\frac{\beta-\alpha}{2}\right)}{\cos\left(\frac{\beta}{2}\right)}$$

or $\cos\left(\frac{\beta-\alpha}{2}\right) = \mu \cos\frac{\beta}{2}$ **Hence proved.**

25. $\theta = 90° - i$



$$\tan\theta = \cot i$$

or $$\frac{dy}{dx} = \cot i \qquad \ldots\text{(i)}$$

$$\mu_0 \sin i_0 = \mu_P \sin i_P$$

(i) $\sin 90° = (\sqrt{1+ay}) \sin i$

$\therefore$ $$\sin i = \frac{1}{\sqrt{1+ay}}$$

$\therefore$ $$\cot i = \sqrt{ay} = \frac{dy}{dx}$$

$\therefore$ $$\int_0^y \frac{dy}{\sqrt{ay}} = \int_0^x dx \quad \text{or} \quad x = 2\sqrt{\frac{y}{a}}$$

Substituting $y = 2$ m,

and $a = 2.0 \times 10^{-6}\ \text{m}^{-1}$

We get $x_{\max} = 2000\ \text{m} = 2\ \text{km}$ **Ans.**

26. Applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ twice, we have

$$\frac{\mu}{v_1} - \frac{1}{-2R} = \frac{\mu - 1}{R}$$

$\therefore$ $$v_1 = \frac{2\mu R}{2\mu - 3}$$

Further, $$\frac{1}{v_2} - \frac{\mu}{-(3R - v_1)} = \frac{1-\mu}{R/2}$$

$\therefore$ $$v_2 = \frac{R(9 - 4\mu)}{(10\mu - 9)(\mu - 2)}$$ **Ans.**

Final image is real if, $v_2 > 0$.

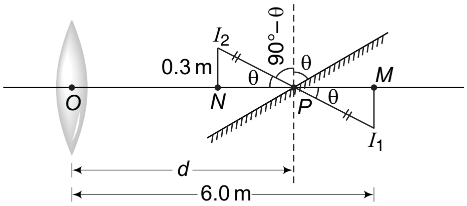
As $10\mu - 9$ is always positive ($\mu > 1$). Therefore, for $v_2 > 0$, either $(9 - 4\mu)$ and $(\mu - 2)$ both should be greater than zero or both should be less than zero. For the first condition (when both > 0) $2 < \mu < 2.25$ and for the second condition (when both < 0), $\mu < 2$ and $\mu > 2.25$ which is not possible. Hence, μ should lie between 2 and 2.25.

27. For the lens, $u = -2.0$ m, $f = +1.5$ m

$\therefore$ $$\frac{1}{v} - \frac{1}{-2.0} = \frac{1}{1.5}$$

or $$v = 6.0\ \text{m}$$

$$m = \frac{6.0}{-2.0} = -3.0$$



Therefore, y-coordinate of image formed by lens is $m(0.1) = -0.3$ m.

$$\frac{0.3}{NP} = \tan\theta = 0.3$$

$\therefore$ $$NP = MP = 1.0\ \text{m}$$

or $$d = 6.0 - 1.0$$
$$= 5.0\ \text{m}$$ **Ans.**

and x-coordinate of final image I_2 is

$$x = d - 1.0 = 4.0\ \text{m}$$ **Ans.**

32. Interference and Diffraction of Light

INTRODUCTORY EXERCISE 32.1

1. $\dfrac{I_1}{I_2} = \dfrac{9}{16}, \dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{\sqrt{I_1/I_2} + 1}{\sqrt{I_1/I_2} - 1}\right)^2$

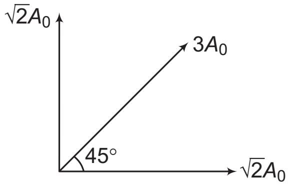
2. (a) $\dfrac{A_{\max}}{A_{\min}} = \dfrac{A_1 + A_2}{A_1 - A_2}$

(b) $\dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{A_{\max}}{A_{\min}}\right)^2$

3. $\dfrac{\lambda}{4}$ path difference is equivalent to 90° phase difference

$$A_{\text{net}} = 5A_0$$

$$\Rightarrow \quad I_{\max} = 25\, I_0$$



INTRODUCTORY EXERCISE 32.2

3. $I = I_{\max} \cos^2 \dfrac{\phi}{2}$

$$\frac{3}{4} I_{\max} = I_{\max} \cos^2 \frac{\phi}{2} \quad \text{or} \quad \cos\frac{\phi}{2} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \frac{\phi}{2} = \frac{\pi}{6}$$

$$\therefore \quad \phi = \frac{\pi}{3} = \left(\frac{2\pi}{\lambda}\right)(\Delta x) = \frac{2\pi}{\lambda}\left(\frac{yd}{D}\right)$$

$$\therefore \quad y = \frac{\lambda D}{6d} = \frac{(600 \times 10^{-19})\,(1.2)}{(6)\,(0.25 \times 10^{-2})}$$

$$= 48 \times 10^{-6}\,\text{m} = 48\,\mu\text{m}$$

4. $A_1 = 3A_2$

So, $\quad I_1 = 9I_2$

Let $\quad A_2 = A_0 \quad$ and $\quad I_2 = I_0$

Then, $\quad A_1 = 3A_0 \quad$ and $\quad I_1 = 9I_0$

$$A_{\max} = (A_1 + A_2) = 4\,A_0$$

and $\quad I_{\max} = 16I_0$

Now, $\quad I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\phi$

$$= 9I_0 + I_0 + 2\sqrt{9I_0 \times I_0}\, \cos\phi$$

$$= 10I_0 + 6I_0 \cos\phi$$

$$= 10 \times \frac{I_{\max}}{16} + 6\left(\frac{I_{\max}}{16}\right) \cos\phi$$

$$= \frac{5}{8} I_{\max} + \frac{3}{8} I_{\max} \cos\phi$$

$$= \frac{5}{8} I_{\max} + \frac{3}{8} I_{\max}\left(2\cos^2\frac{\phi}{2} - 1\right)$$

$$= \frac{1}{4} I_{\max} + \frac{3}{4} I_{\max} \cdot \cos^2\frac{\phi}{2}$$

$$= \frac{I_{\max}}{4}\left(1 + 3\cos^2\frac{\phi}{2}\right)$$

5. (a) $\quad I = I_{\max} \cos^2 \dfrac{\phi}{2}$

$$= I_0 \cos^2 30° \qquad (\text{as } \phi = 60°)$$

$$= \frac{3}{4} I_0$$

(b) 60° phase difference is equivalent to $\dfrac{\lambda}{6}$ path difference.

6. $a_R = 2a\cos\dfrac{\phi}{2}$

(i) For $a_R = 2a, \phi = 0°$

(ii) For $a_R = \sqrt{2}a, \phi = 90°$ etc

Exercises

LEVEL 1

Assertion and Reason

1. $I = 4I_0 \cos^2 \dfrac{\phi}{2} \qquad \ldots\text{(i)}$

$$\phi = \frac{2\pi}{3} \text{ or } 120° \quad \Rightarrow \quad \frac{\phi}{2} = 60°$$

Substituting in Eq. (i), we get

$$I = I_0 \quad \Rightarrow \quad \Delta x = \left(\frac{\lambda}{2\pi}\right)\phi$$

$$\text{For } \phi = \frac{2\pi}{3}, \Delta x = \frac{\lambda}{3}$$

2. The whole fringe pattern will shift upwards.

3. No reflected ray reaches below O.

4. $I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$ and

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

When $\quad I_1 = I_2 = I_0$, then

$$I_{\max} = 4I_0 \text{ and } I_{\min} = 0$$

When slit of one width is slightly increased, then intensity due to that slit becomes greater than I_0. In that case, we can see that

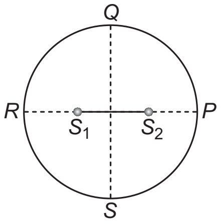
$$I_{\max} > 4I_0 \text{ and } I_{\min} > 0$$

6. Locus of points of equal path difference in the shown case is circle.

7. At centre, path difference is maximum and this is equal to S_1S_2. Then, path difference decreases as we move away on the screen. So, order of fringe also decreases. Hence, 11th order maxima occurs before 10th order maxima.

8. At points P and R

$|\Delta x| = S_1S_2 = 4\lambda$, therefore maxima.



At Q and S

$\Delta x = 0$, therefore again maxima. Then, three maxima in each quadrant (between P and Q or between Q and R etc.) corresponding to $|\Delta x| = \lambda$, 2λ and 3λ. Therefore, there are total 16 maxima.

In each quadrant there are four minima corresponding to $|\Delta x| = 0.5\lambda$, 1.5λ, 2.5λ and 3.5λ. Hence, there are total 16 minima.

9. $d\sin\theta = 2\lambda$ (for second order maxima)

$$\sin\theta = 2\frac{\lambda}{d} = 2\times\frac{1}{4} = \frac{1}{2}$$

$$\therefore \qquad \theta = 30^\circ$$

For maxima, $\quad d\sin\theta = n\lambda$

$$\therefore \qquad n_{\max} = \frac{d}{\lambda} = 4 \qquad \text{(for } \theta = 90^\circ)$$

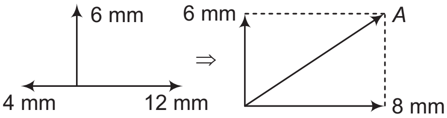
So, there are total 7 maxima corresponding to $n = 0, \pm 1, \pm 2$ and ± 3.

We cannot take $n = 4$, as it is for $\theta = 90^\circ$, which is out of screen.

10. Shift $= \dfrac{(\mu - 1)\,tD}{d}$ is independent of λ.

Objective Questions

1. $A = \sqrt{(8)^2 + (6)^2} = 10\,\text{mm}$



2. $\dfrac{I_1}{I_2} = \beta^2$

So let, $\quad I_2 = 1$ unit, then $I_1 = \beta$

$$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2 = (1+\beta)^2$$

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2 = (1-\beta)^2$$

$$= I_{\max} - I_{\min} = 4\beta$$

$$I_{\max} + I_{\min} = 2(1+\beta^2)$$

$\therefore$ The asked ratio is $\dfrac{2\beta}{1+\beta^2}$.

3. $d\sin\theta = \dfrac{\lambda}{2}$

$$\therefore \qquad \theta = \sin^{-1}\left(\frac{\lambda}{2d}\right) = \sin^{-1}\left(\frac{5460\times10^{-10}}{2\times0.1\times10^{-3}}\right)$$

$$= 0.16^\circ$$

4. 6th dark fringe distance in vacuum = 10th bright fringe distance in liquid.

$$\therefore \qquad 5.5\,\omega = 10\,\omega'$$

$$\text{or} \qquad \frac{\omega}{\omega'} = \mu = \frac{10}{5.5} = 1.81$$

5.

$$\text{Shift} = \frac{(\mu - 1)tD}{d}$$

$$= \frac{(1.5-1)\,(10\times10^{-6})\,(1.0)}{2.5\times10^{-3}}$$

$$= 2\times10^{-3}\text{ m}$$

$$= 2\text{ mm}$$

6. Distance $= \dfrac{\omega}{2} = \dfrac{(\lambda D/d)}{2} = \dfrac{\lambda D}{2d}$

7. Path difference at centre is always zero. Hence, all wavelengths under all conditions always interfere constructively.

8. $\omega = \dfrac{\lambda D}{d} \quad$ or $\quad \omega \propto \lambda$

ω will increase $\dfrac{6000}{4000}$

or $\quad$ 1.5 times.

Hence, number of fringes in same distance will decrease 1.5 times.

9. $\omega = \frac{\lambda D}{d}$

$$\Rightarrow \quad \Delta\omega = \frac{\lambda(\Delta D)}{d}$$

$$\therefore \quad \lambda = \frac{(d)(\Delta\omega)}{\Delta D} = \frac{(10^{-3})(3 \times 10^{-5})}{5 \times 10^{-2}}$$

$$= 0.6 \times 10^{-6} \text{ m} = 6000 \text{ Å}$$

10. $\frac{I_{\max}}{I_{\min}} = \frac{(\sqrt{I_1/I_2} + 1)^2}{(\sqrt{I_1/I_2} - 1)^2} = \frac{49}{9}$

Solving, we get $\frac{I_1}{I_2} = \frac{25}{4}$

11. $y = \frac{3\lambda D}{d}$

$$\therefore \quad \lambda = \frac{yd}{3D} = \frac{(7.5 \times 10^{-3})(0.2 \times 10^{-3})}{(3)(1)}$$

$$= 500 \times 10^{-9} \text{ m} = 500 \text{ nm}$$

12. $\frac{n_1\lambda_1 D}{d} = \frac{n_2\lambda_2 D}{d}$

or $\frac{n_1}{n_2} = \frac{\lambda_1}{\lambda_2} = \frac{5200}{6500} = \frac{4}{5}$

$\therefore$ 4th maxima of λ_1 coincides with 5th maxima of λ_2.

$$y_{\min} = \frac{4\lambda_1 D}{d}$$

$$= \frac{4 \times 6500 \times 10^{-10} \times 1.2}{2 \times 10^{-3}}$$

$$= 1.56 \times 10^{-3} \text{ m} = 0.156 \text{ cm}$$

13. Only fringe pattern will shift. Number of fringes on screen will remain unchanged.

Subjective Questions

1. $A_{\max} = 5 + 3 = 8$ units

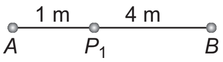
$$A_{\min} = 5 - 3 = 2 \text{ units}$$

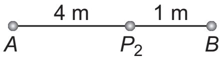
$$\frac{A_{\max}}{A_{\min}} = 4$$

$$\therefore \quad \frac{I_{\max}}{I_{\min}} = (4)^2 = 16$$

2. (a) At centre path difference is zero. Therefore, construction interference will be obtained.

(b) $\frac{\lambda}{2} = 3$ m. At a distance, where path difference is $\frac{\lambda}{2}$ or 3 m destructive interference will be obtained.

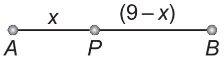




At P_1 $\quad BP_1 - AP_1 = 3\text{m} = \frac{\lambda}{2}$

At P_2 $\quad AP_2 - BP_2 = 3\text{m} = \frac{\lambda}{2}$

3. $\because \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{120 \times 10^6} = 2.5$ m



$$\Delta x = (BP - AP) = (9 - 2x) = n\lambda$$

$$\therefore \quad x = \frac{9 - n\lambda}{2} = \frac{9 - 2.5n}{2}$$

Now, substituting $n = 1, 2, \cdots$ etc.

We can find different values of x.

$x_1 = 3.25$ m for $n = 1$

$x_2 = 2.0$ m for $n = 2$

and $x_3 = 0.75$ m for $n = 3$

Similarly, we will get three points at same distance from other point B.

4. $\because \quad \omega = \frac{\lambda D}{d}$

$$\therefore \quad \lambda = \frac{\omega d}{D} = \frac{(2.82 \times 10^{-3})(0.46 \times 10^{-3})}{2.2}$$

$$= 0.589 \times 10^{6} \text{ m}$$

$$\approx 0.590 \text{ nm}$$

5. $\theta = \frac{\lambda}{d} = \frac{500 \times 10^{-9}}{2.0 \times 10^{-3}}$ radian

$$\approx 0.014°$$

6. Wavelength in water, $\lambda' = \frac{\lambda}{\mu}$

Fringe width, $\omega = \frac{\lambda' D}{d} = \frac{\lambda D}{\mu d}$

$$= \frac{(700 \times 10^{-9})(0.48)}{(4/3)(0.25 \times 10^{-3})}$$

$$= 10^{-3} \text{m}$$

$$= 1 \text{ mm}$$

7. Distance $= \frac{3\lambda_2 D}{d} - \frac{3\lambda_1 D}{d}$

$$= \frac{3(\lambda_2 - \lambda_1)D}{d}$$

$$= \frac{3 \times (600 - 480) \times 10^{-9} \times 1.0}{5.0 \times 10^{-3}}$$

$$= 7.2 \times 10^{-5} \text{ m}$$

$$= 0.072 \text{ mm}$$

8. The required distance = one fringe width

$$= \omega = \frac{\lambda D}{d}$$

$$= \frac{(500 \times 10^{-9})\,(0.75)}{(0.45 \times 10^{-3})}$$

$$= 8.33 \times 10^{-4} \text{ m}$$

$$= 0.83 \text{ mm}$$

9. $y = \omega_1 = \frac{\omega_2}{2}$

$$\therefore \quad \frac{\lambda_1 D}{d} = \frac{\lambda_2 D}{2d}$$

$$\Rightarrow \quad \lambda_2 = 2\lambda_1 = 1200 \text{ nm}$$

10. $d \sin \theta_1 = \frac{\lambda}{2}$

$$\sin \theta_1 = \frac{\lambda}{2d} = \frac{550 \times 10^{-9}}{2 \times 1.8 \times 10^{-6}}$$

$$\therefore \quad \theta_1 = 8.78°$$

$$\frac{y_1}{D} = \tan \theta_1$$

$$\therefore \quad y_1 = D \tan \theta_1$$

$$= 35 \tan 8.78°$$

$$= 5.41 \text{ cm}$$

$$d \sin \theta_2 = \frac{3\lambda}{2}$$

$$\therefore \quad \sin \theta_2 = \frac{3\lambda}{2d} = \frac{3 \times 550 \times 10^{-9}}{2 \times 1.8 \times 10^{-6}}$$

$$\therefore \quad \theta_2 = 27.27°$$

$$\therefore \quad y_2 = D \tan \theta_2$$

$$= 35 \tan 27.27°$$

$$= 18 \text{ cm}$$

$$\Delta y = y_2 - y_1 = 12.6 \text{ cm}$$

11. $\lambda(\text{in Å}) = \sqrt{\frac{150}{E(\text{in eV})}}$

(de-Broglie wavelength of electron)

$$= \sqrt{\frac{150}{100}} = 1.22 \text{ Å}$$

$$\omega = \frac{\lambda D}{d} = \frac{(1.22 \text{ Å})\,(3\text{m})}{(10 \text{ Å})}$$

$$= 0.366 \text{ m}$$

$$= 36.6 \text{ cm}$$

12. (a) $\Delta x = \frac{yd}{D}$

and $\phi = \frac{2\pi}{\lambda} \cdot \Delta x$

$$= \frac{(2\pi)\,yd}{\lambda D}$$

$$= \frac{(2 \times 180)\,yd}{\lambda D} \text{ degree}$$

$$= \frac{360 \times 0.3 \times 10^{-3} \times 10 \times 10^{-3}}{546 \times 10^{-9} \times 1.0}$$

$$= 1978°$$

Now, $I = I_0 \cos^2 \frac{\phi}{2}$

$$= 2.97 \times 10^{-4} I_0$$

$$\approx 3.0 \times 10^{-4} I_0$$

(b) $\omega = \frac{\lambda D}{d}$

∴ Number of fringes between central fringe and P

$$N = \frac{y}{\omega} = \frac{yd}{\lambda D}$$

$$= \frac{(10 \times 10^{-3})\,(0.3 \times 10^{-3})}{(546 \times 10^{-9})\,(1.0)}$$

$$= 5.49$$

So, bright fringes are five.

13. Shift $= \frac{(\mu - 1)\,tD}{d}$

$$S_1 = \frac{(1.6 - 1)\,(10 \times 10^{-6})\,(1.5)}{1.5 \times 10^{-3}}$$

$$= 0.6 \times 10^{-2} \text{ m}$$

$$= 0.6 \text{ m}$$

$$S_2 = \frac{(1.2 - 1)\,(15 \times 10^{-6})\,(1.5)}{1.5 \times 10^{-3}}$$

$$= 0.3 \times 10^{-2} \text{ m}$$

$$= 0.3 \text{ cm}$$

$$= \Delta S = S_1 - S_2 = 0.3 \text{ cm}$$

$$= 3 \text{ mm}$$

14. $\frac{\lambda(2D)}{d} = \frac{(\mu - 1)tD}{d}$ (fringe width = shift)

$$\therefore \quad \lambda = \frac{(\mu - 1)t}{2}$$

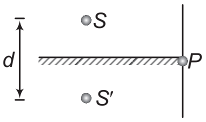
$$= \frac{(1.6 - 1)\,(1.964 \times 10^{-6})}{2} \text{ m}$$

$$= 0.5892 \times 10^{-6} \text{ m} = 589 \text{ nm}$$

15. $\because \quad d = 2\,\text{cm}$

$$\omega = \frac{\lambda D}{d} = \frac{(500 \times 10^{-9})\,(100)}{2 \times 10^{-2}}$$

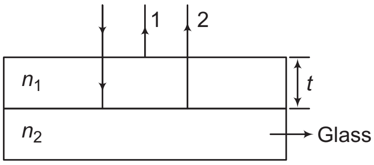
$$= 2.5 \times 10^{-3}\ \text{m} = 2.5\ \text{mm}$$



In this figure, *P* is in a dark fringe, as conditions of maxima and minima are interchanged.

Hence, next dark fringe will be obtained at a distance ω or 2.5 mm from *P*.

16. 1 and 2 both are reflected from denser medium.



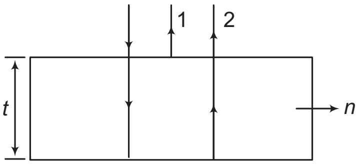
Hence,

$$2n_1 t = \frac{\lambda}{2} \quad \text{for first order minima}$$

or
$$t_{\min} = \frac{\lambda}{4n_1} = \frac{650}{4 \times 1.42}$$

$$= 114\ \text{nm}$$

17. (a) Ray-1 is reflected from a denser medium and ray-2 by a rarer medium.



$$\Delta x = 2nt = (2m-1)\frac{\lambda}{2} \text{ for maximum intensity}$$

$$\therefore \quad \lambda = \frac{4\ nt}{(2m-1)}$$

$$= \frac{(4)(1.53)\,(485\ \text{nm})}{2m-1}$$

$$= \left(\frac{2968}{2m-1}\right) \text{nm}$$

For $m = 1, \lambda = 2968\,\text{nm}$

For $m = 2, \lambda = 989\ \text{nm}$

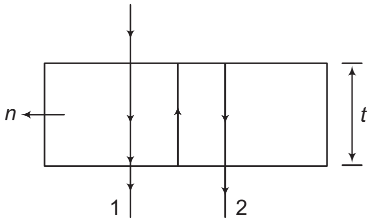
For $m = 3, \lambda = 594\ \text{nm}$

For $n = 4, \lambda = 424\ \text{nm}$

For $n = 5, \lambda = 329\,\text{nm}$

Therefore, two wavelengths lying in the given range are 424 nm and 594 nm.

(b) Ray-1 and ray-2 are in same phase.



Hence,

$$\Delta x = 2nt = m\lambda \qquad \text{(for maximum intensity)}$$

$$\therefore \quad \lambda = \frac{2nt}{m} = \frac{(2)\,(1.53)\,(485\ \text{nm})}{m}$$

$$= \left(\frac{1484}{m}\right) \text{nm}$$

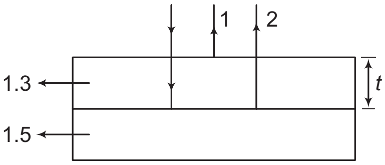
Substituting $m = 1, 2, 3 \cdots$ etc,

$\lambda_1 = 1484\ \text{nm}, \quad \lambda_2 = 742\,\text{nm},$

$\lambda_3 = 495\,\text{nm}, \quad \lambda_4 = 371\,\text{nm}$ etc.

Therefore, only wavelength lying in the given range is 495 nm.

18. Ray-1 and ray-2 both are reflected from denser medium. Hence, they are in phase.

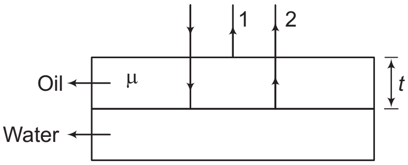


$$\therefore \qquad 2\,\mu t = \frac{\lambda}{2} \text{ for minima}$$

or
$$t = \frac{\lambda}{4\mu} = \frac{600}{4 \times 1.3}$$

$$= 1154\ \text{Å}$$

19. Ray-1 is reflected from denser medium and ray-2 from denser medium.



$\therefore \quad \Delta x = 2\,\mu t = \lambda = 800\,\text{nm}$ for destruction interference.

$\therefore \quad \mu t = 400 \text{ nm}$

For constructive interference,

$$\Delta x = 2\,\mu t = (2n-1)\frac{\lambda}{2}$$

$$\therefore \quad \lambda = \frac{4\mu t}{2n-1} = \frac{1600}{2n-1}$$

For $n = 1, \lambda = 1600 \text{ nm}$

For $n = 2, \lambda = 533 \text{ nm}$

For $n = 3, \lambda = 320 \text{ nm}$

The only wavelength lying in the given range is 533 nm.

20. $2\mu t = \lambda/2$

This is the condition for destructive interference.

$$\therefore \quad t = \frac{\lambda}{4\mu} = \frac{3.0}{4 \times 1.5} = 0.5 \text{ cm}$$

21. Path difference produced by slab,

$$\Delta x = (\mu - 1)t = \frac{\lambda}{2}$$

$\frac{\lambda}{2}$ path difference is equivalent to 180° phase difference. Hence, maxima and minima interchange their positions.

22. $\Delta x = d \sin\theta = n\lambda$

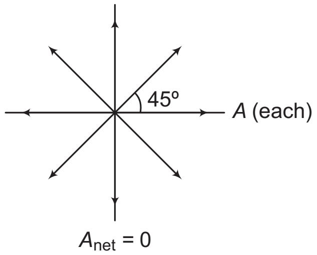
$$n = \frac{d\sin\theta}{\lambda}$$

$$n_{\max} = \frac{d}{\lambda} = \frac{4.0 \times 10^{-6}}{600 \times 10^{-9}}$$

$$= 6.66$$

Highest integer is 6.

23.



LEVEL 2

Single Correct Option

1. $2\,I_0 = 4\,I_0 \cos^2\left(\frac{\theta_1}{2}\right)$

$$\therefore \quad \phi_1 = \frac{\pi}{2} = \left(\frac{2\pi}{\lambda}\right)(\Delta\, x_1) = \left(\frac{2\pi}{\lambda}\right)\left(\frac{y_1 d}{D}\right)$$

$$\therefore \quad y_1 = \frac{\lambda D}{4d} = \frac{\beta}{4}$$

Further, $\quad I_0 = 4I_0 \cos^2\left(\frac{\phi_2}{2}\right)$

$$\therefore \quad \phi_2 = \frac{2\pi}{3} = \left(\frac{2\pi}{\lambda}\right)(\Delta\, x_2) = \left(\frac{2\pi}{\lambda}\right)\left(\frac{y_2 d}{D}\right)$$

$$\therefore \quad y_2 = \frac{\lambda D}{3d} = \frac{\beta}{3}$$

$$\Delta y = y_2 - y_1 = \frac{\beta}{12}$$

2. At path difference λ, we get maximum intensity.

$$\therefore \quad I_{\max} = I$$

$$I_R = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$$

$$\therefore \quad \frac{I}{4} = I \cos^2\left(\frac{\phi}{2}\right)$$

or $\quad \cos\left(\frac{\phi}{2}\right) = \pm\frac{1}{2}$

$$\therefore \quad \frac{\phi}{2} = 60° \text{ or } 120°$$

$$\therefore \quad \phi = 120° \text{ or } 240° \text{ or } \frac{2\pi}{3} \text{ and } \frac{4\pi}{3}$$

From the relation, $\Delta x = \left(\frac{\lambda}{2\pi}\right)\cdot\phi$

We see that,

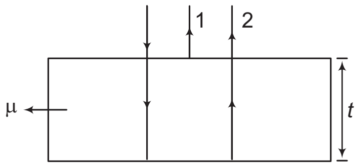
$$\Delta x = \frac{\lambda}{3} \text{ and } \frac{2\lambda}{3}$$

3. Ray-1 is reflected from a denser medium ($\Delta\,\phi = \pi$) while ray-2 comes after reflecting from a rarer medium ($\Delta\,\phi = 0°$).

$\therefore \quad \Delta x = 2\mu t = (2n-1)\frac{\lambda}{2}$ for maximum intensity.

or $\quad \lambda = \frac{4\mu t}{(2n-1)}$

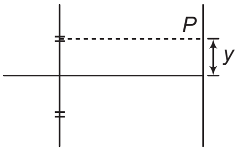
$$= \frac{4 \times 1.5 \times 500}{2n-1} = \left(\frac{3000}{2n-1}\right) \text{nm}$$



Substituting $n = 1, 2, 3 \cdots$ etc, we get $\lambda = 3000$ nm, 1000 nm, 600 nm etc.

$\therefore$ Answer is 600 nm.

4. $\Delta x = \dfrac{yd}{D}$



For destructive interference at P.

$$\Delta x = \frac{yd}{D} = (2n-1)\frac{\lambda}{2}$$

$$\therefore \quad \lambda = \frac{2yd}{(2n-1)D}$$

Substituting $n = 1, 3, 5 \cdots$ etc we get

$$\lambda = \frac{2yd}{D}, \frac{2}{3}\left(\frac{yd}{D}\right), \frac{2}{5}\left(\frac{yd}{D}\right) \text{ etc} \quad \ldots\text{(i)}$$

Here, $\dfrac{yd}{D} = \dfrac{(2\times10^{-3})(0.1\times10^{-3})}{1.0}$

$$= 2\times10^{-7}\,\text{m}$$

$$= 2000\ \text{Å}$$

Substituting in Eq. (i), we get λ = 4000 Å, 2680 Å, 1600Å etc.

So that answer is 4000 Å.

5. $\dfrac{3}{4}I_{\max} = I_{\max}\cos^2\left(\dfrac{\phi}{2}\right)$

$$\therefore \quad \frac{\phi}{2} = \frac{\pi}{12} \text{ and } \frac{5\pi}{6}$$

$$\therefore \quad \phi = \frac{\pi}{6}$$

and $\dfrac{5\pi}{3} = \left(\dfrac{2\pi}{\lambda}\right)(\Delta x) = \left(\dfrac{2\pi}{\lambda}\right)\left(\dfrac{yd}{D}\right)$

$$\therefore \quad y_1 = \frac{\lambda D}{12d} = \frac{(6000\times10^{-10})(1)}{(12)(10^{-3})}$$

$$= 0.05\times10^{-3}\,\text{m} = 0.05\ \text{mm}$$

$$y_2 = 5\left(\frac{\lambda D}{12d}\right) = 5\times0.05\ \text{mm}$$

$$= 0.25\ \text{mm}$$

$$\Delta y = y_2 - y_1 = 0.2\ \text{mm}$$

6. Shift $= \dfrac{(\mu-1)\,tD}{d} = 3.5,\ \omega = \dfrac{3.5\,\lambda D}{d}$

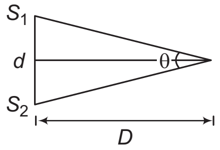
$$t = \frac{3.5\lambda}{\mu - 1}$$

$$= \frac{(3.5)(6000\times10^{-10})}{1.5-1}$$

$$= 4.2\times10^{-6}\text{m}$$

$$= 4.2\ \mu\text{m}$$

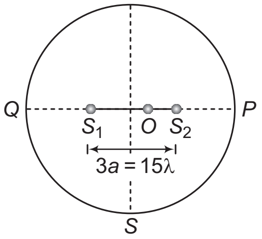
7. $\because \quad \theta = \dfrac{d}{D}$



Third minima,

$$y = \pm\,2.5\,\omega = \pm\,2.5\left(\frac{\lambda D}{d}\right) = \pm\frac{2.5\lambda}{\theta}$$

8. At points P and Q,



$|\Delta x| = 15$, therefore maxima

At points R and S

$\Delta x = 0$, therefore maxima.

Between P and R (and similarly in other three quadrants), we will get 14 maxima corresponding to, $\Delta x = \lambda, 2\lambda \cdots 14\lambda$.

Therefore, total maximas are 60.

9. $\Delta x = \dfrac{yd}{D} = (\omega/4)\dfrac{d}{D} = \dfrac{\lambda}{4}\left(\omega = \dfrac{\lambda D}{d}\right)$

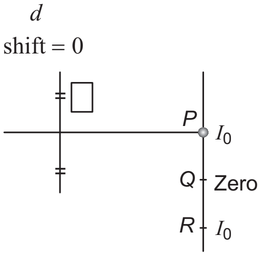
$$\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x = \frac{\pi}{2} \text{ or } 90^\circ$$

$$I = I_{\max}\cos^2\frac{\theta}{2}$$

$$\therefore \quad \frac{I}{I_{\max}} = \cos^2 45^\circ = \frac{1}{2}$$

10. Shift $= \dfrac{(\mu-1)tD}{d}$

At $\mu = 1$, shift $= 0$



Therefore, intensity at centre is maximum or I_0.

As μ increases fringes shift upwards as shown in figure.

So, intensity at P first decreases to zero (as Q reaches at P), then it further increases to I_0 (as point R reaches to P).

11. $\frac{3}{4} I_{\max} = I_{\max} \cos^2 \frac{\phi}{2}$

$$\therefore \quad \frac{\phi}{2} = \frac{\pi}{6} \text{ or } \phi = \frac{\pi}{3} = \left(\frac{2\pi}{\lambda}\right)(\Delta x) = \left(\frac{2\pi}{\lambda}\right)(\mu - 1)\, t$$

$$\therefore \quad t = \frac{\lambda}{6(\mu - 1)} = \frac{6000 \times 10^{-10}}{6(1.5 - 1)}$$

$$= 0.2 \times 10^{-6}\text{m} = 0.2\,\mu\text{m}$$

12. $\Delta x_1 = (\mu_1 - 1),\ \Delta x_2 = (\mu_2 - 1)t$

$$\therefore \quad \Delta x = (\mu_1 - \mu_2)\, t$$

$$= (1.52 - 1.40)\,(10400 \text{ nm})$$

$$= 1248 \text{ nm}$$

For maximum intensity,

$$\Delta x = 1248 = n\lambda$$

$$\therefore \quad \lambda = \frac{1248}{n} \qquad (n = 1, 2, 3 \cdots)$$

For $n = 2,\ \lambda = 624$ nm

and for $n = 3,\ \lambda = 416$ nm

More than One Correct Options

1. λ_v is least. Therefore ω_v is minimum (as $\omega \propto \lambda$). Hence, the fringe next to centre will be violet.

At centre, $\Delta x = 0$ for all wavelengths. Hence, all wavelengths interfere constructively at centre. So, it is white.

2. (c) $\quad I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

When $\quad I_1 = I_2 = I_0$

$$I_{\max} = 4\, I_0 \text{ and } I_{\min} = 0$$

When $\quad I_2 = \frac{I_0}{2}$,

then $\quad I_{\max} < 4I_0$ and $I_{\min} > 0$.

3. $\Delta x_0 = d \sin\theta = (10^{-3})\left(\frac{1}{2}\right)$

$$= 5 \times 10^{-4}\text{m} = (10^3)\,\lambda$$

Since, Δx_0 is integer multiple of λ, it will produce maximum intensity or $4I_0$ at O.

$$\omega = \frac{\lambda D}{d} = \frac{(5 \times 10^{-7})\,(2)}{(10^{-3})}$$

$$= 10^{-3}\text{m} = 1 \text{ mm}$$

At 4 mm, we will get 4th order maxima.

5. Fringe pattern shifts in the direction of slab. But,

$$\text{Shift} = \frac{(\mu - 1)\, tD}{d}$$

So, actual shift will depend on the values of μ, t, D and d.

6. **For overlapping of maxima**

$$\frac{n_1 \lambda_1 D}{d} = \frac{n_2 \lambda_2 D}{d}$$

$$\text{or} \quad \frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} = \frac{7}{5}, \frac{14}{10} \cdots$$

$\Rightarrow$ 14th order maxima of λ_1 will coincide with 10th order maxima of λ_2.

For overlapping of minima

$$\frac{(2n_1 - 1)\,\lambda_1 D}{2d} = \frac{(2n_2 - 1)\,\lambda_2 D}{2d}$$

$$\therefore \quad \frac{2n_1 - 1}{2n_2 - 1} = \frac{\lambda_2}{\lambda_1} = \frac{7}{5}$$

(c) Option with $n_1 = 11$ and $n_2 = 8$ gives this ratio.

Comprehension Based Questions

1. $\lambda = \frac{\lambda_0}{\mu}, \quad \omega = \frac{\lambda D}{d} = \frac{\lambda_0 D}{\mu d}$

$$= \frac{(6300 \times 10^{-10})\,(1.33)}{1.33 \times 10^{-3}}$$

$$= 0.63 \times 10^{-3}\text{ m}$$

$$= 0.63 \text{ mm}$$

2. $\Delta y = 7\omega - 3\omega = 4\omega$

$$= 4 \times 0.63 \text{ mm}$$

$$= 2.52 \text{ mm}$$

3. $\Delta x = \left(\frac{\mu_2}{\mu_1} - 1\right) t = \frac{\lambda}{2}$

$$\therefore \quad t = \frac{\lambda}{2\left(\frac{\mu_2}{\mu_1} - 1\right)}$$

$$= \frac{(6300 \times 10^{-10}/1.33)}{2\left(\frac{1.53}{1.33} - 1\right)}$$

$$= 1.575 \times 10^{-3}\text{m}$$

$$= 1.575 \text{ mm}$$

4. Fringe width remains unchanged by the introduction of glass sheet.

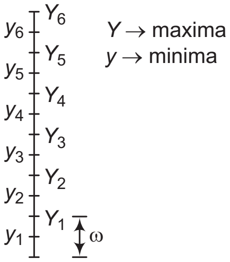
Match the Columns

1. $I = 4I_0 \cos^2\left(\frac{\phi}{2}\right)$

2. $\phi = \frac{2\pi}{\lambda}\Delta x$ and then apply,

$$I = 4I_0 \cos^2\left(\frac{\phi}{2}\right)$$

3.

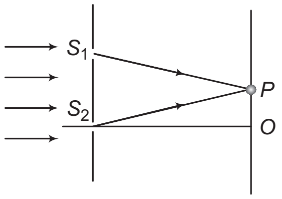


4. (a) Inclined rays and slab both will shift the fringe pattern upwards. So, zero order maxima will definitely lie about O.

(b) Inclined rays will shift the fringe pattern upwards, but slab will shift the fringe pattern downwards. Hence, zero order maxima may lie above O, below O or at O.

(c) Same explanations can be given for this option.

(d)



$$\Delta X_p = 0, \text{ where } S_1P = S_2P$$

5. $I = I_{\max} = 4I_0 \Rightarrow I_0 = \frac{I}{4}$

(a) For $y = \frac{\lambda D}{2d}, \Delta x \frac{yd}{D} = \frac{\lambda}{2}$

∴ $I_{S_3} = I_{S_4} = 0$ for $\Delta x = \lambda/2$

∴ $I_0 = 0$

(b) For $y = \frac{\lambda D}{6d}$, $\Delta x = \frac{yd}{D} = \frac{\lambda}{6}$

$$\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x) = \frac{2\pi}{6} = 60°$$

Now, $I_{S_3} = I_{S_4} = I_{\max}\cos^2\frac{\theta}{2}$

$$= I\cos^2 30° = \frac{3}{4}I$$

∴ $I_0 = 4I_{S_3}$

or $I_{S_4} = 3I$

Same explanations can be given for (c) and (d) options.

6. (a) Fringe pattern will shift in the direction of slab.

(b) No interference pattern will be obtained due to single slit.

∴ $I = I_{S_2} = I_0 \approx$ uniform

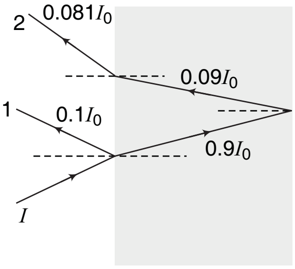
(c) Fringe pattern will shift in the direction of slab.

(d) No interference pattern will be obtained due to two real incoherent sources.

$$I = I_{S_1} + I_{S_2} \approx \text{ uniform}$$

Subjective Questions

1. $I_1 = 0.1I_0$, $I_2 = 0.081I_0$



∴ $\sqrt{\frac{I_1}{I_2}} = \frac{10}{9}$

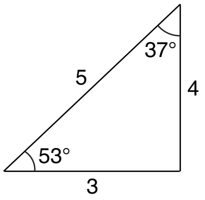
∴ $\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1/I_2}+1}{\sqrt{I_1/I_2}-1}\right)^2$

$= (19)^2$

$= 361$ **Ans.**

2. ∵ $\mu = \frac{\sin i}{\sin r}$

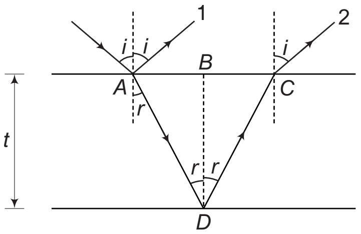
∴ $\frac{4}{3} = \frac{\sin 53°}{\sin r} = \frac{4/5}{\sin r}$



∴ $\sin r = \frac{3}{5}$

∴ $r = 37°$

Refer figure (a)

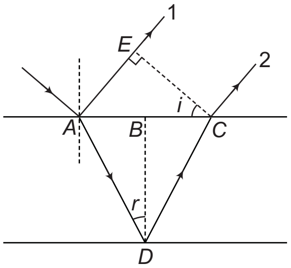


(a)

$$\begin{aligned}\Delta x_1 &= \text{between 2 and 1} = 2(AD)\\ &= 2BD\sec r\\ &= 2t\sec r\end{aligned}$$

Their optical path $\Delta x_1 = 2\mu t \sec r$.

Refer figure (b)



(b)

$$\Delta x_2 = AC\sin i = (2t\tan r)\sin i$$

$$\begin{aligned}\therefore\quad (\Delta x)_{\text{net}} &= \Delta x_1 - \Delta x_2\\ &= 2\mu t\sec r - 2t(\tan r)(\sin i)\\ &= 2\times\frac{4}{3}\times t\times\frac{5}{4} - 2\times t\times\frac{3}{4}\times\frac{4}{5} = \frac{32}{15}t\end{aligned}$$

Phase difference between 1 and 2 is π.

∴ For constructive interference,

$$\frac{32}{15}t = \frac{\lambda}{2} \quad\text{or}\quad t = \frac{15\lambda}{64} = \frac{15\times 0.6}{64}$$

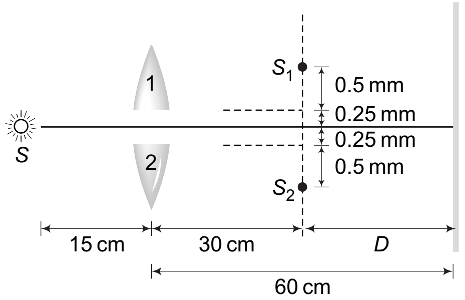
$$= 0.14\ \mu\text{m}$$ **Ans.**

3. Applying lens formula, $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$$\frac{1}{v} + \frac{1}{15} = \frac{1}{10}$$

$$\therefore\qquad v = 30\text{ cm}$$

$$m = \frac{v}{u} = \frac{30}{-15} = -2$$



Distance between two slits,

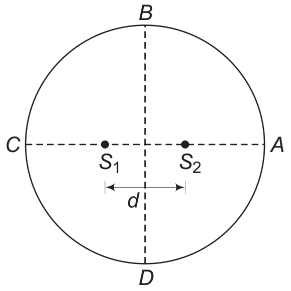
$$d = 1.5\text{ mm},\ D = 30\text{ cm}$$

Fringe width, $\omega = \frac{\lambda D}{d}$

$$= \frac{(5.0\times 10^{-7})(0.3)}{(1.5\times 10^{-3})} = 10^{-4}\text{ m}$$

$$= 0.1\text{ mm}$$ **Ans.**

4. $\lambda = 0.25$ m, $d = 2$ m $= 8\lambda$



At A and C, $\Delta x = d = 8\lambda$, i.e. maximum intensity is obtained.

At B and D, $\Delta x = 0$, i.e. again maximum intensity will be obtained.

Between A and B seven maximas corresponding to $\Delta x = 7\lambda, 6\lambda, 5\lambda, 4\lambda, 3\lambda, 2\lambda$ and λ will be obtained. Similarly, between B and C, C and D, and D and A.

∴ Total number of maximas

$$= 4\times 7 + 4$$

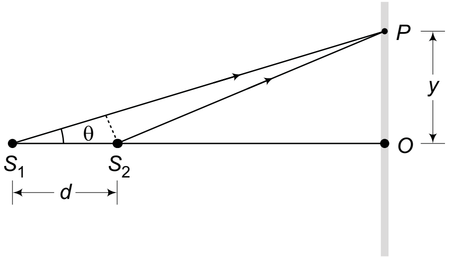
$$= 32$$ **Ans.**

5. (a) ∵ $\Delta x = d\cos\theta$

$$\cos\theta = 1 - \frac{\theta^2}{2} \qquad \text{(When } \theta \text{ is small)}$$

$$\therefore\qquad \Delta x = d\left(1 - \frac{\theta^2}{2}\right)$$

$$= d\left(1 - \frac{y^2}{2D^2}\right)$$



For nth maxima $\Delta x = n\lambda$

$\therefore$ y = radius of nth bright ring

$$= D\sqrt{2\left(1 - \frac{n\lambda}{d}\right)}$$ **Ans.**

(b) $d = 1000\lambda$

At O, $\Delta x = d = 1000\lambda$

i.e. at O, 1000th order maxima is obtained.

Substituting $n = 998$ in,

$$y = D\sqrt{2\left(1 - \frac{n\lambda}{d}\right)}$$

We get the radius of second closest ring

$r = 6.32$ cm **Ans.**

(c) $n = 998$ **Ans.**

6. (a) The optical path difference between the two waves arriving at P is

$$\Delta x = \frac{y_1 d}{D_1} + \frac{y_2 d}{D_2}$$

$$= \frac{(1)\,(10)}{10^3} + \frac{(5)\,(10)}{2 \times 10^3}$$

$$= 3.5 \times 10^{-2}\text{ mm} = 0.035\text{ mm}$$

As, $\Delta x = 70\lambda$

$\therefore$ 70th order maxima is obtained at P.

(b) At O, $\Delta x = \dfrac{y_1 d}{D_1} = 10^{-2}$ mm

$= 0.01$ mm

As $\Delta x = 20\lambda$

$\therefore$ 20th order maxima is obtained at O.

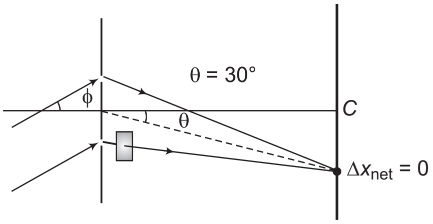
(c) $(\mu - 1)\,t = 0.01$ mm

$\therefore$ $t = \dfrac{0.01}{1.5 - 1} = 0.02\text{ mm} = 20\ \mu\text{m}$ **Ans.**

Since, the pattern has to be shifted upwards, therefore, the film must be placed in front of S_1.

7. (a) $d \sin\phi = \Delta x_1$

$= (50 \times 10^{-4}) \sin 30°$



$= 2.5 \times 10^{-3}$ cm

$\Delta x_2 = (\mu - 1)t$

$$= \left(\frac{3}{2} - 1\right)(0.01)$$

$= 5.0 \times 10^{-3}$ cm

$\Delta x_2 - \Delta x_1 = 2.5 \times 10^{-3}\text{ cm} = \Delta x_1$ (also)

$\therefore$ Central maxima will be obtained at $\theta = 30°$ below C.

(b) **At C** $\Delta X_{\text{net}} = 2.5 \times 10^{-3}\text{ cm} = n\lambda$

$\therefore$ $n = \dfrac{2.5 \times 10^{-3}}{\lambda}$

$$= \frac{2.5 \times 10^{-3}}{500 \times 10^{-7}}$$

$= 50$ **Ans.**

(c) Number of fringes that will pass if we remove the slab

$$= \frac{\text{Path difference due to slab}}{\lambda}$$

$$= \frac{5 \times 10^{-3}}{500 \times 10^{-7}}$$

$= 100$ **Ans.**

8. (a) $(\Delta x)_{\text{net}} = 0$

$\therefore$ $\dfrac{y_1 d}{D_1} = \dfrac{y_2 d}{D_2}$

$\therefore$ $\dfrac{d/2}{1.5} = \dfrac{y}{2.0}$

or $y = \dfrac{d}{1.5}$

$= \dfrac{6}{1.5}$

$= 4$ mm **Ans.**

(b) At O, net path difference,

$$\Delta x = \frac{y_1 d}{D_1} = \frac{(d/2)\,(d)}{D_1} = \frac{(6 \times 10^{-3})^2}{2 \times 1.5} = 12 \times 10^{-6}\ \text{m} = 120 \times 10^{-7}\ \text{m}$$

$$\lambda = 6000\ \text{Å} = 6 \times 10^{-7}\ \text{m}$$

As, $\Delta x = 20\lambda$, therefore at O bright fringe of order 20 will be obtained.

(c) $I = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$

$$\frac{3}{4} I_{\max} = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$$

$$\therefore \quad \frac{\phi}{2} = \frac{\pi}{6}$$

$$\phi = \frac{\pi}{3} = \left(\frac{2\pi}{\lambda}\right)(\mu - 1)\,t$$

$$\therefore \quad t = \frac{\lambda}{6\,(\mu - 1)} = \frac{6000}{6\,(1.5 - 1)} = 2000\ \text{Å}$$ **Ans.**

9. (a) $\left(1 - \frac{1}{\mu}\right) t = \frac{3\lambda}{\mu}$

$$\therefore \quad t = \frac{3\lambda}{(\mu - 1)} = \frac{3 \times 0.78}{1.3 - 1} = 7.8\ \mu\text{m}$$ **Ans.**

(b) **Upwards** $\frac{yd}{D} - \left(1 - \frac{1}{\mu}\right) t = \frac{4\lambda}{\mu}$

Solving, we get $y = 4.2$ mm **Ans.**

Downwards $t\left(1 - \frac{1}{\mu}\right) + \frac{yd}{D} = \frac{4\lambda}{\mu}$

Solving, we get $y = 0.6$ mm **Ans.**

33. Modern Physics - I

INTRODUCTORY EXERCISE 33.1

1. $E = \dfrac{hc}{\lambda}, P = \dfrac{h}{\lambda}$

2. Number of photons emitted per second,

$$N_1 = \frac{\text{Power of source}}{\text{Energy of one photon}} = \frac{P}{(hc/\lambda)} = \frac{P\lambda}{hc}$$

At a distance r, these photons are falling on an area $4\pi r^2$.

$\therefore$ Number of photons incident per unit area per unit time,

$$N_2 = \frac{N_1}{4\pi r^2} = \frac{P\lambda}{(4\pi r^2)\, hc}$$

3. $$\lambda = \frac{h}{\sqrt{2qVm}} \Rightarrow \lambda \propto \frac{1}{\sqrt{qm}}$$

$$\frac{\lambda_P}{\lambda_d} = \sqrt{\frac{(qm)_d}{(qm)_P}} = \sqrt{\frac{1\times 2}{1\times 1}} = \sqrt{2}$$

4. $$\lambda = \frac{h}{\sqrt{2Km}} \Rightarrow \lambda \propto \frac{1}{\sqrt{m}}$$

$$\frac{\lambda_d}{\lambda_\alpha} = \sqrt{\frac{m_\alpha}{m_d}} = \sqrt{\frac{4}{2}} = \sqrt{2}$$

5. Kinetic energy in magnetic field remains unchanged while in electric field it will increase. Further,

$$\lambda \propto \frac{1}{\sqrt{K}}$$

6. $$\lambda = \frac{h}{mv}$$

(a) $$\lambda = \frac{6.63\times 10^{-34}}{(46\times 10^{-3})\,(30)} = 4.8\times 10^{-34}\ \text{m}$$

(b) $$\lambda = \frac{6.63\times 10^{-34}}{9.31\times 10^{-31}\times 10^{7}} = 7.12\times 10^{11}\ \text{m}$$

INTRODUCTORY EXERCISE 33.2

1. $Z = 3$ for doubly ionized atom $E \propto Z^2$

Ionization energy of hydrogen atom is 13.6 eV.

$\therefore$ Ionisation energy of this atom

$$= (3)^2(13.6) = 122.4\ \text{eV}$$

2. $E_n = \dfrac{-13.6}{(n)^2} = -1.51 \Rightarrow n = 3$

Now, $$L_n = n\left(\frac{h}{2\pi}\right)$$

3. The expressions of kinetic energy, potential energy and total energy are

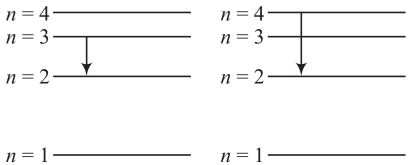
$$K_n = \frac{me^4}{8\varepsilon_0^2 n^2 h^2} \Rightarrow K_n \propto \frac{1}{n^2}$$

$$U_n = \frac{-me^4}{4\varepsilon_0^2 n^2 h^2} \Rightarrow U_n \propto -\frac{1}{n^2} \quad \text{and}$$

$$E_n = \frac{-me^4}{8\varepsilon_0^2 n^2 h^2} \Rightarrow E_n \propto -\frac{1}{n^2}$$

In the transition from some excited state to ground state, the value of n decreases, therefore kinetic energy increases, but potential and total energy decrease.

4.



First line of Balmer series

Second line of Balmer series

For hydrogen or hydrogen type atoms,

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

In the transition from $n_i \longrightarrow n_f$,

$$\therefore \quad \lambda \propto \frac{1}{Z^2\left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2}\right)}$$

$$\therefore \quad \frac{\lambda_2}{\lambda_1} = \frac{Z_1^2\left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2}\right)_1}{Z_2^2\left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2}\right)_2}$$

$$\lambda_2 = \frac{\lambda_1 Z_1^2\left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2}\right)_1}{Z_2^2\left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2}\right)_2}$$

Substituting the values, we have

$$= \frac{(6561\ \text{Å})\,(1)^2\left(\frac{1}{2^2}-\frac{1}{3^2}\right)}{(2)^2\left(\frac{1}{2^2}-\frac{1}{4^2}\right)} = 1215\ \text{Å}$$

$\therefore$ Correct option is (a).

5. The series in U-V region is Lyman series. Largest wavelength corresponds to minimum energy which occurs in transition from $n=2$ to $n=1$.

$$\therefore \qquad 122 = \frac{\frac{1}{R}}{\left(\frac{1}{1^2}-\frac{1}{2^2}\right)} \qquad \ldots\text{(i)}$$

The smallest wavelength in the infrared region corresponds to maximum energy of Paschen series.

$$\therefore \qquad \lambda = \frac{\frac{1}{R}}{\left(\frac{1}{3^2}-\frac{1}{\infty}\right)} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\lambda = 823.5\ \text{nm}$$

$\therefore$ Correct option is (b).

6. The first photon will excite the hydrogen atom (in ground state) to first excited state (as $E_2 - E_1 = 10.2$ eV). Hence, during de-excitation a photon of 10.2 eV will be released. The second photon of energy 15 eV can ionise the atom. Hence, the balance energy, i.e. $(15-13.6)$ eV $= 1.4$ eV is retained by the electron. Therefore, by the second photon an electron of energy 1.4 eV will be released.

$\therefore$ Correct answer is (c).

7. In second excited state $n = 3$,

$$\text{So,} \qquad l_{\text{H}} = l_{\text{Li}} = 3\left(\frac{h}{2\pi}\right)$$

while $E \propto Z^2$ and $Z_{\text{H}} = 1,\ Z_{\text{Li}} = 3$

$$\text{So,} \qquad |E_{\text{Li}}| = 9\,|E_{\text{H}}|$$

$$\text{or} \qquad |E_{\text{H}}| < |E_{\text{Li}}|$$

8. Energy of infrared radiation is less than the energy of ultraviolet radiation. In options (a), (b) and (c), energy released will be more, while in option (d) only, energy released will be less.

9. For hydrogen and hydrogen like atoms,

$$E_n = -13.6\,\frac{(Z^2)}{(n^2)}\ \text{eV}$$

Therefore, ground state energy of doubly ionized lithium atom ($Z = 3$, $n = 1$) will be

$$E_1 = (-13.6)\,\frac{(3)^2}{(1)^2}$$

$$= -122.4\ \text{eV}$$

$\therefore$ Ionization energy of an electron in ground state of doubly ionized lithium atom will be 122.4 eV.

10. Shortest wavelength will correspond to maximum energy. As value of atomic number (Z) increases, the magnitude of energy in different energy states gets increased. Value of Z is maximum for doubly ionised lithium atom ($Z = 3$) among the given elements. Hence, wavelength corresponding to this will be least.

$\therefore$ Correct option is (d).

11. $5E - E = hf \quad \Rightarrow \quad E = \frac{hf}{4}$ $\qquad \ldots$(i)

Between 5*E* and 4*E*

$$5E - 4E = hf_1$$

$$\therefore \qquad f_1 = \frac{E}{h} = \frac{f}{4} \qquad \text{[from Eq. (i)]}$$

Between 4*E* and *E*

$$4E - E = hf_2$$

$$\therefore \qquad f_2 = \frac{3E}{h} = 3\left(\frac{f}{4}\right) = \frac{3}{4}f$$

12. Longest wavelength means minimum energy.

$$(\Delta E)_{\min} = E_3 - E_2$$

$$= -\frac{13.6}{9} + \frac{13.6}{4} = 1.9\ \text{eV}$$

$$\lambda\ (\text{in Å}) = \frac{12375}{1.9} = 6513$$

or $\qquad \lambda \approx 651$ nm

INTRODUCTORY EXERCISE 33.3

1. K_α transition takes place from $n_1 = 2$ to $n_2 = 1$

$$\therefore \qquad \frac{1}{\lambda} = R(Z-b)^2\left[\frac{1}{(1)^2}-\frac{1}{(2)^2}\right]$$

For K-series, $b = 1$

$$\therefore \qquad \frac{1}{\lambda} \propto (Z-1)^2$$

$$\Rightarrow \qquad \frac{\lambda_{\text{Cu}}}{\lambda_{\text{Mo}}} = \frac{(Z_{\text{Mo}}-1)^2}{(Z_{\text{Cu}}-1)^2} = \frac{(42-1)^2}{(29-1)^2}$$

$$= \frac{41\times 41}{28\times 28} = \frac{1681}{784} = 2.144$$

2. Cut off wavelength depends on the applied voltage not on the atomic number of the target. Characteristic wavelengths depend on the atomic number of target.

3. $\dfrac{1}{\lambda} \propto (Z-1)^2$

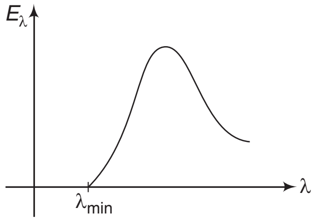
$$\therefore \quad \frac{\lambda_1}{\lambda_2} = \left(\frac{Z_2-1}{Z_1-1}\right)^2 \quad \text{or} \quad \frac{1}{4} = \left(\frac{Z_2-1}{11-1}\right)^2$$

Solving this, we get $Z_2 = 6$

$\therefore$ Correct answer is (a).

4. Wavelength λ_k is independent of the accelerating voltage (V), while the minimum wavelength λ_c is inversely proportional to V. Therefore, as V is increased, λ_k remains unchanged whereas λ_c decreases or $\lambda_k - \lambda_c$ will increase.

5. The continuous X-ray spectrum is shown in figure.



All wavelengths $>\lambda_{min}$ are found, where

$$\lambda_{min} = \frac{12375}{V \text{ (in volt)}} \text{ Å}$$

Here, V is the applied voltage.

6. $\Delta E = h\nu = Rhc\,(Z-b)^2\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$

For K-series, $b = 1$

$$\therefore \quad \nu = Rc\,(Z-1)^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Substituting the values,

$$4.2 \times 10^{18} = (1.1 \times 10^7)\,(3 \times 10^8)\,(Z-1)^2\left(\frac{1}{1} - \frac{1}{4}\right)$$

$$\therefore \quad (Z-1)^2 = 1697$$

$$\text{or} \quad Z - 1 \approx 41$$

$$\text{or} \quad Z = 42$$

INTRODUCTORY EXERCISE 33.4

1. $K_{min} = 0$

and $\quad K_{max} = E - W$

$$= \frac{12375}{2000} - 3.0$$

$$\approx 3.19 \text{ eV}$$

2. $K_{max} = E - W = hf - hf_0$

$$= h\,(f - f_0)$$

$$\therefore \quad K_{max} \propto (f - f_0)$$

3. $K_{max} = E - W$

$$1.2 = E - W \qquad \text{...(i)}$$

$$4.2 = 1.5\,E - W \qquad \text{...(ii)}$$

Solving these equations, we get

$$W = 4.8 \text{ eV} = hf_0$$

$$\therefore \quad f_0 = \frac{4.8 \times 1.6 \times 10^{-19}}{6.63 \times 10^{-34}}$$

$$= 1.16 \times 10^{15} \text{ Hz}$$

4. Energy corresponding to 248 nm wavelength

$$= \frac{1240}{248} \text{ eV} = 5 \text{ eV}$$

Energy corresponding to 310 nm wavelength

$$= \frac{1240}{310} \text{ eV} = 4 \text{ eV}$$

$$\frac{\text{KE}_1}{\text{KE}_2} = \frac{u_1^2}{u_2^2} = \frac{4}{1}$$

$$= \frac{5 \text{ eV} - W}{4 \text{ eV} - W}$$

$$\Rightarrow \quad 16 - 4W = 5 - W$$

$$\Rightarrow \quad 11 = 3W$$

$$\Rightarrow \quad W = \frac{11}{3} = 3.67 \text{ eV}$$

$$\cong 3.7 \text{ eV}$$

5. Saturation current is proportional to intensity while stopping potential increases with increase in frequency.

Hence, $\quad f_a = f_b$ while $I_a < I_b$

Therefore, the correct option is (a).

6. $\lambda \text{ (in Å)} = \dfrac{12375}{W \text{ (eV)}}$

$$= \frac{12375}{4.0} \text{ Å} \approx 3093 \text{ Å}$$

$$\text{or} \quad \lambda \approx 309.3 \text{ nm}$$

$$\text{or} \quad \approx 310 \text{ nm}$$

Note $\lambda(\text{in Å}) = \dfrac{12375}{W\text{ (eV)}}$ *comes from* $W = \dfrac{hc}{\lambda}$

7. Stopping potential is the negative potential applied to stop the electrons having maximum kinetic energy. Therefore, stopping potential will be 4 V.

Exercises

LEVEL 1

Assertion and Reason

2. $E = \frac{hc}{\lambda}$ and $P = \frac{h}{\lambda}$

$\therefore \quad E$ and $P \propto \frac{1}{\lambda}$

Speed of all wavelengths (in vacuum) is c.

3. Intensity = energy incident per unit area per unit time or $I = n(hf)$.

Here, n = number of photons incident per unit area per unit time.

f = frequency of incident photons. Hence, $I \propto nf$.

Hence, intensity can be increased either by increasing n or f. But saturation current only depends on n $(I_s \propto m)$.

By increasing n and decreasing f, we can increase the saturation current even without increasing the intensity.

5. Let us take $n = 3$ as $N = 1$. Then, $n = 6$ means $N = 4$.

So, total number of emission lines between $N = 1$ and $N = 4$ are

$$\frac{N(N-1)}{2} = \frac{4 \times 3}{2} = 6$$

6. $eV_0 = h\nu - h\nu_0$

or
$$V_0 = \frac{h}{e}(\nu - \nu_0)$$
$$V_0' = \frac{h}{e}(2\nu - \nu_0) + \frac{h\nu_0}{e}$$
$$= 2V_0 = \frac{h\nu_0}{e} > 2V_0$$

7. Energy of X-ray > 13.6 eV

8. $\Delta\lambda = \lambda_{K_\alpha} - \lambda_{\min}$

Here, $\lambda_{\min}$ (in Å) $= \frac{12375}{V \text{ (in volts)}}$

If V is increased, $\lambda_{\min}$ decreases. Therefore, $\Delta\lambda$ increases.

9.
$$\left.\begin{aligned} E_2 &= -3.4 \text{ eV} \\ U_2 &= -6.8 \text{ eV} \end{aligned}\right\}$$
$$\left.\begin{aligned} E_1 &= -13.6 \text{ eV} \\ U_1 &= -27.2 \text{ eV} \end{aligned}\right\}$$
$$E_2 > E_1 \text{, similarly } U_2 > U_1$$

10. In X-ray spectrum, all wavelengths greater than $\lambda_{\min}$ are obtained.

Objective Questions

1. $K_{\max} = hf - W$

$K_{\max}$ *versus* f graph is a straight line of slope h (a universal constant)

2. $v_1^H = 2.19 \times 10^6 \text{ m/s} \approx \frac{c}{137}$

3. Let α-particles are n and β-particles are m. Then,

$$86 - 2n + m = 84 \qquad \ldots\text{(i)}$$
$$222 - 4n = 210 \qquad \ldots\text{(ii)}$$

Solving these two equations, we get $n = 3$ and $\beta = 4$

4. $\lambda_{\min} = \frac{12375}{V \text{ (in volts)}}$ in Å

$$= \frac{12375}{20 \times 10^3} = 0.62 \text{ Å}$$

5.
$$K_{\max} = 18 \times 10^3 \text{ eV} = \frac{1}{2} m v_{\max}^2$$
$$\therefore \quad v_{\max} = \sqrt{\frac{2 \times 18 \times 10^3 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}}$$
$$= 8 \times 10^7 \text{ m/s}$$

6. $(2\pi r_2) = 2\lambda_2$

$(2\pi r_3) = 3\lambda_3$

$$\therefore \quad \frac{\lambda_2}{\lambda_3} = \frac{3r_2}{2r_3} \qquad \ldots\text{(i)}$$

Now, $\quad r \propto n^2$

$$\therefore \quad \frac{r_2}{r_3} = \left(\frac{2}{3}\right)^2$$

Substituting in Eq. (i), we get

$$\frac{\lambda_2}{\lambda_3} = \frac{2}{3}$$

7. $E \propto Z^2$

$$\therefore \quad (-13.6)(Z^2) = -122.4$$
$$\therefore \quad Z = 3$$

8. $\lambda_{\min} \propto \frac{1}{V} \propto V^{-1}$

% change in $\lambda_{\min}$ = (– 1) (% change in V) for small % changes

9. $E \propto \frac{Z^2}{n^2}$

$$\therefore \qquad \frac{(-13.2)\times(Z)^2}{(2)^2} = -13.6$$

$$\therefore \qquad Z = 2$$

10. $\lambda_{\min}\ (\text{in Å}) = \frac{12375}{V\ (\text{in volts})}$

For $\lambda_{\min} = 1\text{Å}$

$$V = 12375\ \text{V}$$
$$\approx 12.4\ \text{kV}$$

11. $P = \frac{h}{\lambda}$

12. $\lambda = \frac{12375}{1.6} = 7734\ \text{Å}$

13. $\Delta E = \frac{12375}{1085} = 11.4\ \text{eV}$

Third Balmer line is corresponding to the transition,

$$n = 5 \ \text{ to } \ n = 2.$$

$$E_5 - E_2 = 11.4$$

$$\therefore \qquad \frac{E_1}{(5)^2} - \frac{E_1}{(2)^2} = 11.4$$

$$E_1 = -54.28\ \text{eV}$$

$$\therefore \qquad |E_1| = 54.28\ \text{eV}$$

14. Otherwise $U_1 = -27.2$ eV. Therefore, we have increased it by 27.2 eV. It implies that we have increased it by 27.2 eV in all states.

$$U_2 = -6.8\ \text{eV}$$

$$\therefore \qquad U_2' = (-6.8 + 27.2)\ \text{eV} = +20.4\ \text{eV}$$

$$E_2' = U_2' + K_2 = (20.4 + 3.4)\ \text{eV}$$
$$= 23.8\ \text{eV}$$

15. $eV_0 = \frac{hc}{\lambda_1} - W \qquad \ldots\text{(i)}$

Similarly, $\quad e(2V_0) = \frac{hc}{\lambda_2} - W \qquad \ldots\text{(ii)}$

Subtracting Eq. (i) from Eq. (ii), we get

$$eV_0 = \frac{hc(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2}$$

$$\therefore \qquad V_0 = \frac{hc(\lambda_1 - \lambda_2)}{e\lambda_1\lambda_2}$$

$$= \frac{(6.63\times10^{-34})\,(3\times10^{8})\,(110)}{1.6\times10^{-19}\times330\times220\times10^{-9}}$$

$$\approx 1.8\ \text{volt}$$

16. $\because \qquad E = \frac{hc}{\lambda} - W \qquad \ldots\text{(i)}$

$$4E = \frac{hc}{\lambda/3} - W \qquad \ldots\text{(ii)}$$

Solving these equations, we get

$$W = \frac{hc}{3\lambda}$$

17. $\sqrt{f} \propto (Z-1)$ for K-series

$$\therefore \qquad f \propto (Z-1)^2$$

$$\frac{f_2}{f_1} = \frac{(Z_2-1)^2}{(Z_1-1)^2}$$

$$\therefore \qquad f_2 = \left(\frac{Z_2-1}{Z_1-1}\right)^2 f_1$$

$$= \left(\frac{51-1}{31-1}\right)^2 f$$

$$= \frac{25}{9} f$$

18. $\sqrt{f} \propto (Z-b)\sqrt{\left(\frac{1}{n_1^2} - \frac{1}{n_2}\right)^2}$

$$\therefore \qquad \text{Slope} \propto \sqrt{\frac{1}{n_1^2} - \frac{1}{n_2^2}}$$

$$\therefore \qquad \frac{\text{Slope}_1}{\text{Slope}_2} = \frac{\sqrt{1-1/9}}{\sqrt{1-\frac{1}{4}}} = \sqrt{\frac{32}{27}}$$

For K_β, $\quad n_1 = 1,\ n_2 = 3$

For K_α, $\quad n_1 = 1,\ n_2 = 2$

19. $\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) = R\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5R}{36}$

$$\therefore \qquad \lambda = \frac{36}{5R}$$

20. Energy of electrons = 10000 eV

$$\therefore \qquad \lambda_1\ (\text{in Å}) = \sqrt{\frac{150}{10000}} = 0.122$$

$$\lambda_2\ (\text{in Å}) = \frac{12375}{10000} \approx 1.2$$

$$\therefore \qquad \frac{\lambda_1}{\lambda_2} \approx 0.1$$

21. $\qquad e\,(5V_0) = \frac{hc}{\lambda} - W \qquad \ldots\text{(i)}$

$$eV_0 = \frac{hc}{3\lambda} - W \qquad \ldots\text{(ii)}$$

Solving these equations, we get

$$W = \frac{hc}{6\lambda}$$

22. $eV_0 = 2h\nu_0 - h\nu_0$

$\therefore \qquad h\nu_0 - eV_0$

In second condition,

$$eV = 3h\nu_0 - h\nu_0 = 2h\nu_0 - 2eV_0$$

or $V = 2V_0$

23. Frequency $\propto$ energy and energy $\propto Z^2$

24. $E \propto \frac{Z^2}{n^2}$

$\therefore \qquad \frac{Z^2}{n^2} = 1$

or $\qquad n = Z = 3$

25. $\frac{1}{2}mv_1^2 = h\nu_1 - W$

$$\frac{1}{2}mv_2^2 = h\nu_2 - W$$

From these two equations, we can see that

$$v_1^2 - v_2^2 = \frac{2h}{m}(\nu_1 - \nu_2)$$

26. Longest wavelength of Lyman series means, minimum energy corresponding $n = 2$ to $n = 1$.

$\therefore \qquad (E_2 - E_1)_{\text{H}} = (E_n - E_2)_{\text{He}^+}$

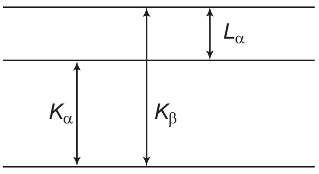
$\therefore \qquad \frac{-13.6}{(2)^2} + \frac{13.6}{(1)^2} = \frac{-13.6(Z)^2}{n^2} + \frac{13.6(Z)^2}{(2)^2}$

Putting $Z = 2$, we get $n = 4$.

27. $\frac{1}{\lambda} = R\,(Z-1)^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$

For K_α-line, $n_1 = 1$ and $n_2 = 2$

28.



$$E_{K_\beta} = E_{K_\alpha} + E_{L_\alpha}$$

$\therefore \qquad h\gamma_2 = h\gamma_1 + h\gamma_3$

$\therefore \qquad \gamma_2 = \gamma_1 + \gamma_3$

29. $\because \quad \lambda = \frac{h}{\sqrt{2qVm}}$ or $\lambda \propto \frac{1}{\sqrt{qm}}$

$\therefore \qquad \frac{\lambda_P}{\lambda_\alpha} = \frac{(\sqrt{qm})_\alpha}{(\sqrt{qm})_P} = \frac{\sqrt{2\times 4}}{\sqrt{1\times 1}} = 2\sqrt{2}$

30. $\lambda = \frac{h}{\sqrt{2Em}}$

where, $\qquad \lambda = \text{constant}$

$\therefore \qquad Em = \text{constant}$

or $\qquad E \propto \frac{1}{m}$

31. $U_1 = -27.2\,\text{eV}$

U_1 is assumed zero. Therefore, it is increased by 27.2 eV at all points.

$E_{\propto} = 0$ under normal conditions.

Hence, in charged conditions it is 27.2 eV.

32. Number of photons emitted per second

$$= \frac{\text{Energy radiated per second}}{\text{Energy of one photon}} = \frac{P}{hf}$$

$$= \frac{1000}{6.63\times 10^{-34}\times 880\times 10^3}$$

$$= 1.7\times 10^{30}$$

33. $E = \frac{12375}{3000} = 4.125\,\text{eV}$

Since, $E < W$ no photoelectric effect will be observed.

34. $E \propto \frac{1}{n^2}$

$\therefore \qquad E_5 = \frac{-13.6}{(5)^2}$

$$= -0.544\text{ eV}$$

35. $K_{\text{max}} = E - W = 2\text{ eV} = 3.2\times 10^{-19}\text{ J}$

36. $m_e v_e = \frac{h}{\lambda}$

$\therefore \qquad v_e = \frac{h}{\lambda m_e}$

$$= \frac{6.63\times 10^{-34}}{(5200\times 10^{-10})\times 9.1\times 10^{-31}}$$

$$\approx 1400\text{ m/s}$$

37. $K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = E - W$

$\therefore \qquad U_{\text{max}} = \sqrt{\frac{2(E-W)}{m}}$

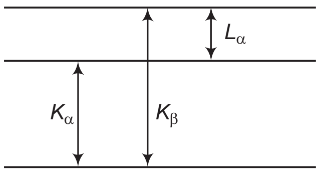
Here, $\qquad E = \frac{12375}{3000} = 4.125\text{ eV}$

$$= \sqrt{\frac{2\times 3.125\times 1.6\times 10^{-19}}{9.1\times 10^{-31}}}$$

$$\approx 1.09\times 10^6\text{ m/s}$$

Subjective Questions

1.



$$E_{L_\alpha} = E_{K_\beta} - E_{K_\alpha}$$

$$\therefore \quad \frac{hc}{\lambda_{L_\alpha}} = \frac{hc}{\lambda_{K_\beta}} - \frac{hc}{\lambda_{K_\alpha}}$$

or
$$\lambda_{L_\alpha} = \frac{\lambda_{K_\alpha}\lambda_{K_\beta}}{\lambda_{K_\alpha} - \lambda_{K_\beta}}$$

$$= \frac{0.71 \times 0.63}{0.71 - 0.63} = 5.59 \text{ nm}$$

2. $\lambda_{K_\alpha} = 0.71 \text{ nm} = 7.1 \text{ Å}$

$$E_2 - E_1 = \frac{127375}{7.1} = 1743 \text{ eV}$$

$$\therefore \quad E_1 = E_2 - 1743$$
$$= -2870 - 1743$$
$$= -4613 \text{ eV}$$

$$\lambda_{K_\beta} = 0.63 \text{ nm}$$
$$= 6.3 \text{ Å}$$

$$E_3 - E_1 = \frac{12375}{6.3}$$
$$= 1964 \text{ eV}$$

$$\therefore \quad E_3 = E_1 + 1964$$
$$= -4613 + 1964$$
$$= -2649 \text{ eV}$$

3. Number of photons incident per second

$$= \frac{\text{Power}}{\text{Energy of one photon}}$$

$$= \frac{P}{(hc/\lambda)} = \frac{P\lambda}{hc}$$

Number of electrons emitted per second

$$= 0.1\% \text{ of } \frac{P\lambda}{hc} = \frac{P\lambda}{1000\, hc}$$

$\therefore$ Current = Charge (on photoelectrons per second)

$$= \frac{P\lambda e}{1000\, hc}$$

$$= \frac{(1.5 \times 10^{-3})(400 \times 10^{-9})(1.6 \times 10^{-19})}{(1000)(6.63 \times 10^{-34})(3 \times 10^{8})}$$

$$= 0.48 \times 10^{-6} \text{ A} = 0.48\ \mu\text{A}$$

4. (a) $p = \dfrac{E}{c} \quad \Rightarrow \quad E = pc$

(b) $$\lambda = \frac{12375}{E \text{ (in eV)}}$$

$$= \frac{12375}{1.54} = 8035 \text{ Å}$$
$$\approx 804 \text{ nm}$$

So, this wavelength lies in ultraviolet region.

5. (a) $f = \dfrac{c}{\lambda}$

(b) $\because$ Number of photons emitted per second

$$N_1 = \frac{\text{Power of source}}{\text{Energy of one photon}} = \frac{P}{hc/\lambda}$$

$$N_1 = \frac{P\lambda}{hc}$$

$$= \frac{(75)(600 \times 10^{-9})}{(6.63 \times 10^{-34})(3 \times 10^{8})}$$

$$\approx 2.3 \times 10^{20} \text{ photons/s}$$

6. (a) $E = hf \quad \Rightarrow \quad f = \dfrac{E}{h}$

(b) $\lambda = \dfrac{c}{f}$

7. (a) $K = \dfrac{p^2}{2m}$ or $K \propto p^2$

If momentum is doubled, kinetic energy becomes four times.

(b) $p = \dfrac{E}{c}$ or $E \propto p$ (for a photon)

If p is doubled, E will also become two times.

8. (a) Number of photons incident per second
= number of photons absorbed per second

$$= \frac{\text{Power}}{\text{Energy of one photon}}$$

$$N = \frac{P}{hc/\lambda} = \frac{P\lambda}{hc}$$

(b) Force = Rate of change of momentum = (Number of photons absorbed per second) × (momentum of one photon)

$$= \left(\frac{P\lambda}{hc}\right)\left(\frac{h}{\lambda}\right) = \frac{P}{c} \qquad (\because P = \text{power})$$

9. See the hint of above example.

If surface is perfectly absorbing, force is $\dfrac{P}{c}$. If surface is perfectly reflecting, then force will be $\dfrac{2P}{c}$.

In this case,

$$F = \frac{0.7P}{c} + \frac{0.3\,(2P)}{c} = \frac{1.3P}{c}$$

10. Force = Rate of change of momentum

$$= 2\,[N]\left[\frac{h}{\lambda}\right]\cdot \cos 60^\circ$$

N = number of photons striking per second

$\frac{h}{\lambda}$ = momentum of one photon.

11. $\lambda = \frac{h}{p} = \frac{h}{mv}$

Since, wavelength is too short, it does not behave wave like property.

12. $\lambda = \frac{h}{mv}$

13. (a) $\lambda = \frac{h}{p} \quad \Rightarrow \quad p = \frac{h}{\lambda}$

(b) $\lambda = \frac{\lambda}{\sqrt{2Km}} \Rightarrow \quad K = \frac{h^2}{2m\lambda^2}$

14. $v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = v$ (say)

and $\qquad \lambda = \frac{h}{mv} = \frac{h}{m}\sqrt{\frac{M}{3RT}}$

Substituting the values, we get

$$\lambda = \frac{6.63\times 10^{-34}}{(2/6.02\times 10^{23})\times 10^{-3}}\sqrt{\frac{2\times 10^{-3}}{3(8.31)\,(20+273)}}$$

$$= 1.04\times 10^{-4}\ \text{m} = 1.04\ \text{Å}$$

15. KE = | Total energy | = 3.4 eV

$$\lambda\ (\text{in Å}) = \sqrt{\frac{150}{\text{KE (in eV)}}} \quad \text{for an electron}$$

$$= \sqrt{\frac{150}{3.4}}$$

$$= 6.6\text{Å}$$

16. (a) $\qquad E_1 = -\,13.6\ \text{eV}$

$\therefore \qquad \text{KE} = |E_1| = 13.6\ \text{eV}$

$$\lambda_1\ (\text{in Å}) = \sqrt{\frac{150}{\text{KE (in eV)}}} \quad \text{for an electron}$$

$$= \sqrt{\frac{150}{13.6}} = 3.32\text{Å}$$

$$2\pi r_1 = 2\pi(0.529\text{Å}) \approx 3.32\ \text{Å}$$

(b) $E_4 = \frac{E_1}{(4)^2} = \frac{-\,13.6}{(4)^2} = -\,0.85\ \text{eV}$

$$\lambda_4 = \sqrt{\frac{150}{0.85}} \quad \text{as KE} = |E_4| = 0.85\ \text{eV}$$

$$\approx 13.3\ \text{Å}$$

$$2\pi r_4 = 2\pi(n)^2 r_1 \qquad (\text{as } r \propto n^2)$$

$$= 53.15\ \text{Å} \approx 4\lambda_4$$

17. Energy required to remove first electron is 24.6 eV. After removing first electrons from this atom, it will become He$^+$.

$$E_1 = -\,(13.6)(2)^2 \quad (\text{as } E \propto Z^2 \text{ and } Z = 2)$$

$$= -\,54.4\ \text{eV}$$

$\therefore$ Energy required to remove this second electron will be 54.4 eV.

$\therefore$ Total energy required to remove both electrons

$$= 24.6 + 54.4$$

$$= 79\ \text{eV}$$

18. $E_n = E_1 = \frac{12375}{1023}$

or $\qquad \frac{-\,13.6}{(n)^2} + 13.6 = 12.1$

Solving this equation, we get $n = 3$

$\therefore$Total possible emission lines $= \frac{n\,(n-1)}{2} = 3$

Longest wavelength means, minimum energy, which is corresponding to $n = 3$ to $n = Z$

$$\therefore \qquad \lambda_{\max} = \frac{12375}{E_3 - E_2}$$

Here, $\qquad E_3 - E_2 = -\frac{13.6}{(3)^2} + \frac{13.6}{(4)^2} = 1.9\ \text{eV}$

$$\therefore \qquad \lambda_{\max} = \frac{12375}{1.9} = 6513\,\text{Å}$$

19. In hydrogen atom,

$$\Delta E_1 = E_3 - E_1 = 12.1\ \text{eV}$$

$$E \propto Z^2$$

$\therefore \quad$ For $\qquad Z = 3,$

$$\Delta E_2 = (3)^2\,\Delta E_1 = (3)^2\,(12.1)$$

$$= 108.9\text{Å}$$

$$\therefore \qquad \lambda = \frac{12375}{108.9} \approx 113\ \text{Å}$$

20. $E_n - E_1 = \Delta E_1 + \Delta E_2$

$$\therefore \quad \left[-\frac{13.6}{n^2} + 13.6\right](Z)^2 = \frac{12375}{1085} + \frac{12375}{304}$$

Putting $Z = 2$ for He$^+$, we get $n = 5$

21. (a) Let these two levels are n_1 and n_2. Then,

$$-0.85 = -\frac{13.6\,(Z)^2}{n_1^2} \qquad \text{...(i)}$$

$$-0.544 = -\frac{13.6\,(Z)^2}{n_2^2} \qquad \text{...(ii)}$$

Total number of lines between n_2 and $n_1$are given by

$$\frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2} = 10 \qquad \text{...(iii)}$$

Solving these equations, we get

$$Z = 4,\ n_1 = 16 \quad \text{and} \quad n_2 = 20$$

(b) Smallest wavelength means maximum energy.

$$\Delta E_{\max} = E_{n_2} - E_{n_1} = -0.544 + 0.85$$
$$= 0.306 \text{ eV}$$

$$\lambda_{\min} = \frac{12375}{0.306} = 40441 Å$$

22. (a) Ionization energy = 15.6 eV

$\therefore$ Ionization potential = 15.6 V

(b) $E_{\propto} - E_2 = 5.3$ eV

$$\therefore \quad \lambda = \frac{12375}{5.3} = 2335 Å$$

(c) $E_3 - E_1 = 12.52$ eV

$\therefore$ Excitation potential is 12.52 V.

(d) $E_3 - E_1 = 12.52$ eV

$$\therefore \quad \lambda = \frac{12375}{12.52}\ Å$$
$$= 988.41\ Å$$

$$\therefore \quad \text{Wave number} = \frac{1}{\lambda} = \frac{1}{988.41 \times 10^{-10}}\ \text{m}^{-1}$$
$$= 1.01 \times 10^7\ \text{m}^{-1}$$

23. (a) Energy of photon $= \dfrac{12375}{8600}$ eV = 1.44 eV

$\therefore$ Internal energy after absorption

$$= -6.52 + 1.44 = -5.08 \text{ eV}$$

(b) Energy of emitted photon $= \dfrac{12375}{4200} = 2.946$ eV

Internal energy after emission = – 2.68 – 2.946

= – 5.626 eV

24. $K_{\max}$ (in eV) $= E - W$

$$= \frac{12375}{2000} - 4.3$$
$$= 1.9 \text{ eV}$$

Therefore, stopping potential is 1.9 V.

25. $F = -\dfrac{dU}{dr} = -m^2\omega^2 r$

$$\text{Now,} \quad \frac{mv^2}{r} = |F| = m^2\omega^2 r$$

$$\text{or} \quad v \propto r$$

$$\text{or} \quad v = \alpha r$$

$$\text{Now,} \quad m\,vr = n\frac{h}{2\pi}$$

$$\text{or} \quad m\,(\alpha r) r = n\frac{h}{2\pi}$$

$$\therefore \quad r \propto \sqrt{n}$$

26. $\dfrac{1}{\lambda} \propto \left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$

$$\therefore \quad \frac{\lambda_{K_\beta}}{\lambda_{K_\alpha}} = \frac{(1/n_1^2 - 1/n_2^2)_{K_\alpha}}{(1/n_1^2 - 1/n_2^2)_{K_\beta}}$$

$$= \frac{(1 - 1/4)}{(1 - 1/9)} = \frac{27}{32}$$

$$\therefore \quad \lambda_{K_\beta} = \frac{27}{32}\lambda_{K_\alpha} = \frac{27}{32}\lambda_0$$

27. $\Delta E_1 = 50\%$ of 50 keV = 25 keV

$$\therefore \quad \lambda_1 = \frac{12375}{25 \times 10^3} = 0.495\ Å$$
$$= 49.5 \text{ pm}$$

$$\Delta E_2 = 50\% \text{ of } 25 \text{ keV}$$
$$= 12.5 \text{ keV}$$

$$\therefore \quad \lambda_2 = \frac{12375}{12.5 \times 10^3} = 0.99 Å$$
$$= 99 \text{ pm}$$

28. $\dfrac{1}{\lambda_{K_\alpha}} = R\,(Z-1)^2 \left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$

$$\text{or} \quad \frac{1}{0.76 \times 10^{-10}} = 1.097 \times 10^7\ (Z-1)^2 \left(1 - \frac{1}{4}\right)$$

Solving this equation, $Z = 41$

29. 26 pm = 0.26 Å

$$\text{Now,} \quad 0.26 = \frac{12375}{\text{V}} - \frac{12375}{1.5\ \text{V}}$$

30. Similar type of example is given in the theory.

31. Transition if from L-shell to K-shell.

$$\frac{1}{\lambda} = R\,(Z-1)^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

or $$\frac{f}{c} = R\,(Z-1)^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

or $$\frac{4.2 \times 10^8}{3 \times 10^8} = 1.1 \times 10^7\ (Z-1)^2 \left(1 - \frac{1}{4}\right)$$

Solving we get, $Z \approx 42$

32. (a) $P = Vi = (40 \times 10^3)\,(10 \times 10^{-3})$

$= 400$ W

1% of 400 W is 4 W.

(b) Heat generated = 400 W − 4 W

$= 396\ \text{W} = 396\ \text{J/s}$

33. Stopping potential = 10.4 V

$$\therefore \quad K_{\max} = 10.4\ \text{eV}$$
$$E = W + K_{\max} = 1.7 + 10.4 = 12.1\ \text{eV}$$
$$\lambda = \frac{12375}{12.1} = 1022\ \text{Å}$$

12.1 eV is the energy gap between $n = 3$ and $n = 1$ in hydrogen atom.

34. $$E = hf = \frac{(6.63 \times 10^{-34})\,(1.5 \times 10^{15})}{1.6 \times 10^{-19}}\ \text{eV} = 6.21\ \text{eV}$$

$$K_{\max} = E - W = 6.21 - 3.7 = 2.51\ \text{eV}$$

35. $\lambda_0 = 5000$ Å

$$W = \frac{12375}{5000} = 2.475\ \text{eV}$$

Stopping potential is 3V. Therefore, $K_{\max} = 3$ eV

$$E = W + K_{\max} = 5.475\ \text{eV}$$
$$\lambda = \frac{12375}{5.475} = 2260\ \text{Å}$$

36. (a) $f_0 = f_A = 10 \times 10^{14}\ \text{Hz} = 10^{15}\ \text{Hz}$

(b) $W = |K_{\max}|_c = 4$ eV

(c) $W = hf_0 \Rightarrow h = \dfrac{W}{f_0}$

37. $$E_1 = \frac{12357}{3000} = 4.125\ \text{eV}$$

$$E_2 = \frac{12357}{6000} = 2.0625\ \text{eV}$$

Maximum speed ratio is 3 : 1. Therefore, maximum kinetic ratio is 9 : 1.

Now, $$9\,K_{\max} = 4.125 - W \qquad \text{...(i)}$$
$$K_{\max} = 2.0625 - W \qquad \text{...(ii)}$$

Solving these two equations, we get

$W \approx 1.81$ eV. and $K_{\max} = 0.26$ eV

Putting $K_{\max} = \frac{1}{2} m v_{\max}^2$ we can find $v_{\max}$. Here, m is the mass of electron.

38. $$E = \frac{12375}{1800} = 6.875\ \text{eV}$$

$K_{\max}$ or $K = E - W$

$= 4.875$ eV

$$r = \frac{\sqrt{2Km}}{Bq}$$
$$= \frac{\sqrt{2 \times 4.875 \times 1.6 \times 10^{-19} \times 9.1 \times 10^{-31}}}{5 \times 10^{-5} \times 1.6 \times 10^{-19}}$$
$$= 0.148\ \text{m}$$

39. $$f_{\text{higher}} = \frac{8 \times 10^{15}}{2\pi}\ \text{s}^{-1}$$

$$E = \frac{hf}{1.6 \times 10^{-19}}\ \text{eV}$$
$$= \frac{(6.63 \times 10^{-34})\,(8 \times 10^{15})}{1.6 \times 10^{-19} \times 2\pi}$$
$$= 5.27\ \text{eV}$$
$$K_{\max} = E - W$$
$$= 3.27\ \text{eV}$$

40. $$f = \frac{\omega}{2\pi} = \frac{(1.57 \times 10^7)\,c}{2\pi}$$

$$E = \frac{hf}{1.6 \times 10^{-19}}\ \text{eV}$$
$$= \frac{(6.63 \times 10^{-34})\,(1.57 \times 10^7)\,(3 \times 10^8)}{1.6 \times 10^{-19} \times 2 \times \pi}\ \text{eV}$$
$$= 3.1\ \text{eV}$$
$$K_{\max} = E - W = 1.2\ \text{eV}$$

LEVEL 2

Single Correct Option

1. $$\frac{mv^2}{r} = \frac{GMm}{r^2} \qquad \text{...(i)}$$

$$mvr = \frac{h}{2\pi} \qquad (\text{for } n = 1) \qquad \text{...(ii)}$$

Solving these two equations, we can find v and r. Then,

$$E = \frac{1}{2} mv^2 - \frac{GMM}{r}$$

2. $$\frac{M}{L} = \frac{q}{2m} = \text{constant}$$

$$M \propto L$$

and $$L = \frac{nh}{2\pi} \quad \text{or} \quad L \propto n$$

$$\therefore \quad M \propto n$$

3. $E_2 - E_1 = 40.8\,\text{eV}$

$$\therefore \qquad \frac{E_1}{(2)^2} - E_1 = 40.8 \quad \text{eV}$$

$$\text{or} \qquad -\frac{3}{4}E_1 = 40.8 \quad \text{eV}$$

$$\text{or} \qquad E_1 = -54.4 \text{ eV}$$

$$|E_1| = 54.4 \text{ eV}$$

4. $i = qf = q\left(\dfrac{v}{2\pi r}\right)$

$$\text{or} \qquad i \propto \frac{v}{r} \propto \frac{(1/n)}{(n)^2} \quad \text{or} \quad i \propto \frac{1}{n^3}$$

$$\frac{i_1}{i_2} = \left(\frac{n_2}{n_1}\right)^3 = (2)^3 = 8$$

5. $n_f = 6$

$$\text{Total emission lines} = \frac{n_f(n_f - 1)}{2} = 15$$

6. $A = \pi r^2$

$$A \propto r^2 \quad \text{or} \quad A \propto n^4 \qquad (\text{as } r \propto n^2)$$

$$\therefore \qquad \frac{A_n}{A_1} = (n)^4$$

$$\ln\left(\frac{A_n}{A_1}\right) = 4 \ln (n)$$

Therefore, $\ln\left(\dfrac{A_n}{A_1}\right)$ *versus* $\ln n$ graph is a straight line of slope 4.

7. $B = \dfrac{\mu_0 i}{2\pi}$

$$\text{or} \qquad B \propto \frac{i}{r}$$

See the hint of Q.No. 4 of same section.

$$i \propto \frac{1}{n^3} \quad \text{and} \quad r \propto n^2$$

$$\therefore \qquad B = \frac{1}{n^5}$$

$$\therefore \qquad \frac{B_1}{B_2} = \left(\frac{n_2}{n_1}\right)^5 = (2)^5 = 32$$

8. $\dfrac{1}{\lambda} = R\left(1 - \dfrac{1}{4}\right) = \dfrac{3R}{4}$

$$\therefore \qquad p = \frac{h}{\lambda}$$

$$= \frac{3Rh}{4} = \text{momentum of photon}$$

From conservation of linear momentum,

Momentum of photon = Momentum of hydrogen atom

$$\therefore \qquad \frac{3Rh}{4} = Mv \quad \Rightarrow \quad v = \frac{3Rh}{4M}$$

9. $\omega_{\max} = 1.5 \times 10^{15}\ \text{rad/s} = 2\pi f_{\max}$

$$\therefore \qquad f_{\max} = \frac{1.5 \times 10^{15}}{2\pi}\ \text{Hz}$$

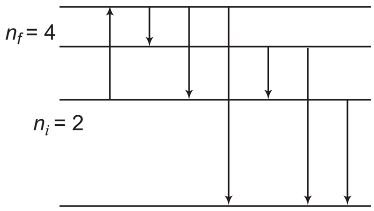
$$E = \frac{hf_{\max}}{1.6 \times 10^{-19}}\ \text{eV}$$

$$= \frac{6.63 \times 10^{-34} \times 1.5 \times 10^{15}}{2\pi \times 1.6 \times 10^{-19}}\ \text{eV}$$

$$\approx 1.0 \text{ eV}$$

Since, $E < W$, no photoemission can take place.

10. $\dfrac{n_f(n_f - 1)}{2} = 6 \quad \Rightarrow \quad n_f = 4$



From $n_i = 2$, energy of six emission lines is either greater than less or equal to the energy of absorption line.

11. $T = \dfrac{2\pi r}{v} \quad \text{or} \quad T \propto \dfrac{r}{v} \quad \text{or} \quad T \propto \dfrac{(n^2/Z)}{(Z/n)}$

$$\therefore \qquad T \propto \frac{n^3}{Z^2}$$

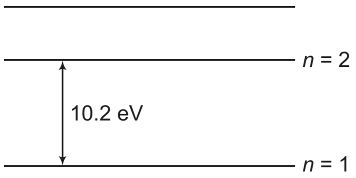
$$\left(\frac{T_1}{T_2}\right) = \left(\frac{n_1}{n_2}\right)^3 = \left(\frac{Z_2}{Z_1}\right)^2$$

$$2 = \left(\frac{1}{2}\right)^3 \left(\frac{Z}{1}\right)^2$$

$$\therefore \qquad Z = 4$$

12. λ_{K_α} will depend on atomic number Z and values Z is same of all three isotopes.

13.



$$\lambda_{2 \to 1} = \frac{12375}{10.2} \approx 1213\,\text{Å}$$

14. For $K \geq 10.2$ eV electrons can excite the hydrogen atom (as $E_2 - E_1 = 10.2$ eV). So, collision may be inelastic.

15. $E_3 - E_1 = \frac{-13/6}{(3)^3} + 13.6 = 12.1$ eV

From momentum conservation,

Momentum of photon = Momentum of hydrogen atom

$$\therefore \quad \frac{E}{c} = mv$$

$$\text{or} \quad v = \frac{E}{mc} = \frac{12.1 \times 1.6 \times 10^{-19}}{1.67 \times 10^{-27} \times 3 \times 10^{8}}$$

$$= 3.86 \text{ m/s}$$

16. $P = Vi = 150 \times 10^3 \times 10 \times 10^{-3} = 1500$ W

$$99\% \text{ of this} = 1500 \times \frac{99}{100} \text{ J/s} = 1485 \text{ J/s}$$

$$= \frac{1485}{4.18} \text{ cal/s}$$

$$\approx 355 \text{ cal/s}$$

17. $E_4 - E_3 = 32.4$

$$\therefore \quad \frac{-13.6\,(Z)^2}{(4)^2} + \frac{13.6\,(Z)^2}{(3)^2} = 32.4$$

Solving, we get

$$Z = 7$$

18. $\because \quad \lambda = \frac{h}{\sqrt{2qVm}}$

$$V = \frac{h^2}{2qm\lambda^2}$$

19. $E_n \propto \frac{1}{n^2}$ and $L_n \propto n$

20. $\frac{M}{L} = \frac{q}{2m}$ (always)

$$\therefore \quad M \propto L$$

$$L = \frac{nh}{2\pi} \Rightarrow L \propto n$$

Hence, $M \propto n$

Third excited state means $n = 4$.

21. $K = \frac{p^2}{2m}$

In collision, momentum p remains constant.

$$\therefore \quad K \propto \frac{1}{\text{mass}}$$

After collision, mass has doubled. So, kinetic energy will remain $\frac{K}{2}$. Hence, loss is also $\frac{K}{2}$.

Now, $\frac{K}{2}$ = minimum excitation energy required.

$$= 10.2 \text{ eV}$$

$$\Rightarrow \quad K = 20.4 \text{ eV}$$

22. KE = $|E|$ = 3.4 eV

$$\lambda \text{ (in Å)} = \sqrt{\frac{150}{\text{KE (in eV)}}}$$

$$= \sqrt{\frac{150}{3.4}}$$

$$= 6.6 \text{ Å}$$

More than One Correct Options

1. λ_{K_α} and λ_{K_β} will remain unchanged. But, λ_0 will decrease $\left(\text{as } \lambda_0 \propto \frac{1}{V}\right)$.

$\therefore \quad \Delta\lambda_1 = \lambda_{K_\alpha} - \lambda_0$ or $\Delta\lambda_2 = \lambda_{K_\beta} - \lambda_0$ will increase.

2. $R \propto n^2$, $v \propto \frac{1}{n}$ and $E \propto \frac{1}{n^2}$

3. $L \propto n$, $r \propto n^2$

$$\text{and} \quad T = \frac{2\pi r}{v} \quad \text{or} \quad T \propto \frac{r}{v} \propto \frac{n^2}{(1/n)}$$

$$\text{or} \quad T \propto n^3$$

4. $\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2Km}}$

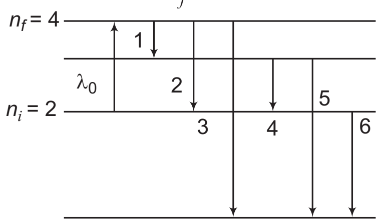
$$\lambda \propto \frac{1}{m} \quad \text{(if } v \text{ is same)}$$

$$\lambda \propto \frac{1}{\sqrt{m}} \quad \text{(if } K \text{ is same)}$$

If change in potential energy is same, then change in kinetic energy is also same. But, this does not mean that kinetic energy is same.

5. $\frac{n_f\,(n_f - 1)}{2} = 6$

$$\Rightarrow \quad n_f = 4$$



λ_1 and λ_4 are longer than λ_0.

λ_3, λ_5 and λ_6 belong to Lyman series.

6. $\sqrt{f} \propto (Z - b)$ ...(i)

If f *versus* Z is a straight line.

$$f = \frac{c}{\lambda}$$

Hence, $\frac{1}{\sqrt{\lambda}}$ *versus* Z is also a straight line.

Comprehension Based Questions

1. $E_1 = \frac{12375}{4950} = 2.5$ eV

K_1 = maximum kinetic energy = 0.6 eV

$W = E_1 - K_2 = 1.9$ eV

2. $K_2 = 1.1$ eV

$$E_2 = W + K_2 = 3.0 \text{ eV}$$

$$\therefore \quad \lambda_2 = \frac{12375}{3.0} = 4125 \text{ Å}$$

3. Magnetic field cannot change the kinetic energy of charged particle.

4. $K_{\max} = E - W = 2$ eV

$$\lambda = \sqrt{\frac{150}{\text{KE (in eV)}}}, \text{ for an electron}$$

$$= \sqrt{\frac{150}{2}} \approx 8.6 \text{ Å}$$

5. $K_{\max}$ is 2 eV. Hence, stopping potential is 2V. Photoemission stops when potential of sphere becomes 2V.

$$\therefore \quad 2 = \frac{q}{4\pi\varepsilon_0 r}$$

$$\therefore \quad q = 8\pi\varepsilon_0 r$$

6. $\because \quad q = ne = 8\pi\varepsilon_0 r$

$$\therefore \quad n = \frac{8\pi\varepsilon_0 r}{e}$$

$$= \frac{2 \times 8 \times 10^{-3}}{9 \times 10^9 \times 1.6 \times 10^{-19}}$$

$$= 1.11 \times 10^7$$

So, this much number of electrons are required to be ejected from the sphere.

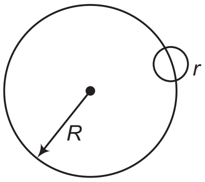
Number of photons emitted per second,

$$N_1 = \frac{\text{Power of source}}{\text{Energy of one photon}}$$

$$= \frac{3.2 \times 10^{-3}}{5 \times 1.6 \times 10^{-19}}$$

$$= 4 \times 10^{15}$$

Number of photons incident on sphere per second,



$$N_2 = \left(\frac{N_1}{4\pi R^2}\right)(\pi r^2)$$

$$= \frac{N_1}{4R^2} r^2$$

$$= \frac{(4 \times 10^{15})(8 \times 10^{-3})^2}{4\,(0.8)^2}$$

$$= 10^{11}$$

Number of photoelectrons emitted per second,

$$N_3 = \frac{N_2}{10^6} = 10^5$$

Now, $\quad N_3 t = n$

$$\therefore \quad t = \frac{n}{N_3} = 111 \text{ s}$$

Match the Columns

2. For He$^+$, $Z = 2$

$$E \propto \frac{Z^2}{n^2}$$

———— $E_2 = -3.4$ eV

———— $E_1 = -13.6$ eV

H-atom

Ionisation energy from first, excited state of H-atom

$= |E_2| = 3.4 \text{ eV} = E$ (given)

(a) $|E_1| = (13.6 \text{ eV})(2)^2$

$= 16\,(3.4 \text{ eV}) = 16E$

(b) $U_2 = 2E_2 = 2\,(-13.6)\frac{(2)^2}{(2)^2}$

$= -8\,(3.4 \text{ eV}) = -8E$

(c) $K_1 = |E_1| = 16E$

(d) $|E_2| = (13.6)\frac{(2)^2}{(2)^2}$

$= 4\,(3.4 \text{ eV}) = 4E$

3. $K_{\max} = hf - W$

$\therefore$ $K_{\max}$ *versus* f graph is a straight line of slope h and intercept $-W$.

$$eV_0 = hf = W$$

$$\therefore \quad V_0 = \left(\frac{h}{e}\right) f - \frac{W}{e}$$

$\therefore$ V_0 *versus* f graph is again a straight line of slope $\frac{h}{e}$ and intercept $\frac{W}{e}$.

4. $T = \dfrac{2\pi r}{v}$ or $T \propto \dfrac{r}{v} \propto \dfrac{n^2/Z}{Z/n}$

$$\therefore \qquad T \propto \frac{n^3}{Z^2}$$

$$L = \frac{nh}{2\pi} \quad \text{or} \quad L \propto n$$

$$v \propto \frac{Z}{n} \quad \text{and} \quad r \propto \frac{n^2}{Z}$$

5. $\dfrac{1}{\lambda} \propto \left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$

$$\text{or} \qquad \frac{\lambda_2}{\lambda_1} = \frac{(1/n_1^2 - 1/n_2^2)_i}{(1/n_1^2 - 1/n_2^2)_f}$$

$$\text{or} \qquad \lambda_2 = \frac{(1/4 - 1/16)\lambda}{(1/n_1^2 - 1/n_2^2)_f}$$

$$= \left(\frac{3}{16}\lambda\right)\left[\frac{1}{(1/n_1^2 - 1/n_2^2)_f}\right]$$

(a) For first line of Balmer series,

$$n_1 = 2, n_2 = 3 \quad \therefore \quad \lambda_2 = \left(\frac{27}{20}\right)\lambda$$

(b) For third line of Balmer series,

$$n_1 = 2, n_2 = 5 \quad \therefore \quad \lambda_2 = \left(\frac{25}{28}\right)\lambda$$

(c) For first line of Lyman series,

$$n_1 = 1, n_2 = 2 \quad \therefore \quad \lambda_2 = \frac{\lambda}{4}$$

(d) For second line of Lyman series,

$$n_1 = 2, n_2 = 3 \quad \therefore \quad \lambda_2 = \left(\frac{27}{128}\right)\lambda$$

7. See the hint of Q.No. 3 of section Assertion and Reason.

Stopping potential increases with increase in maximum kinetic energy of photoelectrons of frequency of incident light.

With increase in distance between cathode and anode, f remains unchanged.

$K_{\max} = hf - W = eV_0$

If W is decreased, $K_{\max}$ and V_0 both will increase.

Subjective Questions

1. (a) Reduced mass of positronium and electron is $\dfrac{m}{2}$,

where, $m =$ mass of electron

$$E \propto m \quad \therefore \quad \lambda \propto \frac{1}{m}$$

m has become half, so λ will become two times or 1312 nm or 1.31μm.

(b) $E \propto Z^2 \quad \therefore \quad \lambda \propto \dfrac{1}{Z^2}$

For singly ionized helium atom $Z = 2$

$$\therefore \qquad \lambda \text{ is } \frac{1}{4}\text{th} \quad \text{or} \quad 164 \text{ nm}$$

2. (a) $f = \dfrac{v}{2\pi r}$

$$f_1 = \frac{(2.2 \times 10^6)}{(2\pi)(0.529 \times 10^{-10})}$$

$$\approx 6.6 \times 10^{15} \text{ Hz}$$

$$v \propto \frac{1}{n} \quad \text{and} \quad r \propto n^2$$

$$f \propto \frac{v}{r} \Rightarrow f \propto \frac{1}{n^3}$$

$$\therefore \qquad f_2 = \frac{f_1}{8}$$

(b) $\Delta E = E_2 - E_1 = 10.2 \text{ eV} = hf$

$$\therefore \qquad f = \frac{10.2 \times 1.6 \times 10^{-19}}{6.6 \times 10^{-34}}$$

$$\approx 2.46 \times 10^{15} \text{ Hz}$$

(c) In option (a), we have found that

$$f_2 = 0.823 \times 10^{15} \text{ Hz}$$

$$T_2 = \frac{1}{f_2}$$

$$N = \frac{t}{T_2} = tf_2 = (10^{-8})(0.823 \times 10^{15})$$

$$= 8.23 \times 10^6 \text{ revolutions}$$

3. (a) $r \propto \dfrac{1}{m}$

$$\therefore \qquad r = \frac{(r_1^H)}{m} = \frac{0.529 \times 10^{-10}}{207}$$

$$= 2.55 \times 10^{-13} \text{ m}$$

(b) $E \propto m$

$\therefore$ Ionization energy of given atom

$= (m)$ (ionization energy of hydrogen atom)

$= (207)(13.6 \text{ eV})$

$= 2815.2 \text{ eV} = 2.81 \text{ keV}$

4. (a) $E_n - E_1 = 12.5$

$$\therefore \qquad -\frac{13.6}{n^2} + 13.6 = 12.5$$

Solving we get, $n = 3.51$

Hence, electron jumps to $n = 3$. So, possible lines are between $n = 3$ to $n = 2$, $n = 3$ to $n = 1$ and between $n = 2$ to $n = 1$.

For $n = 3$ to $n = 2$,

$$\Delta E = E_3 - E_2 = -\frac{13.6}{9} + \frac{13.6}{4}$$

$$= 1.9 \text{ eV}$$

$$\therefore \quad \lambda = \frac{12375}{1.9} \text{ Å} = 6513 \text{ Å}$$

$$\approx 651 \text{ nm}$$

Similarly, other wavelengths can also be obtained.

(b) $n = 3.51$ (in option-*a*)

A photon always transfers its energy completely. So, it cannot excite the ground state electrons to $n = 3$ (like and electrons excited it in part-a).

5. (a) $E = hf = (6.6 \times 10^{-34})(5.5 \times 10^{14})$

$$= 36.3 \times 10^{-20} \text{ J}$$

$$= 2.27 \text{ eV}$$ **Ans.**

(b) Number of photons leaving the source per second,

$$n = \frac{P}{E} = \frac{0.1}{36.3 \times 10^{-20}}$$

$$= 2.75 \times 10^{17}$$ **Ans.**

(c) Number of photons falling on cathode per second,

$$n_1 = \frac{0.15}{100} \times 2.75 \times 10^{17}$$

$$= 4.125 \times 10^{14}$$

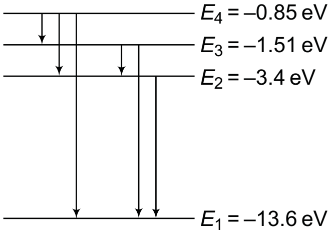
Number of photoelectrons emitting per second,

$$n_2 = \frac{6 \times 10^{-6}}{1.6 \times 10^{-19}} = 3.75 \times 10^{13}$$

$$\therefore \quad \% = \frac{n_2}{n_1} \times 100 = \frac{3.75 \times 10^{13}}{4.125 \times 10^{14}} \times 100$$

$$= 9\%$$ **Ans.**

6. $\dfrac{K_1}{K_2} = 5$



$$\therefore \quad \frac{\Delta E_1 - W}{\Delta E_2 - W} = 5 \quad \ldots(\text{i})$$

Here, $\Delta E_1 = E_4 - E_1 = 12.75$ eV

and $\Delta E_2 = E_3 - E_1 = 12.09$ eV

Substituting in Eq. (i) and solving, we get

$$W = 11.925 \text{ eV}$$ **Ans.**

7. For shorter wavelength

$$\Delta E = E_4 - E_3 = \frac{(-13.6)(3)^2}{(4)^2} - \left[\frac{(-13.6)(3)^2}{(3)^2}\right]$$

$$= 5.95 \text{ eV}$$

$$W = E - K_{\max} = (5.95 - 3.95) \text{ eV} = 2 \text{ eV}$$

For longer wavelength

$$\Delta E = E_5 - E_4 = \frac{(-13.6)(3)^2}{(5)^2} - \left[\frac{(-13.6)(3)^2}{(4)^2}\right]$$

$$= 2.754 \text{ eV}$$

$$\therefore \quad K_{\max} = E - W = 0.754 \text{ eV}$$

or stopping potential is 0.754 V. **Ans.**

8. Magnetic moment, $\mu = NiA = \left(\dfrac{e}{T}\right)(\pi r^2)$

$$\text{or} \quad \mu = \left(\frac{e}{2\pi r/v}\right)(\pi r^2) = \frac{evr}{2} \quad \ldots(\text{i})$$

$$\text{We know that} \quad mvr = \frac{nh}{2\pi} \quad \ldots(\text{ii})$$

Solving Eqs. (i) and (ii), we get

$$\mu = \frac{neh}{4\pi m}$$ **Ans.**

Magnetic induction, $B = \dfrac{\mu_0 i}{2r} = \dfrac{\mu_0 e}{2rT}$

$$\text{or} \quad B = \frac{\mu_0 ev}{(2r)(2\pi r)} = \frac{\mu_0 ev}{4\pi r^2} \quad \ldots(\text{iii})$$

From Newton's second law, $\dfrac{e^2}{4\pi\varepsilon_0 r^2} = \dfrac{mv^2}{r}$

$$\text{or} \quad v^2 = \frac{e^2}{4\pi\varepsilon_0 mr} \quad \ldots(\text{iv})$$

Solving all these equations, we get

$$B = \frac{\mu_0 \pi m^2 e^7}{8\varepsilon_0 h^5 n^5}$$ **Ans.**

9. Energy of electron in ground state of hydrogen atom is – 13.6 eV. Earlier it had a kinetic energy of 2 eV. Therefore, energy of photon released during formation of hydrogen atom,

$$\Delta E = 2 - (-13.6) = 15.6 \text{ eV}$$

$$\therefore \quad \lambda = \frac{12375}{\Delta E} = \frac{12375}{15.6}$$

$$= 793.3 \text{ Å}$$ **Ans.**

10. $U = -1.7$ eV

$$\therefore \quad E = \frac{U}{2} = -0.85 \text{ eV} = \frac{-13.6}{n^2}$$

$$\therefore \quad n = 4$$

Ejected photoelectron will have minimum de-Broglie wavelength corresponding to transition from $n = 4$ to $n = 1$.

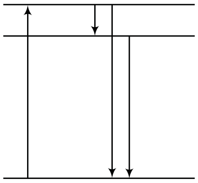
$$\Delta E = E_4 - E_1 = -0.85 - (-13.6)$$
$$= 12.75 \text{ eV}$$
$$K_{\max} = \Delta E - W = 10.45 \text{ eV}$$

$$\therefore \quad \lambda = \sqrt{\frac{150}{10.45}} \text{ Å} \quad \text{(for an electron)}$$
$$= 3.8 \text{ Å}$$ **Ans.**

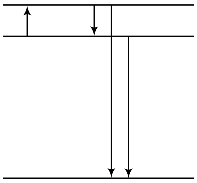
11. (a) $\frac{n(n-1)}{2} = 3$

$$\therefore \quad n = 3$$

i.e after excitation atom jumps to second excited state. Hence, $n_f = 3$. So, n_i can be 1 or 2. If $n_i = 1$, then energy emitted is either equal to or less than the energy absorbed. Hence, the emitted wavelength is either equal to or greater than the absorbed wavelength. Hence, $n_i \neq 1$.



$n_i = 1$



$n_i = 2$

If $n_i = 2$, then $E_e \geq E_a$.

Hence, $\lambda_e \leq \lambda_b$

$$\therefore \quad n_i = 2$$ **Ans.**

(b) $E_3 - E_2 = 68$ eV

$$\therefore \quad (13.6)(Z^2)\left(\frac{1}{4} - \frac{1}{9}\right) = 68$$

$$\therefore \quad Z = 6$$ **Ans.**

(c) $$\lambda_{\min} = \frac{12375}{E_3 - E_1}$$

$$= \frac{12375}{(13.6)(6)^2\left(1 - \frac{1}{9}\right)} = 28.43 \text{ Å}$$ **Ans.**

(d) Ionization energy $= (13.6)(6)^2 = 489.6$ eV **Ans.**

$$\lambda = \frac{12375}{489.6} = 25.3 \text{ Å}$$ **Ans.**

12. Pitch of helical path, $p = (v\cos\theta)\,T = \frac{vT}{2}$.

(as $\theta = 60°$)

$$T = \frac{2\pi m}{Bq} = \frac{2\pi}{B\alpha} \qquad \left(\alpha = \frac{q}{m}\right)$$

$$\therefore \quad p = \frac{\pi v}{B\alpha}$$

$$\text{or} \quad v = \frac{B\alpha p}{\pi} \qquad \text{...(i)}$$

$$\text{KE} = \frac{1}{2}mv^2 = E - W$$

$$\therefore \quad W = E - \frac{1}{2}mv^2 \qquad \text{...(ii)}$$

Substituting value of v from Eq. (i) in Eq. (ii), we get

$$W = 4.9 - \frac{1}{2}\,9.1\times10^{-31}\times(2.5\times10^{-3})^2\,(1.76\times10^{11})^2 \times \frac{(2.7\times10^{-3})^2}{\pi^2\times1.6\times10^{-19}}$$

$$= (4.9 - 0.4) \text{ eV} = 4.5 \text{ eV}$$ **Ans.**

13. (a) $\lambda = 1500\left(\frac{1}{1 - 1/p^2}\right)$

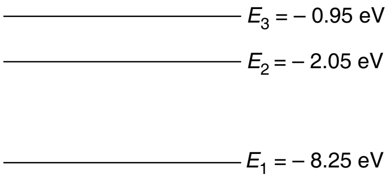
$\lambda_{\max}$ corresponds to least energetic photon with $p = 2$.

$$\therefore \quad \lambda_{\max} = 1500\left(\frac{1}{1 - 1/4}\right) = 2000 \text{ Å}$$ **Ans.**

$\lambda_{\min}$ corresponds to most energetic photon with $p = \infty$

$$\therefore \quad \lambda_{\min} = 1500 \text{ Å}$$ **Ans.**

(b) $\lambda_{\infty - 1} = 1500$ Å



$$\therefore \quad E_\infty - E_1 = \frac{12375}{1500}\text{ eV}$$

$$= 8.25\text{ eV}$$

$$\therefore \quad E_1 = -\,8.25\text{ eV} \qquad (\text{as } E_\infty = 0)$$

$$\lambda_{2-1} = 2000\text{ Å}$$

$$\therefore \quad E_2 - E_1 = \frac{12375}{2000}\text{ eV}$$

$$= 6.2\text{ eV}$$

$$\therefore \quad E_2 = -\,2.05\text{ eV}$$

Similarly, $\lambda_{31} = 1500\left(\dfrac{1}{1-1/9}\right)$

$$= 1687.5\text{ Å}$$

$$\therefore \quad E_3 - E_1 = \frac{12375}{1687.5}\text{ eV} = 7.3\text{ eV}$$

$$\therefore \quad E_3 = -\,0.95\text{ eV}$$

(c) Ionization potential = 8.25 V **Ans.**

14. (a) $K_1 = \dfrac{12375}{3000} - W$ …(i)

$$K_2 = \frac{12375}{1650} - W \qquad \text{…(ii)}$$

$$v_2 = 2v_1 \quad \therefore \quad K_2 = 4K_1 \qquad \text{…(iii)}$$

Solving these equations, we get

$$W = 3\text{ eV}$$

$\therefore$ Threshold wavelength,

$$\lambda_0 = \frac{12375}{3}$$

$$= 4125\text{ Å}$$ **Ans.**

(b) $E_2 = \dfrac{12375}{1650} = 7.5\text{ eV} = 12\times10^{-19}\text{ J}$

Therefore, number of photons incident per second

$$n_2 = \frac{P_2}{E_2} = \frac{5.0\times10^{-3}}{12\times10^{-19}}$$

$$= 4.17\times10^{15}\text{ per second}$$

Number of electrons emitted per second ($\eta = 5.1\%$)

$$= \frac{5.1}{100}\times4.17\times10^{15}$$

$$= 2.13\times10^{14}\text{ per second}$$

$\therefore$ Saturation current in second case

$$i = (2.13\times10^{14})\,(1.6\times10^{-19})\text{ A} \Rightarrow 3.4\times10^{-5}\text{ A}$$

$$= 34\ \mu\text{A}$$ **Ans.**

(c) Energy of photon in first case,

$$= \frac{12375}{3000}$$

$$= 4.125\text{ eV}$$

or $\quad E_1 = 6.6\times10^{-19}\text{ J}$

Rate of incident photons

$$= \frac{P_1}{E_1} = \frac{10^{-3}}{6.6\times10^{-19}}$$

$$= 1.52\times10^{15}\text{ per second}$$

Number of electrons ejected

$$= \frac{4.8\times10^{-3}}{1.6\times10^{-19}}\text{ per second}$$

$$= 3.0\times10^{16}\text{ per second}$$

$\therefore$ Efficiency of photoelectrons generation

$$= \frac{1.52\times10^{15}}{3.0\times10^{16}}\times100$$

$$= 5.1\%$$ **Ans.**

15. **Balmer Series**

$$\lambda_{32} = \frac{12375}{E_3 - E_2} = \frac{12375}{(13.6)\left(\frac{1}{4}-\frac{1}{9}\right)}$$

$$= 6551\text{ Å}$$

$$= 655.1\text{ nm}$$

$$\lambda_{42} = \frac{12375}{E_4 - E_2} = \frac{12375}{(13.6)\left(\frac{1}{4}-\frac{1}{16}\right)}$$

$$= 4853\text{ Å}$$

$$= 485.3\text{ nm}$$

$$\lambda_{52} = \frac{12375}{E_5 - E_2} = \frac{12375}{(13.6)\left(\frac{1}{4}-\frac{1}{25}\right)}$$

$$= 4333\text{ Å}$$

$$= 433.3\text{ nm}$$

First two lie in the given range. Of these λ_{42} corresponds to more energy.

$$E = E_4 - E_2 = (13.6)\left(\frac{1}{4}-\frac{1}{16}\right)$$

$$= 2.55\text{ eV}$$

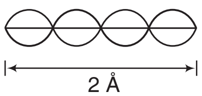
$$\therefore \quad K_{\text{max}} = E - W = (2.55 - 2.0)\text{ eV}$$

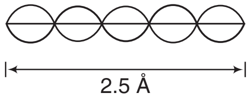
$$= 0.55\text{ eV}$$ **Ans.**

16. From the theory of standing wave, we can say that

$$\frac{\lambda}{2} = (2.5 - 2.0) = 0.5\text{ Å}$$

or $\lambda = 1\text{ Å}$





Therefore, least value of d required will correspond to a single loop.

$$\therefore \quad d_{\min} = \frac{\lambda}{2} = 0.5\text{ Å}$$ **Ans.**

Further for de-Broglie wavelength of an electron,

$$\lambda = \sqrt{\frac{150}{K(\text{in eV})}}\text{ Å}$$

$$\lambda = 1\text{ Å}$$

$$\therefore \quad K = 150\text{ eV}$$ **Ans.**

17. (a) Reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(1837m_e)(207m_e)}{1837m_e + 207m_e}$$

$$= 186m_e$$

$$= 186 \times 9.1 \times 10^{-31}$$

$$= 1.69 \times 10^{-28}\text{ kg}$$ **Ans.**

(b) $E_n \propto m$

Here, reduced mass is 186 times mass of electron. Hence, ground state energy will also be 186 times that of hydrogen atom.

$$\therefore \quad E_1 = 186(-13.6)\text{ eV}$$

$$= -2529.6\text{ eV} \approx -2.53\text{ keV}$$ **Ans.**

(c) $E_2 = 186\,(-3.4)\text{ eV} = -632.4\text{ eV}$

$$\therefore \quad \Delta E_{21} = 1897.2\text{ eV}$$

$$\therefore \quad \lambda_{21} = \frac{12375}{\Delta E_{21}}\text{ Å}$$

$$= \frac{12375}{1897.2}\text{ Å}$$

$$= 6.53\text{ Å} \approx 0.653\text{ nm}$$ **Ans.**

18. Force of interaction between electron and proton is

$$F = -\frac{dU}{dr} = \frac{-k}{r}$$

Force is negative. It means there is an attraction between the particles and they are bound to each other. This force provides the necessary centripetal force for the electron.

$$\therefore \quad \frac{mv^2}{r} = \frac{k}{r} \qquad \text{...(i)}$$

According to Bohr's assumption,

$$mvr = n\frac{h}{2\pi} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$r = \frac{nh}{2\pi\sqrt{mk}}$$

and $$v = \sqrt{\frac{k}{m}}$$

$$\therefore \quad E = U + \frac{1}{2}mv^2 = k\ln r - \frac{k}{2} + \frac{k}{2} = k\ln r$$

Thus, $$r_n = \frac{nh}{2\pi\sqrt{mk}}$$

and $$E_n = k\ln\left\{\frac{nh}{2\pi\sqrt{mk}}\right\}$$ **Ans.**

19. (a) $$r = \frac{mv}{Be} \qquad \text{...(i)}$$

$$mvr = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

Solving these two equations, we get

$$r = \sqrt{\frac{nh}{2\pi Be}}$$

and $$v = \sqrt{\frac{nhBe}{2\pi m^2}}$$

(b) $K = \dfrac{1}{2}mv^2 = \dfrac{nhBe}{4\pi m}$ **Ans.**

(c) $$M = iA = \left(\frac{e}{T}\right)(\pi r^2)$$

$$= \frac{e}{\left(\frac{2\pi r}{v}\right)}(\pi r^2) = \frac{evr}{2}$$

$$= \frac{e}{2}\sqrt{\frac{nh}{2\pi Be}}\sqrt{\frac{nhBe}{2\pi m^2}}$$

$$= \frac{nhe}{4\pi m}$$

$$U = -MB\cos 180°$$

$$= \frac{nheB}{4\pi m}$$

Note *Angle between* **M** *and* **B** *will be 180°. Think why?*

(d) $E = U + K = \dfrac{nheB}{2\pi m}$

(e) $|\phi| = B\pi r^2 = \dfrac{nh}{2e}$

20. (a) and (b) When hydrogen atom is excited, then

$$eV = E_0\left(\frac{1}{1}-\frac{1}{n^2}\right) \qquad \ldots(\text{i})$$

When ion is excited,

$$eV = E_0 Z^2\left[\frac{1}{2^2}-\frac{1}{n_1^2}\right] \qquad \ldots(\text{ii})$$

Wavelength of emitted light,

$$\frac{hc}{\lambda_1} = E_0\left(\frac{1}{1}-\frac{1}{n^2}\right) \qquad \ldots(\text{iii})$$

$$\frac{hc}{\lambda_2} = E_0 Z^2\left(\frac{1}{1}-\frac{1}{n_1^2}\right) \qquad \ldots(\text{iv})$$

Further it is given that

$$\frac{\lambda_1}{\lambda_2} = \frac{5}{1} \qquad \ldots(\text{v})$$

Solving the above equations, we get

$$Z = 2, \quad n = 2, \; n_1 = 4$$

and $\qquad V = 10.2\text{ V}$ **Ans.**

(c) Energy of emitted photon by the hydrogen atom $= E_2 - E_1$

$$= 10.2\text{ eV}$$ **Ans.**

and by the ion $= E_4 - E_1$

$$= (13.6)\,(2)^2\left(1-\frac{1}{16}\right)$$

$$= 51\text{ eV}$$ **Ans.**

21. $0.6 = \dfrac{12375}{4950} - W \qquad \ldots(\text{i})$

$$1.1 = \frac{12375}{\lambda} - W \qquad \ldots(\text{ii})$$

Solving above two equations, we get

$$W = 1.9\text{ eV}$$

and $\qquad \lambda = 4125\text{ Å}$ **Ans.**

22. $E = \dfrac{12375}{4000} = 3.1\text{ eV}$

Number of photoelectrons emitted per second,

$$n = \left(\frac{1}{10^6}\right)\left(\frac{5}{3.1\times 1.6\times 10^{-19}}\right)$$

$$= 1.0\times 10^{13} \quad \text{per second}$$

$\therefore \qquad i = ne$

$$= 1.0\times 10^{13}\times 1.6\times 10^{-19}$$

$$= 1.6\times 10^{-6}\text{ A}$$

$$= 1.6\,\mu\text{A}$$ **Ans.**

23. (a) $E = \dfrac{12375}{4000} = 3.1\text{ eV}$

Energy of electron after first collision

$E_1 = 90\%$ of $E = 2.79$ eV (as 10% is lost)

Energy of electron after second collision

$E_2 = 90\%$ of $E_1 = 2.51$ eV

KE of this electron after emitting from the metal surface

$= (2.51 - 2.2)$ eV $= 0.31$ eV **Ans.**

(b) Energy after third collision,

$E_3 = 90\%$ of $E_2 = 2.26$ eV

Similarly,

$E_4 = 90\%$ of $E_3 = 2.03$ eV

So, after four collisions it becomes unable for the electrons to come out of the metal.

34. Modern Physics-II

INTRODUCTORY EXERCISE 34.1

2. Penetrating power is maximum for γ-rays, then of β-particles and then α-particles because basically it depends on the velocity. However, ionization power is in reverse order.

3. Both the beta rays and the cathode rays are made up of electrons. So, only option (a) is correct.

(b) Gamma rays are electromagnetic waves.

(c) Alpha particles are doubly ionized helium atoms and

(d) Protons and neutrons have approximately the same mass.

Therefore, (b), (c) and (d) are wrong options.

4. During β-decay, a neutron is transformed into a proton and an electron. This is why atomic number (Z = number of protons) increases by one and mass number (A = number of protons + neutrons) remains unchanged during β-decay.

5. Following nuclear reaction takes place

$$_0n^1 \longrightarrow {}_1\text{H}^1 + {}_{-1}e^0 + \bar{\nu}$$

$\bar{\nu}$ is antineutrino.

6. From $R = R_0\left(\frac{1}{2}\right)^n$

we have, $1 = 64\left(\frac{1}{2}\right)^n$

or $n = 6$ = number of half-lives

$\therefore$ $t = n \times t_{t_{1/2}} = 6 \times 2 = 12\text{ h}$

7. Activity of $S_1 = \frac{1}{2}$ (activity of S_2)

or $\lambda_1 N_1 = \frac{1}{2}(\lambda_2 N_2)$

or $\frac{\lambda_1}{\lambda_2} = \frac{N_2}{2N_1}$

or $\frac{T_1}{T_2} = \frac{2N_1}{N_2}$ $\left(T = \text{half-life} = \frac{\ln 2}{\lambda}\right)$

Given, $N_1 = 2N_2$

$\therefore$ $\frac{T_1}{T_2} = 4$

$\therefore$ Correct option is (a).

8. After two half-lives $\frac{1}{4}$ th fraction of nuclei will remain un-decayed. Or, $\frac{3}{4}$ th fraction will decay. Hence, the probability that a nucleus decays in two half-lives is $\frac{3}{4}$.

9. Activity reduces from 6000 dps to 3000 dps in 140 days. It implies that half-life of the radioactive sample is 140 days. In 280 days (or two half-lives) activity will remain $\frac{1}{4}$ th of the initial activity.

Hence, the initial activity of the sample is

4×6000 dps = 24000 dps

Therefore, the correct option is (d).

10. $R = R_0\left(\frac{1}{2}\right)^n$ ...(i)

Here, R = activity of radioactive substance after n half-lives $= \frac{R_0}{16}$ (given)

Substituting in Eq. (i), we get $n = 4$

$\therefore$ $t = (n)\, t_{1/2} = (4)\,(100\ \mu\text{s}) = 400\ \mu\text{s}$

11. Using $N = N_0 e^{-\lambda t}$

where, $\lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{3.8}$

$\therefore$ $\frac{N_0}{20} = N_0 e^{-\frac{\ln 2}{3.8}t}$

Solving this equation with the help of given data we find

$t = 16.5$ days

$\therefore$ Correct option is (b).

12. $1000 = \left(\frac{1}{2}\right)^n 8000$

$\therefore$ $n = 3$ = number of half-lives

These half-lives are equivalent to 9 days. Hence, one half-life is 3 days.

$t_{\text{av}} = 1.44\ t_{1/2} = 1.44 \times 3 = 4.32$ days

13. $R_0 = \lambda N_0 \Rightarrow N_0 = \frac{R_0}{\lambda}$

where, $\lambda = \frac{\ln 2}{t_{1/2}}$

$N = N_0 e^{-\lambda t}$

Find $N_1 = N_0 e^{-\lambda t_1}$

and $N_2 = N_0 e^{-\lambda t_2}$

$\therefore$ Number of nuclei decayed in given time $= N_1 - N_2$

14. (a) $R = R_0 e^{-\lambda t}$

$$R_0 = 20 \text{ MCI}$$
$$R = 8 \text{ MCI}$$
$$t = 4.0 \text{ h}$$

Find λ.

(b) $R_0 = \lambda N_0$. Find $N_0 = \dfrac{R_0}{\lambda}$

(c) Find $R = R_0 e^{-\lambda t}$

15. $R_0 = \lambda N_0$

$$6.0 \times 10^{11} = \lambda\,(10^{15})$$
$$\therefore \quad \lambda = 6.0 \times 10^{-4} \text{ s}$$
$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{6.0 \times 10^{-4}}$$
$$= 1.16 \times 10^{3} \text{ s}$$

16. In 200 minute time,

$$n_1 = \text{number of half-lives of } X$$
$$= \frac{200}{50} = 4$$
$$n_2 = \text{number of half-lives of } Y$$
$$= \frac{200}{100} = 2$$
$$\frac{N_X}{N_Y} = \frac{N_0 (1/2)^4}{N_0 (1/2)^2} = \frac{1}{4}$$

INTRODUCTORY EXERCISE 34.2

1. $4\,({}_2\text{He}^4) = {}_8\text{O}^{16}$

Mass defect, $\Delta m = \{4\ (4.0026) - 15.9994\}$

$$= 0.011 \text{ amu}$$

∴ Energy released per oxygen nuclei

$$= (0.011)\,(931.48) \text{ MeV}$$
$$= 10.24 \text{ MeV}$$

∴ Correct answer is (c).

2. Heavy water is used as moderators in nuclear reactors to slow down the neutrons.

4. During fusion process two or more lighter nuclei combine to form a heavy nucleus.

Hence, the correct option is (c).

5. (a) $m(c)^2 = P \times t$

$$\therefore \quad m = \frac{P \times t}{c^2}$$
$$= \frac{(10^9)\,(24)\,(3600)}{(3 \times 10^8)^2}$$
$$= 9.6 \times 10^{-4} \text{ kg}$$

(b) Number of fissions required per second

$$= \frac{\text{Energy required per second}}{\text{Energy released in one fission}}$$
$$= \frac{10^9}{200 \times 10^6 \times 1.6 \times 10^{-19}}$$
$$= 3.125 \times 10^{19}$$

6. Mass defect $\Sigma m_i - \Sigma m_f = \Delta m$

$$= (238.050784) - (234.043593 + 4.002602)$$
$$= 4.589 \times 10^{-3} \text{ u}$$

Energy released $= \Delta m \times 931.48$ MeV

$$= 4.27 \text{ MeV}$$

8. $Q = (\Delta m \text{ in atomic mass unit}) \times 931.4 \text{ MeV}$

$$= (2 \times \text{mass of } {}_1\text{H}^2 - \text{mass of } {}_2\text{He}^4) \times 931.4 \text{ MeV}$$
$$= (2 \times 2.0141 - 4.0024) \times 931.4 \text{ MeV}$$
$$Q \approx 24 \text{ MeV}$$

9. $2\,{}_1\text{H}^2 \longrightarrow {}_2\text{He}^4$

Binding energy of two deuterons,

$$E_1 = 2\,[2 \times 1.1] = 4.4 \text{ MeV}$$

Binding energy of helium nucleus,

$$E_2 = 4\ (7.0) = 28.0 \text{ MeV}$$

∴ Energy released $\Delta E = E_2 - E_1$

$$= (28 - 4.4) \text{ MeV} = 23.6 \text{ MeV}$$

Exercises

LEVEL 1

Assertion and Reason

1. Huge amount of energy is involved in any nuclear process, which cannot be increased or decreased by pressure or temperature.

2. Some lighter nuclei are also radioactive.

3. By emission of one α-praticle, atomic number decreases by 2 and mass number by 4. But by the emission of one β-particle, atomic number increases by 1 and mass number remains unchanged.

4. In moving from lower energy state to higher energy state electromagnetic waves are absorbed.

5. Neutrino or antineutrino is also produced during β-decay.

6. Total binding energy per nucleon is more important for stability.

9. $(1\text{ amu})(c^2) = 931.48$ MeV

10. α-particles are heaviest. Hence, its ionizing power is maximum.

11. In binding energy per nucleon *versus* mass number graph binding energy per nucleon of daughter nuclei should increase (for release of energy) or the daughter nuclei should lie towards the peak of the graph.

Objective Questions

1. Nuclear density is constant hence, mass $\propto$ volume

or $$m \propto V$$

2. Radius of a nucleus is given by

$$R = R_0 A^{1/3} \qquad (\text{where, } R_0 = 1.25 \times 10^{-15}\text{ m})$$
$$= 1.25\, A^{1/3} \times 10^{-15}\text{ m}$$

Here, A is the mass number and mass of the uranium nucleus will be

$$m \approx Am_p, \text{ where } m_p = \text{mass of proton}$$
$$= A\,(1.67 \times 10^{-27}\text{ kg})$$

$$\therefore \text{ Density, } \rho = \frac{\text{mass}}{\text{volume}} = \frac{m}{\frac{4}{3}\pi R^3}$$
$$= \frac{A\,(1.67 \times 10^{-27}\text{ kg})}{A\,(1.25 \times 10^{-15}\text{m})^3}$$

or $$\rho \approx 2.0 \times 10^{17}\text{ kg/m}^3$$

5. Let $n-\alpha$ particles and $m-\beta$ particles are emitted. Then,

$$90 - 2n + m = 80 \qquad \text{...(i)}$$
$$200 - 4n = 168 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$n = 8 \quad \text{and} \quad m = 6$$

7. By emitting one α-particle, atomic number decreases by 2. By emitting two β-particles, atomic number increases by 2. Hence,

$$Z_A = Z_C$$

or A and C are isotopes.

8. Binding energy $= (\Delta m) \times 931.5$ MeV

$$= [2 \times 1.00785 + 2 \times 1.00866 - 4.00274] \times 931.5$$
$$= 28.2\text{ MeV}$$

9. Number of nuclei left $= \frac{1}{8}$th

$$\text{Now,} \quad \frac{1}{8} = \left(\frac{1}{2}\right)^n$$
$$\therefore \quad n = \text{number of half-lives}$$
$$= 3$$
$$3\text{ half-lives} = 8\text{ s}$$
$$\therefore \quad 1\text{ half-life} = \frac{8}{3}\text{ s}$$

10. Number of atoms disintegrated,

$$N = N_0\,(1 - e^{-\lambda t})$$
$$\therefore \quad \frac{N}{N_0} = 1 - e^{-\lambda t}$$

At t = one mean life $= \frac{1}{\lambda}$

$$\frac{N}{N_0} = 1 - e^{-1} = 1 - \frac{1}{e}$$

11. Decayed fraction is $\frac{3}{4}$th. Therefore, left fraction is $\frac{1}{4}$th.

$$N = N_0\left(\frac{1}{2}\right)^n \text{ or } \frac{N}{N_0} = \left(\frac{1}{2}\right)^n = \frac{1}{4}$$
$$\therefore \quad n = \text{number of half-lives} = 2$$

Two half-lives are equivalent to 15 min. Therefore, one half-life is 7.5 min.

12. $\lambda = \dfrac{\ln 2}{t_{1/2}} = \dfrac{\ln 2}{4}\text{ day}^{-1}$

Now, apply

$$R = R_0 e^{-\lambda t} \quad \Rightarrow \quad 5 = 100 e^{-\lambda t}$$

Substituting value of λ, we can find t.

13. Let N nuclei decay per second. Then,

$$N\,(200 \times 1.6 \times 10^{-13}) = 1.6 \times 10^6$$

Solving we get $N = 5 \times 10^{16}$ per sec

14. Nuclear density is independent of A. It is of the order of 10^7 kg/m^3.

15.
$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{6.93} = \frac{1}{10}\text{ sec}^{-1}$$
$$N = N_0 e^{-\lambda t}$$
$$\therefore \quad \frac{N}{N_0} = e^{-\lambda t} = e^{-\left(\frac{1}{10}\right)(10)}$$
$$= e^{-1} \approx 0.63$$

16. Activity of atoms is 6.25% after four half-lives.

$\therefore$ Four half-lives $\approx$ 2 h = 120 min

$\therefore$ One half-life is 30 min.

17. Probability of survival,

$$P = \frac{\text{Number of nuclei left}}{\text{Initial number of nuclei}} = \frac{N_0 e^{-\lambda t}}{N_0}$$

At $\qquad t = \text{one mean life} = \frac{1}{\lambda}$

$$P = e^{-1} = \frac{1}{e}$$

Subjective Questions

1. (a) $R = R_0 e^{-\lambda t}$

$R = 2700$ per minute, $R_0 = 4750$ per minute

$t = 5$ min

Find λ.

(b) $t_{1/2} = \frac{\ln 2}{\lambda}$

2. $R = \lambda N$

$$6 \times 10^{11} = 1.0 \times 10^{15} \lambda$$

$\therefore \qquad \lambda = 6 \times 10^{-4}$ s

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{6 \times 10^{-4}} \text{ s}$$

$$= 1155 \text{ s} = 19.25 \text{ min}$$

3. $\because \quad R = \lambda N$

$$\therefore \quad N = \frac{R}{\lambda} = \frac{R}{(\ln 2)/t_{1/2}} = \frac{R t_{1/2}}{\ln 2}$$

$$= \frac{(8 \times 3.7 \times 10^{10})\,(5.3)\,(365)\,(24)\,(3600)}{0.693}$$

$$= 7.14 \times 10^{19}$$

$$m = \frac{7.14 \times 10^{9}}{6.02 \times 10^{23}} \times 60 \text{ g}$$

$$= 7.11 \times 10^{-3} \text{ g}$$

4. $R = \lambda N = \left(\frac{\ln 2}{t_{1/2}}\right) N$

$$= \frac{0.693}{4.5 \times 10^{9} \times 365 \times 24 \times 3600} \left(\frac{1}{238}\right) (6.02 \times 10^{23})$$

$$= 1.23 \times 10^{4} \text{dps}$$

5. $\frac{1}{\lambda} = 10$ days

$\therefore \qquad \lambda = 0.1 \text{ day}^{-1}$

Probability of decay

$$= \frac{\text{Number of atoms decayed}}{\text{Initial number of atoms}}$$

$$= \frac{N_0 (1 - e^{-\lambda t})}{N_0}$$

$$= 1 - e^{-\lambda t} = 1 - e^{-0.1 \times 5}$$

$$= 0.39$$

6. $\frac{\text{U}^{238}}{\text{Pb}^{206}} = \frac{3}{1}$

$$N_0 = 3 + 1 = 4$$

$$N = 3$$

$$N = N_0 e^{-\lambda t} \qquad \text{...(i)}$$

$$\lambda = \frac{\ln 2}{t_{1/2}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$t = 1.88 \times 10^{9} \text{ yr}$$

7. $\lambda_1 = \frac{\ln 2}{T_1} \qquad (T = \text{half-life})$

$$\lambda_2 = \frac{\ln 2}{T_2}$$

$$R_{01} + R_{02} = 8 \text{ mCi} \qquad \text{(given)}$$

$\therefore \quad \lambda_1 (4N_0) + \lambda_2 (N_0) = 8$ mCi

From here we can find number

after $t = 60$ yr

$$R = R_1 + R_2$$

$$= (4\lambda_1 N_0) e^{-\lambda_1 t} + (\lambda_2 N_0) e^{-\lambda_2 t}$$

9. (a) $82 + 10 = 92$, $206 + 10 + 20 = 236$

So, this reaction is possible.

(b) $82 + 16 - 6 = 92$, $206 + 32 = 238$

But antineutrino is also emitted with β^{-1} (or electron) decay.

10. Binding energy $= \Delta m \times 931.5$ MeV

$$= (7 \times 1.00783 + 7 \times 1.00867 - 14.00307)\ 931.5$$

$$= 104.72 \text{ MeV}$$

11. Energy released = binding energy

$$= \Delta m \times 931.5 \text{ MeV}$$

$$= (8 \times 1.007825 + 8 \times 1.008665 - 15.994915) \times 931.5$$

$$= 127.62 \text{ MeV}$$

12. (a) Number of nuclei in 1 kg of U^{235},

$$N = \left(\frac{1}{235}\right) (6.02 \times 10^{26})$$

$\therefore \quad$ Total energy released

$$= (N \times 200) \text{ MeV}$$

$$= \left(\frac{1}{235}\right) (6.02 \times 10^{26})\,(200)\,(1.6 \times 10^{-13})$$

$$= 8.19 \times 10^{13} \text{ J}$$

(b) $$m = \frac{8.19 \times 10^{13}}{30 \times 10^3} \text{ g}$$
$$= 2.73 \times 10^9 \text{ g}$$
$$= 2.73 \times 10^6 \text{ kg}$$

13. From momentum of conservation,
$$p_1 = p_2$$
$$\therefore \quad \sqrt{2K_1 m_1} = \sqrt{2K_2 m_2}$$
$$\therefore \quad K_2 = \frac{K_1 m_1}{m_2} = \frac{(6.802)\,(4)}{208}$$
$$= 0.1308 \text{ MeV}$$

14. Number of nuclei in 1 kg of uranium,
$$N = \left(\frac{1}{235}\right)(6.02 \times 10^{26})$$
Now, $$\left(\frac{1}{235}\right)(6.02 \times 10^{26})\,(185 \times 1.6 \times 10^{-13})$$
$$= (100 \times 10^6)t$$
$$\therefore \quad t = 7.58 \times 10^5 \text{ s}$$
$$= \frac{7.58 \times 10^5}{60 \times 60 \times 24} \text{ day}$$
$$= 8.78 \text{ days}$$

15. Q-value $= (\Delta m)\,(931.5)$ MeV

(a) Q-value $= (2 \times 2.014102 - 3.016049 - 1.007825) \times 931.5$
$$= 4.05 \text{ MeV}$$
Similarly, Q-value of other parts can also be obtained.

16. Q-value $= (\Delta m) \times 931.5$ MeV
$$= (2 \times 4.0026 - 8.0053) \times 931.5$$
$$= -\,0.0931 \text{ MeV}$$
$$= -\,93.1 \text{ keV}$$

LEVEL 2

Single Correct Option

1. $\frac{N_0}{16} = N_0 \left(\frac{1}{2}\right)^n \quad \therefore \quad n = 4$

So, $3t$ times is equivalent to four half-lives. Hence, one half-life is equal to $\frac{3t}{4}$.

The given time $\frac{11}{2}t - t = \frac{9}{2}t$ is equivalent to 6 half-lives.
$$\therefore \quad N = N_0 \left(\frac{1}{2}\right)^6 = \frac{N_0}{64}$$

2. $\lambda = \lambda_1 + \lambda_2$
$$\therefore \quad \frac{\ln 2}{T} = \frac{\ln 2}{T_1} + \frac{\ln 2}{T_1} \qquad (T = \text{Half-life})$$
$$\text{or} \quad T = \frac{T_1 T_2}{T_1 + T_2} = 20 \text{ y}$$
$\frac{1}{4}$th sample remains after 2 half-lives or 40 y.

3. Q-value = Final binding energy

Initial binding energy
$$= E_2 N_2 + E_3 N_3 - E_1 N_1$$

4. Energy released = Final binding energy − initial binding energy
$$= 110 \times 8.2 + 90 \times 8.2 - 200 \times 7.4$$
$$= 160 \text{ MeV}$$

5. It means we are getting only 100 MeV of energy by the fission of one uranium nucleus.

Number of nuclei per second
$$= \frac{\text{Energy required per second}}{\text{Energy obtained by one fission}}$$
$$= \frac{16 \times 10^6}{100 \times 1.6 \times 10^{-13}} = 10^{18}$$

6. When the rate production = rate of disintegration, number of nuclei or maximum.
$$\therefore \quad \lambda N = A$$
$$\text{or} \quad \frac{\ln 2}{T} N = A \text{ or } N = \frac{AT}{\ln 2} = \text{maximum}$$

7. $R_0 = 15 \times 200 = 3000$ decay/min from 200 g carbon.

Using $$R = R_0 \left(\frac{1}{2}\right)^n$$
$$\therefore \quad 375 = 3000 \left(\frac{1}{2}\right)^n$$
$\therefore$ n = number of half-lives = 3

$\therefore$ $t = 5730 \times 3 = 17190$ y

8. $A_P = A_0 e^{-\lambda t_1}$
$$A_Q = A_0 e^{-\lambda t_2}$$
$$\lambda_{t_1} = \ln (A_0/A_P)$$
$$\therefore \quad t_1 = \frac{1}{\lambda} \ln (A_0/A_P) = T \ln (A_0/A_P)$$
Similarly, $$t_2 = T \ln (A_0/A_Q)$$
$$\therefore \quad t_1 - t_2 = T \ln \left(\frac{A_Q}{A_P}\right)$$

9. Combining two given equations,
we have, $3\,{}_1\text{H}^2 = 2\text{He}^4 + p + n$

$$\Delta m = 3 \times 2.014 - 4.001 - 1.007 - 1.008$$
$$= 0.026 \text{ u}$$

Energy released by 3 deuterons

$$= 0.026 \times 931.5 \times 1.6 \times 10^{-13}\,\text{J}$$
$$= 3.9 \times 10^{-12}\,\text{J}$$

Now, $(10^{16} \times t) = \left(\dfrac{10^{40}}{3}\right)(3.9 \times 10^{-12})$

Solving we get,

$$t \approx 1.3 \times 10^{12} \text{ s}$$

10. $R_1 = R_2$

$$R_{01}e^{-\lambda_1 t} = R_{02}e^{-\lambda t}$$
$$\therefore \quad \lambda_1 N_0 e^{-\lambda_1 t} = \lambda_2 N_0 e^{-\lambda_2 t} \qquad \ldots\text{(i)}$$
$$\lambda_1 = \frac{\ln 2}{t_1} = \frac{0.693}{t_1}$$
$$\lambda_2 = \frac{0.693}{t_2}$$

Substituting these values in Eq. (i), we can get the *t*.

11. $A = 232 - 4 = 228$

From conservation of momentum,

$$p_\alpha = p_\gamma = \sqrt{2K_\alpha m_\alpha} = \sqrt{2K_\gamma m_\gamma}$$
$$\text{or} \quad \frac{K_\alpha}{K_\gamma} = \frac{m_\gamma}{m_\alpha} = \frac{228}{4}$$
$$\therefore \quad K_\alpha = \left(\frac{228}{228+4}\right)K_{\text{Total}}$$
$$= \left(\frac{228}{232}\right)K_{\text{Total}}$$

12. From momentum of conservation,
momentum of photon = photon of nucleus

$$\therefore \quad \frac{E}{c} = \sqrt{2Km}$$
$$\therefore \quad K = \frac{E^2}{2mc^2}$$
$$= \frac{(7 \times 1.6 \times 10^{-13})^2}{2 \times 24 \times 1.67 \times 10^{-27} \times (3 \times 10^8)^2 \times 1.6 \times 10^{-16}} \text{ keV}$$
$$= 1.1 \text{ keV}$$

13. Activity $A \propto$ Number of atoms

$$A_1 = A_0 e^{-\lambda t_1}$$
$$\therefore \quad t_1 = \frac{1}{\lambda}\ln\left(\frac{A_0}{A_1}\right) = \frac{T}{\ln 2}\ln\frac{A_0}{A_1}$$
$$A_2 = 2A_0 e^{-\lambda t_2}$$
$$t_2 = \frac{T}{\ln 2}\ln\left(\frac{2A_0}{A_2}\right)$$
$$t_1 - t_2 = \frac{T}{\ln 2}\left(\frac{A_0}{A_1} \times \frac{A_2}{2A_0}\right)$$
$$= \frac{T}{\ln 2}\ln\left(\frac{A_2}{2A_1}\right)$$

More than One Correct Options

1. $y = \lambda x = \dfrac{\ln 2}{T} \cdot x$

$$\frac{x}{y} = \frac{1}{\lambda} = \text{constant}$$
$$\frac{x}{y} = \frac{T}{\ln 2} \quad \text{or} \quad \frac{x}{y} > T \, (\text{as } \ln 2 = 0.693)$$

Further,

$$xy = x\,(\lambda x) = \lambda x^2$$

After one half-life, *x* remains half. Hence, x^2 remains $\dfrac{1}{4}$th.

3. By the emission of an α-particle, atomic number decreases by 2 and by the emission of two particles atomic number increases by 2. Hence, net atomic number remains unchanged.

4. At $t = 4T$

Number of half-lives of first $n_1 = 4$ and number of half-lives of second $n_2 = 2$

$$\frac{N_1}{N_2} = x = \frac{N_0\,(1/2)^4}{N_0\,(1/2)^2} = \frac{1}{4}$$
$$y = \frac{R_1}{R_2} = \frac{\lambda_1\,N_0\,(1/2)^4}{\lambda_2\,N_0\,(1/2)^2}$$
$$= \frac{\lambda_1}{4\lambda_2} = \frac{T_2}{4T_1}$$
$$= \frac{2T}{4T} = \frac{1}{2}$$

6. $R = R_0 A^{1/3} \quad \text{or} \quad R \propto A^{1/3}$

Comprehension Based Questions

1. Energy released = (Δm) (931.48) MeV

$$= [2 \times 2.01102 - 3.0160 - 1.007825] \times 931.5$$
$$= 4.03 \text{ MeV} \approx 4 \text{ MeV}$$

2. Let N number of fusion reactions are required, then

$$N \times 4 \times 1.6 \times 10^{-13} = 10^3 \times 3600$$

$$N = 5.625 \times 10^{18}$$

3. In one fusion reaction two ${}^2_1$H nuclei are used. Hence, total number of ${}^2_1$H nuclei are $2N$.

or 1.125×10^{19} Mass in kg

$$= \left(\frac{1.125 \times 10^{19}}{6.02 \times 10^{26}}\right)(2)\ \text{kg}$$

$$= 3.7 \times 10^{-8}\ \text{kg}$$

Match the Columns

1. $\left(-\frac{dN}{dt}\right) = \lambda N$

$$\therefore \quad y = \lambda x \quad \text{or} \quad \lambda = \frac{y}{x}$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = (\ln 2)\,(x/y)$$

$$R = R_0 e^{-\lambda t}$$

$$= y e^{-\lambda(1/\lambda)} = y/e$$

$$R = \frac{y}{e} = \lambda N$$

$$\therefore \quad N = \frac{y}{e\lambda} = \frac{y}{e(y/x)} = \frac{x}{e}$$

2. Energy is released when daughter nuclei lie towards peak of this graph, so that binding energy per nucleon or total binding energy in the nuclear process increases.

3. A will continuously decrease, but C will increase. $(A + B)$ will continuously decrease as C is formed only from A and B.

$$(C + B) = \text{Total} - A$$

A is continuously decreasing. Hence, $(C + B)$ will continuously increase.

4. (a) $Z' = Z - Z + 1 = Z - 1$

$$A' = A - 4$$

(b) $Z' = Z - 2 \times 2 + 1 = Z - 3$

$$A' = A - 2 \times 4 = A - 8$$

(c) $Z' = Z - 2 + 2 \times 1 = Z$

$$A' = A - 4$$

(d) $Z' = Z - 2 \times 2 + 2 \times 2 = Z - 2$

$$A' = A - 2 \times 4 = A - 8$$

5. (a) E is of the order of kT, where k = Boltzmann constant and $T \approx 300$ K

(c) λ to X-rays is of the order of 1Å – 100 Å

$$E = \left[\frac{12375}{\lambda\ (\text{in Å})}\right] \text{in eV}$$

$$= \left[\frac{12.375}{\lambda}\right] \text{in keV}$$

where, λ is in Å.

(d) λ is of the order of 4000 Å – 7000 Å

Now, $\quad E\ (\text{in eV}) = \dfrac{12375}{\lambda\ (\text{in Å})}$

Subjective Questions

1. $N = \dfrac{R}{\lambda} = \dfrac{10^9}{\dfrac{0.693}{14.3 \times 3600}} = 7.43 \times 10^{13}$

Now, $\quad \dfrac{dN}{dt} = q - \lambda N \quad$ or $\quad \displaystyle\int_0^N \frac{dN}{q - \lambda N} = \int_0^t dt$

$$\therefore \quad N = \frac{q}{\lambda}(1 - e^{-\lambda t})$$

Substituting the values,

$$7.43 \times 10^{13} = \frac{2 \times 10^9}{\dfrac{0.693}{14.3 \times 3600}}\left[1 - e^{-(0.693/14.3 \times 3600)t}\right]$$

Solving this equation we get,

$$t = 14.3\ \text{h}$$ **Ans.**

2. (a)

	A	B	C
At $t = 0$	N_0	0	0
At t	N_1	N_2	N_3

Here, $\quad N_1 = N_0 e^{-\lambda t} \quad$ ...(i)

$$\frac{dN_2}{dt} = \lambda\,(N_1 - N_2)$$

or $\quad \dfrac{dN_2}{dt} = \lambda N_0 e^{-\lambda t} - \lambda N_2$

or $\quad dN_2 + \lambda N_2 dt = \lambda N_0 e^{-\lambda t}$

$\therefore \quad e^{\lambda t} dN_2 + \lambda N_2 e^{\lambda t} dt = \lambda N_0 dt$

or $\quad d(N_2 e^{\lambda t}) = \lambda N_0 dt$

$\therefore \quad N_2 e^{\lambda t} = \lambda N_0 t + C$

At $t = 0$, $N_2 = 0$,

$\therefore \quad C = 0$

$\therefore \quad N_2 = \lambda N_0 (t e^{-\lambda t})$

(b) Activity of B is

$$R_2 = \lambda N_2 = \lambda^2 N_0 (t e^{-\lambda t})$$

For maximum activity, $\frac{dR_2}{dt} = 0$

∴ $t = \frac{1}{\lambda}$ **Ans.**

∴ $R_{\max} = \frac{\lambda N_0}{e}$ **Ans.**

3. (a) Let at time t, number of radioactive nuclei are N.

Net rate of formation of nuclei of A

$$\frac{dN}{dt} = \alpha - \lambda N \quad \text{or} \quad \frac{dN}{\alpha - \lambda N} = dt$$

or $$\int_{N_0}^{N} \frac{dN}{\alpha - N} = \int_0^t dt$$

Solving this equation, we get

$$N = \frac{1}{\lambda}[\alpha - (\alpha - \lambda N_0)e^{-\lambda t}] \qquad \text{...(i)}$$

(b) (i) Substituting $\alpha = 2\lambda N_0$ and $t = t_{1/2} = \frac{\ln 2}{\lambda}$

in Eq. (i), we get $N = \frac{3}{2} N_0$

(ii) Substituting $\alpha = 2\lambda N_0$ and $t \to \infty$ in Eq. (i), we get

$$N = \frac{\alpha}{\lambda} = 2N_0$$

or $$N = 2N_0$$

4. (a) Let R_{A_0} and R_{B_0} be the initial activities of A and B. Then,

$$R_{A_0} + R_{B_0} = 10^{10} \text{ dps} \qquad \text{...(i)}$$

Activity of A after time t = 20 days (two half-lives of A) is

$$R_A = \left(\frac{1}{2}\right)^2 R_{A_0} = 0.25\, R_{A_0}$$

Similarly, activity of B after t = 20 days (four half-lives of B) is

$$R_B = \left(\frac{1}{2}\right)^4 R_{B_0} = 0.0625\, R_{B_0}$$

Now, it is given that $R_A + R_B$ = 20% of 10^{10}

or $0.25 R_{A_0} + 0.0625 R_{B_0} = 0.2 \times 10^{10}$ dps ...(ii)

Solving Eqs. (i) and (ii), we get

$$R_{A_0} = 0.73 \times 10^{10} \text{ dps}$$

and $$R_{B_0} = 0.27 \times 10^{10} \text{ dps}$$

(b) $$\frac{R_{A_0}}{R_{B_0}} = \frac{\lambda_A N_{A_0}}{\lambda_B N_{B_0}} = \frac{(t_{1/2})_B}{(t_{1/2})_A} \cdot \frac{N_{A_0}}{N_{B_0}}$$

∴ $$\frac{N_{A_0}}{N_{B_0}} = \left(\frac{R_{A_0}}{R_{B_0}}\right)\frac{(t_{1/2})_A}{(t_{1/2})_B} = \left(\frac{0.73}{0.27}\right)\left(\frac{10}{5}\right)$$

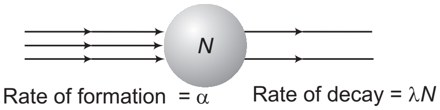
$$= 5.4$$

5. Let N be the number of radio nuclei at any time t. Then, net rate of formation of nuclei at time t is

$$\frac{dN}{dt} = \alpha - \lambda N$$

or $$\int_0^N \frac{dN}{\alpha - \lambda N} = \int_0^t dt$$

or $$N = \frac{\alpha}{\lambda}(1 - e^{-\lambda t})$$



Number of nuclei formed in time $t = \alpha t$ and number of nuclei left after time

$$t = \frac{\alpha}{\lambda}(1 - e^{-\lambda t})$$

Therefore, number of nuclei disintegrated in time

$$t = \alpha t - \frac{\alpha}{\lambda}(1 - e^{-\lambda t})$$

∴ Energy released till time,

$$t = E_0\left[\alpha t - \frac{\alpha}{\lambda}(1 - e^{-\lambda t})\right]$$

But only 20% of it is used in raising the temperature of water.

So, $$0.2\, E_0\left[\alpha t - \frac{\alpha}{\lambda}(1 - e^{-\lambda t})\right] = Q$$

where, $$Q = ms\Delta\theta$$

∴ $\Delta\theta$ = increase in temperature of water $= \frac{Q}{ms}$

∴ $$\Delta\theta = \frac{0.2\, E_0\left[at - \frac{a}{\lambda}(1 - e^{-\lambda t})\right]}{ms}$$

6. We have for B $\frac{dN_B}{dt} = P - \lambda_2 N_B$

$$\Rightarrow \int_0^{N_B} \frac{dN_B}{P - \lambda_2 N_B} = \int_0^t dt$$

$$\Rightarrow \ln\left(\frac{P - \lambda_2 N_B}{P}\right) = -\lambda_2 t$$

$$\Rightarrow N_B = \frac{P(1 - e^{-\lambda_2 t})}{\lambda_2}$$

The number of nuclei of A after time t is

$$N_A = N_0 e^{-\lambda_1 t}$$

Thus, $\dfrac{dN_c}{dt} = \lambda_1 N_A + \lambda_2 N_B$

$\Rightarrow \quad \dfrac{dN_c}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} + P(1 - e^{-\lambda_2 t})$

$\Rightarrow \quad N_c = N_0(1 - e^{-\lambda_1 t}) + P\left(t + \dfrac{e^{-\lambda_2 t} - 1}{\lambda_2}\right)$

7. $^{210}_{84}\text{Po} \longrightarrow {}^{206}_{82}\text{Pb} + {}^{4}_{2}\text{He}$

$$\Delta m = 0.00564 \text{ amu}$$

Energy liberated per reaction $= (\Delta m)931$ MeV $= 8.4 \times 10^{-13}$ J

Electrical energy produced $= 8.4 \times 10^{-14}$ J

Let m g of ^{210}Po is required to produce the desired energy.

$$N = \frac{m}{210} \times 6 \times 10^{23}$$

$$\lambda = \frac{0.693}{t_{1/2}} = 0.005 \text{ per day}$$

$$\left(-\frac{dN}{dt}\right) = \lambda N = \frac{(0.005)(6 \times 10^{23} \text{ m})}{210} \text{ per day}$$

$\therefore$ Electrical energy produced per day

$$= \frac{(0.005)(6 \times 10^{23} \text{ m})}{210} \times 8.4 \times 10^{-14} \text{ J}$$

This is equal to 1.2×10^7 J (given)

$\therefore \qquad m = 10$ g **Ans.**

Activity at the end of 693 days is

$$R = \frac{0.005 \times 6 \times 10^{23} \times 10}{210} = \frac{10^{21}}{7} \text{ per day} = R_0\left(\frac{1}{2}\right)^n$$

Here, n = number of half-lives $= \dfrac{693}{138.6} = 5$

$\therefore \quad R_0 = R(2)^5 = 32 \times \dfrac{10^{21}}{7} = 4.57 \times 10^{21}$ per day

Ans.

8. (i) At $t = 0$, probabilities of getting α and β particles are same. This implies that initial activity of both is equal. Say it is R_0.

Activity after $t = 1620$ s,

$$R_1 = R_0\left(\frac{1}{2}\right)^{1620/405} = \frac{R_0}{16}$$

and $$R_2 = R_0\left(\frac{1}{2}\right)^{1620/1620} = \frac{R_0}{2}$$

Total activity, $R = R_1 + R_2 = \dfrac{9}{16}R_0$

Probability of getting α-particles,

$$= \frac{R_1}{R} = \frac{1}{9}$$

and probability of getting β-particles

$$= \frac{R_2}{R} = \frac{8}{9}$$ **Ans.**

(ii) $R_{01} = R_{02}$

$\therefore \qquad \dfrac{N_{01}}{T_1} = \dfrac{N_{02}}{T_2}$

$\therefore \qquad \dfrac{N_{01}}{N_{02}} = \dfrac{1}{4}$

Let N_0 be the total number of nuclei at $t = 0$.

Then, $N_{01} = \dfrac{N_0}{5}$ and $N_{02} = \dfrac{4N_0}{5}$

Given that $\quad N_1 + N_2 = \dfrac{N_0}{2}$

or $\quad \dfrac{N_0}{5}\left(\dfrac{1}{2}\right)^{t/405} + \dfrac{4N_0}{5}\left(\dfrac{1}{2}\right)^{t/1620} = \dfrac{N_0}{2}$

Let $\qquad \left(\dfrac{1}{2}\right)^{t/1620} = x$

Then, above equation becomes $x^4 + 4x - 2.5 = 0$

$\therefore \qquad x = 0.594 \quad$ or $\quad \left(\dfrac{1}{2}\right)^{t/1620} = 0.594$

Solving, it we get $t = 1215$ s. **Ans.**

9. $N = \dfrac{10^{-3}}{210} \times 6.02 \times 10^{23} = 2.87 \times 10^{18}$

During one mean life period 63.8% nuclei are decayed. Hence, energy released

$$E = 0.638 \times 2.87 \times 10^{18} \times 5.3 \times 1.6 \times 10^{-13} \text{ J}$$

$$= 1.55 \times 10^6 \text{ J}$$ **Ans.**

10. $R_0 = \lambda N = \dfrac{0.693}{14.3 \times 3600 \times 24} \times 6.02 \times 10^{23}$ per sec

$= 3.37 \times 10^{17}$ per sec

After 70 hours activity,

$$R = R_0 e^{-\lambda t} = (3.37 \times 10^{17})\, e^{-(0.693/14.3 \times 24)(70)}$$

$$= 2.92 \times 10^{17} \text{ per sec}$$

In fruits activity was observed 1 μCi or 3.7×10^4 per sec. Therefore, percentage of activity transmitted from root to the fruit.

$$= \frac{3.7 \times 10^4}{2.92 \times 10^{17}} \times 100$$

$$= 1.26 \times 10^{-11} \%$$ **Ans.**

35. Semiconductors

INTRODUCTORY EXERCISE 35.1

1. E_g Carbon = 5.4 eV

E_g Silicon = 1.1 eV

E_g Germanium = 0.7 eV

$\therefore \quad (E_g)_C > (E_g)_{Si} > (E_g)_{Ge}$

INTRODUCTORY EXERCISE 35.2

1. Hole diffusion from p to n side can be viewed as "electron diffusion" from n to p side.

Diffusion occurs due to difference in concentrations in different regions.

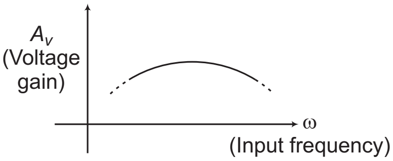
An electron (or hole) diffuses where its concentration is less.

2. Due to forward biasing depletion layer thickness decreases, potential barrier is reduced and diffusion of electrons from n to p side occurs.

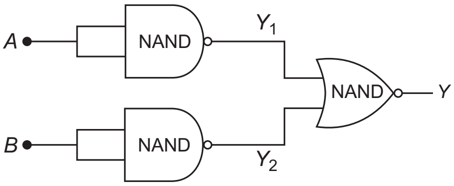
INTRODUCTORY EXERCISE 35.3

1. In a transistor, base must be very thin and lightly doped so that all of the charge carriers are not combined in base and majority of them passes the reverse bias layer to collector side.

2. Voltage gain is maximum and constant for mid frequency range but is less for both low and high frequencies.



(b) We construct truth table to see the logical operation.



A	B	Y_1	Y_2	Y
0	0	1	1	0
1	0	0	1	1
0	1	1	0	1
1	1	0	0	1

Clearly output resembles an 'OR' gate.

3. For given amplifier,

$$V_0 = 2\,\text{V},\ R_0 = 2\ \text{k}\Omega$$
$$\beta_{ac} = 100,\ R_i = 1\ \text{k}\Omega$$

We have, output voltage, $V_0 = I_C R_0$

$\Rightarrow \ I_C$ = collector current

$$= \frac{V_0}{R_0} = \frac{2}{2 \times 10^3} = 10^{-3}\,\text{A}$$
$$= 1\ \text{mA}$$

Also, current amplication $\beta = \dfrac{i_C}{i_B}$

$$\Rightarrow \qquad i_B = \frac{i_C}{\beta} = \frac{10^{-3}}{100}\ \text{A}$$
$$= 10^{-5}\,\text{A} = 10 \times 10^{-6}\,\text{A}$$
$$= 10\ \mu\text{A}.$$

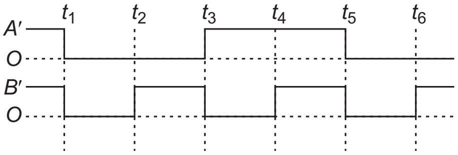
and voltage amplification

$$A_V = \frac{\beta R_0}{R_i} = \frac{V_0}{V_i}$$

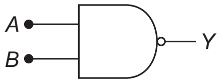
$$\Rightarrow \qquad V_i = \frac{V_0 R_i}{\beta R_0} = \frac{2 \times 1 \times 10^3}{100 \times 2 \times 10^3}$$
$$= 10^{-2}\ \text{volts} = 10\ \text{mV}$$

INTRODUCTORY EXERCISE 35.4

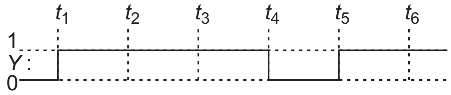
1. Input waveforms are as shown



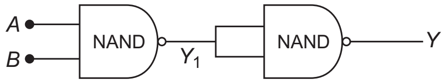
These are fed to an NAND gate,



Output waveform is as shown



2. Truth table for the circuit given is



A	B	Y_1	Y
0	0	1	0
1	0	1	0
0	1	1	0
1	1	0	1

As the output resembles output of a AND gate so, given circuit behaves like an 'AND' gate.

Exercises

1. With increase in temperature, number of electrons reaching conduction band increases but, mean relation time decreases. Effect of decrease in relation time is much less than that of increase in number density of charge carriers.

2. For diode D_1,

$$V_p - V_n = -10 - 0 = -10\,\text{V}$$

So, D_1 is in reverse bias.

For diode D_2,

$$V_p - V_n = 0 - (-10) = 10\ \text{V}$$

So, D_2 is in forward bias.

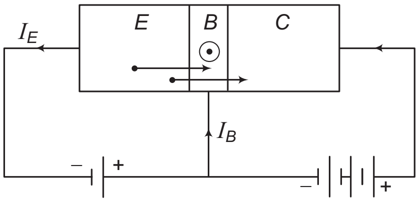
3. Hole is a vacancy created by an electron moving from valance band to conduction band.

4. Capacitor is charged to maximum potential difference

$$\therefore \quad V = V_{\max} = V_{\text{rms}} \times \sqrt{2} = 220\sqrt{2}\ \text{volts}$$

5. Electrons take the energy and move up to neat higher energy level both in conduction and valance band. So, a hole move downwards due to movement of electron.

6. In an n-p-n transistor,



Electrons move from emitter to base and electrons which are not combined with holes in base region, crosses to collector side.

7. In an n-p-n transistor,

$$I_E = I_B + I_C$$

As $I_C = \frac{95}{100} I_E \quad \Rightarrow \quad I_E = \frac{100}{95} \times I_C$

$$= \frac{100}{95} \times 10\ \text{mA} = 10.53\ \text{mA}$$

So, $I_B = 0.53$ mA

8. Depletion zone is formed due to diffusion of e^- from n to p side. Due to diffusion positive immobile ions exists on n side and negative ions on p-side.

Recombination of e^- and holes takes place on p-side. This results in (n-side more positive than p-side) formation of a junction field and potential barrier across the depletion zone.

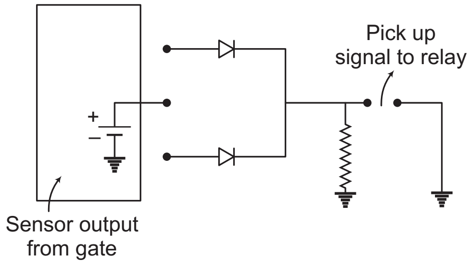
9. When applied voltage approaches zener potential diode breakdown to conduct excess current. This causes a change of current in series resistance.

10. Reverse breakdown may be "Avlanche break down" (breakdown due to high velocity collision of minority carrier) or it may be a zener breakdown (breakdown of bonds due to strong field).

11. No, it cannot be measured by using a voltmeter.

Barrier potential is less than 0.2 V for germanium diode and is less than 0.7 V for silicon. Also there are no free charge carriers which provide 'current' for working of voltmeter.

12. An "OR" gate may be made to operate motor relays for garage gates.



13. Such combination is called "Cascade" combination.

$$\underset{(\text{net})}{A_V} = A_{V_1} \times A_{V_2}$$

$$= 10 \times 20 = 200$$

$$\therefore \quad V_0 = \text{output voltage}$$

$$= \underset{(\text{net})}{A_V} \times V_i$$

$$= 200 \times 0.01$$

$$= 2\,\text{V}$$

14. We use, $E_g = \dfrac{hc}{\lambda} \Rightarrow E_g = \dfrac{1240\ \text{eV}\cdot\text{nm}}{600\ \text{nm}}$

$$\approx 2\ \text{eV}$$

So, photon energy is less than that required (2.8 eV). So, detector is not able to detect this wavelength.

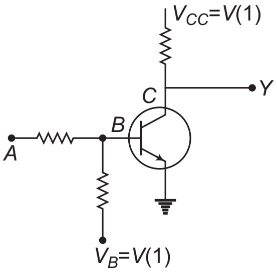
15. (i) Diode is a rectifier diode.

(ii) Point P represents zener breakdown potential.

16. Given is a p-n-p CE configuration, when R_1 is increased, base current decreases as a result current also decreases.

Hence, both ammeter and voltmeter readings decreases.

17. A NOT gate can be formed by proper biasing of a transistor as shown.



If input (A) is low (0), then the transistor is in cut off stage and output (Y) is same as $V_{CC} = V(1)$. If input (A) is high, then transistor current is in saturation and the net voltage at output (Y) is $V(0)$ or is in state 0.

18. Given is an AND gate, its truth table is

A	B	Y
0	0	0
1	0	0
0	1	0
1	1	1

19. Supply voltage may be 7 V and zener breakdown at 5 V.

So, voltage drop across resistance $R = 7 - 5 = 2\,\text{V}$.

Now power rating of zener = 1W.

So, current through zener must not exceed

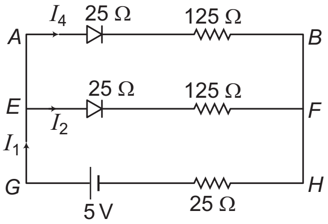
$$I = \frac{P}{V} = \frac{1}{5}\,\text{A}$$

Hence, series resistance

$$R = \frac{V}{i} = \frac{2}{\frac{1}{5}} = 10\,\Omega.$$

20. Current flows only in branches AB and EF as diode of CD branch is in reverse bias. ($\therefore\ I_3 = 0$)

So, given circuit is equivalent to



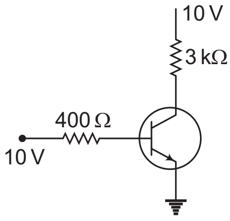
Current through cell is

$$i_1 = \frac{V}{R_{\text{Total}}} = \frac{5}{100} = 0.05\ \text{A}$$

Also by symmetry,

$$I_2 = I_4 = \frac{0.05}{2} = 0.025\ \text{A}$$

21. Given circuit is equivalent to



In base emitter loop,

$$V_B = I_B R_B$$

$$\Rightarrow \quad I_B = \frac{V_B}{R_B} = \frac{10}{400 \times 10^3}$$

$$= 25 \times 10^{-6}\ \text{A}$$

$$= 25\ \mu\text{A}$$

In emitter-collector loop,

$$V_C = I_C R_C$$

$$\Rightarrow \qquad I_C = \frac{10}{3 \times 10^3}$$

$$= 3.33 \text{ mA}$$

and $$\beta_{DC} = \frac{I_C}{I_B} = \frac{3.33 \text{ mA}}{25 \,\mu\text{A}}$$

$$= 133$$

22. Consider the figure given here to solve this problem

$I_C = I_E$ [As base current is very small]

$$R_C = 7.8 \text{ k}\Omega$$

From the figure, $I_C(R_C + R_E) + V_{CE} = 12$

$$(R_E + R_C) \times 1 \times 10^{-3} + 3 = 12$$

$$R_E + R_C = 9 \times 10^3 = 9 \text{ k}\Omega$$

$$R_E = 9 - 7.8 = 1.2 \text{ k}\Omega$$

$$V_E = I_E \times R_E$$

$$= 1 \times 10^{-3} \times 1.2 \times 10^3 = 1.2 \text{ V}$$

Voltage, $V_B = V_E + V_{BE} = 1.2 + 0.5 = 1.7$ V

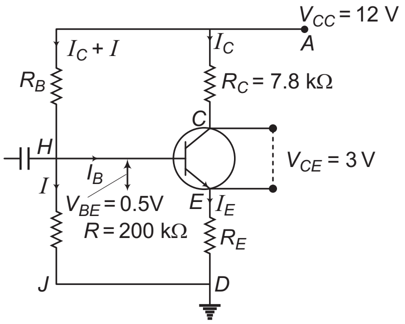
Current, $I = \frac{V_B}{20 \times 10^3} = \frac{1.7}{20 \times 10^3}$

$$= 0.085 \text{ mA}$$

Resistance, $R_B = \frac{12 - 1.7}{\frac{I_C}{\beta} + 0.085} = \frac{10.3}{0.01 + 0.085}$

[Given, β = 100]

$$= 108 \text{ k}\Omega$$



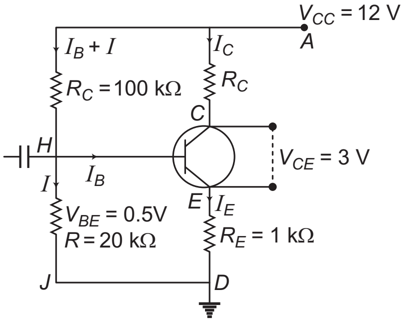
23. Consider the figure to solve this question,

$$I_E = I_C + I_B \text{ and } I_C = \beta I_B \qquad \ldots\text{(i)}$$

$$I_C R_C + V_{CE} + I_E R_E = V_{CC} \qquad \ldots\text{(ii)}$$

$$R I_B + V_{BE} + I_E R_E = V_{CC} \qquad \ldots\text{(iii)}$$

$$I_E = I_C = \beta I_B$$



From Eq. (iii),

$$(R + \beta R_E)\, I_B = V_{CC} - V_{BE}$$

$$\Rightarrow \qquad I_B = \frac{V_{CC} - V_{BE}}{R + \beta R_E}$$

$$= \frac{12 - 0.5}{80 + 1.2 \times 100}$$

$$= \frac{11.5}{200} \text{ mA}$$

From Eq. (ii),

$$(R_C + R_E) = \frac{V_{CE} - V_{BE}}{I_C}$$

$$= \frac{V_{CC} - V_{CE}}{\beta I_B} \qquad (\because\ I_C = \beta I_B)$$

$$(R_C + R_E) = \frac{2}{11.5}(12 - 3) \text{ k}\Omega$$

$$= 1.56 \text{ k}\Omega$$

$$R_C + R_E = 1.56$$

$$R_C = 1.56 - 1 = 0.56 \text{ k}\Omega$$

36. Communication System

Exercises

Single Correct Option

1. Range of frequencies is as follows :

Ground wave : 300 Hz to 300 kHz (it may go upto 3 MHz)

Sky wave : 300 kHz to 3 MHz

Space wave : 3 MHz to 300 GHz.

2. Length of antenna $\geq \frac{\lambda}{4}$

$$\Rightarrow \quad l \geq \frac{\lambda}{4}$$

$$\Rightarrow \quad \lambda \leq 4 \times l$$

$$\text{or} \quad \lambda \leq 400 \text{ m}$$

3. Sideband frequencies are :

Lower sideband

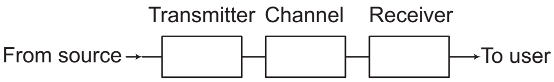
$$\begin{aligned}(\text{LSB}) &= \omega_c - \omega_m \\ &= 1 \times 10^6 \text{ Hz} - 3 \times 10^3 \text{ Hz} \\ &= (1000 - 3) \times 10^3 \text{ Hz} \\ &= 2997 \text{ kHz} \\ \text{and} \quad &= 2.997 \text{ MHz}\end{aligned}$$

Upper sideband

$$\begin{aligned}(\text{USB}) &= \omega_c + \omega_m \\ &= 1 \times 10^6 \text{ Hz} + 3 \times 10^3 \text{ Hz} \\ &= 1003 \text{ kHz} \\ &= 1.003 \text{ MHz}\end{aligned}$$

4. Frequency of amplitude modulated wave is same as that of carrier wave.

5. Basic communication system is



6. Beyond horizon, a signal can reach *via* ionospheric reflection or sky wave mode. Frequency range suitable is 3 MHz to 30 MHz.

7. UHF band is in range of 150 to 900 MHz.

So, suitable mode of communication is "space wave mode".

8. Digital signals employs a discrete values of amplitudes which are coded using binary system.

9. In LOS communication maximum distance upto which a signals can be received from tower is

$$d = \sqrt{2Rh}$$

$$\begin{aligned}&= \sqrt{2 \times 6.4 \times 10^6 \times 240} \\ &\approx 55 \times 10^3 \text{ m} \\ &= 55 \text{ km}\end{aligned}$$

10. Wavelength of signal is

$$\lambda = \frac{c}{F} = \frac{3 \times 10^8}{15 \times 10^3} = 20 \times 10^3 \text{ m}$$

So, size of antenna required

$$\begin{aligned}&= \frac{\lambda}{4} \quad \text{(at least)} \\ &= 5 \times 10^3 \text{ m} = 5 \text{ km}\end{aligned}$$

Also, effective power radiated by antenna is very less.

11.
$$\begin{aligned}\text{LSB} &= \omega_c = \omega_m \\ &= 1.5 \times 10^6 - 3 \times 10^3 \\ &= 1497 \text{ kHz} \\ \text{and} \quad \text{USB} &= \omega_c + \omega_m \\ &= 1503 \text{ kHz}\end{aligned}$$

Bandwidth required

$$\begin{aligned}&= \text{USB} - \text{LSB} = 2\omega_m \\ &= 2 \times 3 \text{ kHz} = 6 \text{ kHz}\end{aligned}$$

12. Due to over modulation $\mu > 1$, sidebands overlaps and fading (loss of information) may occur. Also when $\mu \approx 1$, distortion in carrier waveform occurs.

13. Frequency of signal obtained $= \frac{1}{2\pi\sqrt{\text{LC}}} = 1$ MHz

$$\Rightarrow \quad \text{LC} = \frac{1}{(2\pi \times 10^6)^2}$$

$$\Rightarrow \quad \text{LC} = 0.25 \times 10^{-13} \text{ s}^2$$

$$\Rightarrow \quad \text{LC} = 2.5 \times 10^{-14} \text{ s}^2$$

14. μ = modulation index

$$= \frac{A_m}{A_c}$$

$$\Rightarrow \quad A_m = \mu \times A_c = 0.75 \times 12$$

$$= 9 \text{ volts}$$

15. Vibrating tuning fork and string produces all possible values of displacement hence, these signals are analog.

A light pulse and output of NAND gate gives only discrete values of output. So, signals are digital these.

16. Penetrating power of signal α frequency of signal. So, 3 MHz signal travels longer distance in ionosphere.

17. Modulation index μ,

$$= \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}}$$

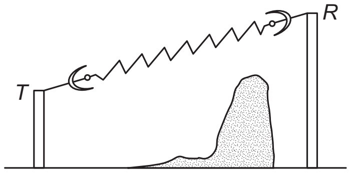
(A = amplitude of modulated wave)

$$\therefore \qquad \mu = \frac{15-3}{15+3} = \frac{12}{18} = \frac{2}{3}$$

18. Man made noises and atmospheric interferences affect only amplitude of a signal.

So, an AM signal is more noisy than a FM signal.

19. In LOS communication, it is not necessary that transmitting antenna and receiving antenna are at same height. Only requirement is that there must not be any obstacle in between.



Given, $h = 81$ m

$\therefore$ Distance upto which transmission can be made

$$= d = \sqrt{2Rh}$$

and, area covered in broadcast

$$= \pi d^2$$
$$= 2\pi Rh$$
$$= 2 \times \pi \times 6.4 \times 10^6 \times 81\,(\text{m}^2)$$
$$= 3258 \text{ km}^2$$

20. (i) When receiver is at ground level, then service area covered

$$= \pi d^2 = \pi(\sqrt{2Rh})^2$$
$$= 2\pi Rh$$
$$= 2 \times \pi \times 6.4 \times 10^6 \times 20$$
$$\approx 804 \text{ km}^2$$

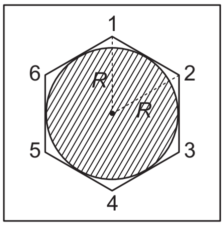
(ii) When receiver is at height of 25 m, area covered

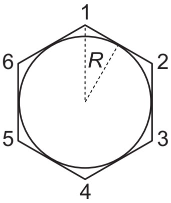
$$= \pi d_1^2 + \pi d_2^2$$
$$= \pi(\sqrt{2Rh_T})^2 + \pi(\sqrt{2Rh_R})^2$$
$$= 2\pi R(h_T + h_R)$$
$$= 2 \times \pi \times 6.4 \times 10^6 \times 45\,(\text{m}^2)$$
$$\approx 3608 \text{ km}^2$$

(iii) Percentage increase in area covered

$$= \frac{A_2 - A_1}{A_1} \times 100$$
$$= \frac{3608 - 804}{804} \times 100$$
$$\approx 349\ \%$$

21. That can be done by using six antennas,





From above figure, side of triangle = $R + h$

Also, altitude of triangle

$$= \frac{\sqrt{3}}{2} \times \text{side} = R$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}(R + h) = R$$

$$\Rightarrow \quad \sqrt{3}\,(R + h) = 2R$$

$$\Rightarrow \quad R + h = \frac{2}{\sqrt{3}} R$$

$$\Rightarrow \quad h = \frac{2}{\sqrt{3}} R - R \approx 0.15\,R$$

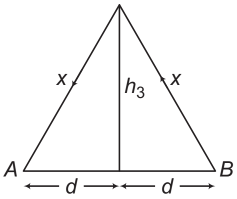
22. From $f_{\max} = 9(N_{\max})^{\frac{1}{2}}$

$$\Rightarrow \quad N_{\max} = \frac{f_{\max}^2}{81}$$

$$\Rightarrow \quad N_{\max}\Big|_{F_1} = \frac{(5 \times 10^6)^2}{81}$$
$$= 0.3086 \times 10^{12} \text{ m}^{-3}$$
$$= 3.086 \times 10^{11} \text{ m}^{-3}$$

and $$N_{\max}\Big|_{F_2} = \frac{(8 \times 10^6)^2}{81}$$
$$= 7.9 \times 10^{11} \text{ m}^{-3}$$

23. Let the receiver is at point A and source is at B.



$$\text{Velocity of waves} = 3 \times 10^8 \text{ m/s}$$

Time to reach a receiver = 4.04 ms = 4.04×10^{-3} s

Let the height of satellite is

$$h_s = 600 \text{ km}$$

$$\text{Radius of earth} = 6400 \text{ km}$$

Size of transmitting antenna = h_T

We know that

$$\frac{\text{Distance travelled by wave}}{\text{Time}} = \text{Velocity of waves}$$

$$\frac{2x}{4.04 \times 10^{-3}} = 3 \times 10^8$$

or $$x = \frac{3 \times 10^8 \times 4.04 \times 10^{-3}}{2}$$

$$= 6.06 \times 10^5 = 606 \text{ km}$$

Using Pythagoras theorem,

$$d^2 = x^2 - h_s^2 = (606)^2 - (600)^2 = 7236$$

or $$d = 85.06 \text{ km}$$

So, the distance between source and receiver = $2d$

$$= 2 \times 85.06$$

$$= 170 \text{ km}$$

The maximum distance covered on ground from the transmitter by emitted EM waves

$$d = \sqrt{2Rh_T}$$

or $$\frac{d^2}{2R} = h_T$$

or size of antenna $$h_T = \frac{7236}{2 \times 6400}$$

$$= 0.565 \text{ km}$$

$$= 565 \text{ m}$$

JEE Main and Advanced

Previous Years' Questions (2018-13)

JEE Main

1. An EM wave from air enters a medium. The electric fields are

$\mathbf{E}_1 = E_{01}\hat{\mathbf{x}}\cos\left[2\pi\nu\left(\frac{z}{c} - t\right)\right]$ in air and

$\mathbf{E}_2 = E_{02}\hat{\mathbf{x}}\cos[k(2z - ct)]$ in medium, where the wave number k and frequency ν refer to their values in air. The medium is non-magnetic.

If ε_{r_1} and ε_{r_2} refer to relative permittivities of air and medium respectively, which of the following options is correct? **(2018)**

(a) $\frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = 4$ (b) $\frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = 2$ (c) $\frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = \frac{1}{4}$ (d) $\frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = \frac{1}{2}$

2. Unpolarised light of intensity I passes through an ideal polariser A. Another identical polariser B is placed behind A. The intensity of light beyond B is found to be $\frac{I}{2}$. Now, another identical polariser C is placed between A and B. The intensity beyond B is now found to be $\frac{1}{8}$. The angle between polariser A and C is **(2018)**

(a) 0° (b) 30° (c) 45° (d) 60°

3. The angular width of the central maximum in a single slit diffraction pattern is 60°. The width of the slit is 1 μm. The slit is illuminated by monochromatic plane waves. If another slit of same width is made near it, Young's fringes can be observed on a screen placed at a distance 50 cm from the slits. If the observed fringe width is 1 cm, what is slit separation distance? (i.e. distance between the centres of each slit.) **(2018)**

(a) 25 μm (b) 50 μm (c) 75 μm (d) 100 μm

4. An electron from various excited states of hydrogen atom emit radiation to come to the ground state. Let λ_n, λ_g be the de-Broglie wavelength of the electron in the nth state and the ground state, respectively. Let Λ_n be the wavelength of the emitted photon in the transition from the nth state to the ground state. For large n, (A, B are constants) **(2018)**

(a) $\Lambda_n \approx A + \frac{B}{\lambda_n^2}$ (b) $\Lambda_n \approx A + B\lambda_n^2$

(c) $\Lambda_n^2 \approx A + B\lambda_n^2$ (d) $\Lambda_n^2 \approx \lambda$

5. If the series limit frequency of the Lyman series is ν_L, then the series limit frequency of the Pfund series is **(2018)**

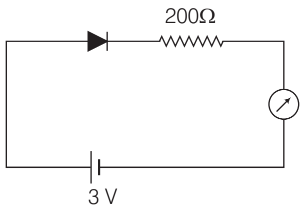
(a) $25\,\nu_L$ (b) $16\,\nu_L$ (c) $\frac{\nu_L}{16}$ (d) $\frac{\nu_L}{25}$

6. It is found that, if a neutron suffers an elastic collinear collision with deuterium at rest, fractional loss of its energy is P_d; while for its similar collision with carbon nucleus at rest, fractional loss of energy is P_c. The values of P_d and P_c are respectively **(2018)**

(a) (.89, .28) (b) (.28, .89)

(c) (0, 0) (d) (0, 1)

7. The reading of the ammeter for a silicon diode in the given circuit is **(2018)**



(a) 0 (b) 15 mA

(c) 11.5 mA (d) 13.5 mA

8. A telephonic communication service is working at carrier frequency of 10 GHz. Only 10% of it is utilised for transmission. How many telephonic channels can be transmitted simultaneously, if each channel requires a bandwidth of 5 kHz? **(2018)**

(a) 2×10^3 (b) 2×10^4
(c) 2×10^5 (d) 2×10^6

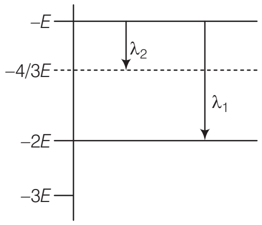
9. A diverging lens with magnitude of focal length 25 cm is placed at a distance of 15 cm from a converging lens of magnitude of focal length 20 cm. A beam of parallel light falls on the diverging lens. The final image formed is **(2017)**

(a) virtual and at a distance of 40 cm from convergent lens
(b) real and at a distance of 40 cm from the divergent lens
(c) real and at a distance of 6 cm from the convergent lens
(d) real and at a distance of 40 cm from convergent lens

10. In a Young's double slit experiment, slits are separated by 0.5 mm and the screen is placed 150 cm away. A beam of light consisting of two wavelengths, 650 nm and 520 nm, is used to obtain interference fringes on the screen. The least distance from the common central maximum to the point where the bright fringes due to both the wavelengths coincide, is **(2017)**

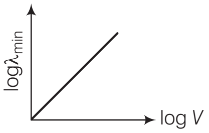
(a) 7.8 mm (b) 9.75 mm (c) 15.6 mm (d) 1.56 mm

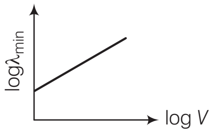
11. Some energy levels of a molecule are shown in the figure. The ratio of the wavelengths $r = \lambda_1 / \lambda_2$ is given by **(2017)**

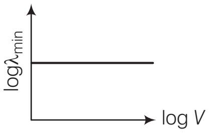


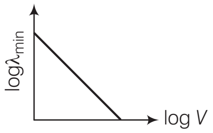
(a) $r = \frac{2}{3}$ (b) $r = \frac{3}{4}$ (c) $r = \frac{1}{3}$ (d) $r = \frac{4}{3}$

12. An electron beam is accelerated by a potential difference V to hit a metallic target to produce X-rays. It produces continuous as well as characteristic X-rays. If $\lambda_{\min}$ is the smallest possible wavelength of X-rays in the spectrum, the variation of log $\lambda_{\min}$ with log V is correctly represented in **(2017)**

(a)


(b)


(c)


(d)


13. A radioactive nucleus A with a half-life T, decays into a nucleus B. At $t = 0$, there is no nucleus B. After sometime t, the ratio of the number of B to that of A is 0.3. Then, t is given by **(2017)**

(a) $t = T \frac{\log 1.3}{\log 2}$ (b) $t = T \log 1.3$
(c) $t = \frac{T}{\log 1.3}$ (d) $t = \frac{T \log 2}{2 \log 1.3}$

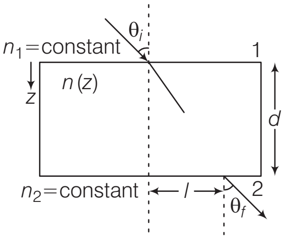
14. A particle A of mass m and initial velocity v collides with a particle B of mass $m/2$ which is at rest. The collision is held on, and elastic. The ratio of the de-Broglie wavelengths λ_A to λ_B after the collision is **(2017)**

(a) $\frac{\lambda_A}{\lambda_B} = 2$ (b) $\frac{\lambda_A}{\lambda_B} = \frac{2}{3}$
(c) $\frac{\lambda_A}{\lambda_B} = \frac{1}{2}$ (d) $\frac{\lambda_A}{\lambda_B} = \frac{1}{3}$

15. In a common emitter amplifier circuit using an n-p-n transistor, the phase difference between the input and the output voltages will be **(2017)**

(a) 90° (b) 135° (c) 180° (d) 45°

16. A transparent slab of thickness d has a refractive index $n(z)$ that increases with z. Here, z is the vertical distance inside the slab, measured from the top. The slab is placed between two media with uniform refractive indices n_1 and $n_2(> n_1)$, as shown in the figure. A ray of light is incident with angle θ_i from medium 1 and emerges in medium 2 with refraction angle θ_f with a lateral displacement l. **(2016)**



Which of the following statement(s) is (are) true?

(a) l is independent on $n(z)$

(b) $n_1 \sin\theta_i = (n_2 - n_1)\sin\theta_f$

(c) $n_1 \sin\theta_i = n_2 \sin\theta_f$

(d) l is independent of n_2

17. In an experiment for determination of refractive index of glass of a prism by i-δ plot, it was found that a ray incident at an angle 35° suffers a deviation of 40° and that it emerges at an angle 79°. In that case, which of the following is closest to the maximum possible value of the refractive index? **(2016)**

(a) 1.5 (b) 1.6 (c) 1.7 (d) 1.8

18. An observer looks at a distant tree of height 10 m with a telescope of magnifying power of 20. To observer the tree appears **(2016)**

(a) 10 times taller (b) 10 times nearer

(c) 20 times taller (d) 20 times nearer

19. The box of a pin hole camera of length L, has a hole of radius a. It is assumed that when the hole is illuminated by a parallel beam of light of wavelength λ the spread of the spot (obtained on the opposite wall of the camera) is the sum of its geometrical spread and the spread due to diffraction. The spot would then have its minimum size (say $b_{\min}$) when **(2016)**

(a) $a = \frac{\lambda^2}{L}$ and $b_{\min} = \left(\frac{2\lambda^2}{L}\right)$

(b) $a = \sqrt{\lambda L}$ and $b_{\min} = \left(\frac{2\lambda^2}{L}\right)$

(c) $a = \sqrt{\lambda L}$ and $b_{\min} = \sqrt{4\lambda L}$

(d) $a = \frac{\lambda^2}{L}$ and $b_{\min} = \sqrt{4\lambda L}$

20. Radiation of wavelength λ, is incident on a photocell. The fastest emitted electron has speed v. If the wavelength is changed to $\frac{3\lambda}{4}$, the speed of the fastest emitted electron will be **(2016)**

(a) $> v\left(\frac{4}{3}\right)^{1/2}$ (b) $< v\left(\frac{4}{3}\right)^{1/2}$

(c) $= v\left(\frac{4}{3}\right)^{1/2}$ (d) $= v\left(\frac{3}{4}\right)^{1/2}$

21. Half-lives of two radioactive elements A and B are 20 min and 40 min, respectively. Initially, the samples have equal number of nuclei. After 80 min, the ratio of decayed numbers of A and B nuclei will be **(2016)**

(a) 1 : 16 (b) 4 : 1 (c) 1 : 4 (d) 5 : 4

22. The temperature dependence of resistances of Cu and undoped Si in the temperature range 300-400 K, is best described by **(2016)**

(a) linear increase for Cu, linear increase for Si

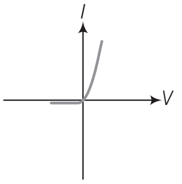
(b) linear increase for Cu, exponential increase for Si

(c) linear increase for Cu, exponential decrease for Si

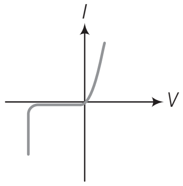
(d) linear decrease for Cu, linear decrease for Si

23. Identify the semiconductor devices whose characteristics are as given below, in the order (a),(b),(c),(d). **(2016)**

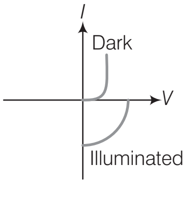
(a)



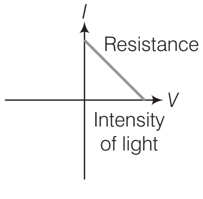
(b)



(c)



(d)



(a) Simple diode, Zener diode, Solar cell, Light dependent resistance
(b) Zener diode, Simple diode, Light dependent resistance, Solar cell
(c) Solar cell, Light dependent resistance, Zener diode, Simple diode
(d) Zener diode, Solar cell, Simple diode, Light dependent resistance

24. Monochromatic light is incident on a glass prism of angle A. If the refractive index of the material of the prism is μ, a ray incident at an angle θ, on the face AB would get transmitted through the face AC of the prism provided **(2015)**

(a) $\theta < \cos^{-1}\left[\mu \sin\left\{A + \sin^{-1}\left(\frac{1}{\mu}\right)\right\}\right]$
(b) $\theta < \sin^{-1}\left[\mu \sin\left\{A - \sin^{-1}\left(\frac{1}{\mu}\right)\right\}\right]$
(c) $\theta > \cos^{-1}\left[\mu \sin\left\{A + \sin^{-1}\left(\frac{1}{\mu}\right)\right\}\right]$
(d) $\theta > \sin^{-1}\left[\mu \sin\left\{A - \sin^{-1}\left(\frac{1}{\mu}\right)\right\}\right]$

25. Assuming human pupil to have a radius of 0.25 cm and a comfortable viewing distance of 25 cm, the minimum separation between two objects that human eye can resolve at 500 nm wavelength is **(2015)**

(a) 30 μm (b) 1 μm (c) 100 μm (d) 300 μm

26. On a hot summer night, the refractive index of air is smallest near the ground and increases with height from the ground. When a light beam is directed horizontally, the Huygens principle leads us to conclude that as it travels, the light beam **(2015)**

(a) becomes narrower
(b) goes horizontally without any deflection
(c) bends upwards
(d) bends downwards

27. A signal of 5 kHz frequency is amplitude modulated on a carrier wave of frequency 2MHz. The frequencies of the resultant signal is/are **(2015)**

(a) 2 MHz only
(b) 2005 kHz 2000 kHz and 1995 kHz
(c) 2005 kHz and 1995 kHz
(d) 2000 kHz and 1995 kHz

28. A red LED emits light at 0.1 W uniformly around it. The amplitude of the electric field of the light at a distance of 1 m from the diode is **(2015)**

(a) 2.45 V/m (b) 1.73 V/m
(c) 5.48 V/m (d) 7.75 V/m

29. As an electron makes a transition from an excited state to the ground state of a hydrogen like atom/ion **(2015)**

(a) kinetic energy, potential energy and total energy decrease
(b) kinetic energy decreases, potential energy increases but total energy remains same
(c) kinetic energy and total energy decrease but potential energy increases
(d) its kinetic energy increases but potential energy and total energy decrease

30. Match Column I (fundamental experiment) with Column II (its conclusion) and select the correct option from the choices given below the list. **(2015)**

	Column I		Column II
A	Franck-Hertz experiment	1.	Particle nature of light
B	Photo-electric experiment	2.	Discrete energy levels of atom
C	Davisson-Germer experiment	3.	Wave nature of electron
		4.	Structure of atom

	A	B	C
(a)	1	4	3
(b)	2	1	3
(c)	2	4	3
(d)	4	3	2

31. A green light is incident from the water to the air-water interface at the critical angle (θ). Select the correct statement. **(2014)**

(a) The entire spectrum of visible light will come out of the water at an angle of 90° to the normal
(b) The spectrum of visible light whose frequency is less than that of green light will come out of the air medium

(c) The spectrum of visible light whose frequency is more than that of green light will come out to the air medium

(d) The entire spectrum of visible light will come out of the water at various angles to the normal

32. A thin convex lens made from crown glass $(\mu = 3/2)$ has focal length f. When it is measured in two different liquids having refractive indices 4/3 and 5/3. It has the focal lengths f_1 and f_2, respectively. The correct relation between the focal length is **(2014)**

(a) $f_1 = f_2 < f$

(b) $f_1 > f$ and f_2 becomes negative

(c) $f_2 > f$ and f_1 becomes negative

(d) f_1 and f_2 both become negative

33. Two beams, A and B, of plane polarised light with mutually perpendicular planes of polarisation are seen through a polaroid. From the position when the beam A has maximum intensity (and beam B has zero intensity), a rotation of polaroid through 30° makes the two beams appear equally bright. If the initial intensities of the two beams are I_A and I_B respectively, then I_A / I_B equals **(2014)**

(a) 3 (b) 3/2 (c) 1 (d) 1/3

34. Hydrogen $({}_1H^1)$, deuterium $({}_1H^2)$, singly ionised helium $({}_2He^4)^+$ and doubly ionised lithium $({}_3Li^8)^{++}$ all have one electron around the nucleus. Consider an electron transition from $n = 2$ to $n = 1$. If the wavelengths of emitted radiation are $\lambda_1, \lambda_2, \lambda_3$ and λ_4, respectively for four elements, then approximately which one of the following is correct? **(2014)**

(a) $4\lambda_1 = 2\lambda_2 = 2\lambda_3 = \lambda_4$

(b) $\lambda_1 = 2\lambda_2 = 2\lambda_3 = \lambda_4$

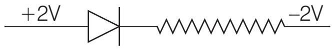
(c) $\lambda_1 = \lambda_2 = 4\lambda_3 = 9\lambda_4$

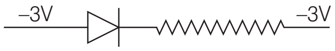
(d) $\lambda_1 = 2\lambda_2 = 3\lambda_3 = 4\lambda_4$

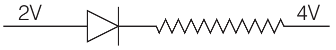
35. The radiation corresponding to $3 \rightarrow 2$ transition of hydrogen atom falls on a metal surface to produce photoelectrons. These electrons are made to enter a magnetic field of 3×10^{-4} T. If the radius of the largest circular path followed by these electrons is 10.0 mm, the work function of the metal is close to **(2014)**

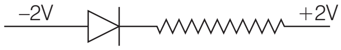
(a) 1.8 eV (b) 1.1 eV (c) 0.8 eV (d) 1.6 eV

36. The forward biased diode connection is **(2014)**

(a) 

(b) 

(c) 

(d) 

37. During the propagation of electromagnetic waves in a medium, **(2014)**

(a) electric energy density is double of the magnetic energy density

(b) electric energy density is half of the magnetic energy density

(c) electric energy density is equal to the magnetic energy density

(d) Both electric and magnetic energy densities are zero

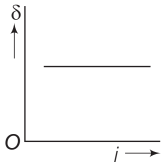
38. Match List I (Electromagnetic wave type) with List II (Its association/application) and select the correct option from the choices given below the lists. **(2014)**

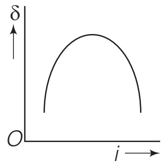
	List I		List II
A.	Infrared waves	1.	To treat muscular strain
B.	Radio waves	2.	For broadcasting
C.	X-rays	3.	To detect fracture of bones
D.	Ultraviolet	4.	Absorbed by the ozone layer of the atmosphere

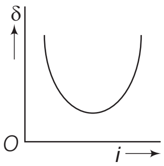
Codes

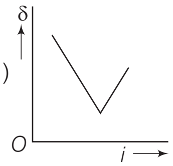
	A	B	C	D
(a)	4	3	2	1
(b)	1	2	4	3
(c)	3	2	1	4
(d)	1	2	3	4

39. The graph between angle of deviation (δ) and angle of incidence (i) for a triangular prism is represented by **(2013)**

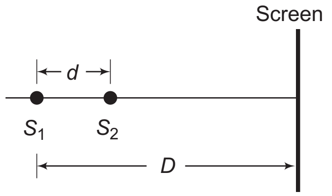
(a)


(b)


(c)


(d)


40. Two coherent point sources S_1 and S_2 are separated by a small distance d as shown. The fringes obtained on the screen will be **(2013)**



(a) points (b) straight lines
(c) semi-circle (d) concentric circles

41. A beam of unpolarised light of intensity I_0 is passed through a polaroid A and then through another polaroid B which is oriented so that its principal plane makes an angle of 45° relative to that of A. The intensity of the emergent light is **(2013)**

(a) I_0 (b) $I_0/2$
(c) $I_0/4$ (d) $I_0/8$

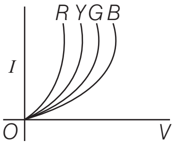
42. Diameter of a plano-convex lens is 6 cm and thickness at the centre is 3 mm. If speed of light in material of lens is 2×10^8 m/s, the focal length of the lens is **(2013)**

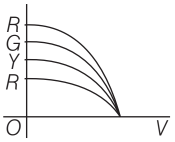
(a) 15 cm (b) 20 cm
(c) 30 cm (d) 10 cm

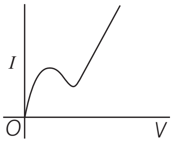
43. In a hydrogen like atom electron makes transition from an energy level with quantum number n to another with quantum number $(n-1)$. If $n >> 1$, the frequency of radiation emitted is proportional to **(2013)**

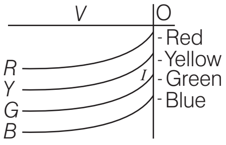
(a) $\frac{1}{n}$ (b) $\frac{1}{n^2}$ (c) $\frac{1}{n^4}$ (d) $\frac{1}{n^3}$

44. The I-V characteristic of an LED is **(2013)**

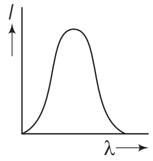
(a)


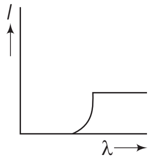
(b)


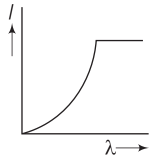
(c)


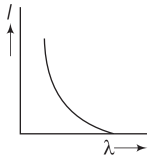
(d)


45. The anode voltage of a photocell is kept fixed. The wavelength λ of the light falling on the cathode is gradually changed. The plate current I of photocell varies as follows **(2013)**

(a)


(b)


(c)


(d)


46. A diode detector is used to detect an amplitude modulated wave of 60% modulation by using a condenser of capacity 250 pico farad in parallel with a load resistance 100 kΩ. Find the maximum modulated frequency which could be detected by it. **(2013)**

(a) 10.62 MHz (b) 10.62 kHz
(c) 5.31 MHz (d) 5.31 kHz

47. The magnetic field in a travelling electromagnetic wave has a peak value of 20 nT. The peak value of electric field strength is **(2013)**

(a) 3 V/m (b) 6 V/m (c) 9 V/m (d) 12 V/m

Answer *with* Explanations

1. **(c)** Speed of progressive wave is given by, $v = \dfrac{\omega}{k}$

As electric field in air is,

$$E_1 = E_{01}\mathbf{x}\cos\left(\frac{2\pi\nu z}{c} - 2\pi\nu t\right)$$

$$\therefore \text{ Speed in air} = \frac{2\pi\nu}{\left(\dfrac{2\pi\nu}{c}\right)} = c$$

Also, $$c = \frac{1}{\sqrt{\mu_0\varepsilon_{r_1}\varepsilon_0}} \qquad \ldots\text{(i)}$$

In medium,

$$E_2 = E_{02}\mathbf{x}\cos(2kz - kct)$$

$$\therefore \text{ Speed in medium} = \frac{kc}{2k} = \frac{c}{2}$$

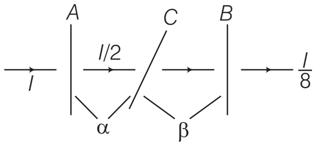
Also, $$\frac{c}{2} = \frac{1}{\sqrt{\mu_0\varepsilon_{r_2}\varepsilon_0}} \qquad \ldots\text{(ii)}$$

As medium is non-magnetic medium, $\mu_{\text{medium}} = \mu_{\text{air}}$

On dividing Eq. (i) by Eq. (ii), we have

$$2 = \sqrt{\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}} \Rightarrow \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = \frac{1}{4}$$

2. **(c)**



Using Malus law, intensity available after C

$$= \frac{I}{2} \times \cos^2\alpha$$

and intensity available after $B = \dfrac{I}{2}\cos^2\alpha \times \cos^2\beta$

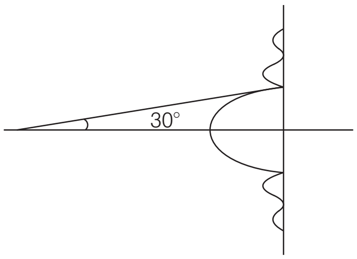
$$= \frac{I}{8} \text{ (given)}$$

So, $\dfrac{I}{2} \times \cos^2\alpha \cdot \cos^2\beta = \dfrac{I}{8} \Rightarrow \cos^2\alpha \cdot \cos^2\beta = \dfrac{1}{4}$

This is satisfied with $\alpha = 45°$ and $\beta = 45°$

So, angle between A and C is 45°.

3. **(a)** Angular width of diffraction pattern = 60°



For first minima,

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}, \text{ [here, } a = 10^{-6} \text{ m, } \theta = 30°]$$

$$\Rightarrow \quad \lambda = 10^{-6} \times \sin 30° \Rightarrow \lambda = \frac{10^{-6}}{2}\text{m}$$

Now, in case of interference caused by bringing second slit,

$$\therefore \text{ Fringe width, } \quad \beta = \frac{\lambda D}{d}$$

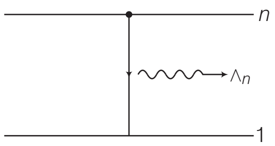
$$\text{[here, } \quad \lambda = \frac{10^{-6}}{2}\text{ m}, \beta = 1\text{ cm} = \frac{1}{100}\text{ m},$$

$$d = ? \text{ and } D = 50\text{ cm} = \frac{50}{100}\text{ m]}$$

So, $$d = \frac{\lambda D}{\beta} = \frac{10^{-6} \times 50}{2 \times \dfrac{1}{100} \times 100} = 25 \times 10^{-6}\text{ m}$$

or $$d = 25\,\mu\text{m}$$

4. **(a)** If wavelength of emitted photon in de-excitation is Λ_n;



Then, $$\frac{hc}{\Lambda_n} = E_n - E_g \quad \frac{hc}{\Lambda_n} = \frac{p_n^2}{2m} - \frac{p_g^2}{2m}, \left[\because E = \frac{p^2}{2m}\right]$$

As energies are negative, we get

$$\frac{hc}{\Lambda_n} = \frac{p_g^2}{2m} - \frac{p_n^2}{2m}$$

$$\Rightarrow \quad = \frac{p_g^2}{2m}\left(1 - \left(\frac{p_n}{p_g}\right)^2\right) = \frac{h^2}{2m\lambda_g^2}\left(1 - \frac{\lambda_g^2}{\lambda_n^2}\right)$$

$$\left[\because p \propto \lambda^{-1}, p = \frac{h}{\lambda}\right]$$

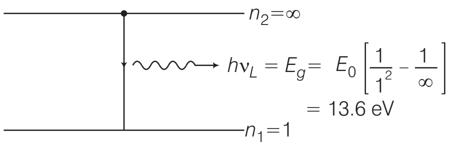
$$\Rightarrow \quad \Lambda_n = \frac{2m\lambda_g^2 c}{h}\left(1 - \frac{\lambda_g^2}{\lambda_n^2}\right)^{-1}$$

$$\Rightarrow \quad \Lambda_n = \frac{2m\lambda_g^2 c}{h}\left(1 + \frac{\lambda_g^2}{\lambda_n^2}\right) \qquad [\because (1 - x)^{-n} = 1 + nx]$$

$$\Rightarrow \quad \boxed{\Lambda_n \simeq A + \frac{B}{\lambda_n^2}}$$

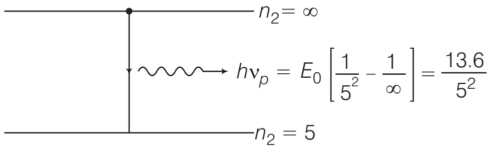
where, $A = \left[\dfrac{2mc\lambda_g^2}{h}\right]$ and $B = \left[\dfrac{2mc\lambda_g^4}{h}\right]$ are constants.

5. **(d)** Series limit occurs in the transition $n_2 = \infty$ to $n_1 = 1$ in Lyman series and $n_2 = \infty$ to $n_1 = 5$ in pfund series. For Lyman series,


$n_2=\infty$
$h\nu_L = E_g = E_0\left[\frac{1}{1^2}-\frac{1}{\infty}\right]$
$= 13.6$ eV
$n_1=1$

$$h\nu_L = 13.6 \quad \ldots\text{(i)}$$

In Pfund series


$n_2=\infty$
$h\nu_p = E_0\left[\frac{1}{5^2}-\frac{1}{\infty}\right]=\frac{13.6}{5^2}$
$n_2 = 5$

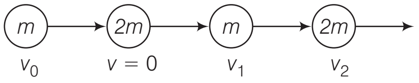
$$h\nu_p = \frac{13.6}{5^2} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$25h\nu_p = h\nu_L$$

$$\therefore \quad \nu_p = \frac{\nu_L}{25}$$

6. (a) Neutron-Deuterium collision;



Momentum conservation gives;

$$mv_0 = mv_1 + 2mv_2$$

$$\Rightarrow \quad v_0 = v_1 + 2v_2 \quad \ldots\text{(i)}$$

Collision given is elastic .

So, coefficient of restitution, $e = 1$

$$\therefore \quad e = 1 = \frac{\text{Velocity of separation}}{\text{Velocity of approach}}$$

$$\Rightarrow \quad 1 = \frac{v_2 - v_1}{v_0 - 0}$$

$$\Rightarrow \quad v_0 = v_2 - v_1 \quad \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2v_0 = 3v_2$$

$$\Rightarrow \quad \frac{2v_0}{3} = v_2$$

So, from Eq. (i), we get

$$v_1 = v_0 - 2v_2$$

$$= v_0 - \frac{4v_0}{3}$$

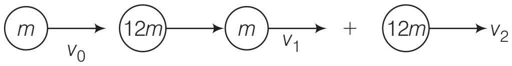
$$\Rightarrow \quad v_1 = -\frac{v_0}{3}$$

Fractional loss of energy of neutron

$$= \left(\frac{-K_f + K_i}{K_i}\right)_{\text{for neutron}}$$

$$= \frac{-\frac{1}{2}mv_1^2 + \frac{1}{2}mv_0^2}{\frac{1}{2}mv_0^2} = \frac{-\frac{v_0^2}{9} + v_0^2}{v_0^2}$$

$$= \left(-\frac{1}{9} + 1\right) = \frac{8}{9} = 0.8\overline{8} = 0.89$$



Similarly, for neutron-carbon atom collision;

Momentum conservation gives;

$$v_0 = v_1 + 12v_2$$

and $\quad e = 1 \Rightarrow v_0 = v_2 - v_1$

So, $\quad v_1 = \frac{11}{13}v_0$

$$\therefore \text{Loss of energy} = \left(-\frac{121}{169} + 1\right) = 0.28$$

So, $P_d = 0.89$ and $P_c = 0.28$

7. **(c)** Potential drop in a silicon diode in forward bias is around 0.7 V.

In given circuit, potential drop across 200 Ω resistor is

$$I = \frac{\Delta V_{\text{net}}}{R} = \frac{3 - 0.7}{200}$$

$$\Rightarrow \quad I = 0.0115\text{A}$$

$$\Rightarrow \quad I = 11.5 \text{ mA}$$

8. **(c)** Only 10% of 10 GHz is utilised for transmission.

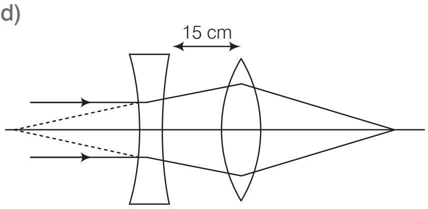
$\therefore$ Band available for transmission

$$= \frac{10}{100} \times 10 \times 10^9 \text{ Hz} = 10^9 \text{ Hz}$$

Now, if there are n channels each using 5 kHz then,

$$n \times 5 \times 10^3 = 10^9 \Rightarrow n = 2 \times 10^5$$

9. **(d)**



Here, $f_1 = -25$ cm, $f_2 = 20$ cm

For diverging lens, $v = -25$ cm

For converging lens, $u = -(15 + 25) = -40$ cm

$$\therefore \quad \frac{1}{v} - \frac{1}{-40} = \frac{1}{+20}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{20} - \frac{1}{40} = \frac{1}{40} \Rightarrow v = 40 \text{ cm}$$

10. **(a)** Let nth bright fringes coincides, then

$$y_{n_1} = y_{n_2}$$

$$\Rightarrow \quad \frac{n_1\lambda_1 D}{d} = \frac{n_2\lambda_2 D}{d} \Rightarrow \frac{n_1}{n_2} = \frac{\lambda_1}{\lambda_2} = \frac{520}{650} = \frac{4}{5}$$

Hence, distance from the central maxima is

$$y = \frac{n_1\lambda_1 D}{d} = \frac{4 \times 650 \times 10^{-9} \times 1.5}{0.5 \times 10^{-3}} = 7.8\,\text{mm}$$

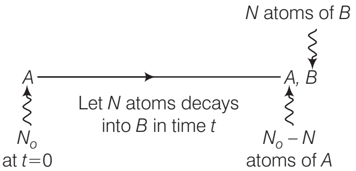
11. **(c)** We have, $\lambda = \dfrac{hc}{\Delta E}$

$$\therefore \quad \frac{\lambda_1}{\lambda_2} = \frac{hc/\Delta E_1}{hc/\Delta E_2} = \frac{\Delta E_2}{\Delta E_1} = \frac{\left(\frac{4}{3}E - E\right)}{2E - E} = \frac{1}{3}$$

12. **(b)** $\lambda_{\min} = \dfrac{hc}{eV}$

$$\log(\lambda_{\min}) = \log\left(\frac{hc}{e}\right) - \log V \Rightarrow y = c - x$$

13. **(a)** Decay scheme is ,



Given, $\dfrac{N_B}{N_A} = 0.3 = \dfrac{3}{10} \Rightarrow \dfrac{N_B}{N_A} = \dfrac{30}{100}$

So, $N_0 = 100 + 30 = 130$ atoms

By using $N = N_0 e^{-\lambda t}$

We have, $100 = 130 e^{-\lambda t}$

$$\Rightarrow \quad \frac{1}{1.3} = e^{-\lambda t} \Rightarrow \log 1.3 = \lambda t$$

$$\Rightarrow \quad \log 1.3 = \frac{\log 2}{T} \cdot t$$

$$\therefore \quad t = \frac{T \cdot \log(1.3)}{\log 2}$$

14. **(a)** For elastic collision,

$$p_{\text{before collision}} = p_{\text{after collision}} .$$

$$mv = mv_A + \frac{m}{2}v_B \Rightarrow 2v = 2v_A + v_B \qquad \dots \text{(i)}$$

Now, coefficient of restitution,

$$e = \frac{v_B - v_A}{u_A - v_B}$$

Here, $u_B = 0$ (Particle at rest) and for elastic collision $e = 1$

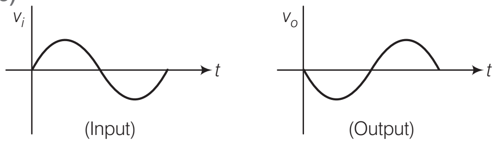
$$\therefore \quad 1 = \frac{v_B - v_A}{v} \Rightarrow v = v_B - v_A \qquad \dots \text{(ii)}$$

From Eq. (i) and Eq. (ii)

$$v_A = \frac{v}{3} \text{ and } v_B = \frac{4v}{3}$$

Hence, $$\frac{\lambda_A}{\lambda_B} = \frac{\left(\frac{h}{mV_A}\right)}{\frac{h}{\frac{m}{2}.V_B}} = \frac{V_B}{2V_A} = \frac{4/3}{2/3} = 2$$

15. **(c)**



In a CE *n-p-n* transistor amplifier, output is 180° out of phase with input.

16. **(a, c, d)** From Snell's law,

$$n\sin\theta = \text{constant}$$

$$\therefore \quad n_1 \sin\theta_i = n_2 \sin\theta_f$$

Further, l will depend on n_1 and $n(z)$. But it will be independent of n_2.

17. **(a)** $\delta = (i_1 + i_2) - A$

$$\Rightarrow \quad 40° = (35° + 79°) - A \Rightarrow A = 74°$$

Now, we know that

$$\mu = \frac{\sin\left(\frac{A + \delta_m}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

It we take the given deviation as the minimum deviation then,

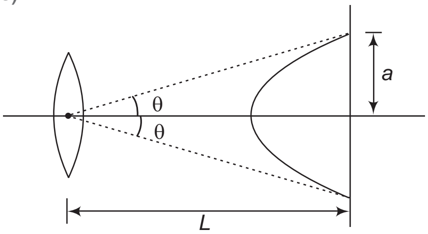
$$\mu = \frac{\sin\left(\frac{74° + 40°}{2}\right)}{\sin\left(\frac{74°}{2}\right)} = 1.51$$

The given deviation may or may not be the minimum deviation. Rather it will be less than this value. Therefore, μ will be less than 1.51.

Hence, maximum possible value of refractive index is 1.51.

18. **(d)** Telescope resolves and brings objects closer. Hence, telescope with magnifying power of 20, the tree appears 20 times nearer.

19. **(c)**



$$a\sin\theta \approx \lambda \Rightarrow a\left(\frac{a}{L}\right) \approx \lambda$$

$$\Rightarrow \quad a = \sqrt{\lambda L}$$

$$\text{Spread} = 2a = \sqrt{4\lambda L}$$

20. (a) According to the law of conservation of energy, i.e. Energy of a photon ($h\nu$) = Work function (ϕ) + Kinetic energy of the photoelectron $\left(\frac{1}{2}mv_{\max}^2\right)$

According to Einstein's photoelectric emission of light

i.e. $$E = (\text{KE})_{\max} + \phi$$

As, $$\frac{hc}{\lambda} = (\text{KE})_{\max} + \phi$$

If the wavelength of radiation is changed to $\frac{3\lambda}{4}$, then

$$\Rightarrow \quad \frac{4}{3}\frac{hc}{\lambda} = \left(\frac{4}{3}(\text{KE})_{\max} + \frac{\phi}{3}\right) + \phi$$

$(\text{KE})_{\max.}$ for fastest emitted electron $= \frac{1}{2}mv'^2 + \phi$

$$\Rightarrow \quad \frac{1}{2}mv'^2 = \frac{4}{3}\left(\frac{1}{2}mv^2\right) + \frac{\phi}{3}$$

i.e. $$v' > v\left(\frac{4}{3}\right)^{1/2}$$

21. (d) **A :** Numbers left : $N \rightarrow \frac{N}{2} \rightarrow \frac{N}{4} \rightarrow \frac{N}{8} \rightarrow \frac{N}{16}$

$\therefore$ Number decayed, $N_A = N - \frac{N}{16} = \frac{15}{16}N$

B : Numbers left : $N \rightarrow \frac{N}{2} \rightarrow \frac{N}{4}$

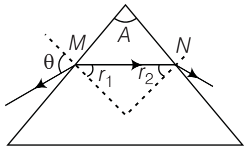
$\therefore$ Numbers decayed, $N_B = N - \frac{N}{4} = \frac{3}{4}N$

Ratio : $\frac{N_A}{N_B} = \frac{(15/16)N}{(3/4)N} = \frac{5}{4}$

22. (c) As, we know Cu is conductor, so increase in temperature, resistance will increase. Then, Si is semiconductor, so with increase in temperature, resistance will decrease.

23. (a) Theoretical question. Therefore, no solution is required.

24. (d)



Applying Snell's law at M,

$$\mu = \frac{\sin\theta}{\sin r_1}$$

$$\therefore \quad r_1 = \sin^{-1}\left(\frac{\sin\theta}{\mu}\right) \text{ or } \sin r_1 = \frac{\sin\theta}{\mu}$$

Now, $$r_2 = A - r_1 = A - \sin^{-1}\left(\frac{\sin\theta}{\mu}\right)$$

Ray of light would get transmitted form face AC if

$$r_2 < \theta_c$$

or $$A - \sin^{-1}\left(\frac{\sin\theta}{\mu}\right) < \theta_c$$

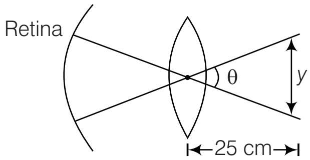
where, $\theta_c = \sin^{-1}\left(\frac{1}{\mu}\right)$

$$\therefore \quad \sin^{-1}\left(\frac{\sin\theta}{\mu}\right) > A - \theta_c \quad \text{or} \quad \frac{\sin\theta}{\mu} > \sin(A - \theta_c)$$

$$\therefore \quad \theta > \sin^{-1}[\mu\sin(A - \theta_c)]$$

or $$\theta > \sin^{-1}\left[\mu\sin\left\{A - \sin^{-1}\left(\frac{1}{\mu}\right)\right\}\right]$$

25. (a)



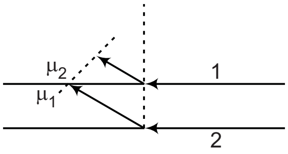
Resolving angle of naked eye is given by

$$\theta = \frac{1.22\lambda}{D}$$

$$\therefore \quad \frac{y}{25 \times 10^{-2}} = \frac{1.22 \times 500 \times 10^{-9}}{0.25 \times 2 \times 10^{-2}}$$

$$\therefore \quad y = 30 \times 10^{-6}\ \text{m} = 30\,\mu\text{m}$$

26. (c)



$$\mu_2 > \mu_1$$

Dotted line is the normal.

According to Huygen's principle, each point on wavefront behaves as a point source of light.

Ray 2 will travel faster than 1 as $\mu_2 > \mu_1$. So, beam will bend upwards.

27. (b) $f_c = 2$ MHz = 2000 kHz, $f_m = 5$ kHz

Resultant frequencies are,

$$f_c + f_m, f_c$$

and $$f_c - f_m$$

or, 2005 kHz, 2000 kHz and 1995 kHz

28. (a) Intensity at a distance r from a point source of power P is given by

$$I = \frac{P}{4\pi r^2} \qquad \text{...(i)}$$

Also, $$I = \frac{1}{2}\varepsilon_0 E_0^2 c \qquad \text{...(ii)}$$

where, E_0 is amplitude of electric field and c the speed of light. Eqs. (i) and (ii) we get

$$E_0 = \sqrt{\frac{2P}{4\pi\varepsilon_0 r^2 c}} = \sqrt{\frac{2 \times 9 \times 10^9 \times 0.1}{(1)^2 \times 3 \times 10^8}}$$

$$= \sqrt{6} = 2.45\ \text{V/m}$$

29. **(d)** The expressions of kinetic energy, potential energy and total energy are

$$K_n = \frac{me^4}{8\varepsilon_0^2 n^2 h^2} \Rightarrow K_n \propto \frac{1}{n^2}$$

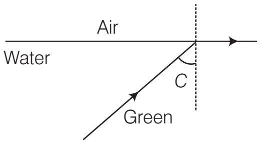
$$U_n = \frac{-me^4}{4\varepsilon_0^2 n^2 h^2} \Rightarrow U_n \propto -\frac{1}{n^2} \text{ and}$$

$$E_n = \frac{-me^4}{8\varepsilon_0^2 n^2 h^2} \Rightarrow E_n \propto -\frac{1}{n^2}$$

In the transition from some excited state to ground state value of n decreases, therefore kinetic energy increases, but potential and total energy decreases.

30. **(b)** No Solution is required

31. **(d)** For total internal reflection of light take place, following conditions must be obeyed.



(*i*) The ray must travel from denser to rarer medium.

(*ii*) Angle of incidence (θ) must be greater than or equal to critical angle (C)

i.e. $$C = \sin^{-1}\left[\frac{\mu_{\text{rarer}}}{\mu_{\text{denser}}}\right]$$

Here, $\sin C = \dfrac{1}{n_{\text{water}}}$ and $n_{\text{water}} = a + \dfrac{b}{\lambda^2}$

If frequency is less ⇒ λ is greater and hence, RI $n_{(\text{water})}$ is less and therefore, critical angle increases. So, they do not suffer reflection and come out at angle less than 90°.

32. **(b)** It is based on lens maker's formula and its magnification.

i.e. $$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

According to lens maker's formula, when the lens in the air.

$$\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f} = \frac{1}{2x} \Rightarrow f = 2x$$

Here, $$\left(\frac{1}{x} = \frac{1}{R_1} - \frac{1}{R_2}\right)$$

In case of liquid, where refractive index is $\frac{4}{3}$ and $\frac{5}{3}$, we get

Focal length in first liquid

$$\frac{1}{f_1} = \left(\frac{\mu_s}{\mu_{l_1}} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \Rightarrow \frac{1}{f_1} = \left(\frac{3/2}{4/3} - 1\right)\frac{1}{x}$$

⇒ f_1 is positive.

$$\frac{1}{f_1} = \frac{1}{8x} = \frac{1}{4(2x)} = \frac{1}{4f} \Rightarrow f_1 = 4f$$

Focal length in second liquid

$$\frac{1}{f_2} = \left(\frac{\mu_s}{\mu_{l_2}} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \frac{1}{f_2} = \left(\frac{3/2}{5/3} - 1\right)\left(\frac{1}{x}\right)$$

⇒ f_2 is negative.

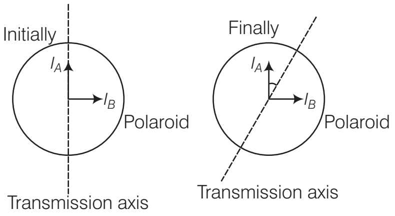
33. **(d)** By law of Malus i.e. $I = I_0 \cos^2\theta$

Now, $I_{A'} = I_A \cos^2 30°$

$I_{B'} = I_B \cos^2 60°$

As, $I_A' = I_B'$

$$I_A \cos^2 30° = I_B \cos^2 60°$$



$$\Rightarrow I_A \frac{3}{4} = I_B \frac{1}{4} \Rightarrow \frac{I_A}{I_B} = \frac{1}{3}$$

34. **(c)** For hydrogen atom, we get

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$

$$\Rightarrow \frac{1}{\lambda_1} = R(1)^2\left(\frac{3}{4}\right) \Rightarrow \frac{1}{\lambda_2} = R(1)^2\left(\frac{3}{4}\right)$$

$$\Rightarrow \frac{1}{\lambda_3} = R(2)^2\left(\frac{3}{4}\right) \Rightarrow \frac{1}{\lambda_4} = R(3)^2\left(\frac{3}{4}\right)$$

$$\Rightarrow \frac{1}{\lambda_1} = \frac{1}{4\lambda_3} = \frac{1}{9\lambda_4} = \frac{1}{\lambda_2}$$

35. **(b)** **Thinking Process** The problem is based on frequency dependence of photoelectric emission. When incident light with certain frequency (greater than on the threshold frequency is focus on a metal surface) then some electrons are emitted from the metal with substantial initial speed.

When an electron moves in a circular path, then

$$r = \frac{mv}{eB} \Rightarrow \frac{r^2e^2B^2}{2} = \frac{m^2v^2}{2}$$

$$\text{KE}_{\max} = \frac{(mv)^2}{2m} \Rightarrow \frac{r^2e^2B^2}{2m} = (\text{KE})_{\max}$$

Work function of the metal (W), i.e. $W = h\nu - \text{KE}_{\max}$

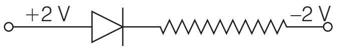
$$1.89 - \phi = \frac{r^2e^2B^2}{2m}\frac{1}{2}\text{eV} = \frac{r^2eB^2}{2m}\text{eV}$$

[$h\nu$ → 1.89 eV, for the transition on from third to second orbit of H-atom]

$$= \frac{100 \times 10^{-6} \times 1.6 \times 10^{-19} \times 9 \times 10^{-8}}{2 \times 9.1 \times 10^{-31}}$$

$$\phi = 1.89 - \frac{1.6 \times 9}{2 \times 9.1} = 1.89 - 0.79 = 1.1 \text{ eV}$$

36. **(a)** For forward bias, *p*-side must be a higher potential than *n*-side.



So, (a) is forward biased.

37. **(c)** Both the energy densities are equal i.e. energy is equally divided between electric and magnetic fields.

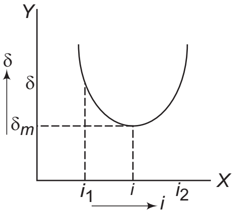
38. **(d)** (a) Infrared rays are used to treat muscular strain.

(b) Radiowaves are used for broadcasting purposes.

(c) X-rays are used to detect fracture of bones.

(d) Ultraviolet rays are absorbed by ozone.

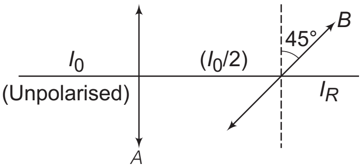
39. **(c)** We know that, the angle of deviation depends upon the angle of incidence. If we determine experimentally, the angles of deviation corresponding to different angles of incidence and then plot *i* (on-*X*-axis) and δ (on-*Y*-axis), we get a curve as shown in figure



Clearly if angle of incidence is gradually increased, from a small value, the angle of deviation first decreases, becomes minimum for a particular angle of incidence and then begins to increase.

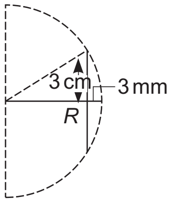
40. **(d)** A fringe is a locus of points having constant path difference from the two coherent sources S_1 and S_2. It will be concentric circle.

41. **(c)** Relation between intensities is



$$I_R = \left(\frac{I_0}{2}\right)\cos^2(45°) = \frac{I_0}{2} \times \frac{1}{2} = \frac{I_0}{4}$$

42. **(c)** By Pythagoras theorem



$$R^2 = (3)^2 + (R - 0.3)^2 \Rightarrow R \approx 15 \text{ cm}$$

Refractive index of material of lens $\mu = \frac{c}{v}$

Here, c = speed of light in vacuum $= 3 \times 10^8$ m/s

v = speed of light in material of lens $= 2 \times 10^8$ m/s

$$= \frac{3 \times 10^8}{2 \times 10^8} = \frac{3}{2}$$

From lens maker's formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Here, $R_1 = R$ and $R_2 = \infty$ (For plane surface)

$$\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{15}\right)$$

$$\Rightarrow \quad f = 30 \text{ cm}$$

43. **(d)** $\Delta E = h\nu$

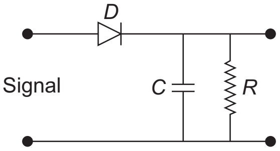
$$\nu = \frac{\Delta E}{h} = k\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right]$$

$$= \frac{k2n}{n^2(n-1)^2} \approx \frac{2k}{n^3} \propto \frac{1}{n^3}$$

44. **(a)** For same value of current higher value of voltage is required for higher frequency.

45. **(d)** As λ is increased, there will be a value of λ above which photoelectron will cease to come out. So, photocurrent will be zero.

46. **(b)**



$$\tau = RC = 100 \times 10^3 \times 250 \times 10^{-12} \text{ s}$$
$$= 2.5 \times 10^{-5} \text{ s}$$

The higher frequency which can be detected with tolerable distortion is

$$f = \frac{1}{2\pi m_a RC}$$

$$= \frac{1}{2\pi \times 0.6 \times 2.5 \times 10^{-5}} \text{ Hz}$$

$$= \frac{100 \times 10^4}{25 \times 1.2\pi} \text{ Hz} = \frac{4}{1.2\,\pi} \times 10^4 \text{ Hz}$$

$$= 10.61 \text{ kHz}$$

47. **(b)** Peak value of electric field

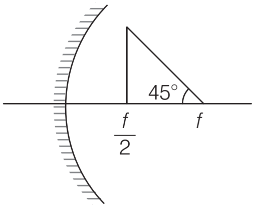
$$E_0 = B_0 c = 20 \times 10^{-9} \times 3 \times 10^8$$
$$= 6 \text{ V/m}$$

JEE Advanced

1. Sunlight of intensity $1.3\,\text{kWm}^{-2}$ is incident normally on a thin convex lens of focal length 20 cm. Ignore the energy loss of light due to the lens and assume that the lens aperture size is much smaller than its focal length. The average intensity of light, in kW m^{-2}, at a distance 22 cm from the lens on the other side is ....

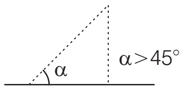
(Numerical Values, 2018)

2. A wire is bent in the shape of a right angled triangle and is placed in front of a concave mirror of focal length f as shown in the figure. Which of the figures shown in the four options qualitatively represent(s) the shape of the image of the bent wire? (These figures are not to scale.)

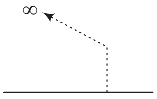


(Single Correct Option, 2018)

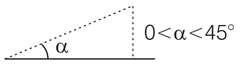
(a)



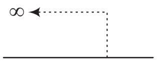
(b)



(c)



(d)



3. In a radioactive decay chain, ${}^{232}_{90}\text{Th}$ nucleus decays to ${}^{212}_{82}\text{Pb}$ nucleus. Let N_α and N_β be the number of α and β-particles respectively, emitted in this decay process. Which of the following statements is (are) true?

(More than One Correct Option, 2018)

(a) $N_\alpha = 5$
(b) $N_\alpha = 6$
(c) $N_\beta = 2$
(d) $N_\beta = 4$

4. In a photoelectric experiment, a parallel beam of monochromatic light with power of 200 W is incident on a perfectly absorbing cathode of work function 6.25 eV. The frequency of light is just above the threshold frequency, so that the photoelectrons are emitted with negligible kinetic energy. Assume that the photoelectron emission efficiency is 100%. A potential difference of 500 V is applied between the cathode and the anode. All the emitted electrons are incident normally on the anode and are absorbed. The anode experiences a force $F = n \times 10^{-4}\text{N}$ due to the impact of the electrons. The value of n is .... (Take, mass of the electron, $m_e = 9 \times 10^{-31}\text{kg}$ and $\text{eV} = 1.6 \times 10^{-19}\,\text{J}$)

(Numerical Value, 2018)

5. Consider a hydrogen-like ionised atom with atomic number Z with a single electron. In the emission spectrum of this atom, the photon emitted in the $n = 2$ to $n = 1$ transition has energy 74.8 eV higher than the photon emitted in the $n = 3$ to $n = 2$ transition. The ionisation energy of the hydrogen atom is 13.6 eV. The value of Z is .... **(Numerical Value, 2018)**

6. For an isosceles prism of angle A and refractive index μ, it is found that the angle of minimum deviation $\delta_m = A$. Which of the following option(s) is/are correct? **(More than One Correct Option, 2017)**

(a) For the angle of incidence $i_1 = A$, the ray inside the prism is parallel to the base of the prism

(b) At minimum deviation, the incident angle i_1 and the refracting angle r_1 at the first refracting surface are related by $r_1 = \left(\frac{i_i}{2}\right)$

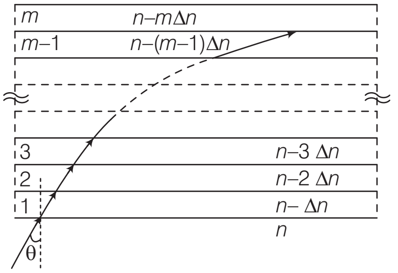
(c) For this prism, the emergent ray at the second surface will be tangential to the surface when the angle of incidence at the first surface is

$$i_1 = \sin^{-1}\left[\sin A\sqrt{4\cos^2\frac{A}{2} - 1} - \cos A\right]$$

(d) For this prism, the refractive index μ and the angle prism A are related as $A = \frac{1}{2}\cos^{-1}\left(\frac{\mu}{2}\right)$

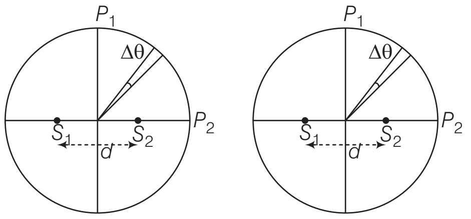
7. A monochromatic light is travelling in a medium of refractive index $n = 1.6$. It enters a stack of glass layers from the bottom side at an angle $\theta = 30°$. The interfaces of the glass layers are parallel to each other.

The refractive indices of different glass layers are monotonically decreasing as $n_m = n - m\,\Delta n$, where n_m is the refractive index of the mth slab and $\Delta n = 0.1$ (see the figure). The ray is refracted out parallel to the interface between the $(m-1)$ th and mth slabs from the right side of the stack. What is the value of m ? **(Single Integer Type, 2017)**



8. Two coherent monochromatic point sources S_1 and S_2 of wavelength $\lambda = 600$ nm are placed symmetrically on either side of the centre of the circle as shown. The sources are separated by a distance $d = 1.8$ mm. This arrangement produces interference fringes visible as alternate bright and dark spots on the circumference of the circle. The angular separation between two consecutive bright spots is $\Delta\theta$. Which of the following option(s) is/are correct?

(More than One Correct Option, 2017)



(a) The angular separation between two consecutive bright spots decreases as we move from P_1 to P_2 along the first quadrant

(b) A dark spot will be formed at the point P_2

(c) The total number of fringes produced between P_1 and P_2 in the first quadrant is close to 3000

(d) At P_2 the order of the fringe will be maximum

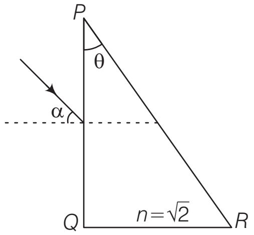
9. ^{131}I is an isotope of Iodine that β decays to an isotope of Xenon with a half-life of 8 days. A small amount of a serum labelled with ^{131}I is injected into the blood of a person. The activity of the amount of ^{131}I injected was 2.4×10^5 Becquerel (Bq). It is known that the injected serum will get distributed uniformly in the blood stream in less than half an hour. After 11.5 h, 2.5 ml of blood is drawn from the person's body, and gives an activity of 115 Bq. The total volume of blood in the person's body, in litres is approximately (you may use $e^2 \approx 1 + x$ for $|x| << 1$ and $\ln 2 \approx 0.7$).

(Single Integer Type, 2017)

10. A photoelectric material having work-function ϕ_0 is illuminated with light of wavelength $\lambda \left(\lambda < \frac{hc}{\phi_0}\right)$. The fastest photoelectron has a de-Broglie wavelength λ_d. A change in wavelength of the incident light by $\Delta\lambda$ results in a change $\Delta\lambda_d$ in λ_d. Then, the ratio $\frac{\Delta\lambda_d}{\Delta\lambda}$ is proportional to **(Single Correct Option, 2017)**

(a) $\frac{\lambda_d^2}{\lambda^2}$ (b) $\frac{\lambda_d}{\lambda}$ (c) $\frac{\lambda_d^3}{\lambda}$ (d) $\frac{\lambda_d^3}{\lambda^2}$

11. A parallel beam of light is incident from air at an angle α on the side PQ of a right angled triangular prism of refractive index $n = \sqrt{2}$. Light undergoes total internal reflection in the prism at the face PR when α has a minimum value of 45°. The angle θ of the prism is **(Single Correct Option, 2016)**

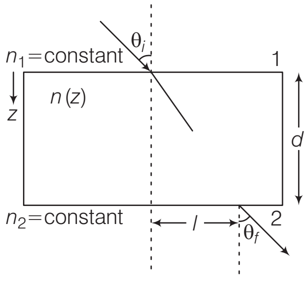


(a) 15° (b) 22.5° (c) 30° (d) 45°

12. A plano-convex lens is made of material of refractive index n. When a small object is placed 30 cm away in front of the curved surface of the lens, an image of double the size of the object is produced. Due to reflection from the convex surface of the lens, another faint image is observed at a distance of 10 cm away from the lens. Which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**

(a) The refractive index of the lens is 2.5
(b) The radius of curvature of the convex surface is 45 cm
(c) The faint image is erect and real
(d) The focal length of the lens is 20 cm

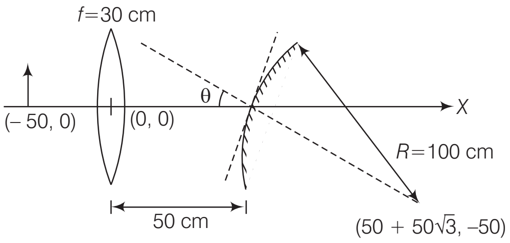
13. A transparent slab of thickness d has a refractive index $n(z)$ that increases with z. Here, z is the vertical distance inside the slab, measured from the top. The slab is placed between two media with uniform refractive indices n_1 and $n_2 (> n_1)$, as shown in the figure. A ray of light is incident with angle θ_i from medium 1 and emerges in medium 2 with refraction angle θ_f with a lateral displacement l.



Which of the following statement(s) is (are) true? **(More than One Correct Option, 2016)**

(a) l is independent on $n(z)$
(b) $n_1 \sin\theta_i = (n_2 - n_1)\sin\theta_f$
(c) $n_1 \sin\theta_i = n_2 \sin\theta_f$
(d) l is independent of n_2

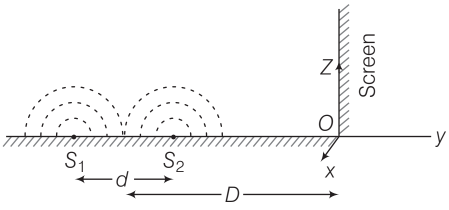
14. A small object is placed 50 cm to the left of a thin convex lens of focal length 30 cm. A convex spherical mirror of radius of curvature 100 cm is placed to the right of the lens at a distance of 50 cm. The mirror is tilted such that the axis of the mirror is at an angle $\theta = 30°$ to the axis of the lens, as shown in the figure. **(Single Correct Option, 2016)**



If the origin of the coordinate system is taken to be at the centre of the lens, the coordinates (in cm) of the point (x, y) at which the image is formed are

(a) $(125/3, 25/\sqrt{3})$
(b) $(50 - 25\sqrt{3}, 25)$
(c) $(0, 0)$
(d) $(25, 25\sqrt{3})$

15. While conducting the Young's double slit experiment, a student replaced the two slits with a large opaque plate in the x-y plane containing two small holes that act as two coherent point sources (S_1, S_2) emitting light of wavelength 600 mm. The student mistakenly placed the screen parallel to the XZ plane (for $z > 0$) at a distance $D = 3$ m from the mid-point of S_1S_2, as shown schematically in the figure. The distance between the source $d = 0.6003$ mm.
The origin O is at the intersection of the screen and the line joining S_1S_2.
(More than One Correct Option, 2016)



Which of the following is (are) true of the intensity pattern on the screen?

(a) Semi circular bright and dark bands centered at point O
(b) The region very close to the point O will be dark
(c) Straight bright and dark bands parallel to the x-axis
(d) Hyperbolic bright and dark bands with foci symmetrically placed about O in the x-direction

16. In a historical experiment to determine Planck's constant, a metal surface was irradiated with light of different wavelengths. The emitted photoelectron energies were measured by applying a stopping potential. The relevant data for the wavelength (λ) of incident light and the corresponding stopping potential (V_0) are given below: **(Single Correct Option, 2016)**

λ (μm)	V_0 (Volt)
0.3	2
0.4	1

Given that

$c = 3 \times 10^8 \text{ms}^{-1}$ and $e = 1.6 \times 10^{-19}$ C,

Planck's constant (in units of J-s) found from such an experiment is)

(a) 6.0×10^{-34} (b) 6.4×10^{-34}
(c) 6.6×10^{-34} (d) 6.8×10^{-34}

17. Highly excited states for hydrogen-like atoms (also called Rydberg states) with nuclear charge Ze are defined by their principle quantum number n, where $n >> 1$. Which of the following statement(s) is (are) true?
(More than One Correct Option, 2016)

(a) Relative change in the radii of two consecutive orbitals does not depend on Z
(b) Relative change in the radii of two consecutive orbitals varies as $1/n$
(c) Relative change in the energy of two consecutive orbitals varies as $1/n^3$
(d) Relative change in the angular momenta of two consecutive orbitals varies as $1/n$

18. A hydrogen atom in its ground state is irradiated by light of wavelength 970Å. Taking $hc/e = 1.237 \times 10^{-6}$ eVm and the ground state energy of hydrogen atom as -13.6 eV, the number of lines present in the emission spectrum is
(Single Integer Type, 2016)

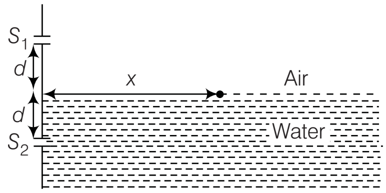
19. An accident in a nuclear laboratory resulted in deposition of a certain amount of radioactive material of half-life 18 days inside the laboratory. Tests revealed that the radiation was 64 times more than the permissible level required for safe operation of the laboratory. What is the minimum number of days after which the laboratory can be considered safe for use?
(Single Correct Option, 2016)

(a) 64 (b) 90
(c) 108 (d) 120

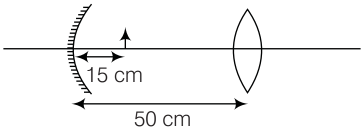
20. The electrostatic energy of Z protons uniformly distributed throughout a spherical nucleus of radius R is given by $E = \frac{3}{5} \frac{Z(Z-1)e^2}{4\pi\varepsilon_0 R}$ The measured masses of the neutron, ${}^1_1$H, ${}^{15}_{7}$N and ${}^{15}_{8}$O are 1.008665 u, 1.007825 u, 15.000109 u and 15.003065 u, respectively. Given that the radii of both the ${}^{15}_{7}$N and ${}^{15}_{8}$O nuclei are same, 1 u = 931.5 MeV/c^2 (c is the speed of light) and $e^2/(4\pi\varepsilon_0) = 1.44$ MeV fm. Assuming that the difference between the binding energies of ${}^{15}_{7}$N and ${}^{15}_{8}$O is purely due to the electrostatic energy, the radius of either of the nuclei is ($1\text{fm} = 10^{-15}$ m) **(Single Correct Option, 2016)**

(a) 2.85 fm (b) 3.03 fm
(c) 3.42 fm (d) 3.80 fm

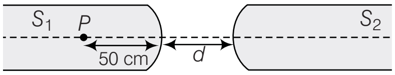
21. A Young's double slit interference arrangement with slits S_1 and S_2 is immersed in water (refractive index = 4/3) as shown in the figure. The positions of maxima on the surface of water are given by $x^2 = p^2 m^2 \lambda^2 - d^2$, where λ is the wavelength of light in air (refractive index = 1), $2d$ is the separation between the slits and m is an integer. The value of p is **(Single Integer Type, 2015)**



22. Consider a concave mirror and a convex lens (refractive index = 1.5) of focal length 10 cm each, separated by a distance of 50 cm in air (refractive index = 1) as shown in the figure. An object is placed at a distance of 15 cm from the mirror. Its erect image formed by this combination has magnification M_1. When the set-up is kept in a medium of refractive index $\frac{7}{6}$, the magnification becomes M_2. The magnitude $\left|\frac{M_2}{M_1}\right|$ is **(Single Integer Type, 2015)**

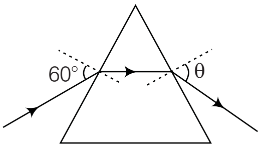


23. Two identical glass rods S_1 and S_2 (refractive index = 1.5) have one convex end of radius of curvature 10 cm. They are placed with the curved surfaces at a distance d as shown in the figure, with their axes (shown by the dashed line) aligned. When a point source of light P is placed inside rod S_1 on its axis at a distance of 50 cm from the curved face, the light rays emanating from it are found to be parallel to the axis inside S_2. The distance d is **(Single Correct Option, 2015)**



(a) 60 cm (b) 70 cm
(c) 80 cm (d) 90 cm

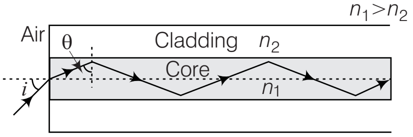
24. A monochromatic beam of light is incident at 60° on one face of an equilateral prism of refractive index n and emerges from the opposite face making an angle θ with the normal (see figure). For $n = \sqrt{3}$ the value of θ is 60° and $\frac{d\theta}{dn} = m$. The value of m is **(Single Integer Type, 2015)**



Passage (Q. Nos. 25-26)

Light guidance in an optical fibre can be understood by considering a structure comprising of thin solid glass cylinder of refractive index n_1 surrounded by a medium of lower refractive index n_2. The light guidance in the structure takes place due to successive total internal reflections at the interface of the media n_1 and n_2 as shown in the figure. All rays with the angle of incidence i less than a particular value i_m are confined in the medium of refractive index n_1. The numerical aperture (NA) of the structure is defined as $\sin i_m$. **(Passage Type, 2015)**

25. For two structures namely S_1 with $n_1 = \frac{\sqrt{45}}{4}$ and $n_2 = \frac{3}{2}$, and S_2 with $n_1 = \frac{8}{5}$ and $n_2 = \frac{7}{5}$ and taking the refractive index of water to be $\frac{4}{3}$ and that to air to be 1, the correct option(s) is/are



(a) NA of S_1 immersed in water is the same as that of S_2 immersed in a liquid of refractive index $\frac{16}{3\sqrt{15}}$

(b) NA of S_1 immersed in liquid of refractive index $\frac{6}{\sqrt{15}}$ is the same as that of S_2 immersed in water

(c) NA of S_1 placed in air is the same as that S_2 immersed in liquid of refractive index $\frac{4}{\sqrt{15}}$

(d) NA of S_1 placed in air is the same as that of S_2 placed in water

26. If two structures of same cross-sectional area, but different numerical apertures NA_1 and $NA_2 (NA_2 < NA_1)$ are joined longitudinally, the numerical aperture of the combined structure is

(a) $\frac{NA_1 NA_2}{NA_1 + NA_2}$ (b) $NA_1 + NA_2$
(c) NA_1 (d) NA_2

27. A nuclear power plant supplying electrical power to a village uses a radioactive material of half life T years as the fuel. The amount of fuel at the beginning is such that the total power requirement of the village is 12.5% of the electrical power available from the plant at that time. If the plant is able to meet the total power needs of the village for a maximum period of nT years, then the value of n is **(Single Integer Type, 2015)**

28. Match the nuclear processes given in Column I with the appropriate option(s) in Column II.

Column I		Column II	
A.	Nuclear fusion	P.	absorption of thermal neutrons by $^{235}_{92}U$
B.	Fission in a nuclear reactor	Q.	$^{60}_{27}Co$ nucleus
C.	β-decay	R.	Energy production in stars via hydrogen conversion to helium
D.	γ-ray emission	S.	Heavy water
		T.	Neutrino emission

(Matching Type, 2015)

29. For a radioactive material, its activity A and rate of change of its activity R are defined as $A = -\frac{dN}{dt}$ and $R = -\frac{dA}{dt}$, where $N(t)$ is the number of nuclei at time t. Two radioactive source P(mean life τ) and Q (mean life 2τ) have the same activity at $t = 0$. Their rate of change of activities at $t = 2\tau$ are R_P and R_Q, respectively. If $\frac{R_P}{R_Q} = \frac{n}{e}$, then the value of n is **(Single Integer Type, 2015)**

30. An electron in an excited state of Li^{2+} ion has angular momentum $\frac{3h}{2\pi}$. The de Broglie wavelength of the electron in this state is $p\pi a_0$ (where a_0 is the Bohr radius). The value of p is **(Single Integer Type, 2015)**

31. A fission reaction is given by $^{236}_{92}U \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + x + y$,

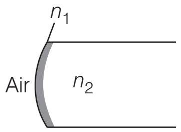
where x and y are two particles. Considering $^{236}_{92}U$ to be at rest, the kinetic energies of the products are denoted by K_{Xe}, K_{Sr}, K_x(2 MeV) and K_y(2 MeV), respectively. Let the binding energies per nucleon of $^{236}_{92}U$, $^{140}_{54}Xe$ and $^{94}_{38}Sr$ be 7.5 MeV, 8.5 MeV and 8.5 MeV, respectively. Considering different conservation laws, the correct options is/are **(Single Correct Option, 2015)**

(a) $x = n$, $y = n$, $K_{Sr} = 129$ MeV, $K_{Xe} = 86$ MeV
(b) $x = p$, $y = e^-$, $K_{Sr} = 129$ MeV, $K_{Xe} = 86$ MeV
(c) $x = p$, $y = n$, $K_{Sr} = 129$ MeV, $K_{Xe} = 86$ MeV
(d) $x = n$, $y = n$, $K_{Sr} = 86$ MeV, $K_{Xe} = 129$ MeV

32. A light source, which emits two wavelengths $\lambda_1 = 400$ nm and $\lambda_2 = 600$ nm, is used in a Young's double-slit experiment. If recorded fringe widths for λ_1 and λ_2 are β_1 and β_2 and the number of fringes for them within a distance y on one side of the central maximum are m_1 and m_2, respectively, then **(More than One Correct Option, 2014)**

(a) $\beta_2 > \beta_1$
(b) $m_1 > m_2$
(c) from the central maximum, 3rd maximum of λ_2 overlaps with 5th minimum of λ_1
(d) the angular separation of fringes of λ_1 is greater than λ_2

33. A transparent thin film of uniform thickness and refractive index $n_1 = 1.4$ is coated on the convex spherical surface of radius R at one end of a long solid glass cylinder of refractive index $n_2 = 1.5$, as shown in the figure.



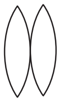
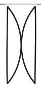
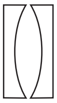
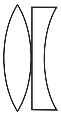
Rays of light parallel to the axis of the cylinder traversing through the film from air to glass get focused at distance f_1 from the film, while rays of light traversing from glass to air get focused at distance f_2 from the film. Then

(More than One Correct Option, 2014)

(a) $|f_1| = 3R$ (b) $|f_1| = 2.8R$

(c) $|f_2| = 2R$ (d) $|f_2| = 1.4R$

34. Four combinations of two thin lenses are given in Column I.The radius of curvature of all curved surfaces is r and the refractive index of all the lenses is 1.5. Match lens combinations in Column I with their focal length in Column II and select the correct answer using the code given below the columns.

	Column I		Column II
P.		1.	$2r$
Q.		2.	$r/2$
R.		3.	$-r$
S.		4.	r

(Matching Type, 2014)

Codes

	P	Q	R	S
(a)	1	2	3	4
(b)	2	4	3	1
(c)	4	1	2	3
(d)	2	1	3	4

35. A metal surface is illuminated by light of two different wavelengths 248 nm and 310 nm. The maximum speeds of the photoelectrons corresponding to these wavelengths are u_1 and u_2, respectively. If the ratio $u_1 : u_2 = 2:1$ and $hc = 1240$ eV nm, the work function of the metal is nearly **(Single Correct Option, 2014)**

(a) 3.7 eV (b) 3.2 eV

(c) 2.8 eV (d) 2.5 eV

36. If λ_{Cu} is the wavelength of K_α, X-ray line of copper (atomic number 29) and λ_{MO} is the wavelength of the K_α, X-ray line of molybdenum (atomic number 42), then the ratio $\lambda_{\text{Cu}}/\lambda_{\text{Mo}}$ is close to

(Single Correct Option, 2014)

(a) 1.99 (b) 2.14 (c) 0.50 (d) 0.48

37. The image of an object, formed by a plano-convex lens at a distance of 8 m behind the lens, is real and is one-third the size of the object. The wavelength of light inside the lens is $\dfrac{2}{3}$ times the wavelength in free space. The radius of the curved surface of the lens is

(Single Correct Option, 2013)

(a) 1 m (b) 2 m (c) 3 m (d) 6 m

38. A ray of light travelling in the direction $\dfrac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$ is incident on a plane mirror. After reflection, it travels along the direction $\dfrac{1}{2}(\mathbf{i} - \sqrt{3}\mathbf{j})$. The angle of incidence is **(Single Correct Option, 2013)**

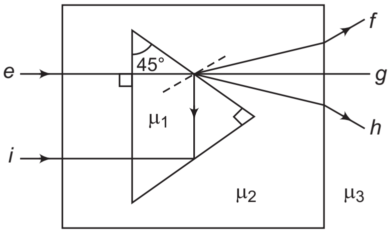
(a) 30°a (b) 45° (c) 60° (d) 75°

39. In the Young's double slit experiment using a monochromatic light of wavelength λ the path difference (in terms of an integer n) corresponding to any point having half the peak intensity is **(Single Correct Option, 2013)**

(a) $(2n+1)\dfrac{\lambda}{2}$ (b) $(2n+1)\dfrac{\lambda}{4}$

(c) $(2n+1)\dfrac{\lambda}{8}$ (d) $(2n+1)\dfrac{\lambda}{16}$

40. A right angled prism of refractive index μ_1 is placed in a rectangular block of refractive index μ_2, which is surrounded by a medium of refractive index μ_3, as shown in the figure, *A* ray of light '*e*' enters the rectangular block at normal incidence. Depending upon the relationships between μ_1, μ_2 and μ_3, it takes one of the four possible paths '*ef*', '*eg*', '*eh*' or '*ei*'.



Match the paths in Column I with conditions of refractive indices in Column II and select the correct answer using the codes given below the columns.

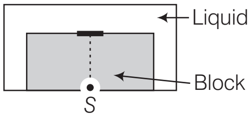
(Matching, Type 2013)

Column I		Column II	
P.	$e \to f$	1.	$\mu_1 > \sqrt{2}\mu_2$
Q.	$e \to g$	2.	$\mu_1 > \mu_1$ and $\mu_2 > \mu_3$
R.	$e \to h$	3.	$\mu_1 = \mu_2$
S.	$e \to i$	4.	$\mu_2 < \mu_1 < \sqrt{2}\,\mu_2$ and $\mu_2 > \mu_3$

Codes

	P	Q	R	S		P	Q	R	S
(a)	2	3	1	4	(b)	1	2	4	3
(c)	4	1	2	3	(d)	2	3	4	1

41. A point source S is placed at the bottom of a transparent block of height 10 mm and refractive index 2.72. It is immersed in a lower refractive index liquid as shown in the figure. It is found that the light emerging from the block to the liquid forms a circular bright spot of diameter 11.54 mm on the top of the block. The refractive index of the liquid is



(Single Correct Option, 2013)

(a) 1.21 (b) 1.30 (c) 1.36 (d) 1.42

42. A pulse of light of duration 100 ns is absorbed completely by a small object initially at rest. Power of the pulse is 30 mV and the speed of light is 3×10^8 ms^{-1}. The final momentum of the object is **(Single Correct Option, 2013)**

(a) 0.3×10^{-17} kg-ms^{-1} (b) 1.0×10^{-17} kg-ms^{-1}
(c) 3.0×10^{-17} kg -ms^{-1} (d) 9.0×10^{-17} kg-ms^{-1}

43. A freshly prepared sample of a radioisotope of half-life 1386 s has activity 10^3 disintegrations per second. Given that ln 2 = 0.693, the fraction of the initial number of nuclei (expressed in nearest integer percentage) that will decay in the first 80 s after preparation of the sample is **(Single Integer Type, 2013)**

44. The radius of the orbit of an electron in a Hydrogen-like atom is 4.5 a_0 where a_0 is the Bohr radius. Its orbital angular momentum is $\frac{3h}{2\pi}$. It is given that h is Planck constant and R is Rydberg constant. The possible wavelength(s), when the atom de-excites, is (are)

(More than One Correct Option, 2013)

(a) $\frac{9}{32R}$ (b) $\frac{9}{16R}$
(c) $\frac{9}{5R}$ (d) $\frac{4}{3R}$

Passage (Q. Nos. 45-46)

The mass of a nucleus ${}^A_Z X$ is less that the sum of the masses of $(A - Z)$ number of neutrons and Z number of protons in the nucleus. The energy equivalent to the corresponding mass difference is known as the binding energy of the nucleus. A heavy nucleus of mass M can break into two light nuclei of masses m_1 and m_2 only if $(m_1 + m_2) < M$. Also two light nuclei of masses m_3 and m_4 can undergo complete fusion and form a heavy nucleus of mass M' only if $(m_3 + m_4) > M'$. The masses of some neutral atoms are given in the table below : **(Passage Type, 2013)**

${}^1_1\text{H}$	1.007825u	${}^2_1\text{H}$	2.014102u	${}^3_1\text{H}$	3.016050u
${}^6_3\text{Li}$	6.01513u	${}^7_3\text{Li}$	7.016004u	${}^4_2\text{He}$	4.002603u
${}^{152}_{64}\text{Gd}$	151.919803u	${}^{206}_{82}\text{Pb}$	205.974455u	${}^{70}_{30}\text{Zn}$	69.925325u
${}^{82}_{34}\text{Se}$	81.916709u	${}^{209}_{83}\text{Bi}$	208.980388u	${}^{201}_{84}\text{Po}$	205.974455u

45. The correct statement is

(a) The nucleus ${}^6_3\text{Li}$ can emit an alpha particle
(b) The nucleus ${}^{210}_{84}\text{Po}$ can emit a proton
(c) Deuteron and alpha particle can undergo complete fusion
(d) The nuclei ${}^{70}_{30}\text{Zn}$ and ${}^{82}_{34}\text{Se}$ can undergo complete fusion

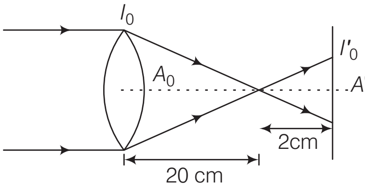
46. The kinetic energy (in keV) of the alpha particle, when the nucleus $^{210}_{84}\text{Po}$ at rest undergoes alpha decay, is

(a) 5316 (b) 5422 (c) 5707 (d) 5818

47. The isotopes $^{12}_{5}\text{B}$ having a mass 12.014 u undergoes β-decay to $^{12}_{6}\text{C}$. $^{12}_{6}\text{C}$ has an excited state of the nucleus ($^{12}_{6}\text{C}^*$) at 4.041 MeV above its ground state. If $^{12}_{5}\text{B}$ decays to $^{12}_{5}\text{C}^*$, the maximum kinetic energy of the β-particle in units of MeV is ($1 \text{ u} = 931.5 \text{MeV}/c^2$, where c is the speed of light in vacuum) **(Single Integer Type, 2013)**

Answer *with* Explanations

1. **(130.0** kW / m^2**)**

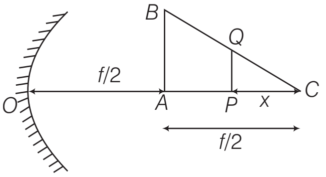


$$\frac{A_0'}{A_0} = \left(\frac{2}{20}\right)^2 = \frac{1}{100} \Rightarrow A_0' = \frac{A_0}{100}$$

$$P = I_0A_0 = I_0'A_0'$$

$$\Rightarrow \quad I_0' = \frac{I_0A_0}{\frac{A_0}{100}} = 100I_0 = 130 \text{ kW/m}^2$$

2. **(d)** Image of point A



$$\frac{PQ}{x} = \frac{AB}{f/2} \Rightarrow PQ = \frac{2(AB)x}{f}$$

For ***A*** **:**

$$\frac{1}{v} + \frac{1}{[-(f/2)]} = \frac{1}{-f} \Rightarrow v = f$$

$$\Rightarrow \quad \frac{I_{AB}}{AB} = -\frac{v}{u} = -\frac{f}{\left(-\frac{f}{2}\right)} \Rightarrow I_{AB} = 2\,AB$$

For height of PQ,

$$\frac{1}{v} + \frac{1}{-[(f-x)]} = \frac{1}{-f}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{(f-x)} - \frac{1}{f} \Rightarrow v = \frac{f(f-x)}{x}$$

$$\Rightarrow \quad \frac{I_{PQ}}{PQ} = -\frac{v}{u} = \frac{f(f-x)}{x[(f-x)]} = \left(\frac{f}{x}\right)$$

$$\Rightarrow \quad I_{PQ} = \frac{f}{x}PQ = \left(\frac{f}{x}\right)\left(\frac{2\,(AB)x}{f}\right)$$

$$\left[\because PQ = \frac{2(AB)x}{f}\right]$$

$$I_{PQ} = 2\,AB$$

(Size of image is independent of x. So, final image will be of same height terminating at infinity)

3. **(a, c)** $^{232}_{90}\text{Th}$ is converting into $^{212}_{82}\text{Pb}$.

Change in mass number $(A) = 20$

$\therefore$ Number of α-particle emitted $= \frac{20}{4} = 5$

Due to 5 α-particles, Z will change by 10 units. Since, given change is 8, therefore number of β-particles emitted is 2.

4. **(24)** $\therefore$ Power $= nhf$

(where, n = number of photons incident per second)

Since, KE = 0, hf = work-funcition W

$$200 = nW = n\,[6.25 \times 1.6 \times 10^{-19}]$$

$$\Rightarrow \quad n = \frac{200}{1.6 \times 10^{-19} \times 6.25}$$

As photon is just above threshold frequency $KE_{\max}$ is zero and they are accelerated by potential difference of 500 V.

$$\therefore \quad \text{KE}_f = q\Delta V$$

$$\frac{P^2}{2m} = q\Delta V \Rightarrow P = \sqrt{2\,mq\,\Delta V}$$

Since, efficiency is 100%, number of electrons emitted per second = number of photons incident per second. As, photon is completely absorbed, force exerted

$$= n(mV) = nP = n\sqrt{2mq\Delta V}$$

$$= \frac{200}{6.25 \times 1.6 \times 10^{-19}} \times \sqrt{2\,(9 \times 10^{-31}) \times 1.6 \times 10^{-19} \times 500}$$

$$= 24$$

5. **(3)** $\Delta E_{2-1} = 13.6 \times Z^2\left[1 - \frac{1}{4}\right] = 13.6 \times Z^2\left[\frac{3}{4}\right]$

$$\Delta E_{3-2} = 13.6 \times Z^2\left[\frac{1}{4} - \frac{1}{9}\right] = 13.6 \times Z^2\left[\frac{5}{36}\right]$$

$$\therefore \qquad \Delta E_2 = \Delta E_{3-2} + 74.8$$

$$13.6 \times Z^2 \left[\frac{3}{4}\right] = 13.6 \times Z^2 \left[\frac{5}{36}\right] + 74.8$$

$$13.6 \times Z^2 \left[\frac{3}{4} - \frac{5}{36}\right] = 74.8$$

$$Z^2 = 9$$

$$\therefore \qquad Z = 3$$

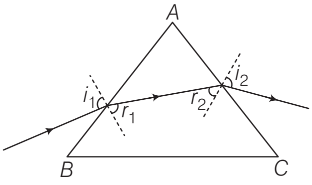
6. **(a,b,c)** The minimum deviation produced by a prism

$$\delta_m = 2i - A = A$$

$\therefore \; i_1 = i_2 = A$ and $r_1 = r_2 = A/2 \Rightarrow r_1 = i_1/2$

Now, using Snell's law

$$\sin A = \mu \sin A/2 \Rightarrow \mu = 2\cos(A/2)$$



For this prism when the emergent ray at the second surface is tangential to the surface

$$i_2 = \pi/2 \Rightarrow r_2 = \theta_c \Rightarrow r_1 = A - \theta_c$$

so, $\qquad \sin i_1 = \mu \sin(A - \theta_c)$

so, $i_1 = \sin^{-1}\left[\sin A\sqrt{4\cos^2\frac{A}{2} - 1} - \cos A\right]$

For minimum deviation through isosceles prism, the ray inside the prism is parallel to the base of the prism if

$$\angle B = \angle C.$$

But it is not necessarily parallel to the base if,

$$\angle A = \angle B \text{ or } \angle A = \angle C$$

7. **(8)** But this value of refractive index is not possible.

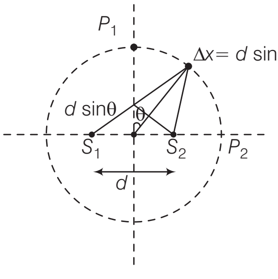
$$1.6 \sin\theta = (n - m\Delta n)\sin 90°$$

$$1.6 \sin\theta = n - m\Delta n$$

$$1.6 \times \frac{1}{2} = 1.6 - m(0.1) \Rightarrow 0.8 = 1.6 - m(0.1)$$

$$m \times 0.1 = 0.8 \Rightarrow m = 8$$

8. **(c,d)**



at $P_1 \qquad \Delta x = 0$

at $P_2 \qquad \Delta x = 1.8 \text{ mm} = n\lambda$

Number of maximas will be

$$n = \frac{\Delta x}{\lambda} = \frac{1.8 \text{ mm}}{600 \text{ nm}} = 3000$$

at $P_2 \qquad \Delta x = 3000\lambda$

Hence, bright fringe will be formed.

At P_2, 3000th maxima is formed.

For (a) option

$$\Delta x = d\sin\theta \Rightarrow d\Delta x = d\cos d\theta$$

$$R\lambda = d\cos\theta R d\theta \Rightarrow Rd\theta = \frac{R\lambda}{d\cos\theta}$$

As we move from P_1 to $P_2 \quad \theta \uparrow \cos\theta \downarrow Rd\theta \uparrow$

9. **(5)** $1^{131} \xrightarrow[T_{1/2} = 8 \text{Days}]{} \text{Xe}^{131} + \beta$

$$A_0 = 2.4 \times 10^5 \text{ Bq} = \lambda N_0$$

Let the volume is $V, t = 0$

$$A_0 = \lambda N_0 \Rightarrow t = 11.5 \text{ h}, \; A = \lambda N$$

$$115 = \lambda\left(\frac{N}{V} \times 2.5\right) \qquad 115 = \frac{\lambda}{V} \times 2.5 \times (N_0 e^{-\lambda t})$$

$$115 = \frac{(N_0\lambda)}{V} \times (2.5) \times e^{-\frac{\ln 2}{8 \text{ day}}(11.5 \text{ h})}$$

$$115 = \frac{(2.4 \times 10^5)}{V} \times (2.5) \times e^{-1/24}$$

$$V = \frac{2.4 \times 10^5}{115} \times 2.5\left[1 - \frac{1}{24}\right]$$

$$= \frac{2.4 \times 10^5}{115} \times 2.5\left[\frac{23}{24}\right]$$

$$= \frac{10^5 \times 23 \times 25}{115 \times 10^2} = 5 \times 10^3 \text{ ml} = 5 \text{ L}$$

10. **(d)** According to photoelectric effect equation

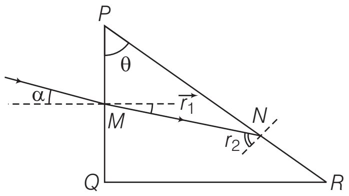
$$\text{KE}_{\max} = \frac{hc}{\lambda} - \phi_0 \Rightarrow \frac{p^2}{2m} = \frac{hc}{\lambda} - \phi_0 \quad [\text{KE} = p^2/2m]$$

$$\frac{(h/\lambda_d)^2}{2m} = \frac{hc}{\lambda} - \phi_0 \qquad [p = h/\lambda]$$

Assuming small changes, differentiating both sides,

$$\frac{h^2}{2m}\left(-\frac{2d\lambda_d}{\lambda_d^3}\right) = -\frac{hc}{\lambda^2}d\lambda \Rightarrow \frac{d\lambda_d}{d\lambda} \propto \frac{\lambda_d^3}{\lambda^2}$$

11. **(a)** According to question,



Applying Snell's law at M,

$$n = \frac{\sin\alpha}{\sin r_1} \Rightarrow \sqrt{2} = \frac{\sin 45^\circ}{\sin r_1}$$

$$\Rightarrow \quad \sin r_1 = \frac{\sin 45^\circ}{\sqrt{2}} = \frac{1/\sqrt{2}}{\sqrt{2}} = \frac{1}{2} \quad r_1 = 30^\circ$$

$$\sin\theta_c = \frac{1}{n} = \frac{1}{\sqrt{2}} \Rightarrow \theta_c = 45^\circ$$

Let us take $r_2 = \theta_c = 45^\circ$ for just satisfying the condition of TIR. In ΔPNM,

$$\theta + 90^\circ + r_1 + 90^\circ - r_2 = 180^\circ$$

or $\quad \theta = r_2 - r_1 = 45^\circ - 30^\circ = 15^\circ$

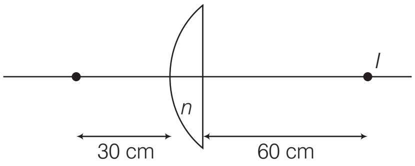
Note If $\alpha > 45^\circ$ (the given value). Then, $r_1 > 30^\circ$ (the obtained value)

$\therefore \quad r_2 > \theta_c$ (as $r_2 - r_1 = \theta$

or $\quad r_2 = \theta + r_1$)

or TIR will take place. So, for taking TIR under all conditions α should be greater than 45° or this is the minimum value of α.

12. (a,d) **Case 1**



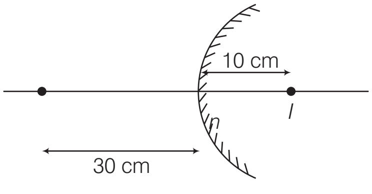
Using lens formula,

$$\frac{1}{60} + \frac{1}{30} = \frac{1}{f_1} \Rightarrow \frac{1}{f_1} = \frac{1}{60} + \frac{2}{60} \Rightarrow f_1 = 20 \text{ cm}$$

$$\text{Further, } \frac{1}{f_1} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right)$$

$$\Rightarrow \quad f_1 = \frac{R}{n-1} = +20 \text{ cm}$$

Case 2



Using mirror formula,

$$\frac{1}{10} - \frac{1}{30} = \frac{1}{f_2} \Rightarrow \frac{3}{30} - \frac{1}{30} = \frac{1}{f_2} = \frac{2}{30}$$

$$f_2 = 15 = \frac{R}{2} \Rightarrow R = 30 \Rightarrow R = 30 \text{ cm}$$

$$\frac{R}{n-1} = +20 \text{ cm} = \frac{30}{n-1}$$

$$\Rightarrow \quad 2n - 2 = 3 \Rightarrow f_1 = +20 \text{ cm}$$

Refractive index of lens is 2.5.

Radius of curvature of convex surface is 30 cm.

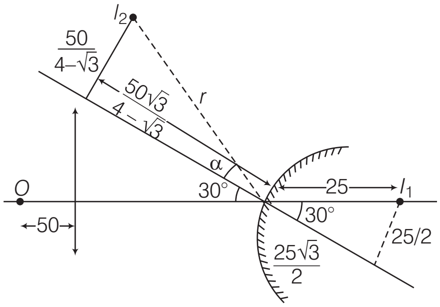
Faint image is erect and virtual. Focal length of lens is 20 cm.

13. (a,c,d) From Snell's law, $n \sin\theta$ = constant

$$\therefore \quad n_1 \sin\theta_i = n_2 \sin\theta_f$$

Further, I will depend on n_1 and $n(z)$. But it will be independent of n_2.

14. (d)



$$\textbf{For Lens} \quad \frac{1}{v} - \frac{1}{u} = \frac{1}{f} \Rightarrow v = \frac{uf}{u+f}$$

$$\Rightarrow \quad v = \frac{(-50)(30)}{-50+30} = 75 \text{ cm}$$

$$\textbf{For Mirror} \quad \frac{1}{v} + \frac{1}{u} = \frac{1}{f} \Rightarrow v = \frac{uf}{u-f}$$

$$\Rightarrow \quad v = \frac{\left(\frac{25\sqrt{3}}{2}\right)(50)}{\frac{25\sqrt{3}}{2} - 50} = \frac{-50\sqrt{3}}{4-\sqrt{3}} \text{ cm}$$

$$m = -\frac{v}{u} = \frac{h_2}{h_1} \Rightarrow h_2 = -\left(\frac{\frac{-50\sqrt{3}}{4-\sqrt{3}}}{\frac{25\sqrt{3}}{2}}\right) \cdot \frac{25}{2}$$

$$h_2 = \frac{+50}{4-\sqrt{3}} \text{ cm}$$

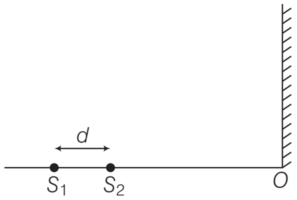
The x-coordinate of the images

$$= 50 - v\cos 30^\circ + h_2 \cos 60^\circ \approx 25 \text{ cm}$$

The y-coordinate of the images

$$= v\sin 30^\circ + h_2 \sin 60^\circ \approx 25\sqrt{3} \text{ cm}$$

15. (a,b)



Path difference at point $O = d = 0.6003$ mm

$= 600300$ nm.

This path difference is equal to $\left(1000\lambda + \frac{\lambda}{2}\right)$.

$\Rightarrow$ Minima is formed at point O.

Line S_1S_2 and screen are $\perp$ to each other so fringe pattern is circular (semi-circular because only half of screen is available)

16. **(b)** $\frac{hc}{\lambda} - \phi = eV_0$ (ϕ = work function)

$$\frac{hc}{0.3 \times 10^{-6}} - \phi = 2e \quad \dots(i)$$

$$\frac{hc}{0.4 \times 10^{-6}} - \phi = 1e \quad \dots(ii)$$

Subtracting Eq. (ii) from Eq. (i)

$$hc\left(\frac{1}{0.3} - \frac{1}{0.4}\right)10^6 = e \Rightarrow hc\left(\frac{0.1}{0.12} \times 10^6\right) = e$$

$$h = 0.64 \times 10^{-33} = 6.4 \times 10^{-34} \text{ J-s}$$

17. **(a,b,d)** As radius $r \propto \frac{n^2}{z}$

$$\Rightarrow \frac{\Delta r}{r} = \frac{\left(\frac{n+1}{z}\right)^2 - \left(\frac{n}{z}\right)^2}{\left(\frac{n}{z}\right)^2} = \frac{2n+1}{n^2} \approx \frac{2}{n} \propto \frac{1}{n}$$

$$\text{Energy } E \propto \frac{z^2}{n^2} \Rightarrow \frac{\Delta E}{E} = \frac{\frac{z^2}{n^2} - \frac{z^2}{(n+1)^2}}{\frac{z^2}{(n+1)^2}}$$

$$= \frac{(n+1)^2 - n^2}{n^2 \cdot (n+1)^2} \cdot (n+1)^2 \Rightarrow \frac{\Delta E}{E} = \frac{2n+1}{n^2} \simeq \frac{2n}{n^2} \propto \frac{1}{n}$$

Angular momentum $L = \frac{nh}{2\pi}$

$$\Rightarrow \frac{\Delta L}{L} = \frac{\frac{(n+1)h}{2\pi} - \frac{nh}{2\pi}}{\frac{nh}{2\pi}} = \frac{1}{n} \propto \frac{1}{n}$$

18. **(6)** Energy of incident light (in eV)

$$E = \frac{12375}{970} = 12.7 \text{ eV}$$

After excitation, let the electron jumps to nth state, then

$$\frac{-13.6}{n^2} = -13.6 + 12.7$$

Solving this equation, we get $n = 4$

$\therefore$ Total number of lines in emission spectrum,

$$= \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$

19. **(c)** Using the relation, $R = R_0\left(\frac{1}{2}\right)^n$

Here, R is activity of radioactive substance, R_0 initial activity and n is number of half lives.

$$1 = 64\left(\frac{1}{2}\right)^n$$

Solving we get, $n = 6$

Now, $t = n(t_{1/2}) = 6(18 \text{ days}) = 108$ days

20. **(c)** Electrostatic energy = Binding energy of N – Binding energy of O

$= \{[7M_H + 8M_n - M_N] - [8M_H + 7M_n - M_O]\} \times C^2$

$= [-M_H + M_n + M_O - M_N]\, C^2$

$= [-1.007825 + 1.008665 + 15.003065 - 15.000109] \times 93.15$

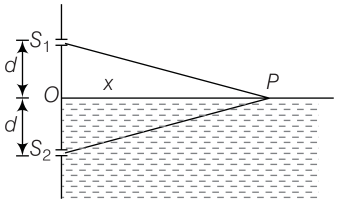
$= +\ 3.5359$ MeV

$$\Delta E = \frac{3}{5} \times \frac{1.44 \times 8 \times 7}{R} - \frac{3}{5} \times \frac{1.44 \times 7 \times 6}{R}$$

$= 3.5359$ MeV

$$R = \frac{3 \times 1.44 \times 14}{5 \times 3.5359} = 3.42 \text{ fm}$$

21. **(3)**



$$\mu(S_2P) - S_1P = m\lambda$$

$$\Rightarrow \mu\sqrt{d^2 + x^2} - \sqrt{d^2 + x^2} = m\lambda$$

$$\Rightarrow (\mu - 1)\sqrt{d^2 + x^2} = m\lambda$$

$$\Rightarrow \left(\frac{4}{3} - 1\right)\sqrt{d^2 + x^2} = m\lambda \quad \text{or} \quad \sqrt{d^2 + x^2} = 3m\lambda$$

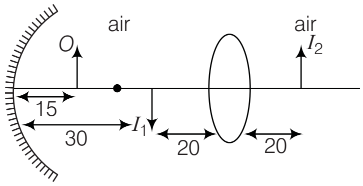
Squaring this equation we get,

$$x^2 = 9m^2\lambda^2 - d^2 \Rightarrow p^2 = 9 \quad \text{or} \quad p = 3$$

22. **(7)** ***Case* I** Reflection from mirror

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \Rightarrow \frac{1}{-10} = \frac{1}{v} + \frac{1}{-15}$$

$$\Rightarrow v = -30$$

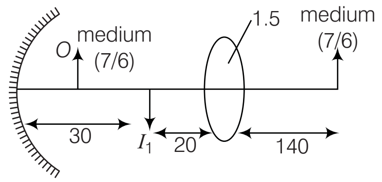


For lens $\frac{1}{f} = \frac{1}{v} - \frac{1}{u} \Rightarrow \frac{1}{10} = \frac{1}{v} - \frac{1}{-20} \Rightarrow v = 20$

$$|M_1| = \left|\frac{v_1}{u_1}\right|\left|\frac{v_2}{u_2}\right| = \left(\frac{30}{15}\right)\left(\frac{20}{20}\right)$$

$= 2 \times 1 = 2$ (in air)

***Case* II** For mirror, there is no change. $v = -30$



$$\text{For lens,}\quad \frac{1}{f_{\text{air}}} = \left(\frac{3/2}{1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_{\text{medium}}} = \left(\frac{3/2}{7/6} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

with $f_{\text{air}} = 10$ cm

$$\text{We get}\quad \frac{1}{f_{\text{medium}}} = \frac{4}{70}\,\text{cm}^{-1} \Rightarrow \frac{1}{v} - \frac{1}{-20} = \frac{4}{70}$$

$$\frac{1}{v} + \frac{1}{20} = \left(\frac{2}{7}\right)\left(\frac{2}{10}\right) = \frac{4}{70} \Rightarrow \frac{1}{v} = \frac{4}{70} - \frac{1}{20}$$

$v = 140,$

$$|M_2| = \left|\frac{v_1}{u_1}\right|\left|\frac{v_2}{u_2}\right| = \left(\frac{30}{15}\right)\left(\frac{140}{20}\right),$$

$$= (2)\left(\frac{140}{20}\right) = 14 \Rightarrow \left|\frac{M_2}{M_1}\right| = \frac{14}{2} = 7$$

23. **(b)** $R = 10$ cm

$$\text{Applying}\quad \frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}\quad\text{two times}$$

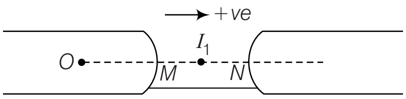
$$\frac{1}{v} - \frac{1.5}{-50} = \frac{1-1.5}{-10}$$

$$\Rightarrow \quad \frac{1}{v} + \frac{1.5}{50} = \frac{0.5}{10}$$

$$\frac{1}{v} = \frac{0.5}{10} - \frac{1.5}{50} = \frac{2.5 - 1.5}{50}$$

$$\Rightarrow \quad v = 50$$

$MN = d$, $MI_1 = 50$ cm $\Rightarrow$ $NI_1 = (d - 50)$ cm



$$\text{Again,}\quad \frac{1.5}{\infty} - \frac{1}{-(d-50)} = \frac{1.5-1}{10}$$

$$\frac{1}{d-50} = \frac{1}{20} \Rightarrow d = 70$$

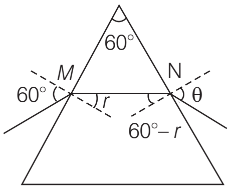
24. **(2)** Applying Snell's law at *M* and *N*,

$$\sin 60° = n \sin r \quad \text{...(i)}$$

$$\sin\theta = n\sin(60 - r) \quad \text{...(ii)}$$

Differentiating we get

$$\cos\theta\,\frac{d\theta}{dn} = -n\cos(60-r)\,\frac{dr}{dn} + \sin(60-r)$$



$$\text{Differentiating Eq. (i),}\ n\cos r\,\frac{dr}{dn} + \sin r = 0$$

$$\text{or}\quad \frac{dr}{dn} = -\frac{\sin r}{n\cos r} = \frac{-\tan r}{n}$$

$$\Rightarrow \cos\theta\,\frac{d\theta}{dn} = -n\cos(60°-r)\left(\frac{-\tan r}{n}\right) + \sin(60°-r)$$

$$\frac{d\theta}{dn} = \frac{1}{\cos\theta}\left[\cos(60°-r)\tan r + \sin(60°-r)\right]$$

Form Eq. (i), $r = 30°$ for $n = \sqrt{3}$

$$\frac{d\theta}{dn} = \frac{1}{\cos 60}(\cos 30 \times \tan 30 + \sin 30) = 2$$

25. **(a)** $\frac{4}{3}\sin i = \frac{\sqrt{45}}{4}\sin(90 - \theta_c) = \frac{\sqrt{45}}{4}\cos\theta_c$

$$\sin\theta_c = \frac{n_2}{n_1} \Rightarrow \cos\theta_c = \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}$$

$$\Rightarrow \frac{4}{3}\sin i = \frac{\sqrt{45}}{4}\,\frac{3}{\sqrt{45}};\ \sin i = \frac{9}{16}$$

$$\text{In second case,}\ \sin\theta_c = \frac{n_2}{n_1} = \frac{7}{8}$$

$$\Rightarrow \cos\theta_c = \frac{\sqrt{15}}{8} \Rightarrow \frac{16}{3\sqrt{15}}\sin i = \frac{8}{5}\sin(90 - \theta_c)$$

Simplifying we get, $\sin i = \frac{9}{16}$

(a) is correct.

26. **(d)** $\sin i_m = n_1 \sin(90 - \theta_c)$

$$\Rightarrow \sin i_m = n_1\cos\theta_c \Rightarrow NA = n_1\sqrt{1 - \sin^2\theta_c}$$

$$= n_1\sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{n_1^2 - n_2^2}$$

Substituting the values we get,

$$NA_1 = \frac{3}{4}\ \text{and}\ NA_2 = \frac{\sqrt{15}}{5} = \sqrt{\frac{3}{4}} \Rightarrow NA_2 < NA_1$$

Therefore, the numerical aperture of combined structure is equal to the lesser of the two numerical aperture, which is NA_2.

27. **(3)** Let initial power available from the plant is P_0. After time $t = nT$ or n half lives, this will become $\left(\frac{1}{2}\right)^n P_0$.

Now, it is given that,

$$\left(\frac{1}{2}\right)^n P_0 = 12.5\% \text{ of } P_0 = (0.125)\, P_0$$

Solving this equation we get, $n = 3$

28. A-(p, q), B- (p, r), C-(p, s) D- (p, q, r)

29. **(2)** Let initial numbers are N_1 and N_2.

$$\frac{\lambda_1}{\lambda_2} = \frac{\tau_2}{\tau_1} = \frac{2\tau}{\tau} = 2 = \frac{T_2}{T_1} \qquad (T = \text{Half life})$$

$$A = \frac{-dN}{dt} = \lambda N$$

Initial activity is same

$$\therefore \qquad \lambda_1 N_1 = \lambda_2 N_2 \qquad \ldots\text{(i)}$$

Activity at time t,

$$A = \lambda N = \lambda N_0\, e^{-\lambda t} \Rightarrow A_1 = \lambda_1 N_1\, e^{-\lambda_1 t}$$

$$\Rightarrow \quad R_1 = -\frac{dA_1}{dt} = \lambda_1^2\, N_1\, e^{-\lambda_1}$$

Similarly, $R_2 = \lambda_2^2\, N_2\, e^{-\lambda_2 t}$

After $t = 2\tau$ $\quad \lambda_1 t = \frac{1}{\tau_1}(t) = \frac{1}{\tau}(2\tau) = 2$

$$\lambda_2 t = \frac{1}{\tau_2}(t) = 1 = \frac{1}{2\tau}(2\tau) = 1$$

$$\frac{R_P}{R_Q} = \frac{\lambda_1^2 N_1 e^{-\lambda_1 t}}{\lambda_2^2 N_2 e^{-\lambda_2 t}} \Rightarrow \frac{R_P}{R_Q} = \frac{\lambda_1}{\lambda_1}\left(\frac{e^{-2}}{e^{-1}}\right) = \frac{2}{e}$$

30. **(2)** Angular momentum $= n\left(\frac{h}{2\pi}\right) = 3\left(\frac{h}{2\pi}\right)$

$$\therefore \qquad n = 3$$

Now, $\quad r_n \propto \frac{n^2}{z} \Rightarrow r_3 = \frac{(3)^2}{3}(a_0) = 3a_0$

Now, $\quad mv_3 r_3 = 3\left(\frac{h}{2\pi}\right) \Rightarrow mv_3(3a_0) = 3\left(\frac{h}{2\pi}\right)$

or $\quad \frac{h}{mv_3} = 2\pi a_0 \Rightarrow \frac{h}{P_3} = 2\pi a_0 \qquad (P = mv)$

or $\quad \lambda_3 = 2\pi a_0 \qquad \left(\lambda = \frac{h}{P}\right)$

$\therefore$ Answer is 2.

31. **(d)** From conservation laws of mass number and atomic number, we can say that $x = n$, $y = n$

$$(x = {}^1_0 n,\ y = {}^1_0 n)$$

$\therefore$ Only (a) and (d) options may be correct.

From conservation of momentum, $|P_{\text{xe}}| = |P_{\text{sr}}|$

From $K = \frac{P^2}{2m} \Rightarrow K \propto \frac{1}{m}$

$$\frac{K_{\text{sr}}}{K_{\text{xe}}} = \frac{m_{\text{xe}}}{m_{\text{sr}}} \Rightarrow K_{\text{sr}} = 129\text{ MeV} \Rightarrow K_{\text{xe}} = 86\text{ MeV}$$

32. **(a, b, c)** Fringe width $\beta = \frac{\lambda D}{d}$ or $\beta \propto \lambda \Rightarrow \lambda_2 > \lambda_1$

So $\qquad \beta_2 > \beta_1$

Number of fringes in a given width

$$m = \frac{Y}{\beta} \text{ or } m \propto \frac{1}{\beta} \Rightarrow m_2 < m_1 \text{ as } \beta_2 > \beta_1$$

Distance of 3rd maximum of λ_2 from central maximum

$$= \frac{3\lambda_2 D}{d} = \frac{1800D}{d}$$

Distance of 5th minimum of λ_1 from central maximum

$$= \frac{9\lambda_1 D}{2d} = \frac{1800D}{d}$$

So, 3rd maximum of λ_2 will overlap with 5th minimum of λ_1

Angular separation (or angular fringe width) $= \frac{\lambda}{d} \propto \lambda$

$\Rightarrow$ Angular separation for λ_1 will be lesser.

33. **(a,c)** $\frac{1}{f_{\text{film}}} = (n_1 - 1)\left(\frac{1}{R} - \frac{1}{R}\right) \Rightarrow f_{\text{film}} = \infty$ (infinite)

$\therefore$ There is no effect of presence of film.

From Air to Glass

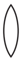
Using the equation $\frac{n_2}{v} - \frac{1}{u} = \frac{n_2 - 1}{R}$

$$\frac{1.5}{v} - \frac{1}{\infty} = \frac{1.5 - 1}{R} \Rightarrow v = 3R \Rightarrow f_1 = 3R$$

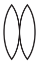
From Glass to Air Again using the same equation

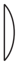
$$\frac{1}{v} - \frac{n_2}{u} = \frac{1 - n_2}{-R} \Rightarrow \frac{1}{v} - \frac{1.5}{\infty} = \frac{1 - 1.5}{-R}$$

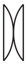
$$\Rightarrow \qquad v = 2R \Rightarrow f_2 = 2R$$

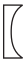
34. **(b)** (P)  $\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{r} + \frac{1}{r}\right) = \frac{1}{r}$

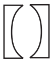
$$\Rightarrow \qquad f = r$$

$\Rightarrow$  $\frac{1}{f_{\text{eq}}} = \frac{1}{f} + \frac{1}{f} = \frac{2}{r} \Rightarrow f_{\text{eq}} = \frac{r}{2}$

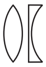
(Q)  $\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{r}\right) \Rightarrow f = 2r$

$\Rightarrow$  $\frac{1}{f} + \frac{1}{f} = \frac{2}{f} = \frac{1}{r} \Rightarrow f_{\text{eq}} = r$

(R)  $\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(-\frac{1}{r}\right) = -\frac{1}{2r}$

$\Rightarrow$  $f = -2r$

$$\Rightarrow \qquad \frac{1}{f_{\text{eq}}} = \frac{1}{f} + \frac{1}{f} = -\frac{2}{2r} \Rightarrow f_{\text{eq}} = -r$$

(S)  $\Rightarrow \frac{1}{f_{\text{eq}}} = \frac{1}{r} + \frac{1}{-2r} = \frac{1}{2r} \Rightarrow f_{\text{eq}} = 2r$

35. **(a)** Energy corresponding to 248 nm wavelength

$$= \frac{1240}{248}\text{ eV} = 5\text{ eV}$$

Energy corresponding to 310 nm wavelength

$$= \frac{1240}{310} \text{ eV} = 4 \text{ eV}$$

$$\frac{\text{KE}_1}{\text{KE}_2} = \frac{u_1^2}{u_2^2} = \frac{4}{1} = \frac{5 \text{ eV} - W}{4 \text{ eV} - W}$$

$$\Rightarrow \quad 16 - 4W = 5 - W \Rightarrow 11 = 3W$$

$$\Rightarrow \quad W = \frac{11}{3} = 3.67 \text{ eV} \cong 3.7 \text{ eV}$$

36. (b) K_α transition takes place from $n_1 = 2$ to $n_2 = 1$

$$\therefore \quad \frac{1}{\lambda} = R(Z - b)^2 \left[\frac{1}{(1)^2} - \frac{1}{(2)^2}\right]$$

For K-series, $b = 1 \Rightarrow \frac{1}{\lambda} \propto (Z - 1)^2$

$$\Rightarrow \quad \frac{\lambda_{\text{Cu}}}{\lambda_{\text{Mo}}} = \frac{(z_{\text{Mo}} - 1)^2}{(z_{\text{Cu}} - 1)^2} = \frac{(42 - 1)^2}{(29 - 1)^2}$$

$$= \frac{41 \times 41}{28 \times 28} = \frac{1681}{784} = 2.14$$

37. (c) $\mu = \frac{\lambda_{\text{air}}}{\lambda_{\text{medium}}} = \frac{1}{(2/3)} = \frac{3}{2}$

Further, $|m| = \frac{1}{3} = \left|\frac{v}{u}\right|$

$$\therefore \quad |v| = \frac{|u|}{3} \Rightarrow u = -24 \text{ m} \qquad \text{(Real object)}$$

$$\therefore \quad v = +8 \text{ m} \qquad \text{(Real image)}$$

Now, $\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = (\mu - 1)\left(\frac{1}{+R} - \frac{1}{\infty}\right)$

$$\therefore \quad \frac{1}{8} + \frac{1}{24} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{R}\right) \Rightarrow R = 3 \text{ m}$$

38. (a) Component along the plane $= \frac{1}{2}$

and component perpendicular to the plane $= \frac{\sqrt{3}}{2}$

$$\therefore \quad \tan i = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$$

$\therefore$ $i = 30° =$ angle of incidence

39. (b) $I = I_{\max} \cos^2 \frac{\phi}{2}$ …(i)

Given, $I = \frac{I_{\max}}{2}$ …(ii)

$\therefore$ From Eqs. (i) and (ii), we have

$$\phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

Or path difference, $\Delta x = \left(\frac{\lambda}{2\pi}\right) \cdot \phi$

$$\therefore \quad \Delta x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots \left(\frac{2n + 1}{4}\right)\lambda$$

40. (d) For $e \to i$

$$45° > \theta_c \Rightarrow \sin 45° > \sin \theta_c$$

$$\frac{1}{\sqrt{2}} > \frac{\mu_2}{\mu_1} \Rightarrow \mu_1 > \sqrt{2\mu_2}, \quad \text{(s-1)}$$

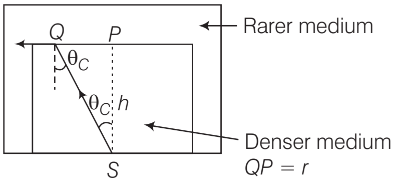
For $e \to f$

angle of refraction is lesser than angle of incidence, so $\mu_2 > \mu_1$ and then $\mu_2 > \mu_3$ (P-2)

For $e \to g$, $\mu_1 = \mu_2$ (Q-3)

for $e \to h$, $\mu_2 < \mu_1 < \sqrt{2}\mu_2$ and $\mu_2 > \mu_3$ (R-4)

41. (c) At point Q angle of incidence is critical angle θ_C, where



$$\sin \theta_C = \frac{\mu_l}{\mu_{\text{block}}}$$

In ΔPQS, $\sin \theta_C = \frac{r}{\sqrt{r^2 + h^2}} \Rightarrow \frac{\mu_l}{\mu_{\text{block}}} = \frac{r}{\sqrt{r^2 + h^2}}$

$$\Rightarrow \mu_l = \frac{\rho}{\sqrt{\rho^2 + \eta^2}} \times 2.72 = \frac{5.77}{11.54} \times 2.72 = 1.36$$

42. (b) Final momentum of object $= \frac{\text{Power} \times \text{time}}{\text{Speed of light}}$

$$= \frac{30 \times 10^{-3} \times 100 \times 10^{-9}}{3 \times 10^8} = 1.0 \times 10^{-17} \text{ kg-m/s}^{-1}$$

43. (4) Number of nuclei decayed in time t,

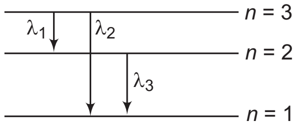
$$N_d = N_0(1 - e^{-\lambda t})$$

$$\therefore \ \% \text{ decayed} = \left(\frac{N_d}{N_0}\right) \times 100 = (1 - e^{-\lambda t} d) \times 100 \quad \ldots\text{(i)}$$

Here, $\lambda = \frac{0.693}{1386} = 5 \times 10^{-4} \text{ s}^{-1}$

$$\therefore \% \text{ decayed} \approx (\lambda t) \times 100 = (5 \times 10^{-4})(80)(100) = 4$$

44. (a, c) $L = 3\left(\frac{h}{2\pi}\right)$



$$\therefore \quad n = 3, \text{ as } L = n\left(\frac{h}{2\pi}\right) \Rightarrow r_n \propto \frac{n^2}{z}$$

$$r_3 = 4.5a_0 \ \therefore \ z = 2$$

$$\frac{1}{\lambda_1} = Rz^2\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = 4R\left(\frac{1}{4} - \frac{1}{9}\right)$$

$$\therefore \quad \lambda_1 = \frac{9}{5R}$$

$$\Rightarrow \quad \frac{1}{\lambda_2} = Rz^2\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = 4R\left(1 - \frac{1}{9}\right)$$

$$\Rightarrow \quad \lambda_2 = \frac{9}{32R}$$

$$\Rightarrow \quad \frac{1}{\lambda_3} = Rz^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = 4R\left(1 - \frac{1}{4}\right)$$

$$\Rightarrow \quad \lambda_3 = \frac{1}{3R}$$

45. (a) ${}_3\text{Li}^7 \rightarrow {}_2\text{He}^4 + {}_1\text{H}^3$

$$\Delta m = [\text{M}_{\text{Li}} - \text{M}_{\text{He}} - \text{M}_{\text{H}^3}]$$
$$= [6.01513 - 4.002603 - 3.016050]$$
$$= -\ 1.003523\ \text{u}$$

Δm is negative so reaction is not possible.

(b) ${}_{84}\text{Po}^{210} \rightarrow {}_{83}\text{Bi}^{209} + {}_1\text{P}^1$

Δm is negative so reaction is not possible.

(c) ${}_1\text{H}^2 \rightarrow {}_2\text{He}^4 + {}_3\text{Li}^6$

Δm is positive so reaction is possible.

(d) ${}_{30}\text{Zn}^{70} + {}_{34}\text{Se}^{82} \rightarrow {}_{64}\text{Gd}^{152}$

Δm is positive so reaction is not possible.

46. (a) ${}_{84}\text{Po}^{210} \longrightarrow {}_2\text{He}^4 + {}_{82}\text{Pb}^{206}$

Mass defect $\Delta m = (m_{\text{Po}} - M_{\text{He}} - m_{\text{Pb}}) = 0.005818$ u

$$\therefore \quad Q = (\Delta m)\,(931.48)\ \text{MeV}$$
$$= 5.4193\ \text{MeV} = 5419\ \text{keV}$$

From conservation of linear momentum, $p_{\text{Pb}} = p_\alpha$

$$\therefore \quad \sqrt{2m_{\text{Pb}}\,k_{\text{Pb}}} = \sqrt{2m_\alpha\,k_\alpha}$$

$$\Rightarrow \quad \frac{k_\alpha}{k_{\text{Pb}}} = \frac{m_{\text{Pb}}}{m_\alpha} = \frac{206}{4}$$

$$\therefore \quad k_\alpha = \left(\frac{206}{206 + 4}\right)(k_{\text{total}})$$
$$= \left(\frac{206}{210}\right)(5419) = 5316\ \text{keV}$$

47. (9) ${}^{12}_{5}B \longrightarrow {}^{12}_{6}C + {}^{0}_{-1}e + \bar{\nu}$

Mass of ${}^{12}_{6}\text{C} = 12.000$ u (by definition of 1 a.m.u.)

Q-value of reaction, $Q = (M_B - M_C) \times c^2$

$= (12.014 - 12.000) \times 931.5 = 13.041$ MeV

4.041 MeV of energy is taken by ${}^{12}_{6}\text{C}$ *

$\Rightarrow$ Maximum KE of β-particle is
$(13.041 - 4.041) = 9$ MeV